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Title: METHOD OF CHARACTERISTICS AND ITS APPLICATION TO BLAST WAVES IN WATER AND TWO-PHASE FLOW

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The theory of characteristics is discussed and its scope has been broadened by its application to the most general form of an unsteady, two-dimensional, compressible, rotational, axial-symmetric fluid flow. Numerical procedure for solving such a problem is outlined. Several boundary conditions which are common in fluid flow problems are considered and for each case a numerical procedure is given. Stability of the numerical schemes is discussed. Two specific problems have come under focus. First, advancement of a two-dimensional shock wave in water is considered. The numerical integrating procedures by the method of characteristics for this case are developed with the aid of a coordinate transformation to the shock coordinates. Equations of state for water and the variation of speed of sound with pressure and temperature are given. Second, propagation of unsteady expansion waves in a two-phase flow is considered. The basic

difference between mixtures of gas and fluid, and vapor and fluid is discussed. By non-dimensionalizing the set of equations a parameter has been obtained which influences the set of equations and thus the method of approach.

Method of Characteristics and Its Application to Blast Waves in Water and Two-Phase Flow

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NOMENCLATURE

English Symbols

a	speed of sound
C_p	specific heat at constant pressure
C_L	specific heat of the liquids
D	depth
$\frac{D}{Dt}$	operator (substantial derivative)
e	internal energy
$e_{\bar{r}}$	unit vector in $\bar{r}$ direction
$e_{\bar{z}}$	unit vector in $\bar{z}$ direction
H	enthalpy
k	a constant
K	mean compressibility
L	latent heat
M	true compressibility (coefficient of compressibility)
n_o	number density of the bubbles
n	characteristic constant in Tait equation of state
p	pressure in the medium (or mixture)
p_g	pressure in the gas (or vapor)
R	radius of a bubble
$\bar{R}$	radius of curvature
r	independent variable in cylindrical coordinates (radial direction)

r^*	dimensionless independent variable (radial direction)
$\bar{r}$	independent variable in the shock coordinates
S	entropy
s	salinity
$\bar{s}$	arc length
t	time
t^*	dimensionless time
T	temperature
T_{sat}	saturation temperature
ΔT_o	$T_o - T_\infty$, amount of superheat
u	velocity component in the radial direction
u^*	dimensionless radial velocity component
$\bar{u}$	radial velocity in shock coordinates
U	velocity of the shock
v	velocity component in the z direction
v^*	dimensionless axial velocity component
$\bar{v}$	velocity component in shock coordinates
V	volume of a bubble
υ	specific volume of the medium (or mixture)
υ_o	specific volume at standard pressure
v_L	specific volume of the liquid
v_g	specific volume of the gas (or vapor)

z	independent variable, axial direction (cylindrical coordinates)
$\bar{z}$	independent variable, normal direction (shock coordinates)
z^*	dimensionless independent variable (axial direction)

Greek Symbols

α	bi-characteristics angle
α	diffusivity of the liquid
$\bar{\beta}$	modulus of compressibility $(\bar{\beta} = \frac{1}{M})$
β	shock angle
γ	ratio of specific heats
δ	shock wave thickness
Δ	difference operator
ϵ	a small parameter
ζ	dimensionless pressure
η	an independent variable
θ	latitude
$\bar{\theta}$	angle
κ	thermal conductivity of the liquid
λ	a parameter in Tumlirz equation of state
$\bar{\lambda}$	a significant length
μ	absolute viscosity (dynamic viscosity)
ν	kinematic viscosity

ξ	an independent variable
ρ	density of a medium (or a mixture)
ρ_L	density of the liquid in a mixture
ρ_g	density of the gas or vapor in a mixture
σ	surface tension
$\bar{\sigma}$	a dimensionless parameter

Subscripts

$\circ$	initial conditions, or standard conditions
∞	very far away from the point of consideration
w	wall
L	liquid
g	gas or vapor
r	partial derivative in r
z	partial derivative in z
t	partial derivative in t

Superscripts

$\cdot$	first derivative in time
$\cdot\cdot$	second derivative in time
$\rightharpoondown$	vector

Other Notations

Bold Face Letters: Vectors

MOC: Method of Characteristics

METHOD OF CHARACTERISTICS AND ITS APPLICATION TO BLAST WAVES IN WATER AND TWO-PHASE FLOW

INTRODUCTION

This work is concerned with the method of characteristics which is a powerful technique for solving problems of the compressible fluid flow. The first chapter is devoted to the development of the theory in a very general form of an unsteady compressible axial-symmetric fluid flow. Different schemes are discussed and the numerical procedure for solving the problem is outlined. Several boundary conditions which are common in fluid flow problems are considered and for each case a numerical procedure is given. Stability of the schemes is discussed.

In the second chapter, we are concerned with two-dimensional moving shock waves in water. The numerical procedure of the method of characteristics is developed and set up with the aid of a coordinate transformation to the shock coordinates. A knowledge of variation of the speed of sound and density as functions of pressure, temperature and salinity is essential to the development of the numerical procedure. The relations are given in the Appendix B. In the third chapter two-phase flow problems have come under focus. The method of characteristics has been used to investigate the propagation of expansion waves in a mixture of fluid and gas bubbles and fluid and vapor bubbles. The basic difference of these two media is discussed and their relations to the resulting systems of equations have been clarified.

By non-dimensionalizing the set of equations, a parameter $\bar{\sigma}$ has been obtained which influences the set of equations and thus the method of approach. More comprehensive and detailed discussions of some of the derivations are given in the Appendix A.

I. THE METHOD OF CHARACTERISTICS

1.1 Introduction

In many branches of engineering, we encounter problems that deal with n first-order partial differential equations of n dependent variables with functions of two, three or four independent variables. Without lack of generality, any higher order set of equations can be reduced to a set of first order equations. Furthermore, a set of equations can be categorized as hyperbolic, elliptic or parabolic.

In compressible fluid flow, by making realistic assumptions, the governing equations are quasi-linear and hyperbolic. The nature of these equations has made them unamenable to any direct analytical treatment. However, this situation has not been a total barrier. As an alternative, many approximate methods have been devised among which the numerical approximation is in the lead. This has been due to the birth and development of digital computers in the past two decades. The new generation of computers has given the engineers opportunity of solving problems of practical application.

Several numerical methods have been suggested and developed to handle the problems of compressible fluid flow. One is the well known and basic method of finite differences in which partial derivatives are replaced by finite differences. Another one is the method of artificial viscosity and finally the method of characteristics.

The method of finite differences is used in incompressible fluid flow frequently and successfully, but there is little application in the area of compressible flow. This method although simple, suffers from several difficulties of which the most important are:

1. Handling the shocks and free boundaries.
2. For accuracy and stability the mesh size has to be confined which badly affects the computer running time.

The method of artificial viscosity was introduced by von Neumann and Richtmyer (75) and was further developed by Lax and Wendroff (48). The essence of the method is a continuous term which is added to the equations to replace the shock waves. We can consider this term like a hump that smoothly dies out. This method, although, is amenable to finite differences and can handle shocks very effectively, smears out the shock over several mesh sizes.

The method of characteristics is potentially the most accurate of all, and gives the best treatment to the singularities and irregularities in the field. This is due to the fact that characteristic surfaces are the paths of wave fronts and discontinuities in higher derivatives propagate along such surfaces. This is why this method has been utilized in the field so extensively to solve complicated problems.

As far as theory is concerned, the method can be considered almost classic, although complicated and hard to grasp. Thus the historical review that follows is pertinent.

1.2 Historical Review

Although the theory of characteristics had its origin in the work of Monge in the early 19th century, the significance of it was not fully realized until 1928 when H. Lewy used it for the case of two independent variables. Lewy's method was generalized by E. W. Titt (72) to the case of three-dimensional problems. Von Mises (52) carried the concept over the field of compressible fluid flows. Coincise discussion of the general theory can be found in references (12) and (53). These earlier attempts that were concerned with the general approach were followed by the endeavor of those who made the method amenable by numerical schemes. This step was a prelude to the actual exploitation of the schemes to solve problems of the field numerically. In this stage of development of the method such work as that of Thornhill (71), Sauer (62), Ferrari (20) Coburn and Dolph (10) Holt (33), Butler (6), Sauerwein (64) and Strom (70), are prominent. However, it should be borne in mind that there has not been any criterion to establish the advantages and efficiency of each scheme. A brief discussion of each method is given in Section 1.6. Among those who have eventually succeeded to solve problems of special importance, or simple problems to demonstrate the feasibility of the numerical schemes, are Butler (6), Tsung (73), Elliott (18), Sauerwein (63), and Strom (70). Again, a brief explanation of their work is given in Section 1.7.

This work is unique in the way that the most general form of unsteady compressible fluid flow with axial symmetry has been considered. The proposed network, in essence, is the one that is used by Butler. The iteration scheme for a point was tried on CDC 3300 and convergence was satisfactory.

1.3 Derivation of General Equations

The following assumptions and conditions prevail:

1. Compressible, rotational flow
2. Unsteady, axial-symmetric configuration
3. Arbitrary thermal and caloric equations of state
4. Temperature and velocity gradients are negligible and dissipation terms and energy sources are absent
5. Chemical and thermodynamical equilibrium exist

Momentum Equations

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial r} + \rho v \frac{\partial u}{\partial z} + \frac{\partial p}{\partial r} = 0 \tag{1-1}$$

$$\rho \frac{\partial v}{\partial t} + \rho u \frac{\partial v}{\partial r} + \rho v \frac{\partial v}{\partial z} + \frac{\partial p}{\partial z} = 0 \tag{1-2}$$

Continuity Equation

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial r} + \rho \frac{u}{r} + \rho \frac{\partial v}{\partial z} + u \frac{\partial \rho}{\partial r} + v \frac{\partial \rho}{\partial z} = 0 \qquad (1\text{-}3)$$

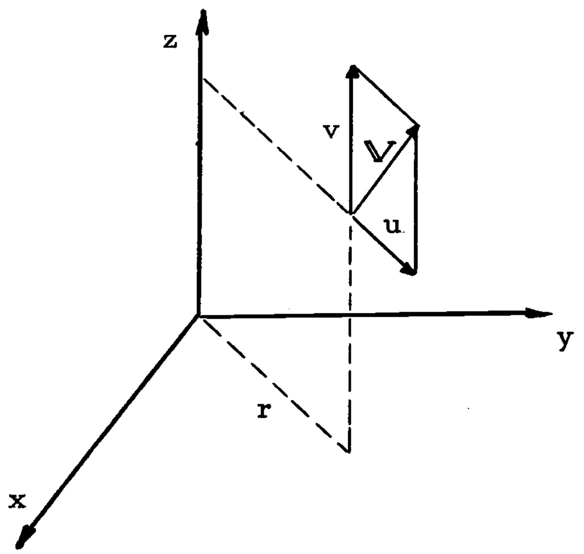

Figure 1-1. Cylindrical coordinate system.

Equation of State

$$p = p(\rho, T), \quad \text{where } T = T(\rho, S) \qquad (1\text{-}4)$$

Caloric Equation of State

$$e = e(\rho, S), \quad \text{where } e \text{ is the internal energy} \qquad (1\text{-}5)$$

Energy Equation

$$\rho\left(\frac{\partial e}{\partial t} + u \frac{\partial e}{\partial r} + v \frac{\partial e}{\partial z}\right) + p \frac{\partial u}{\partial r} + p \frac{\partial v}{\partial z} + p \frac{u}{r} = 0 \qquad (1\text{-}6)$$

In order to find the characteristic surfaces, we multiply the Equation (1-1), (1-2), (1-3), and (1-6) by C_1, C_2, C_3, and C_4, respectively, and add them up. These constants are not known and we will try to find them. The physical and mathematical logic behind this is discussed in the Appendix A, Section A. 1. We form vectors $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, and $\mathbb{A}_4$, as defined below. The result appears in the following fashion.

$$\begin{cases} \mathbb{A}_1 \cdot \nabla \rho + \mathbb{A}_2 \cdot \nabla u + \mathbb{A}_3 \cdot \nabla v + \mathbb{A}_4 \cdot \nabla S = -\rho \frac{u}{r} C_3 - \frac{pu}{r} C_4 \\ \mathbb{A}_1 \equiv (C_3 + \rho \frac{\partial e}{\partial \rho} C_4,\ uC_3 + \frac{\partial p}{\partial \rho} C_1 + \rho u \frac{\partial e}{\partial \rho} C_4,\ vC_3 + \frac{\partial p}{\partial \rho} C_2 + \rho v \frac{\partial e}{\partial \rho} C_4) \\ \mathbb{A}_2 \equiv (\rho C_1,\ \rho C_3 + \rho u C_1 + p C_4,\ \rho v C_1) \\ \mathbb{A}_3 \equiv (\rho C_2,\ \rho u C_2,\ \rho C_3 + \rho v C_2 + p C_4) \\ \mathbb{A}_4 \equiv (\rho \frac{\partial e}{\partial S} C_4,\ \frac{\partial p}{\partial S} C_1 + \rho u \frac{\partial e}{\partial S} C_4,\ \frac{\partial p}{\partial S} C_2 + \rho v \frac{\partial e}{\partial S} C_4) \end{cases} \tag{1-7}$$

Now we seek a surface $\phi(r, z, t) = 0$ so that the vectors $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$ and $\mathbb{A}_4$ are tangent to that surface. This requires the normal $\mathbb{N} \equiv (\phi_r, \phi_z, \phi_t)$ to satisfy

$$\mathbb{A}_1 \cdot \mathbb{N} = 0$$

$$\mathbb{A}_2 \cdot \mathbb{N} = 0$$

$$\mathbb{A}_3 \cdot \mathbb{N} = 0$$

$$\mathbb{A}_4 \cdot \mathbb{N} = 0$$

The only nontrivial solution is the determinant of the coefficients to be zero. The expansion of that determinant gives:

$$\rho^3(\phi_t + u\phi_r + v\phi_z)^2\{(\phi_t + u\phi_r + v\phi_z)^2 - a^2(\phi_r^2 + \phi_z^2)\} = 0 \tag{1-8}$$

where

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_S + \left(\frac{\partial p}{\partial S}\right)_\rho \left(\frac{\frac{p}{\rho^2} - \frac{\partial e}{\partial \rho}}{\frac{\partial e}{\partial S}}\right)$$

introducing the parameter α results in

$$\phi_r = k \cos\alpha$$

$$\phi_z = k \sin\alpha$$

$$\phi_t = -k(a + u\cos\alpha + v\sin\alpha)$$

Now, we can solve for C's; the results appear below

$$C_1 = a \frac{\frac{\partial e}{\partial S}\cos\alpha}{\frac{\partial p}{\partial \rho}\frac{\partial e}{\partial S} - \frac{\partial e}{\partial \rho}\frac{\partial p}{\partial S}} C_3$$

$$C_2 = a \frac{\frac{\partial e}{\partial S}\sin\alpha}{\frac{\partial p}{\partial \rho}\frac{\partial e}{\partial S} - \frac{\partial e}{\partial \rho}\frac{\partial p}{\partial S}} C_3$$

$$C_4 = a \frac{\frac{\partial p}{\partial S}}{\rho\left(\frac{\partial p}{\partial \rho}\frac{\partial e}{\partial S} - \frac{\partial e}{\partial \rho}\frac{\partial p}{\partial S}\right)} C_3$$

Now consider the differentiation along bi-characteristic direction $(1,\ u + a\cos\alpha,\ v + a\sin\alpha)$ which is defined below and depicted in Figure 1-2.

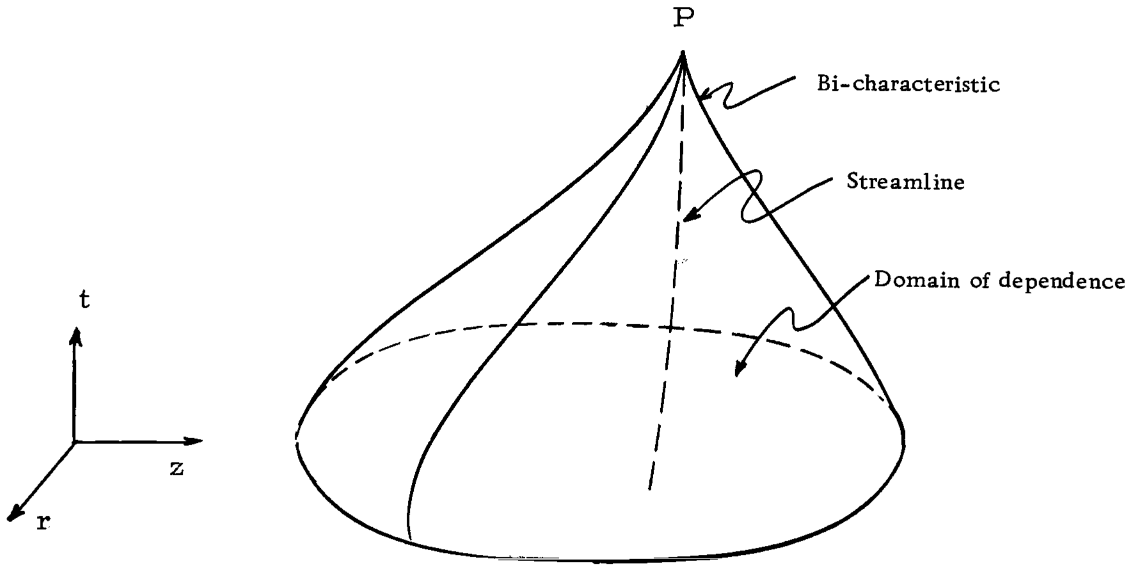

Figure 1-2. Characteristic conoid.

The characteristic surfaces passing through a point envelop a quadratic conoid known as characteristic conoid with vertex at the point of consideration. The lines where characteristic surfaces touch the conoid are called bi-characteristics.

Recalling the definition of differentiation along a curve and its relation to gradient, we can write the differentiation along bi-characteristics as follows:

$$\frac{d}{d\tau} = (1,\ u + a\cos\alpha,\ v + a\sin\alpha)\cdot\nabla$$

After substituting the values of C_1, C_2 and C_4 in the compatability equation and using differentiation along bi-characteristics as was defined before we get the final form of the compatibility equation.

$$\frac{\partial p}{\partial \rho}\frac{\partial e}{\partial S}\frac{d}{dT}\rho + \rho a\frac{\partial e}{\partial S}\cos\alpha\frac{d}{dT}u + \rho a\frac{\partial e}{\partial S}\sin\alpha\frac{d}{dT}v + \frac{\partial e}{\partial S}\frac{\partial p}{\partial S}\frac{d}{dT}S$$

$$= \left(-\rho\frac{\partial p}{\partial \rho}\frac{\partial e}{\partial S} + \rho\frac{\partial e}{\partial \rho}\frac{\partial p}{\partial S} + \rho a^2\frac{\partial e}{\partial S}\cos^2\alpha - \frac{p}{\rho}\frac{\partial p}{\partial S}\right)\frac{\partial u}{\partial r} - \rho a^2\frac{\partial e}{\partial S}\cos\alpha\sin\alpha\frac{\partial u}{\partial z}$$

$$+ \left(-\rho a^2\frac{\partial e}{\partial S}\cos\alpha\sin\alpha\right)\frac{\partial v}{\partial r} + \left(-\rho\frac{\partial p}{\partial \rho}\frac{\partial e}{\partial S} + \rho\frac{\partial e}{\partial \rho}\frac{\partial p}{\partial S} + \rho a^2\frac{\partial e}{\partial S}\sin^2\alpha - \frac{p}{\rho}\frac{\partial p}{\partial S}\right)$$

$$\frac{\partial v}{\partial z} - \frac{\rho u}{r}\left(\frac{\partial p}{\partial \rho}\frac{\partial e}{\partial S} - \frac{\partial e}{\partial \rho}\frac{\partial p}{\partial S}\right) - \frac{pu}{r\rho}\frac{\partial p}{\partial S} \qquad (1\text{-}9)$$

For the case of perfect gas with constant entropy and specific heats, Equation (1-9) will reduce to the simple form of compatibility equation given by Butler (6), except the last two terms which appear because of cylindrical coordinates.

1.4 Numerical Integration

Letus consider one point P_5 on the plane $t = t_o + \Delta t$. The Mach cone from that point cuts the initial surface $t = t_o$ on a circle. Consider four points P_1, P_2, P_3, and P_4 on the circle, each one $\frac{\pi}{2}$ radians apart. Now the pathline from the point P_5 cuts the initial surface at point P_o.

