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A model for an axiom system of projective geometry is developed in order to show these axiom are consistent if linear algebra is consistent.

A PROOF OF THE CONSISTENCY OF PROJECTIVE GEOMETRY

by

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A PROOF OF THE CONSISTENCY OF PROJECTIVE GEOMETRY

INTRODUCTION

This paper is written to show the consistency of an axiomatic system of projective geometry. The need for axiomatic systems has been recognized since the days of Euclid whose Elements came out around 2300 years ago. Less appreciated is the fact that axiom systems should be confirmed not only by intuition and the theorems resulting from the axioms, but also by proof of their consistency. Since absolute consistency apparently is an illusion, I have been forced to a proof of relative consistency. I shall show that the given system of axioms is consistent if linear algebra of real numbers is. Since real linear algebra is consistent if real arithmetic is, the system of axioms is consistent if arithmetic is.

Euclid in formalizing all the geometry known in his day into an axiomatic system, had laid the foundation for a cultural environment that was to free man's intellect to search for new areas of knowledge. This knowledge was to be consistent with the work of Euclid, yet different from it, as demonstrated by such men as Karl Friedrich Gauss, Johann Bolyai, and Nikolai Ivanovich Lobachewsky.

"If you would converse with me, define your terms." These words spoken by Voltaire go a long way in explaining the first step in the system employed by Euclid. (It must be noted that Euclid himself made no explicit statement of his system, which was in fact deficient by modern standards. His definitions were poetic-not exact for he had no primitive terms-and his axioms were not complete.) If you are to carry on an intelligent conversation with anyone you must know the meanings of the words to be used. Here lies part of the problem for we find the meanings of words to be circular. Thus Voltaire was asking the impossible. This gives rise to the first part of an axiomatic system, the primitive or undefined term. The undefined term is one we accept without meaning, and in accepting it we are able to move forward, where as rejection of this idea would leave us in the abominable abyss of trying to define words in terms of themselves.

The second step in establishing an axiomatic system is to set up some statements, or axioms, involving the undefined terms that are to be accepted without proof. This is necessary, for to prove something we must rely on what we already know, so at some point in the proof of a set of statements it is obvious that at least one will have a proof that utilizes itself. It is this system, consisting of undefined terms and axioms, that is so widely used in mathematics today as well as in every aspect of modern society.

A traditional condition put on every axiom system is that of irreducibility. This infers that it is distinct as possible and no axiom can be deduced from the remaining ones. In keeping an axiom system reduced not only is one maintaining at a minimum the facts he must accept without proof, but also as the number of axioms decreases the probability of inadvertent inconsistency lessens. Likewise it should be noted, that one should not reduce an axiom system without proof that the discarded statements are theorems resulting from the remaining axioms, or he may find the system unable to do that for which it was designed.

With the undefined terms and axioms man can, with the laws of logic, deduce new statements and thus increase his knowledge. It is the necessity of these undefined terms and axioms that gave rise to Bertram Russell's famous aphorism that in mathematics one never knows what one is talking about or whether what one is saying is true.

We ask ourselves the question of why we need models that show consistency in the axioms of our system. I believe that there are two major reasons why. First, historically we know that many axioms systems have been inconsistent so we realize that an apparent simple system certainly has the possibility of error.

Second, man has an unending desire for perfection. Man has constantly sought ways to bring perfection into his work. All his physical work although good has little imperfections that he keeps trying to delete. Now in an axiom system he sees a chance to gain this perfection that has so far escaped him. For this reason he feels compelled to try to prove that his axiom system is perfection. To do this he uses a model.

Once we have proved an axiom system to be consistent we can proceed, feeling confident that all theorems that we draw from these axioms will also be true and we have a way of expanding our knowledge. This knowledge we know is true, from the model, so it is a means of gaining more truth relative to our axioms.

Now that we have discussed the reasons for desiring a model for our axiom system let us consider what this model should consist of in terms of prior truth. Since we have no perfection in our work we can only prove consistency if we can put the model back into our logic. We must believe in our logic for if we don't there is no reason to go on. As soon as one doubts logic he can never accept anything and the result is chaotic.

So our model must be resolved to logic which will eliminate the main points of inconsistency, that of the Law of Contradiction and the Law of Excluded Middle. Or it may be shown consistent in another system that is built upon logic.

Our algebra has been built upon logic and is the system in which I will prove the consistency of the axiom system of projective geometry.

A SYSTEM OF PROJECTIVE GEOMETRY

The primitive things of this system are points, planes, and lines.

The primitive relations are the relation of incidence of line and point; the relation of incidence of plane and point; and the relation of separation. The first is a binary relation between the set of lines and the set of points. The second is a binary relation between the set of planes and the set of points. The last is a binary relation on the set of ordered pairs of distinct points and itself.

A set of points is called a collinear set if the same line is incident on each member of the set.

A set of points is called a coplanar set if the same plane is incident on each member of the set.

A point is said to be of or on a line (or of or on a plane) if that line (or plane) is incident upon the point.

A line is in or of a plane if every point of the line is a point of the plane.

A set of lines is called a coplanar set if every line of the set is a line of one and the same plane.

Axioms of Incidence

Axiom 1. If P and Q are distinct points there exists a unique line incident upon them.

Axiom 2. There exist at least two points on a straight line, three non-collinear points in a plane, and at least one plane.

Axiom 3. If P, Q, and R are non-collinear points there exists a unique plane incident upon them.

Axiom 4. If P and Q are points of a plane π and also points of a line l, l is a line of π.

Axiom 5. Any two distinct planes have exactly one line in common.

Axiom 6. There exist at least four non-coplanar points.

Axiom 7. Two distinct coplanar lines have a unique point in common.

Axiom of Order

Axiom 8. Given n collinear points, with n greater than three, there are exactly two ways of naming them, $P_1P_2 \ldots P_n$ or $P_nP_{n-1} \ldots P_1$, so that P_i and P_k separate P_j and P_ℓ if and only if i, j, k, and ℓ are in natural order, or can be obtained from natural order by substitutions from the group generated by $(P_iP_jP_kP_\ell)$ and (P_iP_k).

