AN ABSTRACT OF THE THESIS OF

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(Name) (Degree)

in MATHEMATICS presented on June 26, 1970
(Major) (Date)

TITLE: TOPOLOGICAL VECTOR SPACES AND THEIR INVARIANT MEASURES

Signature redacted for privacy.

Abstract approved: ______________________
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First, topological vector spaces are examined from a partial order structure derived from neighborhood bases of the origin. This structure is used to produce a minimal vector norm for every Hausdorff locally convex space.

Then, topological vector spaces are examined to find translation invariant measures with respect to which functions in the topological dual are integrable. It is shown that every conical measure on a locally convex space E has a unique translationally invariant extension to all of $a(E)$, the Riesz space generated by the real valued continuous affine functions on E. Invariant measures are constructed, characterized, and extended. An integral representation on certain complete weak spaces is found.

Topological Vector Spaces and Their
Invariant Measures

by

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A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

June 1971

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Date thesis is presented June 26, 1970

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ACKNOWLEDGMENT

I wish to thank Professor T.R. Chow for his encouragement, patience, and sound suggestions.

I also wish to acknowledge Clover Redfern for her excellent typing, which gave undeserved clarity to many sections of this work.

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TOPOLOGICAL VECTOR SPACES AND THEIR INVARIANT MEASURES

I. INTRODUCTION

In a series of three papers, 1917-1920, P. J. Daniell initiated the investigation of integration from the point of view of positive linear functionals U on a vector lattice (Riesz space) satisfying "(L) if $f_1(p) \geq f_2(p) \geq \cdots$ and $\lim f_n(p) = 0$ for all p, $\lim U(f_n) = 0$" (Daniell, 1917, p. 280). He also required that the functions in the Riesz space be bounded. In his second paper, Daniell (1918) produced the first examples of integrals for functions defined on an infinite dimensional space which do not reduce to an infinite series or to an integral over a finite number of dimensions. Banach (1937) generalized Daniell's first example by considering as the domain of the functions the unit ball in a separable Hilbert space.

The more standard approach to integration is the Lebesgue theory: a countably additive set function on a sigma ring or sigma algebra is given, the integral is defined on linear combinations of characteristic functions, the integral is then extended by taking sup's or inf's over a set of previously defined integrals. Bochner (1939) produced a Riemann integration theory with respect to positive finitely additive set functions on an algebra of sets by considering limits of partitions.

Tolstov (1962) showed that the Daniell approach to integration theory lead to Lebesgue integrals when the function space included all bounded Borel functions. The necessity for (L) holding in order that a positive linear functional U on the bounded continuous functions be represented by Lebesgue integration with respect to a positive Baire measure had been shown 11 years earlier by Glicksberg (1951). In the same paper, Glicksberg proved that for a completely regular space E, every positive linear functional on the space of bounded continuous functions is represented by a Baire measure if and only if E allows no unbounded continuous functions. This is obviously not the case when E is a Hausdorff topological vector space. This clarified the result of Hewitt (1950), who studied the representation for positive linear functionals I on the space of all continuous functions on a completely regular space. He discovered (Hewitt, 1950, p. 168) that I would then be represented by a Baire measure, with respect to which every continuous function is bounded except on a set of measure zero. In a later paper (Hewitt, 1952), he returned to the more general setting of Daniell: the investigation of positive linear functionals I on some Riesz space L of functions. By using the bounded functions in the Riesz space he found a finitely additive measure which represented I for some functions in L. But he did not succeed in finding reasonable necessary and sufficient conditions in terms of I alone for the representation to hold for all bounded functions in L (Hewitt, 1952, paragraph 3.8).

Glicksberg's 1951 paper and Hewitt's 1952 paper highlighted the difficulties in investigating Riesz spaces of unbounded functions.

The investigation of the Riesz space generated by the dual space of a locally convex Hausdorff topological vector space was initiated by Choquet (1962), who introduced the ideas of conical measures and affine measures. Choquet (1969) proved that conical measures on the complete weak spaces $E(X)$, X denumerable, were localizable by Radon measures. He discovered rotationally invariant conical and affine measures on pre-Hilbert spaces.

Umemura (1965) emphasized the importance of investigating measures--not necessarily sigma additive--on function spaces, both for the theory of stochastic processes and for quantum mechanics. He proved the existence of a continuum of quasi-invariant Borel measures on infinite-dimensional vector spaces, and asked about the existence of essentially different invariant measures.

This thesis is concerned with investigating this question using translation groups, from the Daniell point of view, and seeing how the properties of the resulting measures depend upon properties of the topological vector space structure. We first consider some new properties of a topological vector space, and see how the measures are affected by this structure.

Chapter II recalls and assembles the notation and basic facts needed concerning topological vector spaces and partially ordered sets. We introduce the concept of co-initialness and observe a simple property of it.

Chapter III relates characteristics of the topology of a vector space to certain properties of related posets. There is introduced the equivalence relation .e. and the partial ordering .a. on the equivalence classes, and classifies spaces according to lc-type.

Chapter IV uses .e. to show that every Hausdorff locally convex space is (vectorially) normizable.

Chapter V begins with a representation theorem for Riesz spaces. Although the theorem must be known to at least a few people, neither its statement nor its proof seem to have ever been published. A similar statement could probably be made for the other theorems in the first section. In particular, the proof of Theorem 3 is too short to be unknown. Theorem 4 arose out of an attempt to simplify some of the arguments in Hewitt's 1950 paper. The method of proof reappears in Proposition 4; it also appears in Choquet's 1967 paper and throughout Chapter 8 of his 1969 book. In the second section, Theorem 6 was stated but not proved by Choquet. Theorem 13 was announced by Courrege (Bony and Courrege, 1964). The other results are all original. We show that no non-zero affine measure has an extension which is translation invariant for any translation group. We show that every positive invariant measure on a complete weak space $E(X)$, X denumberable, is a non-localizable extension of a localizable conical measure. We show that invariant measures are not zero as conical measures without being identically zero. We show that although invariant measures are plentiful, they correspond to no even finitely additive set function on a subset of the Baire sets. We show that ever conical measure has a unique translation invariant extension, and produce two characterizations (Theorems 15 and 17) of invariant measures. Finally, for weak spaces $E(X)$, X denumerable, we produce an integral representation.

II. TVS PRELIMINARIES

TVS

The present section recalls and assembles notation and basic facts from the theory of topological vector spaces which are used in this paper. The reader who is familiar with the basic facts of the theory may skip this section and refer to the Notational Index if confronted with any unfamiliar or unusual terminology or notations appearing in the remainder of the paper. For further elucidation of the assembled notations, the reader is referred to Choquest (1969, Chapters 5, 6), Horvath (1966), or Schaefer (1966). The section ends with some terminology from lattice theory.

We shall deal exclusively with topological vector spaces over the reals R. Let us recall that a topological vector space (TVS) is a set E equipped with a real vector space structure as well as a topology such that the vector space operations, addition $E \times E \to E$ and scalar multiplication $R \times E \to E$, are both continuous. E^* denotes the vector space of all linear maps from E into R (the algebraic dual). E' denotes the vector space of all continuous linear maps from E into R (the topological dual).

If F is a second TVS and if $f : E \to F$ is an algebraic isomorphism which is also a homeomorphism, we shall call f a toplinear isomorphism and say that E and F are toplinearly isomorphic.

A base for the neighborhood system at the origin is called a 0-neighborhood base. Let A, B be subsets of E. We say A absorbs B provided there exists a $t > 0$ such that $B \subset sA$ for s in R, $|s| > t$. A is called absorbing provided A absorbs $\{x\}$ for all x in E. A is called balanced provided $sA \subset A$ for $|s| \leq 1$. The subset A of E is bounded provided every 0-neighborhood absorbs A. If A is absorbing, the gauge of A, $g_A(x) = \inf\{t > 0 : x \in tA\}$.

If E has a 0-neighborhood base of convex sets, it is called locally convex. We say that E is an LCS provided it is a locally convex hausdorff TVS.

Suppose A is balanced, convex and absorbing. Then g_A is a semi-norm. We let E_A denote the vector space E furnished with the topology determined by the semi-norm g_A.

A TVS is called locally bounded provided it possesses a bounded 0-neighborhood.

Suppose E is a vector space and F is a subspace of E^*. Then the $\sigma(E, F)$-topology for E is the initial topology with respect to the set of maps F.

An LCS E is called a weak space provided its topology is identical with $\sigma(E, E')$. Notice that in general, $\sigma(E, E')$ is a weaker topology. On any vector space E it is clear that the finest weak topology is $\sigma(E, E^*)$.

For any set X, let $E(X) = R^X$ furnished with the product topology. $E(X)$ is a (complete) weak space whose dual $E(X)' = R^{(X)}$. Actually, it is well known that E is a complete weak space if and only if E is toplinearly isomorphic to some $E(X)$. Given E, a complete weak space, let X be a Hamel basis for E'. Then the map $i : E \to E(X)$ given by point evaluation: $i(y)(x) = x(y)$ for x in X and y in E is the required toplinear isomorphism.[1/]

Posets

We shall also have need of some elementary notions concerning partial orderings. Suppose $(S, \leq)$ is a poset (i.e., a partially ordered set) a subset T of S is said to be left filtering for $\leq$ provided for every u, v in T, there exists a w in T such that $w \leq u$ and $w \leq v$. If S has an element u such that $u \leq s$ for all s in S, then S is said to have a zero element $0 = u$. If S has an element w such that $s \leq w$ for all s in S, then S is said to have a unit element $1 = w$.

We define $u \vee v = \sup(u, v) =$ the zero element of the poset $\{w \in S : u \leq w \text{ and } v \leq w\}$, when it exists. We define

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[1/] See for instance, Choquet, 1969, Volume II, p. 58-61.

$u \wedge v = \inf(u, v) =$ the unit element of the poset $\{w \in S : w \leq u \text{ and } w \leq v\}$, when it exists. A semilattice is a poset in which either of the following two conditions hold:

(i) Every two elements has an inf.

(ii) Every two elements has a sup.

If $(S, \leq)$ is a poset and T is a subset of S, then we say that T is co-initial in S provided $s \in S$ implies there exists a $t \in T$ such that $t \leq s$.

If S is left filtering and T is co-initial in S then T is also left filtering. For let s and t be elements of T. Since S is left filtering there exists an element u of S such that $u \leq s$ and $u \leq t$. By the co-initialness of T there exists an element $w \in T$ such that $w \leq u$ and therefore such that $w \leq s$ and $w \leq t$.

The next chapter relates characteristics of the topology of a TVS to certain properties of related posets. For a presentation of other uses of posets to the investigation of TVS, see Birkhoff (1966).

III. POSET DESCRIPTIONS OF TVS E

E is a TVS

In this section let E be a TVS, and let B be a fixed 0-neighborhood base consisting of balanced sets. In B we introduce the following relations: for U and V in B, let U .a. V provided U is absorbed by V; for U and V in B, let U .e. V provided U .a. V and V .a. U.

Propostion 1. The following properties of elements U, V of B are equivalent.

(a) U .a. V

(b) U is a subset of tV for some $t > 0$.

Proof.

(a) implies (b): by the definition of absorbing.

(b) implies (a): because the elements of B are balanced; $U \subset tV \subset tsV$ for $|s| \geq 1$. Therefore, $U \subset \lambda V$ for all $|\lambda| \geq t$. Q.E.D.

If $U \subset sV$ and $V \subset tW$ then $U \subset stW$ so that U .a. V and V .a. W implies that U .a. W. With the transitivity of .a. established, it is clear that .e. is an equivalence relation.

Corollary. The following properties of elements U, V of B are equivalent.

(a) U .e. V

(b) $U \subset sV \subset tV$ for some $s, t > 0$.

Let $[\cdot] : B \to B/.e.$ be the canonical surjection, so that for U an element of B, $[U] = \{V \in B : V \text{ .e. } U\}$.

If U .e. U' and V .e. V' and U .a. V, then since U' .a. U and V .a. V' it is the case that U' .a. V'. Thus, .a. induces a partial order on $B/.e.$ which will continue to be denoted by .a. . Thus, $[U]$ .a. $[V]$ provided U .a. V for some U in $[U]$ and V in $[V]$.

