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Several lower tolerance limit procedures are compared with regard to efficiency and robustness. The procedures include the nonparametric based on a single order statistic and the maximum likelihood estimates for complete and censored samples from parametric families of distributions. Right-censoring is considered as an approach for improving the robustness of the parametric procedures under departure from the assumed form.

The Pitman asymptotic relative efficiencies of the nonparametric, complete sample and censored sample parametric procedures are compared for the 2-parameter lognormal distribution and the 3-parameter generalized gamma distributions, including the special cases of negative exponential, Weibull and gamma distributions.

Approximate coverage probabilities (A.C.P.'s), based on large sample theory, are evaluated for the parametric procedures under the assumption of a Weibull (lognormal) distribution when instead the true distribution is lognormal (Weibull). The discrepancy between the

A.C.P.'s and the corresponding nominal confidence level is then used as the measure of the robustness of the parametric procedures. A Monte Carlo study using samples of size $n=60$ is conducted to investigate the adequacy of the large sample approximation.

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A COMPARISON OF EFFICIENCY AND ROBUSTNESS FOR LOWER TOLERANCE LIMIT PROCEDURES

1. INTRODUCTION

Consider the problem of setting a lower confidence limit for the p^{th} quantile $x_p = x_p(F)$ of a distribution with continuous cumulative distribution function (c.d.f.) $F(x)$. That is, given p $(0 < p < 1)$ and the confidence level γ a lower confidence limit $L = L(X_1, \ldots, X_n, p, \gamma)$ is specified so that

$$\Pr \{L \leq x_p\} \doteq \gamma \quad \text{for all} \quad F \in \mathcal{F}$$

where $\mathcal{F}$ is some specified family of continuous c.d.f.'s. It is well known that lower confidence limits of quantiles correspond to lower tolerance limits. That is,

$$\Pr \{L \leq x_p\} = \Pr \{F(L) \leq p\} = \Pr \{1-F(L) \geq 1-p\} \geq \gamma$$

shows that the interval (L, ∞) includes at least $(1-p)$ of the distribution with confidence level γ. We are particularly interested in small quantiles $(0.05 \leq p \leq .25)$. Applications would include life-testing and strength of materials. For example, a lumber company may be interested in setting a lower tolerance limit for the breaking stress X for a population of lumber of specified dimension and grade.

This research compares the performance of several lower tolerance limit procedures with respect to efficiency and robustness. The procedures include the nonparametric based on a single order statistics $L = X_{(k)}$ and the parametric based on the maximum likelihood estimates $\hat{x}_p = x_p(\hat{\underset{\sim}{\beta}})$ for the parametric family F. Here we consider the negative exponential, 2-parameter Weibull, 2-parameter gamma, generalized gamma and lognormal families of distributions. Based on the asymptotic multivariate normal distribution for the MLE $\hat{\underset{\sim}{\beta}}$,

$$\hat{\underset{\sim}{\beta}} \sim N\,(\underset{\sim}{\beta}, \frac{1}{n} \cdot I^{-1}(\underset{\sim}{\beta})) ,$$

where $I^{-1}(\underset{\sim}{\beta})$ denotes the inverse of the information matrix for a single observation, we consider

$$L = \hat{x}_p - \Phi^{-1}(\gamma) \cdot \sqrt{\hat{V}ar(\hat{x}_p)} \tag{1.1}$$

where

$$\hat{V}ar(\hat{x}_p) = \frac{1}{n} \cdot \underset{\sim}{D}'(\hat{\underset{\sim}{\beta}}) \cdot I^{-1}(\hat{\underset{\sim}{\beta}}) \cdot \underset{\sim}{D}(\hat{\underset{\sim}{\beta}})$$

and

$$D(\underset{\sim}{\beta}) = [\frac{\partial x_p}{\partial \beta_1}, \frac{\partial x_p}{\partial \beta_2}, \ldots, \frac{\partial x_p}{\partial \beta_s}]'$$

is the gradient vector of x_p w.r.t. $\underset{\sim}{\beta}$.

As the parametric family of distributions is enlarged (e.g., from negative exponential to Weibull to generalized gamma distributions) we would expect the lower tolerance limit procedures (1.1) to become more robust. That is, the nominal probability of coverage, γ, should give a better approximation to the true coverage probability,

$$\Pr\{L \leq x_p \,;\, F\} \tag{1.2}$$

for distributions F not included in the assumed parametric families. However, as the family is enlarged the asymptotic variance $Var(\hat{x}_p)$

increases (efficiency decreases) for distribution F which are included in the smaller families. One possible method for improving the robustness of the parametric lower tolerance limit procedures is to right-censor the larger observations. That is, corresponding to a specified value T (Type I censoring), use the log-likelihood

$$\ell.(\underset{\sim}{\beta}) = \sum_{k=1}^{n} \{\delta_k \ln f(x_k, \underset{\sim}{\beta}) + (1-\delta_k) \ln[1-F(T; \underset{\sim}{\beta})]\} ,$$

where

$$\delta_k = I_{(-\infty, T]}(x_k)$$

indicates whether $(\delta_k=1)$ or not $(\delta_k=0)$ x_k is in the interval $(-\infty, T]$ for the determination of the MLE $\hat{x}_p$ and its asymptotic variance in (1.1). Of course, the censoring will result in some loss in efficiency.

The nonparametric, parametric uncensored, and parametric censored procedures are compared by two large sample performance criteria: the Pitman Asymptotic Relative Efficiency (A.R.E.) and the Approximate Coverage Probability (A.C.P.). The A.R.E. under the assumed parametric family of distributions is calculated as the ratio of the asymptotic variances of the estimaters for the p^{th} quantile x_p. The A.C.P. is defined as the large sample approximation to probability (1.2) when F is not a member of the parametric family of distributions assumed for construction of L.

In Chapter 2, censoring and the corresponding likelihood function are discussed.

In Chapter 3, the efficiencies of the various procedures are investigated. First, we develop the general form of the A.R.E.'s for both the nonparametric and censored parametric procedures relative to the uncensored parametric procedures. Formulas for the A.R.E.'s are then derived for the log-generalized gamma and normal distributions. Numerical values of A.R.E.'s are provided for the parametric families of distributions.

In Chapter 4, we first derive the general form of the A.C.P. The A.C.P.'s are then evaluated for two cases: when the lower tolerance limit is constructed by assuming an extreme value (Weibull) distribution and the true distribution is normal (lognormal) and vice versa.

In Chapter 5, a simulation study is used to investigate the adequacy of the large sample approximations to the true coverage probabilities.

Concerning related research, Habermann and Ethington (1975) conducted a simulation study to investigate the performance of the nonparametric and normal procedures for the lower tolerance limit $x_{.05}$ for uncensored samples of sizes $n=58$ (where $\Pr\{X_{(1)} < x_{.05}\} \doteq .95$) and 93 (where $\Pr\{X_{(2)} < x_{.05}\} \doteq .95$) when the true distribution is either normal, lognormal, gamma or Weibull. They conclude "... the use of nonparametric procedures is conservative while the penalty for an incorrect assumption about the true underlying distribution is possibly severe ...". They also studied the Hanson and Koopman (1964) procedure based on two order statistics from samples of size $n=20$ and 40. The Hanson-Koopman procedure was found to be too conservative. Warren (1974) further studied the sampling distribution of the first order statistic, $L=X_{(1)}$, for the sample size $n=58$ when sampling from the normal, lognormal, 3-parameter gamma, and 3-parameter Weibull distributions. Under standardization of the parent distributions (common mean and common variance), Warren found that the distribution of the first order statistic is highly dependent on the form of the parent distribution.

This research is similar to Habermann and Ethington's in that we are concerned with both the efficiency and robustness of lower tolerance limit procedures. This study is more general in that parametric procedures other than the conventional normal are investigated. The use of censoring for improving robustness of the parametric procedures is also studied.

2. CENSORED SAMPLES

Maximum likelihood estimation for censored data has been widely studied (e.g., see Mann et al., 1974, or Gross and Clark, 1975). For completeness, single right-censoring is discussed briefly here.

A random sample $X_1, \ldots, X_n$ can be singly censored from the right in two different ways:

1) observations larger than a specified value T (also called a truncation point) are censored;

2) corresponding to a specified integer R $(1 \leq R \leq n)$, the $N-R$ largest observations are censored.

These two kinds of censoring are commonly called Type I and Type II censoring, respectively. As an example, suppose it is desired to estimate the average lifetime of light bulbs produced in a factory. For a complete (uncensored) sample, a certain number, n, of light bulbs would be randomly selected and the burn-out (failure) time would be observed for all n bulbs. In order to shorten the duration of the experiment, Type II censoring might be used where the experiment terminated when a fixed number, R, of light bulbs have failed. If Type I censoring was used instead, the experiment would be terminated at a specified number, T, of hours. Note that the number of failures, R, observed in a Type I censored sample will be a random variable. In studies of material strength where the items are tested sequentially in one machine (or a small number of machines) only Type I censoring is convenient. In such applications, the stress need only be increased to the level T. If the item does not fail at level T, then the item may still be usable. That is, for items with $X > T$ the testing is non-destructive.

For the discussion of the likelihood function, let us consider a Type I censored sample first. The likelihood for estimating $\underset{\sim}{\beta}$ based on observations of the random variables

$$Q_i = \text{Min}\ \{X_i,\ T\}$$

and

$$\delta_i = \begin{cases} 1 & \text{for } X_i \le T \\ 0 & \text{for } X_i > T \end{cases}$$

is

$$\begin{aligned} L(\underset{\sim}{\beta}, q_1, \delta_1, q_2, \delta_2, \ldots, q_n, \delta_n) &= \prod_{i=1}^{n} \{f(q_i;\underset{\sim}{\beta})^{\delta_i} \cdot [1-F(q_i;\underset{\sim}{\beta})]^{1-\delta_i}\} \\ &= \prod_{i=1}^{n} \{f(X_i;\underset{\sim}{\beta})^{\delta_i} \cdot [1-F(T;\underset{\sim}{\beta})]^{1-\delta_i}\} \end{aligned} \qquad (2.1)$$

Let

$$R = \sum_{i=1}^{n} \delta_i$$

be the number of observations that fail by T, and let $X_{(1)} \le X_{(2)} \cdots \le X_{(R)} < T$ denote the first R order statistics, then (2.1) becomes

$$L(\underset{\sim}{\beta}, X_{(1)}, \ldots, X_{(R)}, T) = [\prod_{i=1}^{R} f(X_{(i)};\underset{\sim}{\beta})] \cdot [1-F(T;\underset{\sim}{\beta})]^{n-R} \qquad (2.2)$$

Thus, the log-likelihood function for a Type I censored sample can be written as

$$\ell. = \sum_{i=1}^{n} \{\delta_i \cdot \ln f(X_i;\underset{\sim}{\beta}) + (1-\delta_i)\cdot \ln[1-F(T;\underset{\sim}{\beta})]\} \tag{2.3}$$

from (2.1), or

$$\ell. = \sum_{i=1}^{R} \ln f(X_{(i)};\underset{\sim}{\beta}) + (n-R)\cdot \ln [1-F(T;\underset{\sim}{\beta}] \tag{2.4}$$

from (2.2).

For Type II censoring, define $T=X_{(R)}$. Then the log-likelihood function for Type II censoring differs from (2.3) and (2.4) only by the additive constant $\ln(n!/R!(n-R)!)$.

3. ASYMPTOTIC RELATIVE EFFICIENCY

3.1 Derivation

Efficiencies of the nonparametric and the censored sample parametric procedures relative to the complete sample procedure are evaluated for several parametric families of distributions. For convenience of notation, we apply the logarithmic transformation, $Y = \ln(X)$, so that each of the parametric families contains a location parameter. The choice of the scale (x or y) is irrelevant. For all tables that follow the distribution can be identified on either scale; e.g., normal or lognormal, and extreme value or Weibull.

Let $F(y;\underset{\sim}{\beta})$ be a c.d.f. of the assumed parametric family with corresponding density function $f(y;\underset{\sim}{\beta})$, where, in the parameter vector $\underset{\sim}{\beta} = (\beta_1, \beta_2, \ldots, \beta_s) \varepsilon \Omega$, β_1 is the location parameter. The p^{th} quantile of F is then of the form $\theta = \theta_p(\underset{\sim}{\beta}) = \beta_1 + h_p(\beta_2, \beta_3, \ldots, \beta_s)$ where the function h_p depends on the particular family. For example, for the 2-parameter Weibull distribution $x_p = \exp(\theta)$ where $\theta = \beta_1 + \beta_2 \cdot \ln[-\ln[1-p]]$.

Define $\hat{\underset{\sim}{\beta}}_c$ as the M.L.E. for $\underset{\sim}{\beta}$ and $I(\underset{\sim}{\beta}, p_f)$ the information matrix of a single observation from a Type I censored sample where $p_f = \Pr\{X \leq T\} = F(\tau;\underset{\sim}{\beta})$ with $\tau = \ln(T)$. Using the asymptotic normality of the M.L.E. $\hat{\theta}_c = \theta_p(\hat{\underset{\sim}{\beta}}_c)$,

$$\sqrt{n}(\hat{\theta}_c - \theta) \rightarrow N(0, \sigma_c^2(\underset{\sim}{\beta})),$$

where

$$\sigma_c^2(\underset{\sim}{\beta}) = \underset{\sim}{D}'(\underset{\sim}{\beta}) \cdot I^{-1}(\underset{\sim}{\beta}, p_f) \cdot \underset{\sim}{D}(\underset{\sim}{\beta}) \tag{3.1}$$

$$\underset{\sim}{D}'(\underset{\sim}{\beta}) = \left[\frac{\partial\theta}{\partial\beta_1}, \frac{\partial\theta}{\partial\beta_2}, \ldots, \frac{\partial\theta}{\partial\beta_s} \right]', \tag{3.2}$$

yields the lower confidence limit $L_{n,c}$ of asymptotic confidence level γ

$$L_{n,c} = \hat{\theta}_c - Z_\gamma \cdot \hat{\sigma}_c/\sqrt{n}, \tag{3.3}$$

where

$$Z_\gamma = \Phi^{-1}(\gamma)$$

and

$$\hat{\sigma}_c^2 = \sigma_c^2(\hat{\underset{\sim}{\beta}}_c) = \underset{\sim}{D}'(\hat{\underset{\sim}{\beta}}_c) \cdot I^{-1}(\hat{\underset{\sim}{\beta}}_c, \hat{p}_f) \cdot \underset{\sim}{D}(\hat{\underset{\sim}{\beta}}_c), \text{ with } \hat{p}_f = F(\tau; \hat{\underset{\sim}{\beta}}_c), \tag{3.4}$$

is a consistent estimator for $\sigma_c^2(\beta)$. Since a complete sample is a special case with $p_f=1$, the same notation is used for a complete sample except that the subscript c is deleted in the M.L.E. $\hat{\theta}$, asymptotic variance $\sigma^2(\hat{\underset{\sim}{\beta}})$, and lower confidence limit L_n.

