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Title: THE DISTRIBUTION OF CERTAIN SEQUENCES OF n-TUPLES OF NUMBERS MODULO 1

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In this thesis we extend known results concerning the denseness of certain sets S in the n-cube $\mathop{\times}_{i=0}^{n-1} [0, 1]_i$. For a function f, having n derivatives $f^{(1)}, f^{(2)}, \ldots, f^{(n)}$, the set S is defined by $\{(\langle f(i)\rangle, \langle f^{(1)}(i)\rangle, \ldots, \langle f^{n-1}(i)\rangle) \mid i \text{ a natural number}\}$, where $\langle f^{(j)}(i)\rangle$ is the difference between $f^{(j)}(i)$ and the greatest integer less than or equal to $f^{(j)}(i)$.

The thesis addresses itself to functions satisfying the condition $\lim_{x\to+\infty} f^{(n)}(x) = 0$ for some n. In such cases we prove necessary and sufficient conditions that the set S be dense in the n-cube. The results are obtained by an induction on n. In Chapter II we prove necessary and sufficient conditions for the case $n = 1$. Chapter III is devoted to showing that S is dense in $\mathop{\times}_{i=0}^{n-1} [0, 1]_i$ if and only if its projection S' on $\mathop{\times}_{i=1}^{n-1} [0, 1]_i$ is dense in the (n-1)-cube for $n \geq 2$.

In Chapter IV we use the results of Chapter II and Chapter III in an induction that yields necessary and sufficient conditions for S to be dense in the n-cube. In the case $n = 2$, sufficient, but not necessary conditions have previously been shown.

Chapter V is devoted to the examination of certain examples and unsolved problems.

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by

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THE DISTRIBUTION OF CERTAIN SEQUENCES OF n-TUPLES OF NUMBERS MODULO 1

I. INTRODUCTION

The problem to which this thesis addresses itself is an outgrowth of a question asked of Richard Crittenden of Portland State University by Charles S. Rees of the University of New Orleans. In his work in Fourier series Rees found he had need for a sequence of positive integers $\{n_k\}$ that, for k large enough, has the following properties:

(i) $1/4 < n_k \log\log n_k (\text{modulo } 2\pi) < 1/2$

and

(ii) $0 < \log\log n_k (\text{modulo } 2\pi) < 1/4$.

Since then Crittenden, F.S. Cater, and Charles Vanden Eynden [1] have published a paper that proves the existence of such a sequence in a rather general way. This thesis is an extension of the first theorem of that paper, which we state after giving some preliminary definitions.

Definition 1. Let $\langle x \rangle = x - [x]$, where $[x]$ is the greatest integer function.

Henceforth, the symbols R, R^+, N, and Z will be reserved for the sets of real numbers, positive real numbers, natural numbers, and integers, respectively.

Theorem A. Suppose f is defined for $x > a$, $f(x)$ eventually tends monotonically to 0 as x tends to $+\infty$, and $\int_a^x f(t)dt = h(x)$ tends to $+\infty$ as x tends to $+\infty$.

If $\ell(x) = \int_a^x h(t)dt$, then $\{(\langle h(i)\rangle, \langle \ell(i)\rangle) \mid i \in N\}$ is dense in $[0,1] \times [0,1]$.

If we consider the interval $\{x \mid x \geq e\}$ and define

$$\ell(x) = x \log \log x - x + e,$$

$$h(x) = \log \log x + \frac{1}{\log x} - 1,$$

and

$$f(x) = \frac{1}{x \log x} - \frac{1}{x(\log x)^2}$$

then the functions ℓ, h and f satisfy the conditions of Theorem A, and so

$$\{(\langle \ell(i)\rangle, \langle h(i)\rangle) \mid i \in N\}$$

is dense in $[0,1] \times [0,1]$. But since $\lim_{x \to +\infty} \frac{1}{\log x} = 0$, the set

$$S = \{(\langle i \log \log i\rangle, \langle \log \log i\rangle) \mid i \in N\}$$

is also dense in $[0,1] \times [0,1]$. We could have, as well, looked at the functions $\frac{1}{2\pi}\ell$, $\frac{1}{2\pi}h$, and $\frac{1}{2\pi}f$. They also satisfy the conditions of Theorem A and hence

$$S = \{(<\frac{1}{2\pi}\ell(i)>, <\frac{1}{2\pi}h(i)>) \mid i \in N\}$$

is dense in $[0,1] \times [0,1]$; therefore

$$G = \{\left(\ell(i)(\text{modulo } 2\pi), h(i)(\text{modulo } 2\pi)\right) \mid i \in N\}$$

is dense in $[0,2\pi] \times [0,2\pi]$. We see then that the Rees sequence $\{n_k\}$ exists.

In this thesis we take a somewhat different tack. We first make the following definition.

Definition 2. Let $f^{(0)} = f$ and $f^{(n)} = f^{(n-1)'}$ for $n \in N$.

Our approach is to look at a real valued function f defined and having n derivatives for $x > a$, and such that the $\lim_{x \to +\infty} f^{(n)}(x) = 0$, and give necessary and sufficient conditions for the set

$$S = \{(<f(i)>, <f'(i)>, \ldots, <f^{(n-1)}(i)>) \mid i \in N\}$$

to be dense in $\overset{n-1}{\underset{i=0}{\times}} [0,1]_i$. In the language of this thesis, Theorem A would be stated as follows:

Let f be a real valued function, defined, and having a first and second derivative on $\{x \mid x > a\}$ for some a, let $f''(x)$ tend monotonically to 0 as x tends to $+\infty$, and let $\lim_{x \to +\infty} f'(x) = +\infty$. Then the set

$$S = \{(\langle f(i)\rangle, \langle f'(i)\rangle) \mid i \in N\}$$

is dense in $[0, 1] \times [0, 1]$.

Theorem 5 of this thesis restricted to the case $n = 2$ implies Theorem A, but gives necessary and sufficient conditions rather than just sufficient conditions. The monotonicity of f'' is found to be unnecessary and the condition, $\lim_{x \to +\infty} f'(x) = +\infty$, is replaced by the condition, f' is unbounded or $\limsup f' - \liminf f' \geq 1$. Adding this alternative condition for bounded functions allows us to apply the theorem to trigonometric functions which were out of reach of the original theorem. Lemma 3 allows us to conclude that sets like

$$S = \{(\langle i^{(2n+1)/2}\rangle, \langle i^{(2n-1)/2}\rangle, \ldots, \langle i^{1/2}\rangle) \mid i \in N\}$$

are dense in $\displaystyle\mathop{\times}_{i=0}^{n} [0, 1]_i$, while Corollary 7 may be used to show that the set

$$S = \{(\langle i^n \log\log i^w\rangle, \langle \log\log i^w\rangle) \mid i \in N\},$$

where $n \in N$ and $w > 0$, is dense in $[0, 1] \times [0, 1]$.

