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Title: THE APPLICATION OF THE VARIABLE-GRADIENT

        METHOD FOR GENERATING A LIAPUNOV FUNCTION FOR

        THE VAN DER POL EQUATION

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In this paper the systematic method of generating Liapunov

functions for determining stability of the Van der Pol equation is pre-

sented.   The method is based upon the assumption of a variable gradi-

ent function from which both   V   and $\dot{V}$ may be determined. $\dot{V}$

is constrained to be positive semidefinite in the strip $-1 < x_1 < 1$

with the application of Bendixson's negative criterion.   The positive

V-function is derived.   With   V   and $\dot{V}$,   the region of the exist-

ence of limit cycle is also known by means of the second method of

Liapunov.   The limit cycle obtained by analog simulation is pre-

sented.

The Application of the Variable-Gradient Method for
Generating a Liapunov Function for the
Van der Pol Equation

by

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THE APPLICATION OF THE VARIABLE-GRADIENT METHOD FOR GENERATING A LIAPUNOV FUNCTION FOR THE VAN DER POL EQUATION

I. INTRODUCTION

A problem of considerable interest in stability theory is the determination of the domain of attraction of an equilibrium point of an autonomous differential equation. Several approaches based on Liapunov's direct or second method have been presented and have yielded some interesting results. In 1961, Szegö [18] investigated the stability of nonlinear autonomous systems with the non-linearity representable in a polynomial form. A theorem is presented which gives a sufficient condition for the local stability of a nonlinear differential system as the existence of a positive definite function:

$$V = \phi(x) \tag{1}$$

such that

$$\dot{V} = \theta(x)g[\xi(x)] \tag{2}$$

where $\theta(x)$ is a semidefinite function not identically equal to zero on a solution of

$$\dot{x} = X(x) \tag{3}$$

$g(x)$ is such that $g(0) = 0$ and

$$\text{sign } g(u) \neq \text{sign } g(-u) \tag{4}$$

and $\xi(x) = 0$ is a closed surface, or a family of closed surfaces. A procedure for constructing Liapunov functions based upon the use of a generating V function, that is, a V function with some variable coefficients, has been developed. The generating V function chosen has the following form:

$$V(x) = x'A(x)x \qquad (5)$$

where,

$$A(x) = \{a_{ij}(x_i, x_j)\}; \ (a_{ij} = a_{ji}) \qquad (6)$$

Szegö [18] concludes that this method of generating a Liapunov function is more suitable for identifying limit cycles of the Van der Pol equation. In the same year, Szegö [19] also contributed a new procedure for plotting phase plane trajectories of second-order systems by giving one example on the Van der Pol equation.

In 1962, Zubov's construction procedure received further attention in a paper by Margolis and Vogt [14] in which they discuss the control engineering applications of the procedure. They consider the existence of a Liapunov function, the equations for the boundary of the domain of asymptotic stability of the perturbed motion, the necessary and sufficient conditions for global asymptotic stability and an approximate for obtaining the domain of asymptotic stability. The Van der Pol equation has been used to provide computational evidence of the method for obtaining the approximate domain of asymptotic stability.

In 1963, Infante and Clark [8] also gave a method for the deter-mination of the domain of stability of second-order nonlinear auto-nomous systems by giving an example on the Van der Pol equation. The essence of the method is the construction of Liapunov-like func-tions. The result obtained is general, and it is the same as given previously by La Salle [11, 12].

In 1964, another paper on limit cycle construction of the Van der Pol equation using Liapunov functions was also given by Goldwyn and Cox [6]. It also deals with a generalization of the method of Zubov [18, 20] for the construction of Liapunov functions which are useful in estimating the location of stability boundaries.

The author found that the major difficulty in using Liapunov's direct or second method for general stability analysis is in finding a suitable Liapunov function for a given system. By using the work of Schultz and Gibson [17], a Liapunov function can be found. A second-order nonlinear, autonomous system, the Van der Pol equation, is used to illustrate this method. In this paper the problem of stability of a certain equation will be investigated by means of the second method of Liapunov which will also enable us to locate the limit cycle.

II. SYSTEM REPRESENTATION

A description of Liapunov's direct or second method requires the use of the vector state of the dynamic system, that is, a representation of system behavior in terms of a vector or a matrix set whose components are the variables and their derivatives. It is always possible to describe a system by the vector differential equation:

$$\dot{x} = X(x, u(t), t) \qquad (7)$$

u(t) is the external _forcing function_ or _input_, which is assumed to be identically zero. The system is now called _free_ or _unforced system,_ that is,

$$\dot{x} = X(x, t) \qquad (8)$$

The system in Equation (8) with the presence of t as one of the independent functional variables means that the system is _not stationary_ (nonstationary). This system is _time varying._

If the system is both _unforced_ and _time-invariant_, it can be specified by a function X that depends upon x alone and not upon the time t or input u(t), that is,

$$\dot{x} = X(x) \qquad (9)$$

A system of this nature is called _autonomous._ Otherwise, it is

nonautonomous as in Equation (8). The Van der Pol equation [Equation (20)] which will be given next represents a nonlinear, autonomous system, that is, in the form of Equation (9) where the origin is an equilibrium state.

$X(x)$ is a vector function of the vector x which satisfies the conditions for existence, uniqueness, and continuity of the solution of Equation (9) [2, 10].

From Equation (9), we have

$$\dot{x}_1 = X_1(x_1, x_2)$$
$$\dot{x}_2 = X_2(x_1, x_2) \tag{10}$$

Therefore,

$$\frac{\dot{x}_2}{\dot{x}_1} = \frac{X_2(x_1, x_2)}{X_1(x_1, x_2)}$$

$$= f(x_1, x_2) \tag{11}$$

The sufficient conditions for a unique solution are that

1) $f(x_1, x_2)$ is continuous for all values of x_1 and x_2 in the solution, and

2) the Lipschitz condition is satisfied.

The Lipschitz condition is satisfied only when the following inequality holds:

$$\left|f(x_1, x_{2a}) - f(x_1, x_{2b})\right| < K\left|(x_{2a} - x_{2b})\right| \tag{12}$$

where x_{2a} and x_{2b} are any two arbitrary values of x_2 and K is any positive number.

III. DEFINITIONS

Motion

In defining stability, asymptotic stability, and instability we shall frequently use the word "motion." A motion is defined as a trajectory starting from any state or any point in the n-dimensional state space. Here, trajectory, motion, and solution of the differential equations are used interchangeably.