A MODEL OF THE SYSTEM

A model of the system is obtained by means of the following interpretations in the linear algebra of four real numbers: a point is an equivalence class of ordered quadruples of numbers (x, y, z, w), not all zero, such that (x, y, z, w) is equivalent to (x_1, y_1, z_1, w_1) if and only if the rank of

$$\begin{pmatrix} x & y & z & w \\ x_1 & y_1 & z_1 & w_1 \end{pmatrix}$$

is equal to one; a plane is an equivalence class of equations equivalent to

$$Ax + By + Cz + Dw = 0,$$

with $A^2 + B^2 + C^2 + D^2 \neq 0$; and a line is an equivalence class of pairs of equations equivalent to

$$A_1x + B_1y + C_1z + D_1w = 0,$$

$$A_2x + B_2y + C_2z + D_2w = 0,$$

with rank 2 for the matrix

$$\begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \end{pmatrix}.$$

Further interpretations of the system are as follows: a line is said to be incident upon a point if and only if every quadruple of the point is in the solution set of each of the pairs of the line; a plane is said to be incident on a point if and only if each of the quadruples of the point is in the solution set of the equations of the plane.

The final interpretation is that of the relation of separation. Given four collinear points P_1, P_2, P_3, and P_4, the points P_2 and P_4 separate P_1 and P_3 if and only if the cross ratio of the points, $R(P_1P_2, P_4P_3)$, (defined on page 26), is negative.

Now to prove the consistency of the axiom system I need only prove each of the axioms as a theorem in the model described above.

AXIOMS OF INCIDENCE

Theorem 1. If P and Q are distinct points there exists a unique line incident upon them.

Let the two points be represented by $P_1\ (x_1, y_1, z_1, w_1)$ and $P_2\ (x_2, y_2, z_2, w_2)$. To show uniqueness I will assume two lines, l_1 and l_2, are incident upon P_1 and P_2 and show a contradiction. Let l_1 be represented by

$$A_1x + B_1y + C_1z + D_1w = 0,$$
$$A_2x + B_2y + C_2z + D_2w = 0,$$

with rank 2 for the matrix

$$\begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \end{pmatrix}.$$

By symmetry we may assume that $A_1D_2 - A_2D_1 \neq 0$. Let l_2 be represented by

$$A_3x + B_3y + C_3z + D_3w = 0,$$
$$A_4x + B_4y + C_4z + D_4w = 0,$$

with rank 2 for the matrix

$$\begin{pmatrix} A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{pmatrix}.$$

To show that the lines are identical I need only show that their solution sets are equivalent. To do this I will reduce the equations of l_1 to an equivalent set in terms of P_1 and P_2. A similar argument would follow with l_2 which will prove uniqueness.

For l_1 we have,

(1) $A_1x_1 + B_1y_1 + C_1z_1 + D_1w_1 = 0,$

(2) $A_2x_1 + B_2y_1 + C_2z_1 + D_2w_1 = 0,$

(3) $A_1x_2 + B_1y_2 + C_1z_2 + D_1w_2 = 0,$

(4) $A_2x_2 + B_2y_2 + C_2z_2 + D_2w_2 = 0.$

From (1) and (2) we have,

(5) $(A_1D_2-A_2D_1)x_1 + (B_1D_2-B_2D_1)y_1 + (C_1D_2-C_2D_1)z_1 = 0,$

(6) $(A_1D_2-A_2C_1)x_1 + (B_1D_2-B_2C_1)y_1 + (D_1C_2-D_2C_1)w_1 = 0,$

(7) $(A_1B_2-A_2B_1)x_1 + (C_1B_2-C_2B_1)z_1 + (D_1B_2-D_2B_1)w_1 = 0.$

From (3) and (4) we have,

(8) $(A_1D_2-A_2D_1)x_2 + (B_1D_2-B_2D_1)y_2 + (C_1D_2-C_2D_1)z_2 = 0,$

(9) $(A_1C_2-A_2C_1)x_2 + (B_1C_2-B_2C_1)y_2 + (D_1C_2-D_2C_1)w_2 = 0,$

(10) $(A_1B_2-A_2B_1)x_2 + (C_1B_2-C_2B_1)z_2 + (D_1B_2-D_2B_1)w_2 = 0.$

From (5) and (8) we have,

(11) $(A_1D_2-A_2D_1)(x_1z_2-x_2z_1) + (B_1D_2-B_2D_1)(y_1z_2-y_2z_1) = 0,$

(12) $(A_1D_2-A_2D_1)(x_1y_2-x_2y_1) + (C_1D_2-C_2D_1)(z_1y_2-z_zy_1) = 0.$

From (6) and (9) we have,

(13) $(A_1C_2-A_2C_1)(x_1w_2-x_2w_1) + (B_1C_2-B_2C_1)(y_1w_2-y_2w_1) = 0$,

(14) $(A_1C_2-A_2C_1)(x_1y_2-x_2y_1) + (D_1C_2-D_2C_1)(w_1y_2-w_2y_1) = 0$.

From (7) and (10) we have,

(15) $(A_1B_2-A_2B_1)(x_1w_2-x_2w_1) + (C_1B_2-C_2B_1)(z_1w_2-z_2w_1) = 0$,

(16) $(A_1B_2-A_2B_1)(x_1z_2-x_2z_1) + (D_1B_2-D_2B_1)(w_1z_2-w_2z_1) = 0$.

Since the points are distinct we know the rank of the matrix

$$\begin{pmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \end{pmatrix}$$

is two. Now by symmetry we can assume that $x_1z_2-x_2z_1 \neq 0$. But we know that $A_1D_2-A_2D_1 \neq 0$, so from equation (11) we have $B_1D_2-B_2D_1 \neq 0$ and $y_1z_2-y_2z_1 \neq 0$. We will now break the proof into two parts depending on whether or not $z_1w_2-z_2w_1 = 0$.

First let us consider it equal to zero. This gives rise to two conditions, namely $w_2 = 0$ or $w_2 \neq 0$. If $w_2 = 0$, then either $z_2 = 0$, $w_1 = 0$, or $z_2 = w_1 = 0$. With any one of these three conditions being true equations (15) and (16) reduce to the identity, $0 = 0$. If $w_2 \neq 0$, then $z_1 = (z_2w_1)/w_2$. Since we know that $x_1z_2-x_2z_1 \neq 0$ and $y_1z_2-y_2z_1 \neq 0$, we can deduce that $x_1w_2-x_2w_1 \neq 0$ and $y_1w_2-y_2w_1 \neq 0$. Now from equations (13) and (14) either $A_1C_2-A_2C_1$, $D_1C_2-D_2C_1$, and $B_1C_2-B_2C_1$ are all equal to zero or that $A_1C_2-A_2C_1$ and $B_1C_2-B_2C_1$ are not equal to zero. If all equal zero the equations (13) and (14) become the identity, $0 = 0$.