We have chosen to view the sets as being subsets of Euclidean n-space. The topological closure, then, may contain points with some or all co-ordinates equal to 1. Since 1 is not in the range of the function $\langle x \rangle$ this might seem inappropriate. Alternatively we could view the sets as being subsets of the n-dimensional torus $\overset{n}{\underset{i=1}{\times}} (R/Z)_i$. This approach would necessitate identifying numbers with equivalence classes. For this reason and the fact that most people have a more intuitive feeling about a 17-dimensional cube than a 17-dimensional doughnut, we chose the first alternative.

The main result of this thesis, Theorem 6, is proved by an induction. Theorems 1, 2, and 4 of Chapter II constitute the proof of Theorem 6 for the case $n = 1$. Theorem 5 is devoted to showing certain sets are dense in the n-cube, $(n \geq 2)$, if and only if certain other sets are dense in the (n-1)-cube.

Theorem 6 of Chapter IV combines Theorems 1, 2, and 4 with Theorem 5 in an induction that yields the main result.

In Chapter V some examples of the use of Theorem 6 and Corollary 7 are examined.

II. SOME PRELIMINARY THEOREMS

This chapter is devoted to proving three rather short theorems that, taken together, amount to the first step of an induction. Theorem 1 is well known, and in fact if $\lim\limits_{x \to +\infty} f(x) = +\infty$, $\lim\limits_{x \to +\infty} f'(x) = 0$, and $\lim\limits_{x \to \infty} x f'(x) = +\infty$ then the set $S = \{(\langle f(i)\rangle) \mid i \in N\}$ is uniformly dense in $[0, 1]$ [2]. Here a sequence $\{k_i\}$ is defined to be uniformly dense in $[0, 1]$ in the following way. Let $n(a, b)$ equal the cardinality of the set $\{k_i \mid i \leq n \text{ and } k_i \in (a, b)\}$. Then $\{k_i\}$ is uniformly dense in $[0, 1]$ if for all $0 < a < b < 1$, the $\lim\limits_{n \to +\infty} n(a, b)/n = b - a$.

Before we prove Theorem 1 we make the following definition.

<u>Definition 3</u>. For any set S in Euclidean n-space R^n, $\overline{S}$ is the topological closure of S.

<u>Theorem 1</u>. Let f be a real valued function defined on $\{x \mid x > a\}$ for some a which satisfies

(i) $\lim\limits_{x \to \infty} f'(x) = 0$

and

(ii) f is unbounded.

Let $S = \{\langle f(i)\rangle \mid i \in N\}$. Then $\overline{S} = [0, 1]$.

Proof. Let $p \in (0,1)$ and let ε be a positive number satisfying

$$\varepsilon < \min\{p, 1-p\}.$$

Now by part (i) of the hypothesis, there exists a number x_0 such that

$$|f'(x)| < \varepsilon \quad \text{when} \quad x \geq x_0.$$

From the Intermediate Value Theorem, it follows that either $\{y \mid y \geq f(x_0)\}$ or $\{y \mid y \leq f(x_0)\}$ is a subset of $\{f(x) \mid x \geq x_0\}$. In any event there exists a number x_1 such that $x_1 > x_0$ and $\langle f(x_1) \rangle = p$; that is,

$$f(x_1) = j + p$$

where j is an integer.

Put $k = [x_1] + 1$. Then $0 < k - x_1 \leq 1$. Applying the Mean Value Theorem, we obtain a number $\bar{x}$ such that $x_1 < \bar{x} < k$ and

$$f(k) = f(x_1) + f'(\bar{x})(k - x_1).$$

But $\bar{x} > x_1$ and $x_1 > x_0$, and so $|f'(\bar{x})| < \varepsilon$. Since $0 < k - x_1 \leq 1$, then $|f'(\bar{x})(k - x_1)| < \varepsilon$. Thus

$$f(x_1) - \varepsilon < f(k) < f(x_1) + \varepsilon,$$

or

$$j + p - \varepsilon < f(k) < j + p + \varepsilon.$$

Since $\varepsilon < \min\{p, 1-p\}$ then

$$j < j + p - \varepsilon < f(k) < j + p + \varepsilon < j + 1.$$

Therefore,

$$|<f(k)> - p| < \varepsilon.$$

Since p is an arbitrary element of $(0, 1)$, it follows that $\overline{S} = [0, 1]$.

Theorem 2 is proved in much the same way as was Theorem 1 which is as follows.

<u>Theorem 2.</u> Let f be a real valued function defined on $\{x | x > a\}$ for some a which satisfies

(i) $\lim_{x \to \infty} f'(x) = 0$

and

(ii) $\lim\sup f - \lim\inf f \geq 1$.

Let $S = \{<f(i)> | i \in N\}$. Then $\overline{S} = [0, 1]$.

Proof. Put $s = \lim\sup f$ and $i = \lim\inf f$. From (ii) we see that

(2.1) $$i \leq s - 1.$$

Let $p \in (0, 1)$ where $p \neq <s>$ and let ε be a positive number satisfying

$$\varepsilon < \min\{p, 1-p\}$$

By (i) there exists a number x_0 such that

$$|f'(x)| < \varepsilon \quad \text{when} \quad x \geq x_0. \tag{2.2}$$

From (2.1) and the Intermediate Value Theorem, it follows that $(s-1, s) \subseteq \{f(x) | x > x_0\}$. There exists a number y such that $s - 1 < y < s$ and

$$<y> = p. \tag{2.3}$$

Choose $x_1 > x_0$ such that $f(x_1) = y$. Let $k = [x_1] + 1$. Then $0 < k - x_1 \leq 1$. By the Mean Value Theorem, there exists a number $\bar{x}$ such that $x_1 < \bar{x} < k$ and

$$f(k) = f(x_1) + f'(\bar{x})(k-x_1).$$

Since $\bar{x} > x_1$ and $x_1 > x_0$, then $|f'(\bar{x})| < \varepsilon$ by (2.2). Thus $|f'(\bar{x})(k-x_1)| < \varepsilon$, and so

$$y - \varepsilon = f(x_1) - \varepsilon < f(k) < f(x_1) + \varepsilon = y + \varepsilon.$$

From (2.3), we then obtain

$$[y] < [y] + p - \varepsilon < f(k) < [y] + p + \varepsilon < [y] + 1.$$

Therefore,

$$|\langle f(k)\rangle - p| < \varepsilon.$$

and the theorem follows.

After Theorems 1 and 2, Theorem 4 almost states itself, since it is what is needed to produce necessary and sufficient conditions for set $\{(\langle f(i)\rangle) | i \in N\}$ to be dense in $[0, 1]$. However, before we prove Theorem 4 we have need for the following lemma which is used also in Chapters IV and V.