The motion of autonomous systems has the important property that it is invariant under translation in time; that is,

$$\phi(t, x_0, t_0) = \phi(t + T, x_0, t_0 + T), \quad T = \text{constant} \tag{13}$$

for any admissible x_0, t_0, T. Figure 1 illustrates this property.

$$x_2 = \phi(t_2; x_1, t_1) = \phi(t_2; x_0, t_0)$$

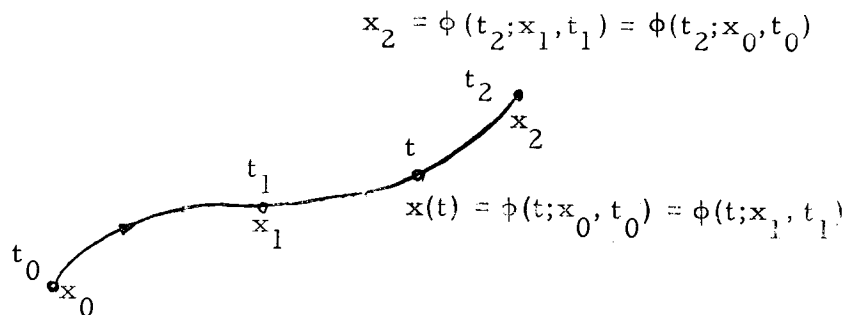$$x(t) = \phi(t; x_0, t_0) = \phi(t; x_1, t_1)$$

Figure 1. Notation for motion.

Equilibrium States

In dealing with stability analysis we are concerned with the

equilibrium states (or points), sometimes called critical points,

which are those states where $X(x) = 0$. For the free, dynamic sys-

tem as in Equation (8), a state x_e is called an equilibrium state if

$$X(x_e, t) = 0, \quad \text{for all} \quad t \tag{14}$$

or, equivalently,

$$\phi(t, x_e, 0) = x_e, \quad \text{for all} \quad t \tag{15}$$

Therefore, a motion passing through an equilibrium state at any time

remains at the same state for all times. If the system under consid-

eration is linear and autonomous, $\dot{x} = Ax$, where A is a (n x n)

matrix with real constant elements. There will be only one equilibri-

um state if A is nonsingular and there will be many equilibrium

states if A is singular. If the system under consideration is non-

linear, there could be many equilibrium points. Note that $\dot{x} = A(x)x$

for nonlinear system.

Since we have defined the equilibrium states of the system, we

should also know an important method to determine the stability of

the equilibrium states of a nonlinear autonomous system as an exam-

ple by applying known linear methods.

From

$$\dot{x} = X(x) \tag{9}$$

The equilibrium points given by $\dot{x} = 0$ are determined from

$$X(x) = 0 \tag{16}$$

Unlike linear systems, nonlinear systems can have more than one equilibrium point (it is not unique), as is clear from Equation (16), which can have more than one solution. We will consider the case where the origin of the state space is an equilibrium point $(x_e = 0)$. Then, expanding each component of $X(x)$ in a Taylor series about the origin and considering only the linear terms, we get the linearized equations:

$$\dot{x} = J(0)x \tag{17}$$

where $J(0)$ is the Jacobian matrix evaluated at the equilibrium point $x = 0$. That is,

$$J(0) = \frac{\partial X}{\partial x} = \begin{bmatrix} \dfrac{\partial X_1}{\partial x_1} & \dfrac{\partial X_1}{\partial x_2} & \cdots & \dfrac{\partial X_1}{\partial x_n} \\[2mm] \dfrac{\partial X_2}{\partial x_1} & \dfrac{\partial X_2}{\partial x_2} & \cdots & \dfrac{\partial X_2}{\partial x_n} \\[2mm] \vdots & \vdots & & \vdots \\[2mm] \dfrac{\partial X_n}{\partial x_1} & \dfrac{\partial X_n}{\partial x_2} & \cdots & \dfrac{\partial X_n}{\partial x_n} \end{bmatrix}_{x=0} \tag{18}$$

Equation (17) is a linear, homogeneous, differential equation that, for the autonomous case under consideration as in Equation (9), has a constant Jacobian matrix J. Therefore, we can determine the stability of the linearized equation from the roots of the characteristic equation:

$$|J(0) - \lambda I| = 0 \tag{19}$$

We will consider the well-known <u>Van der Pol</u> equation as an example for this part. The Van der Pol equation is

$$\ddot{x} - \mu(1-x^2)\dot{x} + x = 0 \tag{20}$$

with $\mu > 0$.

Let

$$x_1 = x$$
$$x_2 = \dot{x}$$

We have

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \mu(1-x_1^2)x_2 \tag{21}$$

Equation (21) is in the form of Equation (9), that is, $\dot{x} = X(x)$. The equilibrium points x_e can be determined from $X(x) = 0$. By substituting $\dot{x}_1 = 0$, it gives $x_2 = 0$ and by substituting $\dot{x}_2 = 0$, it gives $x_1 = 0$. Then,

$$x_e = 0$$

From Equation (18), we have

$$J(0) = \begin{bmatrix} 0 & 1 \\ -1-2\mu x_1 x_2 & \mu(1-x_1^2) \end{bmatrix}_{x = x_e = 0}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}$$

From Equation (17), the linearized equations are

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \mu x_2 \qquad (22)$$

From Equation (19), we have

$$|J(0) - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & \mu-\lambda \end{vmatrix} = 0$$

$$-\lambda(\mu-\lambda) + 1 = 0$$

$$\lambda^2 - \mu\lambda + 1 = 0$$

$$\lambda_{1,2} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2} \qquad (23)$$

From the eigen values obtained, we can conclude that Equation (22) represents an unstable system; all solutions of the linearized system [Equation (22)] leave the equilibrium point as t increases.

Definitions of Stability

Before we use Liapunov's direct method as a tool for stability analysis of the Van der Pol equation, we will study here some principal definitions of stability. Good sources for the following definitions of stability can be found in Kalman and Bertram [10] and LaSalle and Lefschets [12].

Stability in the Sense of Liapunov

For the system as in Equation (9), Kalman and Bertram [10] have defined that the equilibrium point x_e is stable in the sense of Liapunov if for each $\epsilon > 0$ there can be chosen a real number $\delta(\epsilon)$ such that from the following relation

$$\| x(t_0) \| < \delta(\epsilon)$$

it follows that

$$\| x(t) \| < \epsilon$$

Figure 2 illustrates the definition of stability in the sense of Liapunov. The $x_1 - x_2$ plane is for $t = t_0$. If $\| x_0 \|$ is in the interior of the circle of radius δ, the curve $x(t)$ in the motion space remains inside the cylinder of radius ϵ for all future times.

