AN ABSTRACT OF THE THESIS OF

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Title DEFLECTIONS OF SIMPLY SUPPORTED PLYWOOD PLATE
UNDER UNIFORM LOADING BY FINITE-DIFFERENCE
EQUATIONS

This paper offers a practical solution of the complicated equations for the deflections of nonorthotropic plates; in this discussion plywood plates made of Douglas fir are considered. The intricate partial differential equations are expressed in terms of finite differences. The simultaneous algebraic linear equations thus obtained are easily solved.

The theory of obtaining the required elastic constants for the plywood plates is discussed in the introduction. The theory is expanded in Appendix IV by developing the general expression for the deflection of a thin plate. This expression is used in the experimental methods for obtaining the elastic constants.

Part I explains the tests performed on the plywood plates to derive the elastic constants. The plates were cut with the face grain making angles of 0, 22 1/2, 45, 67 1/2, and 90 degrees with the
coordinate axes. Bending and torsion tests were conducted on each plate. Three means of supporting the plates in the bending tests were investigated in order to determine the method which would give the most reliable values for the elastic constants.

The nonorthotropic plate equation is expressed in terms of finite-difference equations for uniformly loaded and simply supported plates in Part II. These equations are simplified considerably by the considerations of the symmetry that is possessed by the plywood plates. The complete derivation of the nonorthotropic plate equation is given in Appendix III.

The theory for the deflections of plywood plates is verified in Part III by conducting experiments on uniformly loaded and simply supported plates. The theoretical and the experimental values of the deflections are compared in a tabulated form; the discrepancy, which varies from 23 to 38 percent, is also indicated.

Sample calculations for the elastic constants are included in Appendix I.

The generalized Hooke’s law is discussed in Appendix II. Also, the stress and strain equations applied to plywood plates are included.
DEFLECTIONS OF SIMPLY SUPPORTED PLYWOOD PLATE UNDER UNIFORM LOADING BY FINITE-DIFFERENCE EQUATIONS

by

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>PART I - INTRODUCTION</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Assumptions</td>
<td>1</td>
</tr>
<tr>
<td>2. Purpose of Thesis</td>
<td>3</td>
</tr>
<tr>
<td>3. General Expression for the Deflection of a Thin Plate</td>
<td>3</td>
</tr>
<tr>
<td>4. Elastic Constants Transferred to New Axis</td>
<td>5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>PART II - EXPERIMENTAL METHODS FOR OBTAINING THE ELASTIC CONSTANTS FOR PLYWOOD</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Bending Tests</td>
<td>6</td>
</tr>
<tr>
<td>I. Simply Supported Plates</td>
<td>7</td>
</tr>
<tr>
<td>II. Clamped Plates</td>
<td>11</td>
</tr>
<tr>
<td>III. Plates Supported on Three Ball Bearings</td>
<td>11</td>
</tr>
<tr>
<td>2. Torsion Tests</td>
<td>13</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>PART III - THE NONORTHOTROPIC PLATE EQUATION IN TERMS OF FINITE-DIFFERENCE EQUATIONS</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Finite-Difference Equations for Uniformly Loaded Plates</td>
<td>34</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>PART IV - EXPERIMENTAL METHODS FOR VERIFYING THE DEFLECTION THEORY FOR PLYWOOD PLATES</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>CONCLUSIONS</td>
<td>56</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td></td>
</tr>
<tr>
<td>APPENDIX I - SAMPLE CALCULATIONS</td>
<td></td>
</tr>
<tr>
<td>1. The Determination of Elastic Constants from the Deflections</td>
<td>68</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>APPENDIX II</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. The Generalized Hooke's Law</td>
<td>69</td>
</tr>
<tr>
<td>2. Hooke's Law Applied to Plywood Plates</td>
<td>72</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>APPENDIX III</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Differential Equation of the Deflection Surface for Nonorthotropic Plate</td>
<td>75</td>
</tr>
</tbody>
</table>
APPENDIX IV

1. Derivation of the General Deflection Equation for a Thin Plate
### LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1(a)</td>
<td>Coordinate axes in the x- and z-directions</td>
<td>4</td>
</tr>
<tr>
<td>1(b)</td>
<td>Coordinate axes in the x- and y-directions</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>Dimensions and the grid system of the plates</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>Side view of the plates in bending</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>The method of mounting the deflection gauges</td>
<td>9</td>
</tr>
<tr>
<td>5</td>
<td>Deflection observations in the bending tests</td>
<td>10</td>
</tr>
<tr>
<td>6</td>
<td>The method of supporting and loading the plates in the torsion tests</td>
<td>14</td>
</tr>
<tr>
<td>7</td>
<td>Deflection observations along the line I in the torsion tests</td>
<td>15</td>
</tr>
<tr>
<td>8</td>
<td>Deflection observations along the line III in the torsion tests</td>
<td>16</td>
</tr>
<tr>
<td>9</td>
<td>Deflection observations along the line IV in the torsion tests</td>
<td>17</td>
</tr>
<tr>
<td>10</td>
<td>Deflection curves for the lines III and IV in the torsion tests</td>
<td>19</td>
</tr>
<tr>
<td>11</td>
<td>Deflection curve along the line I</td>
<td>20</td>
</tr>
<tr>
<td>12</td>
<td>Sample of load-deflection curves for plates in bending</td>
<td>22</td>
</tr>
<tr>
<td>13</td>
<td>The grid for the finite-difference equations</td>
<td>35</td>
</tr>
<tr>
<td>14(a)</td>
<td>The loading pattern for the fictitious deflections</td>
<td>42</td>
</tr>
<tr>
<td>14(b)</td>
<td>Deflection curve in the x-direction</td>
<td>42</td>
</tr>
<tr>
<td>14(c)</td>
<td>Deflection curve in the y-direction</td>
<td>42</td>
</tr>
<tr>
<td>Figure</td>
<td>Title</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>----------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>15</td>
<td>The method of obtaining finite-difference equations for points other than the center</td>
<td>43</td>
</tr>
<tr>
<td>16</td>
<td>The effect of symmetry on plywood plates</td>
<td>45</td>
</tr>
<tr>
<td>17</td>
<td>Deflection gauges and edge supports in the uniform loading tests</td>
<td>59</td>
</tr>
<tr>
<td>18</td>
<td>The frame for holding the uniform load on the plates</td>
<td>60</td>
</tr>
<tr>
<td>19</td>
<td>Uniform load on the plates</td>
<td>61</td>
</tr>
<tr>
<td>20</td>
<td>Deflection observations in the uniform loading tests</td>
<td>62</td>
</tr>
<tr>
<td>21</td>
<td>Sample of load-deflection curves for uniformly loaded plates</td>
<td>63</td>
</tr>
<tr>
<td>22</td>
<td>Stresses on an element of anisotropic material</td>
<td>70</td>
</tr>
<tr>
<td>23</td>
<td>Moments and shearing forces on an element of a plate</td>
<td>75</td>
</tr>
<tr>
<td>24(a)</td>
<td>The positive moments on the middle plane of an element of a plate</td>
<td>79</td>
</tr>
<tr>
<td>24(b)</td>
<td>The positive shearing forces on a plate</td>
<td>79</td>
</tr>
</tbody>
</table>
## LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Deflections w (10^3) in. from Bending Tests</td>
<td>27</td>
</tr>
<tr>
<td>2</td>
<td>Deflection w (10^3) in. from Torsion Tests</td>
<td>28</td>
</tr>
<tr>
<td>3</td>
<td>Observed values of a_{ij}</td>
<td>29</td>
</tr>
<tr>
<td>(a)</td>
<td>Bending</td>
<td>30</td>
</tr>
<tr>
<td>(b)</td>
<td>Torsion</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Observed and calculated values of a_{ij} (unit = 10^6 in^2/lb)</td>
<td>31</td>
</tr>
<tr>
<td>5</td>
<td>Observed and calculated values of B_{ij} (unit = 10^6 lb/in^2)</td>
<td>32</td>
</tr>
<tr>
<td>6</td>
<td>Observed and calculated values of D_{ij} (unit = 10^4 in/lb)</td>
<td>33</td>
</tr>
<tr>
<td>7</td>
<td>Observed and calculated deflections and percent discrepancy</td>
<td>64</td>
</tr>
</tbody>
</table>
INTRODUCTION

Many materials used in engineering have internal structures that differ from the usual assumption of "ideal material," a material that has the same mechanical properties at every point. Plywood plates are a good example of a material of anisotropic character. They are usually constructed so that the grain of the adjacent plies is perpendicular. Experimental studies have shown a great difference in the elastic properties of plywood in the principal directions, along and perpendicular to the grain. It is obvious that the theory of isotropic plates is not applicable for plywood plates but the formulas based on the theory of elasticity for anisotropic bodies must be utilized.

1. Assumptions

In developing the theory and the formulas for plywood plates the following assumptions are made:

(1) Plywood is taken to be an orthotropic material, i.e., a material in which the directions of the grain in adjacent plies are
perpendicular to each other and parallel and perpendicular to edges.

(2) The material in the individual plies is accurately flat-grain, i.e., the directions of the grain and the annual rings are parallel to the faces of the plies.

(3) Each ply is homogeneous. This implies that the variations of the elastic constants from springwood to summerwood are disregarded and average values of the constants are used.

(4) The plate is symmetrical, both geometrically and as to the arrangement and properties of the material, with respect to the plane \( z = 0 \); the axes of coordinates are chosen as in Figure 1.

(5) Deflections \( w \) of a plate are so small in comparison with its thickness \( t \) that direct stresses are not developed to an appreciable extent and therefore the theory of thin plates applies.

(6) Points of a straight line which is normal to the undeformed plane middle surface, \( z = 0 \), of the plate, remain in a straight line which is normal to the middle surface after deformation has taken place.

(7) The pressure on layers parallel to the middle surface (the stress \( \sigma_z \)) is so small by comparison with the stresses \( \sigma_x \), \( \sigma_y \), and \( \tau_{yx} \) that it may be neglected.
2. **Purpose of Thesis**

The purpose of this paper is to express the nonorthotropic plate equations in terms of finite-difference equations for the deflections of simply supported and uniformly loaded Douglas fir plywood plates. The agreement of the theoretical values of the deflections were studied experimentally on five plates with the face grain making angles of 0, 22 1/2, 45, 67 1/2, and 90 degrees with the coordinate axes of the plate.

The elastic constants appearing in the nonorthotropic plate equations were derived from the deflections obtained from bending and torsion tests on each plate. Deflections rather than strains were observed since it was anticipated that the strain readings on plywood would not be reliable as demonstrated by many previous experiments by others.

3. **General Expression for the Deflection of a Thin Plate**

The general expression for the deflection of a thin plate when the plate is subjected to uniform bending moments

\[ M_x \] per unit length of side along AD, BC,

\[ M_y \] per unit length of side along AB, CD, and a uniform twisting moment
\( \frac{M_{xy}}{\text{per unit length along all four sides}} \) is

\[
t^3w = 6M_x \left( a_{11}x^2 + a_{12}y^2 + a_{16}xy \right) + 6M_y \left( a_{12}x^2 + a_{22}y^2 + a_{26}xy \right) + 6M_{xy} \left( a_{16}x^2 + a_{26}y^2 + a_{66}xy \right)
\]

Equation (1a-c) has appeared in (1) but the author of this paper has been unable to find the derivation of this equation; therefore, the complete derivation is worked out in Appendix IV.

In equation (1a-c) \( w \) is the deflection, normal to the plane of the plate, of the point whose coordinates with respect to the center of the plate are \( x, y, \) and \( t \) is the thickness of the plate.

Figure 1. (a) Coordinate axes in the \( x \)- and \( z \)-directions. (b) Coordinate axes in the \( x \)- and \( y \)-directions.

The constants \( a_{16} \) and \( a_{26} \) are zero for specially orthotropic plates; that is, plates whose principal axes are the axes of coordinate system. These constants are not zero for generally orthotropic plates; that is, plates whose principal axes make an angle \( \theta \) with
axes of coordinate system.

When the plate is subjected to uniform bending moments \( M_x \) and \( M_y \) together or alone, the case of pure bending exists. However, on points off the principal axes some twisting will occur because of the presence of the terms \( a_{16}xy \) and \( a_{26}xy \) in equation (1a-c). When uniform twisting moment \( M_{xy} \) is applied to the plate alone, the case of pure torsion exists. Some symmetrical bending will occur, however, caused by the terms \( a_{16}x^2 \) and \( a_{26}y^2 \)

4. **Elastic Constants Transferred to New Axis**

If the elastic constants for the principal directions are known and the axes are rotated counterclockwise through an angle \( \theta \) as shown in Figure 1(b), the elastic constants for the new direction may be obtained from the following equations (2, p. 43):

\[
\begin{align*}
    a_{11}' &= a_{11}\cos^4\theta + (2a_{12} + a_{66})\sin^2\theta \cos^2\theta + a_{22}\sin^4\theta + (a_{16}\cos^2\theta + a_{26}\sin^2\theta)\sin^2\theta \\
    a_{12}' &= a_{11}\sin^4\theta + (2a_{12} + a_{66})\sin^2\theta \cos^2\theta + a_{22}\cos^4\theta - (a_{16}\sin^2\theta + a_{26}\cos^2\theta)\sin^2\theta \\
    a_{66}' &= 4(a_{11} + a_{22} - 2a_{12} - a_{66})\sin^2\theta \cos^2\theta + a_{66} + 2(a_{26} - a_{16})\sin^2\theta \cos^2\theta \\
    a_{16}' &= (a_{22}\sin^2\theta - a_{11}\cos^2\theta + \frac{2a_{12} + a_{66}}{2}\cos^2\theta)\sin^2\theta + a_{16}(\cos^2\theta - 3\sin^2\theta)\cos^2\theta + a_{26}(3\cos^2\theta - \sin^2\theta)\sin^2\theta \\
    a_{26}' &= (a_{22}\cos^2\theta - a_{11}\sin^2\theta - \frac{2a_{12} + a_{66}}{2}\cos^2\theta)\sin^2\theta + a_{16}(3\cos^2\theta - \sin^2\theta)\sin^2\theta + a_{26}(\cos^2\theta - 3\sin^2\theta)\cos^2\theta \\
\end{align*}
\]
PART II

EXPERIMENTAL METHODS FOR OBTAINING THE ELASTIC CONSTANTS FOR PLYWOOD

1. Bending Tests

In order to obtain the best values of the elastic constants for the plywood, three means of supporting the plates were investigated:

I. Simply supported plates along lines 00-40 and 04-44,

II. Plates clamped over the supports along lines 00-40 and 04-44,

III. Plates supported on three ball bearings at points 20, 04, and 44.

Figure 2. Dimensions and the grid system of the plates.
The lines I, II, III, IV and the grid system shown in Figure 2 were marked on the central 18 x 18 in. square of the plates and the deflections along these lines at the grid points were observed by Ames gauges, reading directly to $10^{-3}$ in. The thickness of the plates was 1/2 inch.