Lemma 3. Let $g_1, g_2, \ldots, g_n$ be real valued functions and let $f(x) = kg_j(x) + h(x)$ where $k \in N$ and $\lim_{x\to+\infty} h(x)$ exists and is finite. If

$$\{(\langle g_1(i)\rangle, \langle g_2(i)\rangle, \ldots, \langle g_n(i)\rangle) | i \in N\}$$

is dense in $\mathop{\times}\limits_{i=1}^{n} [0, 1]_i$, then so is

$$\{(\langle g_1(i)\rangle, \ldots, \langle g_{j-1}(i)\rangle, \langle f(i)\rangle, \langle g_{j+1}(i)\rangle, \ldots, \langle g_n(i)\rangle) | i \in N\}.$$

Proof. With no loss in generality, we assume $j = 1$. Put $c = \lim_{x\to+\infty} h(x)$. Let $(r_1, r_2, \ldots, r_n) \in \mathop{\times}\limits_{i=1}^{n} (0, 1)_i$ where $r_1 \neq \langle c\rangle$. Observe that $\langle r_1 - c\rangle \neq 0$, for otherwise $r_1 - c = v \in Z$, $r_1 = v + c$, and $r_1 = \langle r_1\rangle = \langle c\rangle$. Let ε be a positive number satisfying

(2.4) $$\varepsilon < \min\{r_1, 1-r_1, \langle r_1-c\rangle, 1-\langle r_1-c\rangle\},$$

and let t be a number satisfying

(2.5) $$|h(x)-c| < \varepsilon/2 \qquad (x > t).$$

Since $(\frac{1}{k}\langle r_1-c\rangle, r_2, \ldots, r_n) \in \overset{n}{\underset{i=1}{\times}} (0,1)_i$, there is an $i \in N$, $i > t$, for which

$$|\langle g_1(i)\rangle - \frac{1}{k}\langle r_1-c\rangle| < \varepsilon/2k$$

and

$$|\langle g_\ell(i)\rangle - r_\ell| < \varepsilon \qquad (\ell = 2, 3, \ldots, n).$$

Therefore,

$$\frac{1}{k}\langle r_1-c\rangle - \frac{\varepsilon}{2k} < \langle g_1(i)\rangle < \frac{1}{k}\langle r_1-c\rangle + \frac{\varepsilon}{2k}$$

and so

(2.6) $$0 < \langle r_1-c\rangle - \varepsilon/2 < k\langle g_1(i)\rangle < \langle r_1-c\rangle + \varepsilon/2 < 1.$$

Since $kg_1(i) = k[g_1(i)] + k\langle g_1(i)\rangle$ and $k[g_1(i)]$ is an integer, then

$$\langle kg_1(i)\rangle = \langle k\langle g_1(i)\rangle\rangle .$$

From (2.6) we have $\langle k\langle g_1(i)\rangle\rangle = k\langle g_1(i)\rangle$; thus,

$$\langle kg_1(i)\rangle = k\langle g_1(i)\rangle ,$$

and

$$(2.7)\qquad \langle r_1 - c\rangle - \varepsilon/2 < \langle kg_1(i)\rangle < \langle r_1 - c\rangle + \varepsilon/2.$$

Now $\langle r_1 - c\rangle + c = r_1 - c - [r_1 - c] + c = r_1 - [r_1 - c]$; hence,

$$(2.8)\qquad \langle r_1 - c\rangle + c = r_1 + m$$

where $m = -[r_1 - c]$ is an integer. From (2.4), (2.5), (2.7), and (2.8) we have

$$\begin{aligned} f(i) &= kg_1(i) + h(i) \\ &= [kg_1(i)] + \langle kg_1(i)\rangle + h(i) \\ &< [kg_1(i)] + \langle r_1 - c\rangle + \varepsilon/2 + \varepsilon/2 + c \\ &= [kg_1(i)] + m + r_1 + \varepsilon \\ &< [kg_1(i)] + m + 1 \end{aligned}$$

and

$$\begin{aligned} f(i) &= [kg_1(i)] + \langle kg_1(i)\rangle + h(i) \\ &> [kg_1(i)] + \langle r_1 - c\rangle - \varepsilon/2 - \varepsilon/2 + c \\ &= [kg_1(i)] + m + r_1 - \varepsilon \\ &> [kg_1(i)] + m. \end{aligned}$$

From the preceding inequalities we obtain

$$\left| \langle f(i)\rangle - r_1 \right| < \varepsilon.$$

This together with inequality (2.6) yields the result of the lemma.

Theorem 4. Let f be a bounded real valued function defined on $\{x | x > a\}$ for some a. If

$$\lim\sup f - \lim\inf f < 1$$

then $\{<f(i)> | i \in N\}$ is not dense in $[0, 1]$.

Proof. Let $\alpha = \lim\inf f$ and $\beta = \lim\sup f$.

From the preceding lemma we have that $\{<f(i)> | i \in N\}$ is dense in $[0, 1]$ if and only if $\{<g(i)> | i \in N\}$ is dense in $[0, 1]$ where

$$g(x) = f(x) - \frac{\alpha+\beta-1}{2} .$$

Note that

$$\lim\inf g = \alpha - \frac{\alpha+\beta-1}{2}$$

$$= \frac{1-(\beta-\alpha)}{2} > 0$$

and

$$\lim\sup g = \beta - \frac{\alpha+\beta-1}{2}$$

$$= \frac{1+(\beta-\alpha)}{2} < 1.$$

Therefore, with no loss in generality we assume $0 < \alpha < \beta < 1$.

Let $\varepsilon = \min\{\alpha/2, (1-\beta)\}$. Since f is bounded we have from the definition of $\lim\sup f$ and $\lim\inf f$, the existence of a number t for which

$$\alpha - \varepsilon < f(x) < \beta + \varepsilon \qquad (x > t).$$

Thus,

$$\alpha/2 < f(x) < 1 \qquad (x > t),$$

and so

$$\alpha/2 < \langle f(x) \rangle < 1 \qquad (x > t).$$

But then the intersection of $[0, \alpha/2]$ and $\{\langle f(i) \rangle | i \in N\}$ is finite, and the theorem follows.

III. THE INDUCTIVE STEP

Theorem 5, which we now state, is really the heart of the thesis.

Theorem 5. Let $n \geq 2$ and let f be a real valued function with an nth derivative defined on $\{x | x > a\}$ for some a, which satisfies the condition

(i) $\lim_{x \to +\infty} f^{(n)}(x) = 0.$

Let

$$S = \{(<f(i)>, <f'(i)>, \ldots, <f^{(n-1)}(i)>) | i \in N\}$$

and

$$S' = \{(<f'(i)>, \ldots, <f^{(n-1)}(i)>) | i \in N\}.$$

Then

(ii) $\overline{S} = \times_{i=0}^{n-1} [0, 1]_i$

if and only if

(iii) $\overline{S'} = \times_{i=1}^{n-1} [0, 1]_i$.

Since the proof of the if part is rather long and involved a little overview is in order. We first choose a point $Q = (q_0, q_1, \ldots, q_{n-1})$ in $\times_{i=0}^{n-1} (0, 1)_i$. For a given ε we find a new point $Q_1 = (b_0, b_1, \ldots, b_{n-1})$ within a distance ε of Q and which is of the form

$(\frac{a_0}{p}, \frac{a_1}{p}, \frac{a_2}{2^k}, \ldots, \frac{a_{n-1}}{2^k})$ where p is prime, p and k are large depending on ε, and the a's are determined by the q's. We use the fact $\overline{S'} = \overset{n-1}{\underset{i=1}{\times}} [0,1]_i$ to find an integer x_0 which has the property $|\langle f^{(i)}(x_0)\rangle - b_i|$ is small for $1 \leq i < n$.