(a) Motion space   (b) State space
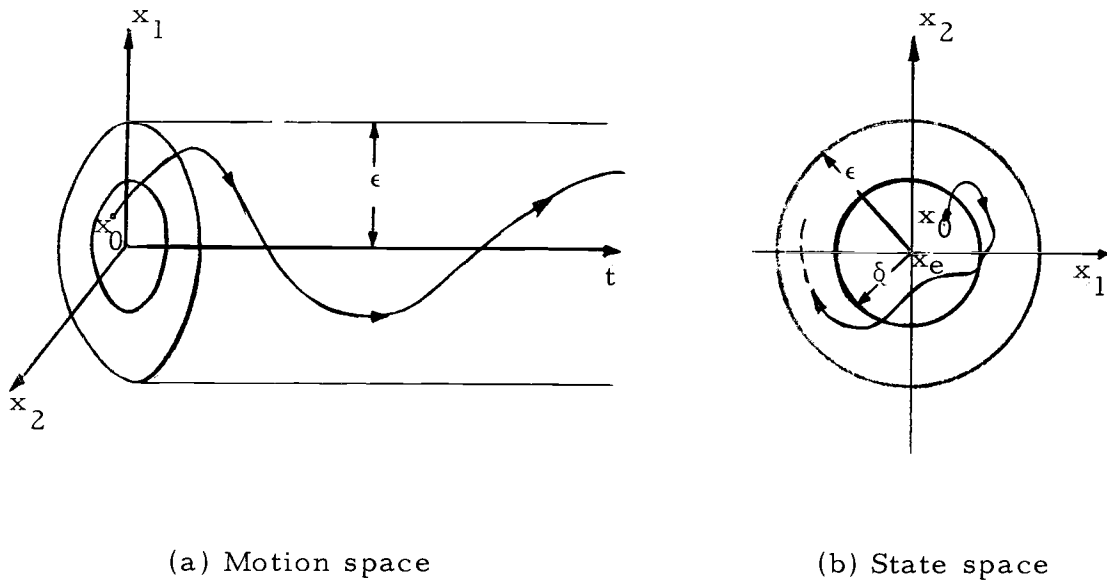
Figure 2.   Definition of stability in the sense of Liapunov (by Kalman and Bertram).

Again, for the same system as in Equation (9), LaSalle and Lefschetz [12] have assumed that the equilibrium state being investigated is located at the origin. They denote S(R) as a spherical region of radius R > 0 around the origin, where S(R) consists of points x satisfying $\| x \|$ < R and H(R) as a spherical region with $\|x\|$ = R. Then, the origin is said to be stable in the sense of Liapunov, or simply stable, if, corresponding to each S(R), there is an S(r) such that solutions starting in S(r) do not leave S(R) as t → ∞. Figure 3 illustrates the definition of stability.
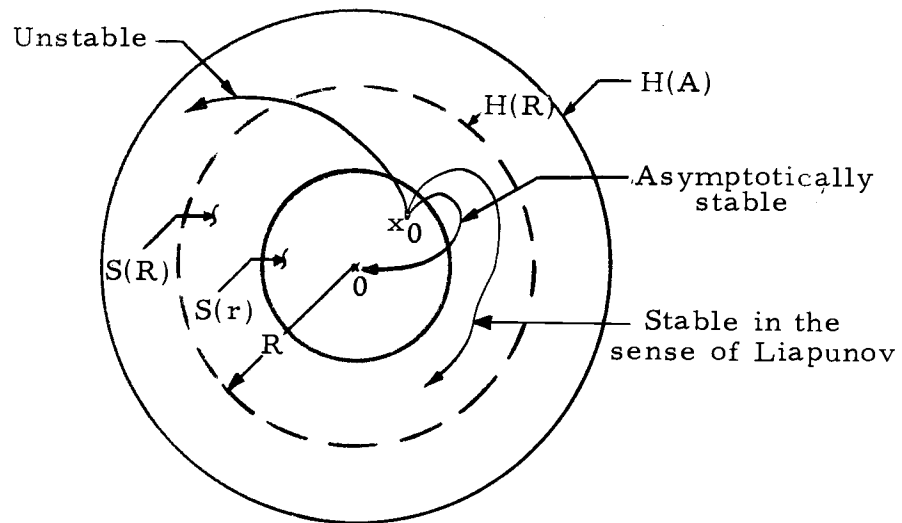
Figure 3.   Definition of stability (by LaSalle and
Lefschetz).

Asymptotic Stability

For the system as in Equation (9), Kalman and Bertram [10]

have defined that if the equilibrium state x_e is stable in the sense

of Liapunov, and also

$$\lim_{t \to \infty} \|x(t)\| = 0, \quad \text{if} \quad \|x(t_0)\| < \delta$$

the equilibrium state x_e is said to be asymptotically stable.

Figure 4 illustrates the definition of asymptotically stable in the

sense of Liapunov.   When $t \to \infty$,   it is seen that the curve in the

motion space approaches the t-axis.

Again, for the same system as in Equation (9), LaSalle and

Letschetz [12] have given that if the origin is stable and, in addition,

every solution in   S(r)   not only stays within   S(R)   but approaches

the origin as time increases indefinitely [t → ∞], then the system

is said to be <u>asymptotically stable</u>. This can be seen from Figure 3.

(a) Motion space   (b) State space
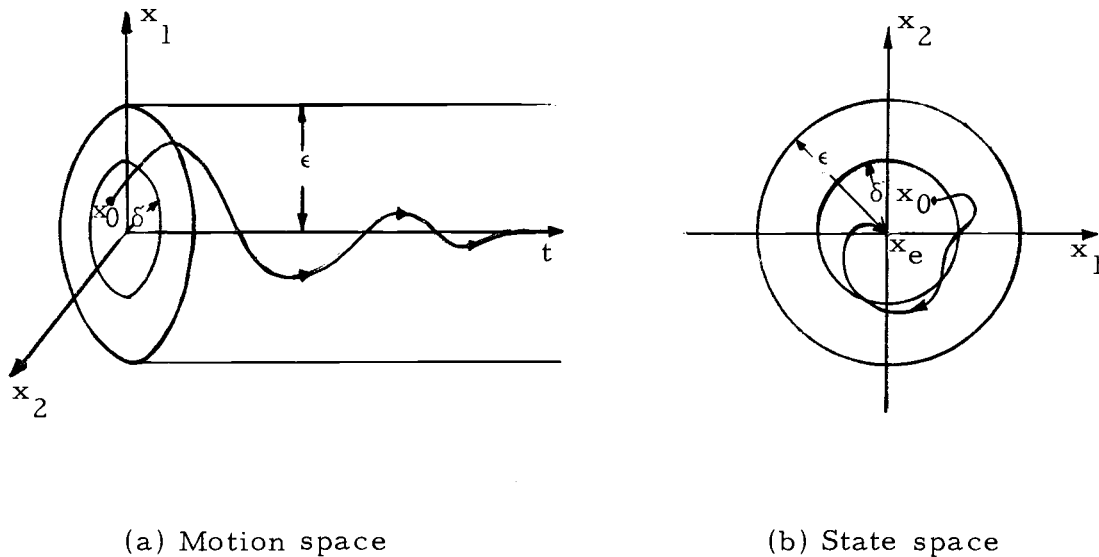
Figure 4. Definition of asymptotic stability in the sense of
Liapunov (by Kalman and Bertram).

Asymptotic Stability in the Large

If asymptotic stability holds for all points in the state space from

which motions originate, the equilibrium state is said to be <u>asymp-</u>

<u>totically stable in the large</u>.

Instability

An equilibrium state is said to be <u>unstable</u> if it is neither stable

nor asymptotically stable. Figure 3 shows an unstable equilibrium

state at the origin of a second order system and a representative

trajectory starting from x_0. As may be seen from Figure 3, in the case of an unstable equilibrium state, for some real number $R > 0$ and any real number $r > 0$, no matter how small r, there is always a point x_0 in the spherical region $S(r)$ [circular region for second order system] such that the motion starting from this point reaches the boundary sphere $H(R)$ [circle for second order system] of $S(R)$.