I. Simply supported plates along lines 00-40 and 04-44.

The plates were supported on 1/8-inch edge along the lines 00-40 and 04-44. The load was applied by means of placing 2 1/2-lb. bags of 5/16-inch diameter leadshot in cans hung at the points marked L in Figure 2. In order to impose a uniform moment along the supports, 3/4 x 3/4 x 1/8 in. angles were placed at the ends of the plates as can be seen in Figure 3. The loading was placed in increments of 20 lbs., i.e., one 2 1/2-lb. bag was placed in each can. The deflections were measured by five Ames gauges mounted on a 1-inch square aluminum rod as shown in Figures 4 and 5. The gauges were moved so that the deflections at the points 00-44 could be read after each increment of loading. The increase of loading was discontinued when the deflections at the center reached the approximate value of $t/10$, where $t$ is the thickness of the plate. After the deflection readings were taken at the grid points up to the maximum loading, the plates were unloaded and the deflection readings were observed at the same load increments as when the plates were
Figure 3. Side view of the plates in bending.
Figure 4. The method of mounting the deflection gauges.
Figure 5. Deflection observations in the bending tests.
loaded. Then the plates were reversed and the deflections were measured again when the plates were loaded and unloaded. The average of the four values of deflections was used in computing the elastic constants. The deflection observations obtained thus are distinguished by the letters A and B in the tables.

II. Plates clamped over the supports. The plates were clamped over the 1/8-inch edges by placing 1 x 1 x 1/8 in. angles on top of the plates over the supports and clamping the angles down with C-clamps. The clamps were placed because it was noticed that the plates were initially warped and did not come to bear uniformly on the supports. The observations were made as in experiment I.

III. Plates supported on three ball bearings. From experiment II it was noticed that the plates tended to curl up along the sides under loading; this curling up was independent of the warp of the plates. As mentioned previously, equation (1a) justifies the curling by the presence of the term $a_{16}xy$. In order to allow the plates to act freely, they were supported on three 1/4-inch ball bearings at the points 20, 04, and 44. The observations on the plates were repeated again as in experiments I and II.

The elastic constants can be obtained from equation (1a-c). If only the bending moment $M_x$ is applied to the plate ($M_y = M_{xy} = 0$),
the elastic constant \( a_{11} \) can be obtained from Equation (1a) which is

\[
t^3w = 6M_x (a_{11}x^2 + a_{12}y^2 + a_{16}xy)
\]

In this experiment \( M_x \) per unit width is

\[
M_x = \frac{P}{2} \left( \frac{7.25}{19} \right)
\]

where \( P \) is the total load applied on the plate, i.e., \( P/2 \) is applied at each end.

**Along line I**

\[
\begin{align*}
x &= 9 \text{ in.} \\
y &= 0 \\
t &= 1/2 \text{ in.}
\end{align*}
\]

\[
t^3w = 6M_x (a_{11}x^2)
\]

\[
a_{11} = \frac{w}{P} \frac{2(19)}{8(6)7.25(x^2)}
\]

\[
= 13.481 \left(10^{-4}\right) \frac{w}{P}
\]

**Along line II**

\[
\begin{align*}
x &= 0 \\
y &= 9 \text{ in.} \\
t &= 1/2 \text{ in.}
\end{align*}
\]

\[
t^3w = 6M_x (a_{12}y^2)
\]

\[
a_{12} = 13.481 \left(10^{-4}\right) \frac{w}{P}
\]

**Along line III**

\[
\begin{align*}
x &= y = 9 \text{ in.} \\
t &= 1/2 \text{ in.}
\end{align*}
\]

\[
t^3w = 6M_x (a_{11}x^2 + a_{12}y^2 + a_{16}xy)
\]

\[
a_{11} + a_{12} + a_{16} = 13.481 \left(10^{-4}\right) \frac{w}{P}
\]

**Along line IV**

\[
\begin{align*}
x &= -y = 9 \text{ in.} \\
t &= 1/2 \text{ in.}
\end{align*}
\]

\[
t^3w = 6M_x (a_{11}x^2 + a_{12}y^2 - a_{16}xy)
\]

\[
a_{11} + a_{12} - a_{16} = 13.481 \left(10^{-4}\right) \frac{w}{P}
\]
For specially orthotropic plates, as has been stated before, $a_{16} = a_{26} = 0$; thus the principal constants are obtained as follows:

\[
\begin{align*}
&\text{from 0° bending test, line I} & a_{11} &= 13.481 \times 10^{-4} \frac{w}{P} \\
&\text{0° bending test, line II} & a_{12} &= 13.481 \times 10^{-4} \frac{w}{P} \\
&\text{90° bending test, line I} & a_{22} &= 13.481 \times 10^{-4} \frac{w}{P}
\end{align*}
\]

2. Torsion Tests

The plates used in the bending tests were squared by cutting the ends off and leaving the 18 x 18-inch central square. The plates were supported at two diagonally opposite corners on 1/4-inch-diameter ball bearings and loaded at the other two diagonally opposite corners by hanging 2 1/2-lb. weights as shown in Figure 6. The deflection readings were measured along the lines I, II, III, and IV at increments of load of 5 lb., i.e., 2 1/2 lb. at each corner. Figures 7, 8, and 9 show the method of observing the deflections along the lines I, III, and IV, respectively. Observations along the line II were made similarly. The maximum load of 10 lbs. was used. At this loading the deflections at the center exceeded $t/10$. Then the plates were unloaded and the deflections were observed again.
This method of loading the plates has been used by Timoshenko (7, p. 45) and Hearmon (1). The theory given by Timoshenko shows that concentrated loads of P/2 at each corner produce a torsional moment of $M_{xy}$ per unit length along each side equal to P/4. Therefore from equation (1c)

$$2t^3w = 3P(a_{16}x^2 + a_{26}y^2 + a_{66}xy)$$

Along line I

$$a_{16} = 1.029 \times 10^{-3} \frac{w}{P}$$

Along line II

$$a_{26} = 1.029 \times 10^{-3} \frac{w}{P}$$

Along line III

$$a_{16} + a_{26} + a_{66} = 1.029 \times 10^{-3} \frac{w}{P}$$

Along line IV

$$a_{16} + a_{26} - a_{66} = 1.029 \times 10^{-3} \frac{w}{P}$$
Figure 7. Deflection observations along the line I in the torsion tests.
Figure 8. Deflection observations along the line III in the torsion tests.
Figure 9. Deflection observations along the line IV in the torsion tests.
The principal constant $a_{66}$ is obtained from the torsion tests as shown below:

from $0^\circ$ and $90^\circ$, line III

$$a_{66} = 1.029 \left(10^{-3}\right) \frac{w}{P}$$

line IV

$$-a_{66} = 1.029 \left(10^{-3}\right) \frac{w}{P}.$$

There should be no deflection along lines I and II of the plates of $0$ and $90$ degrees as shown by Timoshenko (7, p. 44).

In order to obtain the constants from the deflections, the average values of deflections were used. Along each line the deflections were observed at the grid points, first on one side, designated as side A, and then on the reverse side, designated as side B. Since the plates were supported on two ball bearings, the plates were free to swing. However, the plates were balanced by hand in order to make the deflections approximately equal at the outside and inside points along each line. "Outside" deflection designates the difference between the deflections at the center and at the outside points near the edge of the plate, for line III, $w_{22} - \frac{1}{2}(w_{00} + w_{44})$; "inside" deflection designates the difference between the deflections at the center and at the inside or the quarter points, for line III, $w_{22} - \frac{1}{2}(w_{11} + w_{33})$. The deflection curves for lines III and IV for the plate of $0$ degrees are shown in Figure 10. The dashed curves for line IV are the deflection observations made on plate that came to
Figure 10. Deflection curves for the lines III and IV in the torsion tests.

an equilibrium by itself without an effort of trying to balance the deflections at the points 13 and 31, 04 and 40. The average values of the deflections \( w_{13} \) and \( w_{31} \) would give the correct value for the inside point and similarly the average values of \( w_{04} \) and \( w_{40} \) for the outside point. However, in order to avoid confusion in the deflection readings, the plate was balanced by hand so that the deflections \( w_{13} \) and \( w_{31} \) and also \( w_{04} \) and \( w_{40} \) would be approximately equal. In Figure 10 the full line shows the elastic curve for a plate that was balanced by hand. The deflections to be used in computing the elastic constants were obtained as follows:
Along line I
\[ w_{\text{outside}} = w_{22} - \frac{1}{2}(w_{20} + w_{24}) \]
\[ w_{\text{inside}} = 4\left[w_{22} - \frac{1}{2}(w_{21} + w_{23})\right] \]

Along line II
\[ w_{\text{outside}} = w_{22} - \frac{1}{2}(w_{02} + w_{42}) \]
\[ w_{\text{inside}} = 4\left[w_{22} - \frac{1}{2}(w_{12} + w_{32})\right] \]

Along line III
\[ w_{\text{outside}} = w_{22} - \frac{1}{2}(w_{00} + w_{44}) \]
\[ w_{\text{inside}} = 4\left[w_{22} - \frac{1}{2}(w_{11} + w_{33})\right] \]

Along line IV
\[ w_{\text{outside}} = w_{22} - \frac{1}{2}(w_{04} + w_{40}) \]
\[ w_{\text{inside}} = 4\left[w_{22} - \frac{1}{2}(w_{13} + w_{31})\right] \]

Since the deflection curve is a parabola in shape, the inside deflections were multiplied by four in order to convert them to the same magnitude as the outside deflections, see Figure 11.

Figure 11. Deflection curve along the line I.

In Figure 11 w outside equals 4 (w inside).
The observed deflections from the bending and torsion tests are tabulated in Table 1 and Table 2, respectively. As mentioned previously, the deflection readings were taken at increments of 20 lbs. in the bending tests and at 5 lbs. in the torsion tests. For the plate of zero degree the deflections used in computing the elastic constants were selected at the loading of 120 lbs. in the bending test, or the maximum loading on the plate. These deflections were obtained from the load-deflection curves as shown in Figure 12. The deflections at each increment of loading were plotted on this graph; a straight line was fitted through the points thus obtained, the deflection desired being the difference between the abscissas of the ends of this straight line. As can be noticed from Figure 12, the straight line curve does not always pass through the point of origin. The deflection observations at the first increment of load could not be used alone because inaccuracies are introduced by, for example, the plates not being in contact with the supporting edges evenly. The curves through the first and the subsequent points of loading described a straight line very well; this consistency showed that the plates were behaving elastically and that no noticeable creep was detected. Therefore, the deflections at the maximum loading used were representative of the action of the plates. These curves were plotted for each grid point on both sides A and B along the lines I, II,
Figure 12. Sample of Load-Deflection Curves for Plates in Bending.
III, and IV; the resultant deflections are shown in Table 1 together with the average values of those on sides A and B. The deflection on the zero-degree plate at the point 44 on side A was disregarded since it is not consistent with the other deflections. This procedure of obtaining the deflections for each plate was followed and the results are tabulated in Table 1.

As was mentioned previously, the plates tended to curl up along the long edges in the bending tests. This tendency can be seen from the deflections along the line II in Table 1. The deflections shown are positive upward; they are greater on the edges, at points 02 and 42, than at the center, point 22. This differential deflection increases from the plate with 0° angle to the plate with 45° angle and then decreases again on the plates with 67 1/2° and 90°. Therefore, the effect of curling is the greatest on the plate with 45° angle.

The values of deflections in Table 1 are for the plates clamped over the supports. Since the clamping resists the free curling of the plates, it was anticipated and later proved that the observed deflections on plates with 22 1/2, 45, and 67 1/2 angles would not give reliable values for the elastic constants. Therefore, these constants were calculated from the observed deflections on plates with angles of 0 and 90°. Equations (2a-f) were used in calculating the elastic constants for plates with angles other than 0 and 90°.
The deflections in the torsion tests were obtained in the same manner as those in the bending tests; they are shown in Table 2. Deflections were observed at 5-lb. increments of load, that is, 2 1/2 lbs. were placed at the two diagonally opposite corners. The maximum load that could be placed on the plates in the torsion tests before the deflections exceeded the allowable value was 10 lbs. Thus, only two points were obtained on the load-deflection curves. It would have been advantageous to divide the load into smaller increments whereby more reliable load-deflection curves would have been obtained. However, the time did not permit the observations made again.