We then expand the function using Taylor's Theorem, about x_0. A new integer x_{r_0} is then found with the property that $|\langle f^{(j)}(x_{r_0})\rangle - b_j|$ is small for $0 \leq j < n$. The point $Q_0 = (\langle f(x_{r_0})\rangle, \langle f'(x_{r_0})\rangle, \ldots, \langle f^{(n-1)}(x_{r_0})\rangle)$ is within a distance ε of Q_1 and hence close to Q.

Proof. Since S' is the projection of S on $\overset{n-1}{\underset{i=1}{\times}} [0,1]_i$, it is clear that if S is dense in $\overset{n-1}{\underset{i=0}{\times}} [0,1]_i$ then S' is dense in $\overset{n-1}{\underset{i=1}{\times}} [0,1]_i$.

Henceforth, assume $\overline{S'} = \overset{n-1}{\underset{i=1}{\times}} [0,1]_i$ and let $Q \in \overset{n-1}{\underset{i=0}{\times}} (0,1)_i$. Then $Q = (q_0, q_1, \ldots, q_{n-1})$ where $0 < q_i < 1$ for $i = 0, 1, \ldots, n-1$. Let $\varepsilon > 0$ and

$$\varepsilon < q_i/2 \quad \text{and} \quad \varepsilon < (1-q_i)/2 \qquad (i = 0, 1, \ldots, n-1). \tag{3.1}$$

Choose $k \in N$ satisfying the condition

$$1/2^k < \varepsilon/2n. \tag{3.2}$$

Let p be a prime such that

$$p > 2^k n! \ . \tag{3.3}$$

Then from (3.2) and (3.3) we obtain

$$1/p < 1/2^k n! \leq 1/2^k n < \varepsilon/2n^2 \ . \tag{3.4}$$

Next, let $a_i \in N$ $(i = 0, 1, \ldots, n-1)$ be determined by the following inequalities:

$$a_i/p \leq q_i < (a_i+1)/p \qquad (i = 0, 1) \tag{3.5}$$

and

$$a_i/2^k \leq q_i < (a_i+1)/2^k \qquad (i = 2, \ldots, n-1). \tag{3.6}$$

Set $b_0 = a_0/p$, $b_1 = a_1/p$, and $b_i = a_i/2^k$ $(i = 2, \ldots, n-1)$. Let

$$Q_1 = (b_0, b_1, \ldots, b_{n-1})$$

and

$$Q_1' = (b_1, \ldots, b_{n-1}).$$

Since $\lim_{x \to \infty} f^{(n)}(x) = 0$, there exists $t \in R$ such that

$$|f^{(n)}(x)| < \varepsilon/2n^2(p2^k n!)^n \qquad (x > t). \tag{3.7}$$

Since $Q_1' \in \overset{n-1}{\underset{i=1}{\times}} [0, 1]_i$, we see from (iii) that there exists $x_0 > t$ such that $x_0 \in N$ and

(3.8) $$|\langle f^{(i)}(x_0)\rangle - b_i| < \varepsilon/2n^2(p2^k n!)^n \qquad (i = 1, \ldots, n-1).$$

Put

(3.9) $$k_i = [f^{(i)}(x_0)] \qquad (i = 0, 1, \ldots, n-1),$$

where $[x]$ is the greatest integer function, and

(3.10) $$\begin{aligned} h_i &= f^{(i)}(x_0) - k_i - b_i \\ &= \langle f^{(i)}(x_0)\rangle - b_i \qquad (i = 1, \ldots, n-1). \end{aligned}$$

From (3.8) we have

(3.11) $$|h_i| < \varepsilon/2n^2(p2^k n!)^n \qquad (i = 1, \ldots, n-1).$$

Define

(3.12) $$s = 2^k n! \quad \text{and} \quad x_r = x_0 + rs \qquad (r \in N).$$

By using Taylor's Theorem we can express for $i = 1, \ldots, n-1$, and $r = 0$ or $r \in N$,

$$f^{(i)}(x_r) = \sum_{j=i}^{n-1} f^{(j)}(x_0) \frac{(rs)^{j-i}}{(j-i)!} + f^{(n)}(\bar{x}_r) \frac{(rs)^{n-i}}{(n-i)!}$$

where $x_0 \leq \bar{x}_r \leq x_r$. Hence

$$<f^{(i)}(x_r)> = < \sum_{j=i}^{n-1} f^{(j)}(x_0) \frac{(rs)^{j-i}}{(j-i)!} + f^{(n)}(\overline{x}_r) \frac{(rs)^{n-i}}{(n-i)!} > .$$

We now proceed to reduce $f^{(i)}(x_r)$ modulo 1 for $i = 1, \ldots, n-1$, and $r = 0, \ldots, p-1$. From (3.9) and (3.10), we have

$$<f^{(i)}(x_r)> = < \sum_{j=i}^{n-1} (k_j+b_j+h_j) \frac{(rs)^{j-i}}{(j-i)!} + f^{(n)}(\overline{x}_r) \frac{(rs)^{n-i}}{(n-i)!} > .$$

Since $s = 2^k n!$ and $0 \leq j-i < n$, it is the case that

$$(rs)^{j-i}/(j-i)! \in \mathbf{Z},$$

and therefore,

$$\sum_{j=i}^{n-1} k_j \frac{(rs)^{j-i}}{(j-i)!} \in \mathbf{Z} \qquad (i = 1, \ldots, n-1;\ r = 0, \ldots, p-1).$$

Also, when $i < j < n$ and $1 \leq i$ then $b_j = a_j/2^k$, and so

$$\begin{aligned} b_j(rs)^{j-i}/(j-i)! &= (a_j/2^k)(r2^k n!)^{j-i}/(j-i)! \\ &= a_j(r2^k n!)^{j-i-1}(r2^k n!/2^k(j-i)!). \end{aligned}$$

Thus,

$$b_j(rs)^{j-i}/(j-i)! \in \mathbf{Z} \qquad (1 \leq i < j < n;\ r = 0, \ldots, p-1).$$

It follows that for $i = 1, \ldots, n-1$, and $r = 0, \ldots, p-1$,

$$<f^{(i)}(x_r)> = <b_i + \sum_{j=i}^{n-1} h_j \frac{(rs)^{j-i}}{(j-i)!} + f^{(n)}(\overline{x}_r) \frac{(rs)^{n-i}}{(n-i)!} > .$$

From (3.11) and (3.12), we have for $1 \leq i$ and $j = i, \ldots, n-1$, and $r = 0, 1, \ldots, p-1$,

$$\begin{aligned} |h_j(rs)^{j-i}/(j-i)!| &< |h_j(p2^k n!)^n| \\ &< \frac{\varepsilon(p2^k n!)^n}{2n^2(p2^k n!)^n} \\ &= \varepsilon/2n^2 . \end{aligned}$$