Limit Cycles

A limit cycle of the first kind is a closed curve on the phase plane. The determination of the existence and location of limit cycles is an important purpose of any analysis method for the study of non-linear systems. Both the describing-function approach and the phase-plane analysis provide techniques for the investigation of limit cycles. The existence of a limit cycle corresponds to a system oscillation of fixed amplitude and period. The limit cycle may be stable, unstable, or semi-stable. Figure 5a illustrates a stable limit cycle. A limit cycle is called stable if all near trajectories both from the inside and from the outside approach the limit cycle as time approaches infinity. A stable limit cycle corresponds to a stable periodic motion in a physical system. Figure 5b illustrates an unstable limit cycle. It is called unstable if all near trajectories both from the inside and from the outside approach the limit cycle as time approaches minus

infinity, in the other words, all near trajectories move away from the closed curve. Figure 5c illustrates a semi-stable limit cycle. It is called semi-stable if all near trajectories on one side of the limit cycle approach it as time approaches infinity, while those on the other side of the limit cycle leave it. For nonlinear systems, limit cycles do not depend upon the initial conditions.
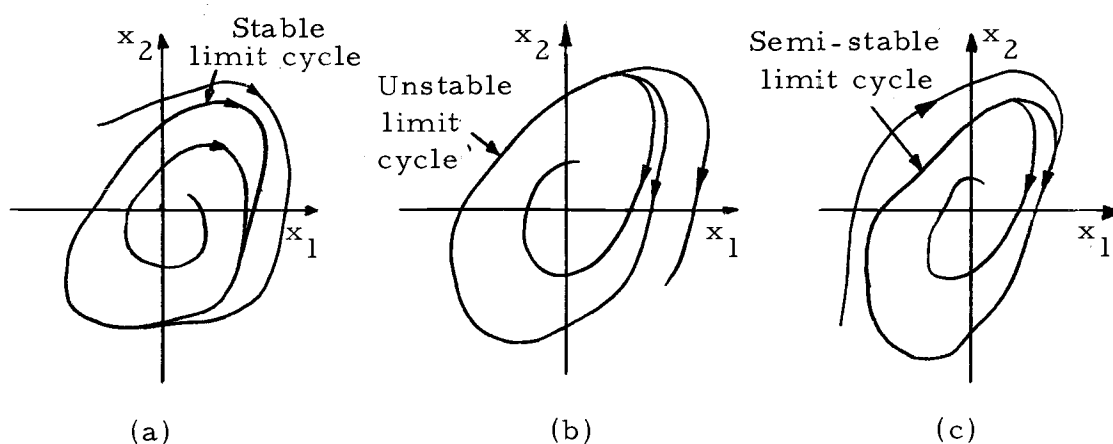
Figure 5. Limit cycles.

There are several theorems which aid the analyst in determining the existence of limit cycles [15]. For example, Poincaré has shown that within any limit cycle the number of node, focus, and center types of singularities must exceed the number of saddle points by one. A second example is Bendixson's second theorem [see Appendix A], which states that, if a path stays inside a finite region D and does not approach a singular point, it must either be a limit cycle or approach a limit cycle asymptotically.

Liapunov's second method has also been applied to the case of systems with limit cycles. There are many papers written on the determination of the limit cycle of the Van der Pol equation with the use of Liapunov's second method. Such papers are due to LaSalle [11, 12], Szegö [18, 19, 20], Infante and Clark [8], Goldwyn and Cox [6], and etc.

Definite and Semidefinite

The Liapunov function is given the symbol $V(x)$. There are two important types of the function $V(x)$ which are the semidefinite and the definite forms. Let $\|x\|$, the norm of x, be the Euclidean length of the vector x, or $\|x\|^2 = x_1^2 + x_2^2 + \ldots + x_n^2$.

Positive [Negative] Definite

The function $V(x)$ is definite in a neighborhood about the origin if it is continuous and has continuous first partial derivatives, and if it has the same sign throughout the neighborhood, and it is nowhere zero, except possibly at the origin. In brief, a scalar function $V(x)$ is positive [negative] definite if, for $\|x\| \leq h$, we have $V(x) > 0$ [$V(x) < 0$] for all $x \neq 0$ and $V(0) = 0$.

Positive [Negative] Semidefinite

The function $V(x)$ is semidefinite in a neighborhood about the

origin if it is continuous and has continuous first partial derivatives, and if it has the same sign throughout the neighborhood, except points at which it is zero. In brief, <u>a scalar function $V(x)$ is positive [negative] semidefinite if, for $\|x\| \leq h$, we have $V(x) \geq 0$ $[V(x) \leq 0]$ for all $x \neq 0$ and $V(0) = 0$.</u>

In the preceding definitions, h may be arbitrarily small, in which case V would be definite in an arbitrarily small region about the origin. If h is infinite, V is definite in the whole state space.

Indefinite

A scalar function $V(x)$ is <u>indefinite</u> if it is neither sign-definite nor sign-semidefinite, and therefore, no matter how small the h is, in the region $\|x\| \leq h$, $V(x)$ may assume both positive and negative values.

In general, the simplest positive definite function can be written as a quadratic form

$$V(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j, \quad a_{ij} = a_{ji} \tag{24}$$

An alternate expression of $V(x)$ is

$$V(x) = x'Ax, \quad a_{ij} = a_{ji} \tag{25}$$

where   x   is a column vector,   x'   is its transpose, and   A   is a square symmetric matrix.   The necessary and sufficient conditions in order that   V(x)   be positive definite were given by J. J. Sylvester. They are that all the leading principal minors of   A   [also called the discriminants of the quadratic form] be positive, that is,

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \ldots, \quad \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} > 0$$

The proof of this theorem can be found in Bellman [1].

In addition, Liapunov's second method requires consideration of the time derivative of   V(x)   along the system trajectories corresponding to Equation (9).   The time derivative of   V(x)   is

$$\dot{V}(x) = \frac{dV}{dt} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 + \ldots + \frac{\partial V}{\partial x_n}\dot{x}_n$$

$$= \frac{\partial V}{\partial x_1}X_1(x) + \frac{\partial V}{\partial x_2}X_2(x) + \ldots + \frac{\partial V}{\partial x_n}X_n(x)$$

$$= (\nabla V)'\dot{x} \tag{26}$$

where ∇V is the gradient of   V(x),   that is,

$$\nabla V = \begin{bmatrix} \dfrac{\partial V}{\partial x_1} \\[1em] \vdots \\[1em] \dfrac{\partial V}{\partial x_n} \end{bmatrix}$$

∇V (grad V)   is a vector with a direction which is normal to the   V

surfaces (defined positive in increasing   V   direction) and a magni-

tude which is a measure of the rate of increase of   V   with respect

to the space coordinates.

