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EMMETT EUGENE CURRAN II for the MASTER OF SCIENCE
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The problem of analyzing the general linear, time-varying system is investigated. The approach starts with four different system representations: the differential equation, transform representations, signal flow graphs, or block diagrams. Given one of these system describing representations, it is shown what the engineer can find out about the system and the methodologies concerned.

The thesis is primarily categorizational, not instructional. It is directed towards those engineers and interested persons who are unfamiliar or inexperienced with linear, time-varying systems and who have an interest in such systems to fulfill. With the annotated bibliography, the thesis is intended to be a starting point in their search for knowledge--a guide to some of the more important works and ideas in this particular field.

Approaches to Linear, Time-Varying Systems Analysis

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Emmett Eugene Curran II

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Professor of Electrical and Electronic Engineering
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Head of Department of Electrical and Electronic Engineering

Dean of Graduate School

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Typed by Donna L. Olson for Emmett Eugene Curran II

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APPROACHES TO LINEAR, TIME-VARYING SYSTEMS ANALYSIS

CHAPTER I

INTRODUCTION

Background

In modern times, a system has become a universal concept that is used in reference to a group of objects or entities interacting with each other and performing some task or function. Engineers are familiar with mechanical, chemical, and electrical systems, where some of the interacting objects might be motors, reaction vats, or amplifiers. Biologists are familiar with ecological systems, and economists are familiar with the world's monetary systems--so the system concept, strictly speaking, is not restricted to engineering. The thesis, however, is directed primarily to systems in engineering.

As technological and scientific knowledge has grown in the last century, the complexity of the systems with which engineers have been concerned has grown tremendously, as, for instance, one familiar with the accomplishments of Apollo 8 and 9 can well testify. Unfortunately, the job of analyzing these increasingly complex systems has become more and more tedious and time consuming, and in some cases impossible. Along with the analysis of modern day systems, problems involving the synthesis and compensation of systems have become increasingly involved. In fact, the modern engineer is concerned with three separate problems, all related, but nevertheless so complex as to justify their being considered separately--namely, system analysis, system synthesis, and system compensation. This thesis will be concerned with the first problem, systems analysis.

Basically there are two types of systems, linear and non-linear. A system is defined to be linear if it obeys the principle of superposition: assume one is given the input to a system as $x_1(t)$, which would result in an output $y_1(t)$, and another input, $x_2(t)$, which correspondingly would result in the output $y_2(t)$, if applied separately. Then, if the input to the system defined by

$$x(t) = a\, x_1(t) + b\, x_2(t),$$

where a and b are arbitrary constants, gives rise to the output

$$y(t) = a\, y_1(t) + b\, y_2(t),$$

then the system is linear. Otherwise, it is considered to be non-linear.

This paper will be concerned with linear systems, of which there are two types: time-invariant and time-varying systems. A linear system is time-invariant if the form (shape) of the response of the system to a particular input does not depend on the time at which the input is applied: for instance, if $x(t)$ is the input in question and $y(t)$ is the corresponding output, then if for any arbitrary t_o, $x(t+t_o)$ gives rise to $y(t+t_o)$ at the output, then the system is time-invariant. If it does not, then it is a time-varying system, which will be the concern of this thesis. A time-varying linear system can usually be recognized by inspection since some parameter in the system will be a function of time.

Aim of This Thesis

Most of the modern textbooks on linear systems analysis are devoted to time-invariant systems since they lend themselves quite easily to relatively simple, straightforward mathematical analysis. There are a few, if any textbooks that concern themselves with time-varying systems primarily. Most of the information on such systems is buried in modern periodicals, or included as a generalizing chapter in books on time-invariant systems.

The interest in LTV (linear, time-varying) systems has grown in recent years for a number of reasons. Practically speaking, no system can be absolutely time-invariant since some system parameters in almost any system change with time as the result of aging, climate, the time of day or sundry other reasons. It has also been discovered that some systems perform better if certain parameters are made to be time dependent in specific manners. A need, therefore, exists for present information on handling LTV systems analysis to be collected and categorized for the use of those engineers involved with such systems--especially those engineers unfamiliar with LTV systems. At present, to the knowledge of the author, no such text or work exists.

The modern engineer, confronted with a LTV system problem has no textbooks to consult, unless he cares to try to dig sufficient information out of the linear, time-invariant systems books which usually have little to say about LTV systems that is of any practical value. There probably is enough information in certain articles for the engineer to at least scratch the surface of his problem--if he knows where to start looking, which in itself can be a somewhat tedious and frustrating search.

The purpose of this thesis, then, is to at least start to fill this need. To do a respectably complete job on the many problems involved with LTV systems would undoubtedly fill many volumes and is beyond the scope of this thesis. The following chapters will therefore be concerned only with part of the analysis problem of LTV systems. It will be assumed that all signal and time functions will be deterministic and that the consideration of noise will be unnecessary. Given a LTV system, the engineer will be shown, or at least introduced to, what he can know about the system, the approaches he might take to obtain such information, and where he must go to learn about the relevant methodologies.

It should be recognized that the following chapters are primarily categorizational and not instructional. They are intended to be the starting point in an analysis problem, from which, with the help of the bibliography (which is annotated) the engineer can decide where he must go to get the detailed information he needs to attack his problem: to give him direction. Consequently, the engineer's efforts in a library are done with purpose, rather than as a random search performed with the hope of coming across some article or book that might help him.

Hereafter, it will be assumed that the engineer has at hand something that represents the system with which he is concerned. This will be called the System Describing Form (SDF) and is not to be confused with the system describing function used in certain types of non-linear system analysis. The SDF can be a differential equation, a transformed representation, a signal flow graph, or a block diagram. The problem, then, is this: given one of the aforementioned SDFs, where does one go to find out certain characteristics of the system? This thesis will be dedicated to answering and discussing this question.

System Characteristics

The next logical question is in what system characteristics might the engineer be interested? The most important, and the ones considered here are: stability, boundedness, system behavior as time approaches infinity (hereafter referred to as asymptoticness), and the response to a specific input. The first three are quite closely related, but not equivalent as will soon become apparant.

There are numerous ways of defining stability--it has been defined in terms of the analyticity of the Laplace transform; in terms of the character of homogeneous solutions; in terms of impulsive response properties; and in terms of the properties of one-sided Green's functions. This is certainly not a complete listing of the ways to define stability. It is felt that the references in the following chapters, however, will cover most, if not all, of the important definitions, since stability is a very important, even criticial, characteristic of engineering systems.

Some engineers might associate stability and boundedness as being equivalent. This is not true. Clearly, stability implies boundedness, but boundedness does not necessarily imply stability, as, for example, in an oscillator, which is certainly bounded, but has no stable state. Very simply, boundedness means that if an input to a system is bounded, then the output will be bounded. It is an important consideration since the engineer is usually concerned about any sensible input not resulting in an output that increases without limit, perhaps damaging his system.

The behavior of the system as time goes to infinity is intimately related to the previous two characteristics. The concern here is usually whether or not any of the transient responses of a system will die out, or go to zero for infinite time. Such a system might be considered to be asymptotically stable. The relationship between asymptoticness and the previous two characteristics, then, suggests that it might be encompassed by stability and boundedness.

In some cases, the engineer may need to know precisely what the response of a given LTV system is to a specified input (the particular solution). Clearly, this is an important, though generally difficult, if not impossible, problem to solve, In a great many cases, the engineer needs to know not just that a positioning system, for example, is stable, but what its exact response is in transit between two positions, since the motion may be the input to another system and must be known in order to determine the output of the second system.

Information Contained in SDFs

In order to determine the characteristics mentioned in the last section, the necessary information must be available in the SDF. This logically brings up the question of how much information is contained in the SDF. Does it contain all the information about the system, or only part--needing supplemental information to complete the total representation of the system and to allow determination of the aforementioned characteristics?

This line of thought raises the question of whether or not what information that the SDF does contain is readily obtainable. It could conceivably be in explicit form; or manipulations, simple or complex, might be necessary to extract it. It could even be completely unobtainable under the present state of the art, though theoretically the information is there, just waiting to be extracted.

For those characteristics that can be analyzed with information practically obtainable from the SDF, the next problem is how to get to it if it is not explicitly found in the SDF. The preceding points are the basic points that will be covered in an introductory manner in each of the following chapters for the different SDFs.

The Following Chapters

The next three chapters will individually investigate the four basic SDFs spoken of here: the differential equation, the transform representations, the signal flow graph, and the block diagram--the last two will be covered in the same chapter since they are quite similar representations.

In each chapter, the information content, obtainability, and the methodologies involved that apply in general to obtaining the information from the different SDFs will be discussed in an introductory manner. Mention will be made of any important highlights or points, and effort will be made to see that everything discussed will be well referenced so that the engineer, when he recognizes something that will be of use to him, can go directly to the relevant reference for more details.