Thus for $i = 1, \ldots, n-1$, or $r = 0, \ldots, p-1$,

$$|\sum_{j=i}^{n-1} h_j \frac{(rs)^{j-i}}{(j-i)!}| < (n-1)\varepsilon/2n^2 . \tag{3.13}$$

Since $\overline{x}_r \geq x_0 > t$, from (3.7) and (3.12) we see that for $i = 1, \ldots, n-1$, and $r = 0, \ldots, p-1$,

$$\begin{aligned} |f^{(n)}(\overline{x}_r) \frac{(rs)^{n-i}}{(n-i)!}| &< \frac{\varepsilon(p2^k n!)^n}{2n^2(p2^k n!)^n} \\ &< \varepsilon/2n . \end{aligned} \tag{3.14}$$

From (3.13) and (3.14) we conclude for $i = 1, \ldots, n-1$, and $r = 0, \ldots, p-1$, that

$$\left| \sum_{j=i}^{n-1} h_j \frac{(rs)^{j-i}}{(j-i)!} + f^{(n)}(\bar{x}_r) \frac{(rs)^{n-i}}{(n-i)!} \right| < \varepsilon/n$$

Set

$$c_{i,r} = \sum_{j=i}^{n-1} h_j \frac{(rs)^{j-i}}{(j-i)!} + f^{(n)}(\bar{x}_r) \frac{(rs)^{n-i}}{(n-i)!} .$$

for $i = 1, \ldots, n-1$, and $r = 0, \ldots, p-1$. Then

$$\langle f^{(i)}(x_r)\rangle = \langle b_i + c_{i,r}\rangle , \tag{3.15}$$

and

$$|c_{i,r}| < \varepsilon/n. \tag{3.16}$$

We now look at $|f^{(i)}(x_r) - b_i|$ for $i = 1, \ldots, n-1$, and $r = 0, \ldots, p-1$. Recall $b_1 = a_1/p$, $a_1/p \leq q_1 < (a_1+1)/p$, and for $i = 2, \ldots, n-1$, $b_i = a_i/2^k$, and $a_i/2^k \leq q_i < (a_i+1)/2^k$. From (3.4) and (3.16) we obtain for $r = 0, \ldots, p-1$,

$$\begin{aligned} |q_1 - (b_1+c_{1,r})| &\leq |q_1 - b_1| + |c_{1,r}| \\ &< 1/p + |c_{1,r}| \\ &< \varepsilon/2n^2 + \varepsilon/n < \varepsilon . \end{aligned}$$

Also, from (3.2) and (3.16) we have for $i = 2, \ldots, n-1$,

$$|q_i - (b_i + c_{i,r})| \le |q_i - b_i| + |c_{i,r}|$$
$$< 1/2^k + |c_{i,r}|$$
$$< \varepsilon/2n + \varepsilon/n < \varepsilon .$$

By (3.1), it follows that

$$0 < q_i - \varepsilon < b_i + c_{i,r} < q_i + \varepsilon < 1$$
$$(i = 1, \ldots, n-1; \; r = 0, \ldots, p-1).$$

Since $0 < b_i + c_{i,r} < 1$, then $\langle b_i + c_{i,r} \rangle = b_i + c_{i,r}$. A substitution in (3.15) gives $\langle f^{(i)}(x_r) \rangle = b_i + c_{i,r}$. Therefore,

$$(3.17) \qquad |\langle f^{(i)}(x_r) \rangle - b_i| < \varepsilon/n$$

for $i = 1, \ldots, n-1$, and $r = 0, \ldots, p-1$.

Now turning our attention to the function f we see from Taylor's Theorem that for $r = 0, \ldots, p-1$,

$$(3.18) \qquad f(x_r) = f(x_0) + \sum_{i=1}^{n-1} f^{(i)}(x_0) \frac{(rs)^i}{i!} + f^{(n)}(\bar{x}_r) \frac{(rs)^n}{n!}$$

where $x_0 \le \bar{x}_r \le x_r$. Let $\bar{a}$ satisfy

$$(3.19) \qquad \bar{a} \in Z, \quad 0 \le \bar{a} < p, \quad \text{and} \quad \bar{a}/p \le \langle f(x_0) \rangle < (\bar{a}+1)/p.$$

Put

$$h_0 = f(x_0) - k_0 - \bar{a}/p \tag{3.20}$$

where $k_0 = [f(x_0)]$. Then (3.4), (3 19), and (3.20) yield the inequality

$$0 \leq h_0 = \langle f(x_0)\rangle - \bar{a}/p < 1/p < \varepsilon/2n^2 . \tag{3.21}$$

Thus, from (3.10), (3.18), and (3.20) it follows that

$$\begin{aligned} f(x_r) &= k_0 + \frac{\bar{a}}{p} + h_0 + \sum_{i=1}^{n-1} (k_i+b_i+h_i) \frac{(rs)^i}{i!} + f^{(n)}(\bar{x}_r) \frac{(rs)^n}{n!} \\ &= \frac{\bar{a}}{p} + \frac{a_1 rs}{p} + \sum_{i=0}^{n-1} k_i \frac{(rs)^i}{i!} + \sum_{i=2}^{n-1} b_i \frac{(rs)^i}{i!} + \sum_{i=0}^{n-1} h_i \frac{(rs)^i}{i!} \\ &\quad + f^{(n)}(\bar{x}_r) \frac{(rs)^n}{n!} \end{aligned} \tag{3.22}$$

for $r = 0, \ldots, p-1$.

We now reduce $f(x_r)$ modulo 1. Since $s = 2^k n!$ then

$$(rs)^i/i! = (r2^k n!)^i/i! \in N \quad (i = 0, \ldots, n-1; \; r = 0, \ldots, p-1).$$

Hence,

$$\sum_{i=0}^{n-1} k_i (rs)^i/i! \in Z \qquad (r = 0, \ldots, p-1). \tag{3.23}$$

For $i = 2, \ldots, n-1$, since $b_i = a_i/2^k$ and $a_i \in N$, then

$$b_i(rs)^i/i! = a_i(r2^k n!)^i/2^k i! \in Z \qquad (i = 2, \ldots, n-1;$$
$$r = 0, \ldots, p-1).$$

Thus

$$(3.24) \qquad \sum_{i=2}^{n-1} b_i(rs)^i/i! \in Z \qquad (r = 0, \ldots, p-1).$$