The confidence intervals $\{\theta^o : \sqrt{n}(\hat{\theta}_c - \theta^o)/\hat{\sigma}_c \leq Z_\gamma\}$ correspond to the acceptance regions $\{\hat{\theta}_c : \sqrt{n}(\hat{\theta}_c - \theta^o)/\hat{\sigma}_c \leq Z_\gamma\}$ for testing the hypothesis

$$H_o : \theta \leq \theta^o \text{ against } H_A : \theta > \theta^o . \tag{3.5}$$

Thus, the power function, denoted by $Pw_{c,n}\{\underset{\sim}{\beta}\}$, corresponds to the probability that θ^o is not contained in the confidence interval

$$Pw_{c,n}\{\underset{\sim}{\beta}\} = \Pr\{\sqrt{n}(\hat{\theta}_c-\theta^o)/\hat{\sigma}_c > Z_\gamma ; \underset{\sim}{\beta}\} = \Pr\{\theta^o \notin [L_{n,c},\infty); \underset{\sim}{\beta}\}.$$

The Pitman Asymptotic Relative Efficiency (A.R.E.) of $\hat{\theta}_c$ to $\hat{\theta}$ for testing the hypotheses (3.5) is then used as the asymptotic relative efficiency of $L_{n,c}$ to L_n.

For an arbitrary positive constant a and parameter values $\underset{\sim}{\beta}^o = (\beta_1^o, \beta_2^o, \ldots, \beta_s^o)$ and $\underset{\sim}{\beta}^{(n)} = (\beta_1^o + a/\sqrt{n}, \beta_2^o, \ldots, \beta_s^o)$, denote $\theta^o = \theta_p(\underset{\sim}{\beta}^o)$ and $\theta^{(n)} = \theta_p(\underset{\sim}{\beta}^{(n)}) = \theta^o + a/\sqrt{n}$. The power function of the test statistic $\sqrt{n}(\hat{\theta}_c - \theta^o)/\hat{\sigma}_c$ for the sequence of alternatives $\theta^{(n)}$ and censoring values $\tau_n = \tau + a/\sqrt{n}$ is

$$\begin{aligned}
Pw_{c,n}\{\beta^{(n)}\} &= \Pr\{\sqrt{n}(\hat{\theta}_c-\theta^o)/\hat{\sigma}_c \geq Z_\gamma ; \underset{\sim}{\beta}^{(n)}\} \\
&= \Pr\{\sqrt{n}(\hat{\theta}_c-\theta^o) \geq Z_\gamma \hat{\sigma}_c ; \beta^{(n)}\} \\
&= \Pr\{\sqrt{n}(\hat{\theta}_c-\theta^{(n)}+a/\sqrt{n}) \geq Z_\gamma \hat{\sigma}_c ; \underset{\sim}{\beta}^{(n)}\} \\
&= \Pr\{\sqrt{n}(\hat{\theta}_c-\theta^{(n)})/\hat{\theta}_c \geq Z_\gamma - a/\hat{\sigma}_c ; \beta^{(n)}\} \\
&= \Pr\{\sqrt{n}(\hat{\theta}_c-\theta^o)/\hat{\sigma}_c \geq Z_\gamma - a/\hat{\sigma}_c ; \underset{\sim}{\beta}^o\} \\
&\doteq 1 - \Phi(Z_\gamma - a/\sigma_c(\underset{\sim}{\beta}^o))
\end{aligned}$$

Note that equality in the next-to-last line above follows from the invariance of $(\hat{\theta}_c-\theta^o)/\hat{\sigma}_c$ under common location transformations of Y_i and τ.

Using a similar argument for the test statistic $\sqrt{n^*}(\hat{\theta}-\theta^o)/\hat{\sigma}$ based on a complete sample of size n^* gives the approximate power

$$Pw_{n^*}\{\underset{\sim}{\beta}^{(n)}\} \doteq 1 - \Phi(Z_\gamma - \sqrt{n/n^*} \cdot a/\sigma(\underset{\sim}{\beta}^o))$$

Thus, $Pw_{n,c}\{\underset{\sim}{\beta}^{(n)}\} \doteq Pw_{n*}\{\underset{\sim}{\beta}^{(n)}\}$ when $n*/n \doteq \sigma^2(\underset{\sim}{\beta}^o)/\sigma_c^2(\underset{\sim}{\beta}^o)$. Hence the asymptotic relative efficiency of $L_{n,c}$ to L_{n*} is the ratio of the asymptotic variances of $\hat{\theta}_c$ and $\hat{\theta}$,

$$\text{A.R.E.}(L_{n,c};\ L_{n*}) = \sigma^2(\underset{\sim}{\beta}^o)/\sigma_c^2(\underset{\sim}{\beta}^o) \quad . \tag{3.6}$$

Now, consider the efficiency of the nonparametric confidence limit procedure, $L_{n**,NP}=Y_{(k)}$, where k satisfies (3.7) below. The value $\theta^o(\theta^o < \tau)$ is contained in the confidence interval $[Y_{(k)}, \infty)$ if and only if $K \geq k$, where

$$K = \text{the number of } Y_i\text{'s} \leq \theta^o$$

has a binomial (n^{**}, p) distribution so that

$$\Pr\{K \geq k\} = \sum_{i=k}^{n^{**}} \binom{n^{**}}{i} p^i (1-p)^{n^{**}-i} \doteq \gamma . \tag{3.7}$$

Thus, $t_{n**} = K/n^{**}$ is a test statistic for the corresponding hypothesis testing problem (3.5). From the results

$$E_\theta(t_{n**}) = F(\theta^o;\ \underset{\sim}{\beta})$$
$$\text{Var}(t_{n**}) = F(\theta^o;\ \underset{\sim}{\beta})\cdot[1-F(\theta^o;\ \underset{\sim}{\beta})]/n^{**}$$
$$\frac{d}{d\theta}E_\theta(t_{n**}) = \frac{d}{d\beta_1}F(\theta^o;\ \underset{\sim}{\beta}) = f(\theta^o;\ \underset{\sim}{\beta})$$
$$\frac{d}{d\theta}E_\theta(t_{n**})]\theta=\theta^{(n)}/\sqrt{n}\cdot\text{Var}(t_{n**})]_{\theta=\theta^o}$$
$$= f(\theta^o;\ \underset{\sim}{\beta}^{(n)})/\sqrt{n/n^{**}}\cdot\ \sqrt{p\ (1-p)}$$

and the asymptotic normality of t_{n**}, we apply Theorem 3.1 in Fraser (1963) to yield the approximate power

$$Pw_{n**,Np}\{\underset{\sim}{\beta}^{(n)}\} \doteq 1-\Phi(Z_\gamma - a\cdot\sqrt{n^{**}/n}\ /[\sqrt{p(1-p)}/f(\theta^o;\underset{\sim}{\beta}^o)]) .$$

Thus,

$$P_{W_{n^{**},Np}}(\underset{\sim}{\beta}^{(n)}) \doteq P_{W_{n^*}}(\underset{\sim}{\beta}^{(n)}) \quad \text{when} \quad n^*/n^{**} \doteq \sigma^2(\underset{\sim}{\beta}^o)/[p(1-p)/f^2(\theta^o;\underset{\sim}{\beta}^o)]$$

The asymptotic efficiency of the nonparametric procedure relative to the maximum likelihood complete sample procedure is then

$$\text{A.R.E.}(L_{n^{**},Np}\ ;\ L_{n^*}) = \sigma^2(\underset{\sim}{\beta}^o)/\sigma^2_{Np}(\underset{\sim}{\beta}^o)\ , \tag{3.8}$$

where

$$\sigma^2_{Np}(\underset{\sim}{\beta}^o) = p(1-p)/f^2(\theta^o;\ \underset{\sim}{\beta}^o) \tag{3.9}$$

is the asymptotic variance of the order statistic $Y_{(n^{**}p)}$.

3.2 Asymptotic Variance for the Log-Generalized Gamma Distribution

The log-generalized gamma distribution with p.d.f.

$$f(y;\underset{\sim}{\beta}) = 1/\beta_2 \cdot 1/\Gamma(\beta_3) \cdot \exp[\beta_3 \cdot (y-\beta_1)/\beta_2 - \exp[(y-\beta_1)/\beta_2)] \tag{3.10}$$

can be obtained from the gamma p.d.f.

$$h_\alpha(z) = z^{\alpha-1} \cdot \exp(-z)/\Gamma(\alpha)$$

by the transformation $Y = \beta_1 + \beta_2 \cdot Z$ with $\beta_3 = \alpha$. The c.d.f. of Z can be written

$$H_\alpha(z) = \Gamma(z;\ \alpha)/\Gamma(\alpha) \tag{3.11}$$

where

$$\Gamma(z;\ \alpha) = \int_0^z t^{\alpha-1}\ e^{-t}\ dt$$

is the incomplete gamma function and $\Gamma(\alpha) \equiv \Gamma(\infty;\alpha)$ is the complete gamma function. The p^{th} quantile of the log-generalized gamma distribution, θ , then depends on the inverse function $H_{\alpha}^{-1}(u)$ of the gamma c.d.f. $(\alpha=\beta_3)$

$$\theta \equiv \theta_p(\underset{\sim}{\beta}) \equiv \beta_1 + \beta_2 \cdot \ell n\, H_{\beta_3}^{-1}(p)$$

The elements of the gradient $\underline{D}$ of θ are then evaluated

$$d_1 = \partial\theta/\partial\beta_1 = 1$$

$$d_2 = \partial\theta/\partial\beta_2 = \ell n\, q_p \tag{3.12}$$

$$d_3 = \partial\theta/\partial\beta_3 = \beta_2[\Gamma'(\beta_3)\cdot p - \Gamma'(q_p,\beta_3)]/[q_p^{\beta_3}\cdot \exp(-q_p)] ,$$

where

$$q_p = H_{\beta_3}^{-1}(p)$$

and $\Gamma'(\beta_3)$ and $\Gamma'(q_p,\beta_3)$ denote respectively the first derivatives of $\Gamma(\beta_3)$ and $\Gamma(q_p,\beta_3)$ w.r.t. β_3. Similar notation, $\Gamma''(\beta_3)$ and $\Gamma''(\cdot,\beta_3)$ is used later for the second derivatives w.r.t. β_3. (See the Appendix for a discussion of the method used for numerical evaluation of the complete and incomplete gamma functions and their derivatives up to the 4^{th} order.)

Recall the p.d.f. for a censored observation

$$g(y;\underset{\sim}{\beta};\delta) = f(y;\beta)^{\delta}\cdot[1-F(\tau;\underset{\sim}{\beta})]^{1-\delta}$$

where $\delta = I_{(-\infty,\tau]}(y)$ is the indicator function on $(-\infty,\tau]$.

The element I_{ij} of the information matrix I can then be written as

$$\begin{aligned} I_{ij} &= -E\{\partial^2 \ell n\, g(y;\underset{\sim}{\beta};\delta)/\partial\beta_i \partial\beta_j\} \\ &= -E\{\delta \cdot \partial^2 \ell n\, f(y;\underset{\sim}{\beta})/\partial\beta_i \partial\beta_j + (1-\delta)\partial^2 \ell n[1-F(\tau;\underset{\sim}{\beta})]/\partial\beta_i \partial\beta_j\} \qquad (3.13) \\ &= -I_{ij}^{(1)} - (1-p_f)\cdot I_{ij}^{(2)} \end{aligned}$$

where

$$I_{ij}^{(1)} = -\int_0^{\tau} [\partial^2 \ell n\, f(y;\underset{\sim}{\beta})/\partial\beta_i \cdot \partial\beta_j]\, f(y;\underset{\sim}{\beta})\, dy$$

and (3.14)

$$I_{ij}^{(2)} = \partial^2 \ell n[1-F(\tau;\underset{\sim}{\beta})]/\partial\beta_i \partial\beta_j \ .$$

From (3.10) the 2nd derivatives of $\ell n\, f(y;\underset{\sim}{\beta})$ are determined

$$\begin{aligned} \partial^2 \ell n f/\partial\beta_1^2 &= (-1/\beta_2^2)\, \exp(y_g) \\ \partial^2 \ell n f/\partial\beta_2^2 &= 1/\beta_2^2 + y_g/\beta_2^2\{2\cdot\beta_3 - y_g \exp(y_g) - 2\cdot\exp(y_g)\} \\ \partial^2 \ell n f/\partial\beta_3^2 &= \partial^2 \ell n\Gamma(\beta_3)/\partial\beta_3^2 = -\Psi'(\beta_3)\ , \text{ the trigamma function,} \\ \partial^2 \ell n f/\partial\beta_1 \partial\beta_2 &= \beta_3/\beta_2^2 - 1/\beta_2^2 \exp(y_g)\cdot\{1+y_g\} \qquad (3.15) \\ \partial^2 \ell n f/\partial\beta_1 \partial\beta_3 &= -1/\beta_2 \\ \partial^2 \ell n f/\partial\beta_2 \partial\beta_3 &= -1/\beta_2 \cdot y_g \end{aligned}$$

where

$$y_g = (y-\beta_1)/\beta_2 \quad .$$

The expectations in (3.13) are evaluated for fixed probability of an observed failure

$$\Pr\{Y \le \tau;\ \underset{\sim}{\beta}\} = p_f$$

Thus,

$$\tau = \beta_1 + \beta_2 \,\ell n\, H_{\beta_3}^{-1} (p_f) \ .$$

To simplify the notation in the expressions that follow, we denote

$$q \equiv q(P_f, \beta_3) = H_{\beta_3}^{-1} (p_f)$$

and

$$q_h \equiv \beta_3 \cdot H_{\beta_3+1} (q) = \beta_3 \cdot p_f - q \cdot h_{\beta_3}(q) \ .$$