The theory of Timoshenko (7, p. 44) that on the plates with 0 and 90° angles there should be no deflection along the lines I and II is verified in these torsion tests. However, slight discrepancies are noticed in the deflections in Table 2 along these lines; these differences can be attributed to mechanical and human errors and to the properties of the plates. Also, the deflections along the lines III and IV on the plates with 0 and 90° angles should be equal at the corresponding points. For example,

\[
\begin{align*}
\text{Line III} & \\
0^\circ \text{ and } 90^\circ & \\
w_{22} & = \frac{1}{2}(w_{04} + w_{40}) - w_{22} \\
w_{22} - \frac{1}{2}(w_{11} + w_{33}) & = \frac{1}{2}(w_{13} + w_{31}) - w_{22} \\
\end{align*}
\]
These relationships can be seen in Figure 10 where the lines III and IV are plotted in their deflected form on the same plane only for better comparison.

On the plate with the face grain at 45 degrees with the axes the deflections along the line I are equal to those along the line II; these equalities can be proved by the symmetry of the face grain about the lines III and IV. The deflections along the lines III and IV do not bear any similarities since the face grain is perpendicular to the line III and parallel to the line IV.

The deflections on the plates with the face grain at angles of 22 1/2 and 67 1/2° exhibit the following similarities:

<table>
<thead>
<tr>
<th>22 1/2° - plate</th>
<th>67 1/2° - plate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Line I</td>
<td>=</td>
</tr>
<tr>
<td>Line II</td>
<td>=</td>
</tr>
<tr>
<td>Line III</td>
<td>=</td>
</tr>
<tr>
<td>Line IV</td>
<td>=</td>
</tr>
</tbody>
</table>

also

<table>
<thead>
<tr>
<th>0° - plate</th>
<th>90° - plate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Line I</td>
<td>=</td>
</tr>
<tr>
<td>Line II</td>
<td>=</td>
</tr>
<tr>
<td>Line III</td>
<td>=</td>
</tr>
<tr>
<td>Line IV</td>
<td>=</td>
</tr>
</tbody>
</table>
In Table 3 the observed values for the elastic constants $a_{ij}$ are calculated. The values for $w$ are obtained in the manner shown in Figure 11. From these deflections the elastic constants $a_{ij}$ are derived according to the equations on pages 12 and 14. A sample calculation is given in Appendix I.

As stated before, the observed values of the elastic constants $a_{ij}$, as calculated in Table 3 and given in Table 4, for the plates with the face grain making angles other than 0 and 90 degrees with the axes did not prove to be reliable. Therefore, these observed values were not used in the subsequent analysis for the plates. Thus, only the principal constants will be of interest henceforth. The manner of obtaining the principal constants was described previously in this paper.

In developing the nonorthotropic plate equation, the constants $a_{ij}$ are expressed with constants $B_{ij}$ as given by equations (23 a-c) in Appendix II and tabulated in Table 5. Furthermore the constants $B_{ij}$ are expressed with constants $D_{ij}$ as given by Equation (24).
<table>
<thead>
<tr>
<th>Deg Side</th>
<th>P lbs</th>
<th>LINE I</th>
<th>LINE II</th>
<th>LINE III</th>
<th>LINE IV</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.1</td>
<td>49.1</td>
<td>62.5</td>
<td>50.2</td>
<td>2.3</td>
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<td>5.3</td>
<td>48.0</td>
<td>65.1</td>
<td>50.7</td>
<td>2.9</td>
</tr>
<tr>
<td>Mean</td>
<td>3.2</td>
<td>48.5</td>
<td>63.8</td>
<td>50.5</td>
<td>2.6</td>
</tr>
<tr>
<td>22 1/2</td>
<td>A</td>
<td>60</td>
<td>4.2</td>
<td>37.2</td>
<td>50.5</td>
</tr>
<tr>
<td></td>
<td>4.8</td>
<td>36.9</td>
<td>51.7</td>
<td>40.3</td>
<td>3.7</td>
</tr>
<tr>
<td>Mean</td>
<td>4.5</td>
<td>37.0</td>
<td>51.1</td>
<td>39.8</td>
<td>3.9</td>
</tr>
<tr>
<td>45</td>
<td>A</td>
<td>40</td>
<td>5.2</td>
<td>41.7</td>
<td>52.3</td>
</tr>
<tr>
<td></td>
<td>4.9</td>
<td>39.6</td>
<td>56.1</td>
<td>43.4</td>
<td>5.4</td>
</tr>
<tr>
<td>Mean</td>
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<td>40.6</td>
<td>54.2</td>
<td>42.4</td>
<td>5.2</td>
</tr>
<tr>
<td>67 1/2</td>
<td>A</td>
<td>40</td>
<td>7.7</td>
<td>57.0</td>
<td>70.5</td>
</tr>
<tr>
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<td>50.2</td>
<td>66.3</td>
<td>53.9</td>
<td>7.5</td>
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<tr>
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<td>68.4</td>
<td>51.1</td>
<td>6.7</td>
</tr>
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<td>A</td>
<td>40</td>
<td>6.6</td>
<td>48.2</td>
<td>61.9</td>
</tr>
<tr>
<td></td>
<td>2.5</td>
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<tr>
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<td>47.5</td>
<td>60.8</td>
<td>46.5</td>
<td>3.5</td>
</tr>
<tr>
<td>Deg. Side P lbs</td>
<td>LINE I $w_{22}$</td>
<td>LINE II $w_{20}+w_{24}$ $w_{21}+w_{23}$</td>
<td>LINE III $w_{02}+w_{42}$ $w_{12}+w_{32}$</td>
<td>LINE IV $w_{00}+w_{44}$ $w_{11}+w_{33}$ $w_{04}+w_{40}$ $w_{13}+w_{31}$</td>
<td></td>
</tr>
<tr>
<td>----------------</td>
<td>-----------------</td>
<td>-----------------</td>
<td>-----------------</td>
<td>-----------------</td>
<td></td>
</tr>
<tr>
<td>0 A 10</td>
<td>80.4 80.0 80.0</td>
<td>80.0 79.4 73.8</td>
<td>79.6 3.3 60.7</td>
<td>81.0 162.8 101.4</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>80.0 82.0 81.4</td>
<td>82.6 83.4 83.6</td>
<td>82.0 3.9 64.5</td>
<td>79.6 160.8 96.4</td>
<td></td>
</tr>
<tr>
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<td>80.2 81.0 80.7</td>
<td>81.3 81.4 78.7</td>
<td>80.8 3.6 62.6</td>
<td>80.3 161.8 98.9</td>
<td></td>
</tr>
<tr>
<td>$22\frac{1}{2}$ A 10</td>
<td>64.6 51.1 62.2</td>
<td>64.5 68.6 65.8</td>
<td>63.3 3.6 52.6</td>
<td>65.6 109.6 76.0</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>65.2 51.3 61.2</td>
<td>64.3 70.2 67.4</td>
<td>66.7 4.4 46.6</td>
<td>64.0 108.4 78.4</td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>64.9 51.2 61.7</td>
<td>64.4 69.4 66.6</td>
<td>65.0 4.0 49.6</td>
<td>64.8 109.0 77.2</td>
<td></td>
</tr>
<tr>
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<td>48.6 43.0 47.6</td>
<td>49.4 42.2 47.6</td>
<td>48.8 2.4 32.6</td>
<td>49.4 71.6 54.3</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>49.8 43.8 48.2</td>
<td>49.8 42.6 47.2</td>
<td>50.0 2.8 33.4</td>
<td>48.6 71.6 55.7</td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>49.2 43.4 47.9</td>
<td>49.6 42.4 47.4</td>
<td>49.4 2.6 33.0</td>
<td>49.0 71.6 55.0</td>
<td></td>
</tr>
<tr>
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<td>65.7 72.2 69.2</td>
<td>64.8 51.4 62.4</td>
<td>67.0 1.7 53.8</td>
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<td></td>
</tr>
<tr>
<td>B</td>
<td>64.3 67.8 64.6</td>
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<td>64.2 2.3 50.2</td>
<td>63.4 106.7 75.4</td>
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<tr>
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<td>64.9 51.6 60.0</td>
<td>65.6 2.0 52.0</td>
<td>65.4 109.6 77.0</td>
<td></td>
</tr>
<tr>
<td>90 A 10</td>
<td>80.7 77.7 74.7</td>
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<td>81.6 4.2 63.2</td>
<td>80.4 154.8 98.8</td>
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<tr>
<td>B</td>
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<td>80.2 154.2 96.4</td>
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</tr>
<tr>
<td>Angle</td>
<td>Line</td>
<td>P (lb)</td>
<td>w(unit = 10^-3 in.)</td>
<td>Observed $a_{ij}$ (unit = 10^-6 in^2/lb)</td>
<td></td>
</tr>
<tr>
<td>-------</td>
<td>------</td>
<td>--------</td>
<td>--------------------</td>
<td>----------------------------------------</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>I</td>
<td>120</td>
<td>60.9 57.2 59.0</td>
<td>$a_{11} = 0.663$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>120</td>
<td>-2.6  -0.8 -1.7</td>
<td>$a_{12} = -0.019$</td>
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</tr>
<tr>
<td></td>
<td>III</td>
<td>120</td>
<td>56.4  56.8 56.6</td>
<td>$a_{11} + a_{12} = 0.636$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>IV</td>
<td>120</td>
<td>58.0  63.6 60.8</td>
<td>$a_{11} + a_{12} = 0.683$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>from 90°</td>
<td></td>
</tr>
<tr>
<td>22 1/2</td>
<td>I</td>
<td>60</td>
<td>46.9  50.8 48.8</td>
<td>$a_{11}' = 1.096$</td>
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</tr>
<tr>
<td></td>
<td>II</td>
<td>60</td>
<td>-7.1  -6.4 -6.8</td>
<td>$a_{12}' = -0.153$</td>
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<tr>
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<td>III</td>
<td>60</td>
<td>43.4  64.4 53.9</td>
<td>$a_{11}' + a_{12}' + a_{16}' = 1.211$</td>
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</tr>
<tr>
<td></td>
<td>IV</td>
<td>60</td>
<td>44.5  56.0 50.3</td>
<td>$a_{11}' + a_{12}' - a_{16}' = 1.130$</td>
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</tr>
<tr>
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<td></td>
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<td></td>
<td>$a_{22}' = a_{11}' = 2.117$</td>
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</tr>
<tr>
<td>45</td>
<td>I</td>
<td>40</td>
<td>49.1  50.8 49.9</td>
<td>$a_{11}'' = 1.682$</td>
<td></td>
</tr>
<tr>
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<td>II</td>
<td>40</td>
<td>-12.0 -13.6 -12.8</td>
<td>$a_{12}'' = -0.431$</td>
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<td>III</td>
<td>40</td>
<td>46.3  47.6 46.9</td>
<td>$a_{11}'' + a_{12}'' + a_{16}'' = 1.581$</td>
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</tr>
<tr>
<td></td>
<td>IV</td>
<td>40</td>
<td>46.3  47.2 46.7</td>
<td>$a_{11}'' + a_{12}'' - a_{16}'' = 1.574$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$a_{22}'' = a_{11}'' = 1.682$</td>
<td></td>
</tr>
<tr>
<td>67 1/2</td>
<td>I</td>
<td>40</td>
<td>61.7  64.0 62.8</td>
<td>$a_{11}''' = 2.117$</td>
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<tr>
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<td>II</td>
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<td>-3.4  -4.4 -3.9</td>
<td>$a_{12}''' = -0.131$</td>
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<td>III</td>
<td>40</td>
<td>60.8  60.0 60.4</td>
<td>$a_{11}''' + a_{12}''' + a_{16}''' = 2.036$</td>
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<tr>
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<td>IV</td>
<td>40</td>
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<td>$a_{11}''' + a_{12}''' - a_{16}''' = 1.951$</td>
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<tr>
<td></td>
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<td></td>
<td></td>
<td>$a_{22}''' = a_{11}''' = 1.096$</td>
<td></td>
</tr>
<tr>
<td>90</td>
<td>I</td>
<td>40</td>
<td>56.8  55.2 56.0</td>
<td>$a_{22} = 1.887$</td>
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<tr>
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<td>II</td>
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<td>1.2   8.4  4.8</td>
<td>$a_{21} = a_{12} = 0.162$</td>
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<td>III</td>
<td>40</td>
<td>54.8  56.0 55.4</td>
<td>$a_{22} + a_{12} = 1.866$</td>
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<tr>
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<td>IV</td>
<td>40</td>
<td>54.6  61.6 58.1</td>
<td>$a_{22} + a_{12} = 1.956$</td>
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### TABLE 3 (continued)

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<thead>
<tr>
<th>Angle deg</th>
<th>Line</th>
<th>P</th>
<th>w (unit = 10(^3) in.)</th>
<th>Observed (a_{ij}) (unit = 10(^{-6}) in(^2)/lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
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<td>outside</td>
<td>inside</td>
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<td>II</td>
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<td>0.6</td>
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<td>III</td>
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<td>77.2</td>
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<td>IV</td>
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<td>-81.5</td>
<td>-74.4</td>
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<td>I</td>
<td>10</td>
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<td>10</td>
<td>-5.0</td>
<td>-8.8</td>
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<td>IV</td>
<td>10</td>
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* See sample calculations in Appendix I
PART III

THE NONORTHOTROPIC PLATE EQUATION IN TERMS OF
FINITE-DIFFERENCE EQUATIONS

The nonorthotropic plate equation

\[ \frac{D_{11}}{\partial x^4} + 4D_{16} \frac{\partial w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66}) \frac{\partial^2 w}{\partial x^2 \partial y^2} + 4D_{26} \frac{\partial^3 w}{\partial x \partial y^3} + D_{22} \frac{\partial^4 w}{\partial y^4} = q \]  

given in (2, p. 141) and derived in Appendix III will be expressed in terms of finite-difference equations.