If $A_1C_2-A_2C_1$ and $B_1C_2-B_2C_1$ do not equal zero we find that equations (11) and (12) are equivalent to

$$(17) \qquad \frac{x_1z_2-x_2z_1}{y_1z_2-y_2z_1} = -\frac{B_1D_2-B_2D_1}{A_1D_2-A_2D_1},$$

$$(18) \qquad \frac{x_1y_2-x_2y_1}{z_1y_2-z_2y_1} = -\frac{C_1D_2-C_2D_1}{A_1D_2-A_2D_1},$$

and that equations (13) and (14) are equivalent to

$$(19) \qquad \frac{x_1w_2-x_2w_1}{y_1w_2-y_2w_1} = -\frac{B_1C_2-B_2C_1}{A_1C_2-A_2C_1},$$

$$(20) \qquad \frac{x_1y_2-x_2y_1}{w_1y_2-w_2y_1} = -\frac{D_1C_2-D_2C_1}{A_1C_2-A_2C_1}.$$

Referring back to the general equations of l_1 we can write the following equations that are equivalent to the original ones

$$(21) \quad (A_1D_2-A_2D_1)x + (B_1D_2-B_2D_1)y + (C_1D_2-C_2D_1)z = 0.$$

$$(22) \quad (A_1C_2-A_2C_1)x + (B_1C_2-B_2C_1)y + (D_1C_2-D_2C_1)w = 0.$$

Now equations (21) and (22) can be rewritten into another pair of equations which are listed below.

$$(23) \quad x + \frac{B_1D_2-B_2D_1}{A_1D_2-A_2D_1}\, y + \frac{C_1D_2-C_2D_1}{A_1D_2-A_2D_1}\, z = 0,$$

$$(24) \quad x + \frac{B_1C_2-B_2C_1}{A_1C_2-A_2C_1}\, y + \frac{D_1C_2-D_2C_1}{A_1C_2-A_2C_1}\, w = 0.$$

From equations (17), (18), and (23) we may derive equation

(25). Likewise from equations (19), (20), and (24) we may derive equation (26).

$$(25) \quad x - \frac{x_1z_2-x_2z_1}{y_1z_2-y_2z_1}y + \frac{x_1y_2-x_2y_1}{y_1z_2-y_2z_1}z = 0.$$

$$(26) \quad x - \frac{x_1w_2-x_2w_1}{y_1w_2-y_2w_1}y + \frac{x_1y_2-x_2y_1}{y_1w_2-y_2w_1}w = 0.$$

Equations (25) and (26) are equivalent to the equations of l_1 which shows uniqueness for part one.

In the second part of this proof we must consider $z_1w_2-z_2w_1 \neq 0$. Now upon examination of equations (15) and (16) we find that either $A_1B_2-A_2B_1 = D_1B_2-D_2B_1 = 0$, or both differ from zero. We already know that $D_1B_2-D_2B_1 \neq 0$. Therefore both must differ from zero. With this information we may rewrite equations (15) and (16) into the forms (27) and (28) given below.

$$(27) \quad \frac{x_1w_2-x_2w_1}{z_1w_2-z_2w_1} = \frac{C_1B_2-C_2B_1}{A_1B_2-A_2B_1},$$

$$(28) \quad \frac{x_1z_2-x_2z_1}{w_1z_2-w_2z_1} = \frac{D_1B_2-D_2B_1}{A_1B_2-A_2B_1}.$$

Referring back to the general equation for l_1 we can deduce equation (29) which then may be written in the form of equation (30).

$$(29) \quad (A_1B_2-A_2B_1)x + (C_1B_2-C_2B_1)z + (D_1B_2-D_2B_1)w = 0,$$

(30) $$x + \frac{C_1B_2-C_2B_1}{A_1B_2-A_2B_1} z + \frac{D_1B_2-D_2B_1}{A_1B_2-A_2B_1} w = 0.$$

From equations (27), (28), and (30) we are able to derive equation (31).

(31) $$x - \frac{x_1w_2-x_2w_1}{z_1w_2-z_2w_1} z + \frac{x_1z_2-x_2z_1}{z_1w_2-z_2w_1} w = 0.$$

Now equations (25) and (31) are another pair of equations equivalent to l_1 which will show uniqueness. Here it should be noted that equation (25) is independent of the value of $z_1w_2-z_2w_1$.

To show existence of such a line I need only show a pair of equations that are satisfied by the given quadruples of numbers. (32) and (33) are such equations.

(32) $$(y_1z_2-y_2z_1)x + (x_2z_1-x_1z_2)y + (x_1y_2-x_2y_1)z = 0.$$

(33) $$(y_1w_2-y_2w_1)x + (x_2w_1-x_1w_2)y + (x_1y_2-x_2y_1)w = 0.$$

Lemma 1. Given points, P_1 (x_1, y_1, z_1, w_1), P_2 (x_2, y_2, z_2, w_2), and P_3 (x_3, y_3, z_3, w_3), and the matrix formed by these quadruples,

$$W = \begin{pmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{pmatrix}$$

(1) if $r(W) = 1$, then the quadruples all represent the same point;

(2) r(W) = 2, then the points are collinear; (3) r(W) = 3, then the points are non-collinear.

Part one follows as a direct consequence of the definition of a point.

In part two since r(W) = 2, the system

$$Ax_1 + By_1 + Cz_1 + Dw_1 = 0 ,$$

$$Ax_2 + By_2 + Cz_2 + Dw_2 = 0 ,$$

$$Ax_3 + By_3 + Cz_3 + Dw_3 = 0 ,$$

has exactly two linearly dependent solutions, hence the points are collinear.

In part three r(W) = 3, hence the system of equations above has an infinite number of solutions but each is dependent on a single solution. Hence the points are non-collinear.

Corollary to Lemma 1. Given three non-collinear points the matrix formed by the quadruples that represent these points has a rank of three.