IV. LIAPUNOV'S STABILITY THEOREMS

Liapunov's direct method is a mathematical approach to derive stability information about a system without solving the differential equations. It is a generalization of Lagrange's theorem of minimum potential energy to establish a condition of stable equilibrium. A large number of theorems which are related to Liapunov's direct method have been written by many authors, such as Hahn [7], Ingwerson [9], Kalman and Bertram [10], LaSalle and Lefschetz [12], etc. Those theorems are defined on the same principal ideas and theorems of Liapunov, and some theorems are developed. The following Liapunov's stability theorems which will be given here without proofs are concerned with the system of Equation (9) whose equilibrium state is at $x_e = 0$. They are based on the notation of sign-definite functions. The proofs of the theorems can be seen in the works of LaSalle and Lefschetz [12] and Kalman and Bertram [10].

Stability Theorem

For a system of Equation (9), if a V function of definite sign can be selected such that its time derivative $\frac{dV}{dt}$ is merely semi-definite and opposite in sign, then the system is stable but not necessarily asymptotically stable. This is applicable only in an arbitrarily small region about the origin.

Asymptotic Stability Theorem

From the above stability theorem, whenever $\frac{dV}{dt}$ is actually

negative definite in the region, then the origin is asymptotically

stable. Schultz and Gibson [17] have also defined that if there exists

a scalar function $V(x)$ with continuous first partials so that

1) $V(x) > 0$ for all $x \neq 0$ (positive definite)

2) $\dot{V}(x) < 0$ for all x (at least negative semidefinite)

3) $\dot{V}(x)$ is not identically zero on a solution of the system

other than the origin $(x = 0)$.

Then the system described by Equation (9) is <u>asymptotically stable</u>.

And if,

4) $V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$

then the system described by Equation (9) is <u>asymptotically stable in</u>

<u>the large</u> or <u>globally asymptotically stable</u>.

Instability Theorem

For a system of Equation (9), if there exists a positive definite

function $V(x)$ in a neighborhood R of the origin where $V(0) = 0$,

and if $\dot{V}$ is positive definite on R, then the origin is <u>unstable</u>.

V. THE APPLICATION OF THE VARIABLE-GRADIENT METHOD FOR GENERATING A LIAPUNOV FUNCTION FOR THE VAN DER POL EQUATION

The problem of stability of the Van der Pol equation from a certain equation will be investigated by means of the second method of Liapunov which will enable us to locate the limit cycle.

When the second-order system (special case of Lienard's equation)

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -g(x) - f(x)x_2 \tag{27}$$

is considered, the existence of the limit cycle of the first kind in a phase-plane has been shown by Levinson and Smith [13] with the conditions as follows:

1. f(x)   is an even function such that for the odd function

$$F(x) = \int_0^x f(x)dx \tag{28}$$

there exists an x_0 with $F(x) < 0$ for $0 < x < x_0$, and $F(x) > 0$ and monotonically increasing for $x > x_0$.

2. g(x)   is an odd-differentiable function such that $g(x) > 0$ for $x > 0$

3. $$\int_0^\infty f(x)dx = \int_0^\infty g(x)dx = \infty \tag{29}$$

To satisfy the above conditions, a certain equation of second-order system can be written in the following form:

$$\dot{x}_1 = ax_2 \qquad\qquad = X_1(x_1, x_2)$$

$$\dot{x}_2 = -bx_1 + \mu(c-x_1^r)^p x_2^q = X_2(x_1, x_2) \qquad (30)$$

where r is positive even integer, p is positive odd integer and q is introduced and assumed to be positive odd integer for further purpose; a, b, c and μ are positive numbers. Here, for the Van der Pol equation, we have $a = 1$, $b = 1$, $c = 1$, $r = 2$, $p = 1$ and $q = 1$.

By applying Bendixson's negative criterion [see Appendix A] to Equation (22), we have

$$\frac{\partial X_1}{\partial x_1} = 0$$

$$\qquad\qquad\qquad\qquad\qquad\qquad (31)$$

$$\frac{\partial X_2}{\partial x_2} = \mu q(c-x_1^r)^p x_2^{q-1}$$

Therefore,

$$\left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} \right) = \mu q(c-x_1^r)^p x_2^{q-1} \qquad (32)$$

We can see that the expression of Equation (32) does not change sign in the strip $-|\sqrt[r]{c}| < x_1 < |\sqrt[r]{c}|$. According to Bendixson's negative criterion, we can conclude that this strip does not contain the (stable)

limit cycle. From here, we introduce Liapunov's second method

which requires consideration of the time derivative of $V(x)$ along

the system trajectories corresponding to Equation (30). We let

$$\dot{V}(x) = \left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2}\right)P(x) \tag{33}$$

where $P(x)$ is a proper <u>positive</u> or <u>negative definite</u> (semidefinite)

function to be chosen. We have already proved that the expression

$\left(\dfrac{\partial X_1}{\partial x_1} + \dfrac{\partial X_2}{\partial x_2}\right)$ does not change sign in the strip $-\left|\sqrt[r]{c}\right| < x_1 < \left|\sqrt[r]{c}\right|$,

then, by multiplying this expression with a function $P(x)$, we will

obtain the function which is positive or negative definite (semidefinite)

according to that of $P(x)$. Let $P(x) = x_2^2$, then from Equation

(32) and Equation (33), we obtain

$$\dot{V}(x) = \mu q(c - x_1^r)^p x_2^{q+1} \tag{34}$$

which is positive semidefinite in the strip $-\left|\sqrt[r]{c}\right| < x_1 < \left|\sqrt[r]{c}\right|$ and

negative semidefinite outside of this strip.

Now we apply the variable-gradient method for generating

Liapunov functions [see Appendix B] to Equation (30) as follows:

$$\dot{V}(x) = (\nabla V)'\dot{x}$$

$$= [\frac{\partial V}{\partial x_1} \quad \frac{\partial V}{\partial x_2}] \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

$$= [\nabla V_1 \quad \nabla V_2] \begin{bmatrix} ax_2 \\ -bx_1 + \mu(c-x_1^r)^p x_2^q \end{bmatrix} \tag{35}$$

Then $V(x)$ can be obtained uniquely from a line integral of a vector function ∇V under the condition that the curl equation

$$\frac{\partial \nabla V_1}{\partial x_2} = \frac{\partial \nabla V_2}{\partial x_1}$$

must be satisfied for this second-order system. Equation (35) becomes

$$\dot{V}(x) = (\nabla V_1)ax_2 - (\nabla V_2)bx_1 + (\nabla V_2)\mu(c-x_1^r)^p x_2^q \tag{36}$$

Comparing Equation (36) with Equation (34), we have

$$\nabla V_2 = qx_2 \tag{37}$$

and

$$(\nabla V_1)ax_2 - (\nabla V_2)bx_1 = 0 \tag{38}$$

Solving Equation (37) and Equation (38), we obtain

$$\nabla V_1 = \frac{(\nabla V_2)bx_1}{ax_2}$$

$$= \frac{(qx_2)bx_1}{ax_2}$$

$$= \frac{qbx_1}{a} \tag{39}$$

With $\nabla V_1 = \dfrac{qbx_1}{a}$ and $\nabla V_2 = qx_2$, the curl equation

$$\frac{\partial \nabla V_1}{\partial x_2} = \frac{\partial \nabla V_2}{\partial x_1}$$

is satisfied.