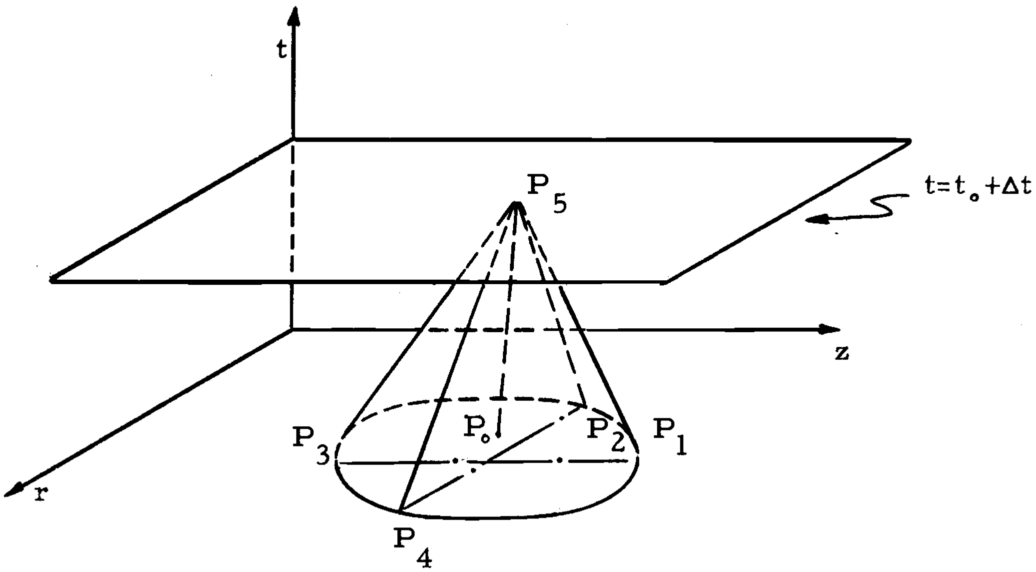

Figure 1-3. Bi-characteristics used for numerical integration.

In order to find the unknowns of the flow at point P_5, we follow these steps:

1. Make a guess about the unknowns at point P_5.
2. Locate four points P_1, P_2, P_3 and P_4 by finite differencing the relations

$$dr = (u + a\cos\alpha)dt \qquad (1\text{-}10)$$

$$dz = (v + a\sin\alpha)dt \qquad (1\text{-}11)$$

3. Make surface fits to flow conditions around these four points.
4. Locate point $P_\circ$.
5. Finite difference the compatibility equation (Equation 1-9) along paths $\overline{P_5P_1}$, $\overline{P_5P_2}$, $\overline{P_5P_3}$ and $\overline{P_5P_4}$; and the continuity equation along the pathline $\overline{P_5P_\circ}$; keeping in mind that along

a pathline the following relations hold

$$u = \frac{dr}{dt} \qquad (1\text{-}12)$$

$$v = \frac{dz}{dt} \qquad (1\text{-}13)$$

Thus the continuity equation along a pathline reads as:

$$d\rho + \rho\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right)dt + \frac{\rho u}{r}\,dt = 0 \qquad (1\text{-}14)$$

6. Solve the resulting five equations and five unknowns.

7. Repeat the procedure, using the obtained data as initial values, until the convergence is reached.

1.5 Points on the Axis

In the neighborhood of the centerline, a different treatment is required. This is due to the term $\frac{u}{r}$ which appears in the compatability equation. To handle this term, we find the limit as follows:

$$\left\{\frac{u}{r}\right\}_{r\to 0} = \frac{0}{0} = \lim \left\{\frac{\frac{\partial u}{\partial r}}{\frac{\partial r}{\partial r}}\right\}_{r\to 0} = \left(\frac{\partial u}{\partial r}\right)_{r=0}$$

Thus on the axis, the term $\frac{u}{r}$ should be replaced by $\left(\frac{\partial u}{\partial r}\right)$

1.6 Numerical Schemes

Numerical schemes that have been developed in previous investigations (regardless of being utilized in solving practical problems) are as follows:

I. Thornhill (71) has proposed two types of networks as shown in Figures 1-4 and 1-5.

a. Hexahedral network of general characteristic surfaces.

b. Hexahedral network of bi-characteristic curves.

These designations are also used in the literature:

a. Tetrahedral characteristic surface network.

b. Tetrahedral characteristic line network.

Scheme (a) uses three points P_1, P_2 and P_3 on the initial plane. Characteristic surfaces through lines $\overline{P_1P_2}$, $\overline{P_1P_3}$ and $\overline{P_2P_3}$ intersect each other at one point P_4. The Mach forecone from P_4 is tangent to these surfaces along $\overline{P_4P_5}$, $\overline{P_4P_6}$ and $\overline{P_4P_7}$. These three lines and the pathline from point P_4 are utilized in finite differencing the proper equations; and the flow properties at P_5, P_6 and P_7 are obtained by direct interpolation between points P_1, P_2 and P_3.

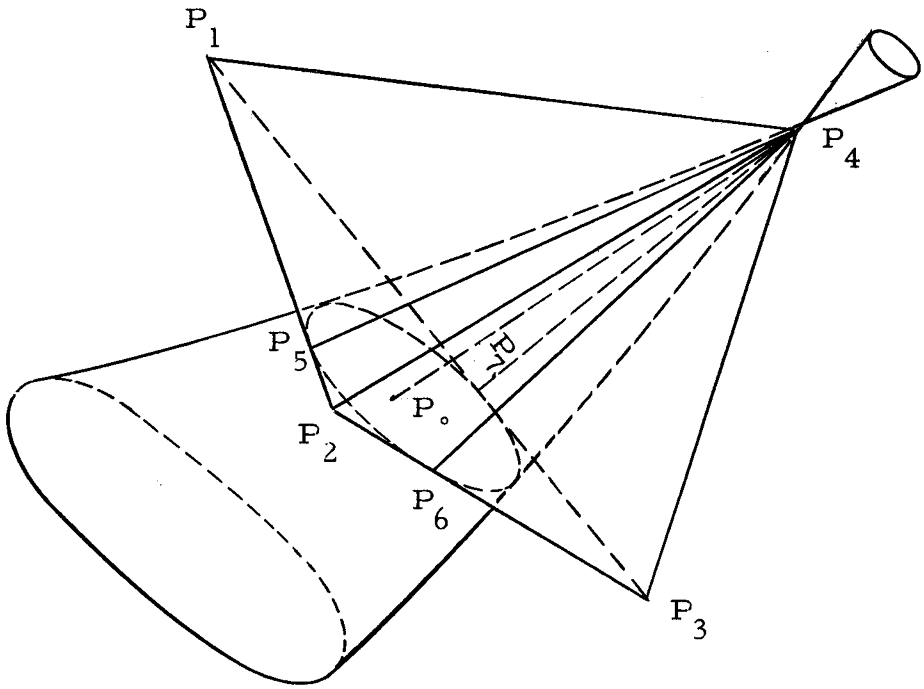

Figure 1-4. Tetrahedral characteristic surface network.

In scheme (b), that is, tetrahedral characteristic line network, the new point P_4 is obtained as a result of intersection of Mach cones from three points P_1, P_2 and P_3 on the initial plane. Thus the lines $\overline{P_1P_4}$, $\overline{P_2P_4}$ and $\overline{P_3P_4}$ represent approximations to the bi-characteristics that can be used for numerical integration. The pathline $\overline{P_4P_\circ}$ provides another line to be used for the same purpose. In this fashion, the flow properties can be obtained by iteration. Sauerwein and Sussman (65) applied this scheme to solve unsteady flow problems and showed that the method was numerically unstable. The reason is that the domain of dependence of the differential equations, that is, the circle passing through points P_1, P_2 and P_3, is not contained within the domain of dependence of the difference equations, that is, triangle $P_1P_2P_3$.

In 1964, Sauerwein modified the method which is known in the literature as "Modified tetrahedral characteristic line network," (Figure 1-6). In this scheme, three points, P_5, P_6 P_7, are used that represent the points of tangency of the circle inscribed within the original triangle $P_1P_2P_3$. These points are chosen as new initial points. The properties at points P_5, P_6 and P_7 are determined by interpolation.

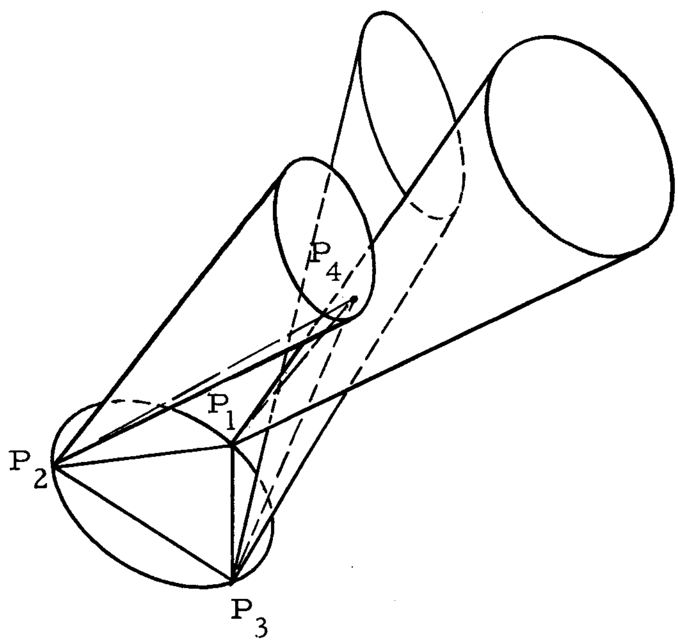

Figure 1-5. Tetrahedral characteristic line network.

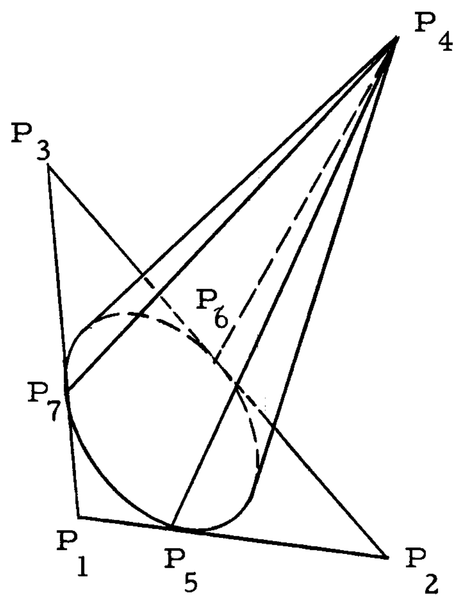

Figure 1-6. Modified tetrahedral characteristic line network.

II. Another scheme that has been utilized by Ferrari (20), Sauer (62) and Ferri (21) makes use of characteristic surfaces and two sets of orthogonal reference planes. This scheme is in fact a combination of the method of characteristics and regular finite difference method (Figure 1-7). Four points, P_1, P_2, P_3 and P_4, on the initial surface are used to calculate the flow properties at the new points P_5 and P_6 which are located at the intersections of characteristic surface passing through $\overline{P_1P_2}$ and another one through $\overline{P_3P_4}$ and a plane parallel to the coordinate surface, x = constant = k. The flow properties at P_5 are obtained from equations for the variations of flow properties along the lines $\overline{P_1P_5}$ and $\overline{P_3P_5}$

in the characteristic surface. The third equation is obtained by considering the variation in the initial surface along $\overline{P_1P_2}$ or $\overline{P_3P_4}$ and relating it to the variations along $\overline{P_1P_5}$ and $\overline{P_3P_5}$. The flow properties at P_6 are obtained similarly. The base point P_4 will, in general, have to be moved in an iteration to P_8 in order to cause P_6 to fall on x = constant = k coordinate plane. This might require interpolation or extrapolation in the initial surface in order to find the flow properties at P_8.

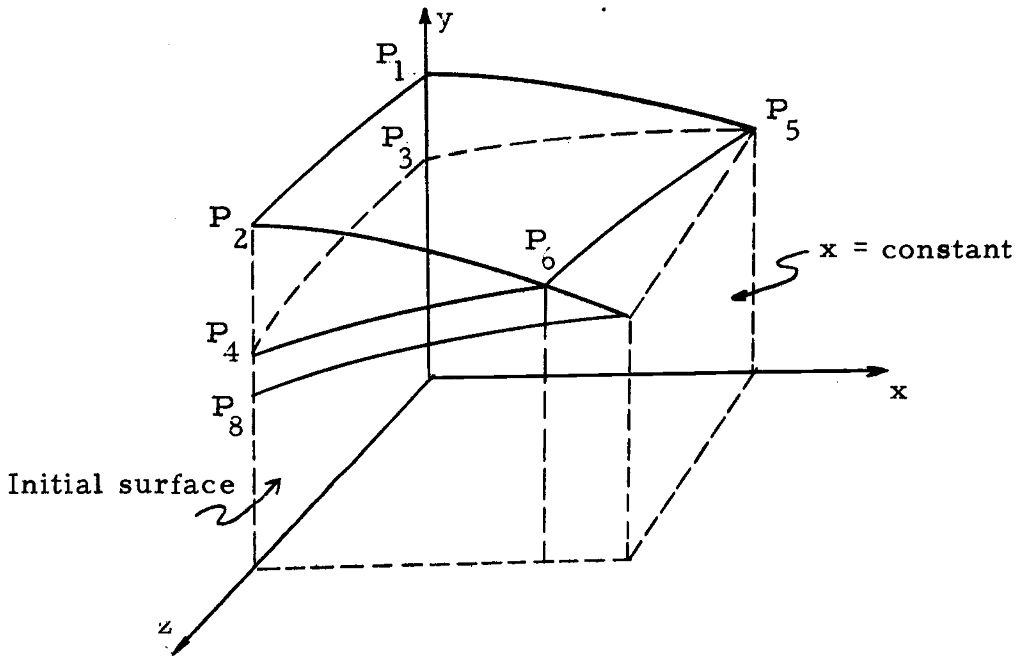

Figure 1-7. Network of intersection of reference planes with characteristic surfaces.

III. Another scheme which is known as prismatic network of characteristic surfaces proposed by Holt (33) who used the work of Coburn and Dolph (10). This method is similar to the network of intersection of reference planes with characteristic surfaces. In this method bi-characteristics are used to perform the integration rather than the lines which are intersections of the characteristic surfaces with reference planes (Figure 1-8). The method consists of choosing the two bi-characteristic directions $\overline{P_1P_5}$ and $\overline{P_3P_5}$ as two coordinate directions at a general point. The third direction is provided by the line $\overline{P_1P_2}$ in the initial plane. In general, this method requires that the flow properties be known on some surface $P_4P_2P_6$ other than the initial surface. This might happen to be the plane of symmetry. Thus this last requirement restricts the application of the scheme.

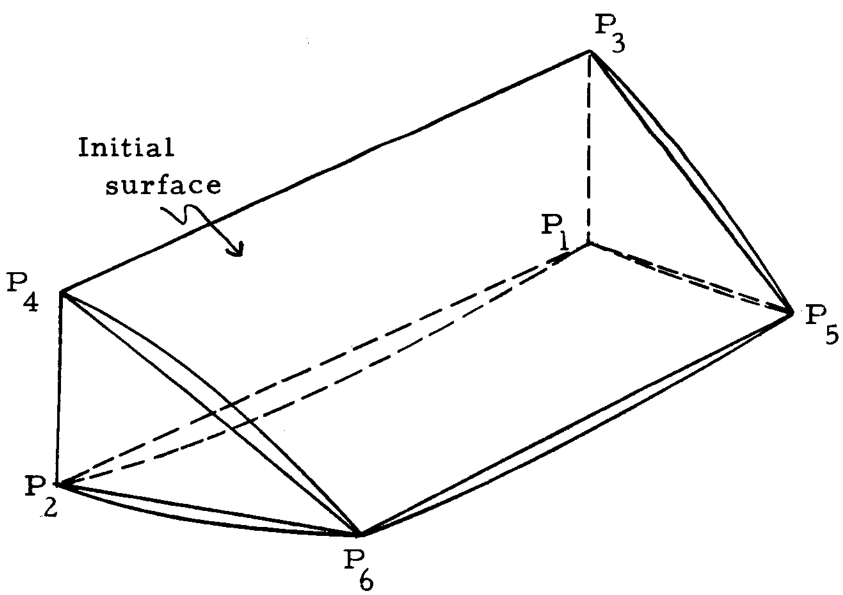

Figure 1-8. Prismatic network of characteristic surfaces.

IV. A scheme introduced by Butler (6) is known as the bi-characteristics method. This method could be considered the extension of the Hartree's method (28) in three dimensions. In this method (Figure 1-9) one point, P_5, is chosen on a plane $t = t_o + \Delta t$ and it is fixed throughout the iteration process. To find the flow properties at this point the compatibility equation (Equation 1-9) is written along four lines known as bi-characteristics $\overline{P_5P_1}$, $\overline{P_5P_2}$, $\overline{P_5P_3}$ and $\overline{P_5P_4}$. The fifth equation consists of continuity equation written along the pathline that passes through the point P_5. Properties of the flow at points P_1, P_2, P_3 and P_4 that are located on the initial plane are determined by interpolation. In this scheme four bi-characteristics, one more than the minimum, are required. The locations of bi-characteristics are not arbitrary, but are determined by the condition for the elimination of the partial derivatives in the compatibility equation (Equation 1-9). In this manner the complication that arises from determining the location of bi-characteristics is traded for the simplification of the equations.

V. The method that was introduced by Strom (70) is not much different from Butler's scheme (Figure 1-9). He called it "Network of intersections of streamlines with reference planes." In this method point P_o is chosen on the initial

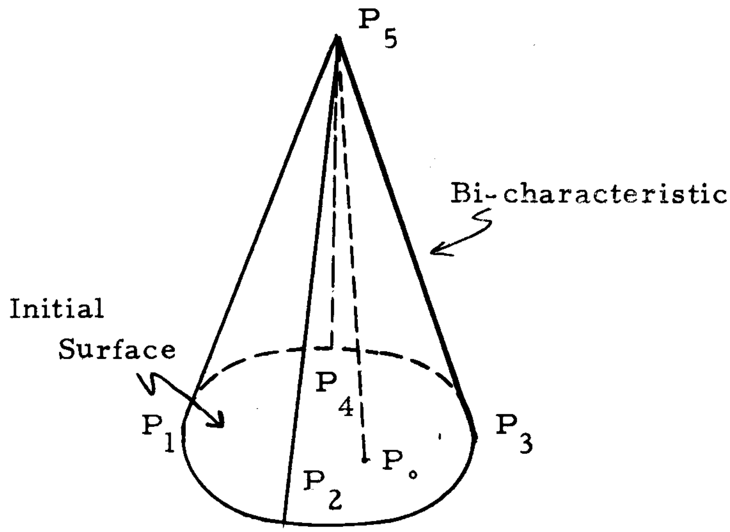

Figure 1-9. Bi-characteristic network.

plane, then letting the streamline that is passing through point P_o cut the plane $t = t_o + \Delta t$ at point P_5. From point P_5 four bi-characteristics $\overline{P_5P_1}$, $\overline{P_5P_2}$, $\overline{P_5P_3}$ and $\overline{P_5P_4}$ are passed back to the initial surface. Properties at point P_5 are guessed and improved by iteration. In this method point P_o is fixed in the iteration process and other points drift around until convergence is reached. The rest of the equations, and finite differencing of them, are the same as Butler's scheme. It is clear that the obtained information on plane $t = t_o + \Delta t$ would not be regular, i. e., on a planned and desired order. Thus it requires involved search schemes. He has improved the interpolation by writing a surface fit to the properties of the adjacent points. His claim that this method avoids complications and unnecessary interpolations is not quite justified and is discussed further in Section 1.12. Thus in overall picture, this method does not present any outstanding advantages over Butler's scheme.

1.7 Practical Application of Schemes

The following list is of the major contributions to the field by those who have used the method of characteristics:

I. Tsung (73) has considered the following cases:

   a. Pressure distribution over a delta wing, which shows a good improvement over the results obtained from linearized theory and agrees well with the experimental data of NACA.

   b. Supersonic flow past a conical boattail with elliptic cross section.

II. Feldman (19) presents his numerical results based on the method of characteristics for an inviscid hypersonic flow in local thermal equilibrium about a long hemispheric cylinder. His conclusion is criticized by Cheng and Chang (8).

III. Sauerwein (64) has considered four problems as listed below:

   a. Steady supersonic flow over a circular cylinder with its axis normal to the free stream. The flow has a detached shock wave which is the boundary of the field on one side.

b. In this case, the flow conditions remained the same as case (a), and the body shape was warped symmetrically from circular cross-section to an elliptic cross-section in a short length of time.

c. In this case, the flow conditions remained the same as case (a), and the cylinder was flattened into an ellipse symmetrically in a manner that the final elliptic section was at an angle of attack of 20°.

d. In this case, the flow conditions remained the same as case (a), and the cylinder was oscillating vertically with a time dependency of $(1 - \cos \omega t)$.

IV. Strom (70) has used the MOC to solve the following problems:

a. Flow field about a spherically blunted 15° cone at zero angle of attack. He has five initial meridional ray located 22.5° apart. It took 205 steps to reach a distance of 8 nose radii down the body. The computer run time has been 90 minutes on the IBM 7094.

b. The flow field about the same body as (a) with 10° angle of attack. He has used 16 meridional rays located 11.33° apart. Because of computing time limitations, the computation has progressed a distance of 3.1 nose radii with the rate of one nose radius per hour on the IBM 7094.

c. The flow field around a spherically blunted elliptical afterbody at zero angle of attack. The required time has been 90 minutes on an IBM 7094 to calculate 3 nose radii using an average step of 0. 03 radii. Strom has given consideration to real gas flow, but, because of computation time and the cost involved, he has run the perfect gas model. His schemes can be utilized for solving chemically frozen and finite-rate reacting flows.

1. 8 Stability and Convergence

In any numerical approach, the solution to a set of partial differential equations is represented by the values of the dependent variables at certain discrete values of the independent variables. In general, there are three types of errors which would cause the numerical values to be different from those which would be obtained from the exact solution of the set of equations. These are truncation errors, round-off errors, and errors which appear in the initial and boundary values. Cumulative effects of these errors upon the solution can be controlled by refining the mesh size. If the reduction of mesh size to zero results in growth of departure of the numerical solution from the exact solution, the scheme is unstable. Thus, a differential scheme is called stable if solutions of the differential equation are uniformly bounded functions of the initial data for all sufficiently small mesh sizes.

