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Robust stability problem for nominally linear systems with nonlinear, time-variant structured uncertainties is considered. The systems are in the form

$$\dot{x}=A_n x+\sum_{i=1}^{q} p_i A_i x \ .$$

The Lyapunov direct method is utilized to determine the robustness bounds for nonlinear, time-variant uncertainies p_i. Determination of the robustness bounds consists of two principal steps: (i) generation of a Lyapunov function and (ii) determination of the bounds based on the generated Lyapunov function. Presently in robustness investigations, a Lyapunov function is generated by inserting the nominal matrix to the Lyapunov equation and setting Q as identity matrix. The objective of this study is to utilize structural features of the uncertainties to develop a recursive algorithm for the generation of the globally optimal quadratic Lyapunov function. The proposed method is seemingly an improvement with respect to those reported in recent literature in three senses: i) ease of application, given an interactive program which requires only system matrices as inputs; ii) provision of improved estimates of the robustness bounds; and iii) extendability of the procedure to the design of robust controllers. The algorithm and the program prepared (in MATLAB) are presented. Several examples are considered for purposes of the comparison of robustness bounds estimates. Examples are demonstrated to show the superiority of the robustness bounds estimated by the proposed method over those obtained by small gain theorem. In a number of cases, the estimated robustness bounds are proven to be the exact robustness bounds.

Robustness of Uncertain Systems: Globally Optimal Lyapunov Function

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CHAPTER 1

INTRODUCTION

The design and analysis of a controller is based on the mathematical model of the physical plant directed for control. This model is obtained following simplification such as lumped parameter approximations, simplified relations, linearizations about operating points, and neglecting the instrumentation uncertainties and changes in the system component properties due to time and environmental influences. Most often the nominal model will end in the form of a linear autonomous system in which uncertainties are modeled as perturbation elements. The perturbation elements can exist in form of structured or unstructured perturbations [1].

In controller design, these perturbations can be accommodated by the use of either adaptive or robust controllers. If the bounds of the perturbations are known, robust controllers are often utilized for reason of the practical advantages they offer. These are the basic facts which have motivated the design of robust controllers for multivariable linear systems [2]-[4].

The fundamental requirement for the design of robust controllers is the ability to analyze system stability and robustness. Stability analysis is concerned with the state trajectories for perturbations of an initial condition from its equilibrium point or reference trajectories. The analysis of robustness is concerned with the determination of the bounds for perturbation elements in which the system stays stable. These bounds are referred as robustness bounds.

There are two basic approaches to the analysis of system robustness, including the time domain approach, based upon state equations, and the frequency domain approach, based upon system transfer functions [5]. The most important developments in robust stability analysis and control have been achieved in H_2 and H_∞ theories within the frequency domain, where the nonsigularity of a matrix is the criterion developing the robustness bounds. Barett [6] presented a useful summary and comparison of the different robustness tests that are available with respect to their conservatism [7]. The underlying concept for these theories was explored by Zames [8], who introduced the concept of the "small gain principle,". This method is concerned only with nominally linear systems.

For the time domain problem in robust design, the application of the Lyapunov direct method has been widely investigated. This interest has been prompted by the fact that this approach provides ready accommodation for both nonlinear and time-varying systems. The drawback in the application of these methods is that their estimates of the robustness bounds are generally conservative, and conservative estimates often result in controller design distinguished by poor performance. Thus, the development of procedures for the improvement of estimation is highly desirable.

The analysis of the robust stability based upon the application of Lyapunov theorems consists of two principal steps: (i) generation of a Lyapunov function, and (ii) determination of the robustness bounds based on the generated Lyapunov function.

Mohler [9] and Schultz [10] have reviewed a variety of techniques to generate the Lyapunov function, including the Aizerman method, the variable gradient method, and the Zubov method. Based upon the Lyapunov direct method, Siljak [11] and Patel et al. [12] established procedures for the estimation of the robustness bounds. Lee [13], Yedavalli [14], Yedavalli and Liang [15], and Zhou and Khargonekar [16] further contributed and improved the robustness bounds of the perturbation elements. Siljak [4] has demonstrated that the estimation of the robustness bounds of perturbation is strongly dependent upon the selection of the system state space. Matrosov [17] and Bellman [18] introduced the concept of the vector Lyapunov functions. Further, Olas and Ahmadkhanlou [19] proposed and presented the algorithm for generation of piecewise

Lyapunov functions. Chen and Chen [20] used an optimization technique to formulate the necessary and sufficient conditions for quadratic stabilizability and found better gains for linear state feedback than those previously reported.

Based upon the small gain theorem, Qiu and Davison [5] obtained estimates of robustness bounds for the perturbation elements. Subsequently, Peterson [21] demonstrated that the conditions of small gain theorems for unstructured perturbations are necessary and sufficient for the existence of the Lyapunov function. Becker and Grimm [22] have shown that the application of the small gain theorem to systems with unstructured perturbations provides robustness bounds which are in every case greater or equal than those reported for the state transformations by Yedavalli and Liang [15].

It should be noted that in the case of structured perturbations, the conservatism of the estimates was principally caused by the failure to consider the structural features of the uncertainties when generating a Lyapunov function. The Lyapunov function was obtained by inserting the nominal matrix to the Lyapunov equation and setting Q as identity matrix.

For the current investigation, the properties of quadratic functions, system linearity, and the structure of perturbations were used to prove a theorem which has enabled the development of a recursive algorithm for the generation of a globally optimal Lyapunov function. The program developed requires only a single solution for the Lyapunov equation, followed by the recursive determination of the eigenvectors and eigenvalues for a symmetric matrix. The robustness bounds obtained by the development of these functions cannot be contained and extended by any other quadratic functions. Further improvement may be possible by consideration of piecewise Lyapunov functions [19].

In the mathematical models, the perturbations elements are modeled for different mathematical or physical causes. For example, one perturbation element may have been modeled due to a change in weight, whereas another may have been modeled due to the imprecision of its sensors. In each case, these perturbation elements were not generally with the same range of magnitude, and one magnitude range may have been significantly greater than others. Thus, to develop the algorithm, the desirable ratios for the estimated robustness bounds were selected. The proposed method provides three distinct advantages: 1) Ease of application, given an interactive program requiring only system matrices as inputs (i.e., for this study, both the algorithm and the MATLAB program are considered); 2) provision of improved means to estimate the robustness bounds; and 3) the extendibility of this procedure to the design of robust controllers. Several examples are considered to demonstrate the advantages of this method for the estimation of robustness bounds with respect to previously reported methods. Examples are demonstrated to show superiority of the robustness bounds estimated by the proposed method over those obtained by small gain theorem. In number of cases, the estimated robustness bounds are proven to be the exact robustness bounds.

Presentation of the results of this investigation is organized as follows. Chapter 2 presents a discussion of the issues of stability, and robustness is considered in Chapter 3. The program developed for the optimization of the Lyapunov function is presented in Chapter 4, followed by consideration of possible applications for the recommended approach in Chapter 5. Conclusions and recommendations are included in Chapter 6.

CHAPTER 2

STABILITY ANALYSIS

2.1 Introduction

Stability of the system is the fundamental requirement in design of control systems. In general, issues of stability are concerned with the state trajectory, when the system is perturbed from the equilibrium point or a reference trajectory. There are a number of different definitions of stability, and the underlying concept which is common to each may be described as follows: Employ some measure called the norm, which characterizes the state at any desired time; let the state whose stability is under investigation be perturbed, then define measures for perturbation as well as for the norm. From this concept, it follows that stability may be defined as follows: If the perturbation does not exceed the defined measure, then the unperturbed state is stable when the change in the norm caused by the perturbation does not exceed its established measure. The specific definition of Lyapunov stability for an equilibrium point is given in section 2.3.

From engineering point of view, these analyses are important because of state perturbations caused by the existence of such external disturbances as noise and environmental changes around the equilibrium points [23]. In nonlinear time-variant systems, one of the tasks in stability analysis is to determine the region of stability. So long as the system is operated within this region, the stability of the system is assured.

2.2 Stability Analysis for Time-Variant Systems

Stability analysis may be conducted in either the time or frequency domains. If the system is linear time-invariant, stability analysis may by established by the Routh-Hurwits or Nyquist criteria. Unfortunately, methods that have proved to be so useful for the estimation of stability for autonomous linear systems cannot be applied directly to nonlinear time-variant cases. The following examples provided by DeCarlo in [24] demonstrate this fact:

Example 1. Consider the time-varying linear system

$$\dot{x} = A(t)\ x(t)$$

with

$$A(t) = \begin{bmatrix} -1-9\cos^2(6t) + 12\sin(6t)\cos(6t) & 12\cos^2(6t) + 9\sin(6t)\cos(6t) \\ -12\sin^2(6t) + 9\sin(6t)\cos(6t) & -1-9\sin^2(6t)-12\sin(6t)\cos(6t) \end{bmatrix}$$

where the eigenvalues of A(t) are -1 and -10 for all t. However the state transition matrix is

$$\phi(t,0) = 0.2\begin{bmatrix} \cos(6t) + 2\sin(6t) & 2\cos(6t) - \sin(6t) \\ 2\cos(6t) - \sin(6t) & -\cos(6t) - 2\sin(6t) \end{bmatrix}\begin{bmatrix} e^{2t} & 2e^{2t} \\ 2e^{-13t} & -e^{-13t} \end{bmatrix}$$

For an initial condition (e.g., $x(0) = (1,0)^T$), the term e^{2t} causes an unbounded zero-input response. Thus, the location of the eigenvalues of the matrix A in the left-half plane do not imply stability for time-variant cases.

Example 2. Consider the time-varying linear system

$$\dot{x} = A(t)\ x(t)$$

with

$$A(t) = \begin{bmatrix} -5.5 + 7.5\sin(12t) & 7.5\cos(12t) \\ 7.5\cos(12t) & -5.5-7.5\sin(12t) \end{bmatrix}$$

where the eigenvalues of A(t) are 2 and -13 for all t. However the state transition matrix is

$$\phi(t,0) = \begin{bmatrix} \cos(6t) + 3\sin(6t) & \cos(6t) - 3\sin(6t) \\ 3\cos(6t) - \sin(6t) & -3\cos(6t) - \sin(6t) \end{bmatrix} \begin{bmatrix} 0.5e^{-t} & (1/6)e^{-t} \\ 0.5e^{-10t} & -(1/6)e^{-10t} \end{bmatrix}.$$

Thus, the presence of the positive eigenvalue at 2 cannot imply instability. In the following section, the various concepts of Lyapunov stability are examined.

2.3 Stability in the Sense of Lyapunov

If the solutions for the state equations are available, it is easy to determine stability for a particular case. However, solving the nonlinear differential equations is frequently a difficult or impossible task. The objective of Lyapunov stability theorems is to analyze system stability in the absence of the knowledge of solutions to the system differential equations. In theory, an isolated (i.e., zero-input) system remains in the equilibrium state if that is where it initially started. In this sense, Lyapunov stability is concerned with the behavior of the system trajectories when the initial state is near the equilibrium point. As mentioned earlier, the results of this analysis are important because of the existence of such external disturbances as noise and environmental influences. Initially, Lyapunov stability theorems have been established for perturbations of initial condition near an equilibrium point. However, as explained in following chapter on issues of robustness, these theorems can be extended and thus applied in the case of system parameter perturbations.

The underlying concept for the Lyapunov theorems is as follows: Consider a system with no external forces acting upon it. If **0** is one of the system equilibrium points, it may be assumed that it is possible to define a function which represents the total energy of the system, such that it is equal to zero at the point of origin and positive elsewhere; if the system dynamics are such that the energy of the system is nonincreasing over time, dependent upon the nature of the energy function, the stability of equilibrium point **0** may be implied. The virtue of the Lyapunov theorem has been to employ this concept in a mathematical form [23].

2.3.1 Basic Definitions

Consider the vector differential equation

$$\dot{x} = f(x, t) . \tag{2-1}$$

Then, assume that 0 is an equilibrium point of the system equation (2-1), which may be done since the equilibrium point can always be transferred by a simple transformation of the states. As described by Vidyasagar [25], the basic definitions of stability for the equilibrium points are as follows:

Definition 1: The equilibrium point 0 at time t_o is said to be stable if, for each $\varepsilon > 0$, there exists a $\delta(t_o,\varepsilon)>0$, such that

$$\| x(t_o)\| <\delta(t_o,\varepsilon) \Rightarrow \| x(t)\| <\varepsilon \ , \qquad \forall t\geq t_o.$$

It is said to be uniformly stable over $[t_o,\infty)$ if, for each $\varepsilon>0$, there exists $\delta(\varepsilon)>0$ such that

$$\| x(t_1)\| <\delta(\varepsilon),\ t_1\geq t_o \Rightarrow \| x(t)\| <\varepsilon \ , \qquad \forall t\geq t_1.$$

Definition 2: The equilibrium point 0 at time t_o is unstable if it is not stable at t_o.

Definition 3: The equilibrium point 0 at time t_o is asymptotically stable at t_o if (1) it is stable at time t_o, and (2) there exists a number $\delta_1(t_o)>0$ such that

$$\| x(t_o)\| <\delta_1(t_o) \Rightarrow \| x(t)\| \dashrightarrow 0 \ , \qquad \text{as } t\dashrightarrow\infty.$$

It is uniformly asymptotically stable over $[t_o,\infty)$ if (1) it is uniformly stable over $[t_o,\infty)$, and (2) there exists a number $\delta_1>0$ such that

$$\| x(t_1)\| <\delta_1,\ t_1\geq t_o \Rightarrow \| x(t)\| \dashrightarrow 0 \ , \qquad \text{as } t\dashrightarrow\infty.$$

Definition 4: The equilibrium point 0 at time t_o is globally asymptotically stable if it is asymptotically stable for all initial states (i.e.,

$x(t) \to 0$ as $t \to \infty$ regardless of $x(t_o)$); thus, if **0** is a globally asymptotically stable equilibrium point at time t_o for a given system, then it should be the only equilibrium point at time t_o.

2.3.2 Lyapunov Stability Theorems

The basic stability theorems for the Lyapunov direct method, as formulated by Vidyasagar [25], are as follows. Let

$$\dot{x} = f(x,t), \quad \text{where} \quad f(0,t) = 0 \;\; \forall\, t$$

describe a given system equation. It follows that:

Theorem 2.1: The equilibrium point **0** at time t_o is stable if there exists a continuously differentiable local positive definite function (l.p.d.f.) $V(x,t)$ such that

$$\dot{V}(x,t) \le 0, \quad \forall\, t \ge t_o, \quad \forall\, x \in B_r \text{ for some ball } B_r.$$

If $V(x,t)$ is a decrescent locally positive definite function in Theorem 2.1, the equilibrium point **0** at time t_o is said to be uniformly stable over $[t_o,\infty)$.