1. **Finite-Difference Equations for Uniformly Loaded Plates**

The finite-difference equations will be written for the central 18 x 18 in. square of the plates as shown in Figure 2.

The application of finite-difference equations to the solution of difficult structural problems is in large measure comparable to the technique now used to surmount mathematical difficulties in the solution of indeterminate structures. Generally, due either to variations in the shape and size of the members or irregularities in the loading pattern, the integration of the moment curves involve lengthy and difficult mathematical manipulations. To avoid such intricate operations the required integrations of the various functions are approximated by the summation of a number of specific values which represent the average area for a finite segment. This substitution of the
numerical procedure of summation for the more precise operation of integration is similar to the replacement of the differential equation by the approximate finite-difference equation.

Just as the replacement of an integral by the summation procedure involves the use of average values of the ordinates, so the replacement of a derivative by finite difference is based on taking the difference of average values of the ordinates. Therefore it is evident from geometrical considerations of Figure 13 that if $w = f(x)$, then

\[
\begin{array}{cccccc}
40 & 41 & 42 & 43 & 44 \\
30 & 31 & 32 & 33 & 34 \\
20 & 21 & 22 & 23 & 24 \\
10 & 11 & 12 & 13 & 14 \\
00 & 01 & 02 & 03 & 04 \\
\end{array}
\]

\[w \rightarrow f(x)\]

Figure 13. The grid for the finite-difference equations.
the slope at point 22 may be approximated by (4, p. 64-81)

\[
\frac{\partial w}{\partial x}_{22} = \frac{1}{h} (w_{23} - w_{22}) \quad \text{the first forward difference quotient}
\]

\[
\frac{\partial w}{\partial x}_{22} = \frac{1}{h} (w_{22} - w_{21}) \quad \text{the first backward difference quotient}
\]

As an alternate and improved approximation the mean value of the forward and backward difference quotients may be obtained; this value is called the first central difference quotient.

\[
\frac{\partial w}{\partial x}_{22} = \frac{1}{2h} \left( \frac{w_{23} - w_{22}}{h} + \frac{w_{22} - w_{21}}{h} \right) = \frac{1}{2h} (w_{23} - w_{21}) \tag{4}
\]

The first central difference quotient will be used in this paper.

To approximate the second derivative at point 22, choice can be made of the three forms above. By the first central difference quotient the second derivative may be approximated by taking the first central difference quotient of the first central difference quotient as follows:

\[
\frac{\partial^2 w}{\partial x^2}_{22} = \frac{1}{2h} \left( \frac{w_{24} - w_{22}}{2h} - \frac{w_{22} - w_{20}}{2h} \right) = \frac{1}{4h^2} (w_{24} - 2w_{22} + w_{20})
\]

This form, however, involves values of \( w \) at points which are two grid points away from the point of interest 22. Since the second derivative at 22 will depend more on the points 23 and 21 than on the
points farther removed, the second derivative may be approximated
by taking the first forward difference quotient of the first backward
difference quotient. The second derivative may also be taken as the
rate of change of slope at 22; thus

$$\left[ \frac{\partial^2 w}{\partial x^2} \right]_{22} = \frac{1}{h} \left( \frac{w_{23} - w_{22}}{h} - \frac{w_{22} - w_{21}}{h} \right) = \frac{1}{2h} (w_{23} - 2w_{22} + w_{21}) \quad (5)$$

Similarly, the third derivative may be approximated by

$$\left[ \frac{\partial^3 w}{\partial x^3} \right]_{22} = \frac{1}{2h} \left[ \left( \frac{\partial w}{\partial x} \right)_{23} - 2 \left( \frac{\partial w}{\partial x} \right)_{22} + \left( \frac{\partial w}{\partial x} \right)_{21} \right]$$

$$= \frac{1}{2h} \left( \frac{w_{24} - w_{22}}{2h} - \frac{w_{23} - w_{21}}{h} + \frac{w_{22} - w_{20}}{2h} \right)$$

$$= \frac{1}{2h} (w_{24} - 2w_{23} + 2w_{21} - w_{20}) \quad (6)$$

and the fourth derivative by

$$\left[ \frac{\partial^4 w}{\partial x^4} \right]_{22} = \frac{1}{h^2} \left[ \left( \frac{\partial^2 w}{\partial x^2} \right)_{23} - 2 \left( \frac{\partial^2 w}{\partial x^2} \right)_{22} + \left( \frac{\partial^2 w}{\partial x^2} \right)_{21} \right]$$

$$= \frac{1}{h^4} \left[ (w_{24} - 2w_{23} + w_{22}) - 2(w_{23} - 2w_{22} + w_{21}) + (w_{22} - 2w_{21} + w_{20}) \right]$$

$$= \frac{1}{h^4} (w_{24} - 4w_{23} + 6w_{22} - 4w_{21} + w_{20}) \quad (7)$$

Similarly in the y-direction derivatives may be approximated by

$$\left[ \frac{\partial w}{\partial y} \right]_{22} = \frac{1}{2h} (w_{12} - w_{32}) \quad (8)$$
\[
\left[ \frac{\partial^2 w}{\partial y^2} \right]_{22} = \frac{1}{h^2} (w_{12} - 2w_{22} + w_{32}) \quad (9)
\]

\[
\left[ \frac{\partial^4 w}{\partial y^4} \right]_{22} = \frac{1}{h^4} (w_{02} - 4w_{12} + 6w_{22} - 4w_{32} + w_{42}) \quad (10)
\]

The mixed second derivative may be approximated by

\[
\left[ \frac{\partial^2 w}{\partial x \partial y} \right]_{22} = \left[ \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial y} \right) \right]_{22} = \frac{1}{2h} \left[ \left( \frac{\partial w}{\partial x} \right)_{12} - \left( \frac{\partial w}{\partial x} \right)_{32} \right]
\]

\[
= \frac{1}{2h} \left( \frac{w_{13} - w_{11}}{2h} - \frac{w_{33} - w_{31}}{2h} \right)
\]

\[
= \frac{1}{4h^2} (w_{13} - w_{11} - w_{33} + w_{31}) \quad (11)
\]

and the mixed fourth derivative by

\[
\left[ \frac{\partial^4 w}{\partial x^2 \partial y^2} \right]_{22} = \left[ \frac{\partial^2}{\partial x^2} \left( \frac{\partial^2 w}{\partial y^2} \right) \right]_{22} = \frac{1}{h^2} \left[ \left( \frac{\partial^2 w}{\partial x^2} \right)_{12} - 2 \left( \frac{\partial^2 w}{\partial x^2} \right)_{22} + \left( \frac{\partial^2 w}{\partial x^2} \right)_{32} \right]
\]

\[
= \frac{1}{h^2} \left[ \frac{w_{13} - 2w_{12} + w_{11}}{h^2} - 2 \left( \frac{w_{23} - 2w_{22} + w_{21}}{h^2} \right)
\]

\[
+ \frac{w_{33} - 2w_{32} + w_{31}}{h^2} \right]
\]

\[
= \frac{1}{h} \left[ \frac{w_{33} + w_{13} + w_{11} + w_{31} - 2(w_{32} + w_{12} + w_{23} + w_{21})}{h^2} + 4w_{22} \right] \quad (12)
\]
and the mixed fourth derivative with the third derivative in the x-direction by

\[
\left[ \frac{\partial^4 w}{\partial x^3 \partial y} \right]_{22} = \left[ \frac{\partial^3 w}{\partial y \partial x^3} \right]_{22} = \frac{1}{2h^3} \left[ \frac{\partial w}{\partial y} \right]_{24} - 2 \left( \frac{\partial w}{\partial y} \right)_{23} + 2 \left( \frac{\partial w}{\partial y} \right)_{21} - \left( \frac{\partial w}{\partial y} \right)_{20}
\]

\[
= \frac{1}{2h^3} \left[ \frac{w_{14} - w_{34}}{2h} - 2 \left( \frac{w_{13} - w_{33}}{2h} \right) + 2 \left( \frac{w_{11} - w_{31}}{2h} \right) - \frac{w_{10} - w_{30}}{2h} \right]
\]

\[
= \frac{1}{4h^4} \left[ \frac{w_{14} - w_{34} - w_{10} + w_{30}}{2h} - 2(w_{13} - w_{33} - w_{11} + w_{31}) \right] \tag{13}
\]

The mixed fourth derivative with the third derivative in the y-direction becomes

\[
\left[ \frac{\partial^4 w}{\partial x \partial y^3} \right]_{22} = \left[ \frac{\partial^3 w}{\partial x \partial y^3} \right]_{22} = \frac{1}{2h^3} \left[ \frac{\partial w}{\partial x} \right]_{02} - 2 \left( \frac{\partial w}{\partial x} \right)_{12} + 2 \left( \frac{\partial w}{\partial x} \right)_{32} - \left( \frac{\partial w}{\partial x} \right)_{42}
\]

\[
= \frac{1}{2h^3} \left[ \frac{w_{03} - w_{01}}{2h} - 2 \left( \frac{w_{13} - w_{11}}{2h} \right) + 2 \left( \frac{w_{33} - w_{31}}{2h} \right) - \frac{w_{43} - w_{41}}{2h} \right]
\]

\[
= \frac{1}{4h^4} \left[ \frac{w_{03} - w_{01} - w_{43} + w_{41}}{2h} - 2(w_{13} - w_{11} - w_{33} + w_{31}) \right] \tag{14}
\]

Since the partial differential equation to be solved is

\[
D_{11} \frac{\partial^4 w}{\partial x^4} + 4D_{16} \frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26} \frac{\partial^4 w}{\partial x \partial y^3} + D_{22} \frac{\partial^4 w}{\partial y^4} = q, \tag{3}
\]

the required partial derivatives at point 22 may be expressed in terms of the deflections at 22 and the neighboring points as follows:
\[
\begin{align*}
\frac{\partial^4 w}{\partial x^4} & \bigg|_{22} = \frac{1}{h^4} (w_{24} - 4w_{23} + 6w_{22} - 4w_{21} + w_{20}) \\
\frac{\partial^4 w}{\partial x^2 \partial y^2} & \bigg|_{22} = \frac{1}{h^4} \left[ w_{14} - w_{34} - w_{10} + w_{30} + 2(2w_{13} - w_{33} - w_{11} + w_{31}) \right] \\
\frac{\partial^4 w}{\partial x^2 \partial y^2} & \bigg|_{22} = \frac{1}{h^4} \left[ w_{33} + w_{13} + w_{11} w_{31} - 2(w_{32} + w_{12} + w_{23} + w_{21}) + 4w_{22} \right] \\
\frac{\partial^4 w}{\partial x \partial y^3} & \bigg|_{22} = \frac{1}{4h^4} \left[ w_{03} - w_{01} - w_{43} + w_{41} - 2(w_{13} - w_{11} - w_{33} + w_{31}) \right] \\
\frac{\partial^4 w}{\partial y^4} & \bigg|_{22} = \frac{1}{h^4} \left( w_{02} - 4w_{12} + 6w_{22} - 4w_{32} + w_{42} \right)
\end{align*}
\]

Since a differential equation expresses a functional relationship valid at any point, while a finite-difference equation expresses a relation for only one point, the replacement of a differential equation by its finite-difference equivalent requires the writing of finite-difference equations for each point. In the case of a beam, slab, and a plate, approximations of the loads, shears, and moments at each point are expressed as linear functions of the deflections of the neighboring points. Thus the problem of solving a complicated differential equation is reduced to one of solving a number of simultaneous linear algebraic equations. These simultaneous equations are written for each grid intersection or node point. The number of equations may be reduced if the plate possesses symmetry.
The grid system may be made infinitely small; the smaller the grid spacing, the greater will be the number of simultaneous equations derived, and the more closely will the solution of the equations approximate that of a rigorous mathematical solution of the differential equation.

Grid systems of 3 x 3 in. and 4 1/2 x 4 1/2 in. were investigated in order to determine the size of grid that must be used on the plates to obtain reasonable predictions of structural behavior. The results were found to vary only by 0.5%; therefore, the grid of 4 1/2 x 4 1/2 in. will be used here.

The finite-difference equations for the neighboring points were obtained by drawing the grid on a transparent paper the central point of which was placed over the point on the plate where the equation was desired and by replacing the deflections w in the finite-difference equation for the central point 22 by the deflections on the plate at the points that fell under the original deflections in the equation for point 22 on the transparent paper as shown in Figure 15.

The fictitious deflections at the points that fall outside of the plate can be expressed in terms of real ones if the plate in question is imagined to be a part of an infinitely large plate loaded with a checker pattern. The large plate is imagined to be supported on a grid of the size of the smaller plate. Then the deflections at any
point on one plate is equal to the deflection at a certain point on the neighboring plate. The determination of the corresponding points is shown in Figure 14.

Figure 14. (a) Loading pattern for fictitious deflections. (b) Deflection curve in the x-direction. (c) Deflection curve in the y-direction.
Figure 14(a) shows the loading pattern for obtaining the deflections for the points that fall outside of a simply supported plate; moment is zero along the edges. The deflection curves in the x-direction and in the y-direction are shown in Figure 14(b) and (c), respectively. From (a) it is seen that the deflection at point 19 is equal to the negative deflection at point 23 rather than at point 21; this phenomenon becomes quite obvious for plates which have the face plies at oblique angles with the axes.

Figure 15. The method of obtaining finite-difference equations for points other than the center.
Figure 15 shows the pattern from which the required finite-difference equations may be obtained easily at point 11 since the central point 22 of the transparent paper is over point 11 of the plate.