Proposition 2. $B/.e.$ is a left filtering partial order.

Proof. It is clear that $(B/.e., .a.)$ is a partial order. Suppose $[U]$, $[V]$ are elements of $B/.e.$. Then since B is a neighborhood base, there exists W in B such that $W \subset U \cap V$. But then $[W]$ .a. $[U]$ and $[W]$ .a. $[V]$. Q.E.D.

Proposition 3. If B is closed under finite intersections, then $B/.e.$ is a semilattice under $\wedge$.

Proof. Let $[U]$, $[V]$ be elements of $B/.e.$. We show that $[U] \wedge [V] = [U \cap V]$. For, $U \cap V$ .a. U and $U \cap V$ .a. V so that $[U \cap V]$ .a. $[U]$ and $[U \cap V]$ .a. $[V]$. Now suppose $[W]$ .a. $[U]$ and $[W]$ .a. $[V]$. Then $W \subset sU$ and $W \subset tV$ so that $W \subset sU \cap tV \subset \max(x,t)U \cap V$. Thus W .a. $U \cap V$ so $[W]$ .a. $[U \cap V]$. Q.E.D.

Recall that a subset T of the poset $(B/.e., .a.)$ is co-initial provided that for all $[U]$ in $B/.e.$ there exists $[V]$ in T such that $[V]$ .a. $[U]$.

Notice that $B/.e.$ itself is co-initial, so that the set of all co-initial subsets of $B/.e.$ is nonempty. Let $lc(B) = \min\{\text{cardinality } T : T \subset B/.e. \text{ is co-initial}\}$.[2/] The cardinals are well-ordered, so that $lc(B)$ is well defined. Indeed there must exist a co-initial subset T such that $lc(B)$ = the cardinality of T.

Proposition 4. Co-initial sets are left filtering.

Proof. This follows from Proposition 2, and the remarks at the end of the preceding section. Q.E.D.

Theorem 1. If B' is another zero-neighborhood base of balanced sets for E, then $lc(B) = lc(B')$.

Proof. Let $T \subset B/.e.$ be co-initial in $B/.e.$.

[2/] lc is a typewritten version of the small script aleph as written in Hebrew.

$M \subset B'/.e.$ co-initial in $B'/.e.$ is constructed as follows: Let $[U] \in B/.e.$. Since B' is a neighborhood base, we may choose a $V \subset U$, $V \in B'$. Let $m([U])$ equal $[V]$, the equivalence class in $B'/.e.$ of that chosen V. Performing this choice for every $[U]$ in $B/.e.$ defines a map $m : B/.e. \to B'/.e.$. Let $M = m(T)$. Then M is co-initial in $B'/.e.$. For let $[W]$ be an element of $B'/.e.$. Since B is a neighborhood base, there exists a U in B such that $U \subseteq W$. But then $m([U])$ .a. $[U]$ .a. $[W]$.

Since $m : T \to M$ is surjective, the cardinality of $M \leq$ cardinality T. Thus, for each co-initial subset T of $B/.e.$ there exists a co-initial subset M of $B'/.e.$ of as low cardinality. Thus, $lc(B') \leq lc(B)$. Since the argument with respect to B and B' is symmetric, also $lc(B) \leq lc(B')$, and thus by the Schröder-Bernstein theorem, $lc(B) = lc(B')$. Q.E.D.

Thus, lc is independent of the choice of neighborhood base, and so we may set $lc(E) = lc(B)$, and consider lc as a function of E.

Theorem 2. The following are equivalent statements about E.

(i) $lc(E) = 1$.

(ii) The partially ordered set $B/.e.$ has a 0 element.

(iii) E is locally bounded.

Proof. (i) implies (ii): Since $lc(E) = lc(B)$, $B/.e.$ has a co-initial subset $\{[U]\}$. So $[U]$ .a. $[V]$ for all $[V]$ in $B/.e.$. This means that $[U]$ is a 0 element for $B/.e.$.

(ii) implies (i): The 0 element furnishes a singleton co-initial subset for $B/.e.$ so $lc(E) = lc(B) = 1$.

(ii) implies (iii): Let $[U]$ be the 0 element of $B/.e.$. We show that U is a bounded neighborhood of the origin in E. Let V be any neighborhood of the origin. Then there exists an element V' of B such that $V' \subset V$. But $[U]$ .a. $[V']$ so that U .a. V' .a. V and thus U is absorbed by V.

(iii) implies (ii): Let U be a bounded neighborhood of the origin in E. Then there exists an element U' of B such that $U' \subset U$. We show that $[U']$ is a 0 element of $B/.e.$ Let $[V]$ be an element of $B/.e.$. Then U' .a. U .a. V so that $[U']$ .a. $[V]$. Q.E.D.

Theorem 3. The following are equivalent statements about E.

(i) $lc(E) = 1$ or $lc(E) = \aleph_0$.

(ii) E has a countable zero-neighborhood base.

(iii) E is semi-metrizable.

Proof. (ii) implies (iii) and (iii) implies (ii) are well known theorems.

Suppose E is semi-metrizable. Then there is a semi-metric

$d: E \times E \to R^+$ whose open balls form a base for the topology of E. Then $\{\{x \in E : d(x,0) < n^{-1}\} : n \in N\}$ is a countable zero-neighborhood base. Now suppose that E has a countable zero-neighborhood base. We sketch one possible construction of a semi-metric: Given a countable neighborhood base $B = \{B_n : n \in N\}$, form a sequence $\{V_n : n \in N\}$ of balanced zero-neighborhoods inductively as follows. Choose as V_1 any balanced zero-neighborhood such that $V_1 \subset B_1$. Having chosen V_{n-1}, let V_n be any balanced zero neighborhood such that $V_n + V_n \subset V_{n-1} \cap B_n$. Then $\{V_n : n \in N\}$ is a zero neighborhood base, $V_n + V_n \subset V_{n-1}$, and $V_n \searrow \{0\}$. Define $d: E \times E \to R^+$ by

$$d(x,y) = \inf_{H \in F(N)} \{ \sum_{n \in H} 2^{-n} : x-y \in \sum_{n \in H} V_n \}$$

where $F(N) = \{H \subset N : H \text{ finite}\}$. The semi-metric d generates the topology of E. For details see Schaefer (1966, p. 28-29).

(ii) implies (i): Suppose E has a countable zero-neighborhood base. Then clearly $1 \leq lc(E) \leq \aleph_0$. Suppose T is a co-initial subset of $B/.e.$ of finite cardinality. $T = \{[U_1], [U_2], \ldots, [U_n]\}$. There is some element V of B such that $V \subset \bigcap_{i=1}^{n} U_i$. But then $[V] \text{ .a. } [U_i]$ for all $i = 1, \ldots, n$. Let $[U]$ be an element of $B/.e.$. Then for some i $[U_i] \text{ .a. } [U]$ so that $[V] \text{ .a. } [U]$. Thus, $\{[V]\}$ is co-initial so

$lc(E) = 1$.

(i) => (ii): Suppose $lc(E) = 1$. Then there exists a balanced bounded zero-neighborhood C in E. So $B' = \{n^{-1}C : n \in N\}$ is a countable zero-neighborhood base. For if U is a balanced zero-neighborhood in E, then by the boundedness of C, $C \subset tU$ for some $t > 0$. So $n^{-1}C \subset U$ for $t \leq n$. Suppose $lc(E) = \aleph_0$ and let $\{[U_n] : n \in N\}$ be a co-initial subset in $B'/.e.$, where B' is the collection of all balanced zero-neighborhoods of E. Then $B'' = \{m^{-1}U_n : n \in N, m \in N\}$ is a countable zero-neighborhood base for E. For let U be a balanced zero-neighborhood. Then there exists $V \in B'$ such that $V \subset U$. Also, there exists n such that $[U_n]$ .a. $[V]$ so $[U_n]$ .a. $[U]$, and thus, $U_n \subset tU$ for some $t \in R^+$. So for $m > t$, $m^{-1}U_n \subset U$. Thus U a zero-neighborhood implies there exists $V \in B''$ such that $V \subset U$. Now suppose $m^{-1}U_n$, $r^{-1}U_s \in B''$. We need only show that there exists $q^{-1}U_t \in B''$ such that $q^{-1}U_t \subset m^{-1}U_n \cap r^{-1}U_s$ to conclude that B'' is a zero-neighborhood base. By Proposition 4, there exists $[U_t]$ such that $[U_t]$ .a. $[U_n]$ and $[U_t]$ .a. $[U_s]$. Thus, $U_t \subset MU_n$ and $U_t \subset SU_s$ for some $M, S \in N$. So $(mM)^{-1}U_t \subset m^{-1}U_n$ and $(rS)^{-1}U_t \subset r^{-1}U_s$. Let $q = \max(mM, rS)$. Then $q^{-1}U_t \subset m^{-1}U_n \cap r^{-1}U_s$.

Thus if $lc(E) = 1$ or $lc(E) = \aleph_0$, there exists a countable base B''. Q.E.D.

In the proof of the preceding theorem, we have shown the following: if $1 \leq lc(E) \leq \aleph_0$, then $lc(E) = 1$ or $lc(E) = \aleph_0$ so that $lc(E)$ never equals 2, 3, 4, etc. Are there any other cardinals which $lc(E)$ avoids? The answer to this question is no, as shown by the following.

Theorem 4. Suppose X is a set of cardinality $\geq \aleph_0$. Then there exists an LCS E such that $lc(E)$ = cardinality of X.

Proof. Because of Theorems 2 and 3, we need only assume that cardinality $(X) > \aleph_0$. Let $E = E(X)$, that is, R^X, equipped with the product topology. We have seen in the preceding chapter that E is locally convex Hausdorff, and that its topology corresponds to $\sigma(E, E')$. Let $B' = \{B_{S,n} : S \in F(X), n \in N\}$ where $F(X) = \{S \subset X : S \text{ is finite}\}$ and

$$B_{S,n} = \{f \in E : -n^{-1} \leq f(s) \leq +n^{-1} \quad \text{for all} \quad s \in S\}.$$

B' is a zero-neighborhood base for the topology of E. Now $B_{S,n}$ .a. $B_{\hat{S},\hat{n}}$ if and only if $S \supset \hat{S}$. So

$$[B_{S,n}] = \{B_{\hat{S},\hat{n}} : B_{\hat{S},\hat{n}} \text{ .e. } B_{S,n}\} = \{B_{\hat{S},\hat{n}} : \hat{S} = S\} = \{B_{S,j} : j \in N\}.$$

If we set $B_S = [B_{S,n}]$, then $B'/.e. = \{B_S : S \in F(X)\}$. Clearly cardinality $(B'/.e.)$ = cardinality $F(X)$. But as is well known, cardinality $(F(X))$ = cardinality (X). For let us write $F(X) = \bigcup_{n=1}^{\infty} F_n(X)$ where $F_n(X) = \{A \subset X : A$ has exactly n elements$\}$. This is a countable disjoint union of sets of cardinality the same as that of X. For clearly cardinality $(X) \leq$ cardinality $(F_n(X)) \leq$ cardinality (X^n) = cardinality (X). So indeed cardinality $(F(X))$ = cardinality (X). The next construction shows that by removing sets from $F(X)$, we are not in danger of lowering cardinality so long as X remains covered.

Let T be a co-initial subset of $B'/.e.$ of minimal lc-value. Then clearly cardinality $(T) \leq$ cardinality $(B'/.e.)$ = cardinality (X). So to conclude the proof of the theorem we need only show that cardinality $(X) \leq$ cardinality (T). Because in that case cardinality (X) = cardinality (T) = $lc(B') = lc(E)$.

Let $\Sigma = \{S \subset X : B_S \in T\}$. Clearly cardinality (Σ) = cardinality (T). Also

$$X = U\Sigma \quad (*)$$

Because by the co-initialness of T, for all $x \in X$ there exists $B_S \in T$ such that B_S .a. $[B_{\{x\},1}]$ which is the case if and only if $x \in S$.

