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NONLINEAR DEGENERATE EVOLUTION EQUATIONS IN MIXED FORMULATION*

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Abstract. We develop the theory of degenerate and nonlinear evolution systems in mixed formulation. It will be shown that many of the well-known results for the stationary problem extend to the nonlinear case and that the dynamics of a degenerate Cauchy problem is governed by a nonlinear semigroup. The results are illustrated by a Darcy–Stokes coupled system with multiple nonlinearities.

Key words. mixed formulation, nonlinear evolution equation, monotone systems, coupled Darcy–Stokes

AMS subject classifications. Primary, 47J35, 47H05, 35F61; Secondary, 35Q35, 76S05

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1. Introduction.

Consider the system of evolution equations in the *mixed formulation*

$$\mathbf{u}(t) \in V,\ p(t) \in Q:$$
$$\frac{d}{dt}\mathcal{E}_1\mathbf{u}(t) + \mathcal{A}\mathbf{u}(t) + \mathcal{B}'p(t) = \mathbf{f}(t) \text{ in } V', \tag{1a}$$
$$\frac{d}{dt}\mathcal{E}_2 p(t) - \mathcal{B}\mathbf{u}(t) + \mathcal{C}p(t) = g(t) \text{ in } Q' \text{ for } t > 0, \tag{1b}$$

where V and Q are Hilbert spaces and the operators $\mathcal{E}_1, \mathcal{A}$ map V to V', $\mathcal{E}_2, \mathcal{C}$ map Q to Q', and $\mathcal{B}$ maps V to Q'. The analysis of the stationary case,

$$\mathbf{u} \in V,\ p \in Q:$$
$$\mathcal{A}\mathbf{u} + \mathcal{B}'p = \mathbf{f} \text{ in } V', \tag{2a}$$
$$\mathcal{B}\mathbf{u} - \mathcal{C}p = -g \text{ in } Q', \tag{2b}$$

most often with $\mathcal{C} = 0$, has been developed extensively when all the operators are linear. These studies include the continuous problem in function space (as here) and more substantially the finite dimensional approximations that correspond to *mixed finite element methods.* The system (2) is regarded as a saddle-point problem when $\mathcal{A}$ and $\mathcal{C}$ are symmetric and positive, and the key ingredient in most of these results is the condition that $\mathcal{B}$ has *closed range*, that is, it satisfies the *LBB-condition* [14, 7, 9]. Such a condition is *necessary* to obtain existence and estimates of p in (2) when $\mathcal{C} = 0$ and in (1) when also $\mathcal{E}_2 = 0$, and it is likewise appropriate in the general case in order to obtain estimates that are independent of $\mathcal{C}$ and $\mathcal{E}_2$.

We study here the corresponding system of evolution equations (1). Classical time-dependent models of Darcy flow with slightly compressible fluid or medium and of Stokes flow lead to the complementary situations in which *one* of $\mathcal{E}_1$ or $\mathcal{E}_2$ is zero and the other is multiplication by a positive function of position. We start by recovering these two classical linear cases which have been discussed already elsewhere [6, 15]. These will be obtained as evolutions described by analytic semigroups on the respective component space.

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Our objective here is to permit the linear and symmetric operators $\mathcal{E}_1$ and $\mathcal{E}_2$ to be *degenerate*, that is, they are nonnegative but may have substantial kernels, e.g., components that vanish. Thus, they need not necessarily contribute any lower order terms to facilitate the estimates and consequential analysis. Even the simplest examples show that such degeneracy is not an exceptional situation; rather, it is ubiquitous. Moreover, both $\mathcal{A}$ and $\mathcal{C}$ are permitted to be *nonlinear*: we require that each is *maximal monotone* and that they satisfy together with the linear coupling operator $\mathcal{B}$ a combination of growth and closed range conditions. These general conditions substantially overlap many earlier works on (2) and show that linearity plays a minor role in the analysis. We shall show that the Cauchy problem for the system (1) is described by a semigroup of nonlinear operators [8, 25] on a space determined by $\mathcal{E}_1$ and $\mathcal{E}_2$.

Nonlinear evolution systems with either $\mathcal{E}_1$ or $\mathcal{E}_2$ being null were reformulated in [1, 2, 3] as a single equation. Nonlinear parabolic problems for (1) with $\mathcal{E}_1 = 0$ were developed in [11], and in [17, 22] is proved the existence and uniqueness of the *approximations* of an assumed solution when $\mathcal{C} = \mathcal{E}_1 = 0$ but with $\mathcal{E}_2$ nonlinear. (See [19] and [21] for similar results for (2).) A fully dynamic case in which both $\mathcal{E}_1$ and $\mathcal{E}_2$ are present but $\mathcal{C} = 0$ was covered in [28], and there it was observed again that the linearity of $\mathcal{A}$ plays a rather minor role. In all of these treatments, the addition of the lower order terms from $\mathcal{E}_1 = I$ or $\mathcal{E}_2 = I$ actually makes the analysis *easier*, a fact that was highlighted in [20], and frequently the presence of such terms makes the closed range condition unnecessary. This was the case in [26], where a similar system was developed; there an additional inelasticity operator provided the coercivity in the second component.

Degenerate or nonlinear systems arise in many applications to coupled multiphysics models, and our results will be illustrated with examples from fluid mechanics. We use boldface letters to indicate vectors in $\Re^3$ and Greek letters for symmetric second-order tensors. Repeated indices are summed, so the scalar product of two vectors is $\mathbf{v} \cdot \mathbf{w} = v_i w_i$, and that of two second-order tensors is $\sigma : \tau = \sigma_{ij}\tau_{ij}$. The gradient of $p(x)$ is $\boldsymbol{\nabla} p$ with components $\partial_j p(x)$, and the divergence of a vector function $\mathbf{v}(x)$ is $\boldsymbol{\nabla} \cdot \mathbf{v}(x) = \partial_i v_i(x)$. The symmetric derivative of $\mathbf{v}(x)$ is the tensor $\boldsymbol{\epsilon}_{ij}(\mathbf{v}) \equiv \frac{1}{2}(\partial_i v_j + \partial_j v_i)$.

General linear models of fluid flow through porous media are written as [5, 30]

$$\frac{\partial}{\partial t}(\rho_1 \mathbf{u}) + \kappa^{-1}\mathbf{u} + \boldsymbol{\nabla} p = \mathbf{f}\,, \tag{3a}$$

$$\frac{\partial}{\partial t}(c_1\, p) + \boldsymbol{\nabla} \cdot \mathbf{u} = g, \tag{3b}$$

where $\mathbf{u}(x,t)$ is the fluid flux, $p(x,t)$ is the pressure, κ^{-1} is the resistance of the porous medium, i.e., the reciprocal of permeability, $\mathbf{f}(x)$ represents the gravity force, and g accounts for fluid sources. The term $\rho_1 \geq 0$ is usually null but permits a dynamic form of Darcy's law,[1] and $c_1 \geq 0$ arises similarly from compressibility. Here and below, a factor of density has been deleted or incorporated in other coefficients.

The slow flow of fluid in an open region is described by the *slightly compressible Stokes system* [29, 24]

$$\frac{\partial}{\partial t}(\rho_2 \mathbf{u}) - \boldsymbol{\nabla} \cdot \sigma + \boldsymbol{\nabla} p = \mathbf{0}\,, \tag{4a}$$

$$\frac{\partial}{\partial t}(c_2 p) + \boldsymbol{\nabla} \cdot \mathbf{u} = 0\,, \tag{4b}$$

[1] See [27] for analogous Maxwell–Cattaneo models of heat conduction.

where $\sigma_{ij} = \lambda_2 \delta_{ij} \boldsymbol{\epsilon}(\mathbf{u})_{kk} + 2\mu_2 \boldsymbol{\epsilon}(\mathbf{u})_{ij}$ specifies the relationship between fluid stress σ and the deformation rate $\boldsymbol{\epsilon}(\mathbf{u})$. In the case of an *incompressible* fluid, we have $c_2 = \lambda_2 = 0$, and (4) reduces to classical Stokes flow (see (10) below).

We shall illustrate the abstract results with a prototype composite multiphysics system which describes the fluid exchange and stress balance between the porous medium and a contiguous fluid-filled chamber and show that this model leads to a mathematically well-posed initial-boundary-value problem. The interface coupling conditions include the continuity of the normal fluid flux and of stress. Additional interface conditions prescribe the Darcy flux dependence on the pressure increment and the tangential component of stress dependence on the velocity increment at the interface, i.e., the slip condition of *Beavers–Joseph–Saffman*. Moreover, we shall include nonlinear versions of each of these. See [4, 18] for the linear stationary Darcy–Stokes system and [13] for nonlinear extensions.

2. Two classical linear cases. The unbounded linear operator $L : D \to H$ with domain D in the Hilbert space H is *accretive* if

$$(Lx, x)_H \geq 0 , \qquad x \in D ,$$

and it is m-*accretive* if, in addition, $I + L$ maps D onto H. Sufficient conditions for the initial-value problem to be well posed are provided by the Hille–Yosida–Phillips theorem [16].

THEOREM 2.1. *Let the operator $L : D \to H$ be m-accretive on the Hilbert space H. Then for every $v^0 \in D(A)$ and $h \in C^1([0,\infty), H)$, there is a unique solution $v \in C^1([0,\infty), H)$ of the initial-value problem*

$$\frac{d}{dt} v(t) + Lv(t) = h(t) \text{ for } t > 0, \quad v(0) = v^0 \ . \tag{5}$$

If additionally L is self-adjoint, then for each $v^0 \in H$ and Hölder continuous $h \in C^\beta([0,\infty), H)$, $0 < \beta < 1$, there is a unique solution $v \in C([0,\infty), H) \cap C^1((0,\infty), H)$ of (5).

This general linear result will be applied to two classical cases.

2.1. Darcy flow. The constitutive law of Darcy takes the form

$$a(x)\, \mathbf{u} + (\, \nabla p - \mathbf{f}(x)\,) = \mathbf{0}, \tag{6a}$$

where $a(x)$ is the resistance of the porous medium, i.e., the reciprocal of permeability, and the conservation law is

$$c(x)\, \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{u} = g(x,t). \tag{6b}$$

The problem is to find a solution $\mathbf{u}$, p of the system (6) with $p = 0$ on the boundary $\partial\Omega = \Gamma$ and prescribed initial-value $p(x,0) = p_0(x)$ on Ω. This initial-boundary-value problem is described by Theorem 2.1. To see this, define on the spaces $V = \mathbf{L}^2_{div}(\Omega) \equiv \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \ \boldsymbol{\nabla} \cdot \mathbf{v} \in L^2(\Omega)\}$ and $Q = L^2(\Omega)$ the multiplication operators $\mathcal{A} : V \to V'$, $\mathcal{E}_2 : Q \to Q'$ by $\mathcal{A}\mathbf{u} = a(\cdot)\, \mathbf{u}$, $\mathcal{E}_2 p = c(\cdot)\, p$, and define $\mathcal{B} : V \to Q'$ by $\mathcal{B}\mathbf{u} = -\boldsymbol{\nabla} \cdot \mathbf{u}$. The mixed formulation of the *evolution system* (6) with appropriately given $\mathbf{f} \in V'$ and $g(t) \in Q'$ takes the form

(7) $\quad \mathbf{u}(t) \in V,\ p(t) \in Q$ for $t > 0$:

$$\begin{aligned} \mathcal{A}\mathbf{u}(t)(\mathbf{v}) + \mathcal{B}' p(t)(\mathbf{v}) &= \mathbf{f}(\mathbf{v}), \\ \mathcal{E}_2 \tfrac{d}{dt} p(t)(q) - \mathcal{B}\mathbf{u}(t)(q) &= g(t)(q), \quad \mathbf{v} \in V,\ q \in Q, \end{aligned}$$

together with the initial condition $p(0) = p_0$.