Theorem 2.2: The equilibrium point **0** at time t_o for the system is asymptotically stable over the interval $[t_o,\infty]$ if there exists a continuously differentiable l.p.d.f. $V(x,t)$ such that $-\dot{V}(x,t)$ is a l.p.d.f.

Theorem 2.3: The equilibrium point **0** at time t_o is globally asymptotically stable if there exists a continuously differentiable decrescent p.d.f. $V(x,t)$ such that

$$\dot{V}(x,t) \le -G(\|x\|) \quad \forall\ t \ge t_o, \quad \forall\ x \in R^n ,$$

where G is a function belonging to class K.

Theorem 2.4: The equilibrium point **0** at time t_o is unstable if there exists a continuously differentiable decrescent function $V(x,t)$ such that (i) $\dot{V}(x,t)$ is an l.p.d.f., and (ii) $V(\mathbf{0},t)=0$, and there exists points x arbitrary close to **0** such that $V(x,t_o)>0$.

Clearly, the advantage of the Lyapunov stability theorems is that they do not require solution of the state equations; in contrast, they are disadvantaged in that only sufficient condition are provided. If a particular function fails to satisfy all of the conditions, then no conclusions can be drawn and another function candidate should be attempted. For this reason, a function is referred to as a Lyapunov candidate when subject to testing under the conditions described above; if all of the conditions for one of the theorems can be satisfied, then it may be termed a Lyapunov function [23]. Thus, the principal drawback of the Lyapunov theory is that there is no general procedure for generating a Lyapunov function. However, though it is difficult to find a Lyapunov function for a given system, the choice of a Lyapunov function is relatively easy in the case of linear or weakly nonlinear systems.

The objective of the current investigation is to present a simple algorithm for the generation of the globally optimal Lyapunov function for nominally linear autonomous systems with nonlinear, time-varying structured perturbations.

2.4 Linear Autonomous Systems

Consider the following linear autonomous system

$$\dot{x} = A\, x.$$

Usually the selected Lyapunov function is in the quadratic form

$$V(x) = x^T S\, x\ , \tag{2-2}$$

where S is a positive definite symmetric matrix. The class of quadratic Lyapunov functions is often used as a part of the Lyapunov function vector, or as the function itself. This results from the properties of quadratic functions and the fact that general energy functions are of the form of quadratic functions, such as "Kinetic Energy = 1/2 mv^2." The derivative of V(x) along the solution of the system is

$$\begin{aligned} \dot{V} &= \dot{x}^T S x + x^T S \dot{x} = (Ax)^T S x + x^T S A x \\ &= x^T A^T S x + x^T S A x \\ &= x^T(A^T S + SA)x. \end{aligned}$$

Consider the equation

$$A^T S + S A = - Q . \tag{2-3}$$

Theorem 2.5: Equation (2-3) provides a unique solution for S corresponding to every $Q \in R^{nxn}$ if and only if

$$\lambda_i + \lambda_j^* \neq 0 , \quad \forall \ i,j ,$$

where $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of A and * denotes a complex conjugate.

Theorem 2.6: The system is asymptotically stable if and only if for every positive definite matrix Q there exists a unique solution for S and this solution is positive definite.

The proofs for Theorems 2.5 and 2.6 may be found, respectively, in Chen [26] and Vidyasagar [25]. It is common to select Q as a positive identity matrix, I, and to solve for S. The system is asymptotically stable if and only if the solution of S is unique and is positive definite. Therefore, it is evident that the conditions for the Lyapunov theorem 2.6 are necessary and sufficient in the linear autonomous case [9]. This conclusion plays a key role in the generation of the Lyapunov function and the design of robust controllers.

2.5 Lyapunov Function Generation

Of the different techniques for the generation of the Lyapunov function, the most important factor is to determine a function which provides the least conservative results. In the case of stability analysis, conservatism of results is referred to the estimated size of regions of stability for state perturbations around equilibrium points or reference trajectories. However, for the analysis of robust stability, the conservatism refers to the estimated size of the robustness bounds. To determine less conservative estimates, the nominal part of the system as well as the structure of the perturbation elements must be considered when generating the Lyapunov functions. Mohler [9] and Schultz [10] have demonstrated a variety of techniques for the generation of a Lyapunov function, two of which are considered in the following sections.

2.5.1 The Aizerman Method

In the Aizerman method, the system is first linearized at **0**, the linear part of system is then used to generate the Lyapunov function. For small perturbations, the linear part is dominant. Hence, the generated Lyapunov function can then be used to determine the range of the perturbations so that Q remains positive definite. Since the structure of the perturbation elements is not considered for this method, the results generated are generally conservative. Despite of this, in the case of linear autonomous systems with unstructured as well as structured perturbations, this method has been given considerable practice.

2.5.2 Variable-Gradient Method

In 1962, Schultz [10] and Schultz and Gibson [27] illustrated this method for the construction of nonlinear autonomous system Lyapunov functions. The methodological concept is to start from a general variable gradient function. A positive definite Lyapunov function can then be determined from the constraints on the $\dot{V}$ and the n(n-1)/2 generalized curl equations. In autonomous nonlinear systems, $\dot{V}$ is equal to

$$\dot{V} = \frac{dV}{dt} = \frac{\partial V}{\partial t} \,.\, f(x) = \nabla^T V\, f(x) \;.$$

Substituting $\dot{x}$ for f(x), V(x) can be obtained from

$$V = \int_0^x \nabla^T V(\sigma)\, d\sigma \;.$$

The complete procedure is as follows:

1) Assume ∇V to be a column vector whose coefficients are functions of the states such that $\nabla V = D(x)\, x$, where D(x) is an nxn matrix whose elements are

$$D_{ij}(x) = \alpha_{ijk} + \alpha_{ijv}(x) \;,$$

and where α_{ijk} is the constant and $\alpha_{ijv}(x)$ is the variable part of $D_{ij}(x)$. Since any constant V(x) represents a closed surface, D(.) is chosen independent of x_n by setting α_{nnv} at zero.

2) V(x) is obtained from the line integral of ∇V . This integral can be calculated independent of the path of integration, the simplest of which is

$$V(x) = \int_0^{x_1} \nabla_1 V \, d\sigma_1 + \int_0^{x_2} \nabla_2 V \, d\sigma_2 + \ldots + \int_0^{x_n} \nabla_n V \, d\sigma_n ,$$

where the following curl equations should satisfy

$$\nabla_i V = \frac{\partial V(x)}{\partial (x_i)} \; ; \; \frac{\partial \nabla_i V}{\partial x_j} = \frac{\partial \nabla_j V}{\partial x_i} \; , \; i \neq j \; , \qquad i.j = 1, \ldots, n \; .$$

3) Check that $\dot{V}(x)$ is constrained to be at least semi-definite, with a definiteness opposite to that of V(x).

CHAPTER 3

ROBUST STABILITY ANALYSIS

3.1 Introduction

The design and analysis of a controller is based on the mathematical model of the physical plant directed for control. As mentioned earlier, the mathematical model is not exact. Inaccuracies are modeled in the form of structured or unstructured perturbations. In controller design, these perturbations can be accommodated by the use of either adaptive or robust controllers. If the bounds of the perturbations are known, robust controllers are often utilized for reason of the practical advantages they offer. One of the fundamental requirement in the design of robust controllers is the ability to analyze system robustness.

The most important developments in robust stability analysis and control are the H_2 and H_∞ theories, developed from the "small gain principle" introduced by Zames [8]. Levine and Reichert [28] have provided an introduction to H_∞ system control design, and Francis [29] has also contributed an excellent introduction to H_∞ theory. Most of the investigations of this subject have been based upon transfer function representation. Qui and Davison [5] used the small gain theorem to consider the robust stability of linear time-invariant systems for state space models, formulating estimates of the robustness bounds for both structured and unstructured perturbations. This method was subsequently extended in their two later reports [30] and [31].

For the time domain, the Lyapunov direct method has been widely used for the investigation of system robustness. The interest has been prompted by the fact that this method provides ready accommodation for both nonlinear and time-variant systems. It should be noted that the small gain theorem is concerned only with nominally linear systems. This is an important factor since the solution of nonlinear differential equations can be a difficult to impossible task.

For robustness, the application of the Lyapunov direct method consists of two principal steps: (i) generation of the Lyapunov function and (ii) the determination of the robustness bounds based on the generated Lyapunov function. In Section 2.5, a selection of methods for the generation of the Lyapunov function was reviewed. Patel and Toda [12] considered linear autonomous systems with nonlinear, time-varying unstructured vector perturbations and unstructured perturbations, and formulated estimates for the robustness bounds. Lee [13] improved the unstructured robustness bounds for the systems with stable polar decomposition. Yedavalli [14] improved the accuracy of these estimates when considering structured perturbations. The bounds obtained by the application of these methods were not directly dependent on the structure of the nominal matrix. Yedavalli and Liang [15] obtained improvements for the estimation of the bounds by transformation of the states. For the case of structured perturbations, Zhou and Khargonekar [16] improved the robustness bounds by separating independent perturbation elements within the perturbation matrix. Juang [32] considered robustness for linear time variant systems, including linear autonomous systems with time-varying perturbations as a special case. Siljak [4] suggested the use of a vector Lyapunov function introduced by Matrosov [17] and Bellman [18], to reduce the conservatism of the estimates.

However, the referred estimates remained more conservative than those obtained from the use of the small gain theorem [5] and [22]. In the case of structured perturbations, the conservatism of the estimates was principally caused by the failure to consider the structural features for the perturbations when generating a Lyapunov function. The Lyapunov function was generated by inserting the nominal matrix to the Lyapunov equation and setting Q as identity matrix. Further Radziszewski [33], in the examination of two-dimensional structured systems, discussed the candidature of quadratic forms as a class of Lyapunov functions and determination of the best Lyapunov functions. It was determined that the estimates of the robustness bounds obtained by the best Lyapunov function of this class were still less than those obtained by some other methods for evaluation of two dimensional systems. Olas and Ahmadkhanlou [19] have proposed a piecewise Lyapunov function for the improvement of estimates for the robustness bounds.

The remainder of this chapter is organized as follows: Section 3.1 provides explanations of the basic differences between the analyses of robustness for structured and unstructured perturbations; Sections 3.2 and 3.3 provide explanation for applications of, respectively, the Lyapunov direct method and the small gain theorem to problems of robust stability analysis.

3.2 Structured and Unstructured Perturbations

In general, structured and unstructured perturbations are the two types of perturbations distinguished in the development of robustness theory. Their existence is dependent upon the physics of the physical plant under consideration. A state matrix for the plant perturbation may be represented by the summation of a nominal fixed matrix and a perturbation matrix, that is,

$$A(t) = A_N + \Delta A(t) \ . \tag{3-1}$$

Most engineering plants such as aircraft or robot can be described with known dynamical equations. The design uncertainties which exist are with regard to the values of specific physical system parameters. Examples of structured perturbations in aircraft models include the parameter values for the spring constant, mass, inertia, aerodynamic coefficients, and changes in air pressure. These values cannot be considered as constant known values, but they affect only specific system parameters [1].

Unstructured perturbations are modeled in cases such as unmodeled dynamics. Modeling continuous systems as finite lumped masses is one of the examples of unmodeled dynamics. In the unstructured perturbations, only the norm ΔA is specified. When possible, perturbations elements should be modeled as structured perturbations since less conservative estimates may be then obtained.

These are the basic facts which have motivated growing interest in the robust control of systems with structured perturbation [1]. In the case of structured perturbations, the system matrix is usually is written in the following form:

$$\dot{x} = A_N x + \sum_{i=1}^{q} p_i A_i x_i , \tag{3-2}$$

where p_i is a perturbation element also called parameter perturbation and A_i is a constant matrix called i^{th} perturbation matrix. The advantage of this form is that it separates each of the independent perturbation parameters from the others (see section 3.3.5).

3.3 Application of the Lyapunov Direct Method

As mentioned earlier, the Lyapunov stability theorems have been first established for the perturbations of initial conditions near an equilibrium point. These theorems were subsequently extended for application to perturbations of the system parameters [23].

The following conclusion may be derived from the Lyapunov direct method:

> **CONCLUSION:** If for a system, there exists a single Lyapunov function for all choices of the perturbation parameters within a compact bounded set, the stability of **0** equilibrium of the system for the nonlinear, time-variant perturbations so bounded is insured.

The remainder of this section reviews selected research on the robust stability analysis of nominally autonomous linear systems with nonlinear, time-variant perturbations.

3.3.1 Robustness Bounds, Patel and Toda

Patel and Toda [12], in an extension of their paper on robustness analysis for linear state feedback design [34], considered nonlinear unstructured vector perturbations and unstructured perturbations for nominally autonomous linear systems.

1) Nonlinear unstructured vector perturbations:

The following system

$$\dot{x} = A_N x + f(x,t) \quad , \tag{3-3}$$

where A_N is a time-invariant asymptotically stable matrix and f(x,t) is a time varying nonlinear vector function of x(t), representing the nonlinear unstructured vector perturbations within the system and f(0,t)= 0 for all times, was considered. It was demonstrated that the system would be stable if

$$\frac{\|f(x,t)\|}{\|x\|} \le \frac{\min\lambda(Q)}{\max\lambda(S)} \triangle \mu_p \ , \tag{3-4}$$

where S is the unique positive definite solution of the Lyapunov equation

$$A_N^T S + S A_N = -2\,Q \ , \tag{3-5}$$

where Q is a positive definite matrix. Further, it was proved that for unstructured perturbations, the robustness bound μ_p in equation (3-4) is maximum when the matrix Q=I, thus

$$\mu_p = \frac{1}{\sigma_{\max}(S)} \ , \tag{3-6}$$

where $\sigma_{max}(S)$ is the maximum singular value of S. It was then demonstrated that

$$\mu_p = \frac{1}{\sigma_{\max}(S)} \le \min \mid \lambda(A_N) \mid \tag{3.7}$$

and that equality is maintained when A_N is a normal matrix. This indicates that the robustness bounds of the unstructured perturbations is less than or equal to the dominant eigenvalue of the system's nominal matrix.

2) Unstructured perturbations:

For the case of unstructured perturbations where

$$\dot{x}(t) = A_N x(t) + \Delta A(t)\, x(t) \,, \tag{3-8}$$

it was reported that the system is stable if

$$|\Delta A_{ij}(t)| \le \frac{1}{n\, \sigma_{\max}(S)} \triangleq \mu_{PU} \,, \tag{3-9}$$

where $\Delta A_{ij}(t)$ is the $(i,j)^{th}$ element of $\Delta A(t)$.