For each $S \in \Sigma$ construct a map $f_S : N \to X$ as follows. Enumerate S so that $S = \{s_1, \ldots, s_n\}$. Then let

$$f_S(j) = \begin{cases} s_j, & \text{if } 1 \leq j \leq n \\ s_1, & \text{if } n < j \end{cases} .$$

Clearly

$$f_S(N) = S \qquad (**)$$

Now we can consider the map $p : \Sigma \times N \to X$ defined by $p(S, n) = f_S(n)$.

By (*) and (**) p is surjective. So

$$\text{cardinality } (X) \leq \text{cardinality } (\Sigma \times N) = \max(\aleph_0, \text{cardinality } (\Sigma)).$$

But $\aleph_0 <$ cardinality (X), so

$$\text{cardinality } (X) \leq \text{cardinality } (\Sigma) = \text{cardinality } (T). \qquad \text{Q.E.D.}$$

<u>E is an LCS</u>

We shall now suppose in this section that E is an LCS, and that B is a fixed 0-neighborhood base consisting of convex balanced sets and closed under finite intersection.

<u>Proposition 1'.</u> The following properties of sets $U, V \in B$ are equivalent:

(a) U .a. V

(b) $U \subset \lambda V$ for some λ in R^+

(c) the identity map $E_U \xrightarrow{id} E_V$ is continuous, where E_U

is the seminormable locally convex space on the set E generated by the gauge of U, g_U.

Proof. (a) implies (b): by Proposition 1.

(b) implies (c): Suppose (b) holds. Consider the map $E_U \xrightarrow{id} E_V$. Because $g_V(x) \leq \lambda$ for all x in $id(U)$, id is bounded on the unit ball of E_U, so that id is a bounded linear mapping of normed spaces, and thus (c) follows.

(c) implies (a): if id is continuous then $id(U)$ is bounded in E_V so $id(U)$ .a. V and thus U .a. V. Q.E.D.

Corollary to the Proposition. The following properties of sets U, V in B are equivalent:

(a) U .e. V

(b) $U \subset \lambda_1 V \subset \lambda_2 U$ for some λ_1, λ_2 in R^+

(c) $E_U = E_V$ as topological vector spaces.

Theorem 2'. The following are equivalent statements about E.

(i) $lc(E) = 1$.

(ii) $B/.e.$ is a semilattice under $\wedge$ with 0.

(iii) E is normable.

Proof. (iii) implies (ii): Suppose E is normable. Then the norm is the gauge of a bounded zero-neighborhood, say C. Thus C .a. V for every zero-neighborhood V. There exists $U \subset C$

such that $U \in B$. Then $[U]$ is the 0 of $B/.e.$. So by Proposition 3, (ii) holds.

(ii) implies (i): Follows from Theorem 2.

(i) implies (iii): By the statement preceding Theorem 1, we may choose a co-initial set of minimal cardinality, say $\{[C']\}$. Let U be any zero-neighborhood, and V in B such that $V \subset U$. Then $[C'] .a. [V]$, so that $C' .a. V .a. U$, and thus $C' .a. U$ so that C' is bounded. Choose a zero-neighborhood W which is balanced convex and absorbing, such that $W \subset C'$. The identity map $E \to E_W$ is continuous because W is a zero-neighborhood. The inverse map $E_W \to E$ is continuous since W is bounded. Therefore E is toplinearly isomorphic to the normed space E_W. Q.E.D.

Recall that there exists nonlocally convex spaces E for which $lc(E) = 1$ but condition (iii) of Theorem 2' does not hold. For instance, let $E = L^p[0,1]$ where $0 < p < 1$. Then

$$U = \{f \in E : \int_0^1 |f(x)|^p dx \leq 1\}$$

is a bounded zero-neighborhood in the topology generated by the pseudo-norm

$$f \mapsto \int_0^1 |f(x)|^p dx.$$

So $lc(E) = 1$. But E is not locally convex. Indeed, E' is empty. So E is not normable. Thus, Theorems 2 and 2' are essentially different.

E is a Separable Space

This section will be concerned with separable spaces.

Theorem 5. Suppose E is a separable Hausdorff topological vector space. Then $lc(E) \leq 2^{\aleph_0}$.

Proof. Suppose D is dense in E, and cardinality $(D) = \aleph_0$. Let $P(D) = \{A : A \subset D\}$ = the power set of D. Let Γ be an open zero-neighborhood base. For each $A \in P(D)$ let $\Gamma(A) = \{G \in \Gamma : G \cap D = A\}$. Let $\Lambda = \{A \in P(D) : \Gamma(A) \neq \emptyset\}$. For each $\lambda \in \Lambda$ choose a $U_\lambda \in \Gamma(\lambda)$. Finally, let $B' = \{U_\lambda : \lambda \in \Lambda\}$. Then cardinality $(B') \leq$ cardinality $(\Lambda) \leq$ cardinality $(P(D)) = 2^{\aleph_0}$. Moreover, B' is a zero-neighborhood base. For suppose V is an open neighborhood of the origin. Then there exists a $G \in \Gamma$ such $G \subset \overline{G} \subset V$ because Hausdorff topological vector spaces are regular. For $\lambda = G \cap D$ there exists a $U_\lambda \in B'$. But $U_\lambda \subset \overline{\lambda} \subset \overline{G} \subset V$. The first inclusion holds because any open set not containing a point of λ cannot meet U_λ, since their intersection would be an open set without a point from D. So B' is, indeed, a zero-neighborhood base. Therefore

$lc(E) = lc(B') \leq \text{cardinality } (B') \leq 2^{\aleph_0}$. Q.E.D.

That the conclusion of the previous theorem cannot be strengthened may be seen from the following example of a separable Hausdorff locally convex space E, with $lc(E) = 2^{\aleph_0}$.

Let $E = E([0,1])$. By Theorem 4, $lc(E) = 2^{\aleph_0}$, so that E is certainly nonmetrizable. Because it is possible to pass a polynomial through any finite set of points in the plane, it is clear that the continuous functions $C([0,1])$ are dense in E. Now $C([0,1])$ is separable with respect to the stronger uniform topology, so a fortiori is separable with respect to the weaker relative topology.

For the possibility of set theory of the real line without the continuum hypothesis, see Gödel (1964). The following proposition may present a stronger result than the preceding paragraph.

<u>Proposition 5.</u> Let $X \subset [0,1]$ be such that cardinality $(X) > \aleph_0$. Then $E(X)$ is a nonmetrizable separable Hausdorff locally convex space with lc value equal to the cardinality of X.

<u>Proof.</u> Theorems 3 and 4 establish all but the separability of the space. Let D be a countable dense subset of $E([0,1])$. Let $D_X = \{f \in R^X : f = g|X \text{ for some } g \in D\}$. We claim that $\bar{D}_X = E(X)$. Now $B = \{B_{S,n}(f) : f \in E(X),\ n \in N,\ S \subset X \text{ finite}\}$ is a base for the

topology of $E(X)$, where

$$B_{S,n}(f) = \{g \in E(X) : |g(s) - f(s)| < n^{-1} \text{ for all } s \in S\}.$$

So let $B_{S,n}(f) \in B$. Choose any $\overline{f} \in E([0,1])$ such that $\overline{f}|S = f|S$. Then since D is dense in $E([0,1])$, there exists a $d \in D$ such that $|d(s) - \overline{f}(s)| < n^{-1}$ for all $s \in S$. Therefore $d|X \in D_X$ and $d|X \in B_{S,n}(f)$. Q.E.D.

Theorem 6. Suppose E is a complete weak space. Then E is separable if and only if E is toplinearly isomorphic to some $E(X)$ where cardinality of $X \leq 2^{\aleph_0}$.

Proof. By our remarks in the preceding chapter and the previous theorems, it is sufficient to show that $E(X)$ is separable whenever cardinality $(X) \leq 2^{\aleph_0}$. Suppose cardinality $(X) \leq 2^{\aleph_0}$. Then there exists an injection $i : X \to [0,1]$. Choose D a countable dense subset in $E([0,1])$. Let

$$D_X = \{f \in E(X) : \text{there exists } d \in D \text{ such that } d \circ i = f\}.$$

Clearly, cardinality $(D_X) \leq$ cardinality $D = \aleph_0$. Consider any $B_{S,n}(\varphi) \subset E(X)$, where S is finite, $n \in N$, $\varphi \in E(X)$, and

$$B_{S,n}(\varphi) = \{f \in E(X) : |f(s) - \varphi(s)| < n^{-1} \text{ for all } s \in S\}.$$

To conclude the proof, we need only find an element of D_X lurking in $B_{S,n}(\varphi)$. There exists a $d \in D$ such that $|d(i(s)) - \varphi(s)| < n^{-1}$ for all $s \in S$. So let $\bar{d} = d \circ i$. Then $d \in D_X$ and $|\bar{d}(s) - \varphi(s)| < n^{-1}$ for all $s \in S$ so that $\bar{d} \in B_{S,n}(\varphi)$. Q.E.D.

Bornology

Let E be a TVS.

We shall now consider the .a. relation on the power set of E; that is, on all subsets of E. We shall denote the power set of E by $P(E)$.

If $A, B \in P(E)$, then A .a. B provided A is absorbed by B. A .e. B provided A .a. B and B .a. A. We use $[\cdot]$ again for the canonical surjection

$$[\cdot] : P(E) \to P(E)/.e.$$

Let $\gamma \subset P(E)$ be the set of all balanced zero-neighborhoods of E. Let $\beta \subset P(E)$ be the set of all bounded subsets of E.

<u>Proposition 6.</u> A subset X of E is bounded if and only if $[X]$ is a lower bound for $[\gamma]$ in the poset $(P(E)/.e., .a.)$.

<u>Proof.</u> X is bounded iff X .a. V for all $V \in \gamma$ iff $[X]$ .a. $[V]$ for all $[V] \in [\gamma]$. Q.E.D.

$[\gamma]$ sits in $P(E)/.e.$ as an "upper segment." That is, $[G] \in [\gamma]$, $[H] \in P(E)/.e.$ and $[G] .a. [H]$ implies that $[H] \in [\gamma]$.

$[\beta]$ sits in $P(E)/.e.$ as a "lower segment." That is, $[X] \in [\beta]$, $[Y] \in P(E)/.e.$ and $[Y] .a. [X]$ implies that $[Y] \in [\beta]$.

We may define other notions from the theory of topological vector spaces in terms of this partial order. The subset A of E is absorbing if and only if $[A]$ is an upper bound for $\{[x] : x \in E\}$. A is bornivorous if and only if $[A]$ is an upper bound for $[\beta]$. An LCS E is bornological if and only if every balanced convex set G satisfies the following property, $(P) : [G] \in [\gamma]$ if and only if $[G]$ is an upper bound for $[\beta]$.

Let us consider spaces E which satisfy (P) but which are not necessarily locally convex.

Call a subset X of E a C-set provided $X + X \subset \lambda X$ for some $\lambda \geq 0$. Taking $\lambda = 2$, we see that every convex set is a C-set. Call a TVS a C-space provided it has a zero-neighborhood base of balanced C-sets. By Theorem 2 of Chapter III, if $lc(E) = 1$ then E is a C-space.

Theorem 7. A C-space E satisfies (P) for every balanced C-set G, if and only if for every TVS F such that $lc(F) = 1$, every linear map $f : E \to F$ taking bounded sets into bounded sets is continuous.

Proof. (=>): Suppose E is a C-space satisfying (P) for balanced C-sets. Let V be a balanced C-set zero-neighborhood in F. Let A be a bounded set in E. Then $f(A)$ bounded implies A .a. $f^{-1}(V)$. So $[f^{-1}(V)]$ is an upper bound for $[\beta]$. Also, $f^{-1}(V)$ is a balanced C-set, since

$$f^{-1}(V) + f^{-1}(V) = f^{-1}(V+V) \subset f^{-1}(\lambda V) = \lambda f^{-1}(V).$$

Therefore, by (P), $f^{-1}(V) \in \gamma$.

(<=): Now let V be a balanced C-set of E such that $[V]$ is an upper bound for $[\beta]$. Then E_V is a TVS such that $lc(E_V) = 1$. Let $f: E \to E_V$ be the identity map. Then $f(V) = V$ is bounded in E_V so that by assumption f is continuous. Therefore $V = f^{-1}(V) \in \gamma$. Thus, (P) is satisfied for V. Q.E.D.