A differential scheme is called convergent if the solution of the difference equation tends to that of the differential equation as mesh size tends to zero. In view of the definitions of convergence and stability, it is clear that convergence means stability. Lax and Richtmyer (47) succeeded to prove that the converse is also true; that is, stability gives convergence. Hence, we need a criterion that assures stability. For a set of first order partial differential equations with constant coefficients; Courant, Friedrichs, and Lewy (13) determined a necessary condition for stability (that is, C-F-L stability condition) of a difference scheme, that is, the domain of dependence of the difference scheme must contain the domain of influence of the differential equation. The domain of dependence of the difference scheme is called the "convex hull" of the difference scheme and is obtained by connecting with straight-line segments the outermost points used in solving the difference equations. Hahn (27), using the work of Lax (46) has shown that the C-F-L condition is a sufficient condition for stability of all simplical networks. By simplical networks is meant that in the initial value surface of dimension n; $n+1$ points are required to determine a new point. The network proposed by Holt, Ferrari, and Butler are nonsimplical. This is evident by referring to Figures 1-7, 1-8 and 1-9. In these networks four points on the initial surface are used to calculate the new point. Since the initial manifold is two dimensional, thus n+1 is three. However, the C-F-L condition is a necessary and sufficient condition for the stability of simplical networks but it is only necessary condition for nonsimplical networks (31). For the latter, further test should be used against von Neumann stability criteria (31).

A necessary condition for stability of linear difference equations with constant coefficients, attributed to von Neumann, is that the eigenvalues of the amplification matrix of the difference equations must be less than one in absolute value. Richtmyer (61) reported that von Neumann condition has turned out to be sufficient as well as necessary for the stability of all of the difference schemes that he has investigated. Thus the von Neumann condition is a stronger condition than the C-F-L condition. Lax and Richtmyer (47) have shown as modified form of the von Neumann condition to be sufficient criterion for the stability of linear partial differential equations with constant coefficients.

Friedrichs (25) has shown that the sufficient condition for stability of linear symmetric hyperbolic partial differential equations with constant coefficients is the positiveness of the coefficient matrices in the difference scheme. Lax (46) extended Friedrichs condition to nonsymmetric equations. Also Strang (69) has shown that the convergence of the nonlinear equations depend uniquely upon the stability of the linear equations for the case where the solutions of the equations possess enough continuous derivatives.

The linearized equations of steady supersonic or unsteady compressible flow with consideration of certain assumptions fall in this category. Hence, stability and convergence criteria can be applied locally. If we assure the smoothness of solution, this restriction that the stability criteria to be met locally can be removed. However, the smoothness of solution and derivatives cannot be proved until actually the problem has been solved. Thus these two depend on each other and there is no way out of it. But we resort to the intuition from the physical configuration and conditions of the problem.

Out of the networks that were mentioned earlier, Sauerwein and Sussman (65) found that the calculation scheme associated with the tetrahedral characteristic line network was unstable. Their approach has been the actual utilization of the scheme for solving the flow of a perfect gas about a circular cylinder with its axis normal to a Mach 5 free stream. The instability occurrred in their results. As they reduced the mesh size, the calculated pressure started to fluctuate and grow beyond a limit. Thus, according to the definition, the scheme is unstable. Instead, they proposed modified tetrahedral characteristic line network.

The tetrahedral characteristic surface network is simplical and it is a stable scheme. Tsung (73) has utilized this scheme for solving his problem and he has not reported any instability in the scheme.

In nonsimplical networks such as the network of intersections of reference planes with characteristic surfaces, Butler's network, and Strom's network, all of the base points or one of them drift around in the initial surface as iteration process goes on. These cases require some discipline of interpolation in the initial surface. The interpolation schemes should be in such manner that restrict the domain of dependence of the difference scheme to the extent that it contains the domain of influence of the differential equation.

1.9 Points on a Body

Calculation of the flow properties for the points located on a body is not much different from regular points which we call them field points, and follows the same routine with some modification. For the sake of generality, we assume that the body surface can be described by the function $f(r, z) = 0$, in a plane θ = constant.

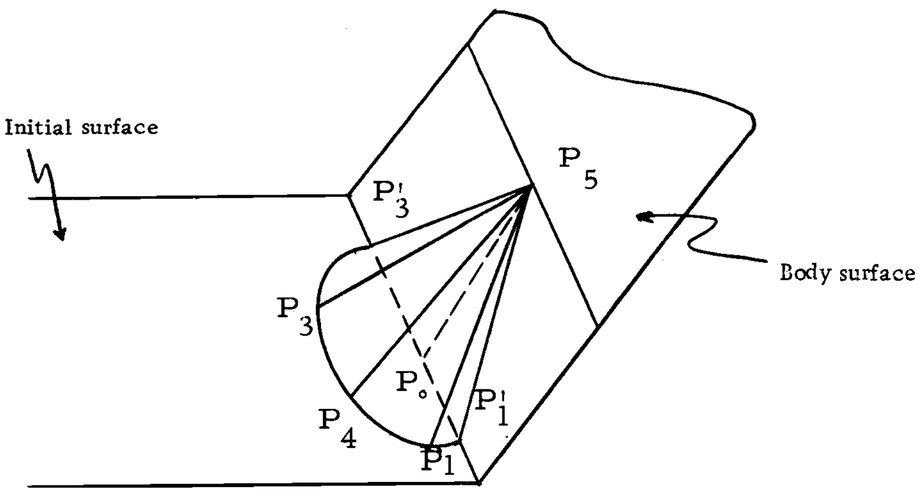

Figure 1-10. A body point.

Steps are as follows:

1. Make a guess about the unknowns at point P_5 on the body.

2. Locate three points, P_1, P_3 and P_4, by finite differencing the relations

$$dr = (u + a\cos\alpha)dt \tag{1-15}$$

$$dz = (v + a\sin\alpha)dt \tag{1-16}$$

3. Make a surface fit to the flow conditions on the initial surface around these three points including P_1' and P_3' points on the body.

4. Locate point P_o along the pathline. Considering the points P_5 and P_o on the body, we can write the following auxiliary equation

$$\frac{v_5}{u_5} = \left(\frac{f_r}{f_z}\right)_5$$

5. Finite differencing the compatibility equation (Equation 1-9) along $\overline{P_5P_3}$, $\overline{P_5P_4}$, $\overline{P_5P_1}$ and the continuity equation (Equation 1-14) along the pathline $\overline{P_5P_o}$.

6. Solve five equations, resulting from steps 4 and 5, for five unknowns.

7. Repeat the procedure, using the obtained data as a rough guess, until the convergence is reached.

1.10 Points on a Free Surface

One of the good features of the MOC is that it can handle free surfaces very easily, even easier than the regular field points. This is due to the fact that the pressure usually does not change with time on the free surfaces. Again, steps to be followed are the same as the field points which is explained below:

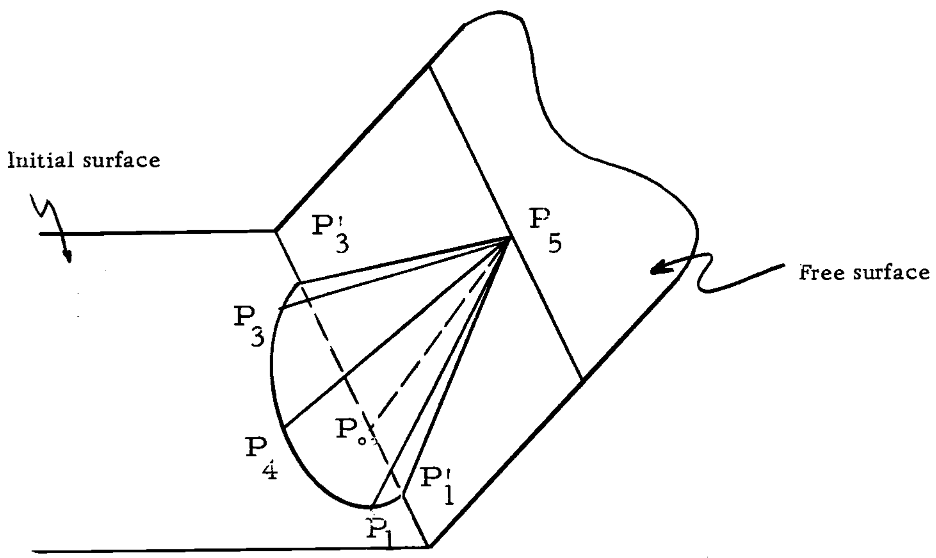

Figure 1-11. A point on a free surface.

1. Make a rough guess about the unknowns at point P_5.
2. Locate three points, P_3, P_4 and P_1, on the initial surface by finite differencing the relations

$$dr = (u + a\cos\alpha)dt$$

$$dz = (v + a\sin\alpha)dt$$

3. Make surface fits to the flow conditions on the initial plane around the points P_1, P_3 and P_4 including P'_1 and P'_3, points on the free surface.

4. Locate point P_o by tracing back point P_5 to the initial plane.

5. Finite differencing the compatibility equation (Equation 1-17)

$$dp + a\rho \cos\alpha du + a\rho \sin\alpha dv$$

$$= \rho a^2 dt\left[-\sin^2\alpha\frac{\partial u}{\partial r} - \cos^2\alpha\frac{\partial v}{\partial z} + \sin\alpha\cos\alpha\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right) - \frac{u}{r}\right]$$

(1-17)[A-34]

which is derived in Appendix A, Section A. 2, along the bi-characteristics $\overline{P_5P_1}$, $\overline{P_5P_4}$, $\overline{P_5P_3}$ and the continuity equation along the pathline $\overline{P_5P_o}$.

6. The condition on the free surface provides an auxiliary equation

$$p_5 = p_o$$

7. Solve five equations, resulting from steps 5 and 6, for five unknowns.

8. Repeat the procedure, using the obtained data as a rough guess, until the convergence is reached.

1.11 Points on a Shock Wave

The procedure that should be followed for calculation of the shock points varies due to the fact that the shock might be either stationary or a moving shock that advances into the fluid with or without preservation of its shape. Since the latter is discussed as a separate problem by itself in the second chapter, we proceed with the points located on a moving shock with preserved shape, and a stationary shock.

The process of calculation of the points on a shock is similar to the ones discussed before. However, shocks are discontinuities in the fluid flow and require more calculations and more complex treatments. Due to the nature of a shock wave, the flow in front of it is supersonic with respect to the shock. For calculations of the flow conditions on the plane $t = t_o + \Delta t$, all points in front of the shock including the points on the shock can be considered as field points and can be treated similarly. The flow behind the shock is subsonic with respect to the shock so that the conoid there intersects the shock surface. The pathline from a point on the shock cuts the initial plane at a point in front of the shock. Thus the part conoid behind the shock does not contain a streamline.

To find the unknowns, we have to use the Rankine-Hugoniot shock conditions that relate the flow conditions in front of the shock to the flow conditions behind the shock at point P_5. However, another

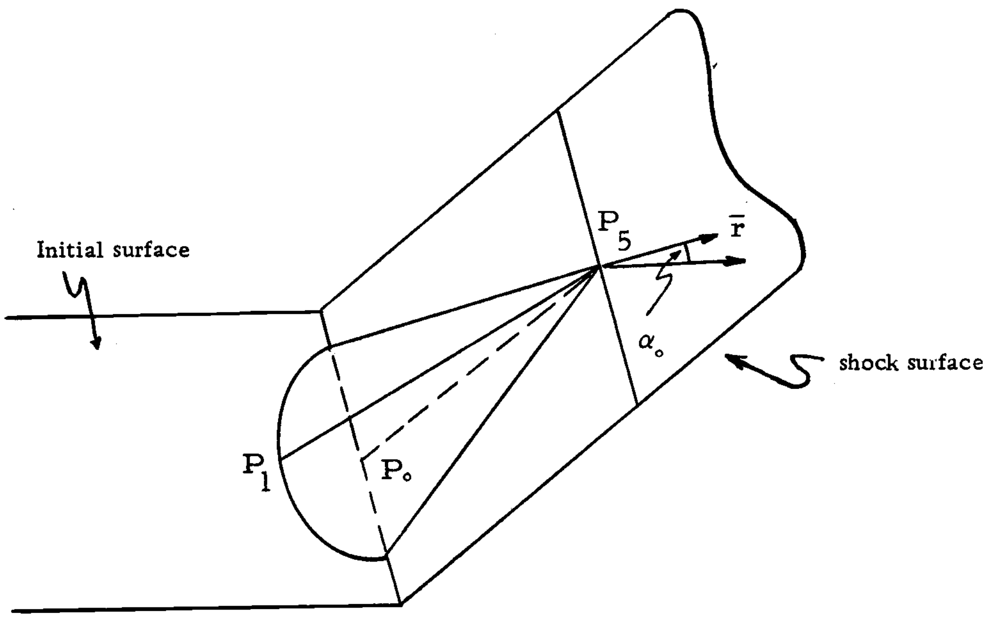

Figure 1-12. A point on a shock surface.

unknown, namely the speed of progress of the shock, is added to the system. The ensuing steps should be followed:

1. Pick up point P_o on the initial surface on the shock curve.
2. Rotate the coordinate system by an angle α_o. α_o is the angle which the normal to the shock surface at point P_5 makes with r axis. To designate the variables in the new coordinate system, we add a bar on the top of the old ones. In the new system the compatibility equation is written

$$dp + \rho a \cos(\alpha-\alpha_o) d\bar{u} + \rho a \sin(\alpha-\alpha_o) d\bar{v}$$

$$= \rho a^2 dt\left[-\sin^2(\alpha-\alpha_o)\frac{\partial \bar{u}}{\partial \bar{r}} - \cos^2(\alpha-\alpha_o)\frac{\partial \bar{v}}{\partial \bar{z}} + \sin(\alpha-\alpha_o)\cos(\alpha-\alpha_o)\left(\frac{\partial \bar{u}}{\partial \bar{z}}+\frac{\partial \bar{v}}{\partial \bar{r}}\right) - \frac{u}{r}\right] \tag{1-18}$$

$$\bar{u} = u \cos \alpha_o + v \sin \alpha_o \tag{1-19}$$

$$\bar{v} = -u \sin \alpha_o + v \cos \alpha_o \tag{1-20}$$

Details of derivation are given in Chapter II.

3. Find the new location of P_o, that is, P_5 on the plane $t = t_o + \Delta t$ by finite differencing

$$\left(\frac{d\bar{r}}{dt}\right)_{shock} = \bar{u} + U \tag{1-21}$$

4. Finite differencing the following relations to find the location of P_1, with a rough guess of flow conditions at point P_5.

$$\frac{d\bar{r}}{dt} = \bar{u} + a \cos(\alpha - \alpha_o) \tag{1-22}$$

$$\frac{d\bar{z}}{dt} = \bar{v} + a \sin(\alpha - \alpha_o) \tag{1-23}$$

5. Make surface fits to flow conditions around P_o.

6. Finite differencing the compatibility equation (Equation 1-18) along $\overline{P_5 P_1}$.

7. Since the curvature of the shock is the same at points P_o and P_5, we can write,

$$\left(\frac{d\bar{r}}{d\bar{z}}\right)_o = \left(\frac{d\bar{r}}{d\bar{z}}\right)_5 , \text{ or}$$

$$\left(\frac{\frac{d\bar{r}}{dt}}{\frac{d\bar{z}}{dt}}\right)_{\circ} = \left(\frac{\frac{d\bar{r}}{dt}}{\frac{d\bar{z}}{dt}}\right)_5 ,$$ which is the same as

$$\left(\frac{U + \bar{u}}{\bar{v}}\right)_{\circ} = \left(\frac{U + \bar{u}}{\bar{v}}\right)_5 .$$

8. Write the Rankine-Hugoniot relations across the shock at point P_5. Denoting the flow conditions behind the shock at the point P_5 by $*$, we get,

$$\rho_5(U - \bar{u}_5) = \rho_* U$$

$$p_5 + \rho_5(U - \bar{u}_5)^2 = p_* + \rho_* U^2$$

$$e_5 + \frac{p_5}{\rho_5} + \frac{(U-\bar{u}_5)^2}{2} = e_* + \frac{p_*}{\rho_*} + \frac{U^2}{2}$$

$$\bar{v}_5 = \bar{v}_*$$

9. Solve six equations resulting from steps 6, 7 and 8 for six unknowns e_5, ρ_5, $\bar{u}_5$, $\bar{v}_5$, $\left(\frac{\partial \bar{v}}{\partial \bar{r}}\right)$ and U.

10. Iterate for convergence.

Stationary shocks are very simple to handle and they require only use of Rankine-Hugoniot relations across the shock.

1.12 Concluding Remarks

Many two-dimensional unsteady problems of the fluid flow can be solved numerically by utilizing and repeating unit processes that were outlined in this chapter. In regard to these procedures the following remarks are valid:

1. Since the error analysis for different schemes and their relations to the computer running time and economical factors is non-existent, we cannot evaluate the merits and shortcomings of the schemes and give preference to one of them For treatment of simple problems, the author has suggested the Butler's scheme. This is due to the fact that the discrete mesh points on the plane $t = t_o + \Delta t$ appear regular and evenly distributed. This distribution cuts the computer time that is likely to be spent for searching. However, for treatment of complicated problems with irregular boundary conditions, the Strom's network has been suggested and used. In this scheme, the obtained data scatter irregularly on the plane $t = t_o + \Delta t$. Thus, one has to interpolate between data on the new plane and this increases the amount of computer time and reduces the accuracy.

2. Interpolation is required for all of the schemes at the initial data surface, except "tetrahedral bi-characteristic line

network," which, as was mentioned earlier is unstable. For more accuracy, higher order fitting procedures than linear are necessary. Thus, the method of interpolation is of critical importance, in regard to the computer running time, accuracy of results and the stability.

3. The C-F-L stability criterion is a sufficient condition to insure stability of the numerical integrating schemes in most of the cases.

4. Errors associated with non-simplical networks are less, and such schemes are recommended for more accurate results.

5. The bi-characteristic networks have the advantage of following the pathlines step by step. This is especially very useful when the flow field and the manner in which the particles move are needed to be visualized.

High costs of operation of present generation of digital computers are detrimental to tackling many desired problems. This simply means that the cost is a factor which cannot be overlooked, and feasibility of solving certain problems should be weighed against the economical considerations.

II. THE METHOD OF CHARACTERISTICS IN LIQUIDS

2.1 Introduction

At shock fronts resulting from underwater explosions where the liquid is subjected to very high pressures, or in operation of high pressure valves for delivery of liquids in pipes or in liquid fuel rocket engines, compressibility and elastic behavior of liquids manifest themselves. Thus, the respective fluid flow in such instances should be regarded compressible and should be treated according to the principles involved.

There is a tremendous amount of literature and research available on different aspects of one-dimensional study of the flow of liquids as they are considered compressible, especially in the related problems of underwater explosion. However, to the best knowledge of the author, two-dimensional problems that arise in such cases have not been dealt with. While in the air, several practical problems, as was discussed in the first chapter, have been considered and solved. Due to the need in technology, the two-dimensional problems in liquids and especially in sea water will have to be considered also. Thus, this is an attempt to set up the scheme, procedure and successive steps in order to treat such problems.

The problem which has been considered and treated is a two-dimensional pressure wave as it progresses in water. The approach is the method of characteristics as was explained in the first chapter. In the early stages of study, a knowledge of an equation of state, caloric equation of state and the variation of speed of sound with temperature and pressure for water was felt to be essential. The results of the search which has been used are presented in the Appendix B, Sections B. 1 and B. 2 respectively.

The numerical procedure is very well adaptable to the computers and is discussed in Section 2. 6.

2. 2 Two-Dimensional Shock Waves in Water

In many instances we have to consider liquids as compressible media. This simply means that the pressure or expansion applied at a point of the liquid will be transmitted as a wave disturbance to other points with a velocity, though large, which is finite.

To study the effects of damage or prevention of damage from underwater explosions, it is necessary to analyze the situation in the light of the fact that the sea water is a compressible medium. When an explosion takes place, it sets a shock wave in motion. In proximity of the explosion the pressure may reach more than 2×10^6 lb/inch^2. The velocity of propagation is several times the value of the speed of sound. The velocity of propagation of the pressure pulse decreases as the pressure front travels outward until it becomes the velocity of sound (about 5000 ft/sec.).

One of the oldest methods of handling of the one-dimensional moving shocks is the use of Riemann function. This method is limited to plane waves and cannot be used for the case of spherical-symmetric flow. Another analytical approach to shock wave propagation in water, has been developed by Kirkwood and Bethe (39); and extended by Kirkwood and co-workers (40, 41). They have made calculations for pressures up to 50 kilobars in pure water and pressures up to 90 kilobars in sea water (0.7 molal solution NaCl) at 0°, 20°, and 40° C. The work of Richardson _et al_ (60) and the compilation of Cole (11) are among the main contributions to the field. In recent years electromagnetic energy has been utilized to study experimentally the shock waves progress in water with pressure fronts in excess of 200 kilobars. Neodymium and ruby lasers were used to generate shocks with velocities as high as 2.3×10^4 ft/sec (1). It appears that lasers can serve as useful tools for extending the range of existing shock data in water. Use of laser promises an extension of present data well into the megabar region.