The matrix cannot have a rank of zero or all points would vanish. If we assume it has a rank of one or two we have a contradiction of lemma one. But the rank of the matrix must be three or less, so it must be three.

Theorem 2. There exist at least two points on a straight line, three non-collinear points in a plane and at least one plane.

This theorem I will break down into three separate parts and prove each one individually.

Part 1. There exist at least two points on a straight line.

Let the line be represented by

$$A_1x + B_1y + C_1z + D_1w = 0,$$

$$A_2x + B_2y + C_2z + D_2w = 0,$$

with rank 2 for the matrix

$$\begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \end{pmatrix}.$$

Since the rank is two one of the following must be true:

$A_1B_2-A_2B_1 \neq 0$, $A_1C_2-A_2C_1 \neq 0$, $A_1D_2-A_2D_1 \neq 0$, $B_1C_2-B_2C_1 \neq 0$, $B_1C_2-B_2C_1 \neq 0$, $B_1D_2-B_2D_1 \neq 0$, or $C_1D_2-C_2D_1 \neq 0$.
By symmetry we may assume that $A_1B_2-A_2B_1 \neq 0$. Thus we can get solutions with $z = 0$ and $w = 1$; and with $z = 1$ and $w = 0$, the values of $x + y$ being then uniquely determined. These solutions will yield unique points. If any of the six determinants is not equal to zero a parallel proof may be used and since one of them must be non-zero, the result follows.

Part 2. There exist at least three non-collinear points in a plane.

Let the given plane be represented by

$$Ax + By + Cz + Dw = 0,$$

with $A^2 + B^2 + C^2 + D^2 \neq 0$. We may assume $A \neq 0$ by symmetry. This allows us to rewrite the equation of the plane as given below.

$$x = -\frac{B}{A}y - \frac{C}{A}z - \frac{D}{A}w$$

Then there is complete freedom in the assigning of values of y, z, and w. We may take these to be 0, 0, 1; 0, 1, 0; and 1, 0, 0; and get three corresponding values of x. Then these three ordered quadruples, (-D/A, 0, 0, 1), (-C/A, 0, 1, 0), and (-B/A, 1, 0, 0), are representatives of points since each has a non-zero member and since the rank of

$$\begin{pmatrix} -D/A & 0 & 0 & 1 \\ -C/A & 0 & 1 & 0 \\ -B/A & 1 & 0 & 0 \end{pmatrix}$$

is 3, for

$$\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} \neq 0 ,$$

the above three points are non-collinear. This fact was demonstrated in lemma 1.

Part 3. There exists at least one plane.

To verify this property I need only show that at least one equation with non-zero coefficients exist. Such an equation is $y = 0$.

Lemma 2. Given a line l that is in a plane π, then l may be represented by a pair of equations that includes a representative equation of π.

Let l be represented by

$$A_1x + B_1y + C_1z + D_1w = 0,$$
$$A_2x + B_2y + C_2z + D_2w = 0,$$

with rank 2 for the matrix

$$\begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \end{pmatrix},$$

and π be represented by

$$A_3x + B_3y + C_3z + D_3w = 0.$$

We need now only consider the rank of the following matrix.

$$N = \begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \end{pmatrix}.$$

Since each of the planes represented above contains the line l, the system

$$A_1x + B_1y + C_1z + D_1w = 0,$$
$$A_2x + B_2y + C_2z + D_2w = 0,$$
$$A_3x + B_3y + C_3z + D_3w = 0,$$

has exactly two linearly independent solutions. Hence the rank of N is 2. Since the rank of

$$\begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \end{pmatrix}$$

is 2, the row (A_3, B_3, C_3, D_3) is linearly dependent upon (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2). Then π and at least one of the other planes determine a line. This line and l are one and the same as l is common to all of the above planes.

Corollary to Lemma 2. Two distinct coplanar lines can be represented by pairs of equations that include a representative of the common plane.

This is an immediate consequence of lemma 2.

Theorem 3. If P, Q, and R are non-collinear points, there exists a unique plane incident upon them.

Let the points be represented by $P\ (x_1, y_1, z_1, w_1)$, $Q\ (x_2, y_2, z_2, w_2)$ and $R\ (x_3, y_3, z_3, w_3)$. Now by the corollary of lemma 1 the matrix formed by the above quadruples has a rank of three, hence the system

$$Ax_1 + By_1 + Cz_1 + Dw_1 = 0,$$

$$Ax_2 + By_2 + Cz_2 + Dw_2 = 0,$$

$$Ax_3 + By_3 + Cz_3 + Dw_3 = 0,$$

has a solution set all of whose members are linearly dependent on some given non-zero solution.

Theorem 4. If P and Q are points of a plane π and also points of a line l, l is a line of π.

Let P and Q be the points on l, with l represented by

$$A'x + B'y + C'z + D'w = 0,$$

$$A''x + B''y + C''z + D''w = 0,$$

with rank 2 for the matrix

$$\begin{pmatrix} A' & B' & C' & D' \\ A'' & B'' & C'' & D'' \end{pmatrix},$$

and P and Q also on π, with π represented by

$$Ax + By + Cz + Dy = 0.$$

Now consider the rank of the following matrix.

$$U = \begin{pmatrix} A' & B' & C' & D' \\ A'' & B'' & C'' & D'' \\ A & B & C & D \end{pmatrix}$$

Since the rank of

$$\begin{pmatrix} A' & B' & C' & D' \\ A'' & B'' & C'' & D'' \end{pmatrix}$$

is 2, the rank of U is at least 2. If the rank of U is 2, then row (A, B, C, D) is linearly dependent upon rows (A', B', C', D') and (A'', B'', C'', D''). Hence every solution of the system

$$A'x + B'y + C'z + D'w = 0,$$

$$A''x + B''y + C''z + D''w = 0,$$

is a solution of π. Since every point of l is a solution of the above system every point of l is in π.

The rank of U cannot be three for then the system

$$A'x + B'y + C'z + D'w = 0,$$

$$A''x + B''y + C''z + D''w = 0,$$

$$Ax + By + Cz + Dw = 0,$$

would be dependent on a given non-zero solution and we were given that it has two linearly independent solutions.

Theorem 5. Two distinct planes have exactly one line in common.