Let's resolve the system (7). First find a pair $\mathbf{u}_f,\ p_f$ which is the solution of a stationary problem

$$\begin{aligned} \mathbf{u}_f \in V,\ p_f \in Q:\ \mathcal{A}\mathbf{u}_f + \mathcal{B}' p_f &= \mathbf{f}, \\ -\mathcal{B}\mathbf{u}_f &= 0. \end{aligned}$$

Subtract $\mathbf{u}_f$ from $\mathbf{u}(t)$ and p_f from $p(t)$ to get the problem (7) but with $\mathbf{f} = \mathbf{0}$ and the initial condition $\mathcal{E}_2 p(0) = \mathcal{E}_2 p_0 - \mathcal{E}_2 p_f$ for the translates $\mathbf{u}(t) - \mathbf{u}_f$ and $p(t) - p_f$. Thus, we want to solve the initial-value problem for the reduced evolution system

$$\begin{aligned} \mathbf{u}(t) \in V,\ p(t) \in Q:\ \mathcal{A}\mathbf{u}(t) + \mathcal{B}' p(t) &= \mathbf{0} \text{ in } V', \\ \mathcal{E}_2 \tfrac{d}{dt} p(t) - \mathcal{B}\mathbf{u}(t) &= g(t) \text{ in } Q' \text{ for } t > 0. \end{aligned} \tag{8}$$

Suppose $\mathcal{E}_2(\cdot)(\cdot)$ is a continuous scalar product on Q, and denote the completion of this space by Q_2; this space is the weighted L^2 with the measure $dy = c(x)\,dx$. The corresponding extension $\mathcal{E}_2 : Q_2 \to Q_2'$ is the Riesz map for this Hilbert space. Define the domain $D \equiv \{p \in Q : \text{there exists a } \mathbf{u} \in V :\ \mathcal{A}\mathbf{u} + \mathcal{B}'p = \mathbf{0},\ -\mathcal{B}\mathbf{u} = \mathcal{E}_2 \bar{g} \text{ for some } \bar{g} \in Q_2\}$, and then set $Lp = \bar{g}$. This determines the operator L in Q_2 with domain $D \subset Q \subset Q_2$.

LEMMA 2.2. *If the matrix operator* $\left(\begin{smallmatrix} \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & \mathcal{E}_2 \end{smallmatrix}\right)$ *maps* $V \times Q$ onto *the product* $\{0\} \times Q_2'$, *then* L *is m-accretive and self-adjoint on* Q_2.

Proof. Let $p,\, q \in D$ with corresponding $\mathbf{u},\, \mathbf{v} \in V$. Then $(Lp, q)_{Q_2} = \mathcal{E}_2\, Lp(q) = -\mathcal{B}\mathbf{u}(q) = -\mathcal{B}' q(\mathbf{u}) = \mathcal{A}\mathbf{v}(\mathbf{u})$, so we have $(Lp, p)_{Q_2} \geq 0$. Also $(I + L)p = \bar{g}$ is equivalent to the system

$$\begin{aligned} \mathbf{u} \in V,\ p \in Q:\ \mathcal{A}\mathbf{u} + \mathcal{B}'p &= \mathbf{0} \text{ in } V', \\ -\mathcal{B}\mathbf{u} + \mathcal{E}_2 p &= \mathcal{E}_2 \bar{g} \text{ in } Q'. \qquad \square \end{aligned} \tag{9}$$

The preceding surjectivity condition on the matrix is essentially always true. In fact, the matrix operator is usually onto $V' \times Q_2'$.

Finally, we note that $p(t)$ is a solution of (5) with $h(t) = \bar{g}(t)$ in $H = Q_2$ if and only if (8) holds with $g(t) = \mathcal{E}_2 \bar{g}(t)$ in Q_2'.

COROLLARY 2.3. *Assume* $a(\cdot),\ c(\cdot) \in L^\infty(\Omega)$ *with* $a(x) \geq a_0 > 0$ *and* $c(x) > 0$ *for each* $x \in \Omega$. *For each* $p_0 \in Q_2$ *and Hölder continuous* $g \in C^\beta([0, \infty), Q_2')$, $0 < \beta < 1$, *there is a unique solution* $p \in C([0, \infty), Q_2) \cap C^1((0, \infty), Q_2)$ *of* (8) *with* $p(0) = p_0$.

Remark 2.1. This is the *parabolic* situation of Theorem 2.1. It does not require the *closed range condition* on $\mathcal{B}$ because of the presence of $\mathcal{E}_2$. However, this condition together with $c(x) \geq c_0 > 0$ will give additional estimates on $\|u(t)\|_{H^1}$ [20].

The same problem can be formulated on the spaces $V \equiv \mathbf{L}^2(\Omega)$ and $Q = H_0^1(\Omega)$ for which we define $\mathcal{A}$ and $\mathcal{E}_2$ as above, but

$$\mathcal{B}\mathbf{u}(q) = \int_\Omega \mathbf{u}(x)\, \boldsymbol{\nabla} q(x)\, dx, \quad \mathbf{u} \in V,\ q \in Q.$$

The space Q_2 for the evolution is the same as before.

2.2. Stokes flow. The *Stokes problem* is to find a pair of functions $\mathbf{u}$, p on the smoothly bounded region Ω in $\Re^3$ for $t > 0$ which satisfy the initial-boundary-value problem

$$\rho_0(x) \tfrac{\partial}{\partial t} \mathbf{u}(t) - \mu \Delta \mathbf{u}(t) + \boldsymbol{\nabla} p(t) = \mathbf{f}(t)\,, \tag{10a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u}(t) = 0 \text{ in } \Omega \tag{10b}$$

with $\mathbf{u}(t) = \mathbf{0}$ on $\Gamma = \partial\Omega$ for each $t > 0$ and the initial condition $u(x, 0) = u_0(x)$ in Ω. The vector-valued function $\mathbf{u}(x, t)$ is the velocity and $p(x, t)$ the pressure of an incompressible fluid flowing (slowly) in the region Ω.

We define the spaces $V = \mathbf{H}_0^1(\Omega)$ and $Q = \mathbf{L}^2(\Omega)$, continuous linear operators $\mathcal{A},\ \mathcal{E}_1 : V \to V'$ and $\mathcal{B} : V \to Q'$ by $\mathcal{E}_1\mathbf{u} = \rho_0(\cdot)\mathbf{u}\,,\ \mathcal{B}\mathbf{u} = -\boldsymbol{\nabla}\cdot\mathbf{u}$,

$$\mathcal{A}\mathbf{u}(\mathbf{v}) = \int_\Omega (\mu\boldsymbol{\nabla} u_i(x) \cdot \boldsymbol{\nabla} v_i(x))\, dx\,, \quad \mathbf{u},\ \mathbf{v} \in V,$$

and the linear functional $\mathbf{f}(\cdot)$ in V' by $\mathbf{f}(t)(\mathbf{v}) = \int_\Omega \mathbf{f}(x, t) \cdot \mathbf{v}(x)\, dx\,, \quad \mathbf{v} \in V$, where $\mathbf{f}(\cdot, t) \in \mathbf{L}^2(\Omega)$ is given for each $t > 0$. Then the initial-boundary-value problem for (10) is to find

$$\mathbf{u}(t) \in V,\ p(t) \in Q :$$

$$\mathcal{E}_1 \tfrac{d}{dt}\mathbf{u}(t) + \mathcal{A}\mathbf{u}(t) + \mathcal{B}'p(t) = \mathbf{f}(t) \text{ in } V', \tag{11a}$$

$$-\mathcal{B}\mathbf{u}(t) = 0 \text{ in } Q' \text{ for } t > 0 \tag{11b}$$

with $\mathbf{u}(0) = \mathbf{u}_0$.

Suppose $\mathcal{E}_1(\cdot)(\cdot)$ is a continuous scalar product on V; this is so if ρ_0 is bounded and everywhere positive. Denote the completion of this space by V_1; this is the weighted L^2 with the measure $dy = \rho(x)\, dx$, and $\mathcal{E}_1 : V_1 \to V_1'$ is the Riesz isomorphism. Then we define the domain $D \equiv \{\mathbf{u} \in V :$ there exists a $p \in Q :\ \mathcal{A}\mathbf{u} + \mathcal{B}'p = \mathcal{E}_1\bar{f} \in V',\ \mathcal{B}\mathbf{u} = 0$ for some $\bar{f} \in V_1\}$ and set $L\mathbf{u} = \bar{f}$. This defines the operator L in V_1 with domain $D \subset V \subset V_1$.

Lemma 2.4. *If the matrix operator* $\left(\begin{smallmatrix} \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & 0 \end{smallmatrix}\right)$ *maps* $V \times Q$ onto *the product* $V_1' \times \{0\}$, *then* L *is m-accretive and self-adjoint on* V_1.

Proof. If $\mathbf{u}, \mathbf{v} \in D$ with corresponding components $p, q \in Q$, then we have $(L\mathbf{u}, \mathbf{v})_{V_1} = \mathcal{E}_1\, L\mathbf{u}(\mathbf{v}) = \mathcal{A}\mathbf{u}(\mathbf{v}) + \mathcal{B}'p(\mathbf{v}) = \mathcal{A}\mathbf{u}(\mathbf{v})$, so L is self-adjoint and accretive. Also $(I + L)\mathbf{u} = \bar{f}$ is equivalent to the system

$$\mathbf{u} \in V,\ p \in Q :\ \mathcal{E}_1\mathbf{u} + \mathcal{A}\mathbf{u} + \mathcal{B}'p = \mathcal{E}_1\bar{f} \text{ in } V', \tag{12a}$$

$$-\mathcal{B}\mathbf{u} = 0 \text{ in } Q'. \quad \square \tag{12b}$$

Finally, we note that $\mathbf{u}(t)$ is a solution of (5) in $H = V_1$ if and only if (11) holds with $\mathcal{E}_1 h(t) = \mathbf{f}(t)$.

Corollary 2.5. *Assume* $\mu > 0$, $\rho_0 \in L^\infty(\Omega)$, *and* $\rho_0(x) > 0$ *for each* $x \in \Omega$. *Then for each* $\mathbf{u}_0 \in V_1$ *and Hölder continuous* $\mathbf{f} \in C^\beta([0, \infty), V_1')$, $0 < \beta < 1$, *there is a unique solution* $\mathbf{u} \in C([0, \infty), V_1) \cap C^1((0, \infty), V_1)$ *of* (11) *with* $\mathbf{u}(0) = \mathbf{u}_0$.