From the results

$$E[\exp(y_g)] = \Gamma(q,\beta_3+1)/\Gamma(\beta_3) = q_h$$

$$E[y_g] = \Gamma'(q,\beta_3)/\Gamma(\beta_3) = p_f \cdot \Psi(q, \beta_3)$$

$$E[y_g \cdot \exp(y_g)] = \Gamma'(q,\beta_3+1)/\Gamma(\beta_3) = q_h \cdot \Psi(q,\beta_3+1)$$

$$E[y_g \ \exp(y_g)] = \Gamma''(q,\beta_3+1)/\Gamma(\beta_3) = q_h \cdot \overline{\Psi}(q,\beta_3+1) \ ,$$

where

$$\Psi(q_f,\beta_3) = \Gamma'(q, \beta_3)/\Gamma(q,\beta_3)$$

$$\overline{\Psi}(q_f,\beta_3) = \Gamma''(q,\beta_3)/\Gamma(q,\beta_3) \quad ,$$

we find

$$I_{11}^{(1)} = q_h/\beta_2^2$$

$$I_{22}^{(1)} = -p_f/\beta_2^2 - 1/\beta_2^2\{2\cdot p_f\cdot\beta_3\cdot\Psi(q,\beta_3) - q_h\cdot\overline{\Psi}(q,\beta_{3+1}) - 2\cdot q_h\cdot\Psi(q,\beta_{3+1})\}$$

$$I_{33}^{(1)} = -\ p_f\cdot\Psi'(\beta_3)$$

$$I_{12}^{(1)} = -p_f\cdot\beta_3/\beta_2^2 + 1/\beta_2^2\cdot q_h\{1+\Psi(q,\beta_{3+1})\} \qquad (3.16)$$

$$I_{13}^{(1)} = p_f/\beta_2$$

$$I_{23}^{(1)} = p_f/\beta_2 \cdot \Psi(q,\beta_3) \quad .$$

Evaluation of the components I_{ij} in (3.13) gives

$$I_{11}^{(2)} = 1/\beta_2^2\cdot q_f\{q-\beta_3 - q_f\}$$

$$I_{22}^{(2)} = -\ 1/\beta_2^2\cdot q_f\{\beta_3(\ell n\ q)^2 + q\cdot\ell n\ q + 2\cdot\ell n\ q - q_f\}$$

$$I_{3}^{(2)} = [\Psi'(\beta_3) + \Psi(\beta_2)^2 - \overline{\Psi}(q,\beta_3)\cdot p_f]/(1-p_f) - \{[\Psi(\beta_3) - p_f\cdot\Psi(q,\beta_3)]/(1-p_f)\}^2 - \Psi'(\beta_3)$$

$$I_{12}^{(2)} = -\ 1/\beta_2^2\cdot\ q_f\{1 - q_f\cdot\ell n\ q\} \qquad (3.17)$$

$$I_{13}^{(2)} = 1/\beta_2\cdot q_f\{\ell n\ q - [\Psi(\beta_3) - p_f\cdot\Psi(q,\beta_3)]/(1-p_f)\}$$

$$I_{23}^{(2)} = I_{12}^{(2)} \cdot \ell n\ q \quad ,$$

where

$$q_f = q \cdot h_{\beta_3}(q)/(1-p_f) \quad .$$

The corresponding terms in (3.16) and (3.17) are then used in (3.13) to give the information matrix $I(\underset{\sim}{\beta}, p_f)$. Finally the asymptotic variance of $\hat{\theta}_c$ is evaluated by (3.1), (3.2), (3.12) and (3.13).

To determine the asymptotic variance, $\sigma^2(\underset{\sim}{\beta})$, for $\hat{\theta}$ in the complete sample case, the information matrix is first determined. Only equations (3.16) are needed for the determination if information matrix I . In this special case with $p_f=1$, the reductions $q_h=\beta_3$, $\Psi(q,\beta_3) = \Psi(\beta_3)$ and $\overline{\Psi}(q,\beta_3) = \Gamma''(\beta_3)/\Gamma(\beta_3)$ yield

$$
\begin{aligned}
I_{11} &= \beta_3/\beta_2^2 \\
I_{22} &= 1/\beta_2^2 \, \{1 + \Gamma''(\beta_3+1)/\Gamma(\beta_3)\} \\
I_{33} &= \Psi'(\beta_3) \\
I_{12} &= 1/\beta_2^2 \cdot \Gamma'(\beta_3 + 1)/\Gamma(\beta_3) \\
I_{13} &= 1/\beta_2 \\
I_{23} &= 1/\beta_2 \cdot \Psi(\beta_3)
\end{aligned}
\tag{3.18}
$$

The asymptotic variance

$$\sigma^2(\underset{\sim}{\beta}) = \underset{\sim}{D}'(\underset{\sim}{\beta}) \cdot I^{-1} \cdot \underset{\sim}{D}(\underset{\sim}{\beta})$$

is then evaluated using expressions (3.12) and (3.18) .

3.3 Asymptotic Variances for the Extreme Value, Log-Gamma, and Log-Negative Exponential Distributions

The log-generalized gamma family of distributions includes the extreme value (log-Weibull, $\beta_3=1$), log-gamma ($\beta_2=1$) , and log-negative exponential ($\beta_3=\beta_2=1$) distributions as special cases.

The asymptotic variances $\sigma_c^2(\underset{\sim}{\beta})$ and $\sigma^2(\underset{\sim}{\beta})$ in these special cases are simply those found by deleting the element(s) of $\underset{\sim}{D}$ in (3.12) and the row(s) and column(s) of I in (3.16), (3.17) and (3.18) corresponding to the fixed value(s) for β_2 and/or β_3.

3.4 Asymptotic Variance for the Normal Distribution

For a normal distribution $\beta_1 = E(Y)$ and $\beta_2^2 = Var(Y)$. The p^{th} quantile of Y depends on the inverse of the standard normal c.d.f. $\Phi(\cdot)$

$$\theta = \theta_p(\beta_1, \beta_2) = \beta_1 + \Phi^{-1}(p)\cdot\beta_2 \quad .$$

The gradient of θ is then

$$(1\,,\ \Phi^{-1}(p))' \quad .$$

Evaluation of the components (3.13) of the information matrix for the normal distribution yield

$$\begin{aligned}
I_{11}^{(1)} &= p_f/\beta_2^2 \\
I_{22}^{(1)} &= p_f/2\cdot\beta_2^4 - 1/\beta_2^4\cdot\Phi^{-1}(p_f)\cdot\phi(\Phi^{-1}(p_f)) \\
I_{12}^{(1)} &= -1/\beta_2^3\cdot\phi(\Phi^{-1}(p_f)) \\
I_{11}^{(2)} &= 1/\beta_2^2\cdot h(\Phi^{-1}(p_f))[h(\Phi^{-1}(p_f)) - \Phi^{-1}(p_f)] \\
I_{22}^{(2)} &= 3/4\ \beta_2^4\cdot\Phi^{-1}(p_f)\cdot h(\Phi^{-1}(p_f)) + \\
&\qquad 1/4\beta_2^4\cdot[\Phi^{-1}(p_f)]^2\cdot h(\Phi^{-1}(p_f))\cdot[h(\Phi^{-1}(p_f))-\Phi^{-1}(p_f)] \\
I_{12}^{(2)} &= 1/2\beta_2^3\cdot h(\Phi^{-1}(p_f)) + 1/2\beta_2^3\ h(\Phi^{-1}(p_f))[h(\Phi^{-1}(p_f)-\Phi^{-1}(p_f)]
\end{aligned} \tag{3.20}$$

where

$$h(\cdot) = \phi(\cdot)/1-\Phi(\cdot)$$

is the hazard function for the standard normal distribution.

The asymptotic variance of $\hat{\theta}_c$ is then obtained using (3.1), (3.2), (3.12) and (3.13). For the complete sample case it is well-known that

$$I = \begin{pmatrix} 1/\beta_2^2 & 0 \\ 0 & 1/2\beta_2^4 \end{pmatrix} .$$

Hence,

$$\sigma^2(\underset{\sim}{\beta}) = \beta_2^2\{1 + \frac{1}{2}\cdot[\Phi^{-1}(p)]^2\} .$$

3.5 Numerical Results

Recall (3.6) and (3.8)

$$\text{A.R.E.}(L_{n,c}; L_{n*}) = \sigma^2(\underset{\sim}{\beta}^o)/\sigma_c^2(\underset{\sim}{\beta}^o)$$
$$\text{A.R.E.}(L_{n**,NP}; L_{n*}) = \sigma^2(\underset{\sim}{\beta}^o)/\sigma^2_{NP}(\underset{\sim}{\beta}^o) ,$$

where the A.R.E.'s of the censored sample parametric $(L_{n,c})$ and the nonparametric $(L_{n**,NP})$ procedures relative to the complete sample parametric procedures (L_{n*}) are seen to depend only on the asymptotic variances $\sigma_c^2(\underset{\sim}{\beta}^o)$, $\sigma^2_{NP}(\underset{\sim}{\beta}^o)$ and $\sigma^2(\underset{\sim}{\beta}^o)$. For convenience of notation denote $\underset{\sim}{\beta}^o$ by $\underset{\sim}{\beta}$ throughout this section. From Sections 3.2 and 3.3, these asymptotic variances are independent of the location parameter β_1 when the expected proportion of uncensored observations p_f is held fixed. Moreover, the scale parameter β_2 (if present) appears in the expressions of these asymptotic variances only as the factor $1/\beta_2^2$. Therefore, the A.R.E.'s for the various families of distributions depend only on the shape parameter β_3 , if present, and $\beta_3 = 1$ otherwise. Of course, the A.R.E.'s may depend on the quantile

p and p_f. Thus, we denote

$$e_1(p_f, p, \beta_3) = \text{A.R.E. } (L_{n,c}; L_{n*}) = \sigma^2(\underset{\sim}{\beta})/\sigma_c^2(\underset{\sim}{\beta})$$

and (3.21)

$$e_2(p,\beta_3) = \text{A.R.E. } (L_{n**,NP}; L_{n*}) = \sigma^2(\underset{\sim}{\beta})/\sigma_{NP}^2(\underset{\sim}{\beta})$$

The values of the asymptotic relative efficiencies $e_1(p_f, p, \beta_3)$ and $e_2(p, \beta_3)$ when $p_f > p$ are given in Table 1 for $p = 0.5$, .1, and .25. Table 2 gives the values of $\beta_2^2\sigma^2(\underset{\sim}{\beta})$ and $\beta_2^2\sigma_{NP}^2(\underset{\sim}{\beta})$. Values of the corresponding $\beta_2^2\, \sigma_c^2\, (\underset{\sim}{\beta})$ can then be obtained from Table 1 by using (3.12).

It is of interest to see how $e_1(p_f, p, \beta_3)$ varies as the family of distribution is enlarged from the log-negative exponential to the extreme value $(\beta_2 = 1)$, or to the log-gamma $(\beta_3 = 1)$, and then to the log-generalized gamma $(\beta_2 = \beta_3 = 1)$. For these cases with $p_f = .5$, the values of $e_1(p_f, p, \beta_3)$ in Table 1 increase from .5 to .66, or to .82, and then to .96 for $p = .05$, and increase from .5 to .76, or to .91, and then to .95 for $p = .1$, and increase from .5 to .95, or to .97, and then decrease to .79 for $p = .25$. Next consider the changes in the asymptotic variances $\beta_2^2\sigma^2(\underset{\sim}{\beta})$. From Table 2, the corresponding parametric complete sample asymptotic variances are seen to increase from 1 to 7.99 or to 10.65 and then to 13.04. It is also of interest to compare $e_1(p_f, p, \beta_3)$ with $e_2(p, \beta_3)$ in Table 1 for the various assumed distributions. These two A.R.E.'s are approximately equal when $p_f = p$ for each distribution. That is, if the expected proportion of uncensored observations p_f is p, then there is no significant improvement in the relative efficiencies of the censored parametric procedure over the nonparametric procedure. Table 3 is also consistent with the well-known fact that the log-gamma distributions tend to the standard normal distribution as the parameter β_3 tends to infinity by noting the values of $e_1(p_f, p, \beta_3)$ for the log-gamma distributions approach those for the normal distributions. As expected, each column in Table 1 is an increasing function of p_f for all p. Finally, $e_1(p_f, p, \beta_3) = p_f$ is independent of p for the log-negative exponential family.

