The Partition Function and Ramanujan Congruences

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Chapter 1

Introduction

The partition function, $p(n)$, for a positive integer n is the number of non-increasing sequences of positive integers which sum to n. Specifically,

$$p(n) = \left| \left\{ (a_0, a_1, \cdots, a_k) | \text{ where } a_i \geq a_{i+1} \text{ for all } 0 \leq i \leq k, \text{ and } \sum_{i=0}^{k} a_i = n \right\} \right|.$$

The study into specific properties of the partition function has been a rich topic for number theorists for many years. Much of the current work involving the arithmetic properties of the partition function find their seed in some keen observations of Ramanujan.

In particular he discovered what are referred to as the ***Ramanujan Congruences*** of $p(n)$. These are appropriately named because Ramanujan was the first to notice these interesting properties of the partition function, [Ram00b],[Ram00d],[Ram00a],[Ram00c]. He found that for all n $\in \mathbb{Z}$,

$$p(5n+4) \equiv 0 \pmod 5 \tag{1.1}$$
$$p(7n+5) \equiv 0 \pmod 7 \tag{1.2}$$
$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.3}$$

In addition to noticing these peculiar relations, he conjectured that (1.1), (1.2), and (1.3) were the only congruences of this form. In particular, these ***Ramanujan Congruences*** are the only congruences of the form

$$p(ln+\beta) \equiv 0 \pmod l$$

for all $n \in \mathbb{Z}$, l prime, and some fixed $\beta \in \mathbb{Z}$.

In his own words, *"It appears that there are no equally simple properties for any moduli involving primes other than these three."* It is this question put forth by Ramanujan that we will consider in this paper. In particular, the main topic of this paper will be the following theorem of Ahlgren and Boylan [AB03]:

Theorem. 1 *Suppose that l is prime. If there is a Ramanujan congruence modulo l, then the congruence must be one of (1.1), (1.2), or (1.3).*

Historically around the same time as results for the non-existence of Ramanujan congruences were being found, mathematicians were searching for non-Ramanujan congruences that did exist. Before the work of Algren and Boylan the existence of non-Ramanujan congruences, families of the form $p(An + B) \equiv 0 \pmod{l}$, for the partition function was shown by Ono [Ono00]. Infinitely many were found and this idea was extended to moduli M coprime to 6 by Ahlgren [Ahl00]. Since then Boylan has continued in this direction by proving of the existence of Ramanujan congruences in powers of the partition generating function.

This paper will focus on providing the proof presented by Ahlgren and Boylan [AB03] for Ramaujan's conjecture. In Chapter 2 we will establish some of the background information that will be necessary to understand the partition function more clearly. After which, in Chapter 2 will we develop some essential information regarding modular forms. At first consideration, this seems out of topic when discussing $p(n)$, but we will see that the partition function is closely related to modular forms and therefore we can use some modular form theory to prove that Ramanujan was accurate in his conjecture. Once we have established these basic concepts around both the partition function and modular forms, in Chapter 3 we will follow a proof of Ramanujan's conjecture presented originally by Ahlgren and Boylan [AB03]. In Chapter 4 we will then briefly look at more recent work regarding Ramanujan Congruences.

Chapter 2

Preliminaries

2.1 The Partition Function

In this section we begin by presenting some interesting properties of the partition function, $p(n)$. The partition function, $p(n)$, is very important in number theory. It is connected to representation theory by enumerating the irreducible representations of the symmetric group S_n. We also see the partition function when dealing with combinatorics and counting arguments. In addition there are some connections to physics. $p(n)$ arises when discussing quantum field theory. Recall, that it is defined as the number of ways of writing n as a non-increasing sum of positive integers. For example,

$$\begin{aligned} 3 &= 3 \\ &= 2+1 \\ &= 1+1+1 \end{aligned}$$

$$\begin{aligned} 5 &= 5 \\ &= 4+1 \\ &= 3+2 \\ &= 3+1+1 \\ &= 2+2+1 \\ &= 2+1+1+1 \\ &= 1+1+1+1+1 \end{aligned}$$

therefore $p(3) = 3$ and $p(5) = 7$.

One of the more useful tools when working with the partition function is Euler's infinite product form for the generating function of $p(n)$,

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)}. \tag{2.1}$$

Equation (2.1) follows from considering the expansion of the infinite product using the geometric series,

$$\begin{aligned} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} &= \left(\frac{1}{1-q}\right) \cdot \left(\frac{1}{1-q^2}\right) \cdot \cdots \\ &= (1+q^1+q^{1+1}+\cdots) \cdot (1+q^2+q^{2+2}+\cdots) \cdots . \end{aligned}$$

Now we can see that the coefficient of q^n in this q-series is the number of different ways we can group the exponents as to make their sum n. This is exactly the the number of partitions of n, $p(n)$.

2.1.1 Additional Combinatorial Structures

There are many functions which are defined similarly in nature to the partition function that are of interest in combinatorics and number theory. We will define two additional combinatorial structures that will not be used for our primary result, but is relevant to some current work that will be introduced in the final chapter of this paper.