The surjectivity condition on the matrix is essentially always true. The *closed range condition* on $\mathcal{B} = -\boldsymbol{\nabla}\cdot :\ V \to Q'$ is necessary to obtain and estimate $p \in Q$ [29]. The system (11) is also *parabolic.*

3. The degenerate nonlinear system. A *multivalued operator* $\mathbb{N}$ from the Hilbert space H to its dual H' is a *relation* on $H \times H'$. We identify it with its graph, so $\mathbb{N} \subset H \times H'$, and we write $[\mathbf{v}, f] \in \mathbb{N}$ if and only if $f \in \mathbb{N}(\mathbf{v})$. The domain of $\mathbb{N}$ is $\mathrm{Dom}(\mathbb{N}) = \{\mathbf{v} \in H :\ \mathbb{N}(\mathbf{v}) \neq \emptyset\}$, and the range of $\mathbb{N}$ is the set $\mathrm{Rg}(\mathbb{N}) = \{f \in H' : [\mathbf{v}, f] \in \mathbb{N}$ for some $\mathbf{v} \in \mathrm{Dom}(\mathbb{N})\}$. The relation $\mathbb{N}$ is called *monotone* if $f_j \in \mathbb{N}(\mathbf{v}_j)$ for $j = 1, 2$ implies that $(f_1 - f_2)(\mathbf{v}_1 - \mathbf{v}_2) \geq 0$. It is called *maximal monotone* if there is no monotone relation in $H \times H'$ that strictly extends $\mathbb{N}$. A monotone operator is maximal if and only if $\mathcal{R} + \mathbb{N}$ is *onto* H', where $\mathcal{R}$ is the Riesz map of the Hilbert space H onto its dual H'. If we identify H with H' by the Riesz map, then we see that m-accretive operators on H are maximal monotone. A special class of maximal monotone operators arises as derivatives of convex functions. Let $\varphi : \mathbf{H} \to [0, +\infty]$ be proper, convex, and lower semicontinuous. The *subgradient* $\partial\varphi \subset H \times H'$ is a maximal monotone operator defined by a *variational inequality*

$$\partial\varphi(\mathbf{u}) = \{\mathbf{f} \in H' : \ \mathbf{f}(\mathbf{v} - \mathbf{u}) \le \varphi(\mathbf{v}) - \varphi(\mathbf{u}) \text{ for all } \mathbf{v} \in H\}.$$

Since φ is proper, this implies that $\varphi(\mathbf{u}) \in \Re$, so $\mathbf{u} \in \mathrm{Dom}(\varphi)$. With φ as given, the *dual convex function* $\varphi^* : H' \to (-\infty, +\infty]$ defined by $\varphi^*(f) = \sup_{\mathbf{v}\in H}\{f(\mathbf{v}) - \varphi(\mathbf{v})\}$ is proper, convex, and lower semicontinuous. Furthermore, $\partial\varphi^*$ is the inverse to $\partial\varphi$, and the following are equivalent: $f \in \partial\varphi(\mathbf{u})$, $\mathbf{u} \in \partial\varphi^*(f)$, and $f(\mathbf{u}) = \varphi(\mathbf{u}) + \varphi^*(f)$. See [8, 12, 25].

Let V and Q be real Hilbert spaces, and let there be given the operators $\mathcal{E}_1,\ \mathcal{A} : V \to V',\ \mathcal{B} : V \to Q',\ \mathcal{E}_2,\ \mathcal{C} : Q \to Q'$. As before, assume that $\mathcal{E}_1$ and $\mathcal{E}_2$ are continuous, linear, symmetric, and nonnegative and that $\mathcal{B}$ is continuous and linear. Let $\mathcal{A}$ be a multivalued operator from V to V' and $\mathcal{C}$ be a multivalued operator from Q to Q'. From these we construct the matrix operators

$$\mathbb{M} \equiv \begin{pmatrix} \mathcal{E}_1 & 0 \\ 0 & \mathcal{E}_2 \end{pmatrix}, \quad \mathbb{N} \equiv \begin{pmatrix} \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & \mathcal{C} \end{pmatrix}$$

from the Hilbert product space $\mathbb{H} \equiv V \times Q$ into its dual $\mathbb{H}' = V' \times Q'$. Then the system (1) is an implicit evolution equation of the form

$$\tfrac{d}{dt}\big(\mathbb{M}w(t)\big) + \mathbb{N}\big(w(t)\big) \ni h(t),\ \ t > 0, \tag{13}$$

with $h(t) = \binom{\mathbf{f}(t)}{g(t)}$ and $w(t) = \binom{\mathbf{u}(t)}{p(t)}$. The continuous linear, symmetric, and nonnegative operator $\mathbb{M} : \mathbb{H} \to \mathbb{H}'$ determines the semiscalar product

$$m(x, y) = \mathbb{M}x(y) = \mathcal{E}_1\mathbf{u}(\mathbf{v}) + \mathcal{E}_2 p(q), \quad x = (\mathbf{u}, p),\ y = (\mathbf{v}, q) \in \mathbb{H}\ ,$$

and we denote the corresponding seminorm space by $\mathbb{H}_m$. Its continuous dual $\mathbb{H}'_m$ is a Hilbert space, and we have the continuous imbedding $\mathbb{H}'_m \subset \mathbb{H}'$. A *solution* on $[0, T]$ of the semilinear equation (13) in $\mathbb{H}'_m$ is a function $w : [0, T] \to \mathbb{H}$ for which $\mathbb{M}w \in C([0, T], \mathbb{H}'_m)$, $\mathbb{M}w(\cdot)$ is absolutely continuous on each $[\delta, T]$, $0 < \delta < T$, hence, differentiable a.e., and (13) holds a.e. on $(0, T)$. The *Cauchy problem* for (13) is to find a solution w for which $(\mathbb{M}w)(0) = w_0$ is specified in $\mathbb{H}'_m$. We have the following result from Theorem IV.6.1 of [25].

THEOREM 3.1. *Let the linear, symmetric, and monotone operator* $\mathbb{M}$ *be given from the real vector space* $\mathbb{H}$ *to its algebraic dual* $\mathbb{H}^*$*, and let* $\mathbb{H}'_m$ *be the Hilbert space which is the continuous dual of* $\mathbb{H}$ *with the seminorm*

$$|x|_m = \mathbb{M}x(x)^{1/2}\ , \qquad x \in \mathbb{H}\ .$$

Let $\mathbb{N} \subset \mathbb{H} \times \mathbb{H}'_m$ *be a* relation *with domain* $\mathrm{Dom}(\mathbb{N}) = \{x \in \mathbb{H} : \mathbb{N}(x) \ne \emptyset\}$.

Assume $\mathbb{N}$ *is monotone. If* w_j *is a solution of*

$$\tfrac{d}{dt}\big(\mathbb{M}w_j(t)\big) + \mathbb{N}\big(w_j(t)\big) \ni h_j(t)\ , \qquad 0 < t < T\ ,$$

for $j = 1, 2$*, then it follows that*

$$|w_1(t) - w_2(t)|_m \le |w_1(0) - w_2(0)|_m + \int_0^t \|h_1(s) - h_2(s)\|_{\mathbb{H}'_m}\, ds,\ \ 0 \le t \le T\ .$$

If $h_1 = h_2$ *and* $\mathbb{M}w_1(0) = \mathbb{M}w_2(0)$*, then* $\mathbb{M}w_1(t) = \mathbb{M}w_2(t)$ *for all* $0 \le t \le T$*. Furthermore, if* $\mathbb{M} + \mathbb{N}$ *is strictly monotone, then there is at most one solution of the Cauchy problem for* (13)*.*

Assume $\mathbb{N}$ *is monotone and* $Rg(\mathbb{M} + \mathbb{N}) = \mathbb{H}'_m$*. Then for each* $w_0 \in D(\mathbb{N})$ *and each* $f \in W^{1,1}(0, T; \mathbb{H}'_m)$*, there is a solution* w *of* (13) *with* $\mathbb{M}w \in W^{1,\infty}(0, T; \mathbb{H}'_m)$*,* $w(t) \in D(\mathbb{N})$ *for all* $t \in [0, T]$*, and* $\mathbb{M}w(0) = \mathbb{M}w_0$*.*

The pair of operators $\mathcal{E}_1$, $\mathcal{E}_2$ determines the space $\mathbb{H}'_m$ for the dynamics of the system (1). The degeneracy of either $\mathcal{E}_1$ or $\mathcal{E}_2$ leads to *multivalued* operators or relations. Even when all operators are linear, the presence of a nontrivial kernel of the evolution operator $\mathbb{M}$ in the implicit evolution equation (13) necessitates the use of multivalued operators.

3.1. The range condition. In order to verify the range condition in our application of Theorem 3.1 to (1), we shall show that the resolvent equation

$$\mathbb{M}v + \mathbb{N}(v) \ni h$$

with $h = \binom{\mathbf{f}}{g} \in \mathbb{H}'_m$ and $v = \binom{\mathbf{u}}{p} \in \mathbb{H}$ is solved in the mixed form (17) below. As we shall see, the linearity of $\mathcal{A}$ and $\mathcal{C}$ does not play a role in the proofs: we permit them to be maximal monotone operators. Also we recover many of the estimates for the linear case which are delicate because they are independent of the properties of the operator $\mathcal{C}$. These estimates exploit the closed range condition on the coupling operator $\mathcal{B}$, and they always include the (most degenerate) case $\mathcal{C} = \mathcal{E}_2 = 0$.

THEOREM 3.2. *Let* V *and* Q *be real Hilbert spaces, and let there be given the operators* $\mathcal{E}_1,\ \mathcal{A} : V \to V',\ \mathcal{B} : V \to Q',\ \mathcal{E}_2,\ \mathcal{C} : Q \to Q'$*. Assume* $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *are continuous, linear, symmetric, and nonnegative,* $\mathcal{B}$ *is continuous and linear, and* $\mathcal{A}$ *and* $\mathcal{C}$ *are maximal monotone with (for simplicity)* $\mathcal{A}(0) \ni 0,\ \mathcal{C}(0) \ni 0$*. Denote by* V_1 *and* Q_2 *the spaces* V *and* Q *with the respective semiscalar products arising from* $\mathcal{E}_1$ *and* $\mathcal{E}_2$*, and let their (Hilbert space) duals be designated by* V_1' *and* Q_2'*. Assume the following:*

(i) *The operators* $\mathcal{E}_1$*,* $\mathcal{A}$*, and* $\mathcal{B}$ *satisfy*

$$\lim_{\substack{\|\mathbf{u}\|_V + \|\xi\|_{V'} \to +\infty \\ \text{with } \xi \in \mathcal{A}(\mathbf{u})}} \left(|\mathbf{u}|^2_{V_1} + \xi(\mathbf{u}) + \|\mathcal{B}\mathbf{u}\|^2_{Q'} \right) = +\infty. \tag{14}$$

(ii) *The operator* $\mathcal{B} : V \to Q'$ *has* closed range.
(Then the same holds for $\mathcal{B}' : Q \to V'$*.)*

(iii) *The operators* $\mathcal{E}_2$ *and* $\mathcal{C}$ *satisfy*

$$\lim_{\substack{\|\eta\|_{Q'} \to +\infty \\ \text{with } \eta \in \mathcal{C}(p)}} \left(|p|^2_{Q_2} + \eta(p) \right) = +\infty. \tag{15}$$

(iv) *There is a constant* K *such that for every* $h \in Q'$ *and* $\varepsilon > 0$*, the condition*

$$p \in \operatorname{Ker} \mathcal{B}' : \ (\varepsilon \mathcal{R}_2 + \mathcal{E}_2) p(q) = h(q) \ \text{for all } q \in \operatorname{Ker} \mathcal{B}' \tag{16}$$

implies that $\|p\|_Q \le K \|h\|_{Q'}$*.*

Then the resolvent system

$$\mathbf{u} \in V,\ p \in Q :$$

$$\mathcal{E}_1 \mathbf{u} + \xi + \mathcal{B}' p = \mathbf{f},\ \xi \in \mathcal{A}(\mathbf{u}) \ \text{in } V', \tag{17a}$$

$$\mathcal{E}_2 p - \mathcal{B}\mathbf{u} + \eta = g,\ \eta \in \mathcal{C}(p) \ \text{in } Q' \tag{17b}$$

has a solution for each pair $\mathbf{f} \in V_1'$ *and* $g \in Q_2'$.