3.3.2 Robustness Bounds, Lee

Lee [11] considered unstructured perturbations as follows. Assuming the existence of a stable orthogonal matrix U for the polar decomposition of A,

$$A_N = U H_R \qquad \text{or} \qquad A_N = H_L U \,, \tag{3-10}$$

it was proved that the system is stable if

$$\sigma_{\max}(\Delta A) < -\sigma_{\min}(A) \cos(\theta_{\min}) \triangleq \mu_{LU} \,, \tag{3-11}$$

where θ_{min} is the smallest principal phase of A measured counter-clockwise from the positive real axis. Principal phase of A in (3-10) is defined as the arguments of the eigenvalues of U in (3-10). If A is normal, then

$$\mu_{LU} = \min |\, Re\, \lambda(A)\, |. \tag{3-12}$$

3.3.3 Robustness Bounds, Yedavalli

Yedavalli [14] considered structured as well as unstructured perturbations.

1) Structured perturbations:

Estimates for the robustness bounds were improved for the case of structured perturbations. It was demonstrated that the system will remain stable if

$$\varepsilon \leq \frac{1}{\sigma_{\max}[\,|S|\,U_a]_s} \triangleq \mu_{ys} , \tag{3-13}$$

where | . | is the modulus matrix, $[\cdot]_s = [\ (\cdot)+(\cdot)^T\ ]^{1/2}$ is the symetric part of the matrix, and U_a is an n x n matrix whose entries are such that

$$U_{aij} = \frac{\Delta a_{ij}}{\varepsilon} \qquad and \qquad \varepsilon \triangleq \max\ (\Delta a_{ij}) \ . \tag{3-14}$$

2) Unstructured perturbations:

Yedavalli further proved that for unstructured perturbations, the bound

$$\sigma_{\max}(\Delta A) < \sigma_{\min}(A_N) \triangleq \mu_{yu} \tag{3-15}$$

insures the stability of the system. Note that the advantage of this expression is that it does not require solution of the Lyapunov equation. In contrary, for the case when A_N is not normal, the methods considered in Sections 3.3.1 and 3.3.2 require solution of the Lyapunov equation or a requirement that the polar decomposition U is stable.

Estimates for the upper bounds of perturbation elements presented by Patel and Toda [12] and Yedavalli [14] were not directly related to the structure of the nominal system matrix (e.g., equations (3-9) and (3-13)), rather they were indirectly affected through the matrix S.

3.3.4 Robustness Bounds, Yedavalli and Liang

Based upon the fact that the stability of a system is invariant with respect to nonsingular linear transformation, Yedavalli and Liang [15] transformed the state vector through M, $x = M\hat{x}$ for the new system

$$\dot{\hat{x}} = \hat{A}(t)\ \hat{x}(t) \qquad \text{where} \quad \hat{A} = M^{-1}\, A(t)\ M\,. \tag{3-16}$$

By changing the system matrix to $\hat{A}$ in the Lyapunov equation while maintaining Q = I, $\sigma(S)$ may be reduced which results in improvement of robustness bounds μ_P (see equations (3-9) and (3-13)). Examples were presented to demonstrate improvement of the bounds, with respect to those achieved by Patel and Toda [12], for the structured as well as unstructured perturbations. However, with the exception of a special case limited to the diagonal transformation matrix, the question of generating the matrix M remained unsolved.

3.3.5 Robustness Bounds, Zhou and Khargonekar

Zhou and Khargonekar [16] considered the structured perturbations. Previous research had been based upon the assumption that the various elements of the system matrix were perturbed independently. However, it was considered that for the practical case in many systems, the entries for the perturbation matrix may be dependent. For example, if

$$\Delta A = \begin{bmatrix} p_1 + \alpha\, p_2 & \beta\, p_1 \\ \gamma\, p_2 & 0 \end{bmatrix},$$

where α, β, and γ are constants, ΔA_{11} is a linear combination of ΔA_{12} and ΔA_{21}. Thus, the following general form was suggested:

$$\dot{x} = A_N\, x + \sum_{i=1}^{q} p_i\, A_i\, x,$$

where p_i is a perturbation element (parameter perturbation) and A_i represents a constant matrix called i^{th} perturbation matrix.

1) Structured perturbations:

For the case where p_i varies in the interval around zero, that is, $p_i \in [-p^-_i\ ,\ p^+_i\ ]$, it was demonstrated that the system was stable if one of the following conditions is true:

$$i) \qquad \sum_{i=1}^{q} p_i^2 < \frac{1}{\sigma^2_{\max}(S_e)},$$

$$ii) \qquad \sum_{i=1}^{q} |\, p_i \,|\ \sigma_{\max}(S_i) < 1\ ,$$

$$iii) \qquad |\, p_j \,| < \frac{1}{\sigma_{\max}(\sum_{i=1}^{q} |\ S_i\ |)}, \qquad j=1,2.....q\ ,$$

$$(3\text{-}17)$$

where

$$S_i = (A_i^T S + S A_i) / 2 , \qquad i = 1,2,\ldots q .$$
$$S_e \triangleq [S_1 \quad S_2 \ldots\ldots S_q]$$

These bounds were less conservative than those considered prior to this formulation. Yedavalli's [14] bounds were a special case of the condition (iii).

3.3.6 Robustness Bounds, Juang

Juang [30] considered the system

$$\dot{X} = \sum_{i=1}^{q} p_i(t)\; A_i\; X$$
$$\text{where } p_i^- \le p_i(t) \le p_i^+ , \qquad \sum_{i=1}^{q} |p_i(t)| \ne 0 \tag{3-18}$$

as a special case for the robustness analysis of autonomous linear systems with structured perturbations. Note that for $p_i(t)=1$, the system is identical to that represented by Zhou and Khargonekar [16].

1) Structured perturbations:

Defining

$$\nu_k = \sum_{i=1}^{q} p_i(t)\; A_i \,\Big|_{p_i(t)\,=p_i^+ \text{ or } p_i^-} , \qquad k=1,2,\ldots 2^q , \tag{3-19}$$

it was demonstrated that the system is stable if an invertible matrix S existed such that $\mu_2(S\ \upsilon_k\ S^{-1}) < 0$ for all $k=1,\ldots 2^q$, where $\mu_2(.)$ denotes the matrix measure [25] corresponding to a 2- norm. As before, this approach left the issue of the generation of Lyapunov function an open question, where the Lyapunov function is $V(x) = x^T S^* S x$ and S^* is the complex conjugate of S.

3.3.7 Vector Lyapunov Functions

The concept of vector Lyapunov function was introduced by Matrosov [17] and Bellman [18]. This concept associates several scalar functions with a given dynamic system in such a way that each function determines a desirable stability property in a part of the state where others do not. These scalar functions are considered as components of a vector Lyapunov function. The next step is to determine stability for the entire system, or its "connective stability." For more detail see Siljak [35].

Siljak [35] considered the system

$$S_E : \quad \dot{x}_i = A_i x_i + \sum_{j=1}^{N} p_{ij} A_{ij} x_j , \qquad i = 1, . . , N, \tag{3-20}$$

to be an interconnection of N subsystems

$$S_i : \qquad \dot{x}_i = A_i x_i , \tag{3-21}$$

where A_i is a negative definite matrix and $|p_{ij}| < p_{ij}^+$ is the perturbation element. (3-21) implies that the couplings between the subsystems consist only of the perturbation elements, and the perturbation within each subsystem is unstructured. Let $V_i(x)$ represent the Lyapunov function for the i^{th} subsystem S_i

$$V_i(X_i) = (X_i^T S_i X_i)^{1/2} , \tag{3-22}$$

where S_i is the symmetric positive definite solution for the Lyapunov equation. The Lyapunov function for the overall system S_E is selected as

$$V(X) = d^T V(X) , \tag{3-23}$$

where $d \in R_+^N$ is a positive vector. It was then demonstrated that the overall system, S_E, is connectively stable if the following matrix is an M matrix, i.e.

$$\begin{bmatrix} w_{11} & w_{12} & \cdots & w_1k \\ w_{21} & w_{22} & \cdots & w_2k \\ \cdots & \cdots & \cdots & \cdots \\ w_{k1} & w_{k2} & \cdots & w_{kk} \end{bmatrix} > 0 , \qquad k = 1, \ldots, N , \tag{3-24}$$

where

$$w_{ij} = \begin{cases} \dfrac{1}{2} \dfrac{\lambda_m(Q_i)}{\lambda_M(H_i)} - p_{ii}^{+} \sigma_i & i = j \\ -p_{ij}^{+} \sigma_{ij} & i \neq j , \end{cases} \tag{3-25}$$

and $\sigma_{ij} = \lambda_M(A_{ij}^T A_{ij})$. As may be seen from (3-25), Q_i may be set as the solution to the following problem to maximize chances of proving stability

$$\begin{aligned} find : \quad & MAX_{Q_i} \frac{\lambda_m(Q_i)}{\lambda_M(S_i)} \\ subject: \quad & A_i^T S_i + S_i A_i = -Q_i . \end{aligned} \tag{3-26}$$

Based upon the assumption that A_i has all distinctive eigenvalues, the maximum value of the ratio found to be

$$\max_{Q_i} \frac{\lambda_m(Q_i)}{\lambda_M(S_i)} = \sigma_M(A_i)\ .$$

For the special case were S_E is reduced to a single subsystem

$$S_E : \quad \dot{x} = A_N\, x + \Delta A\, x\ , \tag{3-27}$$

the system is stable if

$$\sigma\,(\Delta A) < \frac{\lambda_m(Q)}{\lambda_M(S)}\ ,$$

a result identical to that achieved by Patel and Toda [12] for the case of unstructured perturbations.

3.3.8 Piecewise Lyapunov Functions

Olas and Ahmadkhanlou [19] have proposed an algorithm for the generation of piecewise Lyapunov functions, demonstrating by example how this method can be used to improve estimates for the robustness bounds in nominally linear systems with structured perturbations.

With the exception of the report by Olas and Ahmadkhanlou [19], none of the research reviewed in Section 3.3 has used the structural features of the perturbation elements to generate an improved Lyapunov function with less conservative results. However this information was used in determining the robustness bounds based on the generated Lyapunov function.

3.4 Application of the Small Gain Theorem

For the frequency domain, small gain theorem is used extensively to determine the robustness of the nominally linear systems with perturbations elements.

3.4.1 Robustness Bounds, Qiu and Davison

Qiu and Davison [5] demonstrated examples showing that the small gain theorem provides better estimates of the robustness bounds than those developed earlier from the application of Lyapunov direct method.

1) Unstructured perturbations:

The class of systems considered was identical to the systems considered in Section 3.3. For unstructured perturbations,

$$\dot{x} = A\,x + \Delta A\,x\,,$$

it was shown that the system would be stable if

$$\|\Delta A\|_p < \frac{1}{sup_{\omega\geq 0}\|(j\omega I-A)^{-1}\|_p} \triangleq \mu_{QU}\,, \tag{3-28}$$

where $\|\ .\ \|_p$ is the p-norm of (.). The robustness bounds described by equation (3-28) were either larger than or equal to those reported earlier using Lyapunov stability analysis.

2) Structured perturbations:

For structured perturbations, it was assumed that ΔA has the structure

$$\Delta A = S_1 \, \Delta E \, S_2 \, , \tag{3-29}$$

where S_1 and S_2 are known constant matrices, and ΔE represents the perturbation matrix. This structure was selected for the practical reasons, e.g. the perturbation of sensors/actuators for a closed-loop system can be represented in this form. The following bounds,

$$\epsilon < \frac{1}{sup_{\omega \geq 0} \pi \; [|S_2 \; (j \; \omega \; - \; A)^{-1} \; S_1| \; U_e \;]} \; \Delta\mu_{QS}, \tag{3-30}$$

where $\pi(.)$, called the Perron-eigenvalue, is a real, positive eigenvalue of the positive matrix (.) which is greater than or equal to the moduli of all of the other eigenvalues of matrix (.), and U_e is as defined in Section 3.2.3.

EXAMPLE: The following system with structural perturbations ΔE

$$\dot{x} = A \, x + \Delta E \, x \, , \quad \textit{where} \quad A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}$$

was considered for different forms of ΔE. For the case where ΔE is

$$\Delta E = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & 0 \end{bmatrix} \quad \textit{where} \quad | \, \varepsilon_{11} \, | = | \, \varepsilon_{12} \, | = | \, \varepsilon_{21} \, | \leq \varepsilon \, ,$$

the matrix U will thus be

$$U = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} .$$

From equation (3-30), the estimates of the bounds are

$$| \varepsilon_{11} | = | \varepsilon_{12} | = | \varepsilon_{21} | < \varepsilon = 0.6848 ,$$

compared to the bounds 0.317 obtained by Yedavalli [12]. In section 5.3, additional comparisons of the estimates obtained by Yedavalli [7], Qiu and Davison [5], globally optimal Lyapunov functions, and the exact bounds are provided. For examples of unstructured perturbations, see Qiu and Davison [5].

3.4.2 Comparison of Robustness Bounds Obtained by the Small Gain Theorem and the Lyapunov Theorem for Unstructured Perturbations

For unstructured perturbations, Peterson [21] demonstrated that the conditions for the small gain theorem were necessary and sufficient for existence of a Lyapunov function which provides the same robustness bounds. Becker and Grimm [22] considered the same case, demonstrating that the estimates determined by the application of this method would always be larger than or equal to those determined by any state transformations in the time domain presented in Yedavalli and Liang [15].

Khargonekar et al. [36] expanded Peterson's [21] results for quadratic stabilizability, based upon the principle of linear output feedback. The results in [21] were expressed in the form of the following corollary.

COROLLARY: Consider the system with unstructured perturbations as described by Qiu and Davison

[4]. There exists a positive definite matrix S such that

$$(A + \Delta A)^T S + S\,(A + \Delta A) < 0\ , \qquad \forall\, \Delta A\ ,$$

if and only if the conditions of the small gain theorem are satisfied.

This corollary concludes that in the case of unstructured perturbations, there exists no quadratic Lyapunov function which provides better estimate of the robustness bounds than small gain theorem. Note that from this approach, the question of structured perturbations remained an open issue.