A **generalized Frobenius parition**, also called an **F-partition**, is a sequence in which a number n is represented as

$$n = r + \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i$$

where $\{a_i\}$ and $\{b_i\}$ are both strictly decreasing sequences of non-negative integers. It is common to see an F-partition represented as

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}.$$

An F-partition is said to be 2-colored if it is constructed from two copies of the non-negative intergers, written j_0 and j_1 with $j \geq 0$. We say $j_i < t_s$ if $j < t$ or both $j = t$ and $i < s$. Let $c\phi_2(n)$ denote the number of 2-colored F-partitions of n. For example, the nine two-colored Frobenius partitions of 2 are

$$\begin{pmatrix} 0_1 & 0_0 \\ 0_1 & 0_0 \end{pmatrix}, \begin{pmatrix} 1_1 \\ 0_1 \end{pmatrix}, \begin{pmatrix} 1_1 \\ 0_0 \end{pmatrix}, \begin{pmatrix} 1_0 \\ 0_1 \end{pmatrix}, \begin{pmatrix} 1_0 \\ 0_0 \end{pmatrix},$$

$$\begin{pmatrix} 0_1 \\ 1_1 \end{pmatrix}, \begin{pmatrix} 0_1 \\ 1_0 \end{pmatrix}, \begin{pmatrix} 0_0 \\ 1_1 \end{pmatrix}, \text{ and } \begin{pmatrix} 0_0 \\ 1_0 \end{pmatrix}.$$

An **overpartition** of n is a sum of non-increasing positive integers in which the first occurrence of an integer may be overlined. This will create more than $p(n)$ partitions, as there are distinctions between partitions where a specific term is overlined and where it is not. For example, the 14 over partitions of 4 are,

$$4, \bar{4}, 3+1, \bar{3}+1, 3+\bar{1}, \bar{3}+\bar{1}, 2+2, \bar{2}+2,$$

$$2+1+1, \bar{2}+1+1, 2+\bar{1}+1, \bar{2}+\bar{1}+1, 1+1+1+1, \bar{1}+1+1+1.$$

We then define $\bar{p}(n)$ as the number of overpartitions of n.

2.2 Modular Forms for $\mathbf{SL}_2(\mathbb{Z})$

Before we begin our proof for Ramanujan's conjecture regarding Ramanajan congruences we will need a little bit of background information from what seems at first as a distant mathematical topic. However, much of the machinery we will use in the proof of the non-existence of Ramanujan congruences is focused around the study of certain modular forms. We will therefore need to establish some of the basic definitions regarding modular forms before we continue. In this paper our main concern is for modular forms for $\mathrm{SL}_2(\mathbb{Z})$. See [Kob93] and [DS05] for more details.

To understand these functions we consider $\mathrm{SL}_2(\mathbb{Z})$, which is defined as the group of 2×2 matrices with integer entries and determinant 1. This is clearly a subgroup of the group of 2×2 invertible matrices with integer entries since the determinant function is multiplicative. The group $\mathrm{SL}_2(\mathbb{Z})$ is generated by two matrices

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \text{ and } S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

We say that a function $f : \mathcal{H} \mapsto \mathbb{C}$ is **holomorphic** on $\mathcal{H}$ if it is analytic on $\mathcal{H}$.

If f is analytic at ∞ we say that f is **holomorphic** at ∞.

A function $f : \mathcal{H} \mapsto \mathbb{C}$ is said to be **weakly modular of weight k for $\mathbf{SL}_2(\mathbb{Z})$** if

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \tag{2.2}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathcal{H}$.

A function $f : \mathcal{H} \mapsto \mathbb{C}$ is said to be a **modular form of weight k for $\mathbf{SL}_2(\mathbb{Z})$** if it is **holomorphic** on $\mathcal{H}$, **holomorphic** at ∞, and **weakly modular of weight k for $\mathbf{SL}_2(\mathbb{Z})$**. Being a modular form for $\mathrm{SL}_2(\mathbb{Z})$ implies that f has a Fourier series expansion at ∞, of the form

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n \quad (q := e^{2\pi i z}).$$

The set of all holomorphic modular forms of weight k forms a complex vector space. We will denote the vector space of modular forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$ as M_k. We will see a nice basis for M_k in this section.

The fact that S and T generate $\mathrm{SL}_2(\mathbb{Z})$ means we can understand the symmetries of modular forms by considering how these two generators act on points in $\mathcal{H}$. In particular the set $\{z \in \mathbb{C} \mid |z| \geq 1, -\frac{1}{2} \leq \mathrm{Im}(z) < \frac{1}{2}\}$ is the fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$.

If we consider (2.2) for T we see,

$$f\left(\frac{(1)z + (1)}{(0)z + (1)}\right) = ((0)z + (1))^k f(z),$$

$$f(z+1) = f(z).$$

In other words T tells us that modular forms on $\mathrm{SL}_2(\mathbb{Z})$ are periodic via translations, that is we can consider what they do within the strip of $\mathcal{H}$, $\{x + iy \in \mathbb{C} \mid -\frac{1}{2} \leq x \leq \frac{1}{2}\}$, to fully understand these functions on all of $\mathcal{H}$.