Remark 3.1. The condition (14) holds if $\mathcal{A}$ is *bounded*, i.e., maps bounded sets into bounded sets, and satisfies the *growth condition*

$$\lim_{\substack{\|\mathbf{u}\|_V \to +\infty \\ \xi \in \mathcal{A}\mathbf{u}}} (\xi(\mathbf{u}) + \|\mathcal{B}\mathbf{u}\|_{Q'}^2) = +\infty \,.$$

Furthermore, if $\mathcal{A}$ is a subgradient, $\mathcal{A} = \partial\varphi$, then (14) is equivalent to

$$\lim_{\substack{\|\mathbf{u}\|_V + \|\xi\|_{V'} \to +\infty \\ \text{with } \xi \in \mathcal{A}(\mathbf{u})}} (|\mathbf{u}|_{V_1}^2 + \varphi(\mathbf{u}) + \varphi^*(\xi) + \|\mathcal{B}\mathbf{u}\|_{Q'}^2) = +\infty \,.$$

Remark 3.2. The estimate (15) is a growth condition on $\mathcal{C}^{-1}$. It holds trivially if $\mathrm{Rg}(\mathcal{C})$ is bounded, e.g., if $\mathcal{C} = 0$. If $\mathcal{C}$ is a subgradient, $\mathcal{C} = \partial\psi$, then (15) follows from the condition

$$\lim_{\substack{\|\eta\|_{Q'} \to +\infty \\ \text{with } \eta \in \mathcal{C}(p)}} (|p|_{Q_2}^2 + \psi^*(\eta)) = +\infty \,.$$

Moreover, the subgradient $\mathcal{C}$ is bounded (hence, $\mathrm{Dom}(\mathcal{C}) = Q$) if and only if

$$\lim_{\substack{\|\eta\|_{Q'} \to +\infty \\ \eta \in \mathcal{C}(p)}} \frac{\eta(p)}{\|\eta\|_{Q'}} = +\infty \,,$$

and this implies (15).

Remark 3.3. The condition (16) holds if $\mathcal{E}_2$ is Q-coercive on $\operatorname{Ker}\mathcal{B}'$. In particular, it is sufficient for $\operatorname{Ker}\mathcal{B}' = \{0\}$ or for $\operatorname{Ker}\mathcal{B}' = \Re$ and $\mathcal{E}_2(1)(1) > 0$. In the situation of (16), we always have the weaker condition $|p|_{Q_2} \leq \|h\|_{Q_2'}$ for $h \in Q_2'$. If $\mathcal{C} = 0$, then the intermediate condition $\|p\|_Q \leq \|h\|_{Q_2'}$ is sufficient (see Theorem 3.3 below), and this is trivially satisfied if $\mathcal{E}_2 = 0$. But then $g = 0$.

Proof. The diagonal matrix $\left(\begin{smallmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{C} \end{smallmatrix}\right)$ is maximal monotone as an operator from $V \times Q$ to $V' \times Q'$, and the antisymmetric matrix $\left(\begin{smallmatrix} 0 & \mathcal{B}' \\ -\mathcal{B} & 0 \end{smallmatrix}\right)$is Lipschitz continuous, so their sum is maximal monotone. See Lemma IV.2.1 in [25].

Note that $|v|_{V_1}^2 = \mathcal{E}_1 v(v) = \|\mathcal{E}_1 v\|_{V_1'}^2$ for all $v \in V$, and similarly $|w|_{Q_2}^2 = \mathcal{E}_2 w(w) = \|\mathcal{E}_2 w\|_{Q_2'}^2$ for all $w \in Q$. Moreover, we have $\|\mathbf{f}\|_{V'} \leq K_1 \|\mathbf{f}\|_{V_1'}$ for $\mathbf{f} \in V_1' \subset V'$ and $\|g\|_{Q'} \leq K_2 \|g\|_{Q_2'}$ for $g \in Q_2' \subset Q'$ because $\mathcal{E}_1$ and $\mathcal{E}_2$ are continuous.

Let $\mathcal{R}_1$ and $\mathcal{R}_2$ denote the Riesz isomorphisms of the Hilbert spaces V and Q onto their respective duals. For each $\varepsilon > 0$, there is a unique solution $\mathbf{u}_\varepsilon \in V,\ p_\varepsilon \in Q$ of the *regularized system*

$$\mathbf{u}_\varepsilon \in V,\ p_\varepsilon \in Q,\ \xi_\varepsilon \in \mathcal{A}(\mathbf{u}_\varepsilon),\ \eta_\varepsilon \in \mathcal{C}(p_\varepsilon):$$

$$(\varepsilon\mathcal{R}_1 + \mathcal{E}_1)\mathbf{u}_\varepsilon + \xi_\varepsilon + \mathcal{B}' p_\varepsilon = \mathbf{f} \text{ in } V', \tag{18a}$$

$$-\mathcal{B}\mathbf{u}_\varepsilon + (\varepsilon\mathcal{R}_2 + \mathcal{E}_2)p_\varepsilon + \eta_\varepsilon = g \text{ in } Q' \tag{18b}$$

because it is maximal monotone and coercive over $V \times Q$. The solution satisfies the estimate

$$\begin{aligned} &\varepsilon\|\mathbf{u}_\varepsilon\|_V^2 + |\mathbf{u}_\varepsilon|_{V_1}^2 + \varepsilon\|p_\varepsilon\|_Q^2 + |p_\varepsilon|_{Q_2}^2 \\ &\quad \leq (\varepsilon\mathcal{R}_1 + \mathcal{E}_1)\mathbf{u}_\varepsilon(\mathbf{u}_\varepsilon) + \xi_\varepsilon(\mathbf{u}_\varepsilon) + (\varepsilon\mathcal{R}_2 + \mathcal{E}_2)p_\varepsilon(p_\varepsilon) + \eta_\varepsilon(p_\varepsilon) \\ &\qquad \leq \|\mathbf{f}\|_{V_1'}|\mathbf{u}_\varepsilon|_{V_1} + \|g\|_{Q_2'}|p_\varepsilon|_{Q_2} \end{aligned} \tag{19}$$

since $\xi_\varepsilon(\mathbf{u}_\varepsilon) \geq 0$ and $\eta_\varepsilon(p_\varepsilon) \geq 0$, and so every term in (19) is bounded (independent of ε). Then from (15) we see that $\{\|\eta_\varepsilon\|_{Q'}\}$ is bounded, and from (18) we see that

$$\|\mathcal{B}\mathbf{u}_\varepsilon\|_{Q'} \leq \varepsilon\|\mathcal{R}_2 p_\varepsilon\|_{Q'} + \|\mathcal{E}_2 p_\varepsilon\|_{Q'_2} + \|\eta_\varepsilon\|_{Q'} + \|g\|_{Q'} \tag{20}$$

is bounded. From (19) and (20) we conclude that $\{|\mathbf{u}_\varepsilon|^2_{V_1} + \xi_\varepsilon(\mathbf{u}_\varepsilon) + \|\mathcal{B}\mathbf{u}_\varepsilon\|^2_{Q'}\}$ is bounded uniformly in ε, and so from (14) we conclude that $\{\|\mathbf{u}_\varepsilon\|_V\}$ and $\{\|\xi_\varepsilon\|_{V'}\}$ are uniformly bounded.

Next we seek estimates on $\|p_\varepsilon\|_Q$. Write each of these in Q-orthogonal parts, $p_\varepsilon = p^0_\varepsilon + p^1_\varepsilon$ with $p^0_\varepsilon \in \operatorname{Ker}\mathcal{B}'$ and $p^1_\varepsilon \in (\operatorname{Ker}\mathcal{B}')^\perp$. From the closed range condition, we obtain the estimate

$$\|p^1_\varepsilon\|_Q \leq K_B\|\mathcal{B}'p_\varepsilon\|_{V'} \leq K_B\big(\|\mathbf{f}\|_{V'} + \|\xi_\varepsilon\|_{V'} + |\mathbf{u}_\varepsilon|_{V_1} + \varepsilon\|\mathbf{u}_\varepsilon\|_V\big)\,,$$

and this shows that $\{\|p^1_\varepsilon\|_Q\}$ is bounded. Following [9], we note from (18) that $p^0_\varepsilon \in \operatorname{Ker}\mathcal{B}'$ satisfies

$$(\varepsilon\mathcal{R}_2 + \mathcal{E}_2)p^0_\varepsilon(q) = -\mathcal{E}_2 p^1_\varepsilon(q) + g(q) - \eta_\varepsilon(q) \text{ for all } q \in \operatorname{Ker}\mathcal{B}'.$$

Then from the estimates above, it follows that p^0_ε satisfies (16), and so $\{\|p_\varepsilon\|_Q\}$ is uniformly bounded.

Now we can establish the existence of a solution of (17). By passing to a subsequence, we obtain the weak limits

$$\mathbf{u}_\varepsilon \overset{w}{\rightharpoonup} \mathbf{u} \text{ in } V,\ p_\varepsilon \overset{w}{\rightharpoonup} p \text{ in } Q,\ \xi_\varepsilon \overset{w}{\rightharpoonup} \xi \text{ in } V',\ \eta_\varepsilon \overset{w}{\rightharpoonup} \eta \text{ in } Q'$$

as $\varepsilon \to 0$ which satisfy

$$\mathcal{E}_1\mathbf{u} + \xi + \mathcal{B}'p = \mathbf{f} \text{ in } V', \tag{21a}$$

$$-\mathcal{B}\mathbf{u} + \mathcal{E}_2 p + \eta = g \text{ in } Q'. \tag{21b}$$

Then we use (18), (21), and lower semicontinuity to compute

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\big(\xi_\varepsilon(\mathbf{u}_\varepsilon) + \eta_\varepsilon(p_\varepsilon)\big)\\
&\qquad = \limsup_{\varepsilon\to 0}\{(\mathbf{f} - \varepsilon\mathcal{R}_1\mathbf{u}_\varepsilon - \mathcal{E}_1\mathbf{u}_\varepsilon)(\mathbf{u}_\varepsilon) + (g - \varepsilon\mathcal{R}_2 p_\varepsilon - \mathcal{E}_2 p_\varepsilon)(p_\varepsilon)\}\\
&\qquad\qquad \leq f(\mathbf{u}) - \mathcal{E}_1\mathbf{u}(\mathbf{u}) + g(p) - \mathcal{E}_2 p(p) = \xi(\mathbf{u}) + \eta(p).
\end{aligned}$$

Since $\left(\begin{smallmatrix}\mathcal{A} & 0\\ 0 & \mathcal{C}\end{smallmatrix}\right)$ is maximal monotone, this shows that $\mathcal{A}(\mathbf{u}) \ni \xi$ and $\mathcal{C}(p) \ni \eta$, and so $\mathbf{u}$, p is a solution of (17). $\square$

Remark 3.4. It is often the case that (14) is obtained from a condition

$$|\mathbf{u}|^2_{V_1} + \xi(\mathbf{u}) + \|\mathcal{B}\mathbf{u}\|^2_{Q'} \geq c_0\|\mathbf{u}\|^2_V - c_1 \text{ for all } \xi \in \mathcal{A}(\mathbf{u}),\ \mathbf{u} \in V.$$

In this case we can replace $\mathcal{A}$ by $\mathcal{A} - \frac{c_0}{2}\mathcal{R}_1$ and $\mathcal{E}_1$ by $\mathcal{E}_1 + \frac{c_0}{2}\mathcal{R}_1$ for which the dual space is given by $V'_1 = V'$. Then Theorem 3.2 applies with $\mathbf{f} \in V'$.

If $\mathcal{C}$ similarly dominates some norm on Q, that part may be moved to $\mathcal{E}_2$ in order to make Q'_2 larger and to facilitate (16).

For the frequently occurring case of $\mathcal{C} = 0$, we obtain the following variation to which Remark 3.4 applies, and also g can be taken as more general.