CHAPTER 4

GLOBALLY OPTIMAL LYAPUNOV FUNCTIONS

4.1 Introduction

Lyapunov stability theorems have been widely used in robust controller design and robust stability analysis. [14] Design of robust controller based upon application of Lyapunov theorems was initiated by Barnett and Story [37], Bellman [38], Desoer et al. [3], Davison [39], Ackerman [40], Barmish [41], and Eslami [42]. Consideration of explicit robustness bounds for linear systems with non-linear, time-variant perturbations was explored by Siljak [11], Patel et al. [34], Patel and Toda [12], Lee [13], Yedavalli [7,14], Yedavalli and Liang [15], Zhou and Khar-konegar [16], Siljak [4], Juang[32], Olas and Ahmadkhanlou [19]. Application of the Lyapunov direct method to robust stability analysis consists of two principal steps: (i) generation of the Lyapunov function and (ii) determination of the robustness bounds based on the generated Lyapunov function.

The most important factor in the selection of the Lyapunov function is to provide the least conservative estimates of the robustness bounds. In the Chapter 3, previous reports on the robust stability of linear systems with nonlinear, time-variant perturbation elements were considered. It was noted that the Aizerman method was used to generate the Lyapunov function. Since structure of the perturbation matrices were not considered, the Lyapunov functions obtained by this method are generally not good choices and result in conservative estimates. However the structural features of the perturbation elements were used in estimating the robustness bounds based on the generated Lyapunov function.

For the system to remain stable, the derivative of Lyapunov function along the solution of states should remain non-positive. Maximum and minimum eigenvalues and singular values for each term of the Lyapunov derivative are commonly used to determine estimates for the perturbation bounds. However, the maximum values for each term of the Lyapunov derivative do not necessarily occur at the same vector. Thus, in the algorithm presented for consideration in this study, the maximum eigenvalues of the derivative of the Lyapunov function (i.e., the matrix Q) for the worst cases of the uncertainties were considered. The properties of the quadratic functions were then employed to generate the globally optimal Lyapunov function and obtain the robustness bounds.

4.2 Problem Statement

Consider a nominally linear system with the structured perturbation

$$\dot{x} = A_N x + \sum_{i=1}^{q} p_i A_i x \tag{4-1}$$

where $p_i = p_i(x,t)$, p_i is assumed to be bounded and to fulfill conditions on existence and uniqueness for solutions to (4-1) and A_N has negative real parts for its eigenvalues. The system may of course be of the closed loop form (A-BK) as in section 5.5. Let $\mathbf{p} \triangleq [p_1,\ldots,p_q]^T$, and $A_U = \sum_{i=1}^{q} p_i A_i$. The general robust stability problem for (4-1) is defined as to determine the set **E** belonging to the parameter space R^q such that if $\mathbf{p}(x,t) \in \mathbf{E}$ for all x,t, then the system (4-1) is stable in the sense of Lyapunov. For most cases, a reduced problem is discussed when in place of the set **E**, a parallelepiped or a ball embedded in **E** is to be determined. For details of this argument, see Siljak [4].

Let Π denote the parallelepiped in R^q

$$\Pi = \{p \in R^q : p^- \le p \le p^+\} , \tag{4-2}$$

and $\mathbf{p}^{(i)}$ i=1,..2^q to be the vector of parameters on the i^{th} vertex of Π. Since for the small parameter perturbations the nominal part will be dominant, it is the common practice to generate a quadratic Lyapunov function

$$V(x) = x^T S x,$$

where S is the solution of Lyapunov equation

$$A_N^T S + S A_N = - I \tag{4-3}$$

The derivative of the generated Lyapunov function along the solutions of equation (4-1), $\dot{V}$, is a quadratic form of x and is linearly dependent upon parameters $p_i(t,x)$. A convenient way to analyze the sign properties of $\dot{V}$ along solutions of equation (4-1) for $x \neq 0$ is to consider the derivative on the unit sphere $\Phi_1 \in R^n$. The relation

$$dV(x)/dt \leq 0 \quad \text{for } x \in \Phi_1$$

implies that

$$dV(x)/dt \leq 0 \quad \text{for } x \in R^n .$$

This is true because x can always be written in the form of its magnitude times the unit vector. Since the vector magnitude is always positive, the sign of $\dot{V}$ will not be affected. Thus, if Lyapunov derivative is negative on the unit sphere $x \in \phi_1$, then it is negative for all $x \in R^n \backslash 0$. The common procedure to obtain the robustness bounds based on the generated Lyapunov function is to substitute the maximum bound of each individual quadratic terms in $\dot{V}$ (e.g. see [16]),

$$- I + | p_1 | \lambda_M(Q_1) + | p_2 | \lambda_M(Q_2) + . . \leq 0$$

where

$$Q_i = A_i^T S + S A_i$$

Since the maximum values of Q_j are not attained on the same value of x, thus it will be less conservative to consider the robust stability problem of the parallelepiped $\Pi \in R^q$ as follow: If the 2^q quadratic forms

$$QF_j \triangleq x^T\left(A_N^T S + SA_N + A_U^T(\mathbf{p}^{(j)})S + SA_U(\mathbf{p}^{(j)})\right) x \,, \quad j = 1,\ldots,2^q \,, \tag{4-4}$$

generated by the values of the parameters on the 2^q vertices, $\mathbf{p}^{(1)},\ldots,\mathbf{p}^{(2^q)}$, are non-positive for all x, then Π is a solution for the problem of robust stability (see [43]). In such a case, it may be said that Lyapunov function V(x) guarantees the solution Π for the robust stability problem.

With respect to the applied Lyapunov function, the parallelepiped Π is called maximal if enlarging it by decreasing a single bound p_i^- or increasing a single bound p_i^+ causes at least one of the forms QF_j to attain a positive value for some x. Such a definition implies that if the parallelepiped is maximal, then for each p_i^-, p_i^+ there is at least one vertex, referred as active vertex, such that the corresponding form QF_j attains the value zero at some point $x \neq 0$. The point $\zeta \in \Phi_1$ is referred to as a root of the form QF_j if $QF_j(\zeta) = 0$.

4.3 Definitions and Lemmas

Let V^* and V be two quadratic Lyapunov functions of the system (4-1). In addition, let $\zeta_1^*,\ldots,\zeta_{k^*}^*$; $\zeta_1,\ldots,\zeta_k$ correspond to the roots of the forms QF_j^*, QF_j for, respectively, V^* and V. The function V^* is called better than function V if either

i) the maximal parallelepiped generated by V^* is larger and contains the one generated by V, that is,

$$\Pi \subsetneq \Pi^*,$$

or

(ii) $\Pi = \Pi^*$ and the number of roots of the forms QF_j^* is less than the number of roots of QF_j.

Since each quadratic form may be associated with a single symmetric matrix, a space of quadratic Lyapunov functions may be introduced with the distance d between the two functions $V_1 = x^T S_1 x$ and $V_2 = x^T S_2 x$ defined by a norm of a difference between the matrices S_1 and S_2, or

$$d = \| S_1 - S_2 \| .$$

The function V is called optimal if all of the neighboring functions at less than some distance d from V are not better than V; finally, the function V is called globally optimal if the entire space of functions does not contain a better choice.

It is of interest to the argument presented to prove the following lemmas related to the properties of quadratic forms.

Lemma 1. Let Q have m repeated maximum eigenvalues, i.e. $\lambda_1 = \lambda_2 = . . = \lambda_m > \lambda_{m+1} \geq . . \geq \lambda_n$. Then

$$x^T Q x \leq \lambda_1 \qquad \forall x \in \phi_1 \tag{4-5}$$

and equality holds if and only if x is an eigenvector associated to λ_i i=1,...m.

Lemma 2. Define a form $x^T Q_1 x$ such that it has a single root $\eta \in \phi_1$, and fulfills an inequality

$$x^T Q_1 x \leq 0 \quad \forall x \in \phi_1 \tag{4-6}$$

(a) if a matrix Q_2 fulfills

(i) $\eta^T Q_2 \eta > 0$, or

(ii) $\eta^T Q_2 \eta = 0$, η is not an eigenvector of Q_1

then for any $\varepsilon > 0$

$$x^T Q_1 x + \varepsilon\, x^T Q_2 x > 0 \quad \textit{for some } x \in \phi_1 \tag{4-7}$$

(b) if the matrix Q_2 fulfills

$\eta^T Q_2 \eta = 0$, η is an eigenvector of Q_1

then there exist an $\varepsilon_1 > 0$ such that for all $0<\varepsilon<\varepsilon_1$

$$x^T Q_1 x + \varepsilon\, x^T Q_2 x \leq 0 \quad \forall x \in \phi_1 \text{ , } 0< \varepsilon < \varepsilon_1 \tag{4-8}$$

(c) if the matrix Q_2 fulfills

$\eta^T Q_2 \eta < 0$

then there exist an $\varepsilon_2 > 0$ such that for all $0<\varepsilon<\varepsilon_2$

$$x^TQ_1x + \varepsilon\, x^TQ_2x < 0 \quad \forall\, x \in \Phi_1 \;,\; 0< \varepsilon < \varepsilon_2 \qquad (4\text{-}9)$$

Proofs for Lemmas 1 and 2 are given in Appendix A.

The following restriction is imposed on the forms QF_j:

Assumption A: Each of the quadratic forms QF_j, corresponding to the active vertices, have a single root. Thus, there are $k \le 2^q$ roots.

For the sake of simplicity, it may be presumed that the forms are numbered so that the roots denoted by $\zeta_1,\ldots,\zeta_k$ correspond to the roots of the first k forms QF_j, $j = 1,2 ,\ldots,k$. Finally, it is further assumed that with respect to the applied Lyapunov function, the parallelepiped Π is a maximal solution for the robust stability problem of the system in equation (4-1).

4.4 Principal Results

Assume that by any procedure a quadratic Lyapunov function

$$V(x) = x^TSx \qquad (4\text{-}10)$$

has been selected and the 2^q quadratic forms QF_j in (4-4) have been generated. Introduce an arbitrary quadratic form $\Delta V(x) = x^T\Delta Sx$ and consider the Lyapunov function $V^* = V + \varepsilon\Delta V$, where $\varepsilon > 0$ is subsequently selected as sufficiently small. Differentiate V^* along solutions of system and enter the values of parameters on vertices of the parallelepiped Π, $p^{(1)},\ldots,p^{(2^q)}$, to the derivative. Denote the 2^q quadratic forms thus obtained as $QF_j + \varepsilon\Delta QF_j$, where QF_j is the previously defined forms generated by the functions V, and similarly ΔQF_j represents the derivative of the form ΔV along solutions of (4-1) evaluated at $\mathbf{p}^{(j)}$.

Theorem. Let Π be the maximal solution for the robust stability problem in system equation (4-1) obtained by Lyapunov function V. Let the assumption A be satisfied. The function V is then globally optimal iff for any matrix ΔS at least one of the following three conditions is satisfied:

(α) there exists $0 < i \le k$ such that the i-th form ΔQF_i fulfills

$$\Delta QF_i(\zeta_i) > 0$$

(β) there exists $0 < i \le k$ such that the i-th form ΔQF_i fulfills

$$\Delta QF_i(\zeta_i) = 0$$

ζ_i is not the eigenvector of the matrix of the form ΔQF_i.

(γ) $\Delta QF_j(\zeta_j) = 0 \qquad j = 1,\ldots,k$

Proof of the Theorem is given in the Appendix A. Alternative formulation of the Theorem can be found in [44].

4.5 Optimization of the Lyapunov Function

A simple technique for the optimization of the Lyapunov function is based upon the Theorem in previous section, utilizing the observation that if there is a matrix ΔS such that the corresponding forms ΔQF_j, $j = 1,\dots,k$ satisfy

$$\Delta QF_j(\zeta_j) < 0 \ , \qquad j = 1,\dots,k \ , \tag{4-11}$$

then for sufficiently small ε, the function $V + \varepsilon\Delta V$ is better than the function V.

Let $\mathbf{p}^{(j)}$ and $\zeta^{(j)}$ $j = 1,\dots k$ be the vector of the parameters on the active vertices and their associated roots for the maximum parallelepiped Π of the Lyapunov function V. Let $\Delta V = x^T \Delta S\, x$. Derivative of ΔV along the solutions of the (4-1) evaluated for the parameters on the k active vertices, $\mathbf{p}^{(j)}$, and $x = \zeta_j$ result in the following k forms of $\Delta QF_j(\zeta_j)$

$$\Delta QF_j(\zeta_j) \triangleq \zeta_j^T\left(A_N^T\Delta S + \Delta S\, A_N + A_U^T(\mathbf{p}^{(j)})\Delta S + \Delta S\, A_U(\mathbf{p}^{(j)})\right)\zeta_j \ , \quad j = 1,\dots,k.$$

To solve for matrix ΔS which satisfy the inequality (4-11), we carry out the following substitution

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ . \\ u_n \\ u_{n+1} \\ . \\ . \\ . \\ u_{n(n+1)/2} \end{bmatrix} \triangleq \begin{bmatrix} \Delta s_{11} \\ \Delta s_{12} \\ . \\ \Delta s_{1n} \\ \Delta s_{22} \\ . \\ . \\ . \\ \Delta s_{nn} \end{bmatrix} \tag{4-12}$$

and introduce the matrix $\mathbf{B}$ such that

$$B\,u = [\Delta QF_1, \ldots, \Delta QF_K]^T.$$

Thus **B** will be the k by [n(n+1)/2] dimensional matrix:

$$\mathbf{B}=\begin{bmatrix}
\zeta_1^{(1)}[a_{11}\ \cdot\ a_{1n}]\zeta^{(1)} & \cdots\cdots & \zeta_1^{(k)}[a_{11}\ \cdot\ a_{1n}]\zeta^{(k)} \\
[\zeta_2^{(1)}\ \zeta_1^{(1)}]\begin{bmatrix}a_{11} & \cdot & a_{1n}\\ a_{21} & \cdot & a_{2n}\end{bmatrix}\zeta^{(1)} & \cdots\cdots & [\zeta_2^{(k)}\ \zeta_1^{(k)}]\begin{bmatrix}a_{11} & \cdot & a_{1n}\\ a_{21} & \cdot & a_{2n}\end{bmatrix}\zeta^{(k)} \\
\cdots & \cdots & \cdots \\
[\zeta_n^{(1)}\ \zeta_1^{(1)}]\begin{bmatrix}a_{11} & \cdot & a_{1n}\\ a_{n1} & \cdot & a_{nn}\end{bmatrix}\zeta^{(1)} & \cdots\cdots & [\zeta_n^{(k)}\ \zeta_1^{(k)}]\begin{bmatrix}a_{11} & \cdot & a_{1n}\\ a_{n1} & \cdot & a_{nn}\end{bmatrix}\zeta^{(k)} \\
\zeta_2^{(1)}[a_{21}\ \cdot\ a_{2n}]\zeta^{(1)} & \cdots\cdots & \zeta_2^{(k)}[a_{21}\ \cdot\ a_{2n}]\zeta^{(k)} \\
[\zeta_3^{(1)}\ \zeta_2^{(1)}]\begin{bmatrix}a_{21} & \cdot & a_{2n}\\ a_{31} & \cdot & a_{3n}\end{bmatrix}\zeta^{(1)} & \cdots\cdots & [\zeta_3^{(k)}\ \zeta_2^{(k)}]\begin{bmatrix}a_{21} & \cdot & a_{2n}\\ a_{31} & \cdot & a_{3n}\end{bmatrix}\zeta^{(k)} \\
\cdots & \cdots & \cdots \\
[\zeta_n^{(1)}\ \zeta_2^{(1)}]\begin{bmatrix}a_{21} & \cdot & a_{2n}\\ a_{n1} & \cdot & a_{nn}\end{bmatrix}\zeta^{(1)} & \cdots\cdots & [\zeta_n^{(k)}\ \zeta_2^{(k)}]\begin{bmatrix}a_{21} & \cdot & a_{2n}\\ a_{n1} & \cdot & a_{nn}\end{bmatrix}\zeta^{(k)} \\
\cdots & \cdots & \cdots \\
\zeta_n^{(1)}[a_{n1}\ \cdot\ a_{nn}]\zeta^{(1)} & \cdots\cdots & \zeta_n^{(k)}[a_{n1}\ \cdot\ a_{nn}]\zeta^{(k)}
\end{bmatrix}^T \tag{4-13}$$

where $\zeta_i^{(j)}$ is the i^{th} element of the vector $\zeta^{(j)}$. The matrix **B** is called the discrimination matrix. Substituting the values for $\mathbf{p}^{(j)}$ and $\zeta^{(j)}$ j = 1,... k in (4-13), we can calculate the matrix **B**, and thus find u from (4-11). The matrix ΔS can then be found. Further steps are explained through the algorithm. The following conclusion will then be used in the algorithm.