The present investigation concerns the study of a two-dimensional axial-symmetric shock wave that moves into the water. One area of application would be blast waves resulting from the explosion of a cylindrical charge. Spherical and plane shock waves are very idealized models of real phenomena. Even in shock tubes when the diaphragm is being ruptured the shock that moves into the fluid is an axial-symmetric one rather than plane. In here we will not discuss the sequence of events that takes place from early stages of explosion to the eventual upward travel of gas bubbles from the explosion. Rather, we will try to investigate a method that is able of tracking the spatial travel history of the shock and the flow field at each point. It would be desirable to consider a type of process which is called detonation. In such a process a detonation wave initiated at one point sweeps through the explosive material with a steep front. Each particle of the explosive as the front passes over it might not undergo a complete chemical change. A rigorous and realistic analysis of explosion, in this manner, should be based on the incorporation of chemical kinetics and the principles of gas dynamics. This is due to the fact that when the disturbance resulting from an explosion reaches the boundary between the explosive and the water a steep pressure wave moves into the water and an expansion wave into the explosive. Such an analysis that takes into account both phenomena will not be executed. The chemical and thermodynamical considerations of the explosion can provide us with the knowledge of the initial conditions. Thus, with the knowledge of the initial conditions, the problem is properly posed and we can proceed with the valid assumptions of the Chapter I.

2. 3 Equations of Motion

Continuity Equation:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + v\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z} + \frac{u}{r}\right) = 0 \qquad (2\text{-}1)$$

Momentum equation in r and z directions:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + v\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial r} \qquad (2\text{-}2)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + v\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial z} \qquad (2\text{-}3)$$

If no dissipation process takes place, except in the shock, and an element moving with the fluid cannot exchange heat with any other element or surroundings, we can write

$$\frac{DS}{Dt} = 0$$

In this fashion, pressure is a single-valued function of density.

2. 4 Method of Solution

The flow in front of the shock is undisturbed and we would like to find the velocity and the shape of the shock and flow conditions behind it as it progresses into the water.

It would be beneficial to replace the partial derivatives of the density in the continuity equation with the partial derivatives of pressure as follows

$$\frac{Dp}{Dt} = a^2 \frac{D\rho}{Dt}$$

Thus, the continuity equation is written in the following fashion

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial r} + v\frac{\partial p}{\partial z} + \rho a^2(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z} + \frac{u}{r}) = 0 \qquad (2\text{-}4)$$

The momentum equation remains unchanged

$$\rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + v\frac{\partial u}{\partial z}) + \frac{\partial p}{\partial r} = 0 \qquad (2\text{-}5)$$

$$\rho(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + v\frac{\partial v}{\partial z}) + \frac{\partial p}{\partial z} = 0 \qquad (2\text{-}6)$$

Now, we have a set of three first-order quasi-linear partial differential equations. To arrive at a solution, we apply the method of characteristics as was explained in Chapter I. If we use bi-characteristics as paths along which numerical integration is performed, the proper compatibility equation for the set of equations will be obtained in the following form

$$dp + \rho a \cos\alpha du + \rho a \sin\alpha dv$$
$$= \rho a^2 d\tau[-\sin^2\alpha\frac{\partial u}{\partial r} - \cos^2\alpha\frac{\partial v}{\partial z} + \cos\alpha\sin\alpha(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}) - \frac{u}{r}]$$

(2-7)[A-34]

The details of derivation of the compatibility equation (Equation 2-7) can be examined in Appendix A, Section A. 2.

We use the modified Tait equation of state (60) for sea water

$$\log\left(\frac{\upsilon(0,S)}{\upsilon(p,S)}\right) = \frac{1}{n}\log\left(1 + \frac{p}{A(S)}\right) \tag{2-5}$$

The speed of sound can be obtained by partial differentiation of the equation of state.

$$a^2 = -\upsilon^2\left(\frac{\partial p}{\partial \upsilon}\right)_S$$

We can perform the differentiation and find the constant A(S) in terms of a(0,S). The result is

$$A(S) = \frac{a^2(0,S)}{n\,\upsilon(0,S)}$$

Thus the modified Tait equation of state, suitable to our needs in here would be

$$\log\frac{\upsilon(0\ S)}{\upsilon(p,S)} = \frac{1}{n}\log\left(\frac{p}{\frac{a^2(0,S)}{n\,\upsilon(0,S)}} + 1\right) \tag{2-9}$$

If we designate a(0, S) and ρ (0, S) by $a_\circ$ and $\rho_\circ$ respectively, we obtain

$$a(p,S) = a_\circ\left\{1 + \frac{p}{\frac{a_\circ^2}{n\,\upsilon_\circ}}\right\}^{\frac{n-1}{2n}} \tag{2-10}$$

$$\rho(p,S)a(p,S) = \rho_\circ\, a_\circ\left\{1 + \frac{p}{\frac{a_\circ^2}{n\,\upsilon_\circ}}\right\}^{\frac{n+1}{2n}} \tag{2-11}$$

Since S remains constant along each pathline, a_o is a constant along each pathline and for each point it can be traced back to the point in the initial plane. In this manner, we can use the relations of Section B.2 to find a_o at each point on the initial plane as is dictated by the initial conditions.

Finally, the compatibility equation can be written in the following form

$$dp + \rho_o a_o \left\{1 + \frac{p}{\frac{\rho_o a_o^2}{n}}\right\}^{\frac{n+1}{2n}} (\cos\alpha du + \sin\alpha dv)$$
$$= \rho_o a_o^2 d\tau \left(1 + \frac{p}{\frac{\rho_o a_o^2}{n}}\right)\left\{-\sin^2\alpha\frac{\partial u}{\partial r} - \cos^2\alpha\frac{\partial v}{\partial z} + \cos\alpha\sin\alpha\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right) - \frac{u}{r}\right\} \qquad (2\text{-}12)$$

Before starting the numerical calculation procedures, for facilitating the process, we make a coordinate transformation. We call the new coordinate system the shock coordinates. At each point $\bar{r}$ lies in the direction of the normal to the curve and $\bar{z}$ makes a right handed coordinate with it. The independent variables in the shock coordinates are denoted by the old ones with a bar on top of them. Thus we can write the new variables in terms of old ones and vice versa. $\bar{\alpha}$ is defined below and the angle β is shown in Figure 2-1.

$$\bar{\alpha} = \alpha - \beta \qquad (2\text{-}13)$$

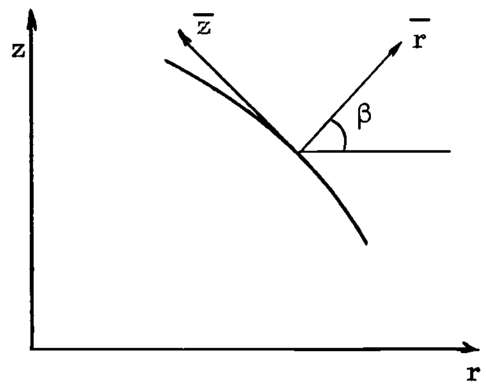

Figure 2-1. Shock coordinates.

Components of the velocity in the old coordinates transform as follows

$$\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \qquad (2\text{-}14)$$

The transformation of the velocity components in the shock coordinates is given below

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \qquad (2\text{-}15)$$

Transformation of $\cos\alpha du + \sin\alpha dv$, using relations (2-13) and (2-15) with the help of trigonometric identities result in the relation (2-16).

$$\cos\alpha du + \sin\alpha dv = \cos\bar{\alpha} d\bar{u} + \sin\bar{\alpha}\, d\bar{v} \qquad (2\text{-}16)$$

Relation (2-16) is obtained by the help of trigonometric manipulation. Also, we can write $\frac{\partial v}{\partial r} + \frac{\partial u}{\partial z}$ in matrix form

$$\frac{\partial v}{\partial r} + \frac{\partial u}{\partial z} = (\frac{\partial}{\partial r}\ \frac{\partial}{\partial z})\binom{v}{u}$$

whose partial transformation is

$$\frac{\partial v}{\partial r} + \frac{\partial u}{\partial z} = \begin{pmatrix} \frac{\partial}{\partial \bar{r}} & \frac{\partial}{\partial \bar{z}} \end{pmatrix} \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix}$$

The complete transformation of $\frac{\partial v}{\partial r} + \frac{\partial u}{\partial z}$ is obtained as follows

$$\frac{\partial v}{\partial r} + \frac{\partial u}{\partial z} = \begin{pmatrix} \frac{\partial}{\partial \bar{r}} & - \frac{\partial}{\partial \bar{z}} \end{pmatrix} \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \sin\beta & \cos\beta \\ \cos\beta & -\sin\beta \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \tag{2-17}$$

Carrying out the matrix multiplications and simplifying the results gives

$$\frac{\partial v}{\partial r} + \frac{\partial u}{\partial z} = \sin 2\beta\, \frac{\partial \bar{u}}{\partial \bar{r}} + \cos 2\beta \frac{\partial \bar{u}}{\partial \bar{z}} + \cos 2\beta \frac{\partial \bar{v}}{\partial \bar{r}} - \sin 2\beta \frac{\partial \bar{v}}{\partial \bar{z}} \tag{2-18}$$

Also, we can write

$$\sin^2\alpha \frac{\partial u}{\partial r} + \cos^2\alpha \frac{\partial v}{\partial z} = \begin{pmatrix} \sin^2\alpha \frac{\partial}{\partial r} & \cos^2\alpha \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{2-19}$$

In the same manner, the complete transformation of $\sin^2\alpha \frac{\partial u}{\partial r} + \cos^2\alpha \frac{\partial v}{\partial z}$ is

$$\sin^2\alpha \frac{\partial u}{\partial r} + \cos^2\alpha \frac{\partial v}{\partial z}$$

$$= \begin{pmatrix} \frac{\partial}{\partial \bar{r}} & \frac{\partial}{\partial \bar{z}} \end{pmatrix} \begin{pmatrix} \sin^2\alpha\cos\beta & \cos^2\alpha\sin\beta \\ -\sin^2\alpha\sin\beta & \cos^2\alpha\cos\beta \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \tag{2-20}$$

Again, carrying out the matrix multiplication and simplifying the results, give

$$\sin^2\alpha\frac{\partial u}{\partial r} + \cos^2\alpha\frac{\partial v}{\partial z}$$

$$= (\sin^2\alpha\cos^2\beta + \cos^2\alpha\sin^2\beta)\frac{\partial \bar{u}}{\partial \bar{z}} + \frac{1}{2}\sin 2\beta\cos 2\beta\frac{\partial \bar{v}}{\partial \bar{r}} + \frac{1}{2}\sin 2\beta\cos 2\beta\frac{\partial \bar{u}}{\partial \bar{z}}$$

$$+ (\cos^2\alpha\cos^2\beta + \sin^2\alpha\sin^2\beta)\frac{\partial \bar{v}}{\partial \bar{z}} \tag{2-21}$$

Substitution of relations (2-18) and (2-21) in the compatibility equation (Equation (2-12), and use of transformed angle $\bar{\alpha}$ with the help of trigonometric identities result in the transformed compatibility equation in the shock coordinates as follows

$$dp + \rho_\circ a_\circ\left(1 + \frac{p}{\frac{\rho_\circ a_\circ^2}{n}}\right)^{\frac{n+1}{2n}}(\cos\bar{\alpha}\,d\bar{u} + \sin\bar{\alpha}\,d\bar{v})$$

$$= \rho_\circ a_\circ^2 d\tau\left(1 + \frac{p}{\frac{\rho_\circ a_\circ^2}{n}}\right)\{(-\sin^2\bar{\alpha}\frac{\partial \bar{u}}{\partial \bar{r}} - \cos^2\bar{\alpha}\frac{\partial \bar{v}}{\partial \bar{z}} + \cos\bar{\alpha}\sin\bar{\alpha})(\frac{\partial \bar{u}}{\partial \bar{z}} + \frac{\partial \bar{v}}{\partial \bar{r}})$$

$$- \frac{\cos\beta\bar{u} - \sin\beta\bar{v}}{\cos\beta\bar{r} - \sin\beta\bar{z}}\} \tag{2-22}$$

2.5 Shock Relations

Actually the shock wave is not a discontinuity, rather a region in which the properties of a fluid flow change sharply within a short distance that can be estimated analytically and can be measured experimentally (26). If we neglect the terms in the hydrodynamical equations involving viscosity, thermal conductivity, and diffusion, the shock wave manifests itself as a mathematical discontinuity. The equations that relate the flow conditions on both sides of this discontinuity are generally known as Rankine-Hugoniot relations. To be consistent with our assumptions, we will use the Rankine-Hugoniot relations. A more accurate discussion, that is, when the dissipative mechanisms are included in a sharp but gradual transition, is given in the Appendix B, Section B. 4, and the shock wave thickness for water is derived.

Another important feature of the transformation that was discussed in the previous section, is that in the new coordinate system, the shock wave progresses only in $\bar{r}$ direction and its velocity in the $\bar{z}$ direction is zero. Denoting the shock speed by U, we can write

Continuity Equation

$$\rho_5(U - \bar{u}_5) = \rho_* U \tag{2-23}$$

Momentum Equation

$$p_5 - p_* = \rho_* U \bar{u}_5 \tag{2-24}$$

Energy Equation

$$H_5 - H_* = \frac{1}{2}(p_5 - p_*)\left(\frac{1}{\rho_5} + \frac{1}{\rho_*}\right) \qquad (2\text{-}25)$$

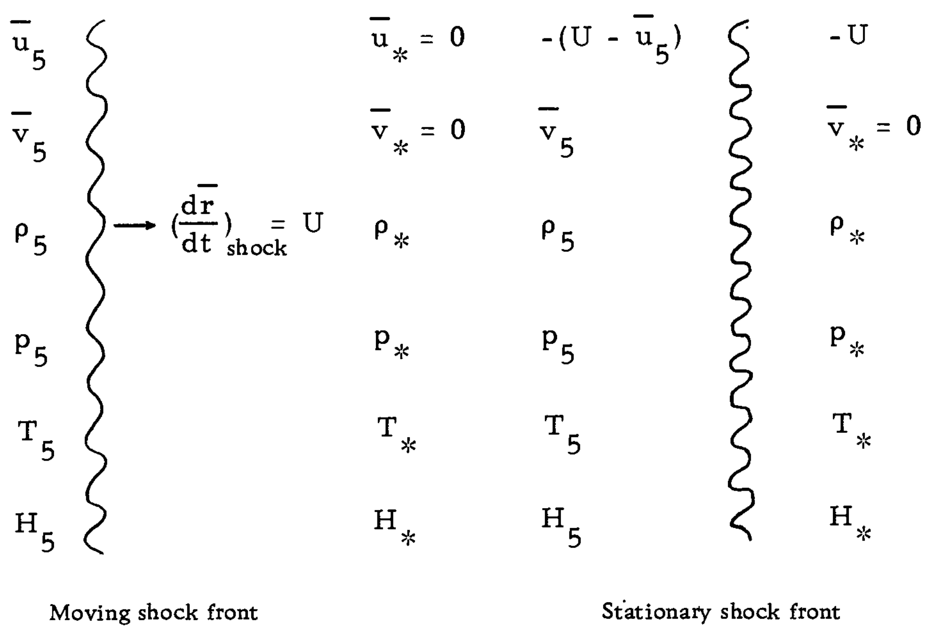

Figure 2-2. Transformation from a stationary reference frame to the reference frame moving with the shock.

Since the tangential velocity component on both sides of the shock remains unchanged, we get

$$\bar{v}_5 = \bar{v}_* \qquad (2\text{-}26)$$

Equation of state as was discussed before is given

$$\rho = \rho(p, T) \qquad (2\text{-}27)$$

The caloric equation of state is given in the Appendix B, Section B. 3. Thus, we have

$$H = H(p, T) \tag{2-28}$$

In here, we have six equations and seven unknowns: T_5, p_5, H_5, $\bar{u}_5$, $\bar{v}_5$, ρ_5, and U. Thus we need another equation that relates these unknowns together. For this purpose, we finite difference the compatibility equation (Equation 2-22) along the bi-characteristic $\overline{P_5P_1}$ (Figure 1-12). P_1 is located on a point where $\bar{\alpha} = 0$, i. e., $\alpha = \beta$. Thus we get

$$(p_5 - p_1) + \rho_\circ a_\circ \left(1 + \frac{p_1}{\frac{\rho_\circ a_\circ^2}{n}}\right)^{\frac{n+1}{2n}} (u_5 - u_1)$$

$$= \frac{\rho_\circ a_\circ^2 \Delta t}{2} \left(1 + \frac{p_1}{\frac{\rho_\circ a_\circ^2}{n}}\right)\left\{-\left(\frac{\partial \bar{v}}{\partial \bar{z}}\right)_1 - \left(\frac{\partial \bar{v}}{\partial \bar{z}}\right)_5 - \left(\frac{\cos\beta\bar{u} - \sin\beta\bar{v}}{\cos\beta\bar{r} - \sin\beta\bar{z}}\right)_1\right\} \tag{2-29}$$

Although we found the seventh relation between the variables, the number of unknowns was added by one, that is, $\left(\frac{\partial \bar{v}}{\partial \bar{z}}\right)_5$. Thus, we have to search for another relation between these eight unknowns. This relation that does not add to the number of unknowns, is differentiation along the shock.

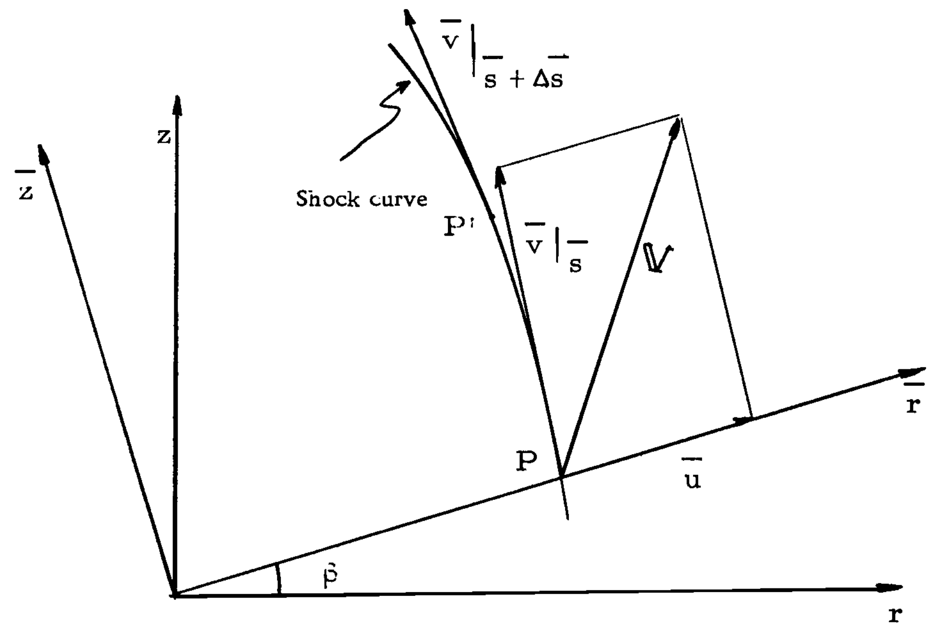

Figure 2-3. Velocity components in the shock coordinates.

We write the relation (2-26) for two points P and P', $\Delta\bar{s}$ apart ($\bar{s}$ in here denotes the length of the curve and should not be confused with the salinity).

$$(\bar{v}\,\vec{e}_{\bar{z}})_5\Big|_{\bar{s}+\Delta\bar{s}} = (\bar{v}\,\vec{e}_{\bar{z}})_*\Big|_{\bar{s}+\Delta\bar{s}} \tag{2-30}$$

$$(\bar{v}\,\vec{e}_{\bar{z}})_5\Big|_{\bar{s}} = (\bar{v}\,\vec{e}_{\bar{z}})_*\Big|_{\bar{s}} \tag{2-31}$$

Deduct relation (2-30) from (2-31) and divide by $\Delta\bar{s}$, then find the limit, we get

$$\frac{\partial}{\bar{R}\partial\bar{\theta}}(\bar{v}\,\vec{e}_{\bar{z}})_* = \frac{\partial}{\bar{R}\partial\bar{\theta}}(\bar{v}\,\vec{e}_{\bar{z}})_5 \tag{2-32}$$

$\bar{R}$ is the radius of curvature and $\Delta\bar{s} = \Delta\bar{z} = \bar{R}\Delta\bar{\theta}$

Since $\frac{\partial}{\partial\bar{\theta}} \vec{e}_{\bar{z}} = -\vec{e}_{\bar{r}}$, we get

$$\left(\frac{\partial\bar{v}}{\partial\bar{z}}\right)_* \vec{e}_{\bar{z}} - \bar{v}_* \vec{e}_{\bar{r}} = \left(\frac{\partial\bar{v}}{\partial\bar{z}}\right)_5 \vec{e}_{\bar{z}} - \bar{v}_5 \vec{e}_{\bar{r}} \tag{2-33}$$

or

$$\left(\frac{\partial\bar{v}}{\partial\bar{z}}\right)_* = \left(\frac{\partial\bar{v}}{\partial\bar{z}}\right)_5 \tag{2-34}$$

Relation (2-34) is the eighth equation that we are after.