In light of this, the bibliography is an annotated one. The titles of articles and books are usually quite ineffective in conveying accurately the topics they discuss. Consequently, annotations have been added to the bibliographical listing in order to more precisely describe the important features and discussions contained in the articles and books. This thesis, then, with the assistance of the bibliography, is meant to be a worthwhile starting point for the engineer who is unfamiliar with LTV systems analysis and who has need to do some work or fulfill some interest in this area. It is not contended that the bibliography contained herein covers completely the field of LTV systems analysis. To the contrary, it does not. But effort has been made to ensure that a number of the important works on the problem of LTV systems analysis discussed here have been included, and that a hopefully representative cross section of the relevant works has been presented.

Appendix I presents and references a few of the common LTV differential equations that have been successfully solved. It is felt that these special cases are quite well covered in the mathematical literature and beyond the scope of this thesis. Nevertheless, a few common cases are mentioned briefly and referenced for the sake of completeness.

Appendix II discusses the uses of computers in LTV systems analysis--again, for the sake of completeness. An article describing a little known use of analog computers is briefly discussed and referenced. The possible uses of digital and hybrid computers are also briefly mentioned.

The uses of computers in the analysis problem is certainly extensive and invaluable. But it is felt that their use is just an extension of the analytical methods, a tool, as it were, rather than an analysis technique in its own right. For this reason, discussion of computers is left to a brief appendix and is considered beyond the scope of this thesis--this is not at all intended to belittle the powerful uses of computers in systems analysis.

CHAPTER II

DIFFERENTIAL EQUATIONS

Introduction

This chapter will be based on the assumption that the engineer has a linear differential equation with time-varying coefficients as his SDF, and which he wants to analyze to determine certain system characteristics. Specifically, he may want to know whether the system described by his equation is stable, bounded, or what its particular solution is.

In order to avoid confusion, it might be best at this point to review the ways differential equations are written. Most everyone is familiar with the standard representation:

$$(1) \quad a_n(t)\frac{d^n y(t)}{dt^n} + a_{n-1}(t)\frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1(t)\frac{dy(t)}{dt} + a_o(t)y(t) = F(t)$$

where the $a_i(t)$ $(i = 0, 1, 2, \ldots, n)$ are the time-varying coefficients; $y(t)$ is the output of the system; and $F(t)$ is the forcing function. The above equation can be written in what is called the operator form:

(2) $$L(p,t)y(t) = F(t),$$

where L (p, t) is the operator defined by

$$a_n(t)\,p^n + a_{n-1}(t)\,p^{n-1}(t) + \cdot\cdot\cdot + a_1(t)\,p + a_o(t)$$

with $p = \frac{d}{dt}$. This is just a shorthand method of writing the first equation--both representations mean the same thing. It should be noted that the forcing function, F(t), is a function of the input and a certain number of its derivatives. For physically realizable systems, the order of the highest derivative of the input in F(t) must be lower than the highest order derivative appearing on the lefthand side of equation (1) or (2). Another way of writing the differential equation is called the matrix, state variable, or vector representation and is represented by the following equation:

(3) $$\frac{dy(t)}{dt} = A(t)y(t) + X(t)$$

where y(t) is an n-vector, A(t) is an nxn matrix (known as the coefficient matrix), and X(t) is another n-vector, which accounts for the forcing function. Ince (1927, p. 73) demonstrates the equivalence of the standard representations (equations (1) or (2)) and the matrix representation (equation (3)). Both representations contain all the information on the system and each representation can be obtained from the other. In some cases, there exist arguments for the preference of one representation over the other, as in the case of time-varying networks (Kinarawala, 1961), but in general, the representations are equivalent.

Given one of these SDFs, what must the engineer do to determine the stability, boundedness, or particular solution of the associated system? For the case of stability, nothing can be determined from the representations themselves. This is quite unfortunate--and it also applies to the determination of boundedness and asymptoticness. If the engineer knows the n homogeneous solutions of his equation, stability and boundedness properties can generally be determined--see Cesari (1959), Bridgland (1963), Bellman (1943, 1946, 1947, and 1953), and Massera (1956) for detailed discussions of the different types of stability and boundedness, and theorems on these characteristics in such a case. The problem of determining a LTV system's stability and boundedness, then, reduces to the determination of its n homogeneous solutions.

The problem of finding the system's particular solution also reduces to the determination of the n homogeneous solutions. Once they are known, application of the variation of parameters method can be used to determine the particular solution.

The following discussions will be devoted, then, to the problem of finding a system's homogeneous solutions--once they have been established, it is a straightforward matter to determine the system characteristics that have been mentioned.

Methods of Solution

Before continuing, it should be pointed out that there is an alternative to having to find the n homogeneous solutions of a system before being able to determine its system characteristics. An equivalent method is to determine the system's impulse response, W(t, T), which is a general expression relating the system's output at time t to an impulse that was applied at time T. Knowing W(t, T), a particular solution can be determined by the superposition integral. Conditions on W(t, T) will also specify system stability or boundedness if they are satisifed. The impulse response, however, is closely related to Zadeh's system function, which is discussed in the next chapter, and is covered there. This chapter, therefore, is devoted to the method of determining the n homogeneous solutions while W(t, T) is covered separately in the next chapter. Both of these approaches are essentially equivalent in the final results they give, and the difficulty involved in their determination.

The Standard Representation

The first-order linear differential equation with time-varying coefficients can always be solved through the use of an integrating factor. Unfortunately, direct solution of the general LTV differential equation is an extremely difficult problem, since there are no known techniques that can be applied in general. Part of the problem is that solutions cannot be determined in the form of some set of elementary functions (Derusso, Roy, and Close, 1965; Sections 2.7 and 2.8). Consequently, discussion here will be of a few techniques that can be of value, but which cannot guarantee solutions--indeed, they are not necessarily applicable in every instance. Ince's book (1927) discusses the general problem of solving differential equations with time-varying coefficients--he investigates some techniques that apply to specific classes of problems, none of which can be applied generally--and is recommended as a good reference book.

In some cases, an appropriate substitution will simplify an equation so that it can be more easily solved, or change it to a time-invariant form which can always be solved. Dasarathy and Srinivasan (1968) discuss a substitution method that can simplify some LTV equations with periodic coefficients whose derivatives can be written in terms of the original coefficients (for example, sine and cosine functions). Derusso, Roy, and Close (1965, p. 69) show how Euler's equation,

$$\sum_{n=0}^{N} a_n (b + ct)^n \frac{d^n y(t)}{dt^n} = 0,$$

where the a_n, b, and c are constants, t is time, and y(t) is the output, can be solved by using the substitution

$$(b + ct) = e^z, \text{ i.e., } z = \ln (b + ct),$$

which reduces the equation to one with constant coefficients. Skylarevich (1961) has shown how, in certain cases, a time-varying operator equation (like equation (3)) may be transformed into the sum of some equations with time-invariant operators, which can be easily solved.

If some number, r, of the n homogeneous solutions are known, it is possible to reduce the order of the equation being solved to one of order n-r. This may drastically simplify the problem--every time the engineer manages to discover one of the homogeneous solutions he can reduce the equation he is working with by one.

Many differential equations with time-varying coefficients have been solved--most of them are equations of a particular form that arise in many different physical problems (e.g., Bessel's and Mathieu's equations). Basically, they were solved by the application of certain techniques that for one reason or another could be applied to them only and not other problems in general. A few of the common such cases are presented and referenced in Appendix I--discussions of such equations can usually be found in any good, comprehensive book on ordinary differential equations (e.g., Ince, 1927). These cases are not covered specifically here because the techniques associated with them are quite specialized to the particular equation and not applicable in general.

Basically, the general approach to LTV systems in differential equation form is somewhat akin to non-linear systems analysis in that each and every problem is different--and so are its methods of solution. There are techniques for handling certain types of problems, but no techniques exist which apply in general. The engineer has to draw heavily on his experience, ingenuity, insight, and sometimes even luck to handle these types of problems.

This sounds quite discouraging--and it is somewhat discouraging. Most books and articles on differential equations abound with discussions of properties of differential equations with time-varying coefficients--under the assumption that the homogeneous solutions are known. But very few of them show techniques for finding the solutions--because there are no general ones. A survey of the literature has shown that no generally applicable method exists for solving differential equations with time-varying coefficients.

The Matrix Representation

Solving the matrix representation problem (consisting of n first order equations in n unknowns) is equivalent to solving the n^{th} order standard representation. Solving either will enable the determination of the particular solution and the determination of stability.