Conclusion: If the system of k linear inequalities

$$\mathbf{B}u < 0\ , \qquad u \in R^{n(n+1)/2}\ , \tag{4-14}$$

has the solution u, then the corresponding matrix ΔS defines the quadratic form ΔV such that for a sufficiently small ε, the function $V + \varepsilon\Delta V$ is better than the function V.

4.6 General Algorithm

Based upon the conclusion provided in Section 4.5, the principal stages of the algorithm are as follows:

1) Solve the Lyapunov equation (4-3) to determine the zero-approximation Lyapunov function matrix;

2) Assign the desired ratios of the uncertainties:

$$\frac{p_2}{p_1}, \ldots, \frac{p_q}{p_1}$$

3) Form the 2^q quadratic forms QF_i, determining the largest eigenvalue Λ_i for each; if all the eigenvalues Λ_i are negative, go to step (7).

4) Select the k forms QF_i with eigenvalues Λ_i larger than or equal to zero; for each Λ_i, determine the eigenvector ζ_i and form the discriminating matrix **B** using (4-13);

5) Solve the inequality equation (4-14), determine the corresponding ΔS; if there is no solution for (4-14), go to step (8);

6) Create a new Lyapunov function with the matrix $S + \varepsilon\Delta S$, where ε is set to be four

times the minimum value of Λ_i found in step (4). If the new matrix S is positive definite or any of the new forms QF_i evaluated at the previous robustness bounds found in step (3) shows to be no longer nonpositive, then ε should be divided by two and the same step repeated.

7) Enlarge the previous step parallelepiped Π_{i-1} by assigning increments of uncertainties, Δp_i, $i = 1,\ldots,q$, then return to step (3).

8) The function is globally optimal with regard to assumed algorithm and the end of the algorithm is reached.

Remark. For the solution of the inequality (4-14) in step (5), the values of the elements u_i should be proportional to their corresponding $\mathbf{B}_{ik}$ coefficients. As will be illustrated in Section 5.6, the bounds resulting from the application of the described procedure are highly dependent upon the way the parallelepiped Π_{i-1} is enlarged in step (7).

4.6 Alternative Algorithm

An alternative algorithm is as follow:

1) At the k^{th} step, let $\Delta S^{(k)} \in R^{nxn}$ be a symmetric matrix corresponding to the vector $u^{(k)}$ as defined in (4-12). Let $u^{(k)}$ to be a vector with all its elements equal to zero except the k^{th} element to be equal to ε. Set k=0 and ε=1.

2) Solve the Lyapunov equation (4-3) to determine the zero-approximation Lyapunov function matrix.

3) Assign the desired ratios of the uncertainties:

$$\frac{p_2}{p_1}, \ldots, \frac{p_q}{p_1}$$

4) Set k=k+1. If $k > n(n+1)/2$, set $\varepsilon = \varepsilon/10$ and k=1. If $\varepsilon < 0.001$, go to step (10).

5) Form the 2^q quadratic forms QF_i, determining the largest eigenvalue Λ_i for each; if all the eigenvalues Λ_i are negative, go to step (9).

6) Select the first form QF_i which has an eigenvalue Λ_i larger than or equal to zero; determine the eigenvector ζ_i.

7) Calculate $\Delta QF_k(\zeta_i)$, where $\Delta S^{(k)}$ is defined in step (1). Set $\varepsilon = -\operatorname{sign}(\Delta QF_k(\zeta_i))\,\varepsilon$.

8) Create a new Lyapunov function with the matrix $S + \varepsilon\Delta S$. If the new matrix S is positive definite or any of the new forms QF_i evaluated at the previous robustness bounds found in step (5) shows to be no longer nonpositive, return to step (4);

9) Enlarge the previous step parallelepiped Π_{i-1} by assigning increments of uncertainties, Δp_i, $i = 1,\ldots,q$, then return to step (5).

10) End of the algorithm.

Remark. This algorithm does not require determination of the matrix **B** and the solution of the inequality (4-14), thus it is faster. However, it may not provide the globally optimal Lyapunov function. This results from the fact that, for instance, in the case where

$$\boldsymbol{B} = \begin{bmatrix} 1 & -2 & 3 \\ -.3 & .2 & -.1 \end{bmatrix}$$

there is no $\Delta S^{(k)}$, as defined in step (1), to provide better Lyapunov function. However by solving the inequality (4-14), the matrix ΔS corresponding to $u = [.1\ .1\ 0]^T$,

$$\Delta S = \begin{bmatrix} .1 & .1 \\ .1 & 0 \end{bmatrix},$$

guarantees improvement of the previous Lyapunov function. The program in Appendix B and the results in Chapter 5 are based on this algorithm.

CHAPTER 5
APPLICATIONS OF GLOBALLY OPTIMAL LYAPUNOV FUNCTIONS

5.1 Introduction

The application of the Lyapunov direct method on explicit robustness bounds for nominally linear systems with nonlinear, time-variant perturbations was explored by Siljak [11], Patel et al. [34], Patel and Toda [12], Lee [13], Yedavalli [7,14], Yedavalli and Liang [15], Zhou and Kharkonegar [16], Siljak [4], Juang[32], Olas and Ahmad-khanlou [19]. Most researches focused on the small gain theorem promoted by the results obtained by Qiu and Davison [5] and Becker and Grimm [22], and the theorems proved by Petersen [21].

In the balance of this chapter, the application of globally optimal Lyapunov functions for systems which have been the focus of various studies is presented. In Section 5.3, examples show that the approach presented in the current investigation provides less conservative estimates than those obtained for the small gain theorem. The examples show improvement in the bounds up to 24 percent. In some cases, the small gain theorem may provide better estimates, but it should be noted that the robustness bounds obtained by GOLF approach may be further improved by piecewise Lyapunov functions. In Sections 5.5 and 5.6, the robustness of actual dynamics of aircraft vertical take-off and landing (VTOL) systems are considered. This example demonstrates the practicality of the approach considered in the current investigation, further indicating that the procedure is not affected by an increase in system dimensions. The program was prepared in MATLAB and has been generalized for systems of all dimensions with any number of perturbation elements. The procedure requires only the standard routines of determining eigenvalues and eigenvectors for symmetric matrices.

5.2 Two-Dimensional System

Radziszewski [33] considered the robust stability problem for the system

$$\dot{x} = A_N x + p_1(x,t)A_1 x, \qquad x \in R^2 , \tag{5-1}$$

where $p_1(x,t)$ is a scalar function, and

$$A_N = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} , \quad A_1 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} .$$

Applying the optimization algorithm presented here, the globally optimal Lyapunov function

$$V(x) = x_1^2 + x_1 x_2 + x_2^2 ,$$

is obtained. The one-dimensional maximal parallelepiped Π was found to have the parameter values $p^{(1)} = -\sqrt{3}/2$, $p^{(2)} = \sqrt{3}/2$ on the active vertices. To carry out the procedure presented in the current study for this problem, the forms QF_j, j = 1,2, associated with $\mathbf{p}^{(1)}$ and $\mathbf{p}^{(2)}$, were analyzed. The two forms QF_j have the corresponding roots $\zeta_1 = [0.939,\ 0.344\ ]^T$ and $\zeta_2 = [0.5907,\ -0.8069]^T$. Using (4-13) and normalizing coefficients,

$$\mathbf{B} = \begin{bmatrix} 1.000 & -1.000 & -.5000 \\ -1.000 & 1.000 & .5000 \end{bmatrix} \tag{5-2}$$

is obtained at an accuracy of four digits.

It is easy to determine that the vector **B** u, where **B** is given by equation (5-2) and u is an arbitrary non-zero three dimensional vector, fulfills the Assumption β as given in the Theorem. It is thus determined that the Lyapunov function is globally optimal for the problem in question. Using methods specific for a two-dimensional case, Radziszewski [33] also demonstrated that there is no quadratic Lyapunov function that can guarantee larger bounds for the parameter $p_1(t,x)$ than the bounds

$$-\sqrt{3}/2 < p_1(t,x) < \sqrt{3}/2\ .$$

However, these bounds may be enlarged when the piecewise quadratic Lyapunov function is used. In Olas and Ahmadkhanlou [19], the bounds

$$-.999 < p_1(t,x) < .999$$

were found; where the Lyapunov function obtained by piecewise Lyapunov function algorithm was

$$V(x,t) = \begin{cases} x^T \begin{bmatrix} 1 & .999 \\ .999 & 1 \end{bmatrix} x & x_1 x_2 \geq 0 \\ x^T \begin{bmatrix} 1 & .333 \\ .333 & 1 \end{bmatrix} x & x_1 x_2 < 0. \end{cases}$$

For the case where p_1 is restricted to attain only constant values, the bound $p_1 > -1$ is found. It should be noted that for the application of the piecewise Lyapunov function, the planes should be assigned such that the first condition of the Theorem given in section 4.4 does not satisfy for the forms ΔQF_j inside any sector.

5.3 Comparison of the GOLF approach and the Small Gain Theorem Approach for a Two-Dimensional System

Qiu and Davison [5] considered the system

$$\dot{x} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} x + p_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x + p_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + p_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x$$

Using the small gain theorem, equation (3-30), the estimates of the robustness bounds were found to be

$$|\, p_1 \,| = |\, p_2 \,| = |\, p_3 \,| \leq 0.6848 \; .$$

Applying the optimization algorithm presented here, results in the GOLF and robustness bounds

$$S_{GOLF} = \begin{bmatrix} 1.644 & 1.103 \\ 1.103 & 3.765 \end{bmatrix},$$

$$|\ p_1\ | = |\ p_2\ | = |\ p_3\ | \le 0.854\ ,$$

which are more than 24 percent greater than those obtained with the small gain theorem. The example indicates that the Lyapunov method may provide less conservative estimates of the robustness bounds for linear systems with nonlinear, time-variant structured perturbations. Table 5.1, where U is as defined in equation (3-14), provides a comparison of the results obtained by Yedavalli [7], small gain theorem as described by Qui and Davison [5], application of GOLF, and the exact bounds for different values of A_j.

Table 5.1 Comparison of robustness bounds for different techniques.

u	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
μ_{exact}	3	2	1	0.6667
μ_{YS}	1.657	1.657	0.655	0.396
μ_{QS}	3	2	1	0.6667
μ_{GOLF}	3	2	1	0.6666
S_{GOLF}	$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 1.8 & .8 \\ .8 & 2.4 \end{bmatrix}$	$\begin{bmatrix} 2.18 & 1.39 \\ 1.39 & 7.49 \end{bmatrix}$	$\begin{bmatrix} .216 & .499 \\ .499 & 1.398 \end{bmatrix}$

Table 5.1 (continue)

u	$\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$	$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$	$\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$
μ_{exact}	2	0.5616	1	1
μ_{YS}	1	0.382	0.48	0.5
μ_{QS}	1.5201	0.5612	0.9150	0.8108
μ_{GOLF}	1.6360	0.5615	0.9740	0.9460
S_{GOLF}	$\begin{bmatrix}1.501 & .725\\.725 & 2.714\end{bmatrix}$	$\begin{bmatrix}.281 & .497\\.497 & 1.487\end{bmatrix}$	$\begin{bmatrix}1.495 & 1.031\\1.031 & 4.011\end{bmatrix}$	$\begin{bmatrix}1.616 & 1.024\\1.024 & 4.248\end{bmatrix}$

Table 5.1 (continue)

u	$\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}$	$\begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}$	$\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}$	$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$
μ_{exact}	0.5	0.4	1	0.4495
μ_{YS}	0.324	0.3027	0.397	0.311
μ_{QS}	0.5	0.4	0.6848	0.4486
μ_{GOLF}	0.498	0.4	0.8540	0.4494
S_{GOLF}	$\begin{bmatrix}.189 & .501\\.501 & 1.5\end{bmatrix}$	$\begin{bmatrix}.4 & .5\\.5 & 1.5\end{bmatrix}$	$\begin{bmatrix}1.644 & 1.103\\1.103 & 3.765\end{bmatrix}$	$\begin{bmatrix}.278 & .502\\.502 & 1.526\end{bmatrix}$

Table 5-1 (continue)

u	$\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$	$\begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}$	$\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$
μ_{exact}	0.3723	0.3542	0.3333
μ_{YS}	0.273	0.256	0.236
μ_{QS}	0.3714	0.3528	0.3333
μ_{GOLF}	0.3723	0.3542	0.3333
S_{GOLF}	$\begin{bmatrix}.492 & .489\\.489 & 1.35\end{bmatrix}$	$\begin{bmatrix}.395 & .501\\.501 & 1.596\end{bmatrix}$	$\begin{bmatrix}1 & 1\\1 & 3\end{bmatrix}$

Analysis of the example leads to an interesting observation, formulated as:

Remark: Consider the system

$$\dot{x} = \begin{bmatrix}-3 & -2+p_2\\1 & 0\end{bmatrix} x.$$

The robustness bound under the restriction that p_2 is constant is $p_2<-2$. For $p_2^*=-2$, the system has a second equilibrium point x_e

$$\begin{bmatrix}-3 & 0\\1 & 0\end{bmatrix} x_e = 0, \; x_e = \begin{bmatrix}0\\1\end{bmatrix}.$$

For any quadratic Lyapunov function $V(x) = x^T Sx$,

$$\dot{V}(x_e) = (A(p*)\ x_e)^T S\ x_e + x_e^T S\ (A(p*)\ x_e)$$
$$= (0)^T S\ x_e + x_e^T S\ (0) = 0$$

By virtue of Lemma 1, for $\dot{V}(x)$ to be non-positive, x_e should be the eigenvector corresponding to eigenvalue zero of Q(p*)

$$Q(p*)\ x_e = (0)\ x_e = 0,$$

where

$$Q(p*) = A^T(p*)\ S + S\ A(p*)$$
$$= \begin{bmatrix} -6S_{11}+2S_{12} & -3S_{12}+S_{22} \\ -3S_{12}+S_{22} & 0 \end{bmatrix}$$

Thus

$$-3S_{12} + S_{22} = 0 \rightarrow S_{22} = 3S_{12},$$

$$Q(p*) = \begin{bmatrix} -6S_{11}+2S_{12} & 0 \\ 0 & 0 \end{bmatrix}.$$

Since Q should be negative semi-definite,

$$S_{11} > \frac{S_{12}}{3},$$

and S_{11} should be selected such that S is non-negative. Thus

$$S = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}.$$

This matrix of the Lyapunov function also provides the bound $|p_2| < 2$. By virtue of Lemma 1, the Lyapunov function which provides maxmimum $|p|$ without iterations was found directly. This would seem to suggest further consideration of this remark.