If we consider (2.2) for S we see the additional transformation,

$$f\left(\frac{(0)z + (-1)}{(1)z + (0)}\right) = ((1)z + (0))^k f(z),$$

$$f\left(-\frac{1}{z}\right) = z^k f(z).$$

We call a modular form, f, **normalized** if the first non-zero coefficient of the Fourier series expansion of $f(z)$ is 1. The vector space structure of M_k allows us to **normalize** our modular forms by dividing by the first non-zero coefficient in the Fourier series expansion, thus assuring that this term becomes 1. Due to the vector space structure we know that the normalized form is again a modular form. We call a modular form f of weight k with a zero constant term in the Fourier expansion of $f(z)$ at ∞, a **cusp form of weight** k.

An example of some of the more commonly seen modular forms for $\mathrm{SL}_2(\mathbb{Z})$ are the

Eisenstein Series. We can construct these modular forms [Kob93] for any even weight $k \geq 4$ by defining

$$E_k(z) = \frac{1}{2} \sum_{\substack{(m,n)=1 \\ m,n \in \mathbb{Z}}} \frac{1}{(mz+n)^k}.$$

If $k = 2$ we can construct an Eisenstein series in a similar fashion, but it is not weakly modular for $\mathrm{SL}_2(\mathbb{Z})$. The Fourier expansion of the Eisenstein series, $E_k(z)$, at ∞ is given by

$$E_k(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where $\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}$ and the B_k are the Bernoulli numbers defined as the coefficients of the series

$$\sum_{k=0}^{\infty} B_k \cdot \frac{t^k}{k!} = \frac{t}{e^t - 1} = 1 - \frac{1}{2}t + \frac{1}{12}t^2 - \ldots.$$

The first way of expressing the Eisenstein series is useful for understanding the analyticity of E_k while the second way allows us to easily see the coefficients of the q-series expansion of E_k. In light of this second way of looking at E_k we see that the following are the first 6 Eisentstein series,

$$\begin{aligned}
E_4(z) &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3 q^n, \\
E_6(z) &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5 q^n, \\
E_8(z) &= 1 + 480 \sum_{n=1}^{\infty} \sigma_7 q^n, \\
E_{10}(z) &= 1 - 264 \sum_{n=1}^{\infty} \sigma_9 q^n, \\
E_{12}(z) &= 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11} q^n, \\
E_{14}(z) &= 1 - 24 \sum_{n=1}^{\infty} \sigma_{13} q^n.
\end{aligned}$$

One interesting aspect of the Eisenstein Series is that they can be used to construct a basis for modular forms of a given weight. We will use this fact later on in Chapter 3. Koblitz [Kob93] shows the following theorem,

Theorem. 2 *Any $f \in M_k$ can be written in the form*

$$f(z) = \sum_{4i+6j=k} c_{i,j} E_4(z)^i E_6(z)^j.$$

Further, a basis for M_k, where k is even and $k \geq 4$ is given by

$$\left\{ E_4(z)^a E_6(z)^b \mid a, b \geq 0 \text{ and } 4a + 6b = k \right\}.$$

In fact, for k even, $dim M_k = \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & \text{if } k \not\equiv 2 \pmod{12}, \\ \lfloor \frac{k}{12} \rfloor & \text{if } k \equiv 2 \pmod{12}. \end{cases}$

One final example of a modular form for $\mathrm{SL}_2(\mathbb{Z})$ that is quite useful for illustrating the structure of M_k is

$$\Delta(z) := q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

This is a cusp form of weight 12, which is the lowest possible weight a non-constant cusp form for $\mathrm{SL}_2(\mathbb{Z})$ can have. Additionally the first non-zero coefficient in its q-series representation is 1, which implies that it is a normalized modular form.

2.3 Filtrations of Modular Forms

For this paper we will consider what happens to Fourier series expansions of modular forms that have integer coefficients when we reduce modulo primes. Let $l \geq 5$ be a prime. Let $f \in M_k \cap \mathbb{Z}[[q]]$, that is f is a modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$ with integer coefficients in the Fourier series expansion of f at ∞. Then we can take the coefficients of f modulo l and define

$$\tilde{f} := f \pmod{l}.$$

With this concept of reducing a function modulo a prime we can define

$$\tilde{M_k} := \{ \tilde{f} : f \in M_k \cap \mathbb{Z}[[q]] \}.$$

We then define the **filtration** of a modular form $f \in M_k \cap \mathbb{Z}[[q]]$ as

$$w(f) := \inf\{ k' : \tilde{f} \in \tilde{M_{k'}} \}.$$

In other words, if we consider all the modular forms which might be congruent to f modulo l regardless of weight, $w(f)$ is then defined to be the infimum of the weights of these forms.

Therefore in order to determine the filtration of a form f, we consider the question of when f is congruent modulo l to a modular form of lesser weight.

Now we consider $f \in M_k \cap \mathbb{Z}[[q]]$ and $g \in M_{k'} \cap \mathbb{Z}[[q]]$, with $\tilde{f} \equiv \tilde{g} \not\equiv 0 \pmod{l}$. Then $k \equiv k' \pmod{l-1}$. (See further [KO92] page 352)

It follows that if $\tilde{f} \not\equiv 0 \pmod{l}$ then $w(f) \equiv k \pmod{l-1}$. Furthermore, we see then that $w(f) = -\infty$ if and only if $\tilde{f} \equiv 0 \pmod{l}$.