The fourth derivative in the x-direction at point 11 is obtained by replacing in Equation (7) the subscripts by those shown in Figure 15 on the right hand side. Thus,

\[
\frac{\partial^4 w}{\partial x^4}_{11} = \frac{1}{h^4} (w_{13} - 4w_{12} + 6w_{11} - 4w_{10} + w_{09})
\]

From Figure 14(a) it is seen that \( w_{09} = -w_{13} \); moreover, the deflections over the supports are equal to zero, i.e., \( w_{10} = 0 \). Thus,

\[
\frac{\partial^4 w}{\partial x^4}_{11} = \frac{1}{h^4} (-4w_{12} + 6w_{11})
\]

Similarly

\[
\frac{\partial^4 w}{\partial x^3 \partial y}_{11} = \frac{1}{4h} \left[ -w_{23} - w_{23} -2(-w_{22}) \right] = \frac{1}{4h} \left( 2w_{22} - 2w_{23} \right)
\]

\[
\frac{\partial^4 w}{\partial x^2 \partial y^2}_{11} = \frac{1}{4h} \left[ w_{22} -2(w_{21} + w_{12}) + 4w_{11} \right] = \frac{1}{4h} \left( 4w_{11} - 2w_{12} - 2w_{21} + w_{22} \right)
\]
\[
\frac{\partial^4 w}{\partial x \partial y^3} = \frac{1}{4h} \left[ -w_{32} - w_{32} - 2(-w_{22}) \right] = \frac{1}{4h} (2w_{22} - 2w_{32})
\]

\[
\frac{\partial^4 w}{\partial y^4} = \frac{1}{h} \left[ -w_{31} + 6w_{11} - 4w_{21} + w_{31} \right] = \frac{1}{h} (6w_{11} - 4w_{21}).
\]

The finite-difference equations will be simplified by the considerations of symmetry as was stated before. Figure 16 shows the effect of symmetry and the points at which the simultaneous equations must be written.

Figure 16. The effect of symmetry on plywood plates.

From symmetry

\[
\begin{align*}
    w_{13} &= w_{31} = w_{33} = w_{11} \\
    w_{23} &= w_{21} \\
    w_{31} &= w_{13} \\
    w_{31} &= w_{13}
\end{align*}
\]
Equations are required at node points

\begin{align*}
0^\circ & \text{ & } 90^\circ \\
22 \ 1/2^\circ & \text{ & } 67 \ 1/2^\circ \\
45^\circ & \\
11, \ 12, \ 21, \ 22 & \\
11, \ 12, \ 13, \ 21, \ 22 & \\
11, \ 12, \ 13, \ 21, \ 22. & \\
\end{align*}

The partial derivatives for various plates at the points where equations are needed as shown in Figure 16 can be expressed in terms of the deflections at the neighboring points as follows:

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<tr>
<th>Angle (deg.)</th>
<th>Node</th>
<th>Equation</th>
</tr>
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| 0            | 22   | \[
\begin{bmatrix} \frac{\partial^4 w}{\partial x^4} \\ \frac{\partial^4 w}{\partial x^3 \partial y} \end{bmatrix} = \frac{1}{h^4} (6w_{22} - 8w_{21}) \\
\frac{\partial^4 w}{\partial y^4} = 0 \\
\frac{\partial^4 w}{\partial x^2 \partial y^2} = \frac{1}{h^4} (4w_{11} - 4w_{12} - 4w_{21} + 4w_{22}) \\
\frac{\partial^4 w}{\partial y^4} = \frac{1}{h^4} (6w_{22} - 8w_{12})
\]
| 0            | 21   | \[
\begin{bmatrix} \frac{\partial^4 w}{\partial x^4} \\ \frac{\partial^4 w}{\partial x^3 \partial y} \end{bmatrix} = \frac{1}{h^4} (6w_{21} - 4w_{22}) \\
\frac{\partial^4 w}{\partial y^4} = 0
\]
Angle (deg.) Node Equation

\[
\begin{align*}
\frac{\partial^4 w}{\partial x^2 \partial y^2} &= \frac{1}{h^4} (2w_{12} - 4w_{11} - 2w_{22} + 4w_{21}) \\
\frac{\partial^4 w}{\partial x \partial y^3} &= 0 \\
\frac{\partial^4 w}{\partial y^4} &= \frac{1}{h^4} (6w_{21} - 8w_{11}) \\
\frac{\partial^4 w}{\partial x^4} &= \frac{1}{h^4} (6w_{12} - 8w_{11}) \\
\frac{\partial^4 w}{\partial x^3 \partial y} &= 0 \\
\frac{\partial^4 w}{\partial x^2 \partial y^2} &= \frac{1}{h^4} (2w_{21} - 2w_{22} - 4w_{11} + 4w_{12}) \\
\frac{\partial^4 w}{\partial x \partial y^3} &= 0 \\
\frac{\partial^4 w}{\partial y^4} &= \frac{1}{h^4} (6w_{12} - 4w_{22}) \\
\frac{\partial^4 w}{\partial x^4} &= \frac{1}{h^4} (6w_{11} - 4w_{12}) \\
\frac{\partial^4 w}{\partial x^3 \partial y} &= \frac{1}{4h^4} (2w_{22} - 2w_{21})
\end{align*}
\]
<table>
<thead>
<tr>
<th>Angle (deg.)</th>
<th>Node</th>
<th>Equation</th>
</tr>
</thead>
</table>
|             |      | \[
\left[ \frac{\partial^4 w}{\partial x^2 \partial y^2} \right] = \frac{1}{4} \frac{1}{h} (w_{22} - 2w_{21} - 2w_{12} + 4w_{11})
\]
|             |      | \[
\left[ \frac{\partial^4 w}{\partial x \partial y^3} \right] = \frac{1}{4} \frac{1}{h} (2w_{22} - 2w_{12})
\]
|             |      | \[
\left[ \frac{\partial^4 w}{\partial y^4} \right] = \frac{1}{4} \frac{1}{h} (6w_{11} - 4w_{21})
\]
| 22 \(1/2\) | 22   | \[
\left[ \frac{\partial^4 w}{\partial x^4} \right] = \frac{1}{4} \frac{1}{h} (6w_{22} - 8w_{21})
\]
|             |      | \[
\left[ \frac{\partial^4 w}{\partial x^3 \partial y} \right] = \frac{1}{4} \frac{1}{h} (4w_{11} - 4w_{13})
\]
|             |      | \[
\left[ \frac{\partial^4 w}{\partial x^2 \partial y^2} \right] = \frac{1}{4} \frac{1}{h} (2w_{11} - 4w_{12} + 2w_{13} - 4w_{21} + 4w_{22})
\]
|             |      | \[
\left[ \frac{\partial^4 w}{\partial x^3 \partial y} \right] = \frac{1}{4} \frac{1}{h} (4w_{11} - 4w_{13})
\]
|             |      | \[
\left[ \frac{\partial^4 w}{\partial y^4} \right] = \frac{1}{4} \frac{1}{h} (6w_{22} - 8w_{21})
\]
| 22 \(1/2\) | 21   | \[
\left[ \frac{\partial^4 w}{\partial x^4} \right] = \frac{1}{4} \frac{1}{h} (6w_{11} - 4w_{22})
\]
|             |      | \[
\left[ \frac{\partial^4 w}{\partial x^3 \partial y} \right] = \frac{1}{4} \frac{1}{h} (2w_{13} - 2w_{11})
\]
<table>
<thead>
<tr>
<th>Angle Node (deg.)</th>
<th>Equation</th>
</tr>
</thead>
</table>
|                  | \[
\left[ \frac{\partial^4 w}{\partial x^2 \partial y^2} \right] = \frac{1}{4h} (-2w_{11} + 2w_{12} - 2w_{13} + 4w_{21} - 2w_{22})
\]
|                  | \[
\left[ \frac{\partial^4 w}{\partial x^3 \partial y} \right] = 0
\]
|                  | \[
\left[ \frac{\partial^4 w}{\partial y^4} \right] = \frac{1}{4h} (-4w_{11} - 4w_{13} + 6w_{21})
\]
| 2 2 1/2 11       | \[
\left[ \frac{\partial^4 w}{\partial x^4} \right] = \frac{1}{4h} (6w_{11} - 4w_{12})
\]
|                  | \[
\left[ \frac{\partial^4 w}{\partial x^3 \partial y} \right] = \frac{1}{4h} (-2w_{21} + 2w_{22})
\]
|                  | \[
\left[ \frac{\partial^4 w}{\partial x^2 \partial y^2} \right] = \frac{1}{4h} (4w_{11} - 2w_{12} - 2w_{21} + w_{22})
\]
|                  | \[
\left[ \frac{\partial^4 w}{\partial x^3 \partial y} \right] = \frac{1}{4h} (-2w_{12} + 2w_{22})
\]
|                  | \[
\left[ \frac{\partial^4 w}{\partial y^4} \right] = \frac{1}{4h} (6w_{11} - 4w_{21})
\]
| 2 2 1/2 12       | \[
\left[ \frac{\partial^4 w}{\partial x^4} \right] = \frac{1}{4h} (-4w_{11} + 6w_{12} - 4w_{13})
\]
|                  | \[
\left[ \frac{\partial^4 w}{\partial x^3 \partial y} \right] = 0
\]
<table>
<thead>
<tr>
<th>Angle (deg.)</th>
<th>Node</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>22</td>
<td>$\left[ \frac{\partial^4 w}{\partial x^2 \partial y^2} \right] = \frac{1}{4h} (-2w_{11} + 4w_{12} - 2w_{13} + 2w_{21} - 2w_{22})$</td>
</tr>
<tr>
<td></td>
<td>1/2</td>
<td>$\left[ \frac{\partial^4 w}{\partial x \partial y^3} \right] = \frac{1}{4h} (-2w_{11} + 2w_{13})$</td>
</tr>
<tr>
<td></td>
<td>13</td>
<td>$\left[ \frac{\partial^4 w}{\partial y^4} \right] = \frac{1}{4h} (6w_{12} - 4w_{22})$</td>
</tr>
<tr>
<td>22 1/2</td>
<td></td>
<td>$\left[ \frac{\partial^4 w}{\partial x^4} \right] = \frac{1}{4h} (-4w_{12} + 6w_{13})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\left[ \frac{\partial^3 w}{\partial x^3 \partial y} \right] = \frac{1}{4h} (2w_{21} - 2w_{22})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\left[ \frac{\partial^2 w}{\partial x \partial y^2} \right] = \frac{1}{4h} (-2w_{12} + 4w_{13} - 2w_{21} + w_{22})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\left[ \frac{\partial^3 w}{\partial x^3 \partial y} \right] = \frac{1}{4h} (2w_{12} - 2w_{22})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\left[ \frac{\partial^4 w}{\partial y^4} \right] = \frac{1}{4h} (6w_{13} - 4w_{21})$</td>
</tr>
<tr>
<td>45</td>
<td>22</td>
<td>$\left[ \frac{\partial^4 w}{\partial x^4} \right] = \frac{1}{4h} (6w_{22} - 8w_{21})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\left[ \frac{\partial^3 w}{\partial x^3 \partial y} \right] = \frac{1}{4h} (4w_{11} - 4w_{13})$</td>
</tr>
<tr>
<td>Angle (deg.)</td>
<td>Node</td>
<td>Equation</td>
</tr>
<tr>
<td>------------</td>
<td>------</td>
<td>----------</td>
</tr>
</tbody>
</table>
| 45         | 21   | \[
\frac{\partial^4 w}{\partial x^2 \partial y^2} = \frac{1}{4h} \left(2w_{12} - 4w_{13} - 2w_{21} + 4w_{22}\right)
\] |
| 45         | 11   | \[
\frac{\partial^4 w}{\partial x^4} = \frac{1}{4h} \left(6w_{11} - 4w_{12}\right)
\] |
| 45         | 21   | \[
\frac{\partial^4 w}{\partial y^4} = \frac{1}{4h} \left(6w_{21} - 4w_{22}\right)
\] |
|           |      | \[
\frac{\partial^4 w}{\partial x^3 \partial y} = \frac{1}{4h} \left(-2w_{11} + 2w_{13}\right)
\] |
|           |      | \[
\frac{\partial^4 w}{\partial x^2 \partial y^2} = \frac{1}{4h} \left(-2w_{11} + 2w_{12} - 2w_{13} + 4w_{21} - 2w_{22}\right)
\] |
|           |      | \[
\frac{\partial^4 w}{\partial x \partial y^3} = 0
\] |
|           |      | \[
\frac{\partial^4 w}{\partial y^4} = \frac{1}{4h} \left(-4w_{11} - 4w_{13} + 6w_{21}\right)
\] |
<table>
<thead>
<tr>
<th>Angle (deg.)</th>
<th>Node</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>45 12</td>
<td>[ \frac{\partial^4 w}{\partial x^2 \partial y^2} = \frac{1}{h^4} (4w_{11} - 2w_{12} - 2w_{21} + w_{22}) ]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[ \frac{\partial^4 w}{\partial x^2 \partial y^3} = \frac{1}{h^4} (2w_{12} - 2w_{12}) ]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[ \frac{\partial^4 w}{\partial y^4} = \frac{1}{h^4} (6w_{11} - 4w_{21}) ]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[ \frac{\partial^4 w}{\partial x^4} = \frac{1}{h^4} (-4w_{11} + 6w_{12} - 4w_{13}) ]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[ \frac{\partial^4 w}{\partial x^3 \partial y} = 0 ]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[ \frac{\partial^4 w}{\partial x^2 \partial y^2} = \frac{1}{h^4} (-2w_{11} + 4w_{12} - 2w_{13} + 2w_{21} - 2w_{22}) ]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[ \frac{\partial^4 w}{\partial x \partial y^3} = \frac{1}{h^4} (-2w_{11} + 2w_{13}) ]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[ \frac{\partial^4 w}{\partial y^4} = \frac{1}{h^4} (6w_{12} - 4w_{22}) ]</td>
<td></td>
</tr>
<tr>
<td>45 13</td>
<td>[ \frac{\partial^4 w}{\partial x^4} = \frac{1}{h^4} (-4w_{12} + 6w_{13}) ]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[ \frac{\partial^4 w}{\partial x^3 \partial y} = \frac{1}{h^4} (2w_{21} - 2w_{22}) ]</td>
<td></td>
</tr>
</tbody>
</table>
Angle (deg.) Node

\[
\begin{align*}
\left[ \frac{\partial^4 w}{\partial x^2 \partial y^2} \right] &= \frac{1}{h^4} (-2w_{12} + 4w_{13} - 2w_{21} + w_{22}) \\
\left[ \frac{\partial^4 w}{\partial x \partial y^3} \right] &= \frac{1}{4h^4} (2w_{12} - 2w_{22}) \\
\left[ \frac{\partial^4 w}{\partial y^4} \right] &= \frac{1}{h^4} (6w_{13} - 4w_{21}).
\end{align*}
\]