Let the two planes be represented by the following two equations:

$$Ax + By + Cz + Dw = 0,$$

$$A'x + B'y + C'z + D'w = 0.$$

Since the planes are distinct we have the rank of

$$\begin{pmatrix} A & B & C & D \\ A' & B' & C' & D' \end{pmatrix}$$

equal to 2. Hence the system

$$Ax + By + Cz + Dw = 0,$$

$$A'x + B'y + C'z + D'w = 0,$$

has a non-empty solution set, i. e. there are points common to the two planes. However, every such a point is on the line represented by

$$Ax + By + Cz + Dw = 0,$$

$$A'x + B'y + C'z + D'w = 0.$$

This is a line for the rank of

$$\begin{pmatrix} A & B & C & D \\ A' & B' & C' & D' \end{pmatrix}$$

is 2.

Theorem 6. There exist at least four non-coplanar points.

Consider the plane represented by

$$Ax + By + Cz + Dw = 0,$$

with $A^2 + B^2 + C^2 + D^2 \neq 0$, and the points P_1 (0, 0, 0, 1), P_2 (0, 0, 1, 0), P_3 (0, 1, 0, 0), and P_4 (1, 0, 0, 0). Assume the four points are coplanar and therefore satisfy the equation of the plane. Then, for P_1, $D = 0$; for P_2, $C = 0$; for P_3, $B = 0$; and for P_4, $A = 0$. But, then $A^2 + B^2 + C^2 + D^2 = 0$, a contradiction, showing that the points are non-coplanar.

Theorem 7. Two distinct coplanar lines have a unique point in common.

By the corollary to lemma 2 we know that two such lines can be represented by pairs of equations that include an equation of the common plane. We will represent these lines as follows

$$l_1: \begin{array}{l} A_1x + B_1y + C_1z + D_1w = 0, \\ Ax + By + Cz + Dw = 0, \end{array}$$

$$l_2: \begin{array}{l} A_2x + B_2y + C_2z + D_2w = 0, \\ Ax + By + Cz + Dw = 0, \end{array}$$

the common plane being represented by

$$Ax + By + Cz + Dw = 0.$$

Since each pair of equations above determines a line the rank of each coefficient matrix is 2.

Now consider the rank of

$$M = \begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A & B & C & D \end{pmatrix}.$$

We need only show that the rank of M is three which will give us a solution set all of whose members are dependent on one non-zero solution.

If the rank of this matrix is less than three it must be two. Since the rank of

$$\begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ A & B & C & D \end{pmatrix}$$

is equal to 2, the row (A_2, B_2, C_2, D_2) is linearly dependent upon (A, B, C, D) and (A_1, B_1, C_1, D_1). But then the second line coincides with the first, a contradiction which proves the rank of M is three.

AXIOM OF ORDER

Lemma 3. Given four collinear points of a line there exists a plane containing one of these points but not any other point of the line.

Let P_1, P_2, P_3, and P_4 be points of the line represented by

$$Ax + By + Cz + Dw = 0,$$

$$A'x + B'y + C'z + D'w = 0,$$

with rank 2 for the matrix

$$\begin{pmatrix} A & B & C & D \\ A' & B' & C' & D' \end{pmatrix}.$$

Since the rank is 2, $Ax + By + Cz + Dw = 0$ has at least one solution that is not a solution of $A'x + B'y + C'z + D'w = 0$, say P_5. Now P_1 and P_5 determine a line and this line is distinct from the line represented above, as P_5 is not a point of the original line. Represent this new line by

$$Ax + By + Cz + Dw = 0,$$

$$A''x + B''y + C''z + D''w = 0,$$

with rank 2 for the matrix

$$\begin{pmatrix} A & B & C & D \\ A'' & B'' & C'' & D'' \end{pmatrix}.$$

Now both lines are common to the plane represented by

$$Ax + By + Cz + Dw = 0,$$

as the quadruple representing the points of these lines satisfies the equation of the plane. But two coplanar lines can have only one point in common, so

$$A''x + B''y + C''z + D''w = 0,$$

represents a plane containing P_1 but no other points of the given line.

Since in the model points are named by an equivalence class of ordered quadruples and planes by an equivalence class of equations we must restrict these representations to a single representative in order to have uniqueness in our discussion of order. The canonical representative for each class is described below.

Points. If $x \neq 0$, we will use the quadruple $(1, y/x, z/x, w/x)$; if $x = 0$ and $y \neq 0$, we will use the quadruple $(0, 1, z/y, w/y)$; if $x = y = 0$ and $z \neq 0$, we will use the quadruple $(0, 0, 1, w/z)$; and if $x = y = z = 0$ and $w \neq 0$, we will use the quadruple $(0, 0, 0, 1)$. Since $(0, 0, 0, 0)$ is not a point in the model, we have a unique canonical representative for each point.

Planes. If $A \neq 0$, we will use the equation $x + (B/A)y + (C/A)z + (D/A)w = 0$; if $A = 0$ and $B \neq 0$, we will use the equation $y + (C/B)z + (D/B)w = 0$; if $A = B = 0$ and $C \neq 0$, we will use the equation $z + (D/C)w = 0$; and if $A = B = C = 0$ and $D \neq 0$, we will use the equation $w = 0$. Again each plane has a unique canonical representative.

With the above restrictions on naming of points and planes, any plane canonically represented by

$$Ax + By + Cz + Dw = 0,$$

will divide the set of all points into three classes which I will define as follows. Positive half space will consist of all points for which

$$Ax + By + Cz + Dw > 0.$$

Negative half space will consist of all points for which

$$Ax + By + Cz + Dw < 0.$$

The third class will consist of all points for which

$$Ax + By + Cz + Dw = 0.$$

Definition. $\pi_i(P_j)$ is a function of π_i and P_j, where π_i is the left side of a canonical representation of the plane containing

$$A_i x + B_i y + C_i z + D_i w = 0,$$

and P_j is the canonical representative of a point containing (x_j, y_j, z_j, w_j).

Definition. The cross ratio of four collinear points P_1, P_2, P_3, and P_4 is $R(P_1P_2, P_4P_3)$. Where

$$R(P_1P_2, P_4P_3) = \frac{\pi_2(P_1)}{\pi_2(P_3)} \div \frac{\pi_4(P_1)}{\pi_4(P_3)},$$

with $\pi_2(P_2) = \pi_4(P_4) = 0$ and $\pi_2(P_1)$, $\pi_2(P_3)$, $\pi_4(P_1)$, and $\pi_4(P_3)$ not equal to zero.