Now that we have some preliminary results and a comfortable feel for filtrations we will discuss a few lemmas and past results that will be useful for the proof of the main theorems. We start by defining the theta operator.

Definition. 3 *The theta operator is defined on a formal power series by,*

$$\Theta\left(\sum_{n=0}^{\infty} a(n)q^n\right) := \sum_{n=0}^{\infty} na(n)q^n.$$

Throughout the paper we will use the following result of Swinnerton-Dyer [SD73], about the theta operator,

Lemma. 4 *The operator Θ maps $\tilde{M}_k$ to $\tilde{M}_{k+l+1}$. Moreover, if $f \in M_k \cap \mathbb{Z}[[q]]$ for some k, and $\tilde{f} \not\equiv 0 \pmod{l}$, then $w(\Theta f) \leq w(f) + l + 1$ with equality if and only if $w(f) \not\equiv 0 \pmod{l}$.*

We must now define an additional operator called the U_l operator. This operator is again defined on the formal power series.

Definition. 5 *The U_l operator is defined on a formal power series by,*

$$\left(\sum_{n=0}^{\infty} a(n)q^n\right)|U_l := \sum_{n=0}^{\infty} a(ln)q^n.$$

Example:

$$\begin{aligned}(1 + 2q + 3q^2 + 4q^3 + 5q^4 + 6q^5 + 7q^6 + \dots)|U_3 \quad &= \\ (1 + 4q + 7q^2 + 10q^3 + 13q^4 + 16q^5 + 19q^6 + \dots\end{aligned}$$

Now we must define one last operator for each prime l.

Definition. 6 *The Hecke operator is defined on a formal power series by,*

$$f|T_l = \sum_{n=0}^{\infty} \left(a(ln) + l^{k-1}a\left(\frac{n}{l}\right)\right) q^n.$$

In addition, $T_l : M_k \mapsto M_k$, or in other words T_l maps modular forms of weight k to modular forms of weight k. Considering the definition of the Hecke operator, we can conclude that both the T_l and the U_l operator both yield the same q-series modulo l. In other words, the images of a modular form when applied to these two operators are congruent modulo l. Hence $U_l : \tilde{M}_k \mapsto \tilde{M}_k$. Moreover, we get the following correspondence,

$$(f|U)^l \equiv f - \Theta^{l-1} f \pmod{l}. \tag{2.3}$$

Lastly we sill need the following result from Serre [Ser73].

Theorem. 7 *If $f \in \tilde{M}_k \cap \mathbb{Z}[[q]]$, then for all $i \in \mathbb{N}$ we have*

$$w(f^i) = iw(f). \tag{2.4}$$

Chapter 3

Proof of Ramanujan's Conjecture

3.1 The Proof Begins

With the essential background information discussed we will now present a proof for Ramanujan's conjecture. For the sake of convenience we will restate the conjecture,

Theorem. 8 *Suppose that l is prime. If there is a Ramanujan congruence modulo l, then the congruence must be one of (1.1), (1.2), or (1.3).*

Recall, that (1.1), (1.2), and (1.3) are the following congruences,

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5 \\ p(7n+4) &\equiv 0 \pmod 7 \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

First we will show there are no Ramanujan congruences modulo 2 or 3.

Let $l = 2$. Then for any β there exists an $n \in \mathbb{Z}$ such that $2n - \beta = 0$ or 1. But in this case $p(0) = 1$ and $p(1) = 1$. Therefore the Ramanujan congruence would not hold for the n which we have found.

Let $l = 3$. Then for any β there exists an $n \in \mathbb{Z}$ such that $3n - \beta = 0, 1$, or 2. But in this case $p(0) = 1, p(1) = 1$, and $p(2) = 2$. Therefore the Ramanujan congruence would not hold for the n which we have found.

Now we will fix a prime l, with $l \geq 5$. It has been shown by Kiming and Olsson [KO92] that if there exists a $\beta \in \mathbb{Z}$ for which there is the congruence

$$p(ln + \beta) \equiv 0 \pmod l \text{ for all } n,$$

then $24\beta \equiv 1 \pmod l$. We define δ_l by

$$\delta_l := \frac{l^2-1}{24}. \tag{3.1}$$

We know δ_l is a positive integer since $l^2 - 1 \equiv 0 \pmod{24}$. To show this we consider the following,

$$l^2 - 1 = (l+1)(l-1).$$

Since l is an odd prime, either $l \equiv 1 \pmod 4$ or $l \equiv 3 \pmod 4$.

If $l \equiv 1 \pmod 4$ then $l - 1 \equiv 0 \pmod 4$ and $l + 1 \equiv 2 \pmod 4$.
Hence $8|l^2 - 1 = (l+1)(l-1)$.

If $l \equiv 3 \pmod 4$ then $l - 1 \equiv 2 \pmod 4$ and $l + 1 \equiv 0 \pmod 4$.
Hence $8|l^2 - 1 = (l+1)(l-1)$.

In either case $8|l^2 - 1$.

Additionally 3 divides one of $l - 1, l$, or $l + 1$. But since $l \geq 5$ is prime we know that 3 divides $l - 1$ or $l + 1$. Since gcd(3,8)=1 we know that $3 \cdot 8 = 24|(l-1)(l+1) = l^2 - 1$. Hence δ_l is in fact an integer.