Table 1. A.R.E.'s of Lower Tolerance Limit Procedures

$e_1(p_f, p, \beta_3)$: censored relative to uncensored parametric

$e_2(p, \beta_3)$: nonparametric relative to uncensored parametric

(a) p=0.05

	p_f	log-N.E.	extreme value	log-gamma			normal	log-generalized gamma		
				β_3=1	2	4		β_3= 1	2	4
	.05	.05	.3998	.5324	.5326	.5309	.5274	.6519	.6466	.6405
	.10	.10	.5447	.7388	.7437	.7427	.7382	.7434	.7507	.7550
	.25	.25	.5638	.7561	.7617	.7615	.7575	.9161	.8951	.8789
	.30	.30	.5792	.7678	.7707	.7697	.7652	.9403	.9276	.9134
	.40	.40	.6175	.7962	.7955	.7930	.7877	.9573	.9640	.9574
$e_1(p_f, p, \beta_3)$	.50	.50	.6614	.8277	.8245	.8213	.8156	.9603	.9783	.9791
	.60	.60	.7097	.8601	.8555	.8518	.8462	.9604	.9825	.9867
	.70	.70	.7625	.8927	.8878	.8841	.8790	.9617	.9831	.9896
	.80	.80	.8217	.9258	.9214	.9189	.9141	.9659	.9834	.9941
	.90	.90	.8920	.9602	.9572	.9553	.9525	.9747	.9859	.9967
	.95	.95	.9358	.9786	.9769	.9760	.9741	.9824	.9894	.9982
$e_2(p, \beta_3)$		.0499	.3988	.5323	.5313	.5277	.5268	.6519	.6466	.6405

Table 1. (continued)

(b) p=0.10

	p_f	log-N.E.	extreme value	log-gamma			normal	log-generalized gamma		
				$\beta_3=1$	2	4		$\beta_3=1$	2	4
	.10	.10	.5340	.6285	.6273	.6264	.6247	.6668	.6547	.6497
	.25	.25	.7380	.9026	.9048	.9038	.9000	.8016	.7855	.7848
	.30	.30	.7382	.9040	.9075	.9068	.9034	.8417	.8137	.8070
	.40	.40	.7460	.9054	.9083	.9074	.9040	.9101	.8728	.8596
$e_1(p_f,p,\beta_3)$	.50	.50	.7645	.9121	.9134	.9122	.9084	.9545	.9216	.9075
	.60	.60	.7912	.9236	.9233	.9215	.9178	.9797	.9561	.9440
	.70	.70	.8247	.9382	.9369	.9350	.9314	.9923	.9787	.9702
	.80	.80	.8654	.9554	.9536	.9524	.9488	.9976	.9919	.9824
	.90	.90	.9166	.9750	.9736	.9687	.9702	.9990	.9984	.9972
	.95	.95	.9496	.9863	.9853	.9848	.9832	.9991	.9997	.9999
$e_2(p,\beta_3)$		.0999	.5339	.6283	.6268	.6262	.6232	.6668	.6547	.6496

Table 1. (continued)

(c) p=0.25

	p_f	log-N.E.	extreme value	log-gamma $\beta_3=1$	log-gamma 2	log-gamma 4	normal	log-generalized gamma $\beta_3=1$	log-generalized gamma 2	log-generalized gamma 4
	.25	.25	.6698	.6488	.6561	.6614	.6668	.6687	.6808	.6857
	.30	.30	.7965	.7741	.7751	.7777	.7813	.7622	.7866	.7962
	.40	.40	.9167	.9165	.9134	.9135	.9147	.7771	.8152	.8357
$e_1(p_f, p, \beta_3)$	.50	.50	.9473	.9724	.9701	.9698	.9700	.7911	.8213	.8405
	.60	.60	.9515	.9917	.9906	.9903	.9900	.8235	.8405	.8518
	.70	.70	.9519	.9972	.9968	.9965	.9958	.8646	.8708	.8776
	.80	.80	.9561	.9981	.9979	.9979	.9967	.9082	.9075	.9151
	.90	.90	.9679	.9982	.9980	.9977	.9970	.9516	.9484	.9492
	.95	.95	.9787	.9987	.9985	.9984	.9977	.9736	.9704	.9714
$e_2(p, \beta_3)$		.2482	.6685	.6472	.6533	.6510	.6610	.6686	.6806	.6854

Table 2. Asymptotic Variances for the Complete Sample Parametric and the Nonparametric Procedures

	p	log-N.E.	extreme value	log-gamma $\beta_3=1$	log-gamma 2	log-gamma 4	normal	log-generalized gamma $\beta_3=1$	log-generalized gamma 2	log-generalized gamma 4
$\beta_2^2\ \sigma^2(\underset{\sim}{\beta})$	.05	1.00	7.99	10.65	3.22	1.14	2.35	13.04	3.92	1.38
	.10	1.00	5.34	6.29	2.04	.73	1.82	6.67	2.13	.80
	.25	1.00	2.69	2.60	.98	.41	1.22	2.69	1.02	.43
$\beta_2^2\ \sigma_{NP}^2(\underset{\sim}{\beta})$	.05	20.00	20.00	20.00	6.06	2.16	4.46	20.00	6.06	2.16
	.10	10.00	10.00	10.00	3.25	1.23	2.92	10.00	3.25	1.23
	.25	4.02	4.02	4.02	1.5	.63	1.85	4.02	1.5	.63

4. ROBUSTNESS

Suppose a lower tolerance limit (3.3) based on the maximum likelihood estimator of the p^{th} quantile x_p $(0.05 \leq p \leq .25)$ for a censored sample is constructed for an assumed family of distributions $\mathcal{F}_\Omega = \{F(y;\underset{\sim}{\beta}) : \underset{\sim}{\beta} \in \Omega\}$. If instead the true distribution is $F_o(y;\underset{\sim}{\alpha}) \notin \mathcal{F}_\Omega$, with the p^{th} quantile

$$\Theta^o = F_o^{-1}(p) \quad ,$$

the true probability of coverage

$$\Pr\{L_{n,c} \leq \Theta^o \; ; \; F_o\} \tag{4.1}$$

is of interest. We approximate the probability (4.1) by large sample theory and call the approximation the Approximate Coverage Probability (A.C.P.). The discrepancy between the A.C.P. and the nominal confidence level γ provides a measure for the robustness of $L_{n,c}$. We expect this discrepancy to decrease as p_f is decreased. That is, the robustness of the procedure should tend to improve with increased censoring of the larger observations. The argument and notation developed here are similar to that used by D. R. Cox (1961) in his "Test of separate families of hypotheses".

4.1 General Construction

The asymptotic distribution of $\hat{\underset{\sim}{\beta}}_c$ and the probability limit of $\hat{\sigma}^2_c$ under $F_o(y;\underset{\sim}{\alpha})$ are required for the large sample approximation of (4.1). We assume that the derivatives of the log-likelihood function (2.3)

$$\partial \ell_{\bullet}/\partial \beta_i = \sum_{k=1}^{n} \{\delta_k \cdot \partial \ell n f(y_k;\underset{\sim}{\beta})/\partial \beta_i + (1-\delta_k)\cdot \partial \ell n\, \overline{F}(\tau;\underset{\sim}{\beta})/\partial \beta_i\} \quad , \quad i=1,\ldots,s$$

yield the M.L.E. $\hat{\underset{\sim}{\beta}}_c$ as the unique solution to the maximum likelihood equations

$$1/n \cdot \partial \ell_{\bullet}/\partial \beta_i \Big|_{\hat{\underset{\sim}{\beta}}_c} = 0 \quad , \quad i = 1,\ldots,s \quad , \tag{4.2}$$

and $\hat{\underset{\sim}{\beta}}_c$ converges in probability as $n \to \infty$ to a limit $\overline{\underset{\sim}{\beta}} = \overline{\underset{\sim}{\beta}}(\underset{\sim}{\alpha})$. Expand (4.2) about $\overline{\underset{\sim}{\beta}}$,

$$0 \doteq 1/n \cdot \partial \ell_{\bullet}/\partial \beta_i \Big|_{\overline{\underset{\sim}{\beta}}} + \sum_{j=1}^{s} 1/n \cdot \partial^2 \ell_{\bullet}/\partial \beta_i \cdot \partial \beta_j \Big|_{\overline{\underset{\sim}{\beta}}} \cdot (\hat{\beta}_{c,j} - \overline{\beta}_j) \quad , \quad i = 1,\ldots,s$$

and apply the Weak Law of Large Numbers to give

$$0 \doteq 1/n \cdot \partial \ell_{\bullet}/\partial \beta_i \Big|_{\overline{\underset{\sim}{\beta}}} + \sum_{j=1}^{s} E_\alpha[\partial^2 \ell/\partial \beta_i \cdot \partial \beta_j \Big|_{\overline{\underset{\sim}{\beta}}} \cdot (\hat{\beta}_{c,j} - \overline{\beta}_j) \quad , \quad i = 1,\ldots,s \tag{4.3}$$

where ℓ is the log-likelihood for a single observation and E_α is the expectation under $F_o(y;\underset{\sim}{\alpha})$. Denote

$$\mathcal{G}_i = 1/n \cdot \partial \ell_{\bullet}/\partial \beta_i \Big|_{\overline{\underset{\sim}{\beta}}} \quad , \qquad G_i = \partial \ell/\partial \beta_i \Big|_{\overline{\underset{\sim}{\beta}}} \quad , \quad G_{ij} = \partial^2 \ell/\partial \beta_i \partial \beta_j \Big|_{\overline{\underset{\sim}{\beta}}} \quad .$$

The solution to (4.3) in matrix notation then is

$$\hat{\underset{\sim}{\beta}}_c \doteq \overline{\underset{\sim}{\beta}} - \underline{M}^{-1} \cdot \underline{\mathcal{G}}$$

where

$$\underline{\mathcal{G}} = [\mathcal{G}_1, \mathcal{G}_2, \ldots , \mathcal{G}_s]'$$

and

$\underline{M}$ is a $s \times s$ matrix with (i,j)th entry $E_\alpha(G_{ij})$. (4.4)

Thus,

$$\mathrm{Cov}(\hat{\underset{\sim}{\beta}}_c) \doteq \underline{M}^{-1} \cdot \mathrm{Cov}(\underline{G}) \cdot \underline{M}^{-1} = 1/n \cdot \underline{M}^{-1} \cdot \underline{C} \cdot \underline{M}^{-1}$$

where

$C = n \cdot \mathrm{Cov}(G)$ is a $s \times s$ matrix with (i,j)th entry

$$E_\alpha(G_i \cdot G_j) \tag{4.5}$$

because

$$E_\alpha(G_i) = E_\alpha(G_i) = 0 \quad , \quad i = 1, \ldots s\,. \tag{4.6}$$

Since $\hat{\underset{\sim}{\beta}}_c$ converges to $\bar{\underset{\sim}{\beta}}$, equation (4.6) can be taken as a definition of $\bar{\underset{\sim}{\beta}}$. Hence, the asymptotic normality of $\hat{\underset{\sim}{\beta}}_c$ under $F_o(y;\ \alpha)$

$$\sqrt{n} \cdot (\hat{\underset{\sim}{\beta}}_c - \bar{\underset{\sim}{\beta}}) \to N(0,\ \sigma_c^2(\bar{\underset{\sim}{\beta}}))\ ,$$

where

$$\sigma_c^2(\bar{\underset{\sim}{\beta}}) = \underline{M}^{-1} \cdot \underline{C} \cdot \underline{M}^{-1}, \tag{4.7}$$

yields the asymptotic normality of $\hat{\theta}_c$ under $F_o(y;\ \underset{\sim}{\alpha})$

$$\sqrt{n} \cdot (\hat{\theta}_c - \bar{\theta}) \to N(0,\ \bar{V}(\bar{\underset{\sim}{\beta}}))\ ,$$

where

$$\bar{\theta} = \theta_p(\bar{\underset{\sim}{\beta}})$$

and

$$\bar{V}(\bar{\underset{\sim}{\beta}}) = \underset{\sim}{D}'(\bar{\underset{\sim}{\beta}}) \cdot \sigma_c^2(\bar{\underset{\sim}{\beta}}) \cdot \underset{\sim}{D}(\bar{\underset{\sim}{\beta}}) \tag{4.8}$$

Next, we determine the probability limit of $\hat{\sigma}_c^2 = \sigma_c^2\ (\hat{\underset{\sim}{\beta}}_c)$ under $F_o(y;\ \underset{\sim}{\alpha})$. Taking the censoring value

$$\tau = \ell n\ T = F_o^{-1}(p_f)$$

and the probability limits $\hat{\underset{\sim}{\beta}}_c \to \bar{\underset{\sim}{\beta}}$ and

$$\hat{p}_f = F(\tau;\hat{\underset{\sim}{\beta}}_c) \to \bar{p}_f = F(F_o^{-1}(p_f)\ ;\ \bar{\underset{\sim}{\beta}}) \tag{4.9}$$

in (3.4) gives

$$\hat{\sigma}_c^2 \to V^*(\bar{\underset{\sim}{\beta}}) \tag{4.10}$$

where

$$V^*(\bar{\underset{\sim}{\beta}}) = \underset{\sim}{D}'(\bar{\underset{\sim}{\beta}}) \cdot I^{-1}(\bar{\underset{\sim}{\beta}}\ ,\ \bar{p}_f) \cdot \underset{\sim}{D}(\bar{\underset{\sim}{\beta}})\ . \tag{4.11}$$

Hence, from (3.3) the probability (4.1) becomes

$$\begin{aligned} &\Pr\{\hat{\theta}_c \le \theta^o + Z_\gamma \cdot \hat{\sigma}_c/\sqrt{n}\,\} \\ &= \Pr\{\sqrt{n}(\hat{\theta}_c-\bar{\theta})/\sqrt{\bar{V}(\bar{\underset{\sim}{\beta}})} \le \sqrt{n}(\theta^o + Z_\gamma\hat{\sigma}_c/\sqrt{n} - \bar{\theta})/\sqrt{\bar{V}(\bar{\underset{\sim}{\beta}})}\} \\ &\doteq \Phi(Z_\gamma \cdot \sqrt{V^*(\bar{\underset{\sim}{\beta}})/\bar{V}(\bar{\underset{\sim}{\beta}})} + \sqrt{n} \cdot (\theta^o - \bar{\theta})/\sqrt{\bar{V}(\bar{\underset{\sim}{\beta}})}). \end{aligned} \tag{4.12}$$

To sum up, we need the $\underline{M}$ and $\underline{C}$ matrices and $\bar{\underset{\sim}{\beta}}$ which are defined respectively in (4.4), (4.5) and (4.6) to evaluate (4.12).