When the constants $D_{ij}$ from Table 6 and also $1/h^4$ that appears in the partial derivatives are replaced by their numerical values, the nonorthotropic plate equation is expressed in terms of finite-difference equations as follows:

Angle (deg.) Node

\[
\begin{align*}
0 & \quad 22 \quad 55.622w_{11} - 163.335w_{12} - 362.313w_{21} + 366.424w_{22} - q = 0 \\
21 & \quad -163.335w_{11} + 27.811w_{12} + 366.424w_{21} - 181.156w_{22} - q = 0 \\
12 & \quad -362.313w_{11} + 366.424w_{12} + 27.811w_{21} - 81.667w_{22} - q = 0 \\
11 & \quad 366.424w_{11} - 181.156w_{12} - 81.667w_{21} - 13.905w_{22} - q = 0 \\
22 & \quad 1/2 \quad 44.773w_{11} - 243.084w_{12} + 108.569w_{13} - 370.479w_{21} + 383.502w_{22} - q = 0 \\
\end{align*}
\]

\[
\begin{align*}
21 & \quad -104.959w_{11} + 76.671w_{12} - 138.125w_{13} + 383.502w_{21} - 185.240w_{22} - q = 0 \\
\end{align*}
\]
<table>
<thead>
<tr>
<th>Angle (deg.)</th>
<th>Node</th>
<th>Simultaneous equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>-185.874w_{11} +383.502w_{12} -184.606w_{13} + 76.671w_{21} -121.542w_{22} - q = 0</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>383.502w_{11} -185.874w_{12} -104.959w_{21} + 22.387w_{22} - q = 0</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>-184.606w_{12} +383.502w_{13} -138.125w_{21} + 54.285w_{22} - q = 0</td>
<td></td>
</tr>
<tr>
<td>45</td>
<td>91.694w_{11} -414.375w_{12} +191.190w_{13} -414.375w_{21} +480.121w_{22} - q = 0</td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>-194.751w_{11} +141.442w_{12} -219.625w_{13} -480.121w_{21} -207.188w_{22} - q = 0</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>-194.751w_{11} +480.121w_{12} -219.625w_{13} +141.442w_{21} -207.180w_{22} - q = 0</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>480.121w_{11} -194.751w_{12} -194.751w_{21} + 45.847w_{22} - q = 0</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>-219.625w_{12} +480.121w_{13} -219.625w_{21} + 95.595w_{22} - q = 0</td>
<td></td>
</tr>
</tbody>
</table>

Since the plates are square, the deflections of the plates with the face grain at 67 1/2 and 90 degrees can be expressed in terms of the deflections of the plates with the face grain at 22 1/2 and 0 degrees, respectively. Thus, only the plates of 0, 22 1/2,
and 45 degrees need be investigated. In the tests the deflections of the plates of 67 1/2 and 90 degrees were included in the deflections of the plates of 22 1/2 and 0 degrees, respectively.
PART IV

EXPERIMENTAL METHODS FOR VERIFYING
THE DEFLECTION THEORY
FOR PLYWOOD PLATES

In order to test the validity of the methods used in computing
the deflections, the deflections were observed in laboratory tests
on simply supported, 18 x 18 in. square plates under uniform load-
ing.

The plates used in this set of tests were the same as used in
the torsion tests. The plates were supported on 1/8-inch edge along
the four sides. For the uniform loading 5/16-inch-diameter lead-
shot was used. To hold the leadshot on the plates, a square 18 x 18
in. and 4 in. high box without a bottom, or a frame, was made of
1/16-inch thick sheet metal as shown in Figure 18. The frame was
stiffened along the four sides by 3/4 x 3/4 x 1/8 in. angles. C-
clamps were used to hold the frame on the plates when the load was
placed.

Timoshenko (7, p. 85) has shown that the corners of a square
plate under uniform loading have the tendency to rise and that this
rising is prevented by concentrated reactions at the corners. In
the tests performed here the corner reactions were provided by
placing point supports at the corners. These point supports were
also placed at the third-points along each side of the frame in order to bring the plates to bear evenly on the edge support, see Figure 18. Placing the frame on the point supports on top of the plates allowed the plates to rotate freely.

The loading of 5/16-inch-diameter leadshot was selected because one layer of 18 x 18 inches weighed 30 lbs. Thus the placing of a uniform loading on the plates was facilitated considerably. Consequently, the load was placed and the deflections were observed at increments of 30 lbs. The maximum loading, the loading that deflected the plates approximately one-tenth of the thickness of the plates, or 0.50/10 in., was 240 lbs; Figure 19 shows the method of loading.

The deflections were observed at the grid points first on one side, or side A, and then the plates were reversed and the deflections were observed in the opposite direction, or on side B. These observations were made at the load increments of 30 lbs. as mentioned above. Figure 17 shows the arrangement of the deflection gauges under the plates at the grid points. The deflection observations under the maximum uniform loading are shown in Figure 20.

The observed deflections from the tests were plotted against the load at increments of 30 lbs. for the uniformly loaded plates. The experimental deflections to be compared against those computed
from the finite-difference equations were obtained from the load-deflection curves as explained before and shown in Figure 21.

Since the number of the simultaneous equations was small, the Doolittle method was employed in solving these equations.

Table 7 gives the observed and the theoretical deflections; the percentage of discrepancy based on the computed deflections is also shown.
Figure 17. Deflection gauges and edge supports in the uniform loading tests.
Figure 18. The frame for holding the uniform load on the plates.
Figure 19. Uniform load on the plates.
Figure 20. Deflection observations in the uniform loading tests.
Figure 21. Sample of Load-Deflection Curves for Uniformly Loaded Plates.
<table>
<thead>
<tr>
<th>Angle deg.</th>
<th>Node</th>
<th>P lbs.</th>
<th>q lbs/in.²</th>
<th>Obs'd* Deflections</th>
<th>Calc'd Deflections</th>
<th>Difference in.</th>
<th>Discrepancy %</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>22</td>
<td>240</td>
<td>240</td>
<td>0.0577</td>
<td>0.0466</td>
<td>0.0111</td>
<td>+23.8</td>
</tr>
<tr>
<td>21</td>
<td></td>
<td></td>
<td></td>
<td>0.0417</td>
<td>0.0334</td>
<td>0.0083</td>
<td>+24.8</td>
</tr>
<tr>
<td>12</td>
<td></td>
<td></td>
<td></td>
<td>0.0422</td>
<td>0.0342</td>
<td>0.0080</td>
<td>+23.4</td>
</tr>
<tr>
<td>11</td>
<td></td>
<td></td>
<td></td>
<td>0.0308</td>
<td>0.0246</td>
<td>0.0062</td>
<td>+25.2</td>
</tr>
<tr>
<td>22¹/₂</td>
<td>22</td>
<td></td>
<td></td>
<td>0.0500</td>
<td>0.0399</td>
<td>0.0101</td>
<td>+25.3</td>
</tr>
<tr>
<td>21</td>
<td></td>
<td></td>
<td></td>
<td>0.0357</td>
<td>0.0286</td>
<td>0.0071</td>
<td>+24.8</td>
</tr>
<tr>
<td>12</td>
<td></td>
<td></td>
<td></td>
<td>0.0367</td>
<td>0.0293</td>
<td>0.0074</td>
<td>+25.3</td>
</tr>
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<td></td>
<td></td>
<td>0.0288</td>
<td>0.0230</td>
<td>0.0058</td>
<td>+25.2</td>
</tr>
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<td></td>
<td></td>
<td></td>
<td>0.0258</td>
<td>0.0205</td>
<td>0.0053</td>
<td>+25.8</td>
</tr>
<tr>
<td>45</td>
<td>22</td>
<td></td>
<td></td>
<td>0.0405</td>
<td>0.0295</td>
<td>0.0110</td>
<td>+37.3</td>
</tr>
<tr>
<td>21</td>
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<td>0.0297</td>
<td>0.0215</td>
<td>0.0082</td>
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</tr>
<tr>
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<td>0.0215</td>
<td>0.0081</td>
<td>+37.7</td>
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<td></td>
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<td>0.0162</td>
<td>0.0057</td>
<td>+35.2</td>
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<td></td>
<td></td>
<td>0.0210</td>
<td>0.0154</td>
<td>0.0056</td>
<td>+36.4</td>
</tr>
</tbody>
</table>

* Observed deflections were obtained from load-deflection curves as shown in Figure 21.

The plus sign in Table 7, column 8 shows that the observed values of the deflections are greater than those calculated.
CONCLUSIONS

(1) The bending tests on plates supported on three ball bearings were anticipated to give the most reliable values for the elastic constants; the plates were free to bend and curl. However, the observed deflections were even more than 100 percent greater than those computed. It was discovered that the bending moment in the x-direction was not uniform transversely across the plates along the lines through ball bearings as was assumed. The overhang of the plates was not long enough in comparison with the thickness of the plates to permit a uniform bending along a line through the point support.

(2) The effect of initial warping of the plywood plates proved undesirable in the bending tests on plates freely supported. Before the warp of the plates was straightened out and the plates were brought to bear uniformly on the supports by initial loading, considerable deflections were imposed. Since a half of the total loading was needed to correct the effect of initial warping, the deflections obtained could not be considered very reliable; the distinction could not be made where the warping ceased and the curling began. However, the differences between the observed and the computed values of deflections diminished considerably over those obtained from the
tests on the plates resting on three ball bearings.

(3) Under the circumstances of the tests, the bending tests on plates clamped over the supports proved to give the most reliable values for the elastic constants to be used in the finite-difference equations for the deflections. As mentioned before, the final elastic constants for the plates were derived from the tests on the plates with the face grain making angles of 0 and 90 degrees with the axes of the plates.

(4) The properties of plywood vary to a great extent from point to point; therefore, the reliability of the experimental values are effected considerably.

(5) In order to derive reliable elastic constants for plywood plates, tests should be repeated on several plates cut from different sheets of plywood. Thus, the discrepancies caused by the nonuniform structure would be minimized. However, such an extensive study is beyond the scope of this paper; only the method of solving problems on plywood plates is intended to be outlined here. Therefore, the results obtained in this paper are to be considered tentative.
BIBLIOGRAPHY


APPENDIX I

SAMPLE CALCULATIONS

1. The Determination of Elastic Constants from the Deflections

$0^\circ$ Plate

$a_{11}$ from bending test

$P = 120$ lbs., along line I, $w$ mean $= 59.0 \times 10^{-3}$ in. (see Table 3)

$$a_{11} = 13.481 \times 10^{-4} \frac{w}{P}$$

$$= 13.481 \times 10^{-4} \times \frac{59.0 \times 10^{-3}}{120}$$

$$= 0.663 \times 10^{-6} \text{ in}^2/\text{lb.}, \text{ see Table 4.}$$

$$E_{11} = \frac{1}{\Delta} (a_{22}a_{66} - a_{26}^2)$$

$$\Delta = a_{11} a_{22} a_{66} - a_{12} a_{22} a_{66} + 2 a_{16} a_{22} a_{66} - a_{12} a_{26} - a_{11} a_{26} - a_{16} a_{26} + a_{16} a_{26} a_{66}$$

substituting the values of $a_{ij}$ from Table 4 gives

$$B_{11} = \frac{1}{9.678 (10^{-6})^3} \left[ 1.887 (10^{-6}) - 7.739 (10^{-6}) \right] = 1.509 (10^6) \text{ lb/in.}^2,$$

see Table 5

$$D_{11} = \frac{t^3}{12} (B_{11})$$

$$= \frac{(0.50)^3}{12} (1.509) (10^6)$$

$$= 1.572 (10^4) \text{ in. lb.}, \text{ see Table 6}$$
1. **The Generalized Hooke’s Law**

An elastic body is called isotropic if its elastic properties are the same in all directions and anisotropic if its elastic properties are different in different directions. In the study of problems of equilibrium of an elastic body under small strains, it is usually assumed that the body follows the generalized Hooke’s law, in agreement in which the strain components are linear functions of the stress components. The generalized Hooke’s law in the most general case of anisotropy can be written in the form (8, p. 25-31):

\[
\begin{bmatrix}
\epsilon_x = a_{11} \sigma_x + a_{12} \sigma_y + a_{13} \sigma_z + a_{14} \tau_{yz} + a_{15} \tau_{xz} + a_{16} \tau_{yx} \\
\epsilon_y = a_{21} \sigma_x + a_{22} \sigma_y + a_{23} \sigma_z + a_{24} \tau_{yz} + a_{25} \tau_{xz} + a_{26} \tau_{yx} \\
\epsilon_z = a_{31} \sigma_x + a_{32} \sigma_y + a_{33} \sigma_z + a_{34} \tau_{yz} + a_{35} \tau_{xz} + a_{36} \tau_{yx} \\
\gamma_{yz} = a_{41} \sigma_x + a_{42} \sigma_y + a_{43} \sigma_z + a_{44} \tau_{yz} + a_{45} \tau_{xz} + a_{46} \tau_{yx} \\
\gamma_{xz} = a_{51} \sigma_x + a_{52} \sigma_y + a_{53} \sigma_z + a_{54} \tau_{yz} + a_{55} \tau_{xz} + a_{56} \tau_{yx} \\
\gamma_{yx} = a_{61} \sigma_x + a_{62} \sigma_y + a_{63} \sigma_z + a_{64} \tau_{yz} + a_{65} \tau_{xz} + a_{66} \tau_{yx}
\end{bmatrix}
\]
Figure 22. Stresses on an element of anisotropic material.