Theorem 3.3. *Let V and Q be real Hilbert spaces, and let there be given the operators $\mathcal{E}_1,\ \mathcal{A} : V \to V',\ \mathcal{B} : V \to Q',\ \mathcal{E}_2 : Q \to Q'$. Assume $\mathcal{E}_1$ and $\mathcal{E}_2$ are* *continuous, linear, symmetric, and nonnegative, $\mathcal{B}$ is continuous and linear, and $\mathcal{A}$ is maximal monotone with $\mathcal{A}(0) \ni 0$. Denote by V_1 and Q_2 the spaces V and Q with the respective semiscalar products arising from $\mathcal{E}_1$ and $\mathcal{E}_2$ and their (Hilbert space) duals by V_1' and Q_2'. Assume the following:*

(i) *For every $\mathbf{u}_0 \in \mathrm{Dom}(\mathcal{A})$, the operators $\mathcal{E}_1$, $\mathcal{A}$, and $\mathcal{B}$ satisfy*

$$\lim_{\substack{\|\mathbf{u}\|_V + \|\xi\|_{V'} \to +\infty \\ \text{with } \xi \in \mathcal{A}\mathbf{u}}} \frac{|\mathbf{u}|_{V_1}^2 + \xi(\mathbf{u}-\mathbf{u}_0) + \|\mathcal{B}\mathbf{u}\|_{Q'}^2}{\|\mathbf{u}\|_V + 1} = +\infty\,. \tag{22}$$

(ii) *The operator $\mathcal{B} : V \to Q'$ has* closed range.

(iii) *There is a constant K such that for every $h \in Q_2'$ and $\varepsilon > 0$, the condition*

$$p \in \mathrm{Ker}\,\mathcal{B}' : \;\; (\varepsilon\mathcal{R}_2 + \mathcal{E}_2)p(q) = h(q) \text{ for all } q \in \mathrm{Ker}\,\mathcal{B}' \tag{23}$$

implies that $\|p\|_Q \le K\|h\|_{Q_2'}$. Then the resolvent system

$$\mathbf{u} \in V,\; p \in Q :$$

$$\mathcal{E}_1\mathbf{u} + \xi + \mathcal{B}'p = \mathbf{f},\;\; \xi \in \mathcal{A}(\mathbf{u}) \text{ in } V', \tag{24a}$$

$$\mathcal{E}_2 p - \mathcal{B}\mathbf{u} = g \text{ in } Q' \tag{24b}$$

has a solution for each pair $\mathbf{f} \in V_1'$ and $g \in \mathcal{B}(\mathrm{Dom}(\mathcal{A})) + Q_2'$.

Proof. We approximate as before with a regularized system but adjust the a priori estimates. Let g be given by $g = -\mathcal{B}(\mathbf{u}_g) + g_2$ with $g_2 \in Q_2'$ for some $\mathbf{u}_g \in \mathrm{Dom}(\mathcal{A})$. Choose $\xi_g \in \mathcal{A}(\mathbf{u}_g)$. Subtract these in the regularized system as indicated to get

$$\begin{aligned} &\mathbf{u}_\varepsilon \in V,\; p_\varepsilon \in Q,\; \xi_\varepsilon \in \mathcal{A}(\mathbf{u}_\varepsilon) : \\ &(\varepsilon\mathcal{R}_1 + \mathcal{E}_1)(\mathbf{u}_\varepsilon - \mathbf{u}_g) + \xi_\varepsilon - \xi_g + \mathcal{B}'p_\varepsilon = \mathbf{f} - \xi_g - (\varepsilon\mathcal{R}_1 + \mathcal{E}_1)\mathbf{u}_g \text{ in } V', \\ &-\mathcal{B}(\mathbf{u}_\varepsilon - \mathbf{u}_g) + (\varepsilon\mathcal{R}_2 + \mathcal{E}_2)p_\varepsilon = g_2 \text{ in } Q'. \end{aligned}$$

Apply the first equation to $\mathbf{u}_\varepsilon - \mathbf{u}_g$ and the second to p_ε, and then add to obtain the estimate

$$\begin{aligned} &\varepsilon\|\mathbf{u}_\varepsilon - \mathbf{u}_g\|_V^2 + |\mathbf{u}_\varepsilon - \mathbf{u}_g|_{V_1}^2 + \varepsilon\|p_\varepsilon\|_Q^2 + |p_\varepsilon|_{Q_2}^2 + (\xi_\varepsilon - \xi_g)(\mathbf{u}_\varepsilon - \mathbf{u}_g) \\ &\quad\le \|\mathbf{f} - \xi_g - \varepsilon\mathcal{R}_1\mathbf{u}_g\|_{V'}\|\mathbf{u}_\varepsilon - \mathbf{u}_g\|_V + \|\mathcal{E}_1\mathbf{u}_g\|_{V_1'}|\mathbf{u}_\varepsilon - \mathbf{u}_g|_{V_1} + \|g_2\|_{Q_2'}|p_\varepsilon|_{Q_2}\,. \end{aligned} \tag{25}$$

Also we have $\|\mathcal{B}(\mathbf{u}_\varepsilon - \mathbf{u}_g)\|_{Q'} \le \|\varepsilon p_\varepsilon\|_Q + |p_\varepsilon|_{Q_2} + \|g_2\|_{Q_2'}$, so adding a small multiple of the square gives the estimate

$$\begin{aligned} &\varepsilon\|\mathbf{u}_\varepsilon - \mathbf{u}_g\|_V^2 + |\mathbf{u}_\varepsilon - \mathbf{u}_g|_{V_1}^2 + \varepsilon\|p_\varepsilon\|_Q^2 + |p_\varepsilon|_{Q_2}^2 \\ &\qquad + (\xi_\varepsilon - \xi_g)(\mathbf{u}_\varepsilon - \mathbf{u}_g) + \delta\|\mathcal{B}(\mathbf{u}_\varepsilon - \mathbf{u}_g)\|_{Q'}^2 \\ &\quad\le \|\mathbf{f} - \xi_g - \varepsilon\mathcal{R}_1\mathbf{u}_g\|_{V'}\|\mathbf{u}_\varepsilon - \mathbf{u}_g\|_V + \|\mathcal{E}_1\mathbf{u}_g\|_{V_1'}|\mathbf{u}_\varepsilon - \mathbf{u}_g|_{V_1} \\ &\qquad + \|g_2\|_{Q_2'}|p_\varepsilon|_{Q_2} + \delta C_0\left(\|g_2\|_{Q_2'}^2 + \|\varepsilon p_\varepsilon\|_Q^2 + |p_\varepsilon|_{Q_2}^2\right). \end{aligned} \tag{26}$$

By taking $\delta > 0$ small enough, subtracting the last two terms terms on the right side, and then dropping some positive terms from the left side, we obtain

$$|\mathbf{u}_\varepsilon - \mathbf{u}_g|_{V_1}^2 + (\xi_\varepsilon - \xi_g)(\mathbf{u}_\varepsilon - \mathbf{u}_g) + \delta\|\mathcal{B}(\mathbf{u}_\varepsilon - \mathbf{u}_g)\|_{Q'}^2 \le C_1\|\mathbf{u}_\varepsilon - \mathbf{u}_g\|_V + C_2\,,$$

and there follows then

$$|\mathbf{u}_\varepsilon|^2_{V_1} + \xi_\varepsilon(\mathbf{u}_\varepsilon - \mathbf{u}_g) + \|\mathcal{B}\mathbf{u}_\varepsilon\|^2_{Q'} \le \tilde{C}_1\|\mathbf{u}_\varepsilon\|_V + \tilde{C}_2\,.$$

From (22) it follows that $\|\mathbf{u}_\varepsilon\|_V$ and $\|\xi_\varepsilon\|_{V'}$ are bounded. From the regularized system we find $\|\mathcal{B}'p_\varepsilon\|_{V'}$ is bounded, and hence, as before writing $p_\varepsilon = p^0_\varepsilon + p^1_\varepsilon$ in orthogonal components, $\{p^1_\varepsilon\}$ is bounded in Q. Then (23) shows that $\{p^0_\varepsilon\}$ is bounded in Q, and we finish as before. □

Remark 3.5. The estimate (22) holds if $\mathcal{A}$ is bounded and

$$\lim_{\substack{\|\mathbf{u}\|_V\to+\infty\\ \text{with } \xi\in\mathcal{A}\mathbf{u}}} \frac{|\mathbf{u}|^2_{V_1} + \xi(\mathbf{u}) + \|\mathcal{B}\mathbf{u}\|^2_{Q'}}{\|\mathbf{u}\|_V} = +\infty. \tag{27}$$

3.2. The Cauchy problem. Let the operators $\mathcal{E}_1,\ \mathcal{E}_2$ be given as in Theorem 3.2 and $\mathcal{B} : V \to Q'$ be linear. Assume $\mathcal{A}$ and $\mathcal{C}$ are relations on $V \times V'$ and $Q \times Q'$, respectively. A *solution* of the system

$$\tfrac{d}{dt}(\mathcal{E}_1\mathbf{u}(t)) + \mathcal{A}\mathbf{u}(t) + \mathcal{B}'p(t) \ni \mathbf{f}(t) \text{ in } V_1', \tag{28a}$$

$$\tfrac{d}{dt}(\mathcal{E}_2 p(t)) - \mathcal{B}\mathbf{u}(t) + \mathcal{C}p(t) \ni g(t) \text{ in } Q_2' \tag{28b}$$

is a pair of functions $\mathbf{u} : [0,T] \to V,\ p : [0,T] \to Q$ for which $\mathcal{E}_1\mathbf{u}(\cdot) \in C([0,T], V_1')$, $\mathcal{E}_2 p(\cdot) \in C([0,T], Q_2')$, $\mathcal{E}_1\mathbf{u}(\cdot)$ and $\mathcal{E}_2p(\cdot)$ are absolutely continuous on each $[\delta, T]$, $0 < \delta < T$, and hence differentiable a.e., and (28) holds a.e. on $(0,T)$. The *Cauchy problem* for (28) is to find a solution $\mathbf{u}(\cdot),\ p(\cdot)$ for which $(\mathcal{E}_1\mathbf{u})(0) = \mathbf{u}^0$ in V_1' and $(\mathcal{E}_2p)(0) = p^0$ in Q_2' are specified.

Define the operator $\mathbb{N}$ by $\binom{\mathbf{f}}{g} \in \mathbb{N}\binom{\mathbf{u}}{p}$ if $\mathbf{u} \in V,\ p \in Q,\ \mathbf{f} \in V_1',\ g \in Q_2'$, $\mathbf{f} - \mathcal{B}'p \in \mathcal{A}(\mathbf{u})$, and $g + \mathcal{B}\mathbf{u} \in \mathcal{C}(p)$. The set of all such pairs $[\mathbf{u},\ p]$ is the *domain* of $\mathbb{N}$, $D(\mathbb{N})$. Then the resolvent equation

$$\mathbb{M}w + \mathbb{N}(w) \ni h$$

with $h = \binom{\mathbf{f}}{g} \in \mathbb{H}_m'$ and $w = \binom{\mathbf{u}}{p} \in \mathbb{H}$ is precisely the resolvent system (17). As a consequence of Theorem 3.2, we have the following theorem.