Definition. The relation of separation for P_2, P_4 and P_1, P_3 will be written $P_2P_4 \text{ V } P_1P_3$, where "V" means separate. "$\not V$" will mean do not separate.

Lemma 4. If P and Q are points of a line canonically represented by

$$A_1x + B_1y + C_1z + D_1w = 0,$$

$$A_2x + B_2y + C_2z + D_2w = 0,$$

with rank 2 for the matrix

$$\begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \end{pmatrix},$$

and π any plane canonically represented by

$$A_3x + B_3y + C_3z + D_3w = 0,$$

so that π is distinct from the planes above, and with $\pi(P) < 0$ and $\pi(Q) > 0$, then there exists a point R of the line such that $\pi(R) = 0$. We note here that π is not the plane canonically represented by $x = 0$, for its negative half space is the empty set.

Since the equations of the planes are canonically represented we know either $A = 1$, $A = 0$ and $B = 1$, $A = B = 0$ and $C = 1$, or $A = B = C = 0$ and $D = 1$. For the discussion here we will use equations of the form

$$x + By + Cz + Dw = 0.$$

If any of the other canonical representations of a line held a parallel proof could be used.

Now the line is represented by

$$x + B_1y + C_1z + D_1w = 0,$$

$$x + B_2 y + C_2 z + D_2 w = 0,$$

and π is represented by

$$x + B_3 y + C_3 z + D_3 w = 0.$$

Since in the canonical representations of point $x = 1$, or $x = 0$, by introducing a parameter t, we can write a set of three equations that are equivalent to the pair of equations that represent the line with respect to the canonical representations of points. Let the parametric representation of the line be given by

$$y = y_o + t\ell,$$

$$z = z_o + tm,$$

$$w = w_o + tn,$$

with $\ell^2 + m^2 + n^2 = 1$. Now if we let t_1 be the value of the parameter for which the parametric equations yield point P and t_2 the value of the parameter for which the parametric equations yield point Q, we can deduce the following pairs of inequalities depending on whether or not x is one or zero. If $x = 1$, we have

(1) $1 + B_3 y_o + C_3 z_o + D_3 w_o + t_1(B_3\ell + C_3 m + D_3 n) < 0$

(2) $1 + B_3 y_o + C_3 z_o + D_3 w_o + t_2(B_3\ell + C_3 m + D_3 n) > 0.$

If $x = 0$, we have

(3) $B_3 y_o + C_3 z_o + D_3 w_o + t_1(B_3\ell + C_3 m + D_3 n) < 0$

(4) $B_3 y_o + C_3 z_o + D_3 w_o + t_2(B_3\ell + C_3 m + D_3 n) > 0.$

Since $B_3\ell + C_3m + D_3n$ can not equal zero (for if it did the expressions $1 + B_3y_o + C_3z_o + D_3w_o$ and $B_3y_o + C_3z_o + D_3w_o$ would have to be both negative and positive) we can write inequalities (1) and (2) into the forms (5) and (6), and inequalities (3) and (4) into the forms (7) and (8).

(5) $$t_1 < -\frac{1 + B_3y_o + C_3z_o + D_3w_o}{B_3\ell + C_3m + D_3n},$$

(6) $$t_2 > -\frac{1 + B_3y_o + C_3z_o + D_3w_o}{B_3\ell + C_3m + D_3n};$$

(7) $$t_1 < -\frac{B_3y_o + C_3z_o + D_3w_o}{B_3\ell + C_3m + D_3n},$$

(8) $$t_2 > -\frac{B_3y_o + C_3z_o + D_3w_o}{B_3\ell + C_3m + D_3n}.$$

Then for either

$$t = -\frac{1 + B_3y_o + C_3z_o + D_3w_o}{B_3\ell + C_3m + D_3n},$$

or

$$t = -\frac{B_3y_o + C_3z_o + D_3w_o}{B_3\ell + C_3m + D_3n},$$

we have a point of the line that is also a point of π.

Since we have a line one of the canonical representations of the line holds and the lemma follows.

Corollary to Lemma 4. If a line has exactly one point in common with a plane then (1) if the plane is canonically represented by

$x = 0$, the line has an infinite number of points in the positive half space determined by $x = 0$ and the negative half space does not exist, (2) if the plane is canonically represented by any other equation, the line has an infinite number of points in each half space determined by the plane.

In case (1) we need only consider the points canonically represented by $(1, y', z', w')$. These points are in the positive half space of $x = 0$ and since we have free choice of y', z', and w' over the field of reals we have an infinite number of possibilities.

In case (2) we can write a parametric representation of the line as shown in lemma 4. Now since there is a point of the line on the plane there exists a value t_o of the parameter t such that the parametric equations yield the point in question for this value of t_o. Then the parameters $t_o + k$ and $t_o - k$, with k greater than zero, are such that the parametric equations will yield an infinite number of points in each half space determined by the plane.

There are twenty-four possible cross ratios of four collinear points. I will divide these into three groups of eight each such that every member of a set will either be positive or negative. The three groups are listed on the following page.

Group One	Group Two	Group Three
$R(P_1P_2, P_4P_3)$	$R(P_1P_3, P_4P_2)$	$R(P_1P_2, P_3P_4)$
$R(P_4P_1, P_3P_2)$	$R(P_4P_1, P_3P_2)$	$R(P_3P_1, P_4P_2)$
$R(P_3P_4, P_2P_1)$	$R(P_2P_4, P_3P_1)$	$R(P_4P_3, P_2P_1)$
$R(P_2P_3, P_1P_4)$	$R(P_3P_2, P_1P_4)$	$R(P_2P_4, P_1P_3)$
$R(P_4P_3, P_1P_2)$	$R(P_4P_2, P_1P_3)$	$R(P_3P_4, P_1P_2)$
$R(P_1P_4, P_2P_3)$	$R(P_1P_4, P_3P_2)$	$R(P_1P_3, P_2P_4)$
$R(P_2P_1, P_3P_4)$	$R(P_3P_1, P_2P_4)$	$R(P_2P_1, P_4P_3)$
$R(P_3P_2, P_4P_1)$	$R(P_2P_3, P_4P_1)$	$R(P_4P_2, P_3P_1)$

Lemma 5. Given four collinear points and their cross ratios, eight and only eight of these cross ratios are less than zero.