For example if $l = 5, 7$, or 11,

$$\begin{aligned}
\delta_5 &= \frac{5^2-1}{24} = \frac{25-1}{24} = 1, \\
\delta_7 &= \frac{7^2-1}{24} = \frac{49-1}{24} = 2, \\
\delta_{11} &= \frac{11^2-1}{24} = \frac{121-1}{24} = 5.
\end{aligned}$$

If we consider the two following sets $\{ln + \beta | n \in \mathbb{N}\}$ and $\{ln - \delta_l | n \in \mathbb{N}\}$, we can see that they are in fact equal if $\beta \equiv -\delta_l \pmod l$. We can see by the way δ_l has been defined together with the [KO92] result that

$$24\delta_l = l^2 - 1 \equiv -1 \pmod l \equiv -24\beta \pmod l.$$

Hence $\delta_l \equiv -\beta \pmod l$ and $\{ln + \beta \mid \forall n \in \mathbb{N}\} = \{ln - \delta_l \mid \forall n \in \mathbb{N}\}$.

Thus in lieu of proving Theorem 8, it suffices to prove the following theorem instead.

Theorem. 9 *If $l \geq 13$ is prime, then*

$$\sum_{n=0}^{\infty} p(ln - \delta_l)q^n \not\equiv 0 \pmod l.$$

Recall from chapter 2 that

$$\Delta(z) := q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

is a cusp form of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$; recall that this means that the constant term of the Fourier series expansion at ∞ is 0. Additionally, for the rest of this paper we can consider our prime l, as $l \geq 13$.

We now define

$$f_l(z) := \Delta^{\delta_l}(z).$$

So by considering equation (3.1), we have

$$f_l(z) = q^{\delta_l}\prod_{n=1}^{\infty}(1-q^n)^{24\delta_l} = q^{\delta_l}\prod_{n=1}^{\infty}\frac{(1-q^n)^{l^2}}{(1-q^n)}.$$

Now we consider that by the binomial theorem, $(a+b)^l = \sum_{n=0}^{l}\binom{l}{n}a^l b^{l-n}$ for all integers a and b. When taken modulo l all of the coefficients of this sum are 0 with the exception of $n = 0$ and $n = l$. Hence

$$(a+b)^l \equiv a^l + b^l \pmod{l}.$$

Using this we can see that,

$$q^{\delta_l}\prod_{n=1}^{\infty}\frac{(1-q^n)^{l^2}}{(1-q^n)} \equiv q^{\delta_l}\prod_{n=1}^{\infty}(1-q^{ln})^l\prod_{n=1}^{\infty}\frac{1}{(1-q^n)} \pmod{l}.$$

Now using (2.1) as a substitution we see that in fact,

$$f_l(z) \equiv q^{\delta_l}\prod_{n=1}^{\infty}(1-q^{ln})^l\sum_{n=0}^{\infty}p(n)q^n \pmod{l}.$$

Then using the q^{δ_l} to perform a change of index in our summation we see that in fact,

$$f_l(z) \equiv \prod_{n=1}^{\infty}(1-q^{ln})^l\sum_{n=0}^{\infty}p(n-\delta_l)q^n \pmod{l}.$$

The U_l operator takes every l^{th} coefficient of the summation and then rescales the powers of q appropriately. Therefore by applying the U_l operator to $f_l(z)$ we are only looking at the coefficients of the summation $p(ln-\delta_l)$, which yields the useful equivalence,

$$f_l|U_l \equiv \prod_{n=1}^{\infty}(1-q^n)^l\sum_{n=0}^{\infty}p(ln-\delta_l)q^n \pmod{l}.$$

From this we see that if there existed a Ramanujan congruence modulo l, then we additionally have that $f_l|U_l \equiv 0 \pmod{l}$. Another way of saying this is that

$$w(f_l|U_l) = -\infty.$$

3.2 Lemmas Regarding Filtrations

We must now consider the filtrations of the forms $\Theta f_l, \Theta^2 f_l, \ldots$. Recall the relation between the U_l operator and the Θ operator modulo l, (2.3). We will use the following two lemmas involving filtrations,

Lemma. 10 *([KO92]). If $m \in \mathbb{N}$, and $l \geq 5$ is prime, then*

$$w(\Theta^m f_l) \geq w(f_l) = \frac{l^2-1}{2}.$$

Proof. First we must consider the filtration of $\Delta(z)$. Since $\Delta(z)$ is a cusp form the constant term is zero, any modular form which $\Delta(z)$ was congruent to modulo l would also have to have a zero constant term modulo l making it a cusp form of a lower weight. But $dim(M_{12}) = 2$ and $dim(M_k) = 1$ for all $k = 4, 6, 8$, or 10. Thus for each of these k, M_k is spanned by E_k which does not have a constant term congruent to 0 modulo l. Additionally if $k = 2$, k is odd, or $k < 0$ then the dimension is zero, and if $k = 0$ then M_0 is spanned by 1. Hence there is no non-constant modular form congruent to $\Delta(z)$ modulo l that has a lesser weight, so $w(\Delta) = 12$, which is the weight of $\Delta(z)$.