THEOREM 3.4. *Let the operators $\mathcal{E}_1,\ \mathcal{E}_2$ be given as in Theorem 3.2. Assume $\mathcal{B} : V \to Q'$ is linear and that $\mathcal{A}$ and $\mathcal{C}$ are monotone operators. If $\mathbf{u}_j(\cdot),\ p_j(\cdot)$ are solutions of (28) with $\mathbf{f}(\cdot) = \mathbf{f}_j(\cdot),\ g(\cdot) = g_j(\cdot)$ for $j = 1, 2$, then it follows that*

$$\begin{aligned}
&|\mathbf{u}_1(t) - \mathbf{u}_2(t)|_{V_1} + |p_1(t) - p_2(t)|_{Q_2} \le \\
&\qquad |\mathbf{u}_1(0) - \mathbf{u}_2(0)|_{V_1} + |p_1(0) - p_2(0)|_{Q_2} + \\
&\qquad \int_0^t \left(\|\mathbf{f}_1(s) - \mathbf{f}_2(s)\|_{V_1'} + \|g_1(s) - g_2(s)\|_{Q_2'}\right) ds\ , \quad 0 \le t \le T\ .
\end{aligned}$$

If $\mathbf{f}_1(\cdot) = \mathbf{f}_2(\cdot),\ g_1(\cdot) = g_2(\cdot),\ \mathcal{E}_1\mathbf{u}_1(0) = \mathcal{E}_1\mathbf{u}_2(0)$, and $\mathcal{E}_2p_1(0) = \mathcal{E}_2p_2(0)$, then $\mathcal{E}_1\mathbf{u}_1(t) = \mathcal{E}_1\mathbf{u}_2(t)$, and $\mathcal{E}_2p_1(t) = \mathcal{E}_2p_2(t)$ for all $0 \le t \le T$. If also $\mathcal{E}_1 + \mathcal{A}$ is strictly monotone, then $\mathbf{u}_1 = \mathbf{u}_2$; if $\mathcal{E}_2 + \mathcal{C}$ is strictly monotone, then $p_1 = p_2$.

Assume all the conditions of Theorem 3.2. Then for each pair $[\mathbf{u}_0,\ p_0]$ in the domain $D(\mathbb{N})$ of $\mathbb{N}$ and each pair $\mathbf{f}(\cdot) \in W^{1,1}(0,T;V_1'),\ g(\cdot) \in W^{1,1}(0,T;Q_2')$, there exists a solution $\mathbf{u} : [0,T] \to V,\ p : [0,T] \to Q$ of (28) for which $\mathcal{E}_1\mathbf{u}(\cdot) \in W^{1,\infty}(0,T;V_1'),\ \mathcal{E}_2p(\cdot) \in W^{1,\infty}(0,T;Q_2'),\ [\mathbf{u}(t), p(t)] \in D(\mathbb{N})$ for all $t \in [0,T]$, and $\mathcal{E}_1\mathbf{u}(0) = \mathcal{E}_1\mathbf{u}_0,\ \mathcal{E}_2p(0) = \mathcal{E}_2p_0$.

4. The Darcy–Stokes coupled system. Let the disjoint regions Ω_1 and Ω_2 in $\Re^3$ share the common *interface*, $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$. The first region Ω_1 is the fully saturated *porous matrix* structure, and the second region Ω_2 is an adjacent *chamber*. The *flux* $\mathbf{u}^1(x,t)$ is the Darcy velocity of seepage flow at $x \in \Omega_1$, and $\mathbf{u}^2(x,t)$ is the velocity of the *fluid* at $x \in \Omega_2$. The *fluid pressure* in Ω_1 is $p^1(x,t)$ and in the adjacent region Ω_2 is $p^2(x,t)$.

We write the constitutive equations together with the conservation equations for momentum and mass as a system of first-order partial differential equations in each of the two regions,

$$\frac{\partial}{\partial t}(\rho_1 \mathbf{u}^1) + \kappa^{-1}\,\mathfrak{a}(\mathbf{u}^1) + \boldsymbol{\nabla} p^1 = \mathbf{f}^1\,, \tag{29a}$$

$$\frac{\partial}{\partial t}\big(c_1\, p^1\big) + \boldsymbol{\nabla}\cdot \mathbf{u}^1 + \mathfrak{c}(p^1) = g^1 \quad \text{ in } \Omega_1 \tag{29b}$$

and

$$\frac{\partial}{\partial t}(\rho_2 \mathbf{u}^2) - \boldsymbol{\nabla}\cdot\, \mathfrak{b}(\boldsymbol{\epsilon}(\mathbf{u}^2)) + \boldsymbol{\nabla} p^2 = \mathbf{f}^2\,, \tag{29c}$$

$$\frac{\partial}{\partial t}(c_2 p^2) + \boldsymbol{\nabla}\cdot \mathbf{u}^2 = g^2 \quad \text{ in } \Omega_2\,, \tag{29d}$$

in which κ^{-1} is the inverse of permeability (times viscosity μ) and the fluid stress tensor is given by $\sigma^2 = \mathfrak{b}(\boldsymbol{\epsilon}(\mathbf{u}^2))$. The nonlinear Darcy's law (29a) describes the fluid flow in the porous medium, possibly of dynamic or Forchheimer type [11, 23, 30], and the fluid may likewise have a nonlinear dependence of stress on strain rate [13]. The term $\mathfrak{c}(p^1)$ is a pressure-dependent distributed loss. We shall assume for the moment that $\mathfrak{a}(\cdot)$, $\mathfrak{b}(\cdot)$, and $\mathfrak{c}(\cdot)$ are (single-valued) functions, but this will be relaxed below.

Interface conditions describe the exchange of fluid and stress between the domains. They are given by

$$u_n^1 = u_n^2\,, \tag{30a}$$

$$\sigma_n^2 - p^2 + p^1 = \alpha(u_n^1)\,, \text{ and} \tag{30b}$$

$$\sigma_T^2 = \sqrt{\kappa^{-1}}\,\beta(\mathbf{u}_T^2) \text{ on } \Gamma. \tag{30c}$$

Let $\mathbf{n}$ be the unit outward normal vector on $\partial\Omega_1 \cup \partial\Omega_2$, except on Γ, where it is directed out of Ω_1. For the vector $\mathbf{u}$ in (30), we have denoted by $u_n = \mathbf{u}\cdot\mathbf{n}$ the normal projection and by $\mathbf{u}_T = \mathbf{u} - u_n\mathbf{n}$ the tangential component. Similar notation is used for tensors. In particular, the normal component of normal stress $\sigma^2(\mathbf{n})$ on $\partial\Omega_2$ is denoted by $\sigma_n^2 = \sigma^2(\mathbf{n})\cdot\mathbf{n}$, and the tangential component of normal stress on Γ is $\sigma_T^2 = \sigma^2(\mathbf{n}) - \sigma_n^2\mathbf{n}$. The conservation of fluid across the interface is (30a). The balance of normal stress is (30b), where the nonlinear $\alpha(u_n^1) \geq 0$ denotes a resistance to normal flow across Γ, and a similar extension of the Beavers–Joseph–Saffman condition (30c) describes a possibly nonlinear resistance to tangential flow along the interface. As before, we shall assume for the moment that $\alpha(\cdot)$ and $\beta(\cdot)$ are functions.

For the exterior boundary conditions, we assume null flux, $\mathbf{u}\cdot\mathbf{n} = 0$, on the Darcy boundary, $\partial\Omega^1 - \Gamma$, and null velocity, $\mathbf{u} = 0$, on the Stokes boundary, $\partial\Omega^2 - \Gamma$.

4.1. The stationary problem. Let $\mathbf{L}^2(\Omega)$ denote the space of vector-valued functions with components in the usual Lebesgue space $L^2(\Omega)$ and $\mathbf{H}^1(\Omega)$ the corresponding vector-valued functions with components in the Sobolev space $H^1(\Omega)$. For functions $u \in L^2(\Omega)$, we denote with superscripts $j = 1, 2$ the restriction of u to Ω_1 or Ω_2, respectively. That is, $u^1 \equiv u|_{\Omega_1}$, $u^2 \equiv u|_{\Omega_2}$, and likewise for vector-valued functions. If $u^1 \in H^1(\Omega_1)$ and $u^2 \in H^1(\Omega_2)$, then each of these has a well-defined *trace*,

$\gamma(u^1) \in H^{\frac{1}{2}}(\partial\Omega_1)$ and $\gamma(u^2) \in H^{\frac{1}{2}}(\partial\Omega_2)$. These are generally different on Γ. On the space $\mathbf{L}^2_{div}(\Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \ \boldsymbol{\nabla} \cdot \mathbf{v} \in L^2(\Omega)\}$, the *normal trace* has a well-defined value. In particular, $\mathbf{v}^1 \in \mathbf{L}^2_{div}(\Omega_1)$ and $\mathbf{v}^2 \in \mathbf{L}^2_{div}(\Omega_2)$, so $\mathbf{v}^1 \cdot \mathbf{n}$ is in the dual space $H^{-\frac{1}{2}}(\partial\Omega_1)$, $\mathbf{v}^2 \cdot \mathbf{n}$ is in $H^{-\frac{1}{2}}(\partial\Omega_2)$, and the condition (30a) holds in $H^{-\frac{1}{2}}(\Gamma)$ for any $\mathbf{u} \in \mathbf{L}^2_{div}(\Omega)$.

We want an appropriate variational formulation of the *stationary* boundary-value problem corresponding to the Darcy–Stokes system (29) with interface conditions (30) and the exterior boundary conditions above. For this purpose, we define the space $\mathbf{V} \equiv \{\mathbf{v} \in \mathbf{L}^2_{div}(\Omega) : \ \mathbf{v}^2 \in \mathbf{H}^1(\Omega_2), \ \mathbf{v}^1 \cdot \mathbf{n} = 0 \text{ on } \partial\Omega_1 - \Gamma, \ \text{ and } \mathbf{v}^2 = 0 \text{ on } \partial\Omega_2 - \Gamma\}$ with norm $\|\mathbf{v}\|_{\mathbf{V}} \equiv (\|\mathbf{v}^1\|^2_{\mathbf{L}^2(\Omega_1)} + \|\boldsymbol{\nabla} \cdot \mathbf{v}^1\|^2_{L^2(\Omega_1)} + \|\mathbf{v}^2\|^2_{\mathbf{H}^1(\Omega_2)})^{\frac{1}{2}}$. The restrictions of $\mathbf{V}$ will be indicated by $\mathbf{V}_1 \equiv \{\mathbf{v}^1 : \ \mathbf{v} \in \mathbf{V})\}$ and $\mathbf{V}_2 \equiv \{\mathbf{v}^2 : \ \mathbf{v} \in \mathbf{V})\}$. We shall write each such $\mathbf{v} = [\mathbf{v}^1, \mathbf{v}^2] \in \mathbf{V}_1 \times \mathbf{V}_2 = \mathbf{V}$. For the second space we use $Q \equiv L^2(\Omega)$, and each $q \in Q$ is similarly written $q = [q^1, q^2] \in L^2(\Omega_1) \times L^2(\Omega_2)$. The boundary-value problem will determine the pressure p up to a constant; this can be determined by the additional requirement that $\int_\Omega p\, dx = 0$. Thus we use the quotient space $L^2(\Omega)/\Re$ with its usual norm, and we note that its dual is characterized by $(L^2(\Omega)/\Re)' \simeq < 1 >^{\perp}$, that is, $h \in (L^2(\Omega)/\Re)'$ if and only if $h \in L^2$ and $(h, 1)_{L^2} = 0$.