Let the four collinear points be canonically represented by P_1, P_2, P_3, and P_4 as above. Now to get a value of a cross ratio it is necessary to introduce the three planes canonically represented by $\pi_2 = 0$, containing P_2 but no other points of the line; $\pi_3 = 0$, containing P_3 but no other point of the line; and $\pi_4 = 0$, containing P_4 but no other point of the line. These planes exist as demonstrated in lemma 3. To find a value of the cross ratio we now must assume a relative position to points P_2, P_3, and P_4 in relation to the planes π_2, π_3, and π_4. We can do this arbitrarily since a change in any one would result in an even number of sign changes in a given cross ratio and hence not affect the final sign. Let us assume P_2 and P_3 in the positive half space of π_4, P_2 in the positive half space of π_3, P_4 in the negative half space of π_3, and P_3 and P_4 in the negative half space of π_2. Then it follows that $\pi_2(P_3)$, $\pi_2(P_4)$, and $\pi_3(P_4)$ are less than zero and $\pi_3(P_2)$, $\pi_4(P_2)$, and $\pi_4(P_3)$ are greater than zero.

Now three distinct planes divide the points of space, not on any of the planes, into eight distinct sets. I will name these sets + + +, + + -, + - +, - + +, + - -, - + -, - - +, and - - -, where + + + indicates that the points are in the positive half space of π_2, the positive half space of π_3, and the positive half space of π_4 respectively. By assumption we have P_2 on π_2 and hence in neither of the sets + + + and - + +; P_3 on π_3 and hence in neither of the sets - + + and - - +; and P_4 on π_4 and hence in neither of the sets - - + and - - -. Then by the corollary to lemma 4 we have a point of the line in each of the sets + + +, - + +, - - +, and - - -. It must be noted here that if one of the above planes π_2, π_3, or π_4 is canonically represented by $x = 0$, by the above assumptions, it must be π_4. But then the line would have no points in the set - - -. Even if this is the case it will have no bearing on the remaining parts of this proof.

I will now show that the line cannot have a point in the remaining four sets of points, + - +, - + -, + + -, and - - +. If it had a point in the set + - +, it would have a point on π_3 which is in neither of the sets + + + and + - +. This is a result of lemma 4. This point is in the positive half space of π_2. But we already have a point P_3 of the line on π_3 and it is in the negative half space of π_2. Therefore the line has two points of π_3 on it which is a contradiction. If we assumed it has a point in each of the remaining three sets mentioned above we will reach a similar contradiction. Therefore we need only consider that P_1 may be in the sets of points + + +, - + +, - - +, and - - -.

Now to prove the lemma I need only show that a cross ratio of one of the above groups of cross ratios is negative if P_1 is in the four sets above and that a cross ratio of each of the other groups is positive. Let $R(P_1P_2, P_4P_3)$ represent group one and $R(P_1P_3, P_4P_2)$ and $R(P_1P_2, P_3P_4)$ represent groups two and three respectively.

If P_1 is in the set + + + we would have the following values of the cross ratios.

$$R(P_1P_2, P_4P_3) = \frac{\pi_2(P_1)}{\pi_2(P_3)} \div \frac{\pi_4(P_1)}{\pi_4(P_3)} < 0$$

$$R(P_1P_3, P_4P_2) = \frac{\pi_3(P_1)}{\pi_3(P_2)} \div \frac{\pi_4(P_1)}{\pi_4(P_2)} > 0$$

$$R(P_1P_2, P_3P_4) = \frac{\pi_2(P_1)}{\pi_2(P_4)} \div \frac{\pi_3(P_1)}{\pi_3(P_4)} > 0$$

If P_1 is in the set - - - we would have the following values of the cross ratios: (From this point on I will just state whether or not the cross ratio is positive or negative and we can check back to the above representations to verify the results.) $R(P_1P_2, P_4P_3) < 0$, $R(P_1P_3, P_4P_2) > 0$, and $R(P_1P_2, P_3P_4) > 0$.

If P_1 is in the set - + + we have the following values of the cross ratios: $R(P_1P_2, P_4P_3) > 0$, $R(P_1P_3, P_4P_2) > 0$, and $R(P_1P_2, P_3P_4) < 0$.

If P_1 is in the set - - + we have the following values of the cross ratios: $R(P_1P_2, P_4P_3) > 0$, $R(P_1P_3, P_4P_2) < 0$, and $R(P_1P_2, P_3P_4) > 0$.

Lemma 6. If $P_2P_4 \; V \; P_1P_3$ then $P_1P_3 \; V \; P_4P_2$, $P_4P_2 \; V \; P_3P_1$, $P_3P_1 \; V \; P_2P_4$, $P_3P_1 \; V \; P_4P_2$, $P_4P_2 \; V \; P_1P_3$, $P_1P_3 \; V \; P_2P_4$, and $P_2P_4 \; V \; P_3P_1$.

Since $P_2P_4 \; V \; P_1P_3$ we have $R(P_1P_2, P_4P_3)$ less than zero, hence all other cross ratios of the group containing $R(P_1P_2, P_4P_3)$ are negative. Since each of the cross ratios that yield the above separations are in this group the lemma follows.

Lemma 7. Given four distinct points of a line either $P_2P_4 \; V \; P_1P_3$, or $P_3P_4 \; V \; P_1P_2$, or $P_2P_3 \; V \; P_1P_4$.

Since each of the cross ratios representing the above separations is in a different group, one must be negative by lemma 5 and the result follows.

Lemma 8. If $P_2P_4 \; V \; P_1P_3$, then $P_3P_4 \; \not{V} \; P_1P_2$ and $P_2P_3 \; \not{V} \; P_1P_4$.

Since the cross ratios representing the above are in different groups only one can be negative as shown in lemma 5. But $P_2P_4 \; V \; P_1P_3$ so $R(P_1P_2, P_4P_3)$ is negative, thus the others are positive and the result follows.

Lemma 9. If $P_2P_4 \ V \ P_1P_3$ and $P_4P_5 \not V \ P_1P_3$ then $P_4P_5 \not V \ P_1P_2$.