From this, if we consider (2.4) then we see $w(f_l) = \delta_l w(\Delta) = \delta_l \cdot (12) = \frac{l^2-1}{2}$. Observe that by expanding out Δ^{δ_l} we see that $f_l = q^{\delta_l} + \cdots$, hence

$$\Theta^m f_l = \delta_l^m q^{\delta_l} + \cdots \not\equiv 0 \pmod{l}. \tag{3.2}$$

Now can now assume that $w(\Theta^m f_l) = k$ and define $d := \dim M_k > 0$. A basis for M_k can be constructed of the form $\{g_0, \ldots, g_{d-1}\}$, where the g_i are modular forms that have integral coefficients and are of the following form,

$$\begin{aligned} g_0 &= 1 + \cdots, \\ g_1 &= q + \cdots, \\ g_2 &= q^2 + \cdots, \\ &\vdots \\ g_{d-1} &= q^{d-1} + \cdots. \end{aligned}$$

This basis is constructed from $\Delta(z)$ and the Eisenstein series of weights 4 and 6 on $\mathrm{SL}_2(\mathbb{Z})$. Considering the number of elements in the basis, we can see from (3.2) that in fact $d \geq \frac{l^2-1}{24} + 1$. With that in mind, we can also conclude by Theorem 2 that $d \leq \frac{k}{12} + 1$. Hence $\frac{l^2-1}{24} + 1 \leq \frac{k}{12} + 1$, and so by solving for k we see that $k \geq \frac{l^2-1}{2}$.

□

Lemma. 11 *Suppose that $l \geq 5$ is prime and let $f_l = \Delta^{\delta l}$. Then either*

$$\text{(1) } w(\Theta^{l-1} f_l) \equiv 0 \pmod{l}, \text{ or}$$
$$\text{(2) } w(\Theta^{l-1} f_l) = w(f_l) = \frac{l^2-1}{2}.$$

Moreover, in the first case we have $w(f_l|U_l) > 0$.

Proof. By applying Fermat's little theorem we can see that $\Theta^l f \equiv \Theta f \pmod{l}$ for all $f \in \mathbb{Z}[[q]]$. From the Lemma preceeding this one we know that $w(f_l) \not\equiv 0 \pmod{l}$ and hence we can use Lemma 4 to see that

$$w(\Theta^l f_l) = w(\Theta f_l) = w(f_l) + l + 1 = \frac{l^2-1}{2} + l + 1.$$

If in fact $w(\Theta^{l-1} f_l) \not\equiv 0 \pmod{l}$, then we can apply Lemma 4 as follows,

$$w(\Theta^l f_l) = w(\Theta\Theta^{l-1} f_l) = w(\Theta^{l-1} f_l) + l + 1.$$

By combining this and the previous result and solving for $w(\Theta^{l-1} f_l)$, we see that when $w(\Theta^{l-1} f_l) \not\equiv 0 \pmod{l}$,

$$w(\Theta^{l-1} f_l) = \frac{l^2-1}{2} = w(f_l).$$

We will now draw our attention to the second claim that occurs in Lemma 11, that in case (1) we have $w(f_l|U_l) > 0$. Combining (2.3) and (2.4), we see that

$$l \cdot w(f_l|U_l) = w((f_l|U_l)^l) = w(f_l - \Theta^{l-1} f_l).$$

Hence,

$$w(f_l|U_l) = \frac{1}{l} w(f_l - \Theta^{l-1} f_l). \tag{3.3}$$

Now consider that we are in case (1) of of Lemma 11, i.e. $w(\Theta^{l-1} f_l) \equiv 0 \pmod{l}$. Then we will complete the proof by way of contradiction, hence assume that $w(f_l|U_l) \leq 0$. Therefore using (3.3) we can observe that $f_l - \Theta^{l-1} f_l$ is in fact constant modulo l. We conclude that this constant must be zero since there is a factor of $\frac{1}{l}$ in (3.3). Therefore we arrive at the conclusion that

$$f_l \equiv \Theta^{l-1} f_l \pmod{l},$$

which cannot occur because $w(f_l) = \frac{l^2-1}{2} \not\equiv 0 \pmod{l}$ and we have assumed that $w(\Theta^{l-1} f_l) \equiv 0 \pmod{l}$. So in fact we can see that our second assertion in Lemma 11 is true, namely $w(f_l|U) > 0$.

□

3.3 Ramanujan's Conjecture

With our two lemmas established we can now return our focus to the proof of Theorem 9. Let $l \geq 13$ be a prime for which

$$\sum_{n=0}^{\infty} p(ln - \delta_l)q^n \equiv 0 \pmod{l}.$$

Then $w(f_l|U) = -\infty$, and therefore by Lemma 11, we know that we are in the second case of Lemma 11, or

$$w(\Theta^{l-1} f_l) = w(f_l) = \frac{l^2 - 1}{2}. \tag{3.4}$$

If it were true that $w(\Theta^{l-2} f_l) \not\equiv 0 \pmod{l}$, then we can utilize Lemma 4 to obtain that

$$= w(\Theta^{l-1} f_l) = w(\Theta^{l-2} f_l) + l + 1.$$

Applying (3.4) would then tell us that,

$$w(\Theta^{l-2} f_l) = w(f_l) - l - 1 < w(f_l).$$

The above result contradicts Lemma 10. To avoid this contradiction then it must be the case that,

$$w(\Theta^{l-2} f_l) \equiv 0 \pmod{l}. \tag{3.5}$$

Considering now that $w(f_l) = \frac{l^2-1}{2}$, we can continually use Lemma 4 to conclude that $w(\Theta^{\frac{l+1}{2}} f_l) \equiv 0 \pmod{l}$. Now using Lemma 4 once more, we see that we can find an $\alpha \geq 1$ such that the following holds,

$$w(\Theta^{\frac{l+3}{2}} f_l) = \frac{l^2 - 1}{2} + \frac{l+3}{2} \cdot (l+1) - \alpha(l-1). \tag{3.6}$$