2.6 Outline of the Computational Procedure

We can proceed with the following steps to solve the problem.

1. Pick a point P_o on the shock curve on the initial surface $t = t_o$.

2. Find the coordinates of the new point P_5 on the surface $t = t_o + \Delta t$, by finite differencing the relation $\left(\frac{d\bar{r}}{dt}\right)_{shock} = U$, or

$$r_5 - r_o = \frac{U_5 + U_o}{2} \Delta t \tag{2-35}$$

   This requires a guess of the shock velocity U at the point P_5.

3. Find the angle β at the point P_5.

4. Find the coordinates of P_1 by finite differencing the

relations

$$\frac{dr}{dt} = u + a\cos\alpha \tag{2-36}$$

$$\frac{dz}{dt} = v + a\sin\alpha \tag{2-37}$$

In here, we transform relations (2-36) and (2-37) to the shock coordinates, we get

$$\frac{d\bar{r}}{dt} = \bar{u} + a\cos\bar{\alpha} \tag{2-38}$$

$$\frac{d\bar{z}}{dt} = \bar{v} + a\sin\bar{\alpha} \tag{2-39}$$

Finite differencing these relations with $\bar{\alpha} = 0$, results in determination of the coordinates of point P_1

$$\bar{r}_5 - \bar{r}_1 = \bar{u}_5\Delta t + a_5\Delta t \tag{2-40}$$

$$\bar{z}_5 - z_1 = \bar{v}_5\Delta t \tag{2-41}$$

To start with, this requires a guess for the flow conditions at point P_5.

5. Make surface fits to the flow conditions around the point P_1, and find the flow conditions for that point.

6. Let the state of the fluid behind the shock on the plane $t = t_\circ + \Delta t$ be p_5 and T_5 and the state of the fluid in front of the shock in the same plane be p_* and T_*. A curve that connects these two points together is indicated in Figure

2-4 by the Hugoniot curve. Consider another path which connects the two states p_* and T_*; p_5 and T_5 together. This path consists of an isobar and an adiabatic process, as is depicted in Figure 2-4.

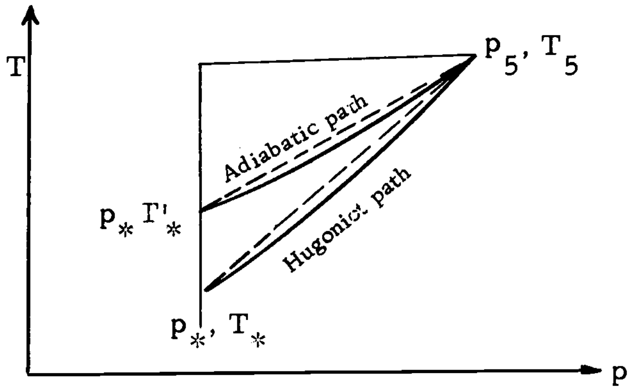

Figure 2-4. Adiabatic and Hugoniot paths in p-T plane.

As a first approximation, we assume that the adiabatic path coincides with the Hugoniot path. Since A(S) does not vary along the adiabatic path, we choose A(S) corresponding to the point P_* and T_*. We call it A_*.

7. Calculate ρ_5 and a_5, with the knowledge of p_5 and T_5.
8. Solve six Equations (2-23), (2-24), (2-25), (2-26), (2-29) and (2-34); for six unknowns: ρ_5, H_5, $\bar{v}_5$, $\bar{u}_5$, U, and $(\frac{\partial \bar{v}}{\partial z})_5$.
9. Try to find the state at the point where $p = p_*$ and $T = T_*$ in Figure 2-4 as follows:

a.

$$\int_{p_*, T_*}^{p_5, T_5} dH = H_5 - H_* = \Delta H$$

b.

$$dH = TdS + \upsilon dp$$

c.

$$\Delta H = \int_{T_*}^{T'_*} C_p(p = p_*, \ T = y)dy + \int_{p_*}^{p_5} \upsilon\,(p, A'_*)dp$$

Step (8. c) is a loop and should be solved by successive approximations. A'_* and n are functions of T and are obtained from the tables given for the modified Tait equation of state.

10. Replace the old values of $\bar{u}_5$, $\bar{v}_5$, p_5, ρ_5, T_5, H_5 and U by the new calculated values; and repeat the procedure until the convergence is reached.

III. THE METHOD OF CHARACTERISTICS IN GAS-LIQUID MIXTURES

3.1 Introduction

The phenomenon of liquid response to a sudden pressure reduction below the saturation pressure corresponding to its temperature has been of great concern to technology and is given various names in the literature, including spontaneous boiling, homogeneous boiling, and flashing. The liquid in this situation is in a metastable condition which will change to a two-phase equilibrium state of saturated liquid and vapor. The phenomenon is of direct interest to design engineers in throttling valve design, in pressure vessel dump operations in hydrodynamic machines, in desalinization equipment design, in cryogenic processes in liquid fuel rocket engines and pumps, in design of safety measures of boilers and nuclear reactors and finally in related area of cavitation. The importance of cavitation phenomenon is well recognized by both the theoretical physicist and design engineers. In particular, the engineers in the field of underwater ordnance are constantly confronted with problems having to do with performance and corrosion of propulsion systems and missles.

Cavitation was recognized as a technical problem during the first decade of this century in connection with an increase in rotational speed of ship propellers. Thus it came under focus and consideration.

The theories that were developed in the early twenties, all based on the classical assumption that cavitation takes place instantaneously when the local fluid pressure drops to the equilibrium vapor pressure of the flowing liquid. The validity of this assumption was questioned for the first time in the mid thirties. War-time torpedo investigations led to detailed investigations of the form and nature of cavitation in the late forties. The noise produced by cavitation became decisive and led to new standards of design for the elimination of cavitation. Modern technological developments and design criteria demanded more knowledge, and created renewed interest in the study of cavitation and bubble development, growth or collapse, as man stepped on the threshold of space age. Soon, it became apparent that the departure from the classical assumptions not only was necessary but inevitable, especially in rocket engines. In this fashion another group of researchers became involved in the study of the field and added to its tempo. The matter, in connection with the development of nuclear reactors attracted the attention of the investigators in the field of heat transfer and it came seriously under their consideration that in turn added to its momentum. Out of this effort, a revitalized field emerged that for a long time had been considered classic and given the title of bubble dynamics.

3.2 Survey of the Work Done

Lord Rayleigh (58) took the first step towards an understanding of the process of bubble growth or collapse and actually he initiated a theory that happened to become the foundation of the analytical research of the future investigators. His formulation is very simple and utilizes the results of unsteady potential flow around a sphere. His equation is given as,

$$\rho_L(R\ddot{R} + \frac{3}{2}\dot{R}^2) = \Delta p \qquad (3\text{-}1)$$

where

$$\Delta p = p_g - p_\infty$$

He neglected the dynamic effects of the gas, as well as the effect of viscosity and surface tension at the gas-liquid interface. By neglecting the pressure inside the bubble and assuming Δp to be time independent, he arrived at the following solution that can be evaluated easily.

$$t = \sqrt{\frac{3\rho_L}{p_\infty}} \int \frac{R^{3/2}dR}{(R_\circ^3 - R^3)^{1/2}} \qquad (3\text{-}2)$$

The model presented by Rayleigh was an oversimplification of a more complex process. Δp = constant, amounts to neglecting the internal pressure of the bubble in the case of collapse or the pressure at infinity for the case of expansion. However, Δp does not remain constant and should be determined according to the laws of thermodynamics which in turn depend on the amount of heat transferred into the system. Moreover, from the well known facts of physics that a fluid exerts a pressure of $\frac{2\sigma}{R}$ inside a cavity, the terms on the right-hand side of the Rayleigh equation should be modified to read $\Delta p - \frac{2\sigma}{R}$. On the other hand, the Clausius-Clapeyron equation connects the pressure difference to the temperature difference, that is, the difference in saturation temperature inside and at great distance from the bubble.

$$\Delta p = \frac{L}{T(\upsilon_g - \upsilon_L)} \Delta T \tag{3-3}$$

When a bubble starts expanding, the temperature of the bubble wall necessarily decreases owing to evaporation at the interface. The temperature of the bubble, for all practical purposes, is the same as the bubble wall temperature which is time dependent. Thus, ΔT becomes time dependent whereby Δp becomes time dependent. Determination of the wall temperature involves the solution of a problem of heat conduction in a medium with specified motion of the boundary. In this manner, the Rayleigh equation is no longer an integrable differential equation, but it is an integrodifferential equation which is not amenable to direct analytical manipulations and should be approached otherwise. The complete form of the differential equation is given as follows

$$R\ddot{R} + \frac{3}{2}\dot{R} + \frac{2\sigma}{\rho_L R} = \frac{L \Delta T_\circ}{\rho_L T(\upsilon_g - \upsilon_L)} - \frac{L^2 \rho_g}{\rho_L^2 T(\upsilon_g - \upsilon_L)(\pi a)^{1/2} C_L}$$

$$\int_0^t \frac{R(\xi)\dot{R}(\xi)}{R(t)(t-\xi)^{1/2}} \cdot \left\{\exp\left[-\frac{(R(t) - R(\xi))^2}{4a(t-\xi)}\right] - \exp\left[-\frac{(R(t) + R(\xi))^2}{4a(t-\xi)}\right]\right\} d\xi \qquad (3\text{-}4)$$

Forster and Zuber (23, 24), and Plesset and Zwick (55, 56) tried to simplify and solve the equation. They have arrived at different solutions on the basis of different arguments. Although the former claims that the assumptions and thus the solution of the latter is in error, their solutions are in agreement for a good range of data. Forster and Zuber simplified the integral by comparison of the terms in Equation (3-4). Their final version of the equation is

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 + \frac{2\sigma}{\rho_L R} = \frac{L \Delta T_\circ}{\rho_L T(\upsilon_g - \upsilon_L)}$$

$$- \frac{L^2 \rho_g}{\rho_L^2 T(\upsilon_g - \upsilon_L)(\pi a)^{1/2} C_L} \int_0^t \frac{\dot{R}(\xi)}{(t-\xi)^{1/2}} \, d\xi \qquad (3\text{-}5)$$

They raised the question whether at superheats as encountered in boiling liquids, the hydrodynamic forces assume great importance. The answer to this question was followed by investigating whether neglect of hydrodynamic forces for small bubbles will result in great initial grow rates. Here, neglect of hydrodynamic forces is the same as the assumption of a liquid with vanishing inertia. First, by nondimensionalizing the equation, they showed that the coefficients of the hydrodynamic forces are much smaller than the other two terms. Second, they established that the neglect of hydrodynamic forces will not bring about a solution with unbounded growth rate. On the basis of such arguments they presented a closed solution which is valid almost in entire range of the validity of the differential equation

$$\frac{R}{R_\circ} + \ln \frac{\frac{R}{R_\circ} - 1}{r_1 - 1} = Ct^{1/2} \qquad (3\text{-}6)$$

$$t > \left(\frac{r_1}{C}\right)^2$$

$$r_1 = 1.015$$

$$C = \frac{\Delta T_\circ C_L \rho_L (\pi\, a)^{1/2}}{L \rho_g R_\circ} \qquad (3\text{-}7)$$

As it appears, C is not a constant and should be determined in subranges for more accuracy.

The solution of Plesset and Zwick (55) is based on a method of successive approximations by perturbation method. The zero-order solution of the problem is

$$\frac{T_\omega - T_\infty}{T_s - T_\infty} = - \frac{\rho_g L}{\rho_L C_L \Delta T_\circ \rho_\circ} \frac{R_\circ}{} \int_0^{\frac{\alpha t}{R_\circ^2}} \frac{R^2(z)\dot{R}(z)\rho_g(T_w)}{\left[\int_z^{\frac{\alpha t}{R_\circ^2}} R^4(y)dy\right]^{1/2}} dz \tag{3-8}$$

They have given the solution for the initial growth and also the asymptotic solution when the rate of growth approaches zero. However, no solution is given to match the two limits. In the asymptotic solution for the radius, the leading term is proportional to $t^{1/2}$. Also, they have shown that the solution in the asymptotic range is not affected by the details of the mathematical model used to describe the behavior of the bubble near $\frac{R}{R_\circ} \simeq 1$. It is interesting to observe that the solution given by Forster and Zuber is also proportional to $t^{1/2}$ in the asymptotic range. Also, both groups of authors claim that their theoretical results are in agreement with the experimental data obtained by Dergarabedian (14, 15).

Looking into the physical process gives us a better understanding of the theoretical results. In the initial stages of the bubble expansion, when the forces are nearly in equilibrium, the rate of growth is very slow. But after a while, it becomes very fast. This is due to the reduction in surface tension which is an opposing force to the growth and acceleration. This process continues until the cooling effect of evaporation becomes effective. The rate of growth hereafter is controlled by the rate of evaporation at the interface. Due to the increasing thickness of boundary layer surrounding the bubble, its growth rate decreases, resulting in a decrease of internal pressure. This process continues while the bubble interface temperature approaches saturation temperature asymptotically. In this phase of growth, the product of $R\dot{R}$ is constant. As the bubble grows larger the buoyancy effects become important and will result in the bubble rising through the liquid. This problem is considered by Wittke and Chao (83). Their problem is associated with the collapse of a bubble rather than expansion. Expansion and collapse of a bubble are defined by the same type of equations and they are usually considered as two sides of one coin. Because of complications in the system of equations, their approach has been numerical integration. Florschuetz and Chao (22) have done experiment on the same problem. Also they have clarified and established the conditions under which a problem becomes heat transfer controlled or inertia controlled. Van Wijngaarden (77, 78) has worked extensively on the collective collapse of a large number of gas bubbles in water. Of some interests in this area are the contributions of Karplus (37); Hsieh and Plessest (35); and Campbell and Pitcher (7).

Those who have done experimental work are by far less in numbers than those who engaged in theoretical work. This is a field that should be exploited further. The data of Dergarabedian (15) that supports the theoretical results of Plesset and Zwick; and Forster and

Zuber are obtained from bubbles that formed on chalk dust (particle size not specified) which was sprinkled into superheated water to cause nucleation. Thus, his bubbles grew on solid particles. Other experimental workers in the field are Hewitt (32), Karplus (37) and Campbell and Pitcher (7).

3.3 Propagation of Expansion Waves in a Two-Phase Flow

Liquids are not generally considered as a compressible media at ordinary pressure range. It is only at very high pressures that the compressibility manifests itself. However, a mixture of liquid and gas becomes a compressible medium even at low pressure. In such a mixture, gas content contributes to the compressibility and liquid contributes to the density of the medium. On the basis of homogeneous and isotropic assumption which is not unrealistic, theories have been developed. The purpose of present investigation is to utilize the method of characteristics to study propagation of expansion waves in such a compressible medium in two-dimensional space and time. Also, consideration is given to the case when the bubble content is vapor.

Hsieh and Plesset (35) have shown that the linearized pressure waves propagate isothermally, although several authors have proceeded with such assumption without being justified. Classical workers have used the adiabatic model. Van Wijngaarden in one of his works (76) has used a polytropic model, to facilitate his analysis. However, the isothermal model has been adopted for this analysis. The details of treatment of the problem are given in the Appendix A. The presence of vapor changes the set of equations from hyperbolic to parabolic. Physically we can consider the presence of the vapor as a dissipating mechanism. We would like to investigate mathematically what causes the system to change from hyperbolic to parabolic. To do this, we assume a relation between the radius of a bubble and pressure as follows

$$\bar{a}_{\circ} + \bar{a}R + \bar{b}p_g + \bar{c}\,\frac{\partial p_g}{\partial t} + \bar{d}\,\frac{\partial p_g}{dx} + \bar{e}\,\frac{\partial R}{\partial t} + \bar{f}\,\frac{\partial R}{\partial x} = 0 \qquad (3\text{-}9)$$

Since the conclusion that we are after is independent of space, we dispense with two-dimensional space and will use only one-dimensional space. The following set of equations in connection with Equation (3-9) should be solved:

Density of the mixture

$$\rho = \rho_L\left(1 - \frac{4}{3}\pi n_{\circ} R^3\right) \qquad (3\text{-}10)$$

Continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \qquad (3\text{-}11)$$

Momentum equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (3\text{-}12)$$

Equation of bubble expansion or contraction

$$\rho_L\{R\frac{\partial^2 R}{\partial t^2} + \frac{3}{2}(\frac{\partial R}{\partial t})^2\} = p_g - p \qquad (3\text{-}13)$$

The following results were obtained. Details are given in the Appendix A.

1. When $\bar{e}$ and $\bar{f}$ are zero, the set of equations is hyperbolic.
2. When $\bar{c}$ and $\bar{d}$ are zero, the set of equations is hyperbolic.
3. When $\bar{d}$, $\bar{e}$ and $\bar{f}$ are zero, the set of equations is hyperbolic.
4. When $\bar{c}$, $\bar{d}$ and $\bar{f}$ are zero, the set of equations is parabolic.
5. When $\bar{c}$, $\bar{d}$ and $\bar{e}$ are zero, the set of equations is parabolic.
6. When $\bar{e}$, $\bar{f}$ and $\bar{c}$ are zero, the set of equations is hyperbolic.
7. When $\bar{e}$, $\bar{f}$, $\bar{c}$ and $\bar{d}$ are zero, the set of equations is hyperbolic.

The physical interpretation of the mathematical conclusion is that when a relation that connects R and p contains either of the derivatives of R (but not all of them), the disturbances that emanate from a point will not create silent zones in a hyperspace in which time is a dimension. Such a case is possible only when the bubble content is vapor for an expansion wave. This relation is derived by solving the energy equation in connection with several simplifying assumptions and can be followed in the Appendix A, Section A. 3.

In contrast, the relation that connects R and p_g for a gas bubble does not contain derivatives of R(Appendix A, Section A. 3). This is a basic difference between a mixture of liquid and vapor, and liquid and gas. We can look more deeply into this argument and will try to analyze why this is so. The vapor bubbles act like pieces of very soft springs with negligible mass connected to stiff rods. A disturbance travels almost instantaneously compared to the system of hard springs with comparable mass connected to stiff rods. After this preliminary discussion, we can set up and use the method of characteristics for solving problems arising in a mixture of liquid and gas. For details consult the Appendix A, Section A. 4.

3. 4 Development of Equations

The equations for an axial-symmetric flow are:

Continuity Equation

$$\frac{\partial \rho}{\partial t} + \rho\left(\frac{u}{r} + \frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right) + u\frac{\partial \rho}{\partial r} + v\frac{\partial \rho}{\partial z} = 0 \qquad (3\text{-}14)$$

Momentum Equation

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + v\frac{\partial u}{\partial z}\right) + \frac{\partial p}{\partial r} = 0 \qquad (3\text{-}15)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + v\frac{\partial v}{\partial z}\right) + \frac{\partial p}{\partial z} = 0 \qquad (3\text{-}16)$$

Equation for the density of the mixture

$$\rho = \rho_L(1 - n_\circ V) \qquad (3\text{-}17)$$

where $n_\circ$ is the number density of the bubbles and V is the volume of a bubble.

$$\frac{\rho_g n_\circ V}{1 - n_\circ V} = \frac{\rho_{g_\circ} n_\circ V_\circ}{1 - n_\circ V_\circ} \tag{3-18}$$

Isothermal expansion of a bubble yields

$$\frac{p_g}{\rho_g} = \frac{p_\circ}{\rho_{g_\circ}} \tag{3-19}$$

Equation of motion for a bubble (Rayleigh equation)

$$p = p_g - \rho_L R_\circ^2 \{ (\frac{p_g}{p_\circ})^{-1/3} \frac{\partial^2}{\partial t^2} (\frac{p_g}{p_\circ})^{1/3} + \frac{3}{2} [\frac{\partial}{\partial t} (\frac{p_g}{p_\circ})^{-1/3}]^2 \} \tag{3-20}$$

We have neglected the mass of the gas in the density of the mixture. An isothermal expansion for the bubbles is considered, this is in conjunction with what was discussed before. By combining Equations (3-17), (3-18) and (3-19), we obtain

$$\rho = \rho_L \frac{\frac{p_g}{p_\circ}}{\frac{p_g}{p_\circ} + \frac{n_\circ V_\circ}{1 - n_\circ V_\circ}} \tag{3-21}$$

3.5 Method of Solution

In order to deal with problems of expansion waves with moderate strength, we linearize the pressure.

$$\frac{p_g}{p_\circ} = 1 + \epsilon \zeta \qquad (3\text{-}22)$$

where ϵ is a small parameter.

Differentiation of Equation (3-21) yields

$$\frac{\partial p_g}{\partial \rho} = \frac{p_\circ(1-n_\circ V_\circ)}{\rho_L n_\circ V_\circ}\left(1 + \epsilon\,\zeta + \frac{n_\circ V_\circ}{1-n_\circ V_\circ}\right) \qquad (3\text{-}23)$$

which can be linearized as follows

$$\frac{\partial p_g}{\partial \rho} = \frac{p_\circ}{\rho_L n_\circ V_\circ (1-n_\circ V_\circ)}\,[1 + 2\epsilon\zeta(1-n_\circ V_\circ) + 0(\epsilon^2)]$$

$$= a_\circ^2 [1 + 2\epsilon\zeta(1-n_\circ V_\circ) + 0(\epsilon^2)] \qquad (3\text{-}24)$$

It is beneficial to make the variables dimensionless by using the following

$$t = \frac{\bar{\lambda} t^*}{a_\circ}, \quad r = r^* \bar{\lambda}, \quad z = z^* \bar{\lambda}, \quad u = \epsilon\, a_\circ n_\circ V_\circ u^*, \quad v = \epsilon\, a_\circ n_\circ V_\circ v^*$$

$\bar{\lambda}$ is a significant length in the problem.