IV. VECTOR VALUED NORMS

For any LCS, we characterize its topology by considering its image in a complete weak space.

Let E be an LCS, B a zero-neighborhood base of balanced, convex sets, a subset T of B such that $\{[U] : U \in T\}$ is a co-initial set of minimal cardinality in $B/.e.$.

Consider the map $n : E \to E(T)$ defined by

$$n(x) = (g_U(x))_{U \in T}$$

Letting $E(T)^+ = \{f : T \to R : f \geq 0\}$, it is clear that $n(E) \subset E(T)^+$. Considering $E(T)$ with the Riesz space structure of $F(T, R)$, we see that $E(T)^+$ is the cone of positive elements in $E(T)$. If $x, y \in E(T)$ then $x \leq y$ if and only if $y - x \in E(T)^+$ if and only if $x(U) \leq y(U)$ for all $U \in T$.

<u>Proposition 1.</u> $x_i \to 0$ in E if and only if $n(x_i) \to 0$ in $E(T)$.

<u>Proof.</u> Suppose $x_i \to 0$. Let $\varepsilon \in T$. We want to find j such that $i \geq j$ implies that $n(x_i)_G \leq \varepsilon$. There exists a J such that $i \geq J$ implies $x_i \in \varepsilon G$ and therefore $n(x_i)_G = g_G(x_i) \leq \varepsilon$. So $n(x_i) \to 0$.

Suppose now that $n(x_i) \to 0$ and G is a zero-neighborhood in E. We want to find j such that $i \geq j$ implies $x_i \in G$. Let $U \in T$ be such that U .a. G, so that $\lambda U \subset G$ for some $\lambda > 0$. $n(x_i) \to 0$ implies $n(x_i)_U = g_U(x_i) \to 0$. Therefore there exists J such that $i \geq J$ implies $g_U(x_i) < \lambda$ implies $x_i \in \lambda U \subset G$. Q.E.D.

Corollary. $x_i \to x$ if and only if $n(x_i - x) \to 0$.

Proposition 2. n is continuous.

Proof. Suppose $x_i \to x$ in E. Will show that for all $G \in T$, $g_G(x_i) \to g_G(x)$. But $|g_G(x_i) - g_G(x)| \leq g_G(x_i - x) \to 0$ by the corollary. Q.E.D.

Proposition 3.

(a) $n(x) = 0$ if and only if $x = 0$,

(b) $n(\lambda x) = |\lambda| n(x)$ for all $\lambda \in R$.

(c) $n(x+y) \leq n(x) + n(y)$.

Proof. (a) follows from the definition. Alternatively, notice that Propositions 1 and 2 imply (a).

(b) follows from the definition of n and scalar multiplication in $E(T)$:

$$n(\lambda x) = (g_G(\lambda x))_{G \in T} = (|\lambda| g_G(x))_{G \in T}$$

$$= |\lambda| (g_G(x))_{G \in T} = |\lambda| n(x).$$

(c) follows from the fact that $g_U(x+y) \leq g_U(x) + g_U(y)$ and the order structure of $E(T)$. Q.E.D.

Proposition 4. n is uniformly continuous.

Proof. Let U be a zero-neighborhood in $E(T)$. Want to find a zero-neighborhood G in E such that $x - y \in G$ implies $n(x) - n(y) \in U$. By definition of the weak topology on $E(T)$, U contains a set $\{f \in E(T) : |f(G_i)| \leq \varepsilon, 1 \leq i \leq n\}$ where $\varepsilon > 0$ and $\{G_i : 1 \leq i \leq n\} \subset T$. Let $G = \cap G_i$. Then εG is the required neighborhood.

For, $x - y \in \varepsilon G$ implies $g_{G_i}(x-y) \leq g_G(x-y) \leq \varepsilon$ for $i = 1, \ldots, n$, which implies that $|g_{G_i}(x) - g_{G_i}(y)| \leq \varepsilon$ for $i = 1, \ldots, n$, which implies that $n(x) - n(y) \in U$. Q.E.D.

Thus, n enjoys all the properties associated with a continuous norm except real-valuedness. The following two theorems highlight the relationship between n and continuous seminorms on E.

Theorem 1. Let f be a positive linear form on $E(T)$. That is, suppose $f \in E(T)'$ and $f(E(T)^+) \subset R^+$. Then $f \circ n$ is a continuous seminorm on E.

Proof. Let f be a positive linear form on $E(T)$. Then since $E(T)' = R^{(T)}$,

$$f = \sum_{i=1}^{n} a_i \Phi(G_i)$$

where $a_i > 0$, and $\Phi(G_i) : E(T) \to R$ evaluation at G_i, $G_i \in T$. Then

$$f \circ n(x) = \sum_{i=1}^{n} a_i g_{G_i}(x).$$

So $f \circ n$ is clearly an element of the cone of continuous semi-norms on E. Q. E. D.

A partial converse is the following.

Theorem 2. For every continuous semi-norm m on E, there exists a continuous semi-norm g on E such that (i) $g = f \circ n$ for some $f \in E(T)'$, and (ii) $m(x) \leq g(x)$ for all $x \in E$.

Proof. If m is a continuous semi-norm on E, then

$$m(x) \leq a \sum_{i=1}^{n} g_{G_i}(x)$$

with $a > 0$ and, we may assume, $G_i \in T$. (This is well-known. For instance, see Horvath (1966 p. 97).) Let

$$f = \sum_{i=1}^{n} a\Phi(G_i)$$

where $\Phi(G_i)$ is as in Theorem 1. Then clearly

$$f \circ n(x) = \sum_{i=1}^{n} a g_{G_i}(x)$$

is a semi-norm which satisfies (i) and (ii). Q. E. D.

Notice that n is a scalar valued norm if and only if $lc(E) = 1$ if and only if E is normizable.

V. INVARIANT MEASURES

Riesz Space Representation

We shall assume the basic properties of Riesz spaces, which can be found, for instance, in Bourbaki (1965, Chapter II) or even Daniell (1917, Chapter 1). If L is a Riesz space, L^+ designates the cone of positive elements of L. $x \vee y = \sup(x, y)$. $x \wedge y = \inf(x, y)$. $L^{*+} = (L^*)^+ = \{f \in L^* : f(L^+) \subset R^+\}$, where L^* is the algebraic dual of L.

If E is a topological space, we let $F(E, R)$ be the set of all real valued functions on E. $F(E, R)$ is a Riesz space which is also an algebra. Let $B(E)$ be the subalgebra of continuous bounded functions.

Let E be a TVS. Then $h(E)$ designates the Riesz subspace of $F(E, R)$ generated by the elements of E', the topological dual of E. Theorem 1 in the present section will show us that $h(E) = \{\bigvee_{i=1}^{n} a_i - \bigvee_{i=1}^{n} b_i : a_i, b_i \in E'\}$. The elements of $h(E)^{*+}$ are called conical measures (Choquet, 1962, 1969).

Let $E' + R$ be the set of all continuous affine functions; $b(E)$ be the Riesz subspace of $F(E, R)$ generated by $E' + R$ consisting of all bounded functions. We shall see by Theorem 1 that $b(E) = \{\bigvee_{i=1}^{n} a_i - \bigvee_{i=1}^{n} b_i \in B(E) : a_i, b_i \in E' + R\}$. The elements of $b(E)^{*+}$ are called affine measures.

We shall be dealing with lattices generated by certain vector subspaces of $F(E, R)$. It would be convenient to have a general representation theorem.

Theorem 1. (General Representation Theorem). Let R be a Riesz space and $S \subset R$ a vector subspace. Then the vector lattice generated by S is $L(S) = \{\bigvee_{i=1}^{n} a_i - \bigvee_{i=1}^{n} b_i : a_i, b_i \in S\}$.

Proof. First it is shown that $L(S)$ is a vector space. Let

$$\bigvee_{i=1}^{n} a_i - \bigvee_{i=1}^{n} b_i,\ \bigvee_{j=1}^{m} \bar{a}_j - \bigvee_{j=1}^{m} \bar{b}_j \in L(S).$$

Then

$$\bigvee_{i=1}^{n} a_i - \bigvee_{i=1}^{n} b_i + \bigvee_{j=1}^{m} \bar{a}_j - \bigvee_{j=1}^{m} \bar{b}_j$$

$$= (\bigvee_{i=1}^{n} a_i) + (\bigvee_{j=1}^{m} \bar{a}_j) - (\bigvee_{i=1}^{n} b_i) - (\bigvee_{j=1}^{m} \bar{b}_j)$$

$$= \bigvee_{i=1}^{n} (a_i + \bigvee_{j=1}^{m} \bar{a}_j) - \bigvee_{i=1}^{n} (b_i + \bigvee_{j=1}^{m} \bar{b}_j)$$

$$= \bigvee_{(i,j)=(1,1)}^{(n,m)} (a_i + \bar{a}_j) - \bigvee_{(i,j)=(1,1)}^{(n,m)} (b_i + \bar{b}_j)$$

is in $L(S)$ since S is closed under addition. Let $\lambda \in R$. Suppose $\lambda \geq 0$. Then

$$\lambda\left(\bigvee_{i=1}^{n} a_i - \bigvee_{i=1}^{n} b_i\right) = \lambda \bigvee_{i=1}^{n} a_i - \lambda \bigvee_{i=1}^{n} b_i$$

$$= \bigvee_{i=1}^{n} \lambda a_i - \bigvee_{i=1}^{n} \lambda b_i \in L(S)$$

since S is closed under scalar multiplication. Suppose $\lambda \leq 0$. Then

$$\lambda\left(\bigvee_{i=1}^{n} a_i - \bigvee_{i=1}^{n} b_i\right) = (-\lambda)\left(\bigvee_{i=1}^{n} b_i - \bigvee_{i=1}^{n} a_i\right)$$

$$= \bigvee_{i=1}^{n} (-\lambda) b_i - \bigvee_{i=1}^{n} (-\lambda) a_i \in L(S).$$

So $L(S)$ is a vector space.

Clearly, $S \subset L(S)$ because $a \in S \Rightarrow a = \bigvee_{i=1}^{1} a - \bigvee_{i=1}^{1} O \in L(S)$.

Since $x \wedge y = -(-x \vee -y)$, in order to show that $L(S)$ is closed under sup's and inf's, it is sufficient to show that $x, y \in L(S) \Rightarrow x \vee y \in L(S)$.

Let $x = \bigvee_{i=1}^{n} a_i - \bigvee_{i=1}^{n} b_i$, $y = \bigvee_{j=1}^{m} \bar{a}_j - \bigvee_{j=1}^{m} \bar{b}_j$.

$$\left(\bigvee_{i=1}^{n} a_i - \bigvee_{i=1}^{n} b_i\right) \vee \left(\bigvee_{j=1}^{m} \bar{a}_j - \bigvee_{j=1}^{m} \bar{b}_j\right)$$

$$= \left[\left(\bigvee_{i=1}^{n} a_i - \bigvee_{i=1}^{n} b_i + \bigvee_{j=1}^{m} \bar{b}_j\right) \vee \left(\bigvee_{j=1}^{m} \bar{a}_j\right)\right] - \bigvee_{j=1}^{m} \bar{b}_j =$$

$$=\{(\bigvee_{i=1}^{n} a_i + \bigvee_{j=1}^{m} \bar{b}_j) \vee (\bigvee_{j=1}^{m} \bar{a}_j + \bigvee_{i=1}^{n} b_i)\} - \bigvee_{j=1}^{m} \bar{b}_j - \bigvee_{i=1}^{n} b_i$$

$$=\{(\bigvee_{(i,j)=(1,1)}^{(n,m)} a_i + \bar{b}_j) \vee (\bigvee_{(i,j)=(1,1)}^{(n,m)} \bar{a}_j + b_i)\} - \bigvee_{j=1}^{m} \bar{b}_j - \bigvee_{i=1}^{n} b_i$$

$$= \bigvee_{(i,j,k)=(1,1,1)}^{(n,m,2)} a_{i,j,k} - \bigvee_{j=1}^{m} \bar{b}_j - \bigvee_{i=1}^{n} b_i \qquad (***)$$

where

$$a_{i,j,k} = \begin{cases} a_i + \bar{b}_j & \text{if } k = 1 \\ \bar{a}_j + b_i & \text{if } k = 2 \end{cases}$$

Because every $\bigvee_{i=1}^{n} s_i = \bigvee_{i=1}^{n} s_i - \bigvee_{i=1}^{n} O$ belongs to $L(S)$ whenever $s_i \in S$, the fact that $L(S)$ is a vector space shows that (***) belongs to $L(S)$. Q.E.D.