Lemma 10 shows us that $w(\Theta^{\frac{l+3}{2}} f_l) \geq \frac{l^2-1}{2}$. Therefore

$$\frac{l^2 - 1}{2} \leq w(\Theta^{\frac{l+3}{2}} f_l) = \frac{l^2 - 1}{2} + \frac{l+3}{2} \cdot (l+1) - \alpha(l-1).$$

Then by rearranging the inequality and subtracting $\frac{l^2-1}{2}$ from both sides we obtain,

$$\alpha(l-1) \leq \frac{l+3}{2} \cdot (l+1).$$

Now dividing through by $(l-1)$ we see that

$$\alpha \leq \frac{l+3}{2(l-1)} \cdot (l+1) = \frac{l+5}{2} + \frac{4}{l-1}. \tag{3.7}$$

Therefore, since $l > 5$, we can see that the second term in (3.7) is less than one. Hence we can conclude that $1 \leq \alpha \leq \frac{l+5}{2}$.

We will now let j be the least integer such that $1 \leq j \leq \frac{l-5}{2}$ and $w(\Theta^{\frac{l+1}{2}+j} f_l) \equiv 0 \pmod{l}$. Notice that such a j must exist since $\frac{l+1}{2} + \frac{l-5}{2} = \frac{2(l-2)}{2} = l-2$ and by (3.5). Then we can conclude from Lemma 4 and (3.6) that

$$\begin{aligned} w(\Theta^{\frac{l+1}{2}+j} f_l) &= \frac{l^2-1}{2} + \left(\frac{l+1}{2} + j\right)(l+1) - \alpha(l-1) \\ &= \frac{(l+1)(l-1)}{2} + \frac{(l+1)(l+1)}{2} + j(l+1) - \alpha(l-1) \\ &= l^2 + l + lj - \alpha l + j + \alpha \\ &\equiv j + \alpha \pmod{l} \\ &\equiv 0 \pmod{l}. \end{aligned}$$

By how α and j are bounded we see that $j + \alpha = l$, not simply a multiple of l.
Then because $1 \leq j \leq \frac{l-5}{2}$, we can see that $\alpha \geq \frac{l+5}{2}$. This with combined with the fact that $1 \leq \alpha \leq \frac{l+5}{2}$ tells us that $\alpha = \frac{l+5}{2}$. Therefore (3.6) can be specified more, now that we know α:

$$w(\Theta^{\frac{l+3}{2}} f_l) = \frac{l^2-1}{2} + \frac{l+3}{2} \cdot (l+1) - \frac{l+5}{2} \cdot (l-1) = \frac{l^2-1}{2} + 4. \tag{3.8}$$

To complete the proof of Ramanujan's conjecture, recall how the Θ operator is defined. This definition tells us that the q-expansion of $\Theta^{\frac{l+3}{2}} f_l$ starts off with the following terms,

$$\Theta^{\frac{l+3}{2}} f_l = \delta_l^{\frac{l+3}{2}} q^{\delta_l} + \cdots = \delta_l^{\frac{l+3}{2}} q^{\frac{l^2-1}{24}} + \cdots . \tag{3.9}$$

Now we recall that

$$E_4(z) := 1 + 240 \sum_{n=1}^{\infty} \sum_{d|n} d^3 q^n = 1 + 240q + \cdots$$

is the Eisenstein series of weight 4 on $\mathrm{SL}_2(\mathbb{Z})$ defined in chapter 2. A basis for $M_{\frac{l^2-1}{2}+4}$ in terms of E_4 and $\Delta(z)$ can be constructed because $\frac{l^2-1}{2} \equiv 0 \pmod{12}$,. This is done by adjusting the basis constructed from E_4 and E_6. The basis looks as follows,

$$\{E_4 \cdot E_4^{\frac{l^2-1}{8}}, E_4 \cdot \Delta \cdot E_4^{\frac{l^2-1}{8}-3}, \cdots, E_4 \cdot \Delta^{\frac{l^2-1}{24}}\}. \tag{3.10}$$

Considering (3.8), (3.9), and (3.10) we see (note that the "last" element in our basis set is in fact $E_4 \cdot f_l$),

$$\Theta^{\frac{l+3}{2}} f_l \equiv \delta_l^{\frac{l+3}{2}} E_4 \cdot f_l \pmod{l}. \tag{3.11}$$