4.2. Coverage Probability of the Extreme Value Procedure When the True Distribution is a Normal

Recall that the extreme value distribution is a special case of the log-generalized gamma distribution ($\beta_3 = 1$) with p.d.f.

$$f(y;\beta_1\ ,\ \beta_2) = 1/\beta_2 \cdot \exp[(y-\beta_1)/\beta_2 - \exp((y-\beta_1)/\beta_2]$$

and c.d.f.

$$F(y;\beta_1,\beta_2) = 1 - \exp(-\exp((y-\beta_1)/\beta_2)) .$$

The log-likelihood for a single observation is then

$$\ell = \delta\cdot[-\ell n\beta_2 + (y-\beta_1)/\beta_2 - \exp((y-\beta_1)/\beta_2)] - (1-\delta)\cdot\exp((\tau-\beta_1)/\beta_2) \tag{4.13}$$

When $Z = (Y-\alpha_1)/\alpha_2$ has a $N(0,1)$ distribution, we write

$$W = (Y - \beta_1)/\beta_2 = A + B\cdot Z$$

$$w_\tau = (\tau - \beta_1)/\beta_2 = A + B\cdot\Phi^{-1}(p_f) ,$$

where

$$A = (\alpha_1 - \beta_1)/\beta_2 \quad , \quad B = \alpha_2/\beta_2$$

since $\tau = \alpha_1 + \alpha_2 \cdot \Phi^{-1}(p_f)$. Using

$$\begin{aligned}
\partial\ell/\partial\beta_1 &= -1/\beta_2\{\delta\cdot[1-\exp(w)] - (1-\delta)\cdot\exp(w_\tau)\} \\
\partial\ell/\partial\beta_2 &= -1/\beta_2\{\delta[1+w-\exp(w)\cdot w] + (1-\delta)\cdot\exp(w_\tau)\cdot w_\tau\} \\
\partial^2\ell/\partial\beta_1^2 &= -1/\beta_2^2\{\delta\cdot\exp(w) + (1-\delta)\cdot\exp(w_\tau)\} \\
\partial^2\ell/\partial\beta_1\cdot\partial\beta_2 &= -1/\beta_2^2\{\delta\cdot[-1+w\cdot\exp(w)+\exp(w)] + (1-\delta)\cdot[\exp(w_\tau)-w_\tau\exp(w_\tau)]\} \\
\partial^2\ell/\partial\beta_2^2 &= -1/\beta_2^2\{\delta[-1-2\cdot w+w^2\cdot\exp(w)+2\cdot w\cdot\exp(w)] \\
&\qquad + (1-\delta)\cdot[w_\tau^2\exp(w_\tau)-2\cdot w_\tau\cdot\exp(w_\tau)]\} ,
\end{aligned} \tag{4.14}$$

we find

$$\begin{aligned}
\beta_2\cdot E_\alpha(\partial\ell/\partial\beta_1) &= J_{12} - p_f + (1-p_f)\cdot J_3 \\
\beta_2\cdot E_\alpha(\partial\ell/\partial\beta_2) &= \ell nJ_1\cdot J_{12} - B\cdot J_3\cdot S_p - A\cdot p_f + B\cdot S_p - p_f + (1-p_f)\cdot\ell nJ_3\cdot J_3 ,
\end{aligned} \tag{4.15}$$

where

$$J_1 = \exp(A + B^2/2) \;;\; J_2 = \Phi(\Phi^{-1}(p_f)-B) \;;\; J_3 = \exp(A+B\cdot\Phi^{-1}(p_f))$$

$$J_{12} = J_1\cdot J_2 \;;\; S_p = \phi(\Phi^{-1}(p_f)$$

are then functions of A and B. Denote the roots of (4.15) as

$$(\bar{A}, \bar{B}) = (\bar{A}(p , p_f) , \bar{B}(p, p_f)) \tag{4.16}$$

from which we can obtain $\bar{\beta}_1 = \alpha_1 - \bar{A}/\bar{B} \cdot \alpha_2$, $\bar{\beta}_2 = \alpha_2/\bar{B}$. Evaluation of the matrices $\underline{M}$ and $\underline{C}$, defined by (4.4) and (4.5), gives

$$\begin{aligned}
\bar{\beta}_2^2\cdot m_{11} &= - p_f \\
\bar{\beta}_2^2\cdot m_{12} &= -p_f - \bar{A}\cdot p_f + \bar{B}\cdot S_p \\
\bar{\beta}_2^2\, m_{22} &= -\{(\bar{A}^2+1)\cdot p_f+[2\cdot\bar{A}\cdot\bar{B}^2+(\bar{B}^2+1)\cdot\bar{B}^2]\cdot J_{12}-[2\cdot\bar{A}\cdot\bar{B}+\bar{B}^2(\bar{B}+\Phi^{-1}(p_f))]\cdot J_3\cdot S_p \\
&\quad + (1-p_f)\cdot[2\cdot\bar{A}\cdot\bar{B}\cdot\Phi^{-1}(p_f) + \bar{B}^2\cdot[\Phi^{-1}(p_f)]^2\cdot J_3]\}
\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}
\bar{\beta}_2^2\cdot C_{11} &= J_4\cdot J_5 + p_f + (1-p_f)\cdot J_6 - 2\cdot J_{12} \\
\bar{\beta}_2^2\, C_{12} &= (\bar{A}+2\cdot\bar{B}^2)\cdot J_4\cdot J_5 - \bar{B}\cdot J_6\cdot S_p - 2\cdot(\bar{A}+\bar{B}^2)\cdot J_{12} + 2\cdot\bar{B}\cdot J_3\cdot S_p - J_{12} \\
&\quad + P_f + (1-p_f)\cdot\ell n\, J_3\cdot J_6 + \bar{A}\cdot p_f - \bar{B}\cdot S_p \\
\bar{\beta}_2^2\, C_{22} &= (\bar{A}^2+4\bar{A}\bar{B}^2+\bar{B}^2+4\bar{B}^2)J_4\cdot J_5-2\bar{A}\,\bar{B}\cdot J_6\cdot S_p-\bar{B}^2\cdot J_6\cdot S_p\cdot J_7-2\bar{A}^2\cdot J_{12}-4\bar{A}\bar{B}^2\cdot J_{12} \\
&\quad + 4\,\bar{A}\,\bar{B}\cdot S_p\cdot J_3 - 2\bar{B}^2(1+\bar{B}^2)J_{12}+2\cdot\bar{B}^2\cdot S_p\cdot J_3\cdot J_8+\bar{A}^2\cdot p_f \\
&\quad - 2\,\bar{A}\,\bar{B}\cdot S_p + \bar{B}^2\cdot J_9 + p_f + (1-p_f)\cdot(\ell n J_3)^2\cdot J_6 - 2\,\bar{A}\cdot J_{12} \\
&\quad - 2\cdot\bar{B}^2 J_{12}+2\bar{B}\cdot J_3\cdot S_p + 2\bar{A}\cdot p_f - 2\bar{B}\cdot S_p ,
\end{aligned} \tag{4.18}$$

where

$$J_4 = \exp(2\bar{A} + 2\cdot\bar{B}^2),\ J_5 = \Phi(\Phi^{-1}(p_f) - 2\bar{B}),\ J_6 = J_3^2,\ J_7 = \Phi^{-1}(p_f)+2\bar{B}$$

$$J_8 = J_7 - \bar{B},\quad J_9 = p_f - \Phi^{-1}(p_f)\cdot S_p \quad .$$

Finally, from (4.11) and (4.9), we have

$$V^*(\bar{\underset{\sim}{\beta}})/\bar{\beta}_2^2 = V^* = \frac{(\bar{p}_f - E_1)(C_{p_f} - C_p)^2 - 2\cdot E_2\cdot C_p + E_1\cdot C_p^2 + E_3 + \bar{p}_f}{\bar{p}_f(\bar{p}_f - E_1)(C_{p_f} - C_p)^2 + \bar{p}_f + \bar{p}_f(E_3 - 2E_2C_p + E_1\cdot C_p^2) - (\bar{p}_f(C_{p_f} - C_p) - E_1C_{p_f} + E_2)^2} \tag{4.19}$$

and

$$\bar{p}_f = 1 - \exp(-J_3) \quad ,$$

where

$E_1 = \Gamma(\quad)$, $E_2 = \Gamma'(u,2)$, $E_3 = \Gamma''(u, 2)$ with $u = -\ln(1-\bar{p}_f)$

Hence, using (4.16) - (4.19) and the fact that

$$\bar{V} = \bar{V}(\underset{\sim}{\beta})/\bar{\beta}_2^2 \tag{4.20}$$

is independent of $\underset{\sim}{\bar{\beta}}$ in (4.8), from (4.12) we have

$$\begin{aligned} \text{A.C.P.} &\doteq \Phi(Z_\gamma\sqrt{V^*(\underset{\sim}{\bar{\beta}})/\bar{V}(\underset{\sim}{\bar{\beta}})} + \sqrt{n}\cdot(\theta^o - \bar{\theta})/\sqrt{\bar{V}(\underset{\sim}{\beta})}\ ) \\ &= \Phi(Z_\gamma\sqrt{V^*/\bar{V}} + \sqrt{n}\,(\alpha_1 + \alpha_2\cdot\Phi^{-1}(p) - \beta_1 - \beta_2\cdot C_p)/\sqrt{\bar{V}}\cdot\bar{\beta}_2 \\ &= \Phi(Z_\gamma\sqrt{V^*/\bar{V}} + \sqrt{n}\,(\bar{A} + \bar{B}\cdot\Phi^{-1}(p) - C_p)\ /\sqrt{\bar{V}}\ ) \quad . \end{aligned} \tag{4.21}$$

4.3 Coverage Probability of the Normal Procedure When the True Distribution is an Extreme Value

Here, we have

$$f(y;\beta_1, \beta_2) = 1/\sqrt{2\pi} \cdot 1/\beta_2 \cdot \exp(-1/2\cdot((y-\beta_1)/\beta_2)^2)$$

with

$$F(y;\beta_1,\beta_2) = \Phi((y-\beta_1)/\beta_2)$$

and

$$\ell = \delta\cdot[\ell n(1/\sqrt{2\pi})-1/2\cdot\ell n\beta_2^2 -1/2\cdot((y-\beta_1)/\beta_2)^2]+(1-\delta)\cdot\ell n[1-\Phi((\tau-\beta_1)/\beta_2)].$$

The evaluations of $\bar{A}$, $\bar{B}$ and $\underline{M}$, $\underline{C}$ are similar to those in the previous section. Analogous to (4.14), we have

$$\partial\ell/\partial\beta_1 = 1/\beta_2\cdot\{\delta\cdot z + (1-\delta)\cdot h(z_\tau)\}$$

$$\partial\ell/\partial\beta_2^2 = -1/2 \cdot 1/\beta_2^2\cdot\{\delta\cdot(1-z^2) - (1-\delta)\cdot h(z_\tau)\cdot z_\tau\}$$

$$\partial^2\ell/\partial\beta_1^2 = -1/\beta_2^2\cdot\{\delta + (1-\delta)\cdot h(z_\tau)\cdot[h(z_\tau) - z_\tau]\}$$

$$\partial^2\ell/\partial\beta_1\cdot\partial\beta_2^2 = -1/\beta_2^3\{\delta\cdot z+1/2\cdot(1-\delta)\cdot h(z_\tau)+1/2\cdot(1-\delta)\cdot z_\tau\cdot h(z_\tau)\cdot[h(z_\tau)-z_\tau]\}$$

$$\partial^2\ell/\partial(\beta_2^2)^2= -1/\beta_2^4\{1/2\cdot\delta+z^2+3/4\cdot(1-\delta)\cdot z_\tau\cdot h(z_\tau)+1/4(1-\delta)\cdot z_\tau^2\cdot h(z_\tau)\cdot [h(z_\tau) - z_\tau]\}$$

where $z = (y-\beta_1)/\beta_2$, $z_\tau = (\tau-\beta_1)/\beta_2$ and $h(\cdot)$ is defined in Section 3.4. Corresponding to (4.15)

$$\beta_2\cdot E_\alpha(\partial\ell/\partial\beta_1) = A\cdot p_f + B\cdot R_1 + (1-p_f)\cdot h(A+B\cdot Cp_f)$$

(4.22)

$$\beta_2\cdot E_\alpha(\partial\ell/\partial\beta_2^2) = A^2\cdot p_f+2\cdot A\cdot B\cdot R_1 + B^2\cdot R_2+(1-p_f)(A+B\cdot Cp_f)h(A+BCp_f)-P_f ,$$

where

$A = (\alpha_1-\beta_1)/\beta_2$, $B = \alpha_2/\beta_2$, $R_1 = \Gamma'(u,1)$, $R_2=\Gamma''(u,1)$, $u = -\ln(1-p_f)$.