Suppose $[\mathbf{u}, p] \in \mathbf{V} \times Q$ is a solution of (29) with the time derivatives deleted. Let $\mathbf{w} \in \mathbf{V}$, multiply Darcy's law (29a) by $\mathbf{w}^1 \in \mathbf{V}_1$ and the momentum equation (29c) by $\mathbf{w}^2 \in \mathbf{V}_2$, integrate, and add to obtain

$$\int_{\Omega_1} \left(\kappa^{-1}\mathfrak{a}(\mathbf{u}_1) \cdot \mathbf{w}^1 - p^1\delta : \boldsymbol{\epsilon}(\mathbf{w}^1)\right) dx + \int_{\Omega_2} \left(\sigma^2 - p^2\delta\right) : \boldsymbol{\epsilon}(\mathbf{w}^2) dx \\ + \int_\Gamma \left(p^1\, \mathbf{n} \cdot \mathbf{w}^1 + \sigma^2(\mathbf{n}) \cdot \mathbf{w}^2 - p^2\, \mathbf{n} \cdot \mathbf{w}^2\right) dS = \int_\Omega \mathbf{f} \cdot \mathbf{w}\, dx\,.$$

Here δ is the identity tensor, so $\delta : \boldsymbol{\epsilon}(\mathbf{w}) = \boldsymbol{\nabla} \cdot \mathbf{w}$ is the trace of $\boldsymbol{\epsilon}(\mathbf{w})$. Since the test functions satisfy the admissibility constraint (30a), i.e, $\mathbf{w}^1 \cdot \mathbf{n} = \mathbf{w}^2 \cdot \mathbf{n}$ on Γ, the interface integral reduces to

$$\int_\Gamma \left(\sigma^2(\mathbf{n}) \cdot \mathbf{w}^2 + (p^1 - p^2)\, \mathbf{n} \cdot \mathbf{w}^2\right) dS\,.$$

Moreover, decomposing the stress terms into their normal and tangential components, we obtain

$$\int_\Gamma \left(\sigma^2_T \cdot \mathbf{w}^2_T + \left(\sigma^2_n + p^1 - p^2\right) \, \mathbf{n} \cdot \mathbf{w}^2\right) dS\,,$$

and then the interface conditions (30c) and (30b) yield

$$\int_\Gamma \sqrt{\kappa^{-1}}\, \beta(\mathbf{u}^2_T) \cdot \mathbf{w}^2_T\, dS + \int_\Gamma \alpha(u^1_n)\, (w^1_n)\, dS\,.$$

Finally, from the two conservation equations, we obtain the second equation of the mixed formulation of the *stationary problem*,

(31a) $\quad \mathbf{u} \in \mathbf{V}, \ p \in Q :$

$$\int_{\Omega_1} \kappa^{-1}\, \mathfrak{a}(\mathbf{u}^1) \cdot \mathbf{w}^1\, dx + \int_{\Omega_2} \mathfrak{b}(\boldsymbol{\epsilon}(\mathbf{u}^2)) : \boldsymbol{\epsilon}(\mathbf{w}^2)\, dx - \int_\Omega p \boldsymbol{\nabla} \cdot \mathbf{w}\, dx \\ + \int_\Gamma \alpha(u^1_n)\, w^1_n\, dS + \int_\Gamma \sqrt{\kappa^{-1}}\, \beta(\mathbf{u}^2_T) \cdot \mathbf{w}^2_T\, dS = \int_\Omega \mathbf{f} \cdot \mathbf{w}\, dx\,,$$

$$\int_{\Omega} \nabla \cdot \mathbf{u}\, q\, dx + \int_{\Omega_1} \mathfrak{c}(p^1)\, q^1\, dx = \int_{\Omega} g\, q\, dx \text{ for all } [\mathbf{w}, q] \in \mathbf{V} \times Q. \tag{31b}$$

We shall define the operators $\mathcal{A} : \mathbf{V} \to \mathbf{V}'$ and $\mathcal{B} : \mathbf{V} \to Q'$ so that they make precise the left sides of (31) and permit the nonlinear functions to be multivalued.

Assumptions 4.1. Five maximal monotone functions are given as follows:

(i) Assume that $\mathfrak{a}(\cdot) : \Re^N \to \Re^N$ is the subgradient of a continuous convex function $\varphi_{\mathfrak{a}} : \Re^N \to \Re$ and that these satisfy

$$\|\mathbf{w}\| \le C(\|\mathbf{v}\| + 1) \text{ if } \mathbf{w} \in \mathfrak{a}(\mathbf{v}), \quad \mathbf{v} \in \Re^N, \tag{32a}$$

$$\varphi_{\mathfrak{a}}(\mathbf{v}) \ge c(\|\mathbf{v}\|^2 - 1), \quad \mathbf{v} \in \Re^N, \tag{32b}$$

for some $c > 0$.

(ii) The vector-valued $\mathfrak{b} : \Re^{N\times N} \to \Re^{N\times N}$ is the subgradient of a continuous convex function $\varphi_{\mathfrak{b}} : \Re^{N\times N} \to \Re$, and these satisfy estimates like (32).

(iii) Assume $\mathfrak{c}(\cdot)$, $\alpha(\cdot)$, and $\beta(\cdot)$ are likewise given as subgradients $\mathfrak{c} = \partial\varphi_{\mathfrak{c}}$, $\alpha = \partial\varphi_\alpha$, and $\beta = \partial\varphi_\beta$ which satisfy the upper bounds (32a), and $\varphi_{\mathfrak{c}},\ \varphi_\alpha : \Re \to \Re$ and $\varphi_\beta : \Re^N \to \Re$ are continuous and convex.

Each of these subgradients may be *multivalued*; the boundedness requirement (32a) simplifies the calculus below. Furthermore, only the first two functions $\varphi_{\mathfrak{a}}$ and $\varphi_{\mathfrak{b}}$ are required to satisfy (32b). There is no such coercive-type requirement on any of the remaining terms, so considerable degeneracy is permitted.

Define the convex function

$$\begin{aligned}\varphi(\mathbf{v}) = \int_{\Omega_1} \kappa^{-1}\varphi_{\mathfrak{a}}(\mathbf{v}^1(x))\, dx + \int_{\Omega_2} \varphi_{\mathfrak{b}}(\boldsymbol{\epsilon}(\mathbf{v}^2(x))\, dx \\ + \int_\Gamma \varphi_\alpha(v_n^1)\, dS + \int_\Gamma \sqrt{\kappa^{-1}}\,\varphi_\beta(\mathbf{v}_T^2)\, dS, \quad \mathbf{v} \in \mathbf{V}.\end{aligned} \tag{33}$$

Note that $\boldsymbol{\epsilon}$, γ_n, and γ_T, are continuous and linear from $\mathbf{V}$ into $L^2(\Omega_2)^{N\times N}$, $L^2(\Gamma)$, and $\mathbf{L}^2(\Gamma)$, respectively, so $\varphi(\cdot)$ is continuous on $\mathbf{V}$. Since it is a sum of compositions of continuous convex functions with continuous linear operators, we can compute the subgradient $\mathcal{A} = \partial\varphi$ term by term and use the chain rule for each term as needed. The computation is standard (see [10, 25], for example), and we summarize it here.

A subgradient at $\mathbf{u} \in \mathbf{V}$ of the first term in (33) is an $\mathbf{h}_1 \in V'$ given by $\mathbf{h}_1(\mathbf{v}) = \int_{\Omega_1} \kappa^{-1}\mathfrak{a}_{\mathbf{u}}(x) \cdot \mathbf{v}^1(x)\, dx$, where $\mathfrak{a}_{\mathbf{u}} \in \mathbf{L}^2(\Omega_1)$ denotes a *measurable selection* $\mathfrak{a}_{\mathbf{u}}(x) \in \partial\varphi_{\mathfrak{a}}(\mathbf{u}(x))$ pointwise a.e. in Ω_1.

A subgradient of the second term is any functional $\mathbf{h}_2 \in V'$ given by $\mathbf{h}_2(\mathbf{v}) = \int_{\Omega_2} \sigma_{\mathbf{u}}(x) : \boldsymbol{\epsilon}(\mathbf{v}(x))\, dx$, where $\sigma_{\mathbf{u}} \in L^2(\Omega_2)^{N\times N}$ is a measurable selection $\sigma_{\mathbf{u}}(x) \in \partial\varphi_{\mathfrak{b}}(\boldsymbol{\epsilon}(\mathbf{u}^2)(x))$ pointwise a.e. in Ω_2. Moreover, its restriction to $C_0^\infty(\Omega_2)$ is $-\nabla \cdot \sigma_{\mathbf{u}}$, and if this belongs to $\mathbf{L}^2(\Omega_2)$, then there is a $\sigma_{\mathbf{u}}(\mathbf{n}) \in \mathbf{H}^{-1/2}(\Gamma)$ for which

$$\mathbf{h}_2(\mathbf{v}) = -\int_{\Omega_2} \nabla \cdot \sigma_{\mathbf{u}}(x) \cdot \mathbf{v}^2(x)\, dx - \sigma_{\mathbf{u}}(\mathbf{n})(\gamma(\mathbf{v}^2)), \quad \mathbf{v} \in V. \tag{34}$$

If $\sigma_{\mathbf{u}}$ is sufficiently smooth, then $\sigma_{\mathbf{u}}(\mathbf{n}) \in \mathbf{L}^2(\Gamma)$ is the normal component.

Finally, due to the growth conditions (32a) on $\alpha(\cdot)$ and $\beta(\cdot)$, we can compute the subgradient of the remaining two terms in (33) as $\mathbf{h}_3 \in \mathbf{V}'$ given by $\mathbf{h}_3(\mathbf{v}) = \int_\Gamma (\alpha_{\mathbf{u}}(s)\, v_n^1(s) + \sqrt{\kappa^{-1}}\beta_{\mathbf{u}}(s) \cdot \mathbf{v}_T^2(s))\, ds$, where $\alpha_{\mathbf{u}} \in L^2(\Gamma)$, $\beta_{\mathbf{u}} \in \mathbf{L}^2(\Gamma)$ are measurable selections $\alpha_{\mathbf{u}}(s) \in \partial\varphi_\alpha(u_n^1(s))$, $\beta_{\mathbf{u}}(s) \in \partial\varphi_\beta(\mathbf{u}_T^2(s))$, pointwise a.e. in Γ. In summary, we have the following lemma.

Lemma 4.1. *The subgradient $\mathcal{A} = \partial\varphi$ is given by $\mathbf{h} \in \mathcal{A}(\mathbf{u})$ for $\mathbf{u} \in \mathbf{V}$ if*

$$\begin{aligned}(35)\quad \mathbf{h}(\mathbf{v}) = \int_{\Omega_1} \kappa^{-1}\mathfrak{a}_{\mathbf{u}}(x)\cdot\mathbf{v}^1(x)\,dx + \int_{\Omega_2} \sigma_{\mathbf{u}}(x) : \boldsymbol{\epsilon}(\mathbf{v}^2(x))\,dx \\ + \int_\Gamma \Big(\alpha_{\mathbf{u}}(s)\,v_n^1(s) + \sqrt{\kappa^{-1}}\beta_{\mathbf{u}}(s)\cdot\mathbf{v}_T^2(s)\Big)\,ds, \quad \mathbf{v} \in \mathbf{V},\end{aligned}$$

for a set of measurable selections $\mathfrak{a}_{\mathbf{u}} \in \partial\varphi_{\mathfrak{a}}(\mathbf{u}^1)$, $\sigma_{\mathbf{u}} \in \partial\varphi_{\mathfrak{b}}(\boldsymbol{\epsilon}(\mathbf{u}^2))$, $\alpha_{\mathbf{u}} \in \partial\varphi_\alpha(u_n^1)$, $\beta_{\mathbf{u}} \in \partial\varphi_\beta(\mathbf{u}_T^2)$ as above.