Since $\rho = \rho(p_g)$, we can write the continuity equation as

$$\frac{\partial p_g}{\partial t} + u\frac{\partial p_g}{\partial r} + v\frac{\partial p_g}{\partial z} + \rho\frac{\partial p_g}{\partial \rho}\left[\frac{u}{r} + \frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right] = 0 \qquad (3\text{-}25)$$

or

$$\frac{\partial \zeta}{\partial t^*} + \epsilon n_\circ V_\circ u^* \frac{\partial \zeta}{\partial r^*} + \epsilon n_\circ V_\circ v^* \frac{\partial \zeta}{\partial z^*} + [1 + \epsilon\zeta(2-n_\circ V_\circ) + 0(\epsilon^2)]\left[\frac{u^*}{r^*} + \frac{\partial u^*}{\partial r^*} + \frac{\partial v^*}{\partial z^*}\right] = 0 \qquad (3\text{-}26)$$

In the same fashion, by combining the Rayleigh equation and the momentum equation in the r direction, we get

$$\frac{\partial u^*}{\partial t^*} + \epsilon n_\circ V_\circ u^* \frac{\partial u^*}{\partial r^*} + \epsilon n_\circ V_\circ v^* \frac{\partial u^*}{\partial z^*} + \frac{p_\circ(1-n_\circ V_\circ \epsilon \zeta)}{a_\circ^2 n_\circ V_\circ \rho_L (1-n_\circ V_\circ)}$$

$$\frac{\partial \zeta}{\partial r^*} + \frac{R_\circ^2(1-n_\circ V_\circ \epsilon\zeta)}{\epsilon(1-n_\circ V_\circ)n_\circ V_\circ \bar{\lambda}^2}\left[\left(\frac{3\epsilon}{9} - \frac{\epsilon^2 \zeta}{9}\right)\frac{\partial^3 \zeta}{\partial t^{*2}\partial r^*} + \frac{\epsilon^2}{3}\frac{\partial \zeta}{\partial r^*}\frac{\partial^2 \zeta}{\partial t \partial r}\right]$$

$$= 0 \qquad (3\text{-}27)$$

By comparing the terms inside of the bracket and neglecting the terms of $0(\epsilon^2)$, and noticing that $\dfrac{p_\circ}{\rho_L a_\circ^2 n_\circ V_\circ (1-n_\circ V_\circ)} = 1$, we get

$$\frac{\partial u^*}{\partial t^*} + \epsilon n_\circ V_\circ u^* \frac{\partial u^*}{\partial r^*} + \epsilon n_\circ V_\circ v^* \frac{\partial u^*}{\partial z^*} + \frac{p_\circ(1-n_\circ V_\circ \epsilon\zeta)}{\rho_L a_\circ^2 n_\circ V_\circ (1-n_\circ V_\circ)}\frac{\partial \zeta}{\partial r^*}$$

$$+ \frac{R_\circ^2(1-n_\circ V_\circ \epsilon\zeta)}{3 n_\circ V_\circ \bar{\lambda}^2(1-n_\circ V_\circ)}\frac{\partial^3 \zeta}{\partial t^{*2}\partial r^*} = 0$$

Giving $\dfrac{R_\circ^2}{\bar{\lambda}^2(1-n_\circ V_\circ)n_\circ V_\circ}$ the designation $\bar{\sigma}$, and dropping *'s, we can write the set of equations as the following

$$\frac{\partial \zeta}{\partial t} + \epsilon n_\circ V_\circ u \frac{\partial \zeta}{\partial r} + \epsilon n_\circ V_\circ v \frac{\partial \zeta}{\partial z} + [1+(2-n_\circ V_\circ)\epsilon\zeta]\left(\frac{u}{r} + \frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right) = 0 \qquad (3\text{-}29)$$

Momentum equations in r and z directions

$$\frac{\partial u}{\partial t} + \epsilon n_\circ V_\circ u \frac{\partial u}{\partial r} + \epsilon n_\circ V_\circ v \frac{\partial u}{\partial z} + (1-n_\circ V_\circ \epsilon\zeta)\frac{\partial \zeta}{\partial r} + \frac{\bar{\sigma}}{3}(1-n_\circ V_\circ \epsilon\zeta)\frac{\partial^3 \zeta}{\partial t^2 \partial r} = 0$$

$$(3\text{-}30)$$

$$\frac{\partial v}{\partial t} + \epsilon n_\circ V_\circ u \frac{\partial v}{\partial r} + \epsilon n_\circ V_\circ v \frac{\partial v}{\partial z} + (1 - n_\circ V_\circ \epsilon\zeta)\frac{\partial \zeta}{\partial z} + \frac{\overline{\sigma}}{3}(1 - n_\circ V_\circ \epsilon\zeta)\frac{\partial^3 \zeta}{\partial t^2 \partial z} = 0 \tag{3-31}$$

This set is equivalent to the following set of first order equations.

$$\left.\begin{aligned}
&\frac{\partial \zeta}{\partial t} + \epsilon n_\circ V_\circ u \frac{\partial \zeta}{\partial r} + \epsilon n_\circ V_\circ v \frac{\partial \zeta}{\partial z} + [1+(2-n_\circ V_\circ)\epsilon\zeta](\frac{u}{r} + \frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}) = 0 \\
&\frac{\partial u}{\partial t} + \epsilon n_\circ V_\circ u \frac{\partial u}{\partial r} + \epsilon n_\circ V_\circ v \frac{\partial u}{\partial z} + (1 - n_\circ V_\circ \epsilon\zeta)\frac{\partial \zeta}{\partial r} + \frac{\overline{\sigma}}{3}(1 - n_\circ V_\circ \epsilon\zeta)\frac{\partial \zeta_2}{\partial r} = 0 \\
&\frac{\partial v}{\partial t} + \epsilon n_\circ V_\circ u \frac{\partial v}{\partial r} + \epsilon n_\circ V_\circ v \frac{\partial v}{\partial z} + (1 - n_\circ V_\circ \epsilon\zeta)\frac{\partial \zeta}{\partial z} + \frac{\overline{\sigma}}{3}(1 - n_\circ V_\circ \epsilon\zeta)\frac{\partial \zeta_2}{\partial z} = 0 \\
&\frac{\partial \zeta}{\partial t} = \zeta_1 \\
&\frac{\partial \zeta_1}{\partial t} = \zeta_2
\end{aligned}\right\} \tag{3-32}$$

If we follow the method given in the first chapter by introducing the characteristic surfaces, we get the following determinant

$$\begin{vmatrix}
\phi_t+\epsilon n_\circ V_\circ(u\phi_r+v\phi_z) & 1 - n_\circ V_\circ \epsilon\zeta)\phi_r & (1 - n_\circ V_\circ \epsilon\zeta)\phi_z & \phi_t & 0 \\
[1+(2-n_\circ V_\circ)\epsilon\zeta]\phi_r & \phi_r+\epsilon n_\circ V_\circ(u\phi_r+v\phi_z) & 0 & 0 & 0 \\
[1+(2-n_\circ V_\circ)\epsilon\zeta]\phi_z & 0 & 0 & \phi_t+\epsilon n_\circ V_\circ(u\phi_r+v\phi_z) & 0 \\
0 & 0 & 0 & 0 & \phi_t \\
0 & \frac{\overline{\sigma}}{3}(1 - n_\circ V_\circ \epsilon\zeta)\phi_r & \frac{\overline{\sigma}}{3}(1 - n_\circ V_\circ \epsilon\zeta)\phi_z & 0 & 0
\end{vmatrix} = 0$$

Expansion of the determinant results in the following characteristics equation

$$\phi_t^2(\phi_z^2 + \phi_r^2)\{\phi_t + \epsilon n_\circ V_\circ(u\phi_r + v\phi_z)\} = 0 \qquad (3\text{-}33)$$

Except the envelope of pathlines, we cannot find any other characteristic surfaces. In this case, the system is again parabolic and we cannot detect silent zones, that is, zones that are not affected by the disturbances in a hyperspace that time is considered as one dimension. This phenomenon is not suprising and we should have been able to discover it earlier, because of third order terms. However, there appears a factor $\bar{\sigma}$ in front of the third order term in the momentum equation which is a function of several parameters. Let us consider a mixture with very small value of $\bar{\sigma}$, this is possible and can be controlled by variation of the parameters involved. In this case we would have the following system of equations

$$\left.\begin{aligned}
&\frac{\partial\zeta}{\partial t} + \epsilon n_\circ V_\circ u\frac{\partial\zeta}{\partial r} + \epsilon n_\circ V_\circ v\frac{\partial\zeta}{\partial z} + [1+(2-n_\circ V_\circ)\epsilon\zeta](\frac{u}{r} + \frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}) = 0\\
&\frac{\partial u}{\partial t} + \epsilon n_\circ V_\circ u\frac{\partial u}{\partial r} + \epsilon n_\circ V_\circ v\frac{\partial u}{\partial z} + (1-n_\circ V_\circ\epsilon\zeta)\frac{\partial\zeta}{\partial r} = 0\\
&\frac{\partial v}{\partial t} + \epsilon n_\circ V_\circ u\frac{\partial v}{\partial r} + \epsilon n_\circ V_\circ v\frac{\partial v}{\partial z} + (1-n_\circ V_\circ\epsilon\zeta)\frac{\partial\zeta}{\partial z} = 0
\end{aligned}\right\} \qquad (3\text{-}34)$$

This set of equations yields the following determinant

$$\begin{vmatrix} \phi_t+\epsilon n_\circ V_\circ(u\phi_r+v\phi_z) & (1-n_\circ V_\circ\epsilon\zeta)\phi_r & (1-n_\circ V_\circ\epsilon\zeta)\phi_z \\ [1+(2-n_\circ V_\circ)\epsilon\zeta]\phi_r & \phi_t+\epsilon n_\circ V_\circ(u\phi_r+v\phi_z) & 0 \\ [1+(2-n_\circ V_\circ)\epsilon\zeta]\phi_z & 0 & \phi_t+\epsilon n_\circ V_\circ(u\phi_r+v\phi_z) \end{vmatrix} = 0$$

The expansion of this determinant results in the characteristics equation of the considered set

$$[\phi_t+\epsilon n_\circ V_\circ(u\phi_r+v\phi_z)]\{[\phi_t+\epsilon n_\circ V_\circ(u\phi_r+v\phi_z)]^2 - [(1-n_\circ V_\circ\epsilon\zeta)][1+(2-n_\circ V_\circ)\epsilon\zeta](\phi_r^2+\phi_z^2)\} = 0 \tag{3-35}$$

A close examination of this characteristics equation reveals its similarity to the characteristics equation that was given before. The rest of the procedure is the same and the numerical schemes can be set up.

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APPENDICES

APPENDIX A

A. 1. Method of Characteristics for a Compressible Fluid Flow

Consider a general system of n-first order, quasi-linear partial differential equations in n-dependent variables u_k and m-independent variables x_i. Using tensor notation, we can write

$$\bar{a}_{ijk} u_{k,i} = \bar{b}_j \tag{A-1}$$

Quasi-linear means that the highest order derivatives that appear in the set are first degree, thus $\bar{b}_j$ and $\bar{a}_{ijk}$ are functions of u_k and x_i only.

The method of characteristics in two dimensions, results in the curves along which a hyperbolic partial differential equation becomes an ordinary differential equation. The method employed and its logic cannot be generalized to more than two dimensions. However, the general method which we describe is applicable to the two-dimensional case also.

A characteristic in m-independent variables can be defined as a subspace of m-1 dimensions, along which the derivatives of the dependent variables are continuously differentiable but across which discontinuities in the derivatives of the dependent variables are allowed to occur.

To find such characteristics for our set of equations which we assume is in three independent variables, we transform to a new set of coordinate systems $\bar{x}_i$ in which $\bar{x}_1 = \phi(x_i) = 0$ is a characteristic. Thus, in the new system of coordinates, on the characteristic $\bar{x}_1 = 0$, the derivatives $\frac{\partial}{\partial \bar{x}_2}$ and $\frac{\partial}{\partial \bar{x}_3}$ are tangential derivatives and therefore continuous, and $\frac{\partial}{\partial \bar{x}_1}$ is normal derivative and can be discontinuous.

Going back to our set of equations (Equations A-1), we can write

$$\bar{a}_{ijk} \frac{\partial u_k}{\partial \bar{x}_1} \frac{\partial \bar{x}_1}{\partial x_i} = \bar{b}_j \qquad \text{(A-2)}$$

Since, u_k and their derivatives $\frac{\partial u_k}{\partial \bar{x}_3}$ and $\frac{\partial u_k}{\partial \bar{x}_2}$ are continuous in the neighborhood of the characteristic manifold, they can be considered known. Thus, we will have a set of equations in $\frac{\partial u_k}{\partial \bar{x}_1}$. However, $\frac{\partial u_k}{\partial \bar{x}_1}$ cannot be determined uniquely in the neighborhood of the characteristic manifold $\bar{x}_1 = 0$. This requires that the set of equations (Equations A-2) should not be solvable in $\frac{\partial u_k}{\partial \bar{x}_1}$ which results in

$$\text{DET}\left(\bar{a}_{ijk} \frac{\partial \bar{x}_1}{\partial x_i}\right) = 0$$

or

$$\text{DET}\left(\bar{a}_{ijk}\, \phi_{,i}\right) = 0$$

Expansion of this determinant gives the characteristic equation.

The implication of the fact that the determinant $\bar{a}_{ijk}\, \phi_{,i}$ is zero, is that the rows or columns in that determinant are linearly dependent which means

$$c_j \bar{a}_{ijk} \phi_{,i} = 0 \ , \qquad c_j \neq 0 \tag{A-3}$$

If we multiply the set of Equations (A-2) by constants c_j, we get

$$c_j \bar{a}_{ijk} \phi_{,i} \frac{\partial u_k}{\partial \bar{x}_1} + c_j a_{ijk} \frac{\partial \bar{x}_2}{\partial x_i} \frac{\partial u_k}{\partial \bar{x}_2} + c_j \bar{a}_{ijk} \frac{\partial \bar{x}_3}{\partial x_i} \frac{\partial u_k}{\partial \bar{x}_3} = c_j \bar{b}_j \tag{A-4}$$

In the light of Equation (A-3), Equation (A-4) can be simplified to

$$c_j \bar{a}_{ijk} \frac{\partial \bar{x}_2}{\partial x_i} \frac{\partial u_k}{\partial \bar{x}_2} + c_j \bar{a}_{ijk} \frac{\partial \bar{x}_3}{\partial x_i} \frac{\partial u_k}{\partial \bar{x}_3} = c_j \bar{b}_j \tag{A-5}$$

In Equation (A-5), as it is apparent, derivatives of the dependent variables with respect to $\bar{x}_1$ do not appear. This equation is called the compatibility equation.

For the special case of the axial-symmetric rotational unsteady flow, we will determine the constants c_1, c_2 and c_3. For this case, the characteristic equation appears in the form

$$\Phi = (\phi_t + u\phi_r + v\phi_z)^2 - a^2(\phi_r^2 + \phi_z^2) = 0 \tag{A-6}$$

Since

$$\frac{dr}{dt} = \frac{\Phi\phi_r}{\Phi\phi_t} = u - \frac{a^2\phi_r}{\phi_t + u\phi_r + v\phi_z} \tag{A-7}$$

$$\frac{dz}{dt} = \frac{\Phi\phi_r}{\Phi\phi_t} = v - \frac{a^2\phi_z}{\phi_t + u\phi_r + v\phi_z} \qquad \text{(A-8)}$$

Elimination of ϕ_t, ϕ_r and ϕ_z results in

$$\left(\frac{dr}{dt} - u\right)^2 + \left(\frac{dz}{dt} - v\right)^2 = a^2 \qquad \text{(A-9)}$$

or

$$(\Delta r - u)^2 + (\Delta z - v)^2 = (a\Delta t)^2 \qquad \text{(A-10)}$$

This equation is called the Monge differential equation.

Geometric interpretation of Equations (A-9) and (A-10) is that if a fluid particle at time t_o is at point P_o with coordinates t_o, r_o and z_o, after a time interval Δt the particle moves to P_1 with coordinates $r = r_o + u\Delta t$, $z = z_o + v\Delta t$ and $t = t_o + \Delta t$. Thus, a disturbance at point P_o moves on a circle of radius $a\Delta t$ at time interval of Δt. Hence, Equation (A-9) defines the Mach conoid

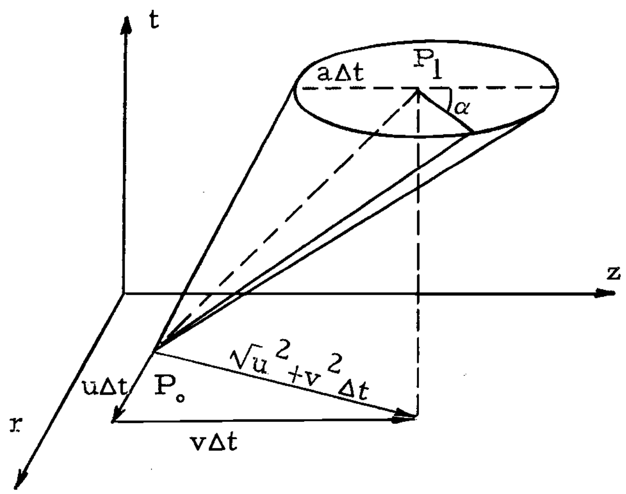

Figure A-1. Mach conoid.

This gives us a hint to use α as a parameter; and the bi-characteristics in parametric form can be written

$$dr = (u + a\cos\alpha)dt \tag{A-11}$$

$$dz = (v + a\sin\alpha)dt \tag{A-12}$$

The direction of the family of bi-characteristics through the vertex of the conoid that lie on the surface of conoid is given by $(1,\ u+a\cos\alpha,\ v+a\sin\alpha)$.

If we substitute relations (A-11) and (A-12) in (A-7) and (A-8), the parametric forms of Equations (A-7) and (A-8) are obtained

$$\frac{a^2\phi_r}{\phi_t + u\phi_r + v\phi_z} = -a\cos\alpha \tag{A-13}$$

$$\frac{a^2\phi_z}{\phi_t + u\phi_r + v\phi_z} = -a\sin\alpha, \tag{A-14}$$

and the parametric form of the characteristic equations is given by

$$\frac{\phi_r}{\cos\alpha} = \frac{\phi_z}{\sin\alpha} = \frac{-\phi_t}{a + u\cos\alpha + v\sin\alpha} \tag{A-15}$$

which in turn results in

$$\phi_r = k\cos\alpha \tag{A-16}$$

$$\phi_z = k\sin\alpha \tag{A-17}$$

$$\phi_t = -k(a + u\cos\alpha + v\sin\alpha) \tag{A-18}$$

In this manner, we solve for constants c_j, in terms of one of them, say c_1, which in turn can be substituted in the compatibility equation.

A. 2. Compatibility Equation Containing Total Derivative of Pressure

Consider the following set of equations of an axial-symmetric flow of a compressible fluid.