Finally, from Equation (B-7) in Appendix B, we obtain

$$V(x) = \int_0^{x_1} \nabla V_1 dx_1 + \int_0^{x_2} \nabla V_2 dx_2$$

$$= \int_0^{x_1} \frac{qbx_1}{a} dx_1 + \int_0^{x_2} qx_2 dx_2$$

$$= \frac{q}{2}[\frac{bx_1^2}{a} + x_2^2] \tag{40}$$

which is positive definite and its time derivative is positive semidefinite in the strip $-|\sqrt[r]{c}| < x_1 < |\sqrt[r]{c}|$.

From Liapunov's Stability Theorems, since $V(x) > 0$ and

$\dot{V}(x) \geq 0$ in the strip $-|\sqrt[r]{c}| < x_1 < |\sqrt[r]{c}|$, we can conclude that the solution of the system of Equation (30) travels in the direction of increasing $V(x)$ inside the strip. But when it leaves the strip region, $\dot{V}(x)$ turns negative, it will travel in the direction of decreasing $V(x)$. We can also conclude that no closed trajectory (limit cycle) remains in the strip $-|\sqrt[r]{c}| < x_1 < |\sqrt[r]{c}|$. In addition, from Liapunov's Instability Theorem, the origin of the system as in Equation (30) is not asymptotically stable.

The results are indicated in Figure 6, 7 and 8 which were taken directly from analog simulation.
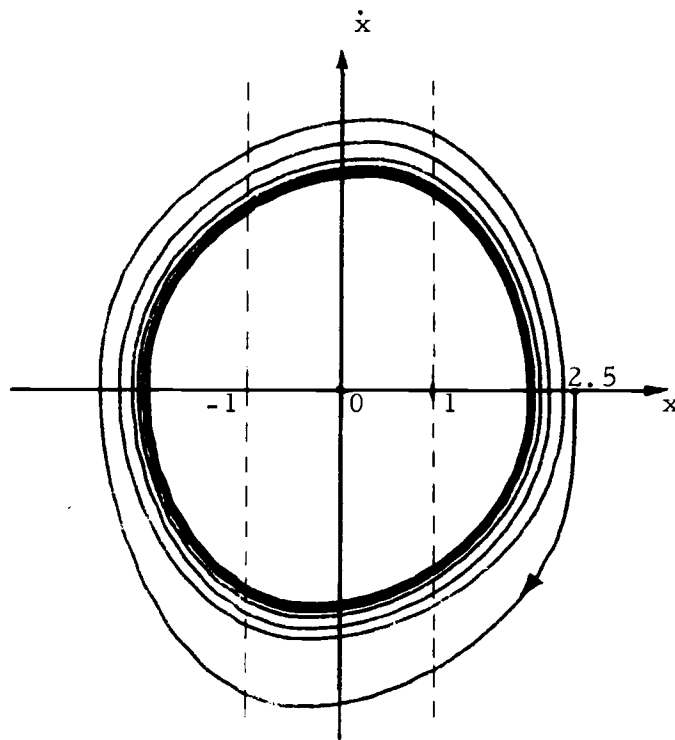
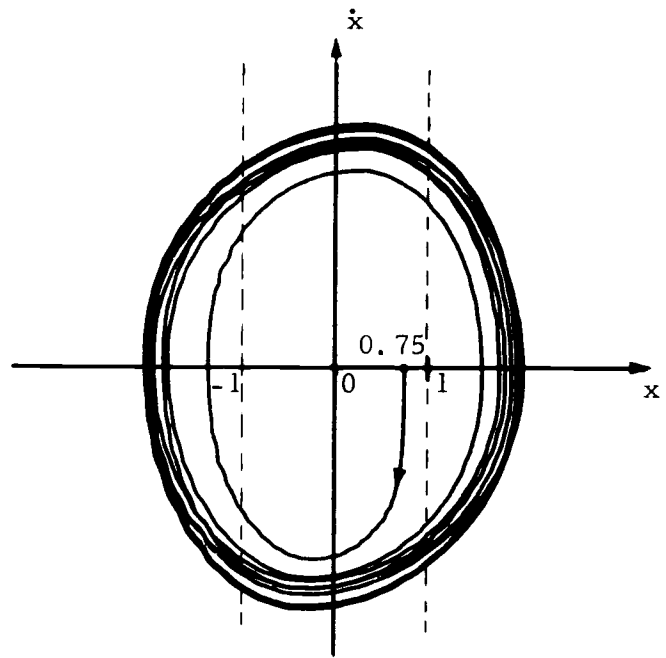Figure 6. x-ẋ curves for $\ddot{x} - \mu(1-x^2)\dot{x} + x = 0$, with $\mu = 0.1$. (Error of the curves is due to the computer.)