In equations (15a-f) the constants \( a_{ij} \) are expressed as follows:

\[
\begin{align*}
a_{11} &= \frac{1}{E} \quad a_{21} = u_y / E \quad a_{31} = u_z / E \quad a_{41} = \frac{1}{G_{yz}} \quad a_{51} = \frac{1}{G_{xz}} \quad a_{61} = \frac{1}{G_{yx}} \\
a_{12} &= \frac{1}{E} \quad a_{22} = \frac{1}{E} \quad a_{32} = u_y / E \quad a_{42} = \frac{1}{G_{yz}} \quad a_{52} = \frac{1}{G_{xz}} \quad a_{62} = \frac{1}{G_{yx}} \\
a_{13} &= \frac{1}{E} \quad a_{23} = u_y / E \quad a_{33} = \frac{1}{E} \quad a_{43} = \frac{1}{G_{yz}} \quad a_{53} = \frac{1}{G_{xz}} \quad a_{63} = \frac{1}{G_{yx}} \\
a_{14} &= \frac{1}{G_{xz}} \quad a_{24} = \frac{1}{G_{yz}} \quad a_{34} = \frac{1}{G_{yz}} \quad a_{44} = \frac{1}{G_{yz}} \quad a_{54} = \frac{1}{G_{yz}} \quad a_{64} = \frac{1}{G_{yz}} \\
a_{15} &= \frac{1}{G_{xz}} \quad a_{25} = \frac{1}{G_{xz}} \quad a_{35} = \frac{1}{G_{xz}} \quad a_{45} = \frac{1}{G_{xz}} \quad a_{55} = \frac{1}{G_{xz}} \quad a_{65} = \frac{1}{G_{xz}} \\
a_{16} &= \frac{1}{G_{yx}} \quad a_{26} = \frac{1}{G_{yx}} \quad a_{36} = \frac{1}{G_{yx}} \quad a_{46} = \frac{1}{G_{yx}} \quad a_{56} = \frac{1}{G_{yx}} \quad a_{66} = \frac{1}{G_{yx}}
\end{align*}
\]

(16a)

By Maxwell's reciprocal theorem the following equalities can be obtained:
and equations (15a-f) can be expressed as follows in 21 independent elastic constants:

\[
\begin{bmatrix}
    a_{21} = a_{12} & a_{43} = a_{34} & a_{61} = a_{16} \\
    a_{31} = a_{13} & a_{51} = a_{15} & a_{62} = a_{26} \\
    a_{32} = a_{23} & a_{52} = a_{25} & a_{63} = a_{36} \\
    a_{41} = a_{14} & a_{53} = a_{35} & a_{64} = a_{46} \\
    a_{42} = a_{24} & a_{54} = a_{45} & a_{65} = a_{56}
\end{bmatrix}
\] (16b)

If there is internal symmetry in the body, then also the elastic properties will exhibit symmetry. The expressions for generalized Hooke's law simplify in the presence of elastic symmetry.

In case of one plane of elastic symmetry; such as, in plywood the plane perpendicular to the z-axis, the general Hooke's law simplifies
(2, p. 14):

\[
\begin{align*}
\epsilon_x &= a_{11} \sigma_x + a_{12} \sigma_y + a_{13} \sigma_z + a_{16} \tau_{yx} \\
\epsilon_y &= a_{12} \sigma_x + a_{22} \sigma_y + a_{23} \sigma_z + a_{26} \tau_{yx} \\
\epsilon_z &= a_{13} \sigma_x + a_{23} \sigma_y + a_{33} \sigma_z + a_{36} \tau_{yx} \\
\gamma_{zy} &= a_{14} \tau_{zy} + a_{15} \tau_{xz} \\
\gamma_{xz} &= a_{45} \tau_{zy} + a_{55} \tau_{xz} \\
\gamma_{yx} &= a_{16} \sigma_x + a_{26} \sigma_y + a_{36} \sigma_z + a_{66} \tau_{yx}
\end{align*}
\]

(18a-f)

2. Hooke's Law Applied to Plywood Plates

Assuming that in plate problems the stress in the z-direction is small compared with the stresses in the x- and y-directions, \( \sigma_z \) can be disregarded in equations (18a-f). The following three equations will be used in developing the ordinary plate theory:

\[
\begin{align*}
\epsilon_x &= a_{11} \sigma_x + a_{12} \sigma_y + a_{16} \tau_{yx} \\
\epsilon_y &= a_{12} \sigma_x + a_{22} \sigma_y + a_{26} \tau_{yx} \\
\gamma_{yx} &= a_{16} \sigma_x + a_{26} \sigma_y + a_{66} \tau_{yx}
\end{align*}
\]

(19a-c)

Lekhnitski (2, p. 136) has shown that when the equations (19a-c) are solved for the stress components and when the strain components are replaced by

\[
\epsilon_x = -z \frac{\partial^2 w}{\partial x^2}, \quad \epsilon_y = -z \frac{\partial^2 w}{\partial x^2}, \quad \gamma_{yx} = -2z \frac{\partial^2 w}{\partial x \partial y}
\]
they yield

\[
\begin{align*}
\sigma_x &= -z \left( B_{11} \frac{\partial^2 w}{\partial x^2} + B_{12} \frac{\partial^2 w}{\partial y^2} + 2B_{16} \frac{\partial^2 w}{\partial x \partial y} \right) \\
\sigma_y &= -z \left( B_{12} \frac{\partial^2 w}{\partial x^2} + B_{22} \frac{\partial^2 w}{\partial y^2} + 2B_{26} \frac{\partial^2 w}{\partial x \partial y} \right) \\
\tau_{yx} &= -z \left( B_{16} \frac{\partial^2 w}{\partial x^2} + B_{26} \frac{\partial^2 w}{\partial y^2} + 2B_{66} \frac{\partial^2 w}{\partial x \partial y} \right)
\end{align*}
\]  

(20a-c)

\[
M_x = \int_{-t/2}^{t/2} \sigma_x z dz \\
M_y = \int_{-t/2}^{t/2} \sigma_y z dz \\
M_{xy} = \int_{-t/2}^{t/2} \tau_{yx} z dz
\]

(21a-c)

The bending and twisting moments are obtained by integrating equations (21a-c) over the thickness of the plate.

\[
\begin{align*}
M_x &= -(B_{11} \frac{\partial^2 w}{\partial x^2} + B_{12} \frac{\partial^2 w}{\partial y^2} + 2B_{16} \frac{\partial^2 w}{\partial x \partial y}) t^3/12 \\
M_y &= -(B_{12} \frac{\partial^2 w}{\partial x^2} + B_{22} \frac{\partial^2 w}{\partial y^2} + 2B_{26} \frac{\partial^2 w}{\partial x \partial y}) t^3/12 \\
M_{xy} &= -(B_{16} \frac{\partial^2 w}{\partial x^2} + B_{26} \frac{\partial^2 w}{\partial y^2} + 2B_{66} \frac{\partial^2 w}{\partial x \partial y}) t^3/12
\end{align*}
\]  

(22a-c)

The constants $B_{ij}$ are given in terms of $a_{ij}$ as follows:

\[
\begin{align*}
B_{11} &= \frac{1}{\Delta} \left( a_{22} a_{66} - a_{26}^2 \right) \\
B_{12} &= \frac{1}{\Delta} \left( a_{16} a_{26} - a_{12} a_{26} \right) \\
B_{16} &= \frac{1}{\Delta} \left( a_{12} a_{26} - a_{16} a_{22} \right) \\
B_{22} &= \frac{1}{\Delta} \left( a_{11} a_{66} - a_{16}^2 \right) \\
B_{26} &= \frac{1}{\Delta} \left( a_{11} a_{26} - a_{12} a_{16} \right) \\
B_{66} &= \frac{1}{\Delta} \left( a_{11} a_{22} - a_{12}^2 \right)
\end{align*}
\]  

(23a-c)
The constants $D_{ij}$, which appear in the nonorthotropic plate equation, are given by the relationship

$$D_{ij} = \frac{t^3}{12} (B_{ij})$$

(24)

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{vmatrix} = a_{11} a_{22} a_{66} + 2a_{12} a_{16} a_{26} - a_{12}^2 - a_{16}^2 - a_{11} a_{26}^2 - a_{12} a_{66}^2$$

(25)
1. Differential Equation of the Deflection Surface for Nonorthotropic Plate

It is assumed that the loading acting on a plate is normal to its surface and that the deflections are small in comparison with the thickness of the plate. At the boundary it is assumed that the edges of the plate are free to move in the plane of the plate; thus the reactive forces are normal to the plate. With these assumptions any strain in the middle plane of the plate during bending can be neglected.

An element cut out of the plate by two pairs of planes parallel to the \(xz\)-\(yz\)-planes, as shown in Figure 23, is considered; the coordinate axes are \(x\) and \(y\) in the middle plane of the plate and the \(z\)-axis perpendicular to that plane.

\[
\begin{align*}
M_y + \frac{\partial M_y}{\partial y} dy - dx \\
M_{yx} + \frac{\partial M_{yx}}{\partial y} dy \\
Q_y + \frac{\partial Q_y}{\partial y} dy \\
M_{xy} + \frac{\partial M_{xy}}{\partial x} dx \\
Q_x + \frac{\partial Q_x}{\partial x} dx
\end{align*}
\]

Figure 23. Moments and shearing forces on an element of a plate.
In addition to the bending moments $M_x$ and $M_y$ and the twisting moment $M_{xy}$, which are considered in the pure bending of a plate, there are vertical shearing forces acting on the sides of the element. The magnitudes of these shearing forces per unit length parallel to the $y$- and $x$-axes are denoted by $Q_x$ and $Q_y$, respectively, so that

$$Q_x = \int_{-t/2}^{t/2} \tau_{zx} \, dz \quad \quad Q_y = \int_{-t/2}^{t/2} \tau_{zy} \, dz \quad (26a-b)$$

Since the moments and shearing forces are functions of the coordinates $x$ and $y$, in discussing the conditions of equilibrium of the element, the small changes of these quantities when the coordinates $x$ and $y$ change by the small quantities $dx$ and $dy$ must be taken into consideration.

The stresses on the $xz$- and $yz$-planes, $\tau_{xz}$ and $\tau_{yz}$, are obtained from the equations of equilibrium (5, p. 76)

$$\begin{bmatrix}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0
\end{bmatrix} \quad (27a-b)$$

since $\tau_{zx} = \tau_{zy} = 0$ at the external surfaces

where $z = \pm t/2$
Differentiating Equation (20a-c) with respect to $x$ and $y$ so that the partial derivatives needed in Equation (27a-b) are obtained gives

$$\frac{\partial \sigma_x}{\partial x} = -z \left( B_{11} \frac{\partial^3 w}{\partial x^3} + B_{12} \frac{\partial^3 w}{\partial y^2 \partial x} + 2B_{16} \frac{\partial^3 w}{\partial x^2 \partial y} \right)$$

$$\frac{\partial \sigma_y}{\partial x} = -z \left( B_{12} \frac{\partial^3 w}{\partial x^2 \partial y} + B_{22} \frac{\partial^3 w}{\partial y^3} + 2B_{26} \frac{\partial^3 w}{\partial x \partial y^2} \right)$$

$$\frac{\partial \tau_{yx}}{\partial y} = -z \left( B_{16} \frac{\partial^3 w}{\partial x^3} + B_{26} \frac{\partial^3 w}{\partial x^2 \partial y} + 2B_{66} \frac{\partial^3 w}{\partial x^2 \partial y} \right)$$

$$\frac{\partial \tau_{yx}}{\partial y} = -z \left( B_{12} \frac{\partial^3 w}{\partial x \partial y^2} + B_{22} \frac{\partial^3 w}{\partial y^3} + 3B_{66} \frac{\partial^3 w}{\partial x \partial y^2} \right)$$

Substituting Equation (28a-d) in Equation (27a-b) gives

$$-\frac{\partial \tau_{zx}}{\partial z} = z \left[ B_{11} \frac{\partial^3 w}{\partial x^3} + B_{12} \frac{\partial^3 w}{\partial y \partial x^2} + 3B_{16} \frac{\partial^3 w}{\partial x^2 \partial y} + B_{26} \frac{\partial^3 w}{\partial y^3} + 2B_{66} \frac{\partial^3 w}{\partial x \partial y^2} \right]$$

$$-\frac{\partial \tau_{zy}}{\partial z} = z \left[ B_{12} \frac{\partial^3 w}{\partial x \partial y^2} + B_{16} \frac{\partial^3 w}{\partial x^3} + B_{26} \frac{\partial^3 w}{\partial x^2 \partial y} + 3B_{66} \frac{\partial^3 w}{\partial x \partial y^2} \right]$$