It has been said that the first order time-varying differential equation is always solvable--the engineer should not be tempted to think that the matrix representation is therefore always solvable since it is comprised of only first order equations. This is not true, since the first order equations are in n unknowns. The unknowns couple the different equations so that they cannot be solved independently--they must be solved simultaneously. The problem, therefore, is of the same magnitude as solving the standard representation.

The matrix representation is investigated quite thoroughly by Zadeh and Desoer (1963)--they refer to it as the state space approach--for the linear time-invariant and the LTV matrix representation problems, though the emphasis is on the former. Some methods of solution are discussed that might be applicable to LTV systems, and the reader is encouraged to read the book to determine if they might be applicable to his particular problem (they are not discussed here since they are quite narrow in scope).

Initially Excited Systems

Consideration so far has been solely given to the unexcited case (no initial conditions). It should be noted that Solodov (1958) has considered the excited case by transforming the initial conditions to various order delta functions. These functions are then applied to the unexcited system, and they give an output equivalent to the output of the excited system--this method assumes knowledge of the impulse response, W(t, T). With knowledge of the n independent homogeneous solutions, the excited case is just a matter of determining the n arbitrary constants--no special procedure is needed to handle such a situation.

Conclusions

Unfortunately, nothing very specific can be said about the problem of finding the homogeneous solutions to the general linear differential equation with time-varying coefficients. It is a potentially discouraging, difficult problem with no generally applicable procedure or technique. Indeed, a survey of the methods for solving the linear time-varying differential equation presents only a study of special cases--examples. To categorize each of these more or less narrow methods here would be beyond the much broader scope of this thesis: a categorization of methods that apply in general.

Hopefully, the engineer confronted with a differential equation with time-varying coefficients has not been discouraged by this chapter into thinking that such equations are hopelessly unsolvable. The main point here has been to illustrate that no generally applicable techniques exist to solve such equations. There are, however, many different methods that may or may not enable the engineer to handle these types of equations to a certain extent--only he can decide if these techniques can apply to his particular problem.

Substitution methods are certainly applicable to a large class of time-varying differential equations--but there are no deterministic rules for implementing such an approach. It has been shown that as solutions are found, the order of the equation can be successively reduced, perhaps simplifying the problem. There is also a set of problems that have been solved and which have been found to appear in several physical problems. It is possible that a given problem may be similar in form to one of these special cases, enabling a solution to be found. There are several other techniques that may be of assistance--a large number of which are found in the literature that has been cited throughout this chapter.

It goes without saying, therefore, that the engineer is going to have to rely heavily on his ingenuity and inventiveness in attempting to solve a differential equation with time-varying coefficients. He may have to make assumptions and approximations to simplify his problem so that he can solve it. Until a generally applicable method is discovered, then, the engineer will have to lean more on his natural talents as an engineer than formulas and procedures listed in textbooks.

CHAPTER III

TRANSFORM TECHNIQUES

Introduction

The use of transform techniques in the solution of problems has developed from a desire to reduce the difficulty involved in solving the given problem. The transformations discussed here will be those that involve the differential equation and those functions that are intimately related to it (i.e., the impulse response and the one-sided Green's function). The philosophy of most transform techniques, then, is to change the given differential equation into an equivalent, much simpler form, thereby facilitating the solution of the equation. For time-invariant systems this is readily observed in the Laplace transform techniques where the problem of solving the relevant differential equation is transformed into that of solving an algebraic equation.

Almost all transforms used in engineering are integral transforms. Each transform usually applies to a certain class of problems and the usefulness of the transform is dictated by the size of the class of problems to which it applies. Laplace transforms apply to all linear time-invariant systems and therefore are used quite extensively--Hinkel and Mellin transforms, for instance, are much more restricted in use (Aseltine, 1954) because they apply to a smaller class of problems.

It should be noted that it is possible to generate successfully a specific transform for a specific problem. There are no general, straightforward techniques to such an approach--it being mainly guesswork--but it is possible, and it should not be ignored in considering the use of a transform approach to a problem.

This chapter will be concerned primarily with the methods developed by Zadeh, since his approach is the only one that applies to general LTV systems, but will also briefly consider some of the other transforms since they do apply to certain classes of problems and should not be ignored. A brief discussion of the limited uses of Laplace transforms, along with why they don't work in general for LTV systems will be presented first, followed by mention of some minor transforms and the general integral transform. The system characteristics that will be investigated from the transform approach will be stability, boundedness, and the response of the system to a specific input.

Laplace, Hinkel, Mellin, and General Integral Transforms

In applying the Laplace transform to LTV systems, the engineer either has the transformed equation, or the differential equation associated with it, as his original SDF. In either case, it is best if he obtains the other form for reasons that will become apparent below. This is usually accomplished quite simply since he almost always has the differential equation initially and must perform the transforming operation to get the transformed equation (i.e., he hardly ever has the case where he must derive the differential equation from the transformed equation--this being potentially more difficult).

The transformed equation may or may not simplify the problem (Derusso, Roy, and Close, 1965; Gerardi, 1959; Pipes, 1955). The engineer may find that he is trading one difficult equation for one that is just as difficult, or possibly even more difficult, to solve--consider the following example:

Example: Consider the differential equation with time-varying coefficients:

$$t^3 \frac{d^2 y(t)}{dt^2} + t \frac{dy(t)}{dt} - y(t) = F(t).$$

Assume the initial conditions are all zero, and take its Laplace transform:

$$-\frac{d^3}{ds^3}\left[s^2Y(s)\right] - \frac{d}{ds} sY(s) - Y(s) = F(s)$$

$$-s^2 \frac{d^3Y(s)}{ds^3} - 6s \frac{d^2Y(s)}{ds^2} - 6 \frac{dY(s)}{ds} - s \frac{dY(s)}{ds} - Y(s)$$

$$- Y(s) = F(s)$$

$$-s^2 \frac{d^3Y(s)}{ds^3} - 6s \frac{d^2Y(s)}{ds^2} - (s+6) \frac{dY(s)}{ds} - 2Y(s)$$

$$= F(s)$$

Note that the transformed equation is more complex by virtue of the fact that the order of the equation has increased--and it is still a differential equation with varying coefficients.

In this case the transform has failed in its purpose to simplify the problem and is therefore essentially useless. This is the reason for having both the differential equation and the transformed equation at hand--to see by comparison if the transformation has really reduced the complexity of the problem.

In some cases, however, the Laplace transform will prove useful as a transform--it will simplify the problem somewhat. In particular, if the time-varying coefficients involved are polynomials in time, with the highest power of t being m, then the transformed equation is of order m (Kaplan, 1962, p. 470). If m is less than the order of the differential equation, the problem has been simplified to that of solving an equation of lower order.

Example: Consider the equation

$$t \frac{d^3 y(t)}{dt^3} + (t+1) \frac{d^2 y(t)}{dt^2} - t \frac{dy(t)}{dt} - t\,y(t) = 0$$

with zero initial conditions. Its transform then is

$$- \frac{d}{ds} [s^3 Y(s)] + s^2 Y(s) - \frac{d}{ds} [s^2 Y(s)] - \frac{d}{ds} [sY(s)]$$

$$- \frac{dY(s)}{ds} = 0$$

$$- s^3 \frac{dY(s)}{ds} - 3s^2 Y(s) + s^2 Y(s) - s^2 \frac{dY}{ds} - 2sY(s)$$

$$- s \frac{dY}{ds} - Y(s) - \frac{dY}{ds} = 0$$

$$(-s^3 - s^2 - s - 1) \frac{dY}{ds} - (2s^2 + 2s - 1)Y(s) = 0,$$

which is simpler than the original equation, since it is of lower order--the order of t to the highest power in the original equation.

In the general case, the Laplace transform finds very limited use since it does not always reduce the order of the equation to be solved. For the same reason, Fourier transforms fail in the general case because they do not always simplify the problem, and may even increase the complexity of the problem (Brown and Nilsson, 1962, p. 259-260).

Gerardi (1959) has shown that Hinkel and Mellin transforms can be applied to small classes of specialized problems--the Euler-Cauchy problem (where all the terms are of the form

$$a_n t^n \frac{d^n y(t)}{dt^n}$$

with a_n a constant and n an integer), and the well known Bessel equation (see Appendix I), respectively. Aseltine (1954) has considered the general integral transformation to some extent and notes that, as has already been mentioned, special transforms can be invented to apply to specific problems. The key difficulty here is alluded to by the word "invented," indicating that guesswork is involved--the assurance of success in this approach is hardly always present. No methodology exists at present for finding such transforms, but the engineer may be lucky and discover a relevant transform that will apply to his particular problem. But then he is faced with having to generate his own table of transforms, conceivably quite a tedious chore. In some cases this might be justified if the system under consideration must be solved for many different forcing functions, or if the importance of solving the problem justifies the work involved. At this point, the reader is referred to the references that have been cited for more details.