The operator $\mathcal{B} = -\boldsymbol{\nabla}\cdot : \mathbf{V} \to Q'$ is defined by

$$(36)\qquad \mathcal{B}\mathbf{v}(q) = -\int_\Omega \boldsymbol{\nabla}\cdot\mathbf{v}(x)\,q(x)\,dx \text{ for all } [\mathbf{v}, q] \in \mathbf{V}\times Q.$$

Lemma 4.2. *The operator $\mathcal{A}$ is bounded from $\mathbf{V}$ to $\mathbf{V}'$, and $\varphi(\cdot) + \|\mathcal{B}(\cdot)\|_{Q'}^2$ is $\mathbf{V}$-coercive.*

Proof. The growth conditions (32a) show $\mathcal{A}$ is bounded. Because of the boundary condition $\mathbf{v}^2 = \mathbf{0}$ on $\partial\Omega^2 - \Gamma$ for each $\mathbf{v}^2 \in \mathbf{V}_2$, Korn's inequality gives the estimate

$$\|\mathbf{v}^2\|_{\Omega_2}^2 + \|\boldsymbol{\nabla}\mathbf{v}^2\|_{\Omega_2}^2 \le C\|\boldsymbol{\epsilon}(\mathbf{v}^2)\|_{\Omega_2}^2, \quad \mathbf{v}^2 \in \mathbf{V}_2,$$

and with the lower estimates (32b), this shows that there is a $c > 0$ for which

$$(37)\qquad \varphi(\mathbf{v}) + \int_{\Omega_1} (\boldsymbol{\nabla}\cdot\mathbf{v}^1(x))^2\,dx \ge \|\mathbf{v}\|_{\mathbf{V}}^2, \quad \mathbf{v} \in \mathbf{V}. \qquad \square$$

Lemma 4.3. *The dual operator $\mathcal{B}'$ is $Q/\Re$-bounding, so $\mathcal{B}$ has closed range,* $\mathrm{Rg}(\mathcal{B}) = (Q/\Re)'$, *and* $\mathrm{Ker}(\mathcal{B}') = \Re$.

Proof. Denote the quotient space $Q/\Re$ by Q_0. Since $\mathbf{H}_0^1(\Omega) \subseteq \mathbf{V}$ and $\|\,\mathbf{v}\,\|_{\mathbf{V}} \le C\,\|\,\mathbf{v}\,\|_{\mathbf{H}_0^1(\Omega)}$, we have

$$\inf_{\varphi\in Q_0}\sup_{\mathbf{v}\in\mathbf{V}} \frac{\mathcal{B}\,v\,(\varphi)}{\|\,v\,\|_{\mathbf{V}}\,\|\,\varphi\,\|_{Q_0}} \ge \frac{1}{C}\inf_{\varphi\in Q_0}\sup_{\mathbf{v}\in\mathbf{H}_0^1(\Omega)} \frac{\mathcal{B}\,v\,(\varphi)}{\|\,v\,\|_{\mathbf{H}_0^1(\Omega)}\,\|\,\varphi\,\|_{Q_0}},$$

and this last term corresponds to the Stokes problem with null boundary conditons; it is known to be bounded below by $c_0 > 0$. This follows, in fact, because the gradient, $\boldsymbol{\nabla} : L^2(\Omega) \to \mathbf{H}_0^1((\Omega)'$, has closed range and its kernel consists of constant functions [29]. $\square$

Define the convex function

$$(38)\qquad \psi(q) = \int_{\Omega_1} \varphi_{\mathfrak{c}}(q^1(x))\,dx, \quad q \in Q.$$

As before, the subgradient $\mathcal{C} = \partial\psi$ is given by $h \in \mathcal{C}(p)$ if $h(q) = \int_{\Omega_1} \mathfrak{c}_p(x)\,q^1(x)\,dx$, where $\mathfrak{c}_p \in L^2(\Omega_1)$ denotes a *measurable selection* $\mathfrak{c}_p(x) \in \partial\varphi_{\mathfrak{c}}(p^1(x))$ pointwise a.e. in Ω_1. Now we apply Theorem 3.3 or Theorem 3.2 with Remark 3.4 to obtain the following theorem.

Theorem 4.4. *Together with the Assumptions 4.1 and $\mathbf{f} \in L^2(\Omega)$, suppose that either*

(i) $\mathfrak{c} = 0$ *and* $g \in (L^2(\Omega)/\Re)'$ *or*

(ii) $\varphi_{\mathfrak{c}}$ *satisfies growth estimates like* (32b) *and* $g \in L^2(\Omega_1)$.

Then there exists a solution of the problem (31), *that is,*

$$\mathbf{u} \in \mathbf{V},\ p \in Q \text{ with selections } \mathfrak{a}_{\mathbf{u}} \in \partial\varphi_{\mathfrak{a}}(\mathbf{u}^1),\ \sigma_{\mathbf{u}} \in \partial\varphi_{\mathfrak{b}}(\boldsymbol{\epsilon}(\mathbf{u}^2)),$$
$$\alpha_{\mathbf{u}} \in \partial\varphi_{\alpha}(u_n^1),\ \beta_{\mathbf{u}} \in \partial\varphi_{\beta}(\mathbf{u}_T^2),\ \mathfrak{c}_p \in \partial\varphi_{\mathfrak{c}}(p^1)$$

as above such that

$$\text{(39a)} \quad \int_{\Omega_1} \kappa^{-1}\, \mathfrak{a}_{\mathbf{u}} \cdot \mathbf{w}^1\, dx + \int_{\Omega_2} \sigma_{\mathbf{u}} : \boldsymbol{\epsilon}(\mathbf{w}^2)\, dx - \int_{\Omega} p \boldsymbol{\nabla} \cdot \mathbf{w}\, dx + \int_{\Gamma} \alpha_{\mathbf{u}}\, w_n^1\, dS + \int_{\Gamma} \sqrt{\kappa^{-1}}\, \beta_{\mathbf{u}} \cdot \mathbf{w}_T^2\, dS = \int_{\Omega} \mathbf{f} \cdot \mathbf{w}\, dx\,,$$

$$\text{(39b)} \qquad \int_{\Omega} \boldsymbol{\nabla} \cdot \mathbf{u}\, q\, dx + \int_{\Omega_1} \mathfrak{c}_p\, q^1\, dx = \int_{\Omega} g\, q\, dx \text{ for all } [\mathbf{w}, q] \in \mathbf{V} \times Q.$$

The first component $\mathbf{u}$ *is unique, and* p *is determined up to a constant. In the second case,* p *is unique.*

4.2. The initial-boundary-value problem. Our last result follows from the above. Let the pair of functions $\rho, c \in L^\infty(\Omega)$ be given, and assume $\rho(x) \geq 0$ and $c(x) \geq 0$ at a.e. $x \in \Omega$. Define the continuous semiscalar products $\mathcal{E}_1\mathbf{u}(\mathbf{v}) = \int_\Omega \rho(x)\mathbf{u}(x) \cdot \mathbf{v}(x)\, dx$, $\mathbf{u}, \mathbf{v} \in \mathbf{V}$, and $\mathcal{E}_2 p(q) = \int_\Omega c(x)p(x)\, q(x)\, dx$, $p, q \in Q$. These determine the corresponding dual spaces $V_1' = \{\rho^{1/2}\mathbf{h} : \mathbf{h} \in \mathbf{L}^2(\Omega)\}$ and $Q' = \{c^{1/2}h : h \in L^2(\Omega)\}$.

THEOREM 4.5. *Let functions* $\mathbf{f}(\cdot) \in W^{1,1}(0,T;\mathbf{L}^2(\Omega))$, $g(\cdot) \in W^{1,1}(0,T;L^2(\Omega))$, *and nonnegative functions* $\rho, c \in L^\infty(\Omega)$ *be given as well as the Assumptions* 4.1. *Define operators* $\mathcal{A} = \partial\varphi : \mathbf{V} \to \mathbf{V}'$, $\mathcal{B} : \mathbf{V} \to Q'$, $\mathcal{C} = \partial\psi : Q \to Q'$ *as above, and assume* $\mathbf{u}_0 \in V$, $p_0 \in Q$ *such that* $\mathbf{f}_0 - \mathcal{B}'p_0 \in \mathcal{A}(\mathbf{u}_0)$, $g_0 + \mathcal{B}\mathbf{u}_0 \in \mathcal{C}(p_0)$ *for some* $\mathbf{f}_0 \in V_1'$, $g_0 \in Q_2'$.

Then there exists a solution $\mathbf{u}$, p *of*

$$\text{(40a)} \qquad \tfrac{d}{dt}(\rho\, \mathbf{u}(t)) + \mathcal{A}\mathbf{u}(t) + \mathcal{B}'p(t) \ni \rho^{1/2}\mathbf{f}(t) \text{ in } V_1',$$

$$\text{(40b)} \qquad \tfrac{d}{dt}(c\, p(t)) - \mathcal{B}\mathbf{u}(t) + \mathcal{C}p(t) \ni c^{1/2}g(t) \text{ in } Q_2',$$

for which $\rho\, \mathbf{u}(\cdot) \in W^{1,\infty}(0,T;V_1')$, $c\, p(\cdot) \in W^{1,\infty}(0,T;Q_2')$ *and* $\rho\, \mathbf{u}(0) = \rho\, \mathbf{u}_0$, $c\, p(0) = c\, p_0$. *Moreover, at every* $t \in [0,T]$, *we have* $\mathbf{u}(t) \in \mathbf{V}$, $p(t) \in Q$ *with measurable selections* $\mathfrak{a}_{\mathbf{u}}(t) \in \partial\varphi_{\mathfrak{a}}(\mathbf{u}^1(t))$, $\sigma_{\mathbf{u}}(t) \in \partial\varphi_{\mathfrak{b}}(\boldsymbol{\epsilon}(\mathbf{u}^2(t)))$, $\alpha_{\mathbf{u}}(t) \in \partial\varphi_{\alpha}(u_n^1(t))$, $\beta_{\mathbf{u}}(t) \in \partial\varphi_{\beta}(\mathbf{u}_T^2(t))$, $\mathfrak{c}_p(t) \in \partial\varphi_{\mathfrak{c}}(p^1(t))$ *as above such that for a.e.* $t \in [0,T]$

$$\int_{\Omega} \tfrac{\partial}{\partial t}(\rho\mathbf{u}(t)) \cdot \mathbf{w}\, dx + \int_{\Omega_1} \kappa^{-1}\, \mathfrak{a}_{\mathbf{u}}(t) \cdot \mathbf{w}^1\, dx + \int_{\Omega_2} \sigma_{\mathbf{u}}(t) : \boldsymbol{\epsilon}(\mathbf{w}^2)\, dx - \int_{\Omega} p(t)\boldsymbol{\nabla} \cdot \mathbf{w}\, dx + \int_{\Gamma} \alpha_{\mathbf{u}}(t)\, w_n^1\, dS + \int_{\Gamma} \sqrt{\kappa^{-1}}\, \beta_{\mathbf{u}}(t) \cdot \mathbf{w}_T^2\, dS = \int_{\Omega} \rho^{1/2}\mathbf{f}(t) \cdot \mathbf{w}\, dx\,,$$

$$\int_{\Omega} \tfrac{\partial}{\partial t}(c\, p(t))\, q\, dx + \int_{\Omega} \boldsymbol{\nabla} \cdot \mathbf{u}(t)\, q\, dx + \int_{\Omega_1} \mathfrak{c}_p(t)\, q^1\, dx = \int_{\Omega} c^{1/2} g(t)\, q\, dx \text{ for all } [\mathbf{w}, q] \in \mathbf{V} \times Q.$$

The first component $\mathbf{u}(t)$ *is unique, and* $p(t)$ *is determined up to a constant at each* $t \in [0,T]$. *If* $c(\cdot) \neq 0$ *in* $L^\infty(\Omega)$ *or* $\mathfrak{c}(\cdot)$ *is strictly monotone, then* $p(t)$ *is unique.*

Proof. It suffices to note that $[\mathbf{u}_0,\ p_0] \in D(\mathbb{N})$ and $\rho^{1/2}\mathbf{f}(\cdot) \in W^{1,1}(0,T;V_1')$, $c^{1/2}g(\cdot) \in W^{1,1}(0,T;Q_2')$, so the result follows from Theorem 3.4. □

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