Corresponding to (4.17) ,

$$\bar{\beta}_2^2\cdot m_{11} = -\, p_f - (1-p_f)\cdot h(T_f)\cdot[h(T_f) - T_f]$$

$$\bar{\beta}_2^2\cdot m_{12} = -\, \bar{A}\cdot p_f - \bar{B}\cdot R_1 - (1-p_f)/2\cdot\{h(T_f) - T_f\cdot h(T_f)\cdot[h(T_f) - T_f]\} \quad (4.23)$$

$$\bar{\beta}_2^2\cdot m_{22} = -\, p_f/2 + \bar{A}^2\cdot p_f - 2\cdot\bar{A}\cdot\bar{B}\cdot\bar{R}_1 - \bar{B}^2\cdot R_2 - 3/4(1-p_f)\cdot T_f\cdot h(T_f)$$

$$- (1-p_f)/4\cdot T_f^2 h(T_f)\cdot[h(T_f) - T_f] ,$$

where

$$T_f = \bar{A} + \bar{B}\cdot Cp_f ,$$

Analogous to (4.18) ,

$$\bar{\beta}_2^2\cdot C_{11} = \bar{A}^2 p_f + 2\cdot\bar{A}\cdot\bar{B}\cdot R_1 + \bar{B}^2\cdot R_2 + (1-p_f)\cdot h^2(T_f)$$

$$\bar{\beta}_2^2\cdot C_{12} = 1/2\cdot\{\bar{A}^3 p_f+3\cdot\bar{A}\cdot\bar{B}\cdot R_1 + 3\bar{A}\cdot\bar{B}^2\cdot R_2 + \bar{B}^3\cdot R_3 - \bar{A}\cdot p_f$$

$$- \bar{B}\cdot R_1 + (1-p_f)\cdot T_f\cdot h^2(T_f)\} \quad (4.24)$$

$$\bar{\beta}_2^2\cdot C_{22} = 1/4\cdot\{\bar{A}^4 p_f + 4\bar{A}\cdot\bar{B}^3\cdot R_1 + 6\cdot\bar{A}^2\cdot\bar{B}^2\cdot R_2 + 4\cdot\bar{A}\cdot\bar{B}^3\cdot R_3 + \bar{B}^4\cdot R_4$$

$$+ (1-p_f)\cdot T_f{}^2 h(T_f) + p_f - 2(\bar{A}^2\cdot p_f + 2\cdot\overline{AB}R_1 + \bar{B}^2\cdot R_2)\} ,$$

where

$$R_3 = \Gamma'''(u,1) , R_4 = \Gamma''''(u,1)$$

Finally, analogous to (4.19) , we have

$$V^*(\underset{\sim}{\bar{\beta}})/\bar{\beta}_2^2 = V^* = \frac{1}{\Delta}\,(i_{22} + 1/4\cdot[\Phi^{-1}(p))]^2\cdot i_{11} - \Phi^{-1}(p)\cdot i_{12}) \quad (4.25)$$

where

$\Delta = i_{11} \cdot i_{22} - i_{12}^2$

and i_{11}, i_{12}, i_{22} can be obtained from the corresponding terms in (3.20) by setting $\beta_2 = 1$ and replacing p_f by

$$\overline{p}_f = \Phi(T_f) \quad .$$

Analogous to (4.21), we then obtain

$$\text{A.C.P.} \doteq \Phi(Z_r \cdot \sqrt{V^*/\overline{V}} + \sqrt{n} \cdot (\overline{A} + \overline{B} \cdot C_p - \Phi^{-1}(p))/\sqrt{\overline{V}}) \ . \tag{4.26}$$

4.4 Numerical Results

From (4.16)-(4.21), (4.22)-(4.26) it is clear that the approximate coverage probabilities A.C.P.=A.C.P.(P_f, P, γ, n) are independent of the parameters $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$. The argument of the Φ-function in (4.21) and (4.26) can be written as

$$Z_\gamma \cdot \sqrt{V^*/\overline{V}} \quad + \sqrt{n}\ (\theta^o - \overline{\theta}) / \ (\sqrt{\overline{V}}\ \overline{\beta}_2)$$

where

$\theta^o = \alpha_1 + \alpha_2 \Phi^{-1}(P)$ and $\overline{\theta} = \overline{\beta}_1 + \overline{\beta}_2 \cdot C_p$ when $F_o = N(\alpha_1, \alpha_2^2)$ and F = the extreme value (β_1, β_2) ;

or

$\theta^o = \alpha_1 + \alpha_2 \cdot C_p$ and $\overline{\theta} = \overline{\beta}_1 + \overline{\beta}_2 \cdot \Phi^{-1}(P)$ when F_o = the extreme value (α_1, α_2) and $F=N(\beta_1, \beta_2^2)$. We call $\theta^o - \overline{\theta}$ the asymptotic bias of $\hat{\theta}_c$ because θ^o is the true p^{th} quantile and $\overline{\theta} = \theta_p(\underset{\sim}{\overline{\beta}})$ is the probability limit of $\hat{\theta}_c$, the estimator for the p^{th} quantile of F, under F_o. Moreover,

$$F_o(\overline{\theta}) - p$$

is the asymptotic bias on probability scale. Both biases provide indices for the robustness of the point estimator $\hat{\theta}_c$ under F_o.

Tables 3a, 3b, and 3c give values of the A.C.P.'s when the assumed distribution is extreme value and the true distribution is normal and vice versa for $p=0.05$, 0.1, and 0.25. Each table contains two values of the nominal confidence level: $\gamma = 0.9$ and $.95$, and four sample sizes: $n = 20$, 40, 60, and 100. In Tables 3a-3c, the values of the A.C.P.'s are closest to the nominal confidence level γ when the expected proportion of uncensored observations p_f is equal to p. The A.C.P.'s are seen to deviate from γ in opposite directions for the two distributions under investigation. For example, in Table 3a, start with $p_f = .1$ and except for $p_f = 1$, the values of the A.C.P.'s are increasing in p_f when (Case 1) F is the extreme value distribution and F_o is the normal distribution and are decreasing in p_f when (Case 2) F is the normal distribution and F_o is the extreme value distribution. Thus, when $p = .05$ the procedures in both Case 1 and Case 2 are less robust as the expected proportion of uncensored observations p_f is increased. We also observe in Table 3a that the deviations between the A.C.P.'s and the nominal confidence level γ increase as the sample size is increased except for $p_f = .25$, $.30$, and $.40$ in Case 2. In Table 3b, the values of the A.C.P.'s are still monotonic in p_f, except for $p_f = 1$, for both Case 1 and 2. The deviations between the A.C.P.'s and γ increase as the sample size

n is increased except for $p_f = .60$, $.70$, $.80$ and $.90$ in Case 2. In Table 3c, the A.C.P.'s are no longer monotonic functions of p_f in both cases. As a matter of fact, the values of the A.C.P.'s decrease and then increase in Case 1 or increase and then decrease in Case 2. The turning point occurs at about $p_f = .5$. The deviations between the A.C.P.'s and γ still get larger as the sample size increases except for $p_f = .8$ and $.9$ in Case 1 and 2.

Table 4 gives values of $\overline{A}$, $\overline{B}$, V^* and $\overline{V}$ for both Cases 1 and 2. By using these values in (4.21) and (4.26), the A.C.P.'s can then be evaluated for any sample size n and confidence level γ. Table 5 gives values of the asymptotic bias on the probability scale $F_o(\overline{\theta}) - p$ (A.B.P.S.) which is also independent of the parameters $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$. It can be seen from Table 5 that the values of the A.B.P.S. corresponding to each p^{th} quantile are of opposite signs for Case 1 and 2 except for a few values of p_f. In Case 1, the smallest absolute value of A.B.P.S. for each p occurred at $p_f = p$. Whereas in Case 2, the smallest absolute value of A.B.P.S. occurred at $p_f = .05$, $.09$ and $.3$ for $p = .05$, $.1$ and $.25$, respectively.

4.5 Remarks Concerning the Choice of a Lower Tolerance Limit Procedure

From Table 1, efficiencies of the censored parametric procedure relative to the complete sample parametric procedure for various distributions were seen to decrease (i.e., more larger observations being censored). But from Table 3, the censoring was also seen to improve the robustness of the parametric procedures for the Weibull and

Table 3. Approximate Coverage Probabilities (A.C.P.'s)

(a) p=0.05

	γ =0.90								γ = 0.95							
	F=extreme value F_o=normal				F=normal F_o=extreme value				F=extreme value F_o=normal				F=normal F_o=extreme value			
p_f \ n	20	40	60	100	20	40	60	100	20	40	60	100	20	40	60	100
.05	.899	.899	.900	.900	.900	.899	.899	.899	.949	.949	.949	.950	.950	.950	.949	.949
.10	.894	.888	.883	.874	.904	.910	.915	.922	.948	.944	.941	.936	.951	.954	.957	.961
.25	.920	.919	.917	.915	.880	.889	.895	.905	.965	.965	.964	.964	.931	.936	.940	.947
.30	.930	.931	.931	.933	.872	.880	.885	.894	.970	.971	.971	.972	.924	.929	.933	.939
.40	.947	.953	.956	.962	.853	.858	.861	.867	.979	.982	.983	.986	.909	.912	.915	.918
.50	.963	.970	.975	.982	.842	.832	.831	.831	.987	.990	.992	.994	.892	.892	.891	.891
.60	.975	.984	.988	.993	.808	.801	.795	.786	.992	.995	.996	.998	.872	.867	.862	.855
.70	.985	.992	.995	.998	.781	.765	.751	.730	.995	.998	.999	.999	.849	.836	.825	.808
.80	.993	.997	.998	.999	.750	.721	.698	.661	.998	.999	.999	1.00	.822	.799	.779	.747
.90	.997	.999	.999	1.00	.712	.668	.634	.576	.999	.999	1.00	1.00	.788	.751	.720	.668
.95	.998	.999	1.00	1.00	.689	.637	.594	.526	1.00	1.00	1.00	1.00	.768	.722	.683	.619
1.0	.993	.998	1.00	1.00	.657	.592	.541	.459	1.00	1.00	1.00	1.00	.737	.679	.631	.551

Table 3. (continued)

(b) p=0.10

$p_f \backslash n$	γ =0.90								γ =0.95							
	F=extreme value F_o=normal				F=normal F_o=extreme value				F=extreme value F_o=normal				F=normal F_o=extreme value			
	20	40	60	100	20	40	60	100	20	40	60	100	20	40	60	100
.10	.899	.900	.900	.900	.899	.898	.898	.897	.949	.949	.950	.950	.950	.949	.949	.948
.25	.889	.877	.866	.849	.911	.922	.931	.942	.946	.939	.933	.923	.953	.960	.965	.971
.30	.895	.884	.875	.859	.907	.920	.929	.942	.951	.944	.939	.930	.949	.957	.963	.970
.40	.911	.905	.899	.890	.897	.912	.922	.939	.961	.957	.954	.949	.941	.951	.957	.966
.50	.929	.938	.927	.926	.886	.901	.911	.926	.970	.970	.970	.969	.932	.942	.949	.958
.60	.946	.950	.953	.957	.873	.887	.897	.911	.979	.981	.982	.984	.922	.931	.938	.948
.70	.963	.970	.974	.980	.858	.870	.878	.891	.987	.990	.991	.994	.909	.918	.924	.933
.80	.978	.985	.989	.994	.840	.848	.855	.864	.993	.995	.997	.998	.895	.901	.906	.913
.90	.989	.995	.997	.999	.818	.821	.823	.827	.997	.998	.999	.999	.876	.879	.880	.883
.95	.994	.998	.999	.999	.804	.803	.803	.802	.998	.999	.999	1.00	.864	.864	.863	.863
1.0	.985	.995	.997	1.00	.784	.778	.773	.766	.994	.998	1.00	1.00	.847	.842	1.838	.832

Table 3. (continued)

(c) p=0.25

	γ = 0.90								γ = 0.95							
	F=extreme value, F_o=normal				F=normal, F_o=extreme value				F=extreme value, F_o=normal				F=normal, F_o=extreme value			
$p_f \backslash n$	20	40	60	100	20	40	60	100	20	40	60	100	20	40	60	100
.25	.900	.901	.902	.903	.876	.892	.889	.885	.949	.950	.951	.951	.948	.946	.944	.942
.30	.882	.875	.869	.859	.914	.917	.919	.923	.939	.934	.931	.925	.958	.960	.962	.964
.40	.860	.837	.819	.787	.932	.943	.951	.961	.926	.912	.900	.878	.968	.974	.978	.983
.50	.854	.823	.798	.752	.948	.953	.962	.974	.924	.905	.888	.857	.970	.978	.983	.989
.60	.860	.828	.800	.752	.939	.956	.967	.979	.930	.910	.892	.859	.969	.979	.984	.991
.70	.874	.846	.822	.779	.937	.956	.968	.981	.940	.924	.909	.881	.967	.978	.984	.991
.80	.896	.876	.860	.830	.933	.955	.967	.981	.954	.943	.934	.916	.963	.976	.983	.991
.90	.926	.918	.912	.904	.928	.951	.964	.979	.971	.967	.964	.958	.959	.973	.981	.990
.95	.945	.944	.943	.942	.924	.948	.962	.977	.980	.980	.979	.979	.956	.971	.979	.988
1.0	.952	.960	.966	.973	.918	.943	.958	.974	.982	.984	.987	.990	.952	.968	.977	.987

Table 4. Quantities $\bar{A}, \bar{B}, V^*, \bar{V}$ Used in the Evaluation of the Approximate Coverage Probabilities

	F=extreme value		F$_o$=normal						F=normal		F$_o$=extreme value					
	$\bar{A}$	$\bar{B}$	V^*			$\bar{V}$			$\bar{A}$	$\bar{B}$	V^*			$\bar{V}$		
p_f			p=.05	.10	.25	.05	.10	.25			.05	.10	.25	.05	.10	.25
.05	1.02	2.43	19.98	30.56	79.48	20.03	28.55	67.79	-.453	.401	4.48	7.62	22.16	4.44	8.36	27.52
.10	.533	2.17	14.68	9.98	20.28	13.61	10.04	18.10	-.284	.445	3.19	2.93	6.98	3.53	2.89	8.22
.25	-.022	1.80	14.18	7.24	3.99	11.25	6.30	4.08	-.034	.525	3.10	2.02	1.87	4.45	2.47	1.80
.30	-.116	1.72	13.79	7.23	3.36	10.63	6.04	3.41	.019	.545	3.08	2.01	1.59	4.64	2.64	1.52
.40	-.252	1.60	12.91	7.16	2.93	9.48	5.59	2.81	.108	.582	2.99	2.01	1.35	4.90	2.95	1.37
.50	-.345	1.49	12.03	6.97	2.84	8.45	5.15	2.54	.181	.615	2.90	2.00	1.27	5.08	3.21	1.43
.60	-.411	1.40	11.18	6.72	2.83	7.53	4.73	2.35	.243	.647	2.81	1.99	1.24	5.23	3.43	1.55
.70	-.459	1.31	10.36	6.43	2.82	6.67	4.29	2.19	.299	.678	2.71	1.96	1.23	5.37	3.63	1.70
.80	-.492	1.23	9.56	6.10	2.81	5.87	3.85	2.02	.351	.709	2.62	1.93	1.23	5.51	3.82	1.87
.90	-.512	1.14	8.75	5.72	2.76	5.12	3.39	1.82	.399	.741	2.52	1.90	1.23	5.66	4.03	2.05
.95	-.515	1.09	8.33	5.52	2.73	4.83	3.17	1.69	.424	.759	2.47	1.88	1.23	5.75	4.14	2.15
1.0	-.500	1.00	7.99	5.34	2.69	9.11	5.38	2.10	.450	.780	2.35	1.82	1.22	5.85	4.26	2.27