It should be pointed out that all the information about a system is contained in the transformed equation for any transform method, since the equation was derived from the original time domain differential equation, which is already known to contain all the information--i.e., the transformed equations completely represent the system. It is merely in a different form, and not necessarily in a form allowing it to be extracted more easily, though hopefully, that is the case.

The Generalized System Function

It has already been said that linear, time-invariant, systems-analysis transform techniques are not adequate to handle LTV systems. The time-invariant techniques must be extended, or generalized, in order to satisfactorily handle the more complex LTV cases. This work was primarily done by Lotfi A. Zadeh and will be discussed here. It is also recommended that the reader investigate any modern book on linear systems analysis for compact presentations of Zadeh's major contributions (e.g., Brown and Nilsson, 1962; or Kaplan, 1962).

Zadeh suggests (1950a) the concept of a time dependent system function that would be a generalization of the time-invariant system function, H(jw). H(jw), it will be recalled, is the Fourier transform of the response of a system to an impulse function applied at $t = 0$. More specifically, the impulse response, written as W(t), is a function of the so-called age variable, which is the time between the instant of observation and time the impulse is applied. In linear time-invariant systems analysis, the input is always assumed to be applied at $t = 0$, so the age variable and time are identical. Nevertheless, W(t) is dependent on the age variable, only. The t for time-invariant systems, therefore, should more precisely be called the age variable.

For time varying systems the impulse response is a function of time and the instant at which the impulse is applied, hereafter called T (the age variable, then, is t-T). If one considers the impulse response, generalized for LTV systems, to be W(t, T), the natural generalization for the system function is H(jw; t), defined by:

$$H(jw; t) = \int_{-\infty}^{\infty} W(t, T)e^{-jw(t-T)}dT$$

(This is the system function in Fourier transform representation--Zadeh (1950c) discusses it in Laplace and operator representations.) Basically, this is what Zadeh proposes, and he proceeds to show (1950a) that the problem of solving a general LTV problem for a specific response, given the input, reduces to one of finding $H(jw; t)$. He shows how to determine the output knowing the input and $H(jw; t)$, and points out that it is generally easier than using the impulse response and the superposition integral. The first method involves algebraic manipulations and the use of transform tables, while the second involves evaluating a potentially difficult integral.

Zadeh (1951) has also considered the problem of stability and boundedness and what can be determined about these characteristics if one knows $H(jw; t)$ or $W(t, T)$. In his approach, it is often easier to consider $W(t, T)$ for stability than $H(jw; t)$, and this seems to be the case in practice--$H(jw; t)$ is used for determining a specific response and $W(t, T)$ is examined for stability (this latter case will be discussed below).

So far, it has been assumed that $H(jw; t)$ is conveniently at hand--indeed, when it is, it is only a straightforward procedure to determine the system characteristics. But this is always the case. It can be analytically evaluated if $W(t, T)$ is available, but this is not always the case either. Zadeh shows an analytical method for determining $H(jw; t)$ which involves solving a differential equation in $H(jw; t)$ of the same order as the original problem (1950a). Immediately it seems as though one difficult problem is being traded for an identically difficult problem, in contradiction to the stated purpose of a transform (to simplify the problem). This is not the only method, however, for finding H(jw; t). Nevertheless, with some simplifying assumptions, Zadeh's method will simplify some systems, as he goes on to show for the case of systems with slowly varying parameters (by slowly varying is meant that the parameters do not change appreciably in value while the system is responding to an impulse). Rudnitski (1961) has successfully attempted to extend the determination of H(jw; t) to systems whose parameters may be rapidly varying--but his work is concerned only with coefficients that are polynomials in t.

The primary drawback with the generalized system function is not in its utilization, but in its determination. Once it is known, the analysis procedure is relatively simple and straightforward. Consideration must be made for the difficulties involved in the determination of H(jw; t) to keep the system function approach of Zadeh in proper perspective--his method works, but there are problems in initially getting the system function. For this reason work has been done on the problem of finding general methods for evaluating H(jw; t). H(jw; t) has already been shown to be intimately related to W(t, T); so determining W(t, T) allows H(jw; t) to be evaluated. The problem can then be considered identical with finding W(t, T). Zadeh (1950b) shows that W(t, T) is related to the one-sided Green's function (G(t, T)--see Miller, 1951) associated with the differential equation describing the system--knowing G(t, T), one can then find

W(t, T) and then evaluate H(jw; t). Zadeh shows a method that determines G(t, T) but his method is restricted to slowly varying systems.

Zadeh (1961) surveys most of the work done on these problems in the ten years following his first paper on the subject. The development of the usefulness of H(jw; t) is quite extensive, but still there have been no major breakthroughs in the problem of how to simply and analytically deduce H(jw; t) in the general case.

It would be clarifying to stop for a moment and consider the implications of what has so far been discussed. The main question seems to be whether or not this extended system function approach has any advantage over the other transform methods, considering the serious drawbacks which have been discussed in determining the system function in the first place. One of the more attractive features of this generalization is that it applies to LTV systems in general whereas other transform methods are limited in applicability to a few specific problems or classes of problems. It is also attractive because it parallels linear time-invariant systems analysis so closely. A major breakthrough in LTV systems analysis will occur when someone works out a simple methodology to determine H(jw; t)--it has not been shown that such a method does exist. Hopefully, as systems analysis techniques are extended and refined, this elusive system function determining method will be discovered. For the purposes of this thesis, emphasis on the extended system function approach is justified since it applies in general and therefore is encompassed by the stated purpose of this thesis. H(jw; t) is the SDF, and, therefore, it deserves to be considered here. Notwithstanding, however, it should be realized that due consideration must be made for the problems involved in determining these associated SDFs (system function, impulse response, and one-sided Green's function), to keep the problem in proper perspective so that the reader will realize that even though Zadeh's approach is powerful once one of the aforementioned SDFs is given, it has the basic, serious drawback in that these SDFs are quite difficult to obtain. But once they are obtained, this approach is a very valuable and informative one in determining system characteristics.

The Impulse Response and Green's Function

Both W(t, T) and G(t, T) have been shown to be directly related to H(jw; t)--knowing either of them, it is a simple matter to determine H(jw; t). For this reason, each of these functions are briefly given separate consideration.

The Impulse Response

Though H(jw; t) is usually easier to work with in finding specific responses, it is sometimes easier to determine W(t, T) and then to determine H(jw; t) from it, instead of finding H(jw; t) directly. Methods do exist to find W(t, T) straight from the differential equation (Derusso, Roy, and Close, 1965; Kaplan, 1962; Borskii, 1959; Stubberud, 1964, Chapter Five) but the problem is equivalent in difficulty to solving the differential equation. Still, in some cases it might be more convenient to find W(t, T) (and therefore H(jw; t)) than to solve the equation directly, therefore justifying its determination. Of course, there is always the possibility that W(t, T) might be determined or at least approximated from experimental or observational data. In any event, knowing W(t, T) allows H(jw; t) to be found. W(t, T) can also be used by itself to find a particular solution by use of the superposition integral.

W(t, T) can be examined directly for stability, instead of converting it to H(jw; t), and then examining this latter function. Youla (1963), Kalman (1962), and Bridgland and Kalman (1963) have discussed quite extensively the conditions that must be satisfied by W(t, T) to insure system stability--they also discuss quite thoroughly the validity of this approach to the determination of system stability and the reader is referred to these articles for very interesting arguments on the subject.

Green's Function

Miller (1955), Miller and Schiffer (1952), and Zadeh (1950b) have shown that the Green's function associated with a differential equation and its impulse response are essentially identical--knowing one, the other can be derived relatively easily--and then the system function can be determined. Miller (1951) has also shown how to obtain G(t, T) from the differential equation, but it is a computationally complex method, though not impossible, and in some cases, a simpler problem than solving the related differential equation directly for a specific response or for the impulse response. In this manner, the problem is simplified--once G(t, T) is known, W(t, T) can be obtained and the procedure from this point is as discussed above for finding H(jw; t) from W(t, T).

Conclusions

Most integral transforms cannot be generally applied to LTV systems since they do not guarantee the simplification of the problem. In some cases, however, they may reduce the order of a differential equation by transforming it to a differential equation of lower order. The standard integral transforms, however, are not generally applicable to LTV systems.

Zadeh has successfully extended the concept of system function in linear time-invariant systems analysis to LTV systems and has shown that it is a generally applicable method. Its use is quite similar to the way the system function for time-invariant systems is used in solving for responses to specific inputs, or in determining stability. It has a very serious handicap in that the time-varying system function is difficult to obtain--and Zadeh's approach is centered around it. At the present time there is no simple, straightforward, practical method for finding H(jw; t) in general--it can be found in certain cases, however, like for slowly varying systems. It can also be found indirectly by first finding W(t, T) or G(t, T)--but those are complicated problems similar in magnitude to finding H(jw; t). Nevertheless, Zadeh's H(jw; t) is the only transform representation that applies in general to LTV systems analysis.