5.4 Three-Dimensional System

Consider the system

$$\dot{x} = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & -1 & -4 \end{bmatrix} x + p_1 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} x + p_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} x \,.$$

Zhou and Khargonekar [16] proved that the system is stable within the following robustness bounds:

i) $| \; p_j(t,x) \; | < 1.55328, \; j = 1,2$

or

ii) $0.60521 \; | \; p_1(t,x) \; | + 0.351205 \; | \; p_2(t,x) \; | < 1.$

or

iii) $p_1^2(t,x) + p_2^2(t,x) < 2.72768$

and Siljak [3] obtained the robustness bounds:

iv) $-0.875 < p_1(t,x) < 1.75$ and $p_2(t,x) < 3$,

whereas the bounds obtained under restriction that the parameters p_1, p_2 be constant are

$$p_1 < 1.75 \quad \text{and} \quad p_2 < 3 \,.$$

The first step to optimize Lyapunov function is to solve the Lyapunov equation

$$A^T S + S A = -I \,,$$

resulting in the matrix

$$S = \begin{bmatrix} .5714 & .0378 & -.1429 \\ .0378 & .3487 & -.0462 \\ -.1429 & -.0462 & .2857 \end{bmatrix}.$$

Utilizing the quadratic Lyapunov function $x^T S x$, the robustness bounds

$$-6.47 < p_1(t,x) < 1.600 \quad \text{and} \quad -13.76 < p_2(t,x) < 2.741$$

are obtained. Applying the optimization algorithm presented, results in GOLF and robustness bounds

$$S_{GOLF} = \begin{bmatrix} 1.0000 & .1407 & -.0190 \\ .1407 & 30916.9163 & -.4867 \\ -.0190 & -.4867 & 1.0266 \end{bmatrix}$$

$$-1500 < p_1(t,x) < 1.7486 \quad \text{and} \quad -3000 < p_2(t,x) < 2.9985 ,$$

which for applications is seemingly the equivalent of the bounds determined while restricting the parameters to a constant value. Figure 5.1 shows the comparison of the robustness bounds.

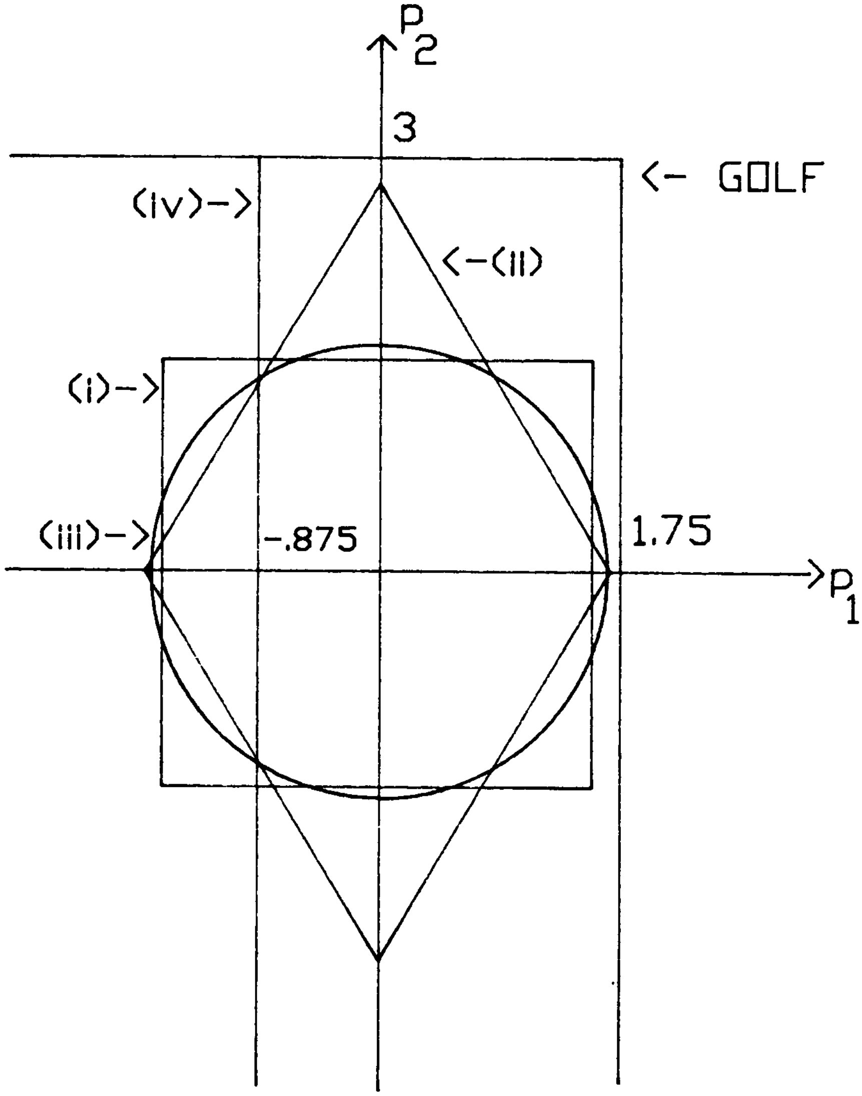

Figure 5.1. Comparison of different robustness bounds.

The following two examples are concerned with the design of a robust controller for the dynamical control of a helicopter in a vertical plane. From Narendra and Tripathi [45], the dynamics are described in the following system

$$\dot{x} = (A + \Delta A)x + (B + \Delta B)u \ ,$$

$$A = \begin{bmatrix} -.0366 & .0271 & .0188 & -.4555 \\ .0482 & -1.0100 & .0024 & -4.0208 \\ .1002 & .2855 & -.7070 & 1.3229 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\Delta A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & p_1 & 0 & p_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} .4422 & .1761 \\ 3.04475 & -7.5922 \\ -5.5200 & 4.49 \\ 0 & 0 \end{bmatrix} \quad \Delta B = \begin{bmatrix} 0 & 0 \\ p_3 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

where the state and the control inputs are:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \textit{horizontal velocity} \\ \textit{vertical velocity} \\ \textit{pitch rate} \\ \textit{pitch angle} \end{bmatrix}, \textit{ and } u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \textit{collective cycle} \\ \textit{longitudenal cycle} \end{bmatrix}$$

For the assigned range of air speed, a significant change was observed in the values of some elements of the matrices A and B, wherein the parameters were subject to variance and their bounds were specified as follows (i.e., following symmetrizing all of the perturbation bounds within the matrices):

$$|p_1| \le 0.2192 \qquad |p_2| \le 1.2031 \qquad |p_3| \le 2.0673 \ . \qquad (5\text{-}3)$$

5.5 Aircraft (VTOL) System

5.5.1 Robust Helicopter Control (I)

Singh and Coelho [46] used a linear quadratic optimization approach to design a robust control for a helicopter, based upon the following feedback gain matrix:

$$K = \begin{bmatrix} -0.8143 & -1.2207 & 0.2660 & 0.8260 \\ -0.2582 & 1.1780 & 0.0623 & -0.2120 \end{bmatrix} . \qquad (5\text{-}4)$$

The following closed loop system was then obtained,

$$\dot{x} = \begin{bmatrix} -.4422 & -.3052 & .1474 & -.1276 \\ -.4708 - .8143p_3 & -13.6703 - 1.2207p_3 & .3393 + .266p_3 & .1037 + .826p_3 \\ 3.4358 & 12.3130 + p_1 & -1.8956 & -4.1885 + p_2 \\ 0 & 0 & 1 & 0 \end{bmatrix} x$$

where p_i is the symmetrized structured perturbation element. However, the stability of system could not be verified analytically throughout an entire range of parameter variations. The estimates obtained were well below the bounds indicated in (5-3). The stability of the system was confirmed only by simulation. Lyapunov function obtained from solving the Lyapunov equation

$$A_N^T S + S A = -I ,$$

provides the robustness bounds

$$|p_1| \le 0.0693 \qquad |p_2| \le 0.3804 \qquad |p_3| \le 0.6536 .$$

Applying the optimization algorithm provides the quadratic Lyapunov function and robustness bounds

$$S_{GOLF} = \begin{bmatrix} 11.003 & -.597 & .117 & -3.896 \\ -.597 & 4.446 & .695 & .001 \\ .117 & .695 & 1.000 & .823 \\ -3.896 & .001 & .823 & 4.371 \end{bmatrix}$$

$$|p_1| \leq 0.2209 \qquad |p_2| \leq 1.2113 \qquad |p_3| \leq 2.0815 \; .$$

Thus, the bounds obtained were higher than the bounds specified in (5-3). It must be noted that since only a limited sequence of possible time-dependent functions may be inserted as varying parameters, simulation is not always a reliable test. In contrast, the bounds determined with the procedure presented in the current study are valid for all of the possible functions considered.

5.5.2 Robust Helicopter Control (II)

Chen and Chen [20] reconsidered the robust controller design developed in the Narendra and Tripathi [45] helicopter model. The task was to obtain a less conservative controller: that is, a controller that could guarantee stability for the parameter bounds in (5-3), with a matrix that would reflect only minimal gains. The Davison-Fletcher-Powell optimization technique was used to determine the gain matrix, obtaining

$$K = \begin{bmatrix} -.1640 & .2699 & .4511 & .4308 \\ .0364 & .1692 & -.1066 & -.4519 \end{bmatrix} . \tag{5-5}$$

For this matrix, the closed loop system assumed the form

$$\dot{x} = \begin{bmatrix} -.1027 & .1762 & .1995 & -.3446 \\ -.7275 - .1640p_3 & -1.4728 + .2699p_3 & 2.1852 + .4511p_3 & .7218 + .4308p_3 \\ 1.1689 & -.4446 + p_1 & -3.6757 & -3.0841 + p_2 \\ 0 & 0 & 1 & 0 \end{bmatrix} x \tag{5-6}$$

To allow for a 70 percent increase in parameter bounds, a new controller, with the gain matrix

$$K = \begin{bmatrix} -0.272 & 0.083 & 0.878 & 1.502 \\ 0.024 & 0.279 & 0.169 & -0.150 \end{bmatrix},$$

was formulated. Note that in comparison to (5-5), some of the gains increased considerably.

To test the gain matrix in (5-5) for conservatism, and to determine if there is a need to increase the gains to ensure robust stability for extended parameter bounds, the GOLF

$$S_{GOLF} = \begin{bmatrix} 4.0530 & .3630 & -.3680 & -1.7870 \\ .3630 & .5180 & .2810 & -.5750 \\ -.3680 & .2810 & 2.2220 & 1.5920 \\ -1.7870 & -.5750 & 1.5920 & 5.9110 \end{bmatrix}$$

was developed. The resulting robustness bounds were

$$|p_1| \le .3888 \qquad |p_2| \le 2.1308 \qquad |p_3| \le 3.6613. \tag{5-7}$$

Comparison with the bounds described in (5-3) indicates that the bounds in (5-7), obtained by the optimal Lyapunov function, average approximately 70 percent larger than those for (5-3). The gain matrix in (5-5) remains conservative and assures robust stability for extended bounds.

5.6 Selection for Desired Parameter Ratios for an Aircraft (VTOL) System

Finally, the system described in (5-6) is considered to investigate different means of enlarging the parallelepiped given in step 7 (Section 4.6) to extend the influence of the algorithm upon the resulting bounds.

a) The bounds for (5-7) were obtained from the ratios of increments described in (5-3):

$$\frac{|\Delta p_2|}{|\Delta p_1|} = \frac{1.2031}{0.2192}\ , \quad \frac{|\Delta p_3|}{|\Delta p_1|} = \frac{2.0673}{0.2192}\ ,$$

$$|p_1| \le .3888 \qquad |p_2| \le 2.1308 \qquad |p_3| \le 3.6613.$$

b) Assigning the ratio

$$\frac{|\Delta p_2|}{|\Delta p_1|} = \frac{|\Delta p_3|}{|\Delta p_1|} = 1.0$$

results in the following matrix S_{GOLF} and the bounds

$$S_{GOLF} = \begin{bmatrix} 4.115 & .452 & -.091 & -2.213 \\ .452 & .687 & .150 & -.583 \\ -.091 & .150 & 1.497 & .911 \\ -2.213 & -.583 & .911 & 4.414 \end{bmatrix},$$

$$|p_1| \le 1.4426 \qquad |p_2| \le 1.4418 \qquad |p_3| \le 1.4418.$$

c) Finally, assigning

$$\frac{|\Delta p_2|}{|\Delta p_1|} = 10.0\ , \quad \frac{|\Delta p_3|}{|\Delta p_1|} = 6.0$$

yields

$$S_{GOLF} = \begin{bmatrix} 6.625 & .257 & -.169 & -1.289 \\ .257 & .668 & .150 & -.579 \\ -.169 & .150 & 1.627 & 1.511 \\ -1.289 & -.579 & 1.511 & 7.217 \end{bmatrix},$$

$$|p_1| \le 0.2783 \qquad |p_2| \le 2.7600 \qquad |p_3| \le 1.6560.$$

Figure 5.2 demonstrates different optimal parallelepipeds.