<u>Theorem 2.</u> Let $F(E, R)$ be the real algebra of all functions from E into R. Let $A \subset F(E, R)$ be a real subalgebra. Let I be a positive linear functional on A; that is, $I \in A^{*+}$. Suppose there exists an f in A such that $f(x) \geq 1$ for all $x \in E$, and $I(f) = 0$. Then $I = 0$.

<u>Proof.</u> Let $\varphi \in A$.

Then since for all $n \in N$, $0 \leq (\varphi - n)^2 = \varphi^2 - 2n\varphi + n^2$ we have $0 \leq \varphi^2 - 2n\varphi + n^2 f$ so

$$2n\varphi \leq \varphi^2 + n^2 f$$

$$\varphi \leq \frac{1}{2n} \varphi^2 + \frac{n}{2} f$$

Therefore

$$I(\varphi) \leq \frac{1}{2n} I(\varphi^2) + \frac{n}{2} I(f) = \frac{1}{2n} I(\varphi^2)$$

Since the above inequality is true for all $n \in N$

$$I(\varphi) \leq 0 \quad \text{for all} \quad \varphi \text{ in } A. \qquad (*)$$

Therefore $I|A^+ = 0$, by the positivity of I. On the other hand,

$$0 \leq (\varphi+1)^2 = \varphi^2 + 2\varphi + 1 \leq \varphi^2 + 2\varphi + f$$

So

$$0 \leq I(\varphi^2) + 2I(\varphi) + I(f) = 2I(\varphi),$$

since $\varphi^2 \in A^+$.

Therefore,

$$0 \leq I(\varphi). \qquad (**)$$

From (*) and (**), it is clear that $I(\varphi) = 0$. Q.E.D.

Notice that if there exists any g in A such that

$$\inf_{x \in E} g(x) > 0 \quad \text{and} \quad I(g) = 0,$$

then the hypothesis is satisfied with $f = \frac{1}{\inf g} g$.

The proof of the theorem is easier, of course, if it is the case that A consist only of bounded functions, or when $A = A^+ - A^+$ which occurs when A is a Riesz space.

Notice that all we have used in the proof of the theorem is the closure of the vector space A under the taking of squares. But this characterizes an algebra, since

$$fg = \frac{(f+g)^2 - f^2 - g^2}{2} \quad .$$

<u>Theorem 3.</u> Suppose A is a Riesz subspace of $F(E,R)$ and $I \in A^{*+}$ is a Daniell integral.[3/] Then if $I(f) = 0$ for some $f \in A$ such that $f(x) \geq 1$ for all $x \in E$, then $I = 0$.

<u>Proof.</u> Let $\varphi \in A^+$. When $\varphi(x) \leq n$, $nf(x) \wedge \varphi(x) = \varphi(x)$. Therefore, it is clear that $\varphi - (nf \wedge \varphi)$ converges monotonically to zero pointwise. So

$$I(\varphi - (nf \wedge \varphi)) = I(\varphi) - I(nf \wedge \varphi) \searrow 0$$

$$I(nf \wedge \varphi) \nearrow I(\varphi)$$

But,

$$I(nf \wedge \varphi) \leq I(nf) = nI(f) = 0.$$

[3/] Recall, this means that I satisfies the property: $\varphi_n \in A^+$, φ_n monotonically decreases to zero pointwise, implies that $I(\varphi_n) \searrow 0$.

Therefore $I(\varphi) = 0$, so $I|A^+ = 0$. Since in a Riesz space $A = A^+ - A^+$, this implies that $I = 0$. Q.E.D.

That the hypothesis that I be a Daniell integral cannot be relaxed may be seen from the following example.

Let $A = a(R)$, the Riesz space generated by the affine functions in R. That is, $\varphi \in A$ if an only if φ is a continuous "piecewise linear" function: there exist real numbers $x_1 < \dots < x_n$ such that $\varphi|(-\infty, x_1], \varphi|[x_1, x_2], \dots, \varphi|[x_{n-1}, x_n]$ and $\varphi|[x_n, +\infty)$ are all "linear" (affine). Let $I : A \to R$ be defined as follows. If $\varphi \in A$ and $\varphi|[x, +\infty) = f|[x, +\infty)$ where $f(t) = at + b$, let $I(\varphi) = a$. That is, $I(\varphi)$ is the slope of the last piece of φ. It is easily verified that I is well defined, $I \in A^{*+}$, I is zero on the constants, and yet $I \neq 0$, since, e.g., $I(f) = 1$ for $f(t) = t$. By the previous theorem, it is not necessary to verify that I is not Daniell.

A condition which ensures that an algebra be a Riesz space is the following.

Theorem 4. Suppose A and B are real algebras of real-valued functions such that the set of positive elements of A, A^+, is closed under taking square roots. Let $\psi : A \to B$ be an algebra homomorphism. Then $\psi(A)$ is a Riesz space and ψ is a lattice homomorphism. In particular, A is also a Riesz space. (Take

ψ = identity map.)

Proof. It is sufficient to show that for all f in A,

$$\psi(|f|) = |\psi(f)|$$

because in that case

$$\psi(f \vee g) = \psi(\tfrac{1}{2}(f+g+|f-g|)) = \tfrac{1}{2}(\psi(f)+\psi(g)+|\psi(f-g)|)$$

$$= \tfrac{1}{2}(\psi(f)+\psi(g)+|\psi(f)-\psi(g)|) = \psi(f) \vee \psi(g)$$

and

$$\psi(f \wedge g) = \psi(\tfrac{1}{2}(f+g\cdot |f-g|)) = \tfrac{1}{2}(\psi(f)+\psi(g)-|\psi(f)-\psi(g)|) = \psi(f) \wedge \psi(g).$$

So let f be an element of A. $|f| = \sqrt{f^2} \in A$ by hypothesis.

Because ψ is an algebra homomorphism, it is order preserving. For g is in A^+ if and only if there exists h in A^+ such that $g = h^2$ in which case $\psi(g) = \psi(hh) = \psi(h)\psi(h) = \psi(h)^2 \geq 0$.

Since $|f|^2 = f^2$, $\psi(|f|)^2 = \psi(f)^2$. So $|\psi(|f|)| = |\psi(f)|$. But by the order preserving property of ψ, $\psi(|f|)$ is positive. Therefore

$$\psi(|f|) = |\psi(|f|)| = |\psi(f)|. \qquad \text{Q.E.D.}$$

Notice that the only properties of ψ that were used in the proof were that ψ should be linear and preserve squares. Thus:

Theorem 5. Suppose A and B are real algebras of real-valued functions such that the set of positive elements of A is closed under the taking of square roots. Let $\psi : A \to B$ be a linear transformation such that $\psi(f^2) = \psi(f)^2$ for all f in A. Then $\psi(A)$ is a Riesz space and ψ is a lattice homomorphism.

Characterizations of Invariant Measures

Let E be a locally convex topological vector space. Let I be a positive measure[4/] on E. Suppose I is invariant under some translation operators, and every functional in E' in integrable with respect to I. We shall study properties of this I. If it were the case that I give rise to a finitely additive measure on a ring of subsets of E, it would be sufficient to require that $I \in (L(E'))^{*+}$; i.e., that I be a positive linear functional on the Riesz space generated by the continuous linear functionals (see Hewitt, 1952).

If $p \in E$, define $T_p : F(E,R) \to F(E,R)$ by $T_p f(x) = f(x+p)$ for $x \in E$.

In the remainder of this chapter, we shall consider a fixed subgroup G of E, $G \neq O$.

Let $a(E)$ be the smallest Riesz subspace of $F(E,R)$ containing E' and closed under the operation of translation in G; that

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[4/] See page 45.

is, such that $p \in G$, $f \in a(E)$ implies that $T_p f \in a(E)$. Such a space certainly exists. $a(E) = \cap\{F : E' \subset F \subset F(E, R),\ F$ a Riesz space, $T_p F \subset F,\ \forall p \in G\}$.

Proposition 1. $a(E)$ contains the constants.

Proof. Choose a $w \in E'$ such that $G \not\subset \ker w$. This is certainly possible by the Hahn-Banach theorem. Then $w(p) \neq 0$ for some $p \in G$. Without loss of generality, we may assume that $w(p) = 1$. $w \in E' \subset a(E) \Rightarrow T_p w \in a(E)$. Therefore, $T_p w - w \in a(E)$.

But $(T_p w - w)(x) = w(x+p) - w(x) = w(p) = 1$ for all $x \in E$. Thus, $1 \in a(E)$. Q.E.D.

Proposition 2. $a(E) = L(E'+R)$, so that the set is independent of G.

Proof. By the previous proposition, $E' + R \subset a(E)$. Since $a(E)$ is a Riesz space, this implies that $L(E'+R) \subset a(E)$. On the other hand, if $\bigvee_{i=1}^{n} a_i - \bigvee_{i=1}^{n} b_i \in L(E'+R)$, where $a_i, b_i \in E' + R$, then

$$T_p\left(\bigvee_{i=1}^{n} a_i - \bigvee_{i=1}^{n} b_i\right) = T_p\left(\bigvee_{i=1}^{n} a_i\right) - T_p\left(\bigvee_{i=1}^{n} b_i\right)$$

$$= \bigvee_{i=1}^{n} (T_p a_i) - \bigvee_{i=1}^{n} (T_p b_i) \in L(E'+R).$$

Thus, $E' \subset L(E'+R) \subset F(E,R)$, $L(E'+R)$ is a Riesz space, and $T_p L(E'+R) \subset L(E'+R)$ for all $p \in G$. So $a(E) \subset L(E'+R)$. Therefore $a(E) = L(E'+R)$. Q.E.D.

So by the General Representation Theorem,

$$a(E) = \{\bigvee_{i=1}^{n} a_i - \bigvee_{i=1}^{n} b_i : a_i, b_i \in E' + R\}.$$

We also have the following representation theorem for $a(E)$, which is modeled after Choquet (1969, Theorem 30.2(ii)).

Theorem 6. Every $f \in a(E)$ has an affine decomposition: E may be written as a finite union of finite intersections of affine half spaces E_i, $1 \leq i \leq m$, such that the intersection of two distinct E_i is a subset of an affine hyperplane, and for each i there exists $f_i \in E' + R$ such that $f|E_i = f_i|E_i$.

Proof. First, supposing affine representations exist for $f, g \in a(E)$, we show it true for $f - g$. Let $E = \bigcup_{i=1}^{m} E_i$, $f|E_i = f_i|E_i$ and $E = \bigcup_{j=1}^{n} F_j$, $g|F_j = g_j|F_j$ where $f_i, g_j \in E' + R$, and E_i, F_j satisfy the hypothesis. Let $E = \bigcup_{(i,j)=(1,1)}^{(m,n)} E_{ij}$ where $E_{ij} = E_i \cap F_j$. Then E_i, F_j being finite intersections of affine half spaces imply the same about each E_{ij}. Since $E_{ij} \cap E_{i'j'} \subset E_i \cap E_{i'} \cap F_j \cap F_{j'}$, two distinct E_{ij} intersect in either two distinct E_i or two distinct F_j, so that their intersection is in an affine hyperplane by hypothesis. Moreover, $f - g|E_{ij} = f_i - g_j|E_{ij}$ and $f_i - g_j \in E' + R$. Therefore, our initial assertion has been proven. So by the previous proposition, we need only prove the theorem for $f = \bigvee_{i=1}^{n} a_i$ where $a_i \in E' + R$. The proof is by induction on n. The assertion is trivial for $n=1$. Suppose it has been proven for $n = k$ and $f = \bigvee_{i=1}^{k+1} a_i$. Let $g = \bigvee_{i=1}^{k} a_i$ have the decomposition $E = \bigcup_{j=1}^{m} E_j$, $g|E_j = \alpha_j|E_j$, $\alpha_j \in E' + R$. Clearly $f = g \vee a_{k+1}$. Let

$$E_j^+ = E_j \cap \{x \in E : a_{k+1}(x) - \alpha_j(x) \geq 0\},$$

$$E_j^- = E_j \cap \{x \in E : a_{k+1}(x) - \alpha_j(x) \leq 0\}.$$

Let

$$\mathcal{E} = \{E_j^+ : E_j^+ \neq \emptyset\} \cup \{E_j^- : E_j^- \neq \emptyset\}.$$

$$E = \cup \mathcal{E} .$$

Each element of $\mathcal{E}$ is a finite intersection of affine half spaces. Any two distinct elements of $\mathcal{E}$ meet in an affine hyperplane since

$$E_j^+ \cap E_j^- \subset \{x \in E : a_{k+1}(x) - \alpha_j(x) = 0\}$$

and if $j_1 \neq j_2$, $E_{j_1}^+ \cap E_{j_2}^+$, $E_{j_1}^- \cap E_{j_2}^+$, $E_{j_1}^- \cap E_{j_2}^-$, $E_{j_1}^+ \cap E_{j_2}^-$ are all subsets of $E_{j_1} \cap E_{j_2}$, and $f|E_j^+ = a_{k+1}|F_j^+$,

$f|E_j^- = a_j|E_j^-$. Q.E.D.