CHAPTER IV

SIGNAL FLOW GRAPHS AND BLOCK DIAGRAMS

Introduction

At least some familiarity with signal flow graphs (SFGs) and block diagrams (BDs) is assumed; nevertheless a very brief description will be given here to avoid any misunderstandings. Both of these SDFs are essentially pictorial representations of a system, derived from either the actual system or from the differential equations describing it. Both the SFG and the BD contain complete information on the system, since they are generated from the system itself, or a differential equation, which is already known to describe the system completely.

Basically, a SFG consists of a collection of nodes and directed branches with associated transmittances. The nodes represent the variables in the system, and the directed branches represent the relationships that exist between the variables. The graph is "read" by the understanding that at each node, the sum of the quantities coming in on the incoming branches is algebraically summed and sent out on all the outgoing branches--and a branch multiplies its transmittance by the variable at its tail, supplying the product at its head. For a more involved description of SFGs and the manipulative techniques associated with them, see Mason's two articles, (1953) and (1956).

A BD consists of blocks with an input and an output, summers, and signal flow lines. The output of each block is equal to the product of the input to it times whatever function is written inside the block, commonly referred to as the block's transfer function. A summer is actually a block whose output equals the sum of its inputs, which may be more than one. The signal flow lines show the relationships between the blocks and summers, and are understood to be directed branches with a transmittance of one. One obvious difference between SFGs and BDs may come to mind at this point, and that is that it is not always clear where to go on a BD to find the value of a particular system variable, while on an SFG, it is just a matter of finding the appropriate node. An advantage of the BD is that it can be a very representative pictorial diagram of a system where each block represents one of the divisable parts of the system (amplifier, motor, oscillator, generator, etc.)--see Brown and Nilsson, 1962, p. 100-108.

There are certain similarities between SFGs and BDs that are strong enough to tempt one to consider the two representations to be equivalent. There are also differences, however, pointing out the fact that they are actually different methods of representing the same thing, one having possible advantages over the other in certain cases. These differences are rather minor, but still substantial enough to justify the consideration of SFGs and BDs as separate representations. The SFG can handle a large number of system variables more easily than the BD can. The BD can represent the actual system on paper more easily than the SFG: the engineer gets a better feel for the system because he can see and visualize how the parts interact with each other, whereas this is sometimes hard to do on a SFG. It should be noted that not all systems are easily divisable into parts whose functions can be assigned to a block on a BD. In such cases, the SFG might prove to be a more feasible and better representation--from which, if desired, an informative BD might be generated after manipulations.

SFGs and BDs, then, are not equivalent representations; rather, they complement each other. The merits and problems of each, as they apply to LTV systems shall now be discussed separately. The primary concern will be in solving for the output to a specific input. Determining stability from SFGs and BDs in the case of LTV systems is practically impossible since no methodology that applies in general exists.

Signal Flow Graphs

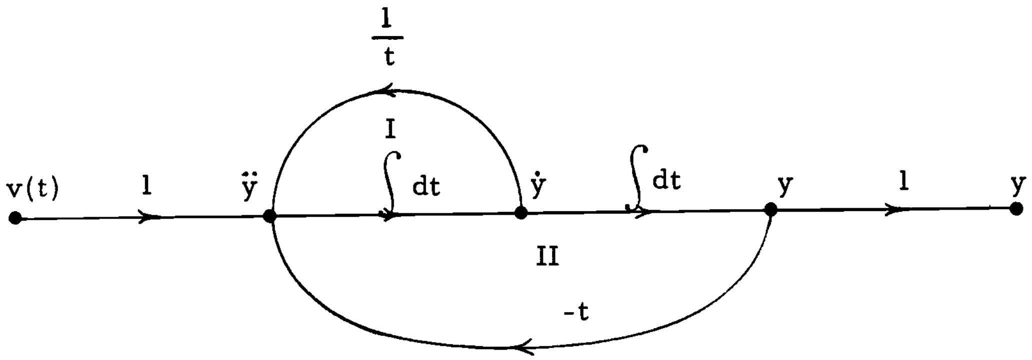
Signal flow graphs apply to linear, time-invariant systems quite naturally. By their very nature, where all transmittances are quantities that multiply the inputs to get the branch ouput, a SFG represents algebraic relationships. Though the differential equation in linear time-invariant analysis does not represent algebraic relationships, its Laplace transform does. Consequently, a SFG lends itself quite naturally to being a pictorial representation of Laplace transformed equations.

There is a problem, however, when SFGs are to be applied to LTV systems. Clearly, the differential equation does not represent simple algebraic relationships--and there still is no general transform method that changes the original equation into an algebraic one. The meaning of this is that at present, there is no transformation which allows the SFG of a LTV system to consist of purely algebraic transmittances. The following example will illustrate the problem (the Laplace transform is used in the example--similar problems will arise for any other transform that might be used).

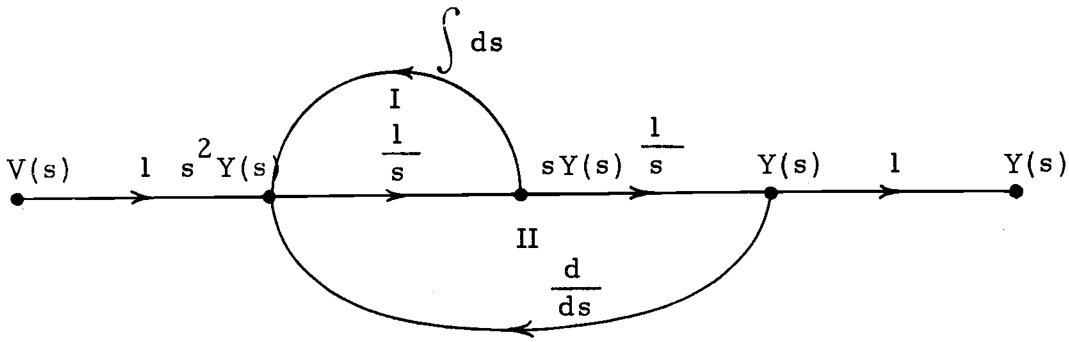
Example: Consider a system described by the following differential equation,

$$\frac{d^2 y(t)}{dt^2} + \frac{1}{t}\frac{d\,y(t)}{dt} - t\,y(t) = v(t),$$

where t is time, and y(t) is the dependent variable (the system output), and v(t) is the forcing function. The corresponding SFG, in the time domain, is this:



The transformed SFG is:



Note that the loops I and II each involve branch transmittances that are operative in form, and not algebraic. What is the loop gain of loop II, for instance? It could be

$$\frac{d}{ds}\,\frac{1}{s^2}\,,$$

$$\frac{1}{s}\,\frac{d}{ds}\,\frac{1}{s}\,,\ \text{or}$$

$$\frac{1}{s^2}\,\frac{d}{ds}\,.$$

Certainly the first two are possibilities, but the last one does not make any sense. Mason's gain formula, therefore, cannot be applied, since it is based on algebraic manipulations. There is no other methodology or set of rules at present that applies to such situations.

At this point, the future of SFGs in LTV systems analysis looks rather bleak. In the general LTV case, most of the branches in the SFG could be operative in nature, rather than algebraic. In such cases, no techniques exist yet for generating or finding the transfer function of the graph, much less its output from an input. Stear and Stubberud (1961) and Stubberud (1964) have made some efforts in developing techniques for manipulating branches representing time-varying functions. Their work is primarily BD oriented, but, it should be recalled, BDs can generate SFGs and therefore their results apply equally to SFGs. Their main results concern the combination of parallel and series branches (with transmittances represented by differential equations) into a single branch. This is principally a manipulative technique, unfortunately, and does not contribute significantly to the problem of solving the SFG.

All is not necessarily lost, however. Some equations can be transformed into time-invariant equations, and some LTV systems are basically time-invariant systems with only a few parameters varying with time. In the first case, the engineer proceeds as in the case of a time-invariant system, using Mason's gain formula to solve the graph. In the second case, it may very well be possible to divide a SFG into several time-invariant parts connected with isolatable time-varying parts. In other words, the system may be divisable into several smaller subsystems, where the SFG techniques of Mason can be applied to the time-invariant ones while the time-varying parts can be solved by some other appropriate method. After each subsystem is solved separately, they can be put together and the main problem solved by assembling the separate solutions appropriately. This cannot be done, however, if the original system is not divisable into time-varying and time-invariant parts--in such a case, SFGs can be only of very limited use in the analysis procedure--mainly as a qualitative visualization technique. This subdividing of the problem could be a tedious method of solution, but the analysis of most LTV systems is a non-trivial matter. If no logical, common sense simplifications can be made, the engineer has no choice but to revert to some other SDF (other than the BD) to attack his problem.