$P_2P_4 \ V \ P_1P_3$ implies by lemma 8 that $P_2P_3 \not V \ P_1P_4$. $P_2P_3 \not V \ P_1P_4$ indicates that

$$(1) \qquad \frac{\pi_2(P_1)}{\pi_2(P_4)} \div \frac{\pi_3(P_1)}{\pi_3(P_4)} > 0.$$

$P_4P_5 \not V \ P_1P_3$ indicates that

$$(2) \qquad \frac{\pi_4(P_1)}{\pi_4(P_3)} \div \frac{\pi_5(P_1)}{\pi_5(P_3)} > 0.$$

We will now proceed with an indirect proof by assuming $P_4P_5 \ V \ P_1P_2$. This implies by lemma 8 that $P_5P_2 \not V \ P_1P_4$. $P_5P_2 \not V \ P_1P_4$ indicates that

$$(3) \qquad \frac{\pi_5(P_1)}{\pi_5(P_4)} \div \frac{\pi_2(P_1)}{\pi_2(P_4)} > 0.$$

Now multiply together inequalities (1) and (3) and get (4).

$$(4) \qquad \frac{\pi_3(P_4)}{\pi_3(P_1)} \div \frac{\pi_5(P_4)}{\pi_5(P_1)} > 0$$

Now inequality (4) means $P_3P_5 \not V \ P_4P_1$. Since we were given that $P_4P_5 \not V \ P_1P_3$, we may conclude by lemma 7 that $P_3P_4 \ V \ P_1P_5$. $P_3P_4 \ V \ P_1P_5$ indicates that

$$(5)\qquad \frac{\pi_3(P_1)}{\pi_3(P_5)} \div \frac{\pi_4(P_1)}{\pi_4(P_5)} < 0.$$

Since we assumed that $P_4P_5 \ V \ P_1P_2$ we know, by lemma 8, that $P_4P_2 \not V \ P_1P_5$ indicates that

$$(6)\qquad \frac{\pi_4(P_1)}{\pi_4(P_5)} \div \frac{\pi_2(P_1)}{\pi_2(P_5)} > 0.$$

Now multiply inequality (5) and (6) and get (7).

$$(7)\qquad \frac{\pi_3(P_1)}{\pi_3(P_5)} \div \frac{\pi_2(P_1)}{\pi_2(P_5)} < 0.$$

Now inequality (7) means $P_3P_2 \ V \ P_1P_5$. By lemma 6 $P_3P_2 \ V \ P_1P_5$ implies $P_2P_3 \ V \ P_1P_5$ and by lemma 8 $P_2P_3 \ V \ P_1P_5$ implies $P_2P_5 \ V \ P_1P_3$, which in turn implies $P_5P_2 \ V \ P_1P_3$ by lemma 6. $P_5P_2 \not V \ P_1P_3$ indicates that

$$(8)\qquad \frac{\pi_5(P_1)}{\pi_5(P_3)} \div \frac{\pi_2(P_1)}{\pi_2(P_3)} > 0.$$

Now if we multiply inequalities (2) and (8) together we get (9).

$$(9)\qquad \frac{\pi_4(P_1)}{\pi_4(P_3)} \div \frac{\pi_2(P_1)}{\pi_2(P_3)} > 0.$$

But inequality (9) means $P_4P_2 \not V \ P_1P_3$, and since we were given that $P_2P_4 \ V \ P_1P_3$ which implies, by lemma 6, that $P_4P_2 \ V \ P_1P_3$ we have a contradiction and the lemma follows.

Definition. A finite set of collinear points are in the order $P_1P_2 . . . P_n$ or $P_nP_{n-1} . . . P_1$ if no two of its points are separated by the pairs P_1P_2, P_2P_3, . . . P_nP_1.

Theorem 8. Given n collinear points, with n greater than three, there are exactly two ways of naming them, $P_1P_2 . . . P_n$ or $P_nP_{n-1} . . . P_1$, so that P_i and P_k separate P_j and P_ℓ if and only if i, j, k, and ℓ are in natural order, or can be obtained from natural order by substitutions from the group generated by $(P_iP_jP_kP_\ell)$ and (P_iP_k).

This proof will proceed along the lines of math induction. The first step is to show that we can order four collinear points. By lemma 5 we know that at least one cross ratio of the four points is negative, say $R(P_1P_2, P_4P_3)$. Then by definition we know that $P_2P_4 \; \mathrm{V} \; P_1P_3$. This result along with lemmas 6 and 7 imply $P_1P_2 \not\mathrm{V} P_3P_4$, $P_2P_3 \not\mathrm{V} P_4P_1$, $P_3P_4 \not\mathrm{V} P_1P_2$, and $P_4P_1 \not\mathrm{V} P_2P_3$. Hence by definition the points are in the order $P_1P_2P_3P_4$ or $P_4P_3P_2P_1$.

We now assume that we can order k such points of a line and hence no two of the k points of the line are separated by the following pairs of points, P_1P_2, $P_2P_3 . . . P_kP_1$.

We now need only show that we can order k + 1 of these points. If we add one point P_{k+1} to the k points discussed above, we will have for one pair of points P_iP_j the fact that $P_iP_j \; \mathrm{V} \; P_{k+1}P_x$, where P_x is any other of the remaining k - 2 points of the line. This fact is a direct result of lemma 5. Since we assumed the k points were ordered we know $P_iP_j \not\mathrm{V} P_xP_y$, where P_x and P_y are any of the remaining k - 2 points of the line. Thus from lemma 9 we can deduce that $P_iP_{k+1} \not\parallel P_xP_y$ and $P_{k+1}P_j \not\parallel P_xP_y$. Hence the points are ordered $P_1P_2 \ldots P_iP_{k+1}P_j \ldots P_k$ or $P_kP_{k-1} \ldots P_jP_{k+1} \ldots P_1$ and the theorem follows.

BIBLIOGRAPHY

1. Blumenthal, Leonard M. A modern view of geometry. San Francisco, W. H. Freeman and Company, 1961. 187 p.

2. Hilbert, David. The foundations of geometry. Tr. by E. J. Tounsend. La Salle, Illinois, The Open Court Publishing Company, 1962. 143 p.

3. Veblen, Oswald and John Wesly Young. Projective geometry. New York, Ginn and Company, 1918. 2 vols.