Notice that this decomposition need not be necessarily unique, since the representation of $f = \bigvee_{i=1}^{n} a_i - \bigvee_{i=1}^{n} b_i$ in terms of the affine functions a_i and b_i, is not necessarily unique.

Definition 4. $\mu \in a(E)^{*+}$ is called a positive measure.

If $\mu \in a(E)^{*+}$ and $\mu(T_p f) = \mu(f)$ for all $p \in G$ and $f \in a(E)$, then μ is called a positive G-invariant measure.

Let $m(E) = a(E)^{*+}$.

Let $m_G(E) = \{\mu \in a(E)^{*+} : \mu(T_p f) = \mu(f)$ for all $p \in G$, for all $f \in a(E)\}$.

Remarks. Since $h(E) = L(E') \subset L(E'+R) = a(E)$ and $b(E) = a(E) \cap B(E) \subset a(E)$, every positive measure is both a conical measure and an affine measure.

An example of a positive measure is the I presented after the proof of Theorem 3. Notice that $I \in m_R(R)$. We shall soon prove a representation theorem for all $I \in m_R(R)$.

Theorem 7. Suppose $I \in m_G(E)$. Then $I|b(E) = 0$.

Proof. It is sufficient to show that $I(1) = 0$. Because if $f \in b(E)$ then $-M \leq f \leq +M$ for some $M \in R$. Then by the positivity of I, $I(-M) \leq I(f) \leq I(M)$ so $-MI(1) \leq I(f) \leq MI(1)$. So $0 \leq I(f) \leq 0$.

Let $w \in E'$ be as in Proposition 1. That is, $w(p) = 1$ for some $p \in G$. Then $I(1) = I(T_p w - w) = I(T_p w) - I(w) = 0$. Q.E.D.

Corollary 1. If $I \in m_G(E)$ is a Daniell integral, then $I = 0$.

Proof. This clearly follows from the preceding theorem combined with Theorem 3. Q.E.D.

Corollary 2. If $I \in m_G(E)$, $I \neq 0$, then I cannot be localized.

Proof. Recall, I is said to be localized if there exists a compact set $K \subset E$ and a positive Radon measure $\mu \in M^+(K)$ such that $I(f) = \mu(f|K)$ for all $f \in a(E)$.

Suppose $f_n \in a(E)$ and f_n monotonically converges to zero pointwise, then $f_n|K$ monotonically converges to zero pointwise $\Rightarrow f_n|K \searrow 0$ uniformly by Dini's Theorem. Therefore, since Radon measures are continuous for the uniform topology, $\mu(f_n|K) \searrow 0$. Thus, $I(f_n) = \mu(f_n|K) \searrow 0$. Therefore, I would be a Daniell integral. This is not allowed, by the previous corollary. Q.E.D.

Remarks. Theorem 7 is actually a nonlocalizability condition. Suppose $I \in m_G(E)$, and $f_1, f_2 \in a(E)$, $f_1|_{E \setminus K} = f_2|_{E \setminus K}$ for some compact set K. Then $I(f_1) = I(f_2)$. Because $f_1 - f_2$ is continuous, and non-zero only on the compact K. It is therefore bounded. So $f_1 - f_2 \in b(E)$. Therefore $I(f_1 - f_2) = 0$ implies $I(f_1) = I(f_2)$.

The previous theorem and its corollaries show that $I \in m_G(E)$ behaves differently than affine measures. The theorem states that invariant measures are zero as affine measures. Hence no nontrivial affine measure has an extension as a G-invariant measure for any subgroup G of E, $G \neq O$.

Let us now examine the situation when E is a complete weak space, so that $E = E(X)$ for some set X. These spaces were considered in detail in Chapter III. On these spaces the behavior of I is essentially different from its restriction $I|h(E)$ as a conical measure. For $E = E(X)$, $I|h(E)$ is Daniell (Choquet, 1969, Theorem 38.13). Corollary 1 states that a G-invariant extension of this conical measure, $I|h(E)$, to all of $a(E)$ remains Daniell only when $I = 0$. When cardinality $X \leq \aleph_0$, $I|h(E(X))$ is localizable (Choquet, 1969, Theorem 38.8). So Corollary 2 implies the following

<u>Proposition 3.</u> Let $E = E(X)$ where cardinality of $X \leq \aleph_0$. Then every positive invariant measure on E is a nonlocalizable extension of a localizable conical measure.

Let us now return to the situation where E is any locally convex TVS. Then Corollary 1 immediately shows that a non-zero invariant measure is not an integral with respect to a σ-additive measure on a σ-algebra of Baire sets in E. Actually, the theorem implies that a non-zero invariant measure cannot have an integral in any of the usual ways--Lebesgue Theory or Riemann Theory--with respect to even a finitely additive measure on an algebra of subsets of E. For if I is generated by a set function μ on an algebra Σ in E, and $A \in \Sigma$, then $0 \leq I(\chi_A) \leq I(1) = 0$, where χ_A is the characteristic function of A. Thus $\mu(A) = 0$ for all $A \in \Sigma$ (see Hewitt, 1952).

The following is a more constructive proof of Corollary 1. Let $p \in G$, $p \neq 0$. Let $F = Rp = \{\lambda p : \lambda \in R\}$ and consider $\pi_1 : F \to R$ by $\pi_1(\lambda p) = \lambda$. $\pi_1 \in F'$, so it may be extended by the Hahn-Banach theorem to a map $\pi \in E'$. For $a < b$ in R, let $\pi_a^b = [\frac{1}{b-a}(\pi - a) \wedge 1] \vee 0$. Clearly $\pi_a^b \in a(\pi)$.

$$\pi_a^b(y) = \begin{cases} 0, & \text{for } y \text{ in the closed half space} \\ & \{y \in E : \pi(y) \leq a\} = \pi^{-1}(-\infty, a] \\ 1, & \text{for } y \text{ in the closed half space} \\ & \{y \in E : \pi(y) \geq b\} = \pi^{-1}[b, +\infty) \\ \dfrac{\pi(y)-a}{b-a}, & \text{for } y \in \pi^{-1}[a, b] \end{cases} \quad . \qquad (+)$$

We shall construct $f_n \in a(E)^+$ such that $f_n \nearrow 1$ pointwise but $I(f_n) = 0$ for all $n \in N$. Let $f_n = \pi_{-n}^{1-n}$. Then f_n are increasing since $\pi_{-n}^{1-n} \leq \pi_{-n-1}^{-n}$.

For if $\pi(y) \leq -n-1$ then $\pi(y) \leq -n$ so

$$0 \leq \pi^{1-n}_{-n}(y) \leq \pi^{1-(n+1)}_{-(n+1)}(y) = 0 \qquad \text{by } (+)$$

If $-n-1 \leq \pi(y) \leq -n$ then

$$0 = \pi^{1-n}_{-n}(y) \leq \pi^{1-(n+1)}_{-(n+1)}(y) = \frac{\pi(y)+n+1}{1}$$

If $-n \leq \pi(y) \leq -n+1$ then

$$\pi(y) + n = \frac{\pi(y)+n}{1} = \pi^{1-n}_{-n}(y) \leq \pi^{1-(n+1)}_{-(n+1)}(y) = 1.$$

If $-n+1 \leq \pi(y)$ then

$$1 = \pi^{1-n}_{-n}(y) \leq \pi^{1-(n+1)}_{-(n+1)}(y) = 1.$$

Thus, f_n is, indeed, increasing. Let $x \in E$. Then $f_n(x) = 1$ when $n \geq 1-x$; that is, when $1-n \leq x$, so $f_n \nearrow 1$. Also, $T_p f_n = f_{n+1}$ so that $T_{mp} f_1 = f_{m+1}$. Thus $I(f_m) = I(T_{mp} f_1) = I(f_1)$ for all $m \in N$.

Now let $g_n = \pi^{n+1}_n = T_{-2np} f_n$. We can similarly show that the g_n are decreasing and $g_n \searrow 0$. If I were Daniell then $I(g_n) \searrow 0$. But each $I(g_n) = I(T_{-2np} f_n) = I(f_1)$. So $I(g_n) \searrow 0 \Rightarrow I(f_1) = 0 \Rightarrow I(1) = 0$. We now see how an invariant measure depends upon its restriction as a conical measure.

Theorem 8. Suppose $I \in m_E(E)$ and $I|h(E) = 0$ then $I = 0$.

Proof. Since $a(E)$ is a Riesz space, $a(E) = a(E)^+ - a(E)^+$. So it is sufficient to show that $I|a(E)^+ = 0$ in order to conclude that $I = 0$.

Let $f \in a(E)^+$, $f = \bigvee_{i=1}^{n} a_i - \bigvee_{i=1}^{n} b_i$ where $a_i, b_i \in E' + R$. Always in a Riesz space

$$0 \leq |\bigvee_{i=1}^{n} a_i| \leq \bigvee_{i=1}^{n} a_i \vee \bigvee_{i=1}^{n} -a_i = \bigvee_{i=1}^{n} (a_i \vee -a_i) = \bigvee_{i=1}^{n} |a_i| .$$

So

$$0 \leq f \leq |f| \leq |\bigvee_{i=1}^{n} a_i| + |\bigvee_{i=1}^{n} b_i| = \bigvee_{i=1}^{n} |a_i| + \bigvee_{i=1}^{n} |b_i| \leq \sum_{i=1}^{n} (|a_i|+|b_i|).$$

Therefore,

$$0 \leq I(f) \leq \sum_{i=1}^{n} I(|a_i|) + I(|b_i|)$$

If we show that each $|a_i| = T_{x_i} \alpha_i$, $|b_i| = T_{y_i} \beta_i$ where $\alpha_i, \beta_i \in h(E)$ and $x_i, y_i \in E$, then the theorem will have been proven. Because

$$0 \leq I(f) \leq \sum_{i=1}^{n} I(T_{x_i} \alpha_i) + I(T_{y_i} \beta_i) = \sum_{i=1}^{n} I(\alpha_i) + I(\beta_i) = 0 ,$$

the last equality holding by the hypothesis on I. Without loss of generality we shall show this only for a_1, assuming that a_1 is not constant. (For I is zero on constants.) Consider the affine hyperplane $E_1 = \{x \in E : a_1(x) = 0\}$. Choose some $x \in E_1$; then since $T_x(a_1)(0) = a_1(x) = 0$, $T_x a_1 \in E'$. So $|T_x(a_1)| = T_x(a_1) \bigvee -(T_x(a_1) \in h(E)$. But $|T_x(a_1)| = T_x|a_1|$. So with $x_1 = -x$, $\alpha_1 = |T_x a_1|$, $|a_1| = T_{x_1} \alpha_1$. Q.E.D.

Theorem 9. Let G be a non-zero subgroup of R. Then there is a one to one correspondence between $m_G(R)$ and $R^+ \times R^+$.

Proof. There exists $p \in G$, $p \neq 0$.