Hence by expanding (3.11) it follows that,

$$\delta_l^{\frac{l+3}{2}} E_4 \cdot f_l \equiv \delta_l^{\frac{l+3}{2}} (1 + 240q + \cdots)(q^{\delta_l} + q^{\delta_l+1} + \cdots) \equiv \delta_l^{\frac{l+3}{2}} q^{\delta_l} + 241 \cdot \delta_l^{\frac{l+3}{2}} q^{\delta_l+1} \cdots \pmod{l}.$$

Now we will consider an alternative way of constructing the polynomial $\Theta^{\frac{l^2+3}{2}} f_l$, consider that

$$f_l = q^{\delta_l}(1-q)^{l^2-1} \cdots \equiv q^{\delta_l} + q^{\delta_l+1} + \cdots \pmod{l},$$

therefore

$$\Theta^{\frac{l+3}{2}} f_l \equiv \delta_l^{\frac{l+3}{2}} q^{\delta_l} + (\delta_l + 1)^{\frac{l+3}{2}} q^{\delta_l+1} + \cdots \pmod{l}. \tag{3.12}$$

Considering the two polynomials constructed in (3.11) and (3.12), we know they must be congruence. By considering the coefficient on the q^{δ_l+1} term we see that,

$$(\delta_l + 1)^{\frac{l+3}{2}} \equiv 241 \cdot \delta_l^{\frac{l+3}{2}} \pmod{l}. \tag{3.13}$$

Because $\frac{1}{\delta_l} \equiv -24 \pmod{l}$, when dividing by $\delta_l^{\frac{l+3}{2}}$ equation (3.13) gives us

$$\begin{aligned} \frac{(\delta_l + 1)^{\frac{l+3}{2}}}{\delta_l^{\frac{l+3}{2}}} &= \left(\frac{\delta_l + 1}{\delta_l}\right)^{\frac{l+3}{2}} \\ &\equiv (1 + (-24))^{\frac{l+3}{2}} \\ &\equiv 241 \pmod{l}. \end{aligned}$$

Hence we can conclude that,

$$(-23)^2 \cdot (-23)^{\frac{l-1}{2}} \equiv 241 \pmod{l}.$$

Now we can utilize Fermat's Little Theorem, to see that

$$((-23)^{\frac{l-1}{2}})^2 \equiv 1 \pmod{l}.$$

This implies that $(-23)^{\frac{l-1}{2}} \equiv -1, 1 \pmod{l}$, which implies that

$$\pm 529 \equiv 241 \pmod{l}.$$

Ramanujan's Conjecture follows since

$$529 + 241 = 770 = 2 \cdot 5 \cdot 7 \cdot 11,$$

and

$$529 - 241 = 288 = 2^5 \cdot 3^2.$$

Notice that in neither the -241 case or the $+241$ case the result is congruent to 0 modulo any prime l other than when $l = 2, 3, 5, 7, 11$. Hence this Ramanjan congruence can only occur if in fact $l = 2, 3, 5, 7, 11$. This concludes the proof of the non-existence of Ramanujan congruences other then (1.1), (1.2), and (1.3).

□

Chapter 4

An Overview of Other Work On Partitions and Modular Forms

The results presented in this paper were first published by Ahlgren and Boylan [AB03] in 2003. In this paper they also presented a related result. Consider the following conjecture originally put forth by Newman [New60],

Conjecture. 1 *If M is a positive integer then for every integer $0 \leq r < M$ there are infinitely many non-negative integers n such that $p(n) \equiv r \pmod{M}$.*

If M itself is prime the conditions to verify the conjecture can be simplified thanks to work of Brunier and Ono [BO04]. Their result has lead to the verification of the conjecture for all primes $M < 2 \times 10^5$. By using a result of [BO04] together with Theorem 1 Ahlgren and Boylan obtain the following theorem.

Theorem. 12 *Newman's Conjecture is true for every prime modulus M with the possible exception of $M = 3$. Moreover, if $l \geq 5$ is prime, then we have*

$$\#\{0 \leq n \leq X : p(n) \equiv r \pmod{l}\} \gg_{r,l} \begin{cases} \frac{\sqrt{X}}{\log X} & \text{if } 1 \leq r < l, \\ X & \text{if } r = 0. \end{cases}$$

In addition to Ahlgren and Boylan's results, there has been recent work generalizing the results about the non-existence of Ramanujan congruences to other types of partitions.

In particular, Dewar [Dew] proved the non-existence of Ramanujan congruences in modular forms of level four. Also, Dewar provides "a general method for investigating sequences related to modular forms and prove the non-existence of Ramanujan congruences for large primes l" for a number of specific combinatorial objects. More specifically, he shows the following two results.

Theorem. 13 *The only Ramanujan congruences* $c\phi_2(ln+a) \equiv \pmod{l}$ *are*

$$\begin{aligned} c\phi_2(2n+1) &\equiv 0 \pmod 2, \\ c\phi_2(5n+3) &\equiv 0 \pmod 5. \end{aligned}$$

Theorem. 14 *There are no Ramanujan congruences* $\bar{p}(ln+a) \equiv 0 \pmod{l}$ *when* $l \geq 3$.

Dewar, a student of Ahlgren, works at applying the study of modular forms to additional combinatorial structures to show the absence of Ramanujan congruences in these structures, much like Ahlgren and Boylan did with the partition function. The non-existence of Ramanujan Congruences is an intriguing in both number theory and combinatorics.

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