By (3.21) and (3.11), respectively,

$$|h_0| < \varepsilon/2n^2$$

and

$$|h_i| < \varepsilon/2n^2(p2^k n!)^n \qquad (i = 1, \ldots, n-1).$$

Thus, for $r = 0, \ldots, p-1$,

$$\sum_{i=0}^{n-1} |h_i| \frac{(rs)^i}{i!} \leq \varepsilon/2n^2 + \sum_{i=1}^{n-1} \frac{\varepsilon}{2n^2(p2^k n!)^n} \frac{(rs)^i}{i!}$$

$$< \varepsilon/2n^2 + \sum_{i=1}^{n-1} \frac{\varepsilon}{2n^2} \frac{(r2^k n!)^n}{(p2^k n!)^n}$$

$$< \varepsilon/2n^2 + \sum_{i=1}^{n-1} (\varepsilon/2n^2)$$

$$= n\varepsilon/2n^2 = \varepsilon/2n .$$

So if we set

$$u_r = \sum_{i=0}^{n-1} h_i (rs)^i / i! ,$$

we have

(3.25) $$|u_r| < \varepsilon/2n \qquad (r = 0, \ldots, p-1)$$

Since $\bar{x}_r \geq x_0 > t$, from (3.7) we see that

$$|f^{(n)}(\bar{x}_r)| < \varepsilon/2n^2(p2^k n!)^n ,$$

from which we obtain

$$|f^{(n)}(\bar{x}_r)(rs)^n/n!| < \varepsilon(r2^k n!)^n/2n^2(p2^k n!)^n$$

$$< \varepsilon/2n^2 \qquad (r = 0, \ldots, p-1).$$

So if we set $\ell_r = f^{(n)}(\bar{x}_r)(rs)^n/n!$ we have

(3.26) $$|\ell_r| < \varepsilon/2n^2 \qquad (r = 0, \ldots, p-1).$$

Returning to (3.22) we see that

$$\langle f(x_r)\rangle = \langle \frac{\bar{a}+a_1 rs}{p} + \sum_{i=0}^{n-1} k_i \frac{(rs)^i}{i!} + \sum_{i=2}^{n-1} b_i \frac{(rs)^i}{i!}$$

$$+ \sum_{i=0}^{n-1} h_i \frac{(rs)^i}{i!} + f^{(n)}(\bar{x}_r) \frac{(rs)^n}{n!} \rangle .$$

Using (3.23), (3.24), (3.25), and (3.26) we see that for $r = 0, 1, \ldots, p-1$,

$$(3.27) \qquad \langle f(x_r)\rangle = \left\langle \frac{\bar{a}+a_1 rs}{p} + u_r + \ell_r \right\rangle$$

where $\bar{a}$, a_1, r, and s are nonnegative integers, $|u_r| < \varepsilon/2n$, and $|\ell_r| < \varepsilon/2n^2$.

Since $q_1 < 1$ and since $\frac{1}{p} < \varepsilon < q_1$ by (3.1) and (3.4), then $0 < a_1 < p$ by (3.5). Also, $0 < s < p$ by (3.3) and (3.12). Hence, the set of all products of the form $a_1 rs$ $(r = 0, 1, \ldots, p-1)$ comprise a complete residue system modulo p; that is, for each integer k there exists an integer r, $0 \leq r < p$, such that $a_1 rs = jp + k$, where j is an integer. We see then that there exists an integer r_0, $0 \leq r_0 < p$, and an integer j for which

$$(3.28) \qquad \bar{a} + a_1 r_0 s = jp + a_0 \,.$$

A substitution using (3.28) in (3.27) gives

$$\langle f(x_{r_0})\rangle = \left\langle \frac{jp+a_0}{p} + u_{r_0} + \ell_{r_0} \right\rangle$$

$$= \left\langle j + \frac{a_0}{p} + u_{r_0} + \ell_{r_0} \right\rangle .$$

Since j is an integer it follows that

$$(3.29) \qquad \langle f(x_{r_0})\rangle = \langle \frac{a_0}{p} + u_{r_0} + \ell_{r_0} \rangle .$$

We now consider $|\langle f(x_{r_0})-a_0/p\rangle|$. In (3.5) we find $a_0/p \leq q_0 < (a_0+1)/p$, and thus $0 < q_0 - a_0/p < 1/p$. From (3.4) we have $1/p < \varepsilon$ and from (3.1) we have $2\varepsilon < q_0$. Thus $2\varepsilon - a_0/p < \varepsilon$, or $\varepsilon < a_0/p$. Using (3.25) and (3.26) we see that

$$(3.30) \qquad |u_{r_0}+\ell_{r_0}| \leq |u_{r_0}| + |\ell_{r_0}| < \varepsilon/n < \varepsilon.$$

Therefore,

$$0 < a_0/p + u_{r_0} + \ell_{r_0} .$$

From (3.1) and (3.5) we have $2\varepsilon < 1 - q_0$ and $a_0/p \leq q_0$; that is,

$$a_0/p + 2\varepsilon < 1.$$

It follows that

$$a_0/p + u_{r_0} + \ell_{r_0} < a_o/p + \varepsilon < 1.$$

Since $0 < a_0/p + u_{r_0} + \ell_{r_0} < 1$, then from (3.29) we obtain

$$\langle f(x_{r_0})\rangle = a_0/p + u_{r_0} + \ell_{r_0} .$$

Using (3.30) and the fact that $b_0 = a_0/p$, we then have

(3.31) $$\left| \langle f(x_{r_0}) \rangle - b_0 \right| < \varepsilon/n.$$

Set

$$Q_0 = (\langle f(x_{r_0}) \rangle, \langle f'(x_{r_0}) \rangle, \ldots, \langle f^{(n-1)}(x_{r_0}) \rangle)$$

and let d_0 be the distance from Q_0 to Q_1, d_1 the distance from Q_1 to Q, and d the distance from Q to Q_0. Then we see that from (3.17) and (3.31), we obtain

$$d_0 = \sqrt{\sum_{i=0}^{n-1} (\langle f^{(i)}(x_{r_0}) \rangle - b_i)^2}$$

$$< \sqrt{\sum_{i=0}^{n-1} (\varepsilon/n)^2} < \varepsilon .$$

By (3.4) and (3.5) we have

$$|q_i - b_i| < 1/p < \varepsilon/n \qquad (i = 0, 1),$$

and by (3.2) and (3.6) we obtain

$$|q_i - b_i| < 1/2^k < \varepsilon/n \qquad (i = 2, 3, \ldots, n-1).$$

Hence,

$$d_1 = \sqrt{\sum_{i=0}^{n-1} (q_i - b_i)^2} < \sqrt{\sum_{i=0}^{n-1} (\varepsilon/n)^2} < \varepsilon .$$

Now $d \leq d_0 + d_1$ and, consequently, $d < 2\varepsilon$. So for any $\varepsilon > 0$ and any point $Q \in \times_{i=0}^{n-1} (0, 1)_i$ we have found a point Q_0 such that $Q_0 \in S$ and the distance from Q to Q_0 is less than 2ε.

Thus

$$\overline{S} = \times_{i=1}^{n-1} [0, 1]_i .$$

IV. THE MAIN THEOREM

We now use the results of Chapters II and III in a proof by induction of the main theorem of the thesis.

Theorem 6. Let $n \geq 1$ and let f be a real valued function defined and having n derivatives on $\{x | x > a\}$ for some a and let $\lim_{x \to +\infty} f^{(n)}(x) = 0$. Then

$$S = \{(<f(i)>, <f'(i)>, \ldots, <f^{(n-1)}(i)>) \mid i \in N\}$$

is dense in $\displaystyle\times_{i=0}^{n-1} [0, 1]_i$ if and only if

(i) $f^{(n-1)}$ is unbounded as $x \to +\infty$

or

(ii) $\lim\sup f^{(n-1)} - \lim\inf f^{(n-1)} \geq 1$.