Recall that $f \in a(R)$ if and only if there exist real numbers $-\infty < x_1 < x_2 < \ldots < x_n < +\infty$ and functions f_i where $f_i(t) = a_i t + b_i$, $0 \leq i \leq n$ such that $f_i|[x_i, x_{i+1}] = f|[x_i, x_{i+1}]$ for $i = 1, \ldots, n-1$ and $f_0|(-\infty, x_1] = f|(-\infty, x_1]$ and $f_n|[x_n, +\infty) = f|[x_n, +\infty)$.

Let $(p, q) \in R^+ \times R^+$.

Define $\mu(p, q) : a(E) \to R$ by $\mu(p,q)(f) = a_0 p + a_n q$.

We will show that $\mu(p, q)$ is a well-defined function. For suppose f has a second affine decomposition: $-\infty < y_1 < y_2 < \ldots < y_m < +\infty$, functions g_j where $g_j(t) = c_j t + d_j$, $0 \leq j \leq m$, where

$$g_j | [y_j, y_{j+1}] = f | [y_j, y_{j+1}] \qquad 1 \leq j \leq m-1$$

$$g_0 | (-\infty, y_1] = f | (-\infty, y_1]$$

$$g_m | [y_m, +\infty) = f | [y_m, +\infty).$$

Then

$$f | (-\infty, y_1 \wedge x_1] = g_0 | (-\infty, y_1 \wedge x_1] = f_0 | (-\infty, y_1 \wedge x_1].$$

Therefore $c_0 = a_0$.

Also,

$$f | [y_m \vee x_n, +\infty) = g_m | [y_m \vee x_n, +\infty) = f_n | (-\infty, y_m \vee x_n]$$

Therefore $d_m = a_n$.

So $a_0 p + a_n q = c_0 p + d_m q$, and thus $\mu(p, q)$ is well defined.

Clearly $\mu(p, q) \in m_G(R)$ since it is positive, linear and invariant. Also, the map $\mu : R^+ \times R^+ \to m_G(R)$ is injective. For suppose $\mu(p, q)(f) = \mu(\bar{p}, \bar{q})(f)$ for all $f \in m_G(R)$. Let $x^-, x^+ : R \to R$ by

$$x^+(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$x^-(t) = \begin{cases} 0, & t \geq 0 \\ t, & t < 0 \end{cases} .$$

Then

$$q = \mu(p, q)(x^+) = \mu(\bar{p}, \bar{q})(x^+) = \bar{q}$$

and

$$p = \mu(p, q)(x^-) = \mu(\overline{p}, \overline{q})(x^-) = \overline{p}.$$

Thus, we need only show that μ is surjective. Let $I \in m_G(R)$. Let $p = I(x^-)$ and $q = I(x^+)$. Suppose $f \in a(R)$, with $-\infty < x_1 < \ldots < x_n < +\infty, a_i, b_i$ as before. Let $F : R \to R$ by

$$F|(-\infty, mp] = T_{mp}(a_0x^-)|(-\infty, mp]$$

$$F|[x_{mp}, x_{sp}] = 0$$

$$F|[x_{sp}, +\infty) = T_{sp}(a_n\ x^+)|[sp, +\infty)$$

where $m, s \in Z$ such that $mp < x_1 < x_n < sp$. Clearly, $F \in a(R)$.

$$f - F|(-\infty, mp] = b_0 + a_0mp$$

$f - F|[mp, sp]$ is piecewise linear.

$$f - F|[sp, +\infty) = b_n + a_nsp$$

$f - F$, continuous on the compact set $[mp, sp]$ is bounded there, so $f - F \in b(R)$. Therefore $I(f-F) = 0$ by Theorem 13. So $I(f) = I(F)$. But

$$\begin{aligned}
I(F) &= I(T_{mp}(a_0x^-) + T_{sp}(a_nx^+)) \\
&= I(T_{mp}(a_0x^-)) + I(T_{sp}(a_nx^+)) \\
&= I(T_p^m(a_0x^-)) + I(T_p^s(a_nx^+)) \\
&= I(a_0x^-) + I(a_nx^+) \\
&= a_0I(x^-) + a_nI(x^+) \\
&= a_0p + a_nq.
\end{aligned}$$

Q.E.D.

We see from the theorem, that the positive cone of R-invariant measures on R has two generators. Each $\mu(p,q) = p(\mu(1,0)) + q(\mu(0,1))$. So the set $\{\mu(1,0), \mu(0,1))\}$ generates the cone and is obviously independent.

Also from the Theorem, we see that $\mu(x,-x)$ generates a cone of invariants in $m(R) - m(R)$ which vanish on $E' + R$.

This cone of measures illustrates the fact that the kernal of an invariant measure need not be a lattice.

Theorem 10. Let E be a locally convex topological vector space, $p \neq 0$, $p \in G \subset Rp \subset E$. Then the (linear) dimension of $m_G(E) = 2$ if and only if E is toplinearly isomorphic to R;

In general, the linear dimension of $m_G(E) \geq$ twice the linear dimension of E.

Proof. By the previous theorem, it is sufficient to prove the last statement. Let I be a Hamel basis for E. Let $x \in I$, $(p,q) \in R^+ \times R^+$. Define $\mu(x;p,q) \in m(E)$ by

$$\mu(x;p,q)(f) = \mu(p,q)(f|Rx) \qquad \text{for all } f \in a(E),$$

where $\mu(p,q)$ is the invariant measure of the last theorem. It is clear that

$$\{\mu(x;0,1) : x \in I\} \cup \{\mu(x;1,0) : x \in I\}$$

are linearly independent. Also, $\mu(x;p,q) \in m_{Rx}(E)$. We shall see by Theorem 15 in the next section that this is enough to insure that $\mu(x;p,q) \in m_G(E)$. Modulo this result, the theorem is proved.

Q.E.D.

We shall now examine the existence of F-invariant measures where F is any finite dimensional subspace of E. If $\mu \in m_{R^n}(R^n)$ and F is n-dimensional, we may consider μ as in $m_F(F)$ and define $\bar{\mu} \in m_F(E)$ by $\bar{\mu}(f) = \mu(f|F)$.

Therefore, it is sufficient to consider $m_{R^n}(R^n)$.

We sketch the construction of $\mu(S;p) \in m_{R^n}(R^n)$ where $S = (e_1, \ldots, e_n)$ is an ordered basis for R^n and $p \in (R^+)^n$.

Let $f \in a(E)$. f has an affine decomposition $E = \bigcup_{i=1}^{n} E_i$, $f|E_i = f_i|E_i$, $f_i \in E' + R$.

Let $t_1 \in R^+$ be defined by

$$t_1 = \bigvee_{i=1}^{n} t_1^i ,$$

where

$$t_1^i = \inf\{t \geq 0 : E_i \cap (te_1 + Re_2 + \ldots + Re_n) \neq \emptyset \quad \text{implies}$$

$$E_i \cap (se_1 + Re_2 + \ldots + Re_n) \neq \emptyset \quad \text{for all} \quad s \geq t\}.$$

t_1 is well defined, since each E_i projects onto Re_1 as an interval. Having defined t_j for some $1 \leq j < n$, define

$$t_{j+1} = \bigvee_{i=1}^{n} t_{j+1}^{i},$$

where

$$t_{j+1}^{i} = \inf\{t \geq 0 : E_i \cap (t_1 e_1 + \ldots + t_j e_j + t e_{j+1} + \ldots + R e_n) \neq \emptyset$$
$$\text{implies that } E_i \cap (t_1 e_1 + \ldots + t_j e_j + s e_{j+1} + \ldots + R e_n) \neq \emptyset$$
$$\text{for all } s \geq t\}$$

Thus we may define inductively $t_1, t_2, \ldots, t_n$.

Considering $f | t_1 e_1 + \ldots + t_{n-1} e_{n-1} + R e_n$ as an element of $a(R)$, it is clear from the construction of t_n that f is affine on $[t_n, +\infty)$. That is

$$f | t_1 e_1 + \ldots + t_{n-1} e_{n-1} + [t_n, +\infty) e_n$$
$$= f_j | t_1 e_1 + \ldots + t_{n-1} e_{n-1} + [t_n, +\infty) e_n$$

for some affine f_j in the decomposition of f. f_j continuous affine on R^n implies that

$$f_j(x) = (x, a) + b \quad \text{for } x \text{ in } R^n$$

where $a, b \in R^n$ and $(\cdot, \cdot)$ is e.g., the usual inner product. Then define $\mu(S; p)(f) = (p, a)$.

We claim that $\mu = \mu(S; p) \in m_{R^n}(R^n)$. Let

$x = x_1 e_1 + \ldots + x_n e_n \in R^n$.

It is easy to verify that $\mu(T_{x_i e_i} f) = \mu(f)$ for $i = 1, \ldots, n$. For here the value of $\mu(f)$ depends only upon the behavior of $f|Re_i$, and our construction reduces to that shown in Theorem 9.

So

$$\mu(T_{x_1 e_1 + \ldots + x_n e_n} f) = \mu(T_{x_1 e_1}(T_{x_2 e_2 + \ldots + x_n e_n} f))$$

$$= \mu(T_{x_2 e_2 + \ldots + x_n e_n} f) = \ldots$$

$$= \mu(T_{x_n e_n} f) = \mu(f).$$

The other properties that μ must have, its well-definedness, linearity, may all be verified as in Theorem 9.

O(E) and Other Extensions

In the remainder of this chapter we shall show that every invariant measure has a unique invariant extension to a subspace of $F(E, R)$ which contains $a(E)$ properly.

Let $O(E)$ be the vector space $a(E) + B(E)$ where $B(E)$ is the subalgebra of all continuous bounded functions in $F(E, R)$.

Theorem 11. $O(E)$ is a Riesz space.

Proof. Since $O(E)$ is a vector space, it is sufficient to show that f_1 in $O_1(E)$ implies that $f^+ = f \vee 0 \in O_1(E)$.

Then $f^- = (-f) \vee 0 \in O_1(E)$ so that $|f| \in O_1(E)$. Letting $f = h + b$, $h \in a(E)$, $b \in B(E)$,

$$f \vee 0 = (h+b) \vee 0 = (h \vee -b) + b.$$

So it is sufficient to show that $h \vee -b \in O_1(E)$.

Since $-b \in B(E)$, there exists $m, M \in R$ such that $m \leq -b \leq M$. Then $h \vee m \leq h \vee (-b) \leq h \vee M$. Now

$$[h \vee M - h \vee m](x) = \begin{cases} M-m, & h(x) \leq m \leq M \\ M-h, & m \leq h(x) \leq M \\ 0\ , & m \leq M \leq h(x) \end{cases}$$

But when $m \leq h(x) \leq M$, $0 \leq M - h(x) \leq M-m$. So that always $0 \leq h \vee M - h \vee m \leq M-m$.

$$h \vee (-b) = h \vee m + [h \vee (-b) - h \vee m],$$

with $h \vee m \in a(E)$ and $0 \leq h \vee (-b) - h \vee m \leq M-m$ so that $(h \vee -b) - (h \vee m) \in b(E)$. Thus $h \vee (-b) \in O_1(E)$. Q.E.D.

Theorem 12. Every $I \in m_G(E)$ has a unique extension to a G-invariant functional $\overline{I} \in O_1(E)^{*+}$.

Proof. Let $f \in O_1(E)$. So $f = h + b$ where $h \in a(E)$ and $b \in B(E)$.

Define $\overline{I}(f) = I(h)$. We shall show that $\overline{I}$ is well defined.

Suppose f is also equal to $h' + b'$ where $h' \in a(E)$, $b' \in B(E)$. Then

$$h - h' = b - b' \in B(E).$$

So $m \leq h - h' \leq M$ for some constants m, M. But $0 = I(m) \leq I(h) - I(h') \leq I(M) = 0$, so $I(h) = I(h') = \overline{I}(f)$. Thus, $\overline{I}$ is well defined.

It is clearly linear.

Suppose $f = h + b \in O_1(E)^+$. We shall find $h' \in a(E)^+$, $b' \in B(E)$ such that $f = h' + b'$. Then, since $I \in a(E)^{*+}$,

$$\overline{I}(f) = I(h') \geq 0.$$

Since $h + b \geq 0$, $h \geq -b \geq -M$ for some $M \in R$. So let $h' = h + M$, $b' = b - M$. Then clearly, $h' \in a(E)^+$, $b' \in B(E)$ and $f = h' + b'$. Thus, we have shown that $\overline{I} \in O_1(E)^{*+}$.