Table 5. Asymptotic Bias on the Probability Scale

	$F_o(\bar{\theta}) - p = \Phi\left(\frac{Cp-\bar{A}}{\bar{B}}\right) - p$			$F_o(\bar{\theta}) - p = 1-\exp(-\exp\left(\frac{\Phi^{-1}(p)-\bar{A}}{\bar{B}}\right)) - p$		
	F=extreme value	F_o=normal		F=normal	F_o=extreme value	
p_f	p=0.05	p=0.10	p=0.25	p=0.05	p=0.10	p=0.25
.05	-.0000331	-	-	.00028	-	-
.10	.0032	-.0001	-	-.0040	.0009	-
.25	.0012	.0084	-.0011	-.0045	-.0112	.0057
.30	-.0007	.0083	.0066	-.0038	-.0120	-.0057
.40	-.0054	.0057	.0172	-.0020	-.0123	-.0206
.50	-.0105	.0011	.0233	.0001	-.0113	-.0295
.60	-.0160	-.0052	.0258	.0026	-.0096	-.0351
.70	-.0217	-.0131	.0250	.0052	-.0075	-.0385
.80	-.0277	-.0228	.0207	.0081	-.0048	-.0402
.90	-.0340	-.0353	.0110	.0114	-.0016	-.0407
.95	-.0376	-.0437	.0020	.0134	.0003	-.0404
1.0	-.0435	-.0600	-.0130	.0160	.0030	-.0390

lognormal distributions. In Figure 1 (2), Curve (a) gives the efficiency of the censored Weibull (lognormal) parametric procedure relative to the complete sample Weibull (lognormal) parametric procedure for $0.05 \leq p_f \leq 1$ and (b) gives the approximate coverage probability when the censored Weibull (lognormal) parametric procedure is used ($\gamma = .95$, $n = 60$, $p = .05$) where the true distribution is lognormal (Weibull) for $0.05 \leq p_f \leq 1$. For example, in Figure 2, for lognormal procedure with $p = .05$, $\gamma = .95$, and $n = 60$ the approximate coverage probability increased from .63 for the complete sample ($p_f = 1$) to .90 for censoring with $p_f = .5$ when the true distribution is the Weibull. In addition to improving robustness, censoring has practical benefits. The censored units may not be destroyed by testing. For the lumber stress example, the boards are not damaged by certain stress tests. In life testing, censoring is used to shorten the duration of the experiment.

An alternative to censoring that might improve the robustness of a parametric procedure would be to choose a larger family of distributions. For example, suppose the true distribution were Weibull ($\beta_3 = 1$) and the generalized gamma family were assumed. In this case, it can be determined from Table 1 that the complete sample parametric procedure based on the 3-parameter generalized gamma distribution is about as efficinet as the Weibull censored parametric procedure with $p_f = .4$. The complete sample procedure for the 3-parameter family might also tend to be more robust than the Weibull censored procedure with $p_f = .4$. For example, if the true distribution were a gamma with $\beta_2 = 1$, the Weibull censored procedure would not have the A.C.P.

exactly equal to the nominal confidence coefficient γ. Moreover, there is less than 5 percent loss in efficiency for censoring with $p_f = .4$ in the generalized gamma family.

Goodness of fit tests or tests for discriminating among various parametric families might also be made prior to adopting a lower tolerance limit procedure. The effects of such preliminary test on robustness of the lower tolerance limit procedures would be of interest for further research.

Figure 1. Efficiency and Robustness of Weibull Procedure

Curve (a): A.R.E. $e_1(p_f, p, \beta_3)$ when the true distribution is Weibull ($p = .05$)

Curve (b): A.C.P. of Weibull procedure when the true distribution is lognormal ($\gamma = .95$, $n = 60$, $p = .05$)

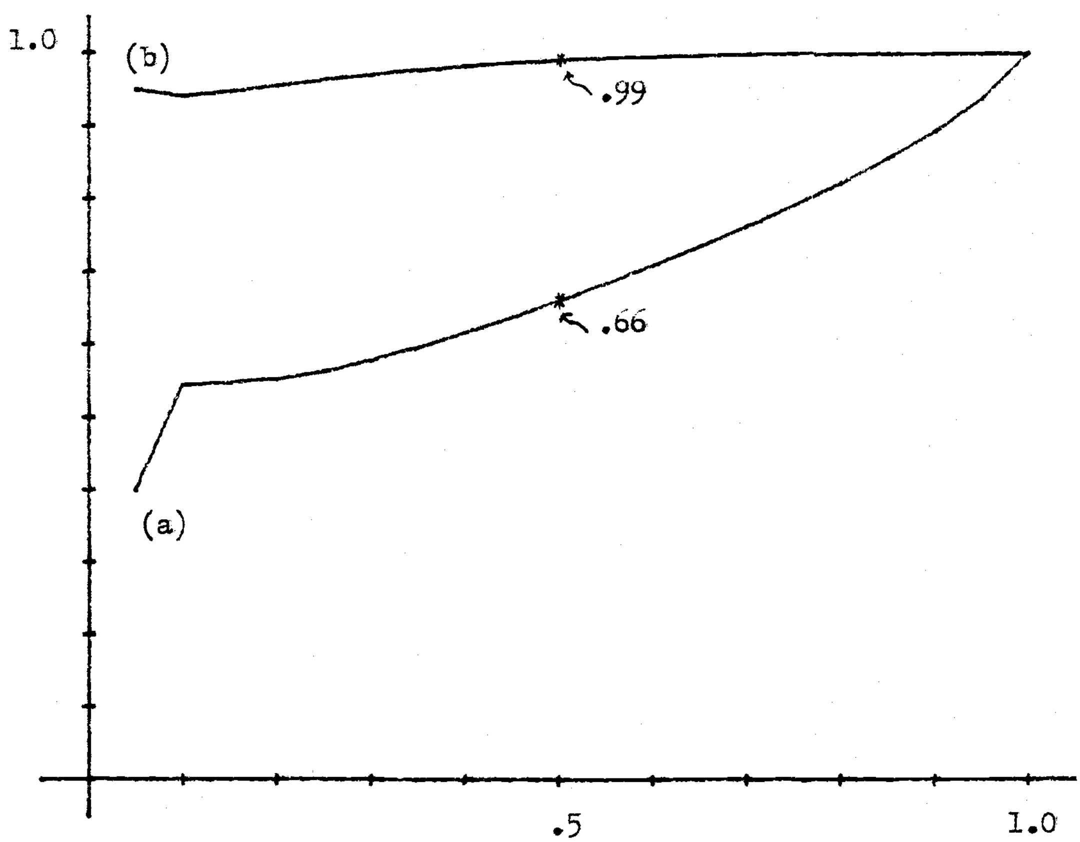
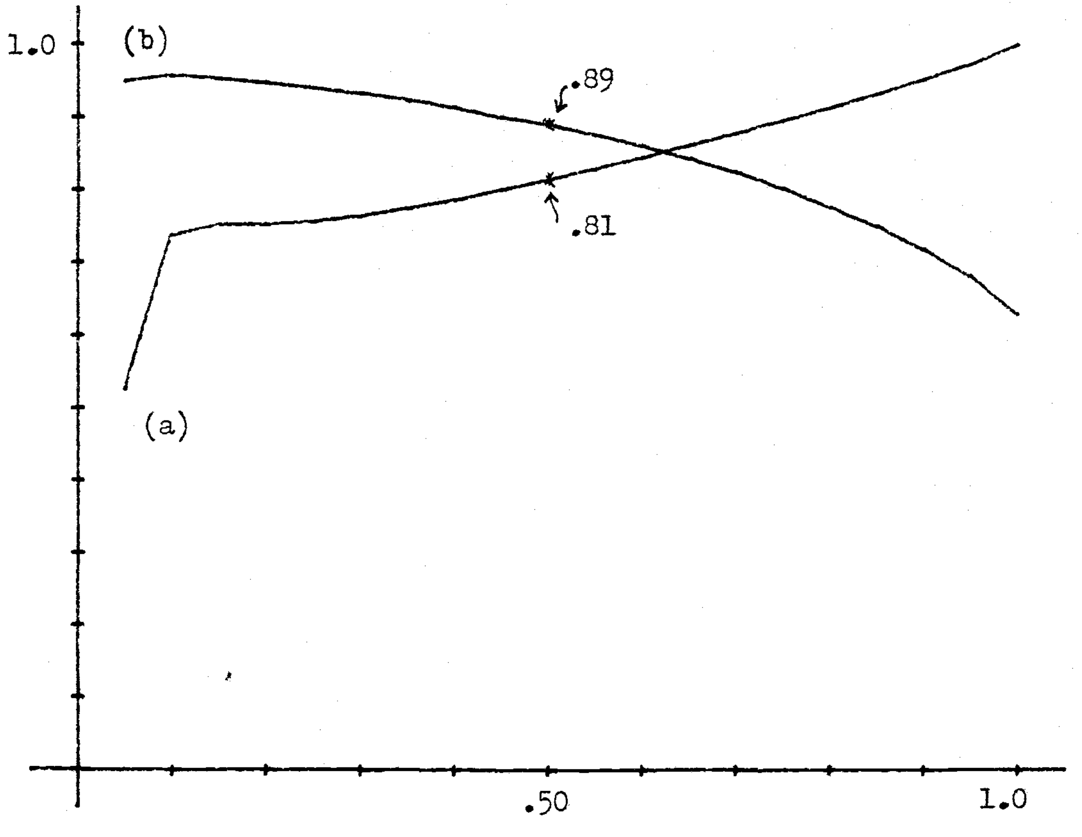


Figure 2. Efficiency and Robustness of Lognormal Procedure

Curve (a): A.R.E. $e_1(p_f, p, \beta_3)$ when the true distribution is lognormal (p = .05)

Curve (b): A.C.P. of lognormal procedure when the true distribution is Weibull (γ = .95 , n = 60 , p = .05)



5. SIMULATION STUDY

A simulation study was conducted to investigate the adequacy of the large sample normal approximations used in Chapters 3 and 4.

Samples of $n = 60$ uniform $(0,1)$ random variates were generated using the subroutine GGUB of the IMSL Library on the Cyber 70/73 Computer at Oregon State University. These variates were then transformed into ordered samples from either the lognormal or the Weibull distributions. Sets of 500 samples were generated for 12 cases comprised from the combinations of the two distributions; the three quantiles $p = .05$, $.10$, $.25$; and the two expected proportions of uncensored observations $p_f = .5$ and 1. Six lower confidence limits for the p^{th} quantile

$$x_p = \exp(y_p) = \exp(\theta)$$

with nominal confidence level $\gamma = .90$ were calculated for each sample. For the non-randomized

$$NP = X_{(k)}$$

and randomized

$$RNP = \begin{cases} X_{(k)} & , \quad \text{with probability } c \\ X_{(k+1)} & , \quad \text{with probability } 1-c \end{cases}$$

nonparametric procedures the integer k and randomization probability c are determined such that

$$P_k = \Pr\{X_{(k)} \leq X_p\} = \sum_{i=k}^{60} \binom{60}{i} p^i (1-p)^{n-i} \geq .90$$

but $\quad P_{k+1} < .90 \quad$ and $\quad c = (.90 - P_{k+1})/(P_k - P_{k+1})$.

Thus,

$$\Pr\{NP \le x_p\} \ge \Pr\{RNP \le x_p\} = .90$$

Weibull lower tolerance limits are constructed from the asymptotic normal distribution of $\hat{\theta} \equiv \hat{y}_p$

$$W_y = \hat{\theta} - 1.282 \cdot \hat{\sigma}/\sqrt{n} \quad ,$$

as described in Section 3.1, and from $\hat{x}_p = \exp(\hat{\theta})$

$$W_x = \exp(\hat{\theta}) \cdot \{1 - 1.282 \cdot \hat{\sigma}/\sqrt{n}\} \quad .$$

The corresponding lognormal lower tolerance limits based on $\hat{\theta} \equiv \hat{y}_p$ and $\hat{x}_p = \exp(\hat{\theta})$ are denoted respectively as LN_y and LN_x. Newton's method was used for iterative solution of the maximum likelihood equations (see Elashoff, p. 63, 1975).

The proportion of the 500 samples for which the lower tolerance limit is less than or equal to the value of the true p^{th} quantile x_p^o (on x-scale) or θ^o (on y-scale) then gives the empirical estimate $\hat{\gamma}$ of the true coverage probability.

The empirical estimators γ of the true coverage probability are given in Table 6. The values of the A.C.P.'s that are within two standard error units, i.e.,

$$\hat{\gamma} - 2 \cdot \sqrt{\hat{\gamma}(1-\hat{\gamma})/500} \le \text{A.C.P.} \le \hat{\gamma} + 2 \cdot \sqrt{\hat{\gamma}(1-\hat{\gamma})/500}$$

are identified in Table 6 by * . Notice that the values of the A.C.P.'s tend to give better approximations for the parametric procedures developed from x_p (x-scale) than that from $\hat{\theta} \equiv \hat{y}_p = \ell n(\hat{x}_p)$. Also note that 45 out of the total 72 (12 cases x 6 procedures) empirical values are less than the corresponding values of the A.C.P.'s. Overall, we found for the sample size $n = 60$ that the A.C.P.'s give reasonably accurate approximations for the true coverage probabilities.

(4.1).