Proof. For $k = 1, 2, \ldots, n$, define

$$S_k = \{(<f^{(n-k)}(i)>, \ldots, <f^{(n-1)}(i)>) \mid i \in N\}.$$

Note $S = S_n$.

Assume (i) or (ii). Then by Theorems 1 and 2, S_1 is dense in $\displaystyle\times_{i=n-1}^{n-1} [0, 1]_i = [0, 1]$ and by Theorem 5, $2 \leq k \leq n$, if S_{k-1} is dense in

$\displaystyle\mathop{\times}_{i=n-(k-1)}^{n-1} [0,1]_i$ then S_k is dense in $\displaystyle\mathop{\times}_{i=n-k}^{n-1} [0,1]_i$. Hence, by induction on k, $S = S_n$ is dense in $\displaystyle\mathop{\times}_{i=0}^{n-1} [0,1]_i$.

Now assume not (i) and not (ii). Then by Theorem 4, S_1 is not dense in $[0,1] = \displaystyle\mathop{\times}_{i=n-1}^{n-1} [0,1]_i$, and by the only if part of Theorem 5 for $2 \leq k \leq n$, S_{k-1} not dense in $\displaystyle\mathop{\times}_{i=n-(k-1)}^{n-1} [0,1]_i$ implies S_k is not dense in $\displaystyle\mathop{\times}_{i=n-k}^{n-1} [0,1]_i$. Using induction on k, it follows that $S = S_n$ is not dense in $\displaystyle\mathop{\times}_{i=0}^{n-1} [0,1]_i$.

Recall now from Chapter I that the original Rees problem amounts to finding a sequence $\{k_i\}$ of integers that placed $<k_i \log\log k_i>$ and $<\log\log k_i>$ within certain bounds. We would now like to extend this result to put $<k_i^n \log\log k_i^t>$ and $<\log\log k_i^t>$, $n \in N$ and $t \in R^+$, within given bounds. For that purpose $<f(i)>$ and $<f^{(n)}(i)>$ are of interest to us, but not $<f^{(j)}(i)>$ where $0 < j < n$. For this example, and many like it, we prove the following corollary.

<u>Corollary 7</u>. Let f and g be functions defined on $\{x \mid x > a\}$ for some a which satisfy

(i) $g(x) = \frac{k_1}{k_2} f^{(j)}(x) + q(x)$ where $k_1 \in N$, $k_2 \in N$, $j \in N$ or $j = 0$, and $\lim_{x \to +\infty} q(x)$ exists and is finite. Also, for some $n > \max\{j, 1\}$

(ii) either $f^{(n-1)}$ is unbounded or

$$\limsup \frac{1}{k_2} f^{(n-1)} - \liminf \frac{1}{k_2} f^{(n-1)} > 1,$$

and

(iii) $\lim_{x \to +\infty} f^{(n)}(x) = 0.$

Then the set $\{(\langle f^{(\ell)}(i)\rangle, \langle g(i)\rangle) \mid i \in N\}$, where $\ell \neq j$ and $0 \leq \ell < n$, is dense in $[0, 1] \times [0, 1]$.

Proof. Let f, q, k_1, k_2, and j satisfy conditions (i), (ii), and (iii). Now the function $\frac{1}{k_2} f$ satisfies the hypothesis of the if part of Theorem 5; therefore, the set

$$S = \{(\langle \frac{1}{k_2} f(i)\rangle, \langle \frac{1}{k_2} f'(i)\rangle, \ldots, \langle \frac{1}{k_2} f^{(n-1)}(i)\rangle) \mid i \in N\}$$

is dense in $\displaystyle\bigtimes_{i=0}^{n-1} [0, 1]$. It follows that the projection of S on the ℓ-j plane

$$S_{\ell, j} = \{(\langle \frac{1}{k_2} f^{(\ell)}(i)\rangle, \langle \frac{1}{k_2} f^{(j)}(i)\rangle) \mid i \in N\}$$

is dense in $[0, 1] \times [0, 1]$. An application of Lemma 3 to the second coordinate function gives

$$\{(<\frac{1}{k_2} f^{(\ell)}(i)>, <\frac{k_1}{k_2} f^{(j)}(i) + q(x)>) \mid i \in N\}$$

is dense in $[0, 1] \times [0, 1]$. A second application of Lemma 3 to the first coordinate function yields the desired result.

V. APPLICATIONS AND UNSOLVED PROBLEMS

We now would like to show, as an example of the use of our results, that the set

$$S = \{(<i^n \log\log i^t>, <\log\log i^t>) \mid i \in N\}$$

where $n \in N$ and $t > 0$ is dense in $[0,1] \times [0,1]$. For that reason we prove the following lemma.

<u>Lemma 8.</u> Let $D^{(k)}(x^n/\log x)$ be the kth derivative of the function $f(x) = x^n/\log x$, $x > 0$. Then

$$\lim_{x \to \infty} D^{(n)}(x^n/\log x) = 0$$

and

$$\lim_{x \to \infty} D^{(n+1)}(x^n/\log x) = 0.$$

Proof. We claim that $D^{(k)}(x^n/\log x)$ is of the form

$$\sum_{j=1}^{k+1} a_j x^{n-k}/(\log x)^j \qquad \text{for} \qquad 1 \le k \le n+1.$$

Observe that

$$D^{(1)}(x^n/\log x) = nx^{n-1}/\log x - x^{n-1}/(\log x)^2.$$

Next assume

$$D^{(i)}(x^n/\log x) = \sum_{j=1}^{i+1} a_j x^{n-i}/(\log x)^j$$

where $1 \leq i < n$. Then

$$D^{(i+1)}(x^n/\log x) = \sum_{j=1}^{i+1} \{(n-i)a_j \frac{x^{n-i-1}}{(\log x)^j} - j a_j \frac{x^{n-i-1}}{(\log x)^{j+1}}\} ;$$

hence,

$$D^{(i+1)}(x^n/\log x) = \sum_{j=1}^{i+2} b_j x^{n-(i+1)}/(\log x)^j$$

where $b_1 = (n-i)a_1$, $b_j = (n-i)a_j - (j-1)a_{j-1}$, $(2 \leq j \leq i+1)$, and $b_{i+2} = -(i+1)a_{i+1}$. The claim follows by mathematical induction.

Thus $D^{(n)}(x^n/\log x)$ has the form $\sum_{j=1}^{n+1} a_j/(\log x)^j$ and $D^{(n+1)}(x^n/\log x)$ has the form $\sum_{j=1}^{n+1} c_i/x(\log x)^{j+1}$, and the result of the lemma is immediate.

<u>Example 1</u>. The set

$$H = \{(\langle i^n \log\log i^t \rangle, \langle \log\log i^t \rangle) \mid i \in N\},$$

where $n \in N$ and t is a positive real number, is dense in $[0, 1] \times [0, 1]$.

Proof. Let $f(x) = x^n \log\log x^t$ and $g(x) = \log\log x^t$ where $n \in N$ and $t \in R^+$.