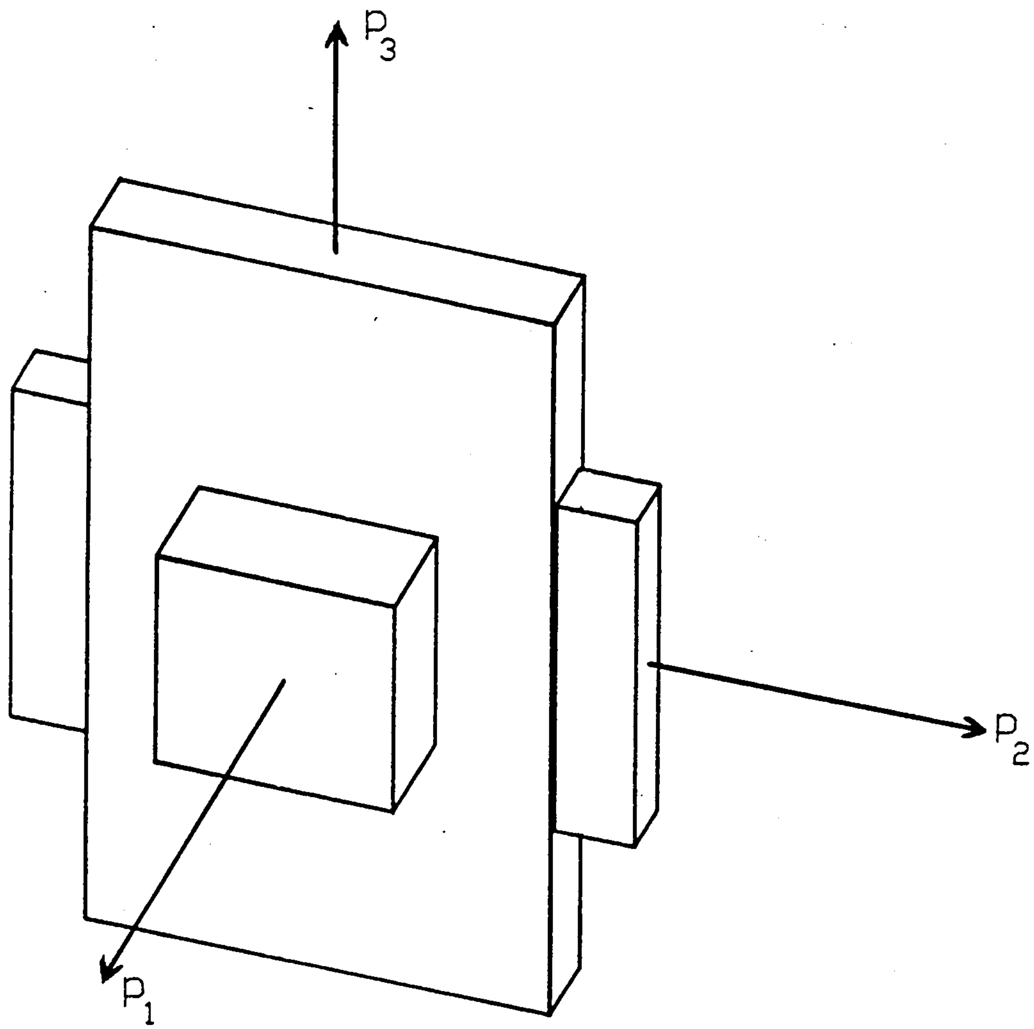

Figure 5.2. Comparison of different optimal parallelepipeds.

CHAPTER 6

CONCLUSION

The subject of the current investigation has been the robust stability of nominally linear systems with nonlinear, time-variant structured perturbations. The Lyapunov direct method and the class of quadratic functions were used to develop the analysis. An original and less conservative method for estimation of the perturbation bounds, in comparison to previously developed approaches, has been presented. The principal advantage provided by the proposed method is that it uses the structural features of the uncertainties to generate the Lyapunov function. Previous investigations in this problem area have ignored these features when generating the Lyapunov function.

The Theorem formulated in Section 4.4, provides a complete solution for the problem described above for the class of quadratic functions. The optimal uncertainty bounds, p_i^{-*}, p_i^{+*}, $i = 1,\ldots,q$ (i.e., as obtained by application of the globally optimal Lyapunov function), cannot be improved by the application of alternative quadratic functions. By the improvement of the bounds p_i^-, p_i^+, it is understood that

$$p_i^- \le p_i^{-*} \ , \ p_i^+ \ge p_i^{+*} \ , \ i = 1,\ldots,q \ ,$$

where at least one of the inequalities remains strong. None of the other techniques restricted to the use of quadratic Lyapunov functions is able to provide comparable or better results. For structured perturbations, as shown in Section 5.3, it was demonstrated that the results obtained through application of the proposed approach are superior to those obtained from application of the small gain theorem. In some cases, small gain theorem may provide better estimates of the robustness bounds, but it should be noted that the robustness bounds obtained by GOLF approach may be further improved by piecewise Lyapunov functions.

As shown in Section 5.6, there are possibly more than one globally optimal Lyapunov function and set of optimal bounds. A unique solution for this problem may be found when the desired proportions between the bounds can be inserted into the input data. In addition, the type of problem considered may be easily expanded to consideration of convex bodies with protruding vertices, rather than consideration of parallelepiped within a parameter space.

The Theorem described in Section 4.4, provides the basis for development of a simple and effective algorithm for Lyapunov function optimization. Only the standard numerical procedures are required, principally those to determine the eigenvalues and the eigenvectors of symmetric matrices. Nonetheless, quadratic Lyapunov functions are not always the option best suited for solution of the robust stability problem. The optimal Lyapunov function used in Example 1 provided the bounds $|p_1(x,t)| < .8660$, whereas the piecewise quadratic Lyapunov function guaranteed the bounds $|p_1(t,x)| < .999$. Since the piecewise Lyapunov functions were constructed by modifying the quadratic Lyapunov function, the role of the optimal Lyapunov function is twofold. First, it may provide satisfactory estimates of the bounds; second, if the estimate is not sufficient, then the function may be considered as the first approximation for the iterative process of designing the piecewise Lyapunov function. In Section 5.4, in the practical sense, the optimal Lyapunov function effectively reached the exact robustness bounds. In Section 5.5, for VTOL aircraft system applications, the practicality of the proposed method was evident, demonstrating that this method can be effectively used to directly analyze the behavior and conservatism of the robust controllers. It further demonstrated that the procedure is not affected by an increase in system dimensions. Further consideration of the direct calculation of S_{GOLF}, as noted in section 5.3, is suggested, as well as the extension of presented algorithm for vector Lyapunov functions, piecewise Lyapunov functions, systems with combined structured and unstructured perturbations, and determination of the region of stability. The most important factor may be the application of the technique developed for the current investigation to the determination of improved gains of robust controllers.

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APPENDICES

Appendix A

Proofs of Lemmas and Theorem

PROOF OF THE LEMMA 1:

Lemma 1. Let Q have m repeated maximum eigenvalues, i.e. $\lambda_1 = \lambda_2 = . . = \lambda_m > \lambda_{m+1} \geq . . \geq \lambda_n$. Then

$$x^T Q x \leq \lambda_1 \qquad \forall\, x \in \phi_1,$$

and equality holds if and only if x is an eigenvector associated to λ_i i=1,...m.

Proof. Let λ_i and η_i be the eigenvalues and the associated eigenvectors of Q such that $\lambda_1 = \lambda_2 = \lambda_m > \lambda_{m+1} . . \geq . . \geq \lambda_n$. Let U be the orthogonal matrix which diagonalize the matrix Q. Thus

$$U = [\eta_1\ ,\ \eta_2\ ,\ .\ ,\ .\ .\ ,\ \eta_n],$$

and

$$U^T Q\, U = D \qquad where \quad D = diag\ (\lambda_1\ \lambda_2\ .\ .\ \lambda_n\ ).$$

Substituting x = U y, we have

$$x^T Q\, x = y^T U^T Q\, U\, y = y^T D\, y$$

$$= y_1^2\ \lambda_1 + y_2^2\ \lambda_2 + . + . . . + y_n^2\ \lambda_n\ .$$

Since y is also on ϕ_1, and $\lambda_1 = \lambda_2 = \lambda_m > \lambda_{m+1} . . \geq . . \geq \lambda_n$

$$\max\ (\ x^T Q\, x\ ) \leq \lambda_1\ (\ y_1^2 + y_2^2 + ... + y_n^2\ ) = \lambda_1 \quad \forall\, x \in \phi_1.$$

Thus, the equality holds iff $y_j = 0$ for $j > m$, therefore

$$x = U\,y = U\begin{bmatrix} y_1 \\ y_2 \\ . \\ y_m \\ 0 \\ . \\ 0 \end{bmatrix} = y_1\,\eta_1 + y_2\,\eta_2 +.+.+ y_m\,\eta_m .$$

Since η_m are the eigenvectors associated to λ_m, thus x is an eigenvector associated to λ_m, and it ends the proof.

PROOF OF THE LEMMA 2:

Lemma 2. Define a form x^TQ_1x such that it has a single root $\eta \in \phi_1$, and fulfills an inequality

$$x^T\,Q_1\,x \le 0 \quad \forall\, x \in \phi_1 .$$

(a) If a matrix Q_2 fulfills

(i) $\eta^TQ_2\eta > 0$, or

(ii) $\eta^TQ_2\eta = 0$, η is not an eigenvector of Q_1,

then for any $\varepsilon > 0$

$$x^TQ_1x + \varepsilon\, x^TQ_2x > 0 \quad \textit{for some } x \in \phi_1 .$$

(b) If the matrix Q_2 fulfills

$\eta^TQ_2\eta = 0$, η is an eigenvector of Q_1,

then there exist an $\varepsilon_1 > 0$ such that for all $0<\varepsilon<\varepsilon_1$

$$x^TQ_1x + \varepsilon\, x^TQ_2x \le 0 \quad \forall\, x \in \phi_1 \ ,\ 0< \varepsilon < \varepsilon_1 .$$

(c) If the matrix Q_2 fulfills

$\eta^TQ_2\eta < 0$,

then there exist an $\varepsilon_2 > 0$ such that for all $0<\varepsilon<\varepsilon_2$

$$x^T Q_1 x + \varepsilon\, x^T Q_2 x < 0 \quad \forall\, x \in \phi_1 \ ,\ 0 < \varepsilon < \varepsilon_2 .$$

Proof. Let ϕ^* represent the sufficiently small subset of ϕ_1 around η, where there is no other eigenvectors than η. Since

$$x^T\, Q_1\, x < 0 \qquad \forall\, x \in \phi_1 / \phi^*,$$

a sufficiently small $\varepsilon_1 > 0$ always exist such that

$$|\max\ (\ x^T\, Q_1\, x\ )\ | > \max\ (\ \varepsilon\ x_T\, Q_2\, x\ ) \qquad \forall\, x \in \phi_1 / \phi^*,\ \varepsilon < \varepsilon_1,$$

thus

$$x^T\, Q_1\, x + \varepsilon\ x^T\, Q_2\, x < 0 \qquad \forall\, x \in \phi_1 / \phi^*,\ \varepsilon < \varepsilon_1 .$$

Therefore to prove the part (b) of the Lemma, we only need to consider the subset ϕ^*.

(a) If (i) is fulfilled for any $\varepsilon > 0$, we have

$$\eta^T\, Q_1\, \eta + \varepsilon\ \eta^T\, Q_2\, \eta = 0 + \varepsilon\ \eta^T\, Q_2\, \eta > 0 .$$

If (ii) is fulfilled, then

$$\eta^T\, Q_1\, \eta + \varepsilon\ \eta^T\, Q_2\, \eta = 0 + \varepsilon\ (0) = 0 .$$

By definition, η is the eigenvector of Q if $Q\ \eta = \lambda\ \eta$, where λ is a scalar (eigenvalue). Since η is the eigenvector of Q_1 associated to zero eigenvalue and is not the eigenvalue of Q_2, we have

$$Q_1 \eta = (0)\ (\eta) = 0 \qquad \textit{and} \qquad Q_2 \eta \neq (0)\ (\eta) = 0$$

$$(Q_1 + \varepsilon Q_2)\ \eta = Q_1 \eta + \varepsilon Q_2 \eta \neq 0.$$

Therefore η is not the eigenvector of $(Q_1 + \varepsilon Q_2)$, associated to zero eigenvalue. Thus by virtue of the Lemma 1, for any $\varepsilon > 0$ we have

$$\max\ (x^T Q_1 x + \varepsilon x^T Q_2 x) > \eta^T Q_1 \eta + \varepsilon \eta^T Q_2 \eta,$$

which ends the proof of part (a)

(b) Again note that for sufficiently small $\varepsilon < \varepsilon_1$, the only values of x where the form may be positive must be located in the neighborhood of η(set ϕ^*). If η is an eigenvector of Q_2, then

$$(Q_1 + \varepsilon Q_2)\ \eta = Q_1 \eta + \varepsilon Q_2 \eta = 0.$$

Thus η is the eigenvector of $(Q_1 + \varepsilon Q_2)$ associated to zero eigenvalue. Therefore for all $\varepsilon < \varepsilon_1$, $(Q_1 + \varepsilon Q_2)$ has a single eigenvalue at zero and (n-1) negative eigenvalues. Thus

$$x^T Q_1 x + \varepsilon x^T Q_2 x \leq 0 \quad \forall x \in \phi_1,\ 0 < \varepsilon < \varepsilon_1,$$

which ends the proof of part (b). Note that if in addition, Q_2 satisfies $x^T Q_2 x \leq 0$ for all $x \in \phi_1$, the above form will be true for all $\varepsilon > 0$.

(c) We can always find an $\varepsilon_2 > 0$ such that

$$|\max\ (x^T Q_1 x)| > \max\ (\varepsilon x_T Q_2 x) \quad \forall x \in \phi_1,\ 0 < \varepsilon < \varepsilon_2,$$

then

$$x^T Q_1 x + \varepsilon x^T Q_2 x < 0 \quad \forall x \in \Phi_1, \; \varepsilon < \varepsilon_2,$$

which ends the proof of part (c).