Table 6. Empirical[1] and Asymptotic Estimates (A.C.P.) for the Probability of Coverage

(a) F_o = Extreme Value Distribution

	$p=0.05$; $x_p=.05129$				$p=0.10$; $x_p=.10536$				$p=0.25$; $x_p=.28768$			
	$p_f=1.0$		$p_f=0.5$		$p_f=1.0$		$p_f=0.5$		$p_f=1.0$		$p_f=0.5$	
	EMP.	A.C.P.	EMP.	A.C.P.	EMP.	A.C.P.	EMP.	A.C.P.	EMP.	A.C.P.	EMP.	A.C.P.
NP	.934	.954	.96	.954	.96	.947	.958	.947	.892	.914	.892	.914
RNP	.872*	.90	.916*	.90	.938	.90	.916*	.90	.868	.90	.876*	.90
W_x	.90*	.90	.926*	.90	.90*	.90	.878*	.90	.886*	.90	.884*	.90
W_y	.82	.90	.854	.90	.854	.90	.824	.90	.846	.90	.858	.90
LN_x	.55*	.542	.842*	.832	.738*	.774	.892*	.911	.948*	.958	.956*	.963
LN_y	.488	.542	.774	.832	.694	.774	.846	.911	.944*	.958	.948*	.963

[1] Based on 500 independent samples of size $n = 60$ for each case

Table 6 (continued)

(b) F_o = Normal Distribution

	$p=0.05$; $x_p=.19304$				$p=0.10$; $x_p=.27760$				$p=0.25$; $x_p=.50940$			
	$p_f=1.0$		$p_f=0.5$		$p_f=1.0$		$p_f=0.5$		$p_f=1.0$		$p_f=0.5$	
	EMP.	A.C.P.	EMP.	A.C.P.	EMP.	A.C.P.	EMP.	A.C.P.	EMP.	A.C.P.	EMP.	A.C.P.
NP	.958	.954	.972	.954	.946	.947	.954	.947	.948	.914	.926	.914
RNP	.904*	.90	.924	.90	.918*	.90	.908*	.90	.936	.90	.912*	.90
W_x	1.00*	1.00	.964*	.976	1.00*	1.00	.898*	.927	.994	.966	.81*	.798
W_y	1.00*	1.00	.938	.976	1.00*	1.00	.864	.927	.964*	.966	.80*	.798
LN_x	.878*	.90	.852	.90	.904*	.90	.856	.90	.914*	.90	.918*	.90
LN_y	.84	.90	.826	.90	.876*	.90	.836	.90	.894*	.90	.896*	.90

BIBLIOGRAPHY

Barlow, R. E. and F. Proschan. (1966). "Tolerance and confidence limits for classes of distributions based on failure rate." Annals of Mathematical Statistics 37(6):1593-1601.

Bickel, P. J. and K. A. Doksum. (1977). Mathematical statistics: basic ideas and selected topics. San Francisco: Holden Day.

Cohen, A. C. (1951). "Estimating parameters of log-normal distribution from censored samples." Journal of the American Statistical Association 46:206-212.

Cox, D. R. (1961). "Tests of separate families of hypotheses." Proceedings of the Fourth Berkeley Symposium 1:105-123.

Cox, D. R. and D. V. Hinkley. (1974). Theoretical statistics. London: Chapman and Hall.

David, H. A. (1970). Order statistics. New York: John Wiley and Sons, Inc.

Elashoff, R. M. (1975), (Editor). Perspectives in biometrics. 1, New York: Academic Press, Inc.

Fraser, D. A. S. (1963). Non-parametric methods in statistics. New York: John Wiley and Sons, Inc.

Gross, A. J. and V. A. Clark. (1975). Survival distributions: reliability applications in the biomedical sciences. New York: John Wiley and Sons, Inc.

Habermann, H. and R. L. Ethington. (1973). "Simulation studies of nonparametric tolerance intervals for describing wood strength." Forest Products Journal 25(11):26-31.

Hanson, D. L. and L. H. Koopmans. (1964). "Tolerance limits for the class of distributions with increasing hazard rate." Annals of Mathematical Statistics 35:1561-1570.

Harter, H. L. (1964). "New tables of the incomplete gamma function ratio and of percentage points of the chi-square and beta distributions." Washington, D.C.: U. S. Government Printing Office.

Johnson, N. L. and S. Kotz. (1972). Distributions in statistics. continuous variate, vol. 1.

Mann, N. R., R. D. Schafer and N. D. Singpurwalla. (1974). Methods for statistical analysis of reliability and life data. New York: John Wiley and Sons, Inc.

Mantel, N. and M. Myers. (1971). "Problems of convergence of maximum likelihood procedures in multi-parameter situations." Journal of the American Statistical Association. 66:484-491.

Prentice, R. L. (1974). "A log gamma model and its maximum likelihood estimation." Biometrika 62(3):607-614.

Stacey, E. W. and G. A. Mihram. (1965). "Parameter estimation for a generalized gamma distribution." Technometrics 7:349-358.

Tiku, M. L. (1967). "Estimating the mean and standard deviation from a censored normal sample." Biometrika 54:155-165.

Tiku, M. L. (1968). "Estimating the parameters of log-normal distribution from censored samples." Journal of the American Statistical Association 63:134-140.

Warren, W. G. (1974). "Comparison of exclusion values for lumber strength." Wood and Fiber 6(3):237-241.

APPENDIX

Recall (3.11)

$$H_\alpha(z) = \Gamma(z;\ \alpha)/\Gamma(\alpha) = \int_0^z t^{\alpha-1} e^{-t}\ dt/\Gamma(\alpha)$$

and the root of

$$H_\alpha(z) - p_f \qquad \text{(A.1)}$$

are required in Chapter 3 to evaluate the asymptotic variance σ_c^2 for the censored generalized gamma procedure. In general there is no simple analytic treatment to find the root. However, if α is a positive integer, then $H_\alpha(z)$ becomes (see Johnson Kotz, 1972)

$$1 - \exp(-z) \cdot \sum_{j=0}^{\alpha-1} z^j/j! \qquad \text{(A.2)}$$

and hence (A.1) can be solved numerically by iteration. Table 7 gives the root for $\alpha = 1,\ 2$ and 4. Moreover, in Chapter 3 we have terms like $\Gamma'(q;\ \alpha+1)$, $\Gamma''(q;\ \alpha+1)$ where q is the root of (A.1). So a general series expansions for these derivatives of the incomplete gamma functions are developed. Recall that

$$\Gamma(x,A) = \int_0^x t^{A-1}\ e^{-t}\ dt \quad , \quad A > 0\ .$$

By repeated use of integration by parts we have

$$\Gamma(x,A) = \exp(-A) \cdot x^A/A \cdot \{\ 1 + x/(A+1) + x^2/(A+1)(A+2) + \ldots\}\ . \qquad \text{(A.3)}$$

Also,

$$\Gamma'(x,A) = \int_0^x t^{A-1}\ e^{-t}\ \ell n\, t\ dt \ .$$

By direct partial differentiation w.r.t. A in (A.3)

$$\frac{\partial\Gamma}{\partial A} = (\ell n\, x - \frac{1}{A})\cdot\Gamma(x,A) - \frac{\exp(-A)\cdot x^A}{A}\cdot\{\frac{x}{(A+1)}\, C_1 + \frac{x^2}{(A+1)(A+2)}\, C_2 + \ldots\} \qquad (A.4)$$

where

$$C_i = \sum_{j=1}^{i} \frac{1}{A+j} \quad , \quad i = 1,\ 2,\ 3,\ \ldots \quad .$$

Moreover,

$$\Gamma''(x,A) = \int_0^x t^{A-1}\ e^{-t}\ (\ell n\, t)^2\ dt \quad , \quad A > 0 \ .$$

By again direct partial differentiation in (A.4)

$$\frac{\partial^2\Gamma}{\partial A^2} = \frac{1}{A^2}\,\Gamma(x,A) + \Gamma'(x,A)(\ell n\, x - \frac{1}{A}) - \frac{\exp(-A)\cdot x^A}{A}\cdot\{\frac{x}{(A+1)}\, C_1 + \frac{x^2}{(A+1)(A+2)}\, C_2 + \ldots\}$$

$$(A.5)$$

$$- \frac{\exp(-A)\cdot x^A}{A}\{x\cdot\frac{d}{dA}\,\frac{C_1}{(A+1)} + x^2\ \frac{d}{dA}\,\frac{C_2}{(A+1)(A+2)} + \ldots\} \ .$$

But

$$\frac{d}{dA}\left(\frac{C_1}{A+1}\right) = -2/(A+1)^3$$

$$\frac{d}{dA}\left(\frac{C_i}{(A+1)(A+2)\ldots(A+i)}\right) = \frac{2}{(A+1)(A+2)\ldots(A+i)}[D_i - C_i^2] \ , \ i=2,3\ldots$$

where

$$D_i = (1 + \frac{1}{2} + \ldots + \frac{1}{i-1})(\frac{1}{A+1} - \frac{1}{A+2}) + (\frac{1}{2} + \ldots + \frac{1}{i-2})(\frac{1}{A+2} - \frac{1}{A+i-1}) \tag{A.6}$$

$$+ \ldots + \frac{1}{k}(\frac{1}{A+k} - \frac{1}{A+k+1}) \quad \text{when} \quad i = 2k, \; k = 1, 2, \ldots \quad \text{and}$$

$$= (1 + \frac{1}{2} + \ldots + \frac{1}{i-1})(\frac{1}{A+1} - \frac{1}{A+2}) + (\frac{1}{2} + \ldots + \frac{1}{i=2})(\frac{1}{A+2} - \frac{1}{A+i-1})$$

$$+ \ldots + (\frac{1}{k} + \frac{1}{k+1})(\frac{1}{A+k} - \frac{1}{A+k+1}) \quad \text{when} \quad i = 2k+1, \; k=1,2,\ldots$$

Therefore, after substituting (A.6) in (A.5) and combining terms, we have

$$\frac{\partial^2 \Gamma}{\partial A^2} = \frac{1}{A^2}\Gamma(x,A) + \Gamma'(x,A)(\ell n\, x - \frac{1}{A}) - (\ell n\, x - \frac{1}{A})^2 \cdot \Gamma(x,A) - \frac{\exp(-A)\cdot x^A}{A} \cdot \tag{A.7}$$

$$\left\{ \frac{-2x}{(A+1)^3} + \frac{2x^2}{(A+1(A+2)} + \left\{ (D_2 - C_2^2) + \frac{x}{(A+3)} \left\{ [D_3 - C_3^2] + \frac{x}{(A+4)} \left\{ \ldots \right\}\right\}\right\}\right\}$$

In Chapter 4, we have to compute $\Gamma'(u,1), \Gamma''(u,1), \Gamma'''(u,1)$ and $\Gamma''''(u,1)$. We can use (A.4) and (A.7) to compute $\Gamma'(u,1)$ and $\Gamma''(u,1)$. As for $\Gamma'''(u,1)$ and $\Gamma''''(u,1)$, another representation of incomplete gamma function based on the Taylor expansion of $\exp(-t)$ is used

$$\Gamma(x,A) = \int_0^x t^{A-1}\, e^{-t}\, dt = \sum_{h=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{n+A}}{n+A} \tag{A.8}$$

From (A.8) we direct differentiate w.r.t. A four times and found

$$\Gamma'(x,A) = \ell n\ x \cdot \Gamma(x,A) - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{n+A}}{(n+A)^2}$$

$$\Gamma'(x,A) = 2 \cdot \ell n\ x \cdot \Gamma'(x,A) - (\ell n\ x)^2 \cdot \Gamma(x,A) + 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{n+A}}{(n+A)^3}$$

$$\text{(A.9)}$$

$$\Gamma'''(x,A) = 3 \cdot \ell n\ x \cdot \Gamma''(x,A) - 3(\ell n\ x)^2\ \Gamma'(x,A) + (\ell n\ x)^3\ \Gamma(x,A)$$
$$- 6 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{n+A}}{(n+A)}$$

$$\Gamma''''(x,A) = 4\ \ell n\ x\ \Gamma'''(x,A) - 6(\ell n\ x)^2\ \Gamma''(x,A)$$
$$+ 4(\ell n\ x)^3\ \Gamma'(x,A) - (\ell n\ x)^4\ \Gamma(x,A)$$
$$+ 24 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{n+A}}{(n+A)^5}$$

When equations (A.3), (A.4), and (A.7) are used to compute the incomplete gamma function and its first and second derivatives, it is found that the first 12 digits of the values obtained by using the first 50 , 100 , 150 , 200 terms in the series expansion(A.3), (A.4), and (A.7) are the same. On the other hand, the error can be bounded analytically for (A.9) because in an alternating decreasing series, the error for the n terms approximation to the whole series will be less than the absolute value of the next term. That is, for example

$$\text{Error} < \frac{x^{n+A}}{n!(n+A)^2}$$

where

$$\text{Error} = s - s_k$$

and

$$s = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{n+A}}{(n+A)} \quad , \quad s_k = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{n+A}}{(n+A)}$$

Table 7. Roots of the function $H_\alpha(z)-p_f$ for $\alpha = 1, 2, 4$

p_f	$\alpha = 1$	$\alpha = 2$	$\alpha_2 = 4$
.05	.0512932943	.3553615108	1.366318398
.10	.1053605157	.5318116084	1.744769594
.15	.1625189295	.6832386131	2.039099548
.20	.2231435513	.8243883188	2.296786806
.25	.2876820725	.9612787632	2.535320212
.30	.3566749493	1.097330533	2.763711043
.35	.4307829161	1.235033575	2.987644562
.40	.5108256380	1.376420537	3.211322778
.45	.5978370008	1.523380674	3.438315333
.50	.6931471806	1.678340731	3.672056688
.55	.7985076962	1.843566915	3.916215561
.60	.9162907319	2.022313143	4.175262726
.65	1.049822124	2.218842854	4.454679274
.70	1.203972804	2.439198247	4.762227200
.75	1.386294361	2.692634523	5.109414292
.80	1.609437912	2.994308082	5.514995322
.85	1.897119985	3.372432505	6.013458699
.90	2.302585093	3.889720139	6.680764999
.95	2.995732274	4.743816045	7.753574496