Block Diagrams

Block diagrams quite naturally apply to time-invariant systems for the same reasons that SFGs do. The blocks perform algebraic operations, and lend themselves to analysis and manipulations rather easily (Brown and Nilsson, 1962, p. 100-108). In general the BD representation is much more meaningful than the set of equations it represents since it gives the engineer a qualitative feel of the system by pictorially showing him how its parts interact with each other. But this is only a qualitative asset and usually only of limited help when quantitative results are desired.

The problems that arise when BDs are applied to LTV systems are identical with those illustrated in the example for SFGs. The variables involved must be consistent with each other, and in the case of most transformations, the blocks may have to perform operations rather than simple multiplications as in time-invariant systems analysis.

As in the SFG case, all is not necessarily lost, however. Granted, this approach cannot apply in general since the general LTV system differential equation may have all time-varying coefficients. But some systems can be changed into time-invariant systems and some systems are only partially time-varying. By appropriate manipulations in this latter case, it might be possible to change the BD into a series of separate, isolatable time-invariant and time-varying subsystems that can be solved separately. The solutions can then be combined appropriately to give the solution of the overall system. As in the case with SFGs, if the engineer cannot simplify the system by some means, he must revert to another SDF for a solution--but he hopefully may have at least gained some valuable insight into the problem by having been able to visualize it on paper.

Conclusions

When the operations described by the blocks in the BD and the branches in SFGs are algebraic in nature, methods do exist for determining the input-output relationships. These methods break down and do not apply when an attempt is made to make such a determination if any of the blocks or transmittances are in operator form, such as arises when these representations are applied to LTV systems. It therefore seems that SFGs and BDs are limited to time-invariant systems analysis--until a method for transforming LTV system equations into algebraic relationships is developed.

Just what, then, could the possible uses of these two SDFs in LTV systems analysis be? Of the four SDFs discussed in this thesis, these two are the only ones that give any physical feeling for the system, through their pictorial representations. By some basic manipulations it might be possible to simplify a given LTV system SFG or BD by illustrating how the system might be broken down into smaller, separable, less complex parts, not all of which would be time-varying.

Needless to say at this point, the use of SFGs and BDs in LTV systems analysis is not quantitatively powerful. Their only strong attribute is the qualitative one--pictorially representing the system. Every problem in LTV systems analysis is a new and different situation, as far as solution methodologies are concerned. These two SDFs can be of some limited assistance to supplement these methodologies to the solution of a problem. In most cases, this small amount of assistance is better than no assistance at all, and sometimes can be the difference between success and failure.

CHAPTER V

CONCLUDING REMARKS

This thesis has presented and discussed the application of some basic SDFs to linear time-varying systems analysis. Theoretically two of the four SDFs covered here, differential equations and the transform representations, can handle LTV systems. The differential equation is solved for its n homogeneous solutions, which then can be used for the determination of a particular solution or system stability and boundedness. Knowing Zadeh's system function, stability, boundedness and particular solutions can be determined. All this sounds very encouraging, at first glance, but there are serious problems between the lines of each of these assertions.

Systems analysis via the differential equation can be successful if the homogenous solutions are found. Unfortunately, there is a serious drawback in that the determination of these solutions is a very difficult problem, in general. The system function is a very useful SDF--except for the fact that arriving at it is also a very difficult problem. All the analytical avenues that terminate with the system function are difficult procedures of large magnitude. The system function may be a very useful SDF, but its utilization is impossible if it cannot be determined.

Quantitatively, signal flow graphs and block diagrams seem to be destined to apply only to time-invariant problems where system relationships can somehow be transformed into algebraic ones. The main drawback of these two pictorial representations is just that--they can only deal with algebraic relationships and LTV systems have not yet been representable in that manner. This does not mean that SFGs and BDs are worthless to the analysis engineer. Indeed they are not--their pictorial representations can be, in some cases, a valuable source of insight to the system that the other SDFs cannot give.

The problem of analyzing the general LTV system is a different kind of problem from the ones the engineer is used to in time-invariant analysis. In this latter case, there always are a certain set of "sure-fire" methods that the engineer can choose from (Laplace transform, solving the differential equation, etc.), each of which the engineer knows beforehand will eventually lead to a solution of his problem. This is not the case with LTV systems. In fact, LTV systems analysis more closely parallels non-linear systems analysis, with which the reader is probably familiar. For any given non-linear system, there is no set procedure the engineer can utilize to solve the analysis problem. There is a set of various techniques that may apply in some cases, but may not for others, from which the engineer can choose. After studying the particular problem, the engineer makes his choice or choices and picks his way through his analysis procedure, most likely running into deadends occasionally and having to alter his attack.

This same type of approach characterizes LTV analysis--there are no set procedures to handle the general problem. There is a set of SDFs, each with its own subset of possible analysis techniques, that he can utilize. Each may or may not help the engineer solve his problem. He must study his problem and make an educated judgement on his course of action. The question of selecting one of possible analysis procedures is not a question of black and white, yes or no, this one or that one. The experience of the engineer is an important factor to draw upon when he studies the different courses of action open to him.

This thesis has been directed primarily to those engineers and interested persons inexperienced or unknowledgeable of the analysis of LTV systems and the problems associated with it. Hopefully the reader now has a feeling for LTV systems analysis and its problems so that if he had such a system to analyze, he would have some idea of the avenues open to him to start his analysis. With the help of this thesis and its annotated bibliography, then, the reader should be at a starting point from which he can approach his problem with some confidence and knowledge of what may lay ahead in the form of success and difficulties, even though he may have never been exposed to a problem of this nature before. If this is the case, this thesis has succeeded in its purpose. It is the sincerest hope of the author that this is indeed the case.

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A discussion of some classical differential equations with periodic coefficients (Mathieu, Hill, Lamé, etc.); their properties and solutions are presented.

Aseltine, J. A. 1954. A transform method for linear time-varying systems. Journal of Applied Physics 25: 761-764.

The general integral transform method is described especially with respect to second order systems. A discussion on generating a specific transform for a particular problem is also presented.

Bellman, R. 1943. The stability of solutions of linear differential equations. Duke Mathematical Journal 10: 643-647.

The article relates the stability of one LTV differential equation to another similar equation whose stability is already known--sufficient conditions for the stability of the first equation are presented.

Bellman, R. 1946. On the stability of systems of differential equations. Proceedings of the National Academy of Sciences 32: 190-193.

Conditions for the stability of a perturbed LTV system, given certain conditions on the unperturbed system, are presented and discussed.

Bellman, R. 1947. The boundedness of solutions of linear differential equations. Duke Mathematical Journal 14: 83-97.

Bellman discusses many conditions and theorems on the boundedness of LTV systems (as expressed in matrix form).

Bellman, R. 1953. Stability theory of differential equations. New York, McGraw-Hill. 166 p.

This book gives a good discussion of the solutions of linear systems with respect to stability, boundedness, and asymptotic behavior, where the system differential equations are written in matrix form. The special case of the second order LTV system, in the standard representation, receives special consideration.

Bennett, W. R. 1950. A general review of linear varying parameter and non-linear circuit analysis. Proceedings of the Institute of Radio Engineers 38:259-263.

Communication engineers may be interested in this review of methods of approaching some simple LTV systems found in their field (for example, oscillators and modulators). The article is mainly discussion and not mathematical.

Borskii, V. 1959. On the properties of impulsive response functions of systems with variable parameters. Automation and Remote Control 20:822-830.

A discussion is presented on the derivation of the differential equation, given the impulse response. The difficulties of such a process are also presented. The reverse process is also discussed along with its difficulties.

Bridgland, T. F., Jr. 1963. Stability of linear signal transmission systems. SIAM Review 5:7-32.

Many definitions and conditions for different types of stability are presented for systems expressed in matrix form. Extensions from simple systems to more complex systems are also presented.

Bridgland, T. F., Jr. and R. E. Kalman. 1963. Some remarks on the stability of linear systems. Transactions on Circuit Theory of the Institute of Electrical and Electronic Engineers CT-10:539-542.

This is a sequel to Kalman (1962)--the discussion centers on the extent one can determine the stability properties of a system from knowledge of the impulse response. Specific conditions and theorems are presented and discussed.

Brown, R. G. and J. W. Nilsson. 1962. Introduction to linear systems analysis. New York, Wiley. 403 p.

Though primarily devoted to time-invariant systems, Chapter Nine discusses the failure of the transforms used in analyzing time-invariant systems to handle LTV systems--a concise presentation of the generalizations of Zadeh is then given.

Cesari, L. 1959. Asymptotic behavior and stability problems in ordinary differential equations. Berlin, Springer-Verlag. 271 p.