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial r} + v\frac{\partial p}{\partial z} + \rho a^2\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z} + \frac{u}{r}\right) = 0 \qquad \text{(A-19), [2-4]}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + v\frac{\partial u}{\partial z}\right) + \frac{\partial p}{\partial r} = 0 \qquad \text{(A-20), [2-5]}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + v\frac{\partial v}{\partial z}\right) + \frac{\partial p}{\partial z} = 0 \qquad \text{(A-21), [2-6]}$$

If we multiply Equation (A-19), (A-20) and (A-21) by C_1, C_2 and C_3, respectively, and add the resulting equations, we get

$$[C_1, C_1u+C_2, C_1v+C_3]\cdot\nabla p + [\rho C_2, \rho a^2 C_1 + \rho u C_2, \rho v C_2]\cdot\nabla u$$

$$+ [\rho C_3, \rho u C_3, \rho a^2 C_1 + \rho v C_3]\cdot\nabla v = -\rho a^2 \frac{u}{r} C_1 \qquad \text{(A-22)}$$

Then, the vectors $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$ and $\mathbb{N}$ are

$$\mathbb{A}_1 = C_1,\ C_1u + C_2,\ C_1v + C_3$$

$$\mathbb{A}_2 = \rho C_2,\ \rho a^2 C_1 + \rho u C_2,\ \rho v C_2$$

$$\mathbb{A}_3 = \rho C_3, \ \rho u C_3, \ \rho a^2 C_1 + \rho v C_3$$

$$\mathbb{N} = \phi_r, \ \phi_z, \ \phi_t$$

The existence of the characteristic surfaces requires

$$\begin{cases} \mathbb{A}_1 \cdot \mathbb{N} = 0 \\ \mathbb{A}_2 \cdot \mathbb{N} = 0 \\ \mathbb{A}_3 \cdot \mathbb{N} = 0 \end{cases} \qquad \text{(A-23)}$$

The Equations (A-23) can be written in the following form

$$\begin{pmatrix} \phi_t + u\phi_r + v\phi_z & \phi_r & \phi_z \\ \rho a^2 \phi_r & \rho(\phi_t + u\phi_r + v\phi_z) & 0 \\ \rho a^2 \phi_z & 0 & \rho(\phi_t + u\phi_r + v\phi_z) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = 0$$

(A-24)

The only nontrivial solution to this set of simultaneous equations is possible when the determinant of the square matrix is zero.

$$\begin{vmatrix} \phi_t + u\phi_r + v\phi_z & \phi_r & \phi_z \\ \rho a^2 \phi_r & \rho(\phi_t + u\phi_r + v\phi_z) & 0 \\ \rho a^2 \phi_z & 0 & \rho(\phi_t + u\phi_r + v\phi_z) \end{vmatrix} = 0$$

Expansion of this determinant results in the characteristic equation (Equation A-25).

$$\rho^2(\phi_t + u\phi_r + v\phi_z)\,[(\phi_t + u\phi_r + v\phi_z)^2 - a^2(\phi_r^2 + \phi_z^2)] = 0 \qquad \text{(A-25)}$$

Introducing parametric bi-characteristic angle α (Figure A-1), results in the parametric form of the characteristic equation

$$\begin{cases} \phi_r = k\cos\alpha \\ \phi_2 = k\sin\alpha \\ \phi_t = -k(a + u\cos\alpha + v\sin\alpha) \end{cases} \qquad \text{(A-26)}$$

Now we can solve for C_1, C_2 and C_3 by substituting Equation (A-26) in Equations (A-24) which are

$$-a_1C_1 + \cos\alpha\, C_2 + \sin\alpha\, C_3 = 0 \qquad \text{(A-27)}$$

$$\rho a^2\cos\alpha\, C_1 - \rho a C_2 = 0 \qquad \text{(A-28)}$$

$$\rho a^2\sin\alpha\, C_1 - \rho a C_3 = 0 \qquad \text{(A-29)}$$

From Equation (A-29), we obtain

$$C_3 = a\sin\alpha\, C_1 \qquad \text{(A-30)}$$

From Equation (A-28), we obtain

$$C_2 = a\cos\alpha\, C_1 \qquad \text{(A-31)}$$

Equations (A-30) and (A-31) satisfy Equation (A-27). Thus they are the nontrivial solution of the set of Equations (A-24). Now, we substitute Equations (A-30) and (A-31) in Equation (A-22); the resulting equation is the compatibility equation that contains the derivative of pressure.

$$[1, u+a\cos\alpha, v+a\sin\alpha]\cdot\nabla p+[\rho a\cos\alpha, \rho a^2+\rho au\cos\alpha, \rho av\cos\alpha].$$

$$\nabla u+[\rho a\sin\alpha, \rho au\sin\alpha, \rho a^2+\rho av\sin\alpha]\cdot\nabla v=-\rho a^2\frac{u}{r} \qquad \text{(A-32)}$$

Defining the bi-characteristics $(1, u+a\cos\alpha, v+a\sin\alpha)$ the paths along which differentiation is performed, yields

$$\frac{dp}{dT}+\rho a\cos\alpha\frac{du}{dT}+\rho a\sin\alpha\frac{dv}{dT}=(-\rho a^2+\rho a^2\cos^2\alpha)\frac{\partial u}{\partial r}+\rho a^2\sin\alpha\cos\alpha\frac{\partial u}{\partial z}$$

$$+\rho a^2\sin\alpha\cos\alpha\frac{\partial v}{\partial r}+(-\rho a^2+\rho a^2\sin^2\alpha)\frac{\partial v}{\partial z}-\rho a^2\frac{u}{r} \qquad \text{(A-33)}$$

which can be written in the following form also

$$dp+\rho a\cos\alpha\, du+\rho a\sin\alpha\, dv=\rho a^2\{-(\sin^2\alpha\frac{\partial u}{\partial r}+\cos^2\alpha\frac{\partial v}{\partial z})$$

$$+\sin\alpha\cos\alpha(\frac{\partial v}{\partial r}+\frac{\partial u}{\partial z})-\frac{u}{r}\}\,dT \qquad \text{(A-34)}$$

A. 3. Physical Significance of the Coefficients in Equation (A-35)

We postulated a general mathematical relation between p and R.

$$\bar{a}_o+\bar{a}R+\bar{b}p_g+\bar{c}\frac{\partial p_g}{\partial t}+\bar{d}\frac{\partial p_g}{\partial x}+\bar{e}\frac{\partial R}{\partial t}+\bar{f}\frac{\partial R}{\partial x}=0 \qquad \text{(A-35), [3-9]}$$

Now, we can consider two phenomena that arise in a physical case:

1. Isothermal expansion of a gas bubble
2. Expansion of the vapor bubble in a superheated liquid

In an isothermal expansion, product of the volume and pressure remains unchanged. Thus

$$pV = p_\circ V_\circ \tag{A-36}$$

Or

$$(R^2/R_\circ^3)R - (p_\circ p^{-2})p = 0 \tag{A-37}$$

which states

$$a_\circ = 0$$
$$\bar{a} = R^2/R_\circ^3$$
$$\bar{b} = -p_\circ p^{-2}$$
$$\bar{c} = \bar{d} = \bar{e} = \bar{f} = 0$$

The analysis of the expansion of a vapor bubble in a superheated liquid is more involved and requires the consideration of energy balance. If we denote the mass of vapor in a bubble by m_v, we can write

$$L(m_v\big|_{t+\Delta t} - m_v\big|_t) = 4\pi R^2 K \frac{\partial T}{\partial r}\Big|_{r=R} \Delta t \tag{A-38}$$

dividing Equation (A-38) by Δt and take the limit, we obtain

$$\frac{L}{3}\frac{d}{dt}(R^3 \rho_v) = R^2 K \frac{\partial T}{\partial r}\Big|_{r=R} \tag{A-39}$$

because of the fast rate of growth; and evaporation of some of the mass of the liquid in the process of expansion, $\dot{R}\rho_v >> R\dot{\rho}_v$. Thus we can write

$$\frac{L}{K}\dot{R}\rho_v = \left.\frac{\partial T}{\partial r}\right|_{r=R} \qquad \text{(A-40)}$$

In order to get $\left.\frac{\partial T}{\partial r}\right|_{r=R}$, we have to solve the energy equation in the spherical coordinate, which is

$$\frac{\partial T}{\partial t} + \dot{R}\frac{R^2}{r^2}\frac{\partial T}{\partial r} = \alpha\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} \qquad r > R \qquad \text{(A-41)}$$

Initial conditions are

$$\begin{aligned} & r(R,\ t=0) = R_o \\ & \dot{r}\ (R,\ t=0) = 0 \\ & p(r > R_o\ \ t=0) = p_o \\ & T_v(r < R_o,\ t=0) = T_o \end{aligned} \qquad \text{(A-42)}$$

In order that the evaporation would be possible, the pressure inside the bubble at each instant must be equal to the pressure corresponding to the saturation temperature. The equation that relates these two variables together is the Clausius-Clapeyron equations:

$$\frac{dp_{sat}}{dT_{sat}} = \frac{S_{lg}}{\upsilon_{lg}} = \frac{H_{lg}}{T_{sat}\,\upsilon_{lg}} \qquad \text{(A-43)}$$

To arrive at an analytical answer, reasonable enough, we have to sacrifice the accuracy and resort to some simplifying assumptions, which are

1. Use of the perfect gas relation for the vapor.
2. $\upsilon_{1g} = \upsilon_g$
3. Amount of superheat is to that extent that allows linearization.
4. The thermal boundary layer is very thin.

Thus we can solve Equation (A-43) to get

$$\frac{p_v}{p_{v_\circ}} = e^{\left(-\frac{L}{R_m} \frac{T_{v_\circ} - T_w}{T_{v_\circ} T_w}\right)} \qquad \text{(A-44)}$$

where

p_v is the saturation pressure at T_w

$p_{v_\circ}$ is the saturation pressure at $T_{v_\circ} = T_\circ$

$T_{v_\circ}$ is the temperature of the vapor at $t = 0$

T_w is the wall temperature dictated by the heat transfer

R_m is the gas constant for the vapor

Linearizing Equation (A-44) results in

$$\frac{p_v - p_{v_\circ}}{p_{v_\circ}} = \frac{L}{R_m T_{v_\circ}^2} (T_w - T_{v_\circ}) \qquad \text{(A-45)}$$

To solve the energy equation, we take the method of integral boundary layer approach. The simplest relation would be

$$T = T_w + (T_\circ - T_w)\left(\frac{r}{\delta_T} - \frac{R}{\delta_T}\right) \qquad \text{(A-47)}$$

$$\left.\frac{\partial T}{\partial r}\right|_{r=R} = \frac{T_\circ - T_w}{\delta_T} = \frac{L}{K}\dot{R}\rho_v \qquad \text{(A-48)}$$

Substituting Equation (A-46) in (A-41), performing the differentiation of the definite integrals with variable limits, and simplifying the results, we get

$$\dot{\delta}_T = \frac{2}{T_w - T_\circ}\frac{La}{K}\rho_v\dot{R} \qquad \text{(A-49)}$$

which can be integrated as follows

$$\delta = \frac{2a}{T_w - T_\circ}\frac{\rho_v L}{K}(R - R_\circ) \qquad \text{(A-50)}$$

Substitution of Equation (A-50) in (A-48) results in the following

$$\frac{(T_\circ - T_w)^2}{2a\,\rho_v L(R - R_\circ)} = \frac{L}{K}\dot{R}\rho_v \qquad \text{(A-51)}$$

Since T_w is a function of time we cannot integrate Equation (A-51). However, we can use Equation (A-45) to eliminate T_w. The results appear in the following fashion

$$\dot{R}(R - R_\circ) - \frac{K}{2a\rho_v^2 L^2}\left\{\left[(T_\circ - T_{v_\circ}) - \frac{R_m T_{v_\circ}^2}{L}\right] + \frac{R_m}{L}\frac{T_{v_\circ}^2}{p_{v_\circ}}p_v\right\}^2 = 0$$

(A-52)

Relation (A-52) is the equation that connects R and p together. Comparing this equation with Equation (A-35) reveals the Constants:

$$\bar{a}_\circ = - \frac{\mathcal{K}}{2a\rho_v^2 L^2} [(T_\circ - T_{v_\circ}) - \frac{R_m T_{v_\circ}^2}{2}]^2 \qquad \text{(A-53)}$$

$$\bar{a}_1 = 0$$

$$\bar{b} = - \frac{\mathcal{K}}{2a\rho_v^2 L^2} \{ \frac{R_m^2 T_{v_\circ}^4}{L^2 p_{v_\circ}^2} p_v + 2[(T_\circ - T_{v_\circ}) - \frac{R_m T_{v_\circ}^2}{L}] (\frac{R_m T_{v_\circ}^2}{L p_{v_\circ}}) \} \qquad \text{(A-54)}$$

$$\bar{c} = 0$$

$$\bar{d} = 0$$

$$\bar{e} = (R - R_\circ)$$

$$\bar{g} = 0$$

A. 4 Derivation of the Characteristic Equations in Conjunction with Equation (3-9)

Since we want to investigate the influence of the coefficients in Equation (3-9) on the system, a manifold of one dimensional space and time would be sufficient. The equations to be considered are

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) = 0 \qquad \text{(A-55), [3-11]}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = - \frac{1}{\rho} \frac{\partial p}{\partial x} \qquad \text{(A-56), [3-12]}$$

$$\rho_L\left(R\frac{\partial R'}{\partial t} + \frac{3}{2}R'^2\right) = p_g - p \qquad \text{(A-57)}$$

$$\frac{\partial R}{\partial t} = R' \qquad \text{(A-58)}$$

$$\bar{a}_\circ + \bar{a}R + \bar{b}p_g + \bar{c}\frac{\partial p_g}{\partial t} + \bar{d}\frac{\partial p_g}{\partial x} + \bar{e}\frac{\partial R}{\partial t} + \bar{f}\frac{\partial R}{\partial x} = 0 \qquad \text{(A-59)}$$

From Equation (3-10), we get

$$\frac{\partial \rho}{\partial t} = -4\pi n_\circ \rho_L R^2 \frac{\partial R}{\partial t} \qquad \text{(A-60)}$$

$$\frac{\partial \rho}{\partial x} = -4\pi n_\circ \rho_L R^2 \frac{\partial R}{\partial x} \qquad \text{(A-61)}$$

We can eliminate derivatives of ρ in Equation (A-55) by substituting relations (A-60) and (A-61) in that equation which results in

$$\rho\frac{\partial u}{\partial x} - 4\pi n_\circ R^2\left(\frac{\partial R}{\partial t} + u\frac{\partial R}{\partial x}\right) = 0 \qquad \text{(A-62)}$$

The set of equations that we are dealing with in this case, are Equations (A-56), (A-57), (A-58), (A-59) and (A-62).

Again, if we follow the method that was explained before, we can obtain the determinants whose expansions yield characteristic equations for the case of very small value of $\bar{\sigma}$. The following cases are considered

1. When $\bar{c}$, $\bar{d}$ and $\bar{f}$ in Equation (3-9) are zero, the pertinent determinant is

$$\begin{vmatrix} \rho\phi_x & \rho(\phi_t+u\phi_x) & 0 \\ 0 & \phi_x & 0 \\ -4\pi n_\circ R^2(\phi_t+u\phi_x) & 0 & \bar{e}\phi_t \end{vmatrix} = 0$$

The expansion of this determinant results in

$$\phi_t\phi_x^2 = 0 \qquad \text{(A-63)}$$

From Equation (A-63), it is concluded that the system is parabolic.

2. When $\bar{c}$, $\bar{d}$ and $\bar{e}$ in Equation (3-9) are zero, the pertinent determinant is

$$\begin{vmatrix} \rho\phi_x & \rho(\phi_t+u\phi_x) & 0 \\ 0 & \phi_x & 0 \\ -4\pi n_\circ R^2(\phi_t+u\phi_x) & 0 & \bar{f}\phi_x \end{vmatrix} = 0$$

The expansion of this determinant results in

$$\phi_x^3 = 0 \qquad \text{(A-64)}$$

From Equation (A-64), it is concluded that the system is parabolic.

3. When $\bar{c}$ and $\bar{d}$ in Equation (3-9) are zero the pertinent determinant is

$$\begin{vmatrix} \rho\,\phi_x & \rho\,(\phi_t+u\phi_x) & 0 \\ 0 & \phi_x & 0 \\ -4\pi n_\circ R^2(\phi_t+u\phi_x) & 0 & \bar{e}\phi_t+\bar{f}\,\phi_x \end{vmatrix} = 0$$

The expansion of this determinant results in

$$\phi_x^2(\bar{e}\phi_t+\bar{f}\,\phi_x) \qquad \text{(A-65)}$$

From Equation (A-65), we conclude that the system is hyperbolic

4. When $\bar{d}$, $\bar{e}$ and $\bar{f}$ in Equation (3-9) are zero, the pertinent determinant is

$$\begin{vmatrix} \rho\,\phi_x & \rho\,(\phi_t+u\phi_x) & 0 \\ 0 & \phi_x & \bar{c}\phi_t \\ -4\pi n_\circ R^2(\phi_t+u\phi_x) & 0 & 0 \end{vmatrix} = 0$$

The expansion of this determinant results in

$$\phi_t(\phi_t+u\phi_x)^2 = 0 \qquad \text{(A-66)}$$

From Equation (A-66), we conclude that the system is hyperbolic.

5. When $\bar{e}$ and $\bar{f}$ in Equation (3-9) are zero, the pertinent determinant is

$$\begin{vmatrix} \rho\,\phi_x & \rho\,(\phi_t+u\phi_x) & 0 \\ 0 & \phi_x & \bar{c}\phi_t+\bar{d}\phi_x \\ -4\pi n_\circ R^2(\phi_t+u\phi_x) & 0 & 0 \end{vmatrix} = 0$$

The expansion of this determinant results in

$$(\phi_t+u\phi_x)^2(\bar{c}\phi_t+\bar{d}\phi_x) = 0 \qquad \text{(A-67)}$$

From Equation (A-67), we conclude that the system is hyperbolic

6. When none of the coefficient in Equation (3-9) is zero, the pertinent determinant is

$$\begin{vmatrix} \phi_t & \rho\,(\phi_t+u\phi_x) & 0 \\ 0 & \phi_x & \bar{c}\phi_t+\bar{d}\phi_x \\ -4\pi n_\circ R^2(\phi_t+u\phi_x) & 0 & \bar{e}\phi_t+\bar{f}\,\phi_x \end{vmatrix} = 0$$

The expansion of this determinant results in a third order equation in $(\frac{dx}{dt})$, which we cannot discuss in generality and we need to know more about the coefficients in specific cases.

7. When $\bar{e}$, $\bar{f}$ and $\bar{c}$ in Equation (3-9) are zero, the pertinent determinant is

$$\begin{vmatrix} \rho\,\phi_x & \rho\,(\phi_t+u\phi_x) & 0 \\ 0 & \phi_x & \bar{d}\phi_x \\ -4\pi n_\circ R^2(\phi_t+u\phi_x) & 0 & 0 \end{vmatrix} = 0$$

The expansion of this determinant results in

$$\phi_x(\phi_t+u\phi_x)^2 \tag{A-68}$$

From Equation (A-68), we conclude that the system is hyperbolic.

APPENDIX B

B. 1 Equation of State for Water

Investigation of the equation of state of water dates back to the 18th centur. Canton (1762), in a paper entitled "Experiments to Prove that Water is not Compressible," explained the results of his experiments. Since then, the crude experimental instruments have been improved considerably to the extent that the density at atmospheric pressure is available better than one unit in five significant figures. Oceanographers are not still satisfied with the present precision, and are trying to get more accurate figures at the expense of high costs, although it is criticized by some scientists (74).

In fresh water we are seeking a relation between p, v, T; while in the ocean another variable joins the other three and the relation becomes more complicated. One can find the results of investigations of the workers in the field in reference (16), although a few have been ignored and recent work has been developed since then. The list of major contributions to the data is given below:

I. E. H. Amagat, 0 to 180° C, 1 to 1000 atm.

II. E. H. Amagat, 0 to 49° C, 1 to 3000 atm.

III. P. W. Bridgman, -20 to 80° C, 1 to 12000 atm.

IV. P. W. Bridgman, -20 to 100° C, 1 to 12000 kg/cm^2

V. G. Tammann & W. Jellinghaus, -14 to 15° C, 1 to 1500 kg/cm^2

VI. L. B. Smith & F. G. Keyes, 0 to 360° C, 1 to 350 atm.

VII. G. Tammann & A. Rühenbeck, 20 to 650° C, 1 to 2500 kg/cm^2

VIII. L. H. Adams, 25° C, 1 to 12000 bars

IX. M. Trautz & H. Steyer, 0 to 370° C, 50 to 300 atm.

X. G. C. Kennedy, W. L. Knight & W. T. Holser (38), 0° to 100° C, 1 to 1400 bars

XI. Sixth International Conference on the Properties of Steam (57) 0° to 350° C, 1 to 1000 bars.

Eckart (17) has made a critical review of the data and believes that Amagat's data is the most accurate of all. This is also confirmed by Zaworski (84).

Several investigators have tried to find a relation that fits the data within a certain range of pressure and temperature. They are as follows:

I. Simplest of all is given by Tumlirz,

$$(p + p_o)(\upsilon - \upsilon_o) = \lambda \qquad \text{(B-1)}$$

another form of this can be written,

$$(p - p_o)(\Delta - \Delta_o) = 10^4 \lambda \qquad \text{(B-2)}$$

where

$$\Delta = 10^4 (1 - \upsilon)$$

$$\Delta_o = 3020$$

$$p_o = 5890 + 38T - 0.375T^2 \qquad \text{(B-3)}$$

$$\lambda = 10^2[17.795 + 0.1125T - 0.000745T^2] \tag{B-4}$$

units:

p Atm. = 1.03323 kg/cm^2 = 1.01325 bar

T Degree Centigrade

υ and Δ ml/gm = 1.000027 cm^3/gm

Range of accuracy:

$0 < T < 40^\circ C$, $p < 3000$ atm.

II. R. J. Zaworski and J. H. Keenan (84),

$$\upsilon = \frac{3.25639}{(p + 2825)^{0.14854}} - 0.18762 \times 10^{-4}(T_m^2)^{0.83230} + \frac{1.4839 \times 10^9}{(6500 - p)^{3.69772}} [(T-T_m)^2]^{0.9 - 0.000096p} \tag{B-5}$$

where units are:

υ Specific volume cc/gm

T Temperature in degree Centigrade

T_m Temperature of maximum density

p Pressure in atm.

Range of accuracy:

$0 < T < 30^\circ C$, $p < 3000$ atm.