A straight forward induction argument yields the following formula for the jth derivative of f $(1 \leq j \leq n)$:

$$f^{(j)}(x) = \frac{n!}{(n-j)!} x^{n-j} \log\log x + \frac{n!}{(n-j)!} x^{n-j} \log t$$

$$+ \sum_{i=1}^{j} \frac{n!}{(n-i+1)!} D^{(j-i)}(x^{n-i}/\log x).$$

Therefore,

$$f^{(n)}(x) = n! \log\log x + n! \log t + \sum_{i=1}^{n} \frac{n!}{(n-i+1)!} D^{(n-i)}\left(\frac{x^{n-i}}{\log x}\right).$$

Define

$$h(x) = \sum_{i=1}^{n} \frac{n!}{(n-i+1)!} D^{(n-i)}(x^{n-i}/\log x).$$

From the preceding lemma, we have

$$\lim_{x \to \infty} h(x) = 0$$

and

$$\lim_{x \to \infty} h'(x) = 0$$

It follows that

$$\lim_{x \to \infty} f^{(n)}(x) = \infty.$$

and

$$\lim_{x \to \infty} f^{(n+1)}(x) = \lim_{x \to \infty} \frac{n!}{x \log x} + h'(x) = 0.$$

Next observe that

$$\begin{aligned} n!g(x) &= n! \log \log x^t \\ &= n! \log (t \log x) \\ &= n! \log t + n! \log \log x \\ &= f^{(n)}(x) - h(x). \end{aligned}$$

Hence, $g(x) = \frac{1}{n!} f^{(n)}(x) - \frac{1}{n!} h(x)$, and by Corollary 7 we have that $\{(<f(i)>, <g(i)>) \mid i \in N\}$ is dense in $[0, 1] \times [0, 1]$.

The following is an example of the use of the condition of Theorem 5 that $\limsup f - \liminf f \geq 1$. It, along with many other examples of functions that are compositions of periodic functions with other functions, satisfies this condition.

<u>Example 2</u>. Let $f(x) = x^3 \cos(\log x)$. Repeated differentiation yields

$$\begin{aligned} f^{(3)}(x) &= -10 \sin(\log x) \\ f^{(4)}(x) &= \frac{-10}{x} \cos(\log x). \end{aligned}$$

We see then that $f^{(3)}$ satisfies (ii) of Theorem 5 and $\lim_{x \to \infty} f^{(4)}(x) = 0$, and so

$$S = \{(<f(i)>, <f^{(1)}(i)>, <f^{(2)}(i)>, <f^{(3)}(i)>) \mid i \in N\}$$

is dense in $\overset{3}{\underset{i=0}{\times}} [0,1]_i$. What is probably of as much interest is the fact that $\{<f(i)> \mid i \in N\}$ is dense in $[0,1]$. As a matter of fact, if m and n are such that $\max\{n, m\} \in N$ then it can be shown in the same way that

$$\{<i^n \sin(\log i^r) + i^m \cos(\log i^s)> \mid i \in N\}$$

is dense in $[0,1]$ if $r > 0$, $s > 0$.

Example 3. Let f be a real valued function defined on R^+ in the following way

$$f(x) = \sum_{i=1}^{m} a_i x^{r_i}$$

where $r_1 > r_i$ for all $1 < i \leq m$, $r_1 > 1$, and $r_1 \notin N$, and set $n = [r_1] + 1$. Then

$$f^{(j)}(x) = a_1 \prod_{i=0}^{n-2} (r_1 - i) x^{<r_1>} + g(x)$$

where $g(x)$ has only powers of x less than $<r_1>$. Hence, the function $f^{(n-1)}$ is unbounded. Differentiation yields

$$f^{(n)}(x)=a_1\left(\prod_{i=0}^{n-1}(r_1-i)\right)x^{<r_1>-1}+g'(x).$$

Since all the powers of x in g' are less than 0; thus

$$\lim_{x\to\infty} f^{(n)}(x) = \lim_{x\to\infty} a_1\left(\prod_{i=0}^{n-1}(r_1-i)\right)x^{<r_1>-1} + \lim_{x\to\infty} g'(x) = 0.$$

Hence by Theorem 5 the functions $f, f', \ldots, f^n$ satisfy the condition of Lemma 3.

Looking at a special case of Example 3 we see the power of Lemma 3. Set

$$f(x) = x^{(2n+1)/2} / \prod_{i=0}^{n-1} ((2n+1)-2i)$$

then

$$f'(x) = x^{(2n-1)/2}/2 \prod_{i=1}^{n-1} ((2n+1)-2i)$$

$$f^{(2)}(x) = x^{(2n-3)/2}/2^2 \prod_{i=2}^{n-1} ((2n+1)-2i)$$

$$\vdots$$

$$f^{(n)}(x) = x^{1/2}/2^n$$

$$f^{(n+1)}(x) = x^{-1/2}/2^{n+1} .$$

And the $\lim_{x\to+\infty} f^{(n)}(x) =$ and $\lim_{x\to+\infty} f^{(n+1)}(x) = 0$. Thus, by Theorem 5 and repeated use of Lemma 3

$$S = \{(<i^{1/2}>, <i^{3/2}>, \ldots, <i^{(2n+1)/2}>) i \in N\}$$

is dense in $\overset{n}{\underset{i=0}{\times}} [0,1]_i$.

Theorem 5 may also be applied in the following way. Given a sequence of integers $\{n_i\}$, and a function f, that for some $\varepsilon > 0$ satisfies

$$n_i - 1/2 + \varepsilon < f(i) < n_i + 1/2 - \varepsilon$$

for all i, and where for some $n \in N$, $\lim_{x \to \infty} f^{(n)}(x) = 0$. Then the function $f^{(n-1)}$ is bounded and fails to satisfy the lim sup-lim inf condition of Theorem 5.

Let us now look at a few questions left unanswered by this thesis. It might be tempting to try, in some way, to move to the infinite case. There may well be some meaningful way to do this. However, there are restrictions. Thus, consider the sequence $\{k_i\}$ where

$$k_i = <i^{(2i+1)/2} + 1/2> .$$

Now we know from the special case of Example 3 that for any n-tuple $(r_1, r_2, \ldots, r_n)$, where $0 < r_j < 1$, $0 < j \leq n$, and any $\varepsilon > 0$, that there exists an integer i such that $|r_j - <i^{(2j+1)/2}>| < \varepsilon$ for all j. Looking at $(k_1, k_2, \ldots)$, we see that for any $i \in N$,

$$|k_i - \langle i^{(2i+1)/2} \rangle| = 1/2 .$$

It may be of interest to look at the case where $\lim_{x \to \infty} f^{(n)}(x) = 0$ and $\lim_{x \to \infty} f^{(n-1)}(x) = c$ for some $c \neq 0$. Then one might try to see, under what circumstances

$$\{(\langle f(i) \rangle, \langle f'(i) \rangle, \ldots, \langle f^{(n-2)}(i) \rangle \mid i \in N\}$$

is dense in the (n-1)-cube.

Finally, the whole area of uniform density in the n-cube is an open question that is of interest.

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