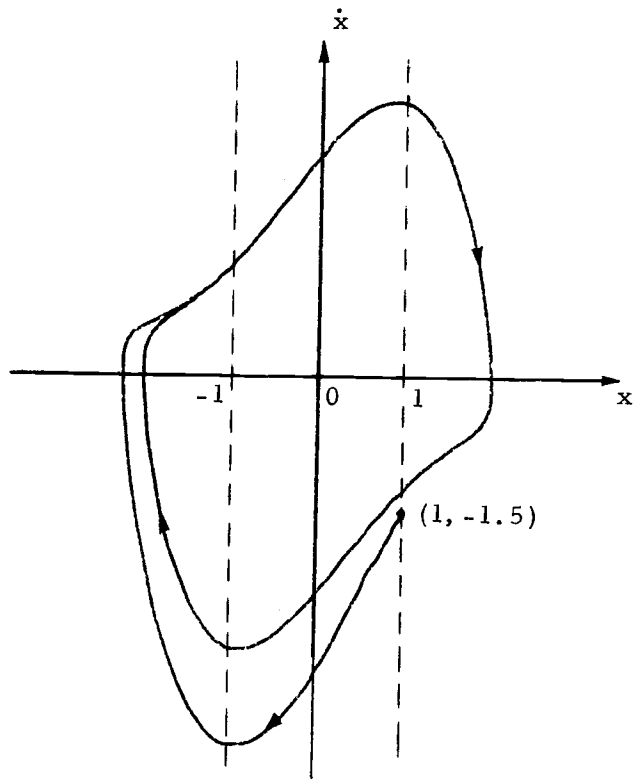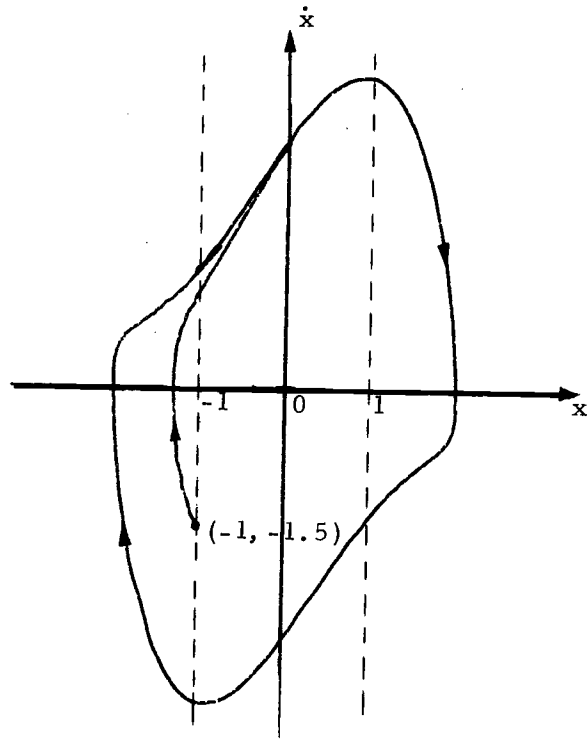Figure 7. x-ẋ curves for $\ddot{x} - \mu(1-x^2)\dot{x} + x = 0$, with $\mu = 1.0$.
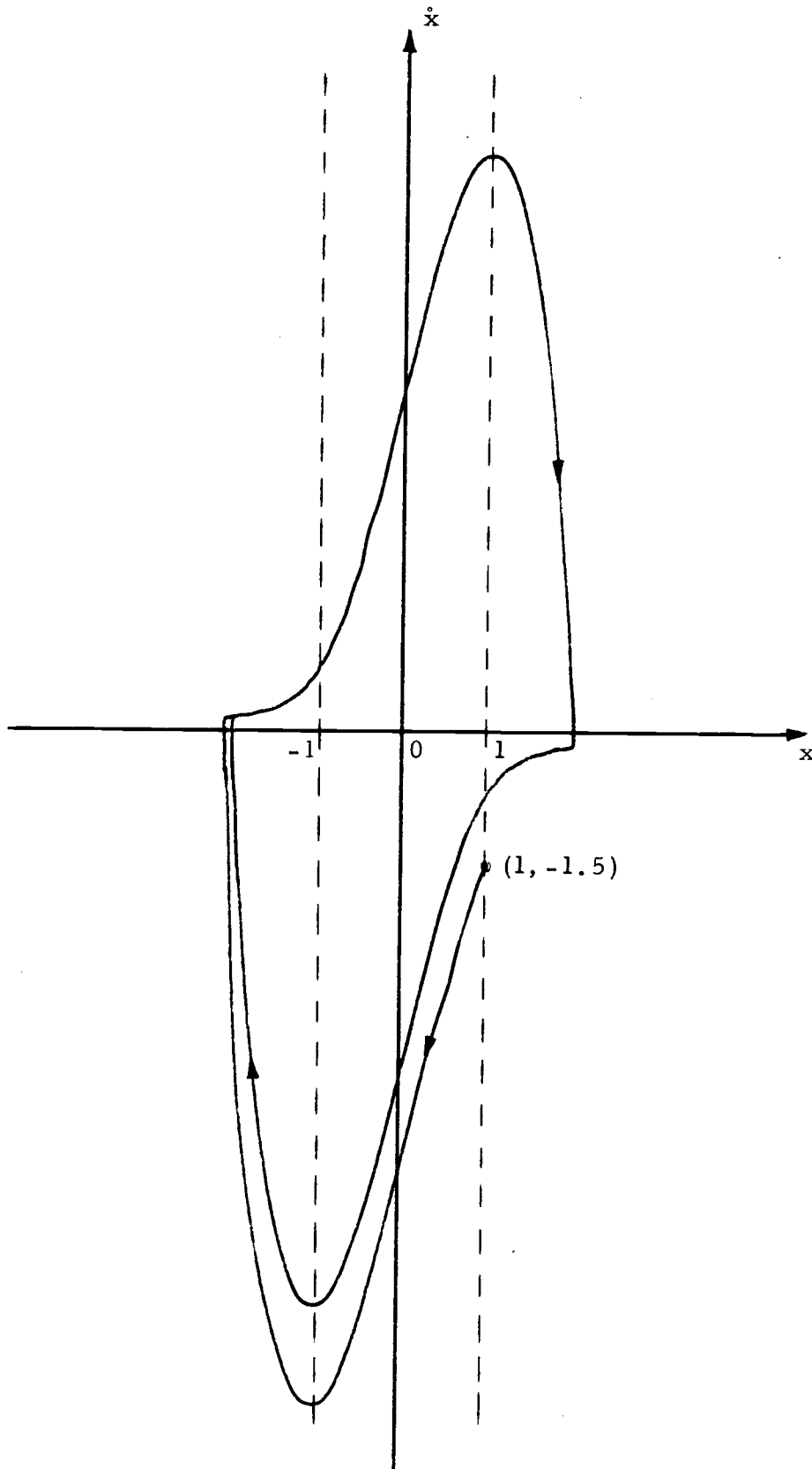(Error of the curves is due to the computer.)

Figure 8. x-$\dot{x}$ curves for $\ddot{x} - \mu(1-x^2)\dot{x} + x = 0$, with $\mu = 4.0$.

VI. CONCLUSION

The Van der Pol equation with the variable-gradient method of generating a Liapunov function [17] for identifying a limit cycle has received particular attention in this paper. The method is based upon the assumption of a variable gradient function with the unknown elements of each of the n components of the variable gradient being determined from constraints on the time derivative of V and $n^{(n-1)}/2$ generalized curl equations. The V-function and its time derivative obtained in this paper are the same as that presented by LaSalle and Lefschetz [12]. LaSalle has first assumed the V-function without systematic derivation but with the systematic method contributed by Schultz and Gibson [17], the V-function can be derived and $\dot{V}$ is constrained to be positive semidefinite in the strip $-1 < x_1 < 1$ by applying Bendixson's negative criterion. The information on the region of asymptotic stability of the Van der Pol equation has been presented by LaSalle [11].

In conclusion, through the study of the stability of the Van der Pol equation with Liapunov's second method, the major difficulty is in finding a suitable Liapunov function. It seems to the author that with the variable-gradient method by Schultz and Gibson [17] this major difficulty is reduced and the great advantage of this method is that it can be applied to higher order systems.

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APPENDIX A

Bendixson's Criteria for the Existence of Limit Cycles

Bendixson's theorem [5, 15] states that if a trajectory re-
mains in a closed bounded domain D without approaching a singular
point (equilibrium position), then the trajectory is either a closed
trajectory or approaches a closed trajectory. This theorem gives
sufficient conditions for existence of a closed trajectory-limit cycle.
Its principal limitation is the difficulty of determining the domain D
satisfying the requirements of the theorem. In the case of the domain
D bounded by two closed curves C_1 and C_2, as illustrated in
Figure 9 below, it is sufficient for the existence of a limit cycle that

1) Trajectories enter (leave) D through the boundary C_1

   and C_2.

2) C_1 and C_2 contains a singularity or singularities but
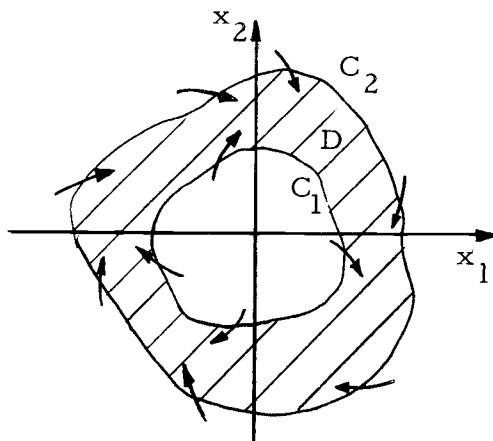
   there are no singular points in D.