This book presents a very comprehensive and mathematical treatment of determining stability in general linear differential equations, which includes LTV systems. Appropriate theorems and sufficient conditions for different types of stability and boundedness are presented. The bibliography is quite extensive--69 p.

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A substitution method for solving LTV differential equations with periodic coefficients whose derivatives can be written in terms of the original coefficients is presented.

Derusso, P. M., R. J. Roy and C. M. Close. 1965. State variables for engineers. New York, Wiley. 608 p.

Chapter Two briefly considers basic approaches to solving LTV parameter differential equations--and finding the impulse response. Chapter Three contains a section on the application of Laplace transforms to LTV systems, while several sections in Chapter Five consider the solution of LTV systems described in the matrix representation.

Gerardi, F. R. 1959. Application of Mellin and Hankel transforms to networks with time-varying parameters. Transactions on Circuit Theory of the Institute of Radio Engineers CT-6:197-208.

The article demonstrates how a transform can be developed to handle a specific problem. The Euler-Cauchy equation is shown to be handled by the Mellin transform, and Bessel's equations is shown to be solved by the Hankel transform. Examples are shown of the application of these two transforms to these equations.

Ince, E. L. 1927. Ordinary differential equations. London, Longmans, Green. 558 p.

This is a very comprehensive book on the solution of ordinary differential equations, including those representing LTV systems. It is quite complete and covers most of the methods that, to a certain extent, handle LTV differential equations.

Kalman, R. E. 1962. On the stability of time-varying linear systems. Transactions on Circuit Theory of the Institute of Radio Engineers CT-9: 420-422.

Kalman shows that a generalization of the conditions commonly applied to the time-invariant impulse response to determine stability are insufficient when applied to LTV systems. He presents the additional conditions necessary to insure stability in such cases, and argues that the impulse response is merely a property of a system, and not necessarily a representation of it.

Kaplan, W. 1962. Operational methods for linear systems. Reading, Massachusetts, Addison-Wesley. 577 p.

Though primarily concerned with time-invariant systems, Kaplan devotes Chapter Eight to LTV systems and methods to determine their characteristics. He presents very clear discussions on several methods (attacking the differential equation, solving the matrix representation, and the methods of Zadeh).

Kinariwala, B. K. 1961. Analysis of time-varying networks. In: Automatic control; circuit theory; information theory. IRE International Convention Record. Part IV. New York, Institute of Radio Engineers. p. 268-276.

Kinariwala contends that the differential equation representation is inadequate from the viewpoint of network analysis. He says the matrix representation is superior, and discusses his reasons. He also illustrates some solution methods.

Litovchenko, Ts. G. 1961. Analytical solutions of linear equations describing one class of dynamical systems with variable parameters. Automation and Remote Control 22: 390-397.

The special class of systems that Litovchenko is concerned with is the one described by a differential equation whose time-varying parameters are linear functions of time. The relevant methodology is presented, along with an applicable example.

McLachlan, N. W. 1947. Theory and application of Mathieu functions. Oxford, Clarendon. 401 p.

The theory and solution of Mathieu equations is presented, along with several practical applications. Hill's equation also receives brief consideration.

McLachlan, N. W. 1955. Bessel functions for engineers. 2d ed. Oxford, Clarendon. 239 p.

McLachlan presents a good treatment of the theory of Bessel equations and their solutions from an engineering standpoint. Very good examples are also presented and discussed.

Mason, S. J. 1953. Feedback theory--some properties of signal flow graphs. Proceedings of the Institute of Radio Engineers 41: 1144-1156.

The basic theory of signal flow graphs is presented, along with methods for simplifying and rearranging the graphs (the time-invariant case).

Mason, S. J. 1956. Feedback theory--further properties of signal flow graphs. Proceedings of the Institute of Radio Engineers 44: 920-926.

Extensions and refinements of the original article (1953) are discussed. The general gain formula for (time-invariant) flow graphs is presented and a proof is given.

Massera, J. L. 1956. Contributions to stability theory. Annals of Mathematics 64: 182-206.

A comprehensive mathematical discussion of stability theory and boundedness of differential equations is presented. The discussions are quite mathematically oriented, but not exceptionally difficult to comprehend.

Matyash, I. 1959. Methods of analog computer solution of linear differential equations with variable coefficients. Automation and Remote Control 20: 813-821.

Two methods for using an analog computer to solve an LTV problem are presented--one from the differential equation, the other from the impulse response.

Miller, K. S. 1951. The one-sided Green's function. Journal of Applied Physics 22: 1054-1057.

Miller describes what the one-sided Green's function is and its relationship with the impulse response. He also presents a method for constructing the Green's function for a system.

Miller, K. S. 1955. Properties of impulsive responses and Green's functions. Transactions on Circuit Theory of the Institute of Radio Engineers CT-2: 26-31.

The article presents a very good discussion of the relationship between, and properties of, the Green's function and impulse response of a system.

Miller, K. S. and M. M. Schiffer. 1952. On the Green's functions of ordinary differential systems. Proceedings of the American Mathematical Society 3: 433-441.

This article studies the relationship between the Green's function of a system and the differential equation describing it (as expressed in operator form). Knowing the Green's function, it is contended that all boundary value problems can be solved.

Pipes, L. A. 1955. Four methods for the analysis of time-variable circuits. Transactions on Circuit Theory of the Institute of Radio Engineers CT-2: 4-12.

Pipes considers four approaches to the solution of LTV systems: attacking the differential equation; matrix representation analysis; the BWK approximation (when the time-varying parameters vary only slightly about a large average value); and the use of Laplace transforms. Simple examples are worked using each approach as applied to electrical circuits.

Rudnitskii, B. E. 1961. Determining the transfer functions for certain systems with varying parameters. Automation and Remote Control 21: 1115-1125.

Rudnitskii attempts to extend the methods of Zadeh to systems with rapidly varying parameters. He presents a procedure for determining the system function for systems whose coefficients can be expressed as polynomials in time. He considers the non-homogeneous case primarily.

Sklyarevich, A. N. 1961. Representing non-stationary linear differential polynomial operators in the form of sums of stationary operators. Automation and Remote Control 22: 255-263.

This is a very interesting article that presents a method for changing a differential equation expressed in operator form (with time-varying coefficients) to a finite sum of time-invariant operators, operating on a modified function derived from the first equation. This is a very promising method which transforms an LTV expression into a time-invariant one, therefore solvable by conventional techniques.

Solodov, A. V. 1958. Conversion of output initial conditions in a linear system with variable parameters into an equivalent input signal. Automation and Remote Control 19: 645-651.

A method is presented whereby an LTV system that is initially excited can be treated as an unexcited system with an appropriate input consisting of various order delta functions. The output transient response is then determinable, if the impulse response is known.

Stear, E. B. and A. R. Stubberud. 1961. Signal flow graph theory for linear time-variable systems. Transactions of the

American Institute of Electrical Engineers 80 (Part I): 695-701.

The article shows how the differential equation can be determined from the impulse response, when the impulse response is expressible in a particular form--broadly, as a signal flow graph. Block diagram manipulations of the differential equation representation of LTV systems are also presented in signal flow graph form.

Stubberud, A. R. 1964. Analysis and synthesis of linear time variable systems. Berkeley, University of California. 108 p.

This monograph examines LTV system analysis and synthesis--primarily the latter. It is quite general, and in places seems to be rather vague, but it does convey some feeling for the problems involved with LTV systems engineering, and could be of some value to the engineer in acquainting him with the analysis of such systems.

Youla, D. C. 1963. On the stability of linear systems. Transactions on Circuit Theory of the Institute of Electrical and Electronic Engineers CT-10: 276-279.

Youla proves a theorem that assures the stability of a LTV system if certain conditions on the impulse response are met.

Zadeh, L. A. 1950a. Frequency analysis of variable networks. Proceedings of the Institute of Radio Engineers 38: 291-299.

This is Zadeh's original work where he presents his generalizations of the system function of time-invariant analysis to LTV systems analysis. He discusses the advantages and disadvantages of his generalized system function, and presents a non-trivial method for finding it, which simplifies in the case of slowly varying systems.

Zadeh, L. A. 1950b. The determination of the impulsive response of variable networks. Journal of Applied Physics 21: 642-645.

A method for determining the impulse response of a system by determining the Green's function of the system is presented. Having done this, the system function can be found. The method seems to be solvable in general only for the case of slowly varying systems.

Zadeh, L. A. 1950c. Circuit analysis of linear varying parameter networks. Journal of Applied Physics 21: 1171-1177.

Zadeh discusses his system function in Fourier, Laplace, and operator form; and he presents various combinatorial expressions for more than one system function. He also discusses the application of his generalized system function approach to network analysis.