III. The relation given by L. B. Smith and F. G. Keyes is quite complicated as follows:

$$\upsilon = \frac{\upsilon_s + B(p - p_s) + D(p - p_s)^2 + E(p - p_s)^{1/2}}{1 + C(p - p_s)} \quad \text{(B-6)}$$

$$\upsilon_s = \frac{\upsilon_c + a_1(T_c - T)^{1/3} + b_1(T_c - T) + C_1(T_c - T)^4}{1 + d_1(T_c - T)^{1/3} + e_1(T_c - T)} \quad \text{(B-7)}$$

$$\log_{10}(-\frac{1}{E}) = a_2 + b_2(T_c - T)^{1/2}$$

$$\log_{10}(-D) = a_3 + b_3(T_c - T)$$

$$\log_{10}(C) = a_4 + b_4(T_c - T)^{2/3} + c_4(T_c - T) + d_4(T_c - T)^{3/2}$$

$$\log_{10}(B) = a_5 + b_5(T_c - T)^{2/3} + c_5(T_c - T) + d_5(T_c - T)^{3/2}$$

where

p_s	Saturation pressure in atmosphere
T_c	Critical temperature; 374.11 °C
υ_c	Critical volume; 3.197500 cc/gram
a_1	-3.151548×10^{-1}
b_1	-1.203374×10^{-3}
c_1	7.48908×10^{-13}
d_1	1.342489×10^{-1}
e_1	-3.946263×10^{-3}
a_2	4.127788×10^{-1}
b_2	3.999207×10^{-1}
a_3	-4.9573810

b_3	2.9752567×10^{-2}
a_4	-1.6141307
b_4	$-5.2849209 \times 10^{-2}$
c_4	$-6.9720421 \times 10^{-3}$
d_4	4.5803994×10^{-4}
a_5	-1.2925743
b_5	$-9.7238056 \times 10^{-2}$
c_5	$-1.2938329 \times 10^{-2}$
d_5	5.8890279×10^{-4}
T	Temperature in degrees Centigrade
p	Pressure in atmosphere

Range of accuracy:

$$0 < T < 360^{\circ} C, \; p < 350 \text{ atm.}$$

IV. In sea water salinity is another factor which is defined by the amount of salt per thousand. The classical and still widely used method to determine the salinity of the sea water is based on Mohr's Cl-titration, where the chlorine content of sea water is determined by chemical processes. There is a linear relationship between chlorinity and salinity that was adopted by an international commission in 1902, and defined salinity, that is, $s ‰ = 0.030 + 1.8050 \, Cl$, Cl is measured parts in thousands. However, in recent years (74), there has been a concern for determination of salinity through electrical conductivity, among scientists.

In the years gone, oceanographers have shown very little interest to find relations that fit $p - \mho - T - s$ data. The reason is due to precision that they need in their calculation. Thus, the tables of Knudsen (42) still serve them very well, although it is very old.

$$\mho_{s,T,p} = \mho_{35,0,p} + \Delta_{s,T} + \delta_{s,p} + \delta_{T,p}$$

Each one of the terms is a function of two variables and can be found from the tables (42).

V. Modified Tumlirz equation of state, as is examined by Eckart (17) gives a good relation that fits the data, as follows:

$$(p + p_\circ)(\mho - \mho_\circ) = \lambda \qquad \text{(B-8)}$$

or

$$(p - p_\circ)(\Delta_\circ - \Delta) = 10^4\lambda \qquad \text{(B-9)}$$

where

$$\Delta_\circ = 3020$$

$$p_\circ = 5890 + 38\,T - 0.375T^2 + 3s$$

$$\lambda = 10^2[17.795 + 0.1125T - 0.000745T^2 - (0.0380 + 0.0001\,T)s]$$

The range of validity of this relation is the same as the one given for fresh water.

VI. Equation of state by Tait is less accurate but it covers a wider range. It is in this form:

$$\frac{\upsilon(0, T) - \upsilon(T, p)}{\upsilon(0, T)} = \frac{1}{n} \log [1 + \frac{p}{B(T)}] \qquad \text{(B-10)}$$

$B(25^{\circ}) = 3.156$ Kilobars

$n = 7.445$ up to 4 kilobars

Modified Tait equation of state, as given by R. E. Gibson:

$$\upsilon(0, T + 273.16) = 0.994150 + 2.929 \times 10^{-4}(T-25) + 3.241 \times 10^{-6}(T-25)^2 \text{ cm}^3/\text{gm} \qquad \text{(B-11)}$$

$$B(T) = 3.134 - 1.65 \times 10^{-3}(T-55) - 1.81 \times 10^{-4}(T-55)^2 + 5.32 \times 10^{-7}(T-55)^3 \qquad \text{(B-12)}$$

B is in kilobars
T is in degrees Centigrade

Range of validity

$$25^{\circ} C < T < 95^{\circ} C, \quad p < 25 \text{ Kb}$$

For $T < 10^{\circ}$ C, $\upsilon(0, T + 273.16)$ is given:

$$\upsilon(0, T+273.16) = 0.991442 + 6.025 \times 10^{-6}(T-3.8)^2 \text{ cm}^3/\text{gm} \qquad \text{(B-13)}$$

T is in degrees Centigrade

B. 2 Speed of Sound in Water

Speed of sound is one of the important characteristics of a medium. Since, in the equations of hydrodynamics, it is the rate of propagation of weak compression waves, it is very important to determine its variation with the properties of the concerned medium. In oceanography, it is necessary to know the speed of sound within fractions of one foot per second. It is often used to locate marine animals, study of the bottom of oceans, and in any theoretical investigation of compressible flow.

Speed of sound in general is defined as

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_S = \gamma \left(\frac{\partial p}{\partial \rho}\right)_T$$

It has a simple form for perfect gas, that is,

$$a^2 = \gamma R T$$

For liquids, the simplest form is given by Newton:

$$a^2 = \frac{1}{M\rho}$$

M is the isothermal compressibility.

Laplace proposed

$$a^2 = \frac{\gamma \bar{\beta}}{\rho}$$

$\bar{\beta}$ is the modulus of compressibility ($\bar{\beta} = \frac{1}{M}$)

γ is the ratio of the specific heat at constant pressure to the specific heat at constant volume. Thus it is vital to evaluate the variation of ρ, $\bar{\beta}$ and γ with temperature, pressure and salinity for sea water.

The first tables were computed by Heck and Service (30) in 1924. Since then, several other tables have been calculated. The precision has improved considerably. In 1927, Matthews (50) published his tables which had been rigorously calculated for both pure and sea water. In 1938, Kuwahara (45) published his tables. In 1938, Matthews (51) published a revision of his previous work. The revised edition is in better agreement with Kuwahara's work than his earlier work. In 1941, Stephenson and Woodsmall (68) obtained a relation for the temperature variation, using the data of Kuwahara's table II. In 1951, Mackenzie (49) developed a complete relation based on Kuwahara's tables. His formula agrees with Kuwahara's tables within 0.1 m/sec. His relation is given below:

$$a_{TsD} = a_{0,35,0} + \Delta a_T + \Delta a_s + \Delta a_D + \Delta a_\theta + \Delta a_{sT} + \Delta a_{sD} + \Delta a_{TD} \qquad \text{(B-14)}$$

$$a_{0,35,0} = 1445.5 \text{ m/sec}$$

$$\Delta a_T = 4.6374T - 5.383 \times 10^{-2} + 2.543 \times 10^{-4} T^3$$

$$\Delta a_s = (1.307 - 0.009T)(s-35)$$

$$\Delta a_D = 0.06 + 1.816 \times 10^{-2} D - 5.695 \times 10^{-12} D^3$$

$$\Delta a_\theta = 1.5 \times 10^{-6} D(\theta-35) + 0.94 \times 10^{-2} D^2 (\theta-35)^2 - 2.94 \times 10^{-18} D^3 (\theta-35)^3$$
$$- 1.214 \times 10^{-3} (\theta-35)$$

$$\Delta a_{sT} = (s-35)[-1.07 \times 10^{-2} T + (5.00 \times 10^{-5} - 4.1 \times 10^{-8} D) T^2]$$

$$\Delta a_{sD} = (s-35)[3.36 \times 10^{-5} D - 4.55 \times 10^{-9} D^2]$$

$$\Delta a_{TD} = D[-1.9 \times 10^{-6} T^2 + 6.35 \times 10^{-8} T^3 + 4.1 \times 10^{-10} T^4]$$
$$+ T[6.95 \times 10^{-6} D - 5.27 \times 10^{-9} D^2 + 2.7 \times 10^{-14} D^3]$$

Units:

T Degrees Centigrade

s Salinity

D Depth in meter

θ Absolute value of latitude in degrees

a m/sec

Kuwahara's data and its proceedings had been used for two decades. It was not until several investigators measured the velocity of sound with more sophisticated devices such as ultrasonic interferometer and sound velocimeter. Results of their research show that the velocity of sound given by Kuwahara is 3-4 m/sec low. The error in Kuwahara's data stems from indirect calculation of mean compressibility of water K, given by Ekman, which is

$$10^5 K = \frac{4886}{1 + 0.183p} - [227 + 28.33T - 0.551T^2 + 0.004T^3]$$
$$+ p[105.5 + 9.50T - 0.158T^2] - 1.5p^2 T$$
$$- 0.1(\sigma_{s,0,0} - 28)[147.3 - 2.72T + 0.04T^2$$
$$- p(32.4 - 0.87T + 0.02T^2)] + 0.01(\sigma_{s,0,0} - 28)^2$$
$$[4.5 - 0.1T - p(1.8 - 0.06T)] \qquad \text{(B-15)}$$

where

$$\upsilon = \upsilon_o(1 - Kp) \qquad \text{(B-16)}$$

K is the mean compressibility between 0 and p bars, in reciprocal bars.

p is in kilobars above atmosphere.

T is in degrees Centigrade

$$\sigma_{s,0,0} = -0.069 + 1.478\,Cl - 0.001570\,Cl^2 + 0.0000398\,Cl^2$$

Cl is the chlorinity

$$M = -\frac{1}{\upsilon}\frac{\partial \upsilon}{\partial p} \qquad \text{(B-17)}$$

and

$$M = \frac{K + p\frac{\partial K}{\partial p}}{1 - Kp} \qquad \text{(B-18)}$$

M is the true coefficient of compressibility.

The relationship between the true compressibility M and the mean compressibility K, follows from elimination of υ between Equation (B-16) and Equation (B-17).

Recently W. D. Wilson (82) gave a more accurate formula. He used 581 sound speeds, and obtained the coefficients in their formula by forming a 20 x 20 matrix. It reads as follows

$$a = 1449.22 + \Delta a_T + \Delta a_p + \Delta a_s + \Delta a_{sTp} \qquad \text{(B-19)}$$

$$\Delta a_T = 4.6233\,T - 5.4585 \times 10^{-2}\,T^2 + 2.822 \times 10^{-4}\,T^3 - 5.07 \times 10^{-7}\,T^4$$

$$\Delta a_p = 1.60518 \times 10^{-1} p + 1.0279 \times 10^{-5} p^2 + 3.451 \times 10^{-9} p^3$$
$$- 3.503 \times 10^{-12} p^4$$

$$\Delta a_s = 1.391 (s-35) - 7.8 \times 10^{-2} (s-35)^2$$

$$\Delta a_{sTp} = (s-35)(-1.197 \times 10^{-2} T + 2.61 \times 10^{-4} p - 1.96 \times 10^{-7} p^2$$
$$- 2.09 \times 10^{-6} pT) + p(-2.796 \times 10^{-4} T + 1.3302 \times 10^{-5} T^2$$
$$- 6.644 \times 10^{-8} T^3 + p^2(-2.391 \times 10^{-7} T + 9.286 \times 10^{-10} T^2)$$
$$- 1.745 \times 10^{-10} p^3 T$$

Units:

a m/sec

p Absolute pressure kg/cm^2

T Degrees Centigrade

s Salinity $^o/oo$

Range of validity:

$$-3^\circ < T < 30^\circ C$$

$$1.033 < p < 1000 \ kg/cm^2$$

$$33 < s < 37 \ ^o/oo$$

Accuracy is more than 0.005 m/sec within the range of validity.

Finally the work of Wilson (81) which is a revised form of his earlier work and covers a wider range of salinity:

$$a = 1449.14 + \Delta a_T + \Delta a_p + \Delta a_s + \Delta a_{sTp} \qquad \text{(B-20)}$$

$$\Delta a_T = 4.5712 - 4.4532 \times 10^{-2} T^2 - 2.6045 \times 10^{-4} T^3 + 7.9851 \times 10^{-6} T^4$$

$$\Delta a_p = 1.60272 \times 10^{-1} p + 1.0268 \times 10^{-5} p^2 + 3.5216 \times 10^{-9} p^3$$
$$- 3.3603 \times 10^{-12} p^4$$

$$\Delta a_s = 1.39799(s-35) + 1.69202 \times 10^{-3} (s-35)^2$$

$$\Delta a_{sTp} = (s-35)(-1.1244 \times 10^{-2} T + 7.7711 \times 10^{-7} T^2 + 7.7016 \times 10^{-5} p$$
$$- 1.2943 \times 10^{-7} p^2 + 3.1580 \times 10^{-8} pT + 1.5790 \times 10^{-9} pT^2)$$
$$+ p(-1.8607 \times 10^{-4} T + 7.4812 \times 10^{6} T^2 + 4.5283 \times 10^{-8} T^3)$$
$$+ p^2(-2.5294 \times 10^{-7} + 1.8563 \times 10^{-4} T^2) + p^3(1.9649 \times 10^{-10} T)$$

Units:

a m/sec

p Absolute pressure kg/cm^2

T Degree Centigrade

s Salinity $^o/oo$ (parts per thousand)

Range of Accuracy:

$$-4 < T < 30^\circ C$$

$$1 < p < 1000\ kg/cm^2$$

$$0 < s < 37\ ^o/oo$$

Recent measurements of Hays (29) show a deviation from the computed values from Wilson's formula. The difference between the two shows a nearly linear increase with depth. This systematic difference between the two values suggests that further investigation is needed to assure an accuracy of order of 0.1 m/sec. However, the relation given by Wilson is the most reliable of all and can be used for our purpose.

B.3 Caloric Equation of State for Water

Because of limited use, a general caloric equation of state that relates H with p and T have not been attempted for water. Instead, there are relations that give C_p as functions of temperature. Here is presented the heat capacity data quoted by Kuwahara (44):

$$C_p = C'_p - 0.0004226\,T + 0.000006321\,T^2 \text{ cal/gm}\,^\circ\text{C} \tag{B-21}$$

where

$$C'_p = 1.005 - 0.004136s + 0.0001098s^2 - 0.000001324s^3$$

In the above relation T is temperature in degrees Centigrade and s is salinity in parts per thousand.

B.4 Thickness of a Shock Wave in Water

The Rankine-Hugoniot equations are generally accepted and universally used to represent the flow conditions at a shock front. While the conclusion of discontinuity in the flow conditions is mathematically valid, it cannot be justified physically. Thus we have to give an exceedingly small length to the region of abrupt change and try to estimate the length.

In macroscopic approach which the fluid is considered as a continuum, we include the dissipative mechanisms in the flow equations. A more accurate analysis would be the microscopic or molecular approach. When the Mach number is very high and a large change occurs in the flow conditions across a shock the assumption of constant transport coefficients is not valid in the narrow region of change. In this approach, we are dealing with a complicated integro-differential equation which should be solved by resorting to different approximate methods. In water, since the Mach number cannot become very high, macroscopic approach suffices. Thus, we will try to make an estimate of the shock wave thickness from the consideration of the Navier-Stokes equations.

Consider a plane wave (Figure B-1) moving with a velocity of a into a stationary fluid.

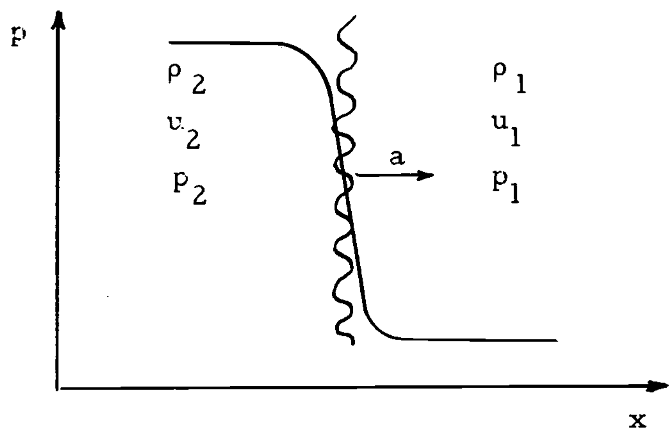

Figure B-1. Moving shock wave.

The equations of motion are:

1. Continuity Equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \tag{B-22}$$

2. Momentum Equation

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} \tag{B-23}$$

The following assumptions are made:

1. $\frac{\partial p}{\partial \rho} = \frac{\bar{\beta}}{\rho}$, with constant $\bar{\beta}$.
2. Speed of the wave is constant.
3. Change of viscosity μ is neglected.

To solve the set of equations, let $\xi = x - at$. The continuity equation in terms of the new variable is written

$$\frac{d\rho}{d\xi} = \frac{1}{a}\frac{d}{d\xi}(\rho u) \tag{B-24}$$

The momentum equation appears in the following form

$$\rho(u-a)\frac{du}{d\xi} = \frac{\bar{\beta}}{\rho}\frac{d\rho}{d\xi} + \mu \frac{d^2 u}{d\xi^2} \tag{B-25}$$

If we integrate Equation (B-24), we get

$$\rho = \frac{\rho_1}{1 - \frac{u}{a}} \tag{B-26}$$

Then, the Equation (B-25) can be written

$$-a\frac{du}{d\xi} = -\frac{\bar{\beta}}{\rho_1}(1-\frac{u}{a})\rho_1 \frac{d}{d\xi}(\frac{1}{1-\frac{u}{a}}) + \mu\frac{d^2u}{d\xi^2} \quad \text{(B-27)}$$

which can be written in a more compact form

$$a\nu_1 \frac{d^2\frac{u}{a}}{d\xi^2} = [\frac{\bar{\beta}/\rho_1}{1-\frac{u}{a}} - a^2]\frac{d\frac{u}{a}}{d\xi} \quad \text{(B-28)}$$

Outside of the region of transition, the Rankine-Hugoniot relations are valid. Thus,

$$a = \frac{\Delta p}{\rho_1 u_2} \quad \text{(B-29)}$$

Combination of Equation (B-26) with (B-29) results in

$$a = -\frac{\bar{\beta}}{\rho_1 u_2}\ln(1-\frac{u_2}{a}) \quad \text{(B-30)}$$

Since $\frac{u_2}{a}$ is less than one in general, and is very much less than one for water, we can expand $\ln(1-\frac{u_2}{a})$ and $\frac{1}{1-\frac{u}{a}}$ in terms of power series of $\frac{u_2}{a}$. If we truncate the series of $\ln(1-\frac{u_2}{a})$ after the quadratic term, the series of $\frac{1}{1-\frac{u}{a}}$ after the linear term, and designate $\frac{u}{a}$ by w, we get

$$a\nu_1 \frac{d^2w}{d\xi^2} = -\frac{\bar{\beta}}{\rho_1}(\frac{w^2}{2} - w)\frac{dw}{d\xi} \quad \text{(B-31)}$$

Since $\bar{\beta}/\rho_1 = a^2$, Equation (B-13) can be written

$$\nu_1 \frac{d^2w}{d\xi^2} = -a(\frac{w^2}{2} - w)\frac{dw}{d\xi} \quad \text{(B-32)}$$

We can integrate this equation once.

$$\nu_1 \frac{dw}{d\xi} + a\frac{w_2 w}{2} - a\frac{w^2}{2} = 0 \qquad \text{(B-33)}$$

Letting $\eta = \frac{\xi a}{2\nu_1}$, and $\frac{w}{w_2} = \frac{u}{u_2} = \omega$, we get

$$\frac{d\omega}{\omega^2 - \omega} = w_2 d\eta \qquad \text{(B-34)}$$

Integrating this equation once more yields

$$\ln \frac{\omega}{\omega - 1} + w_2 \eta = k \qquad \text{(B-35)}$$

One estimate of the shock wave thickness would be to evaluate the following

$$\delta = u_2 / \left(\frac{\partial u}{\partial x}\right)_{max} \qquad \text{(B-36)}$$

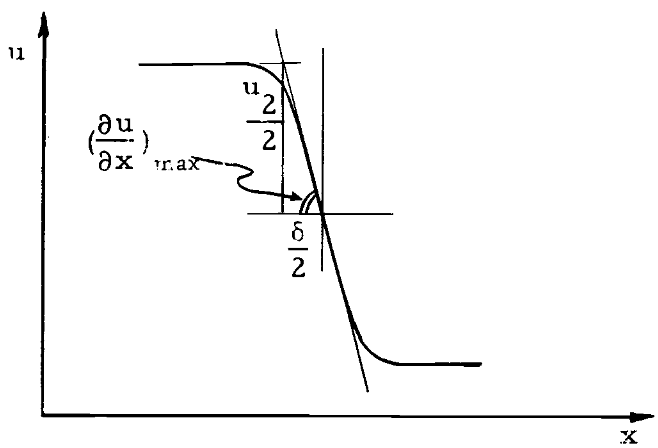

Figure B-2. Shock wave thickness.

Another estimate of the shock wave thickness would be to integrate Equation (B-34) directly, using the concept of boundary layer.

$$\delta = \frac{2\mu a}{\Delta p} \int_{\omega_1}^{\omega_2} \frac{d\omega}{(\omega^2 - \omega)} \qquad \text{(B-37)}$$

ω_1 and ω_2 are the limits of integration and can be considered for example 0.01 and 0.99 respectively. The former yields a straightforward answer

$$\delta = \frac{8\mu a}{\Delta p} \qquad \text{(B-38)}$$

The latter, with limits of $\omega_1 = -0.99$ and $\omega_2 = -0.01$, yields

$$\delta = \frac{7.636\mu a}{\Delta p}$$