PROOF OF THEOREM 1:

Theorem 1. Let Π be the maximal solution for the robust stability problem in system equation (4-1) obtained by Lyapunov function V. Let the assumption A be satisfied. The function V is then globally optimal iff for any matrix ΔS at least one of the following three conditions is satisfied:

(α) there exists $0 < i \le k$ such that the i-th form ΔQF_i fulfills

$$\Delta QF_i(\zeta_i) > 0$$

(β) there exists $0 < i \le k$ such that the i-th form ΔQF_i fulfills

$$\Delta QF_i(\zeta_i) = 0$$

ζ_i is not the eigenvector of the matrix of the form ΔQF_i.

(γ) $\Delta QF_j(\zeta_j) = 0 \;, \qquad j = 1,\ldots,k \;.$

Proof. Note that the two conditions (i) and (ii) in section (4.3) can be reduced to the following condition: The function V^* is better than the function V, if at least one of the forms QF_j^*, j=1,2,.....k, evaluated for the parameters on the active vertices becomes negative definite, while the rest remain non-positive.

Sufficiency. We prove that if at least one of the conditions (α), (β), or (γ) is fulfilled, V is globally optimal; thus all the forms ($QF_j + \varepsilon\Delta QF_j$), j=1,2,.....k, on active vertices remain negative semi-definite or attain positive value for some $x \in \phi_1$. In Lemma 1 we proved that if either of the conditions (α) or (β) is fulfilled, then

$$QF_i + \varepsilon\Delta\ QF_i > 0 \qquad \textit{for some } x \in \Phi_1 \ \ \forall\, \varepsilon > 0;$$

and we also showed that if (γ) is fulfilled, all the k forms ($QF_j + \varepsilon\Delta QF_j(\zeta_j)$) remain negative semi-definite for all sufficiently small ε, and may become non-definite or positive definite for large ε. This ends the proof of sufficiency.

Necessity. We prove that if V is globally optimal, then either (α), (β), or (γ) holds. Since V is assumed to be globally optimal, thus for any $\varepsilon > 0$ and ΔQF_i the k forms of ($QF_j + \varepsilon\Delta QF_j(\zeta_j)$) on active vertices should stay negative semi-definite or attain positive value for some $x \in \phi_1$. If for all $\varepsilon > 0$, the form ΔQF_i attains positive value for some $x \in \phi_1$, then by means of lemma 2, either (α) or (β) is fulfilled. If the k forms ($QF_j + \varepsilon\Delta QF_j(\zeta_j)$) on the active vertices stay negative semi-definite for all sufficiently small $\varepsilon > 0$, then $\Delta QF_i(\zeta) = 0$, and ζ_i should be the eigenvector of ΔQF_i. If the k forms ($QF_j + \varepsilon\Delta QF_j(\zeta_j)$) on the active vertices stay negative semi-definite for all $\varepsilon > 0$, then in addition the k forms ΔQF_i should satisfy $\Delta QF_i \leq 0$ for all $x \in \phi_1$. This ends the proof of necessity.

Appendix B

Computer Programs

```
% oplf.m
% This program finds the globally optimal Lyapunov function
% thus providing the optimal robustness bounds in a n
% dimensional linear system with q nonlinear, time variable
% structured perturbations.

%    Dimension of system
  n=input('please enter the dimension of the system :');

%    Number of parameters
 nk=input('please enter the number of parameters :');

%    Increment of dp and dk
  d=input('please enter the increment size of elements of P
           matrix (1) :');
 dk=input('please enter the increment size of parameters
           (.1) :');
       dc=d;dkc=dk;

%    Number of digits accuracy required for the bounds.
per=input('please enter number of digits required for the
           bounds (2) : ');
```

```
%    Entering the nominal and perturbation matrices
aa=[];
  eigcheck=1;
  while eigcheck==1
aa(1:n,1:n)=input('please enter the nominal matrix : ');
      if max(eig(aa))<0,eigcheck=2;break
      else 'The nominal matrix should be negative
            definite.',eigAn=eig(aa)
        'Please enter proper nominal matrix.', end,
      end
        for i=1:nk;i
        aa(1:n,(i*n+1):((i+1)*n))=input('please enter the
                                 perturbation matrix of ki : ');
        end

%    Entering the desired diameter ratio of the
     parallelepiped
        dr=[1];
        if nk>1;
drk=input('please enter the desired diameter ratios of the
            parallelepiped in the following matrix form :
            [k2/k1  k3/k1 ..... k(nk)/k1] ; ')

dr=[1,drk]; end
```

```
%    Starting from fixed diameters parallelepiped
par=0;

%    Starting with all parameters to be zero.
pkk=[0*(1:nk)];

%    Calculating the nominal matrix ( no perturbations )
[a,pkk]=amat(0,[0*(1:nk)],par,pkk,aa,n,nk,dr);

%    Calculating the P matrix for linear part
  ee=3;
while ee==3
  num=input('do you want to start with a specific P matrix
            other than using Q=I in Lyapunov equation ?
             NO=1 ,YES=2   :');
  if num==1
  q=2*eye(n);
  p=lyap(a(:,1:n)',q), break
  end
    if num==2
    p=input('please input the starting P matrix.'),break
    end,'please enter 1 or 2 '
end

%    Rounding off the P matrix to three decimals. since
%    dp>=.001
     p=round(1000*p)/1000
```

```
%    Starting k =0
     ef=3;
while ef==3
  nun=input('do you want to start k from k=0 ?  YES=1 ,
             NO=2    : ');
  if nun==2, k=input('please enter starting value for k :')
  , break, end;
  if nun==1, k=0, break, end;
end

  tt=1;
for i=1:3;
 while tt==1
 [p,kn,pkk]=optimal(p,k,n,nk,d,dk,par,pkk,per,aa,dr);
  if (kn-k)<5*10^(-(per+1)), d=d/10; end
  if d<.001, break, end
  k=kn;
 end
 d=dc, if nk==1, break, end

%   Now we improve P unilaterally along each parameter.
  if nk>1
for par=1:nk;
  k=abs(pkk(par));
  d=dc; dk=dkc;
 while tt==1
  [p,kn,pkk]=optimal(p,k,n,nk,d,dk,par,pkk,per,aa,dr);
```

```
  if (kn-k)<5*10^(-(per+1)) , d=d/10; end
  if d<.001, break, end
  k=kn;
  end;end
```

```
function [p,k,pkk]=optimal(p,k,n,nk,d,dk,par,pkk,per,aa,dr)
%   This subroutine optimizes the Lyapunov function and
%   determines the robustness bounds.

%   Constructing matrix of the 2^nk vertices
  kkk=edges(nk);
  alr=0;
%   Constructing the dp matrices.
   for iii=1:n;
   for jjj=iii:n;
    dp=0*p;
    dp(iii,jjj)=d;
    dp(jjj,iii)=d;
    vhp=100;saa=0; dd=1;

%    We now check for the  bounds
   zz1=1; zz2=1; ss=0;
while ss==0
  s=0;dkk=0;k=round((10^(per+1))*k)/(10^(per+1));
  if alr==0
```

```
 while s==0
   as=0;
    k=k+dkk
   for ver=1:(2^nk);
   [a,pkk]=amat(k,[kkk(:,ver)],par,pkk,aa,n,nk,dr);
   q=a'*p+p*a;
    if max(eig(a'*p+p*a))>0;
    if dkk>0 ,if dkk<=10^(-(per+1)), [g,h]=eig(q);
               [ll,j]=max(diag(h));
    g=g(:,j)/norm(g(:,j)); h=max(eig(q)); end, end
    if dkk==0; p=p-saa*dd*dp, k=k-dkk, alr=3; s=1; ss=1;
    as=1; break, end;
    k=k-dkk;  g;kkk(:,ver); h; z=-1; as=1; break
    end
if as==1, if dkk>10^(-(per+1)), dkk=dkk/10, else break;
end; end;
 if dkk==0,dkk=dk;end
 if ss==1, break, end
 k,

%    Checking to see if we should add or subtract dp to p.
dd=1; dp
[a,pkk]=amat(k+dkk,[kkk(:,ver)],par,pkk,aa,n,nk,dr);
dq=a'*dp+dp*a;
dv=g'*dq*g;
vhn=dv/h;
 if (vhn/vhp)>.99, if (vhn/vhp)<1.01, break, end, end;
```

```
  vhp=vhn;
  if abs(dv)<h/20, break, end
  if dv<0 ,p=p+dd*dp, alr=0; zz1=-1; saa=+1;
   else p=p-dd*dp, alr=0; zz2=-1; saa=-1;
 end
 if min(eig(p))<=0, p=p-saa*dd*dp, alr=2; break, end
 if zz1*zz2>0, p=p-saa*dd*dp,zz1*zz2, alr=1;break, end
  dvp=dv;
end
[a,pkk]=amat(k,[0*(1:nk)]+1,par,pkk,aa,n,nk,dr)
  k;p
```

```
function kk=edges(nk)
%    This subroutine provides all the combinations of signs
%    of the edges.
s=-1                        ;
kk3=[]                      ;
for i=1:nk                  ;
kk2=[];                     ;
for j=1:(2^(nk-i+1))        ;
s=-s                        ;
kk1=s*[0*(1:2^(i-1))+1]     ;
kk2=[kk2 kk1]               ;
end                         ;
kk3=[kk3;kk2]               ;
```

```
end                    ;
kk=kk3
```

```
function [a,pkk]=amat(k,x,par,pkk,aa,n,nk,dr)
%  amat.m
%  This is the subroutine to find the A matrix for systems
%  with "nk" parameters
if par==0, for i=1:nk;
 kp(i)=x(i)*abs(k)*dr(i); end ;end;
if par>0, for i=1:nk;
  kp(i)=x(i)*abs(pkk(i));
  end ;kp(par)=x(par)*abs(k); end;
pkk=abs([kp(1:nk)]); a=aa(:,1:n);
 for i=1:nk;
 ak=kp(i)*aa(:,(i*n+1):(i+1)*n);
 a=a+ak; end;
```

Appendix C

Geometrical View

For the generation of a globally optimal Lyapunov function which provides less conservative estimates of the robustness bounds, the Lyapunov function must be tailored to the system structure. What effect do changes to the diagonal and off-diagonal elements of the matrix S have upon tailoring the Lyapunov function? This Appendix shows the geometrical view of the effects of the elements of matrix S upon the constant paths of the Lyapunov functions. In figure C.1, the circle drawn with a centered line represents the constant path of the Lyapunov function,

$$V(x) = x^T \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} x \, .$$

Figures C.1(a) and C.1(b) show the effect, respectively, of decreasing or increasing S_{12} on the constant path of the Lyapunov function. Note that an increase of S_{ij} is similar to pushing the closed path from the quadrants, where $x_i\ x_j >$ 0, causing the path to move inside the other quadrants. The Lyapunov functions for figures C.1(a) and C.1(b) are

$$V(x) = x^T \begin{bmatrix} 2 & -1.5 \\ -1.5 & 2 \end{bmatrix} x,$$

and

$$V(x) = x^T \begin{bmatrix} 2 & 1.5 \\ 1.5 & 2 \end{bmatrix} x.$$

Figure C.1(c) shows the effect of increasing S_{11}. In this case, the increase of S_{11} was similar to pushing the closed path through x_1. The Lyapunov function for Figure C.1(c) was

$$V(x) = x^T \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} x.$$

Figure C.1(d) shows the effect of a simultaneous increase of S_{11} and S_{12}; the Lyapunov function is

$$V(x) = x^T \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} x.$$

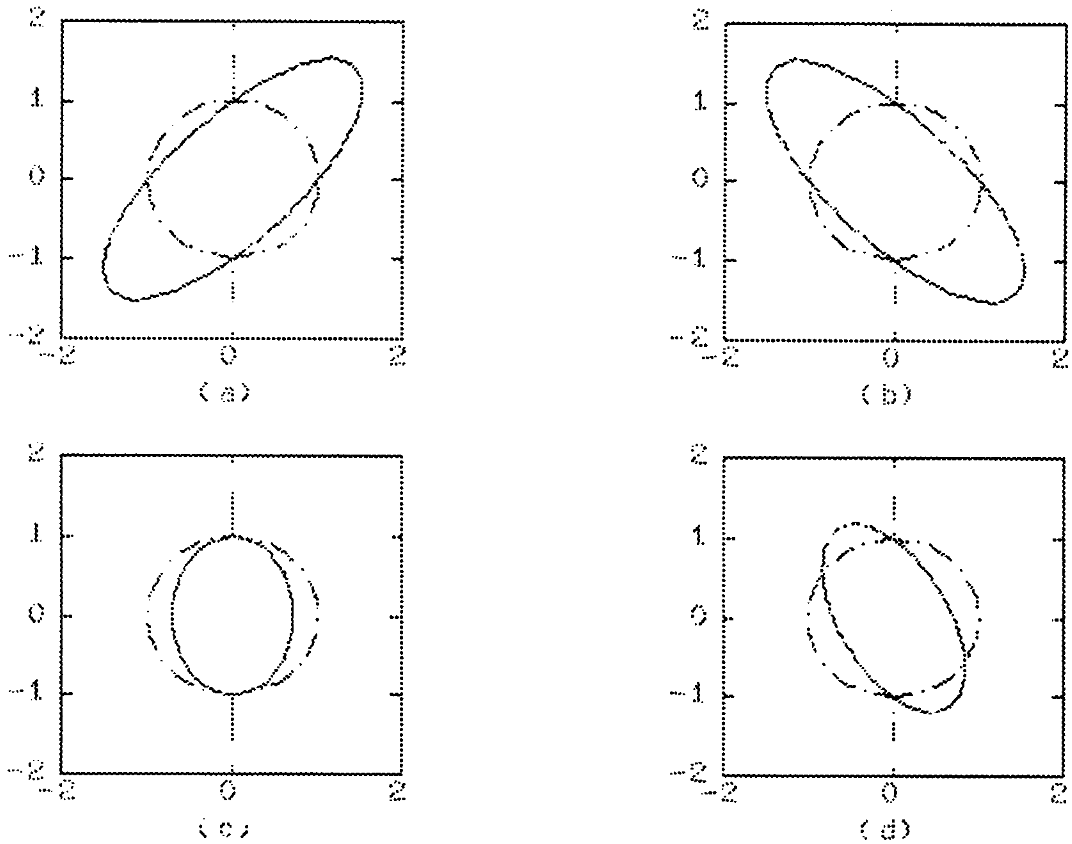

Figure C.1. Geometrical view on the effects of elements of S.