Integrating Equation (29a-b) over the thickness of the plate yields

$$-T_{zx} = \int_{\frac{-L}{2}}^{\frac{L}{2}} \left[ B_{11} \frac{\partial^3 w}{\partial x^3} + B_{12} \frac{\partial^3 w}{\partial y \partial x^2} + 3B_{16} \frac{\partial^3 w}{\partial x^2 \partial y} + B_{26} \frac{\partial^3 w}{\partial y^3} + 2B_{66} \frac{\partial^3 w}{\partial x \partial y^2} \right] z dz$$
\[-T_{zx} = \frac{1}{2} (z^2 - t^2) \left[ B_{11} \frac{\partial^3 w}{\partial x^3} + 3B_{16} \frac{\partial^3 w}{\partial x^2 \partial y} + (B_{12} + 2B_{66}) \frac{\partial^3 w}{\partial x \partial y^2} + B_{26} \frac{\partial^3 w}{\partial y^3} \right] \]

Similarly

\[-T_{zy} = \frac{1}{2} (z^2 - t^2) \left[ B_{16} \frac{\partial^3 w}{\partial x^3} + (B_{12} + 2B_{66}) \frac{\partial^3 w}{\partial x^2 \partial y} + 3B_{26} \frac{\partial^3 w}{\partial x \partial y^2} + B_{22} \frac{\partial^3 w}{\partial y^3} \right] \]

Integrating Equation (26a-b) gives

\[Q_x = \frac{t^3}{12} \left[ B_{11} \frac{\partial^3 w}{\partial x^3} + 3B_{16} \frac{\partial^3 w}{\partial x^2 \partial y} + (B_{12} + 2B_{66}) \frac{\partial^3 w}{\partial x \partial y^2} + B_{26} \frac{\partial^3 w}{\partial y^3} \right] \]

\[Q_y = \frac{t^3}{12} \left[ B_{16} \frac{\partial^3 w}{\partial x^3} + (B_{12} + 2B_{66}) \frac{\partial^3 w}{\partial x^2 \partial y} + 3B_{26} \frac{\partial^3 w}{\partial x \partial y^2} + B_{22} \frac{\partial^3 w}{\partial y^3} \right] \]

The middle plane of the element is represented in Figure 23 (a) and (b) and the directions in which the moments and forces are taken as positive are indicated.

The load distributed over the upper surface of the plate must also be considered. The intensity of this load is denoted by \( q \), so that the load acting on the element is \( q \, dx \, dy \).
Figure 24. (a) The positive moments on the middle plane of an element of a plate.

(b) The positive shearing forces on a plate.

Projecting all forces acting on the element onto the z-axis, the following equations are obtained:

\[ \frac{\partial Q_x}{\partial x} \text{d}x \text{d}y + \frac{\partial Q_y}{\partial y} \text{d}y \text{d}x + q \text{d}x \text{d}y = 0 \]  \hspace{1cm} (32)

from which

\[ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0 \]  \hspace{1cm} (33)
Taking moments of all forces acting on the element with respect to the \(x\)-axis, the equation of equilibrium is obtained:

\[
\frac{\partial M_{xy}}{\partial x} \, dx \, dy - \frac{\partial M_{x}}{\partial y} \, dy \, dx + Q_y \, dy \, dx = 0
\]  

(34)

The moment of load \(q\) and the moment due to change in the force \(Q_y\) are neglected in this equation since they are small quantities of a higher order than those retained. Thus, Equation (34) becomes

\[
\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_{y}}{\partial y} + Q_y = 0
\]  

(35)

In the same manner, by taking moments with respect to the \(y\)-axis,

\[
\frac{\partial M_{yx}}{\partial y} + \frac{\partial M_{y}}{\partial x} - Q_x = 0
\]  

(36)

and

\[
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0
\]  

(33)

where \(q\) is the load per unit area perpendicular to the external surface. By substitution of the shear forces Equations (31a-b) in Equation (33) the equation for the deflection of a nonorthotropic plate is obtained:

\[
D_{11} \frac{\partial^4 w}{\partial x^4} + 4D_{16} \frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x \partial y^2} + 4D_{26} \frac{\partial^4 w}{\partial x \partial y^3} + D_{22} \frac{\partial^4 w}{\partial y^4} = q
\]  

(3)
APPENDIX IV

1. Derivation of the General Deflection Equation for a Thin Plate

Solving Equation (22a-c) from Appendix II

\[
\begin{align*}
\frac{12}{3} M_x &= -(B_1 \frac{\partial^2 w}{\partial x^2} + B_2 \frac{\partial^2 w}{\partial y^2} + 2B_6 \frac{\partial^2 w}{\partial x \partial y}) \\
\frac{12}{3} M_y &= -(B_1 \frac{\partial^2 w}{\partial x^2} + B_2 \frac{\partial^2 w}{\partial y^2} + 2B_6 \frac{\partial^2 w}{\partial x \partial y}) \\
\frac{12}{3} M_{xy} &= -(B_6 \frac{\partial^2 w}{\partial x^2} + B_2 \frac{\partial^2 w}{\partial y^2} + 2B_6 \frac{\partial^2 w}{\partial x \partial y}) \tag{22a-c}
\end{align*}
\]

for \( \frac{\partial^2 w}{\partial x^2} \), \( \frac{\partial^2 w}{\partial y^2} \), and \( \frac{\partial^2 w}{\partial x \partial y} \) gives

\[
\begin{align*}
\frac{\partial^2 w}{\partial x^2} &= \frac{12}{3} \frac{1}{t \Delta} (AM_x + BM_y + CM_{xy}) \\
\frac{\partial^2 w}{\partial y^2} &= \frac{12}{3} \frac{1}{t \Delta} (BM_x + EM_y + FM_{xy}) \tag{37a-c} \\
\frac{\partial^2 w}{\partial x \partial y} &= \frac{12}{3} \frac{1}{t \Delta} (GM_x + FM_y + HM_{xy})
\end{align*}
\]
In Equation (37a-c)

\[
\begin{bmatrix}
A &= B_{22}B_{66} - 2B_{26}^2 \\
B &= B_{16}B_{26} - B_{16}B_{66} \\
C &= B_{12}B_{26} - B_{16}B_{22} \\
D &= -B_{11}B_{22} - 2B_{12}B_{66} - B_{16}B_{22} + B_{26}^2 + B_{16}^2 + B_{22}^2
\end{bmatrix}
\]

(38a-h)

Integrating Equations (37a-c) gives the value of \( w \)

\[
w = \frac{12}{t^3D} \left[ (AM_x + BM_y + CM_{xy})x^2 + (BM_x + EM_y + FM_{xy})y^2 + (GM_x + FM_y + HM_{xy})xy + C_1x + C_2y + C_3 \right]
\]

(39)

The constants of integration \( C_1, C_2, \) and \( C_3 \) define the plane from which the deflections \( w \) are measured. If this plane is taken tangent to the middle surface of the plate at the origin, the constants of integration must be equal to zero and the deflection surface is given by the equation

\[
w = \frac{6}{t^3D} \left[ (AM_x + BM_y + CM_{xy})x^2 + (BM_x + EM_y + FM_{xy})y^2 + (GM_x + FM_y + HM_{xy})xy \right]
\]

(40)
Equation (40) can be expressed as follows:

\[ w = \frac{6}{t^3D} \quad m_x (Ax^2 + By^2 + Gxy) \]

\[ w = \frac{6}{t^3D} \quad m_y (Bx^2 + Ey^2 + Fxy) \]

\[ w = \frac{6}{t^3D} \quad m_{xy} (Cx^2 + Fy^2 + Hxy) \]  \hspace{1cm} (41)

Expressing Equation (41) in terms of \( B_{ij} \) gives

\[
\frac{6}{D} \quad [m_x \left( (B_{22} B_{66} - B_{26}^2)x^2 + (B_{16} B_{26} - B_{12} B_{66})y^2 + (B_{12} B_{26} - B_{16} B_{22})xy \right) + m_y \left( (B_{16} B_{26} - B_{12} B_{66})x^2 + (B_{11} B_{66} - B_{16}^2)y^2 + (B_{12} B_{16} - B_{11} B_{26})xy \right) + m_{xy} \left( (B_{12} B_{26} - B_{11} B_{22} - B_{12} B_{16} - B_{11} B_{26})x^2 + (B_{12} B_{16} - B_{11} B_{26})y^2 + (B_{11} B_{22} - B_{12} B_{16})xy \right)]
\]

\[
\frac{6}{D} \quad \frac{w^3}{D} = \frac{6}{D} \quad [x^2 + y^2 + xy] \quad \text{(42a-c)}
\]

\[
\frac{6}{D} \quad \frac{w^3}{D} = \frac{6}{D} \quad \left( \begin{array}{c}
-B_{12} B_{26} - B_{12} B_{16} - B_{12} B_{26} + B_{16} B_{26} + B_{16} B_{26} + B_{12} B_{26} + B_{11} B_{26} + B_{12} B_{26} \\
11 B_{22} B_{66} - 2B_{12} B_{16} - 12 B_{16} B_{26} + B_{16} B_{26} + B_{16} B_{26} + B_{12} B_{26} + B_{11} B_{26} + B_{12} B_{26}
\end{array} \right)
\]
Substituting \(a_{ij}\) for \(B_{ij}\) in Equation (42a) gives

\[
w_1 t^3 = \frac{6M}{D\Delta} x \left\{ \left[ (a_{11} a_{66} - a_{16} a_{12}^2) (a_{11} a_{22} - a_{12}^2) - (a_{12} a_{16} - a_{11} a_{26})^2 \right] x^2 + [\left( a_{12} a_{26} - a_{16} a_{22} \right) (a_{12} a_{16} - a_{11} a_{26}) - (a_{16} a_{26} - a_{12} a_{66}) (a_{11} a_{22} - a_{12}^2) y^2] \right\}
\]

(43)

Multiplying out and collecting terms in Equation (43) give

\[
w_1 t^3 = \frac{6M}{D\Delta} x \left\{ \left[ (a_{11} a_{22} a_{66} - a_{16} a_{22} - a_{12} a_{66} + 2a_{12} a_{16} a_{26} - a_{11} a_{26}) x^2 + a_{11}^2 \right] + [\left( a_{12} a_{16} a_{26} - a_{16} a_{22} - a_{11} a_{26} + a_{11} a_{22} a_{66} + a_{12} a_{16} a_{26} - a_{12} a_{66} \right) y^2 a_{12} + [\left( a_{12} a_{16} a_{26} - a_{16} a_{22} - a_{11} a_{26} + a_{11} a_{22} a_{66} + a_{12} a_{16} a_{26} - a_{12} a_{66} \right) x y a_{16} \] \right\}
\]

(44)

The substitution of the value of \(\Delta\) in Equation (44) gives

\[
w_1 t^3 = \frac{6M}{D\Delta} x (a_{11} x^2 + a_{12} y^2 + a_{16} xy)
\]

(45)
Letting $k = a_{11}x^2 + a_{12}y^2 + a_{16}xy$ in Equation (45) gives

$$w_1 t^3 = \frac{6M}{D\Delta} k \tag{46}$$

Substituting the values of $a_{ij}$ for $D$ in Equation (46) gives

$$\frac{w_1 t^3}{6M\Delta^2k} = \frac{1}{x^2} \left\{ -\frac{(a_{22}a_{66} - a_{26}a_{12})^2}{a_{11}a_{16} - a_{12}a_{16}} + \frac{(a_{16}a_{26} - a_{11}a_{16})^2}{a_{12}a_{66} - a_{12}a_{16}} \right\} \tag{47}$$

Multiplying out and collecting terms Equation (47) gives

$$\frac{w_1 t^3}{6M\Delta^2k} = \frac{1}{x^2} \left\{ \frac{a_{22}a_{66}^2}{a_{11}a_{16}a_{12}a_{26}a_{66}} + \frac{4a_{11}a_{12}a_{16}a_{22}a_{26}a_{16}}{a_{11}a_{16}a_{22}a_{26}a_{66}} - \frac{2a_{11}a_{16}a_{22}a_{26}a_{66}}{a_{11}a_{16}a_{22}a_{26}a_{66}} + \frac{2a_{11}a_{12}a_{22}a_{26}a_{66}}{a_{11}a_{16}a_{22}a_{26}a_{66}} + \frac{2a_{11}a_{16}a_{22}a_{26}a_{66}}{a_{11}a_{16}a_{22}a_{26}a_{66}} \right\} \tag{48}$$

By squaring the value of $\Delta$, or $(a_{11}a_{22}a_{66} + 2a_{12}a_{16}a_{26} - a_{11}a_{16}a_{22} - a_{12}a_{16}a_{26})^2$,

it is proved that the right hand member of Equation (48) is equal to $\Delta^2$;
Thus Equation (45) becomes

\[ w_1 t^3 = 6M_x (a_{11} x^2 + a_{12} y^2 + a_{16} xy) \]  \hspace{1cm} (49)

Similarly it can be shown that

Equation (42b) yields

\[ w_2 t^3 = 6M_y (a_{12} x^2 + a_{22} y^2 + a_{26} xy) \]  \hspace{1cm} (50)

and Equation (42c) yields

\[ w_3 t^3 = 6M_{xy} (a_{16} x^2 + a_{26} y^2 + a_{66} xy) \]  \hspace{1cm} (51)

Thus the GENERAL EXPRESSION for the deflection of a thin plate when it is subjected to uniform

bending moments \( M_x \) and \( M_y \) and a uniform twisting moment \( M_{xy} \) becomes

\[ w t^3 = 6M_x (a_{11} x^2 + a_{12} y^2 + a_{16} xy) + 6M_y (a_{12} x^2 + a_{22} y^2 + a_{26} xy) \]

\[ + 6M_{xy} (a_{16} x^2 + a_{26} y^2 + a_{66} xy) \]  \hspace{1cm} (1a-c)