Zadeh, L. A. 1951. On stability of linear varying-parameter systems. Journal of Applied Physics 22: 402-405.

Stability of a system is related to certain conditions on the system function and impulse response. Several theorems on stability are presented.

Zadeh, L. A. 1961. Time-varying networks. I. Proceedings of the Institute of Radio Engineers 49: 1488-1503.

Zadeh reviews the significant papers on LTV systems published since his first article came out in 1950. It contains a large bibliography; and Zadeh speaks of a sequel article continuing the discussion he starts but to this author's knowledge, it has never been published.

Zadeh, L. A. and C. A. Desoer. 1963. Linear system theory: the state space approach. New York, McGraw-Hill. 628 p.

This book is primarily concerned with the time-invariant case, but it does devote a chapter to LTV systems and also contains a comprehensive chapter on stability. It is a difficult book to read in any way other than from the beginning straight through (one cannot pick it up and start reading in it anywhere) because of the specific nonstandard and very complex notation adopted by the authors.

APPENDICES

APPENDIX I

SOME SPECIAL CASES IN LTV DIFFERENTIAL EQUATIONS

There are many LTV differential equations of order two or greater that have been solved, one way or another. A few of the more common cases will be presented and referenced here--it is in no way a complete listing. For a more complete listing of this type of problem (with time-varying coefficients) that has been solved, the reader is referred to the mathematical literature, or to a mathematician well-versed in the field of differential equations with time-varying coefficients.

Bessel's equation: This equation first arose in the study of elliptical planetary motion, and since has appeared in the study of many different physical problems. Some of these are: the oscillation of a pendulum whose length varies sinusoidally with time; the oscillations of a circular membrane; the acoustical analysis of loudspeaker horns; and the analysis of current density in a wire carrying AC current. The general form of the equation is:

$$\frac{d^2 y(t)}{dt^2} + \frac{1}{t} \frac{dy(t)}{dt} + \left(1 - \frac{n^2}{t^2}\right) y(t) = 0,$$

where t is time, n is an integer, and y(t) is the dependent variable. For a comprehensive discussion of this equation and the behavior of its solutions, the reader is referred to McLachlan (1955).

Mathieu's equation: Like Bessel's equation, this equation has received comprehensive study, along with its solutions and regions of stability. The general expression is

$$\frac{d^2y(t)}{dt^2} + (a - q \cos 2t)\, y(t) = 0,$$

where a and q are constants. The equation arises naturally in many physical systems--in the analysis of electromagnetic wave-guides, for instance. Arscott (1964) briefly discusses when and why this particular equation appears in analysis problems. McLachlan's book (1947) is recommended as a good, complete reference to the engineer interested in the solutions and properties of this particular equation.

Hill's equation: This equation is directly related to Mathieu's equation--indeed the latter is just a special case of the former. Hill's equation is

$$\frac{d^2y(t)}{dt^2} + J(t)\, y(t) = 0,$$

where J(t) is defined by

$$J(t) = \theta_o + \sum_{r=1}^{\infty} 2\theta_r \cos 2rt,$$

with θ_o, θ_r, constants and r an interger.

Its properties and solutions are discussed by Arscott (1964) and Ince (1927)--and McLachlan (1947), to a certain extent.

Lame's equation: Both Ince (1927) and Arscott (1964) discuss this equation to some extent, though not as much as any of the aforementioned equations. The general form is

$$\frac{d^2y(t)}{dt^2} + (h - n(n+1)\, k^2 \sin^2 t)\, y(t) = 0,$$

where h and k are constants and n is an integer.

Coefficients linear in t: This equation has no name (it is not Euler's equation), and has been studied briefly by Litovchenko (1961). Its general expression is

$$\sum_{n=0}^{N} (a_n + b_n t) \frac{d^n y(t)}{dt^n} = F(t)$$

where a_n and b_n are constants (n = 0,1,2...,N), n is an interger, and F(t) is a forcing function. Litovchenko has developed a methodology for arriving at a solution and in his paper he demonstrates his approach with an applicable example.

The general second order equation: The general expression,

$$\frac{d}{dt}\left[k(t)\,\frac{dy(t)}{dt}\right] + l(t)\,y(t) = 0,$$

where k(t) and l(t) are continuous functions of time, has received close scrutiny by Bellman (1953). He has studied it from a general viewpoint and has made appropriate observations of different methods of arriving at solutions for different special cases. The engineer confronted with such a second order equation should consult Bellman's book for his observations.

Floquet theory: Ince (1927) describes this technique that handles many equations that can be written in the form

$$\frac{d^n y(t)}{dt^n} + p_1(t)\,\frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + p_{n-1}(t)\,\frac{dy(t)}{dt} + p_n(t)\,y(t) = 0,$$

where the coefficients, $p_i(t)$ $(i = 1, 2, \ldots, n)$, are purely periodic. This theory, first published in 1883, is discussed rather briefly by Ince.

It is possible to extend this listing on and on, but not very practical. The purpose here has been to mention a few of the more common special cases of LTV equations that have been solved or where methods exist to handle them to acquaint the reader with the fact that the analysis of time-varying differential equations is primarily a collection of special cases. The problems presented here are just a sampling of this collection of special problems. In any event, the engineer with a differential equation with time-varying coefficients to solve should review the literature, or better yet, consult with a competent mathematician in the field, to see if perhaps his equation might be in, or reducible to, a form that has already been solved.

APPENDIX II

USING COMPUTERS

Introduction

Computers are very powerful analytical tools, and there is no exception to this fact in their application to systems analysis. For the purposes of this thesis, their use is considered to be as stated--a tool--and not an analysis technique in its own right. But since they are such powerful tools, it is felt that a brief discussion of the possible uses of computers is justified here.

Basically, there are three classes of computers: analog computers, digital computers, and the combination of the two, or the so-called hybrid computers. The special purpose computers (computers build for a specific task only) are subclasses of these three basic categories. Each type of computer has its own special uses and potentialities in LTV systems analysis and will be examined separately here very briefly.

Analog Computers

The primary, and best known, use of the electronic analog computer is in the simulation of a system. After a system is electronically simulated, it can be studied, tested, modified, and investigated without ever having to build the actual system. Particular solutions can be generated and studied, enabling the solution of systems that are not solvable analytically.

There is another use of the analog computer in LTV systems analysis that has been studied--see Matyash, 1959. This is the use of the analog computer to solve a non-homogeneous differential equation with time-varying coefficients for the particular solution. Instead of simulating the system, the differential equation is "simulated." Matyash shows two methods for doing this, if the time varying parameters satisfy certain conditions: one using the differential equation itself, and the other using the impulse response. This is another powerful use of the analog computer--comparable in usefulness with the first, and actually complementing it. In both of these methods, essentially all of the system variables are accessible for observation, not just one. Changes and modifications can be made quite easily to solve system problems involving inadequate responses, or instability, and the system can be investigated for every kind of input that can be electronically generated, which probabably includes all that are imaginable. Different initial condition situations can be studied also because they are so easily set up.

There are some minor problems, however, which may possibly increase the complexity of such an analysis--or hinder it drastically. It is possible that some time-varying parameter may be of such a complicated nature that it becomes quite a chore to duplicate electronically--practically speaking, this is reflected in terms of the cost and the complexity of the computer needed to simulate the problem. Theoretically speaking (i.e., cost no object), just about any function can be generated and therefore, just about every system can be simulated on the analog computer. Indeed, the analog computer is a very powerful tool, and may allow the engineer to study a problem that just will not be solved analytically in any way.

Digital Computers

Essentially, everything that can be done on an analog computer can be done on a digital computer. Since a digital computer lends itself naturally to numerical methods, it can be used to solve differential equations numerically. It can even be used to simulate a system--one approach is where a sampled data type of system is used to approximate the real system and the computer is then used to perform the digital simulation. The accuracy desired can be determined by the sampling interval of the approximating sampled data system.

Hybrid Computers

Hybrid computers are actually two separate computers, an analog and a digital computer, with interfacing analog-to-digital and digital-to-analog converters. This type of computer can potentially utilize the best qualities of the two computers. Some of the complicated function-generating needs of the analog computer can be done by the digital computer and then converted to analog form for use by the analog computer. This may reduce the problems of cost and complexity mentioned in the discussion on analog computers.

Conversely, some digital operations may be more easily performed if they are converted to analog form and handled by the analog computer. The hybrid computer, then, is a teaming up of two basic computers--the analog and digital--so that they supplement each other, combining their best qualities.

Conclusions

Computers are indeed a very valuable tool in LTV systems analysis (or any type of systems analysis for that matter). Often, they can be used to implement techniques that will enable an engineer to analyze systems that he would have been unable to handle using purely analytical techniques. The computer, therefore, is a very valuable asset to the systems engineer, and an important part of the overall methodology of LTV systems analysis.