Figure 9. Illustration for Bendixson's Theorem.

There is also a test developed by Bendixson, sometimes called

the negative criterion of Bendixson, which does not give sufficient

conditions for the existence of limit cycles. It says: if the expres-

sion $\left(\dfrac{\partial X_1}{\partial x_1} + \dfrac{\partial X_2}{\partial x_2}\right)$ of equations

$$\dot{x}_1 = X_1(x_1, x_2)$$

$$\dot{x}_2 = X_2(x_1, x_2)$$

$$(A-1)$$

does not change its sign within a region D, then no closed trajec-

tory can exist in D.

This theorem can be proved as follows: from Equation (A-1)

$$\frac{dx_2}{dx_1} = \frac{X_2(x_1, x_2)}{X_1(x_1, x_2)}$$

or

$$X_1 dx_2 - X_2 dx_1 = 0 \qquad (A-2)$$

which describes a relationship that must exist along any solution

trajectory in the state space. Gauss' theorem states that a surface

integral over a complete domain D is related to the line integral

around the closed path C_0 which is the boundary of the origin:

$$\iint_D \left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2}\right) dx_1 dx_2 = \oint_{C_0} (X_1 dx_2 - X_2 dx_1) \qquad (A-3)$$

In a limit cycle C_0 is assumed bounding the region D, the integral is zero everywhere on C_0, this could be true if the integrand of the double integral changed sign within D, that is, $\left(\dfrac{\partial X_1}{\partial x_1} + \dfrac{\partial X_2}{\partial x_2} \right)$ must change sign within D.

Hence, the criterion is proved.

APPENDIX B

<u>The Variable-Gradient Method for Generating</u>
<u>Liapunov Functions</u>

As the name implies, the variable-gradient method is based

upon the assumption of a vector ∇V with n undetermined com-

ponents. The gradient of V, written mathematically as

$$\nabla V = \frac{\partial V}{\partial x_1} + \frac{\partial V}{\partial x_2} + \frac{\partial V}{\partial x_3} + \ldots + \frac{\partial V}{\partial x_n} \qquad (B-1)$$

can be employed to compute both V and $\dot{V}$. As already men-

tioned,

$$\dot{V} = \frac{dV}{dt} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 + \ldots + \frac{\partial V}{\partial x_n}\dot{x}_n \qquad (B-2)$$

or

$$\dot{V} = (\nabla V)'\dot{x} \qquad (B-3)$$

where $(\nabla V)'$ is the transpose of ∇V, and $\dot{x}$ is as defined in

Equation (9).

The Liapunov function V can also be obtained as a line inte-

gral of ∇V, i.e.,

$$V = \int_0^x (\nabla V)'dx \qquad (B-4)$$

where the upper limit of integration implies that the integral is

extended to the arbitrary point $(x_1, x_2, \ldots, x_n)$. It is shown in

standard texts on vector calculus that, for a scalar function V

to be obtained uniquely from a line integral of a vector function ∇V,

the matrix Φ formed by $\partial \nabla V_i / \partial x_i$ must be symmetrical; that is

$$\Phi = \begin{bmatrix} \dfrac{\partial \nabla V_1}{\partial x_1} & \dfrac{\partial \nabla V_2}{\partial x_1} & \cdots & \dfrac{\partial \nabla V_n}{\partial x_1} \\[2em] \dfrac{\partial \nabla V_1}{\partial x_2} & \dfrac{\partial \nabla V_2}{\partial x_2} & \cdots & \cdots \\[1em] \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\[1em] \dfrac{\partial \nabla V_1}{\partial x_n} & \dfrac{\partial \nabla V_2}{\partial x_n} & \cdots & \dfrac{\partial \nabla V_n}{\partial x_n} \end{bmatrix} \qquad (B\text{-}5)$$

must be a symmetrical matrix. The condition of the matrix Φ is

thus a generalized curl requirement for the n-dimensional case, that

is,

$$\frac{\partial \nabla V_i}{\partial x_j} = \frac{\partial \nabla V_j}{\partial x_i} \qquad (i, j = 1, 2, \ldots, n) \qquad (B\text{-}6)$$

The integral of Equation (B-4) can now be written as

$$V = \int_0^{x_1,(x_2=x_3=\ldots=x_n=0)} \nabla V_1 dx_1 + \int_0^{x_2,(x_1=x_1,x_3=x_4=\ldots=x_n=0)} \nabla V_2 dx_2$$

$$+ \ldots + \int_0^{x_n,(x_1=x_1,x_2=x_2,\ldots,x_{n-1}=x_{n-1})} \nabla V_n dx_n$$

(B-7)

where the components of the vector ∇V in the x_i direction are ∇V_i. In actually mechanizing the technique, a gradient of the form

$$\nabla V = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots \quad \ldots \\ \vdots \\ a_{n1}x_1 + \ldots \quad \ldots + 2x_n \end{bmatrix}$$

(B-8)

is chosen. With no loss of generality, the a_{ii} coefficients are restricted to be functions of x_i along, with the exception of a_{nn} which is equated to the constant 2. The a_{ii} elements are chosen positive to increase the probability of obtaining a positive definite V. With the exception of a_{nn}, the remaining a_{ii}'s form undetermined quantities. Further, it is assumed that each a_{ij} consists of a constant part a_{ijk} together with a number of variable terms a_{ijv}, which are functions of the state variables up to and including the (n-1) variable, that is,

$$a_{ij} = a_{ijk} + a_{ijv}(x_1, x_2, \ldots, x_{n-1})$$

(B-9)

Substituting Equation (B-9) into Equation (B-8) yields the most general form of the gradient function. Thus

$$\nabla V = \begin{bmatrix} (a_{11k} + a_{11v})x_1 + (a_{12k} + a_{12v})x_2 + \ldots + (a_{1nk} + a_{1nv})x_n \\ (a_{21k} + a_{21v})x_1 + (a_{22k} + a_{22v})x_2 + \ldots \quad \ldots \\ \vdots \\ (a_{n1k} + a_{n1v})x_1 + \quad \ldots \quad \ldots + 2x_n \end{bmatrix}$$

(B-10)

To this point, then, the steps in the variable-gradient method of determining a suitable Liapunov function are:

1. Assume ∇V is of the form illustrated in Equation (B-10).

2. From ∇V, determine $\dot{V}$ from Equation (B-3).

3. Constrain $\dot{V}$ to be at least negative semidefinite.

4. Use the equations implied by the statement that Φ must be symmetric to compute the remaining unknown coefficients in ∇V.

5. Determine V from Equation (B-4), and verify $\dot{V}$ since the addition of terms required in Step 4 may have altered the original $\dot{V}$.

6. Determine the region of closedness of V.