To conclude the proof, we need only show the uniqueness. Suppose I_1, I_2 are G-invariant functionals in $O_1(E)^{*+}$ and $I_1 | a(E) = I_2 | a(E)$. We shall show that $I_1 = I_2$. We can use the proof of Theorem 13 to conclude that

$$I_1 | B(E) = I_2 | B(E) = 0$$

Let $f = h + b \in O_1(E)$, $h \in a(E)$, $b \in B(E)$. Then

$$I_1(f) = I_1(h+b) = I_1(h) + I_1(b) = I_1(h) = I(h) = I_2(h) = I_2(h) + I_2(b) = I_2(h+b) = I_2(f).$$

Q. E. D.

We may employ the argument used in the proof of Theorem 11 to show that $h(E) + b(E)$ is a Riesz space. For if we substitute $h(E)$ for $a(E)$ and $b(E)$ for $B(E)$, the proof goes through, mutatis mutandis. The sum of two Riesz spaces is not, in general, a Riesz space; however, it is so when one of the Riesz spaces consists of bounded functions.

Theorem 13. $a(E) = h(E) \oplus b(E)$.

Proof. $E' \subset h(E)$, $R \subset b(E)$ so

$$E' + R \subset h(E) + b(E) \subset a(E)$$

$$L(E'+R) \subset L(h(E) + b(E)) \subset L(a(E))$$

But since $h(E) + b(E)$ and $a(E)$ are Riesz spaces,

$$L(E'+R) \subset h(E) + b(E) \subset a(E) = L(E'+R)$$

So

$$a(E) = h(E) + b(E).$$

Let

$$x \in h(E) \cap b(E)$$

Then x agrees with some elements of E' on polyhedral cones through the origin of E. But x bounded implies x agrees with the zero functional. Thus

$$h(E) \cap b(E) = O \, . \qquad \text{Q.E.D.}$$

Theorem 14. If $f \in a(E)$ and $d \in E$, then $T_d f - f \in b(E)$.

Proof. It is sufficient to show that $T_d(\bigvee_{i=1}^{n} h_i) - \bigvee_{i=1}^{n} h_i \in b(E)$ for $h_i \in E' + R$, $1 \leq i \leq n$. Because if that is the case, then $f \in a(E)$ implies $f = \bigvee_{i=1}^{n} h_i - \bigvee_{i=1}^{n} k_i$, with $h_i, k_i \in E' + R$ and then $T_d f - f = (T_d(\bigvee_{i=1}^{n} h_i) - \bigvee_{i=1}^{n} h_i) - (T_d \bigvee_{i=1}^{n} k_i - \bigvee_{i=1}^{n} k_i) \in b(E)$. So let $f = \bigvee_{i=1}^{n} h_i$, $h_i \in E' + R$.

Let $x \in E$. Suppose

$$h_\alpha(x) \geq h_i(x) \quad \text{for all} \quad i \neq \alpha \qquad \text{(A)}$$

$$T_d h_\beta(x) \geq T_d h_i(x) \quad \text{for all} \quad i \neq \beta \qquad \text{(B)}.$$

$$h_\alpha(x) - T_d h_\beta(x) = h_\alpha(x) - h_\beta(x) - h_\beta(d) + h_\beta(0)$$

This expression is greater than or equal to $-h_\beta(d) + h_\beta(0)$ because, by (A), $h_\alpha(x) - h_\beta(x) \geq 0$. On the other hand, the expression is less than or equal to $h_\alpha(0) - h_\alpha(d)$ because, by (B), $T_d h_\beta(x) - T_d h_\alpha(x) \geq 0$ so that

$$h_\alpha(x) + h_\alpha(d) - h_\alpha(0) \leq h_\beta(x) + h_\beta(d) - h_\beta(0).$$

Thus

$$h_\beta(0) - h_\beta(d) \leq h_\alpha(x) - T_d h_\beta(x) \leq h_\alpha(0) - h_\alpha(d).$$

Therefore,

$$\bigwedge_{i=1}^{n} h_i(0) - h_i(d) \leq \bigvee_{i=1}^{n} h_i - T_d \bigvee_{i=1}^{n} h_i \leq \bigvee_{i=1}^{n} h_i(0) - h_i(d).$$

So $T_d f - f \in a(E) \cap B(E) = b(E)$. Q. E. D.

Theorem 15. Let $\mu \in m(E)$. Then the following statements are equivalent:

(a) $\mu \in m_G(E)$ for some subgroup G of E, $G \neq O$.

(b) $\mu \in m_E(E)$.

(c) $\mu | b(E) = 0$.

(d) $\mu(1) = 0$.

Proof. (b) => (a) and (c) => (d) are trivial. (d) => (c) and (a) => (c) have been demonstrated in Theorem 7.

But (c) => (b) follows from Theorem 14. Q. E. D.

Thus any G-invariant measure is automatically E-invariant, and so we may refer to such measures simply as 'invariant measures.'

Theorem 16. Every conical measure has a unique extension to an invariant measure.

Proof. Let $\mu \in h(E)^{*+}$. Define $I: a(E) \to R$ by $I(h+b) = \mu(h)$ for all $h \in h(E)$, $b \in b(E)$. By Theorem 13. I is well defined on $a(E)$. Clearly $I | h(E) = \mu$ and $I | b(E) = 0$ so that by Theorem 15, I is an invariant measure. Q. E. D.

Notice that this yields an alternative proof to Theorem 8. Combining the results of Theorem 12 with Theorem 15 we have

Theorem 17. Let $\mu \in O_1(E)^{*+}$. Then the following statements are equivalent:

(a) μ is G-invariant on $O_1(E)$.

(b) $\mu \in m_E(E)$.

(c) $\mu | b(E) = 0$.

(d) $\mu | B(E) = 0$.

(e) $\mu(1) = 0$.

Theorem 16 shows that all conical measures can become translation invariant at very little extra cost. That extra cost is spelled out in the remarks following Theorem 7.

We close with some observations on the extension problem. We have extended invariant measures to positive linear functionals on $O_1(E)$. Can we extend them much further? Theorem 2 immediately implies that no nontrivial invariant measure may be extended to a positive linear functional on all of $C(E)$. (This, by the way, shows us that the results of Choquet (1967) and Schwarz (1964) are not applicable to the study of invariant measures.)

Let $O_2(E) = \{f \in C(E)^+ : f \leq g$ for some $g \in O_1(E)\}$ where $C(E) \subset F(E, R)$ is the Riesz space of all continuous functions. Let $O(E) = O_2(E) - O_2(E)$. For $E = R$, $O(E)$ corresponds to the set of continuous functions which are $O(x)$, that is, which are 'big oh of x.' The extension theorem of Choquet (1969, p. 289) shows that each invariant measure has an extension to a positive linear functional on

$O(\dot{E})$. However such an extension need not be either unique or invariant, contrasting the situation in Theorem 12. We do have the following

Proposition 4. Let $I \in m_E E$. Suppose I extends to a positive linear functional $\bar{I}$ on a vector space M where $a(E) \subset M \subset F(E, R)$. Then if $f \in M$, $\bar{I}(f) > 0$ and $h \in F(E, R)$, $h \geq f^2$. Then $h \notin M$.

Proof. Assume that $h \in M$.

$$0 \leq f^2 - 2nf + n^2$$

so

$$2nf \leq f^2 + n^2 \leq h + n^2$$

$$2f \leq n^{-1} h + n$$

so

$$0 < 2I(f) \leq n^{-1} I(h) + I(n) = n^{-1} I(h)$$

for all $n \in N$.

This cannot happen. Q.E.D.

Let $V = \{M : M$ is a subspace of $F(E, R)$, $a(E) \subset M$, and every $I \in m_E E$ has a positive linear extension onto $M\}$.

V is clearly a non-empty poset under set inclusion. It is easy to verify that V is inductively ordered. Therefore, by Zorn's lemma we can find a maximal element M of V. The following two facts may be verified.

1) $f \in M$ and $0 \leq g \leq f$ imply $g \in M$.

2) If $g \in F(E,R)^+$, then $g \notin M$ if and only if there exists $0 \leq f_n \leq g$, $f_n \in M$ such that $I(f_n) \nearrow +\infty$ for some $I \in m_E(E)$.

Integral Representation on Weak Spaces

In this section, we consider operations on invariant measures, other than convex combinations.

Let E and F be LCS's, and let $q:E \to F$ be a continuous linear map. Let $\mu \in m_E(E)$. Define $q(\mu):a(F) \to R$ by $q(\mu)(f) = \mu(f \circ q)$. It is clear that $q \in a(F)^{+*}$. However, $q(\mu)(1) = \mu(1 \circ q) = \mu(1) = 0$. Thus, by Theorem 15, $q(\mu)$ is an element of $m_F(F)$. We see that the projection of an invariant measure is an invariant measure.

Let A be a convex cone in an LCS E. For $\mu \in m_E(E)$ we define $\mu_A:a(E) \to R$ by

$$\mu_A(f) = \inf\{\mu(\varphi) : \varphi \in a(E),\ \varphi \geq f \text{ on } A\}$$

for $f \in a(E)^+$. For $f \in a(E)$ let $\mu_A(f) = \mu_A(f^+) - \mu_A(f^-)$. That

μ_A is an element of $a(E)^{+*}$ follows from the Riesz decomposition property applied to $a(E)$ as well as from Proposition II. 2. 1 of Bourbaki (1965, p. 26). See Proposition 30. 8 of Choquet (1965) in which he applies the Riesz decomposition property to $h(E)$ in order to prove a very similar proposition.

We say that μ is carried by X provided if $f \in a(E)$ and $f = 0$ on X then $\mu(f) = 0$. Notice that if μ is carried by X, then if $f \in a(E)$, $f \geq 0$ on X then $\mu(f) \geq 0$. Because if $f \geq 0$ on X, then

$$0 = \mu(f^+ - f) = \mu(f^+) - \mu(f),$$

So that

$$0 \leq \mu(f^+) = \mu(f).$$

Clearly, $\mu_A \leq \mu$. Also, $\mu_A \leq \mu$ if and only if μ is carried by A. We can now show that μ_A is an element of $m_E(E)$. For

$$0 \leq \mu_A(1) \leq \mu(1) = 0.$$

Thus, the restriction of an invariant measure to a convex cone remains invariant.

Let $P = P(E) = \{\mu(x; 1, 0) : x \in E\}$, where it will be recalled from the proof of Theorem 10 that $\mu(x; 1, 0)(f) = \mu(1, 0)(f|Rx)$ for f in $a(E)$. We shall always assume that P carries the initial topology defined by the set of maps $\{f^* : f \in a(E)\}$ where $f^* : P \to R$ by $f^*(\mu) = \mu(f)$. Notice that $\mu(x;1,0)_{Rx} = \mu(x;1,0)$ since $\mu(x;1,0)$ is carried by Rx.

Proposition 5. Let E be an LCS. Then for every compact $K \subset E$, positive Radon measure v on K, and continuous map $p : K \to P$, the positive linear functional

$$\mu = \int_{x \in K} p(x) dv(x)$$

defined by

$$\mu(f) = \int_K p(x)(f) dv(x)$$

is an invariant measure on E.

Proof. Clearly $f^* \circ p$ is continuous on K, so that $x \mapsto p(x)(f)$ is within the domain of v. Also, it is clear that $\mu \in a(E)^{+*}$ by the positive linearity of each $p(x)$. The invariance of μ is also clear since each $p(x) \in m_E(E)$:

$$\int_K p(x)(f) dv(x) = \int_K p(x)(T_y f) dv(x).$$

Q.E.D.

The converse of this proposition holds for certain weak spaces.

Theorem 18. Let E be a complete weak space with $lc(E) \leq \aleph_0$, and let $\mu \in m_E(E)$. Then there exists a compact $K \subset E$, a positive Radon measure v on K, and a continuous map $p: K \to P(E)$ such that

$$\mu = \int_{x \in K} p(x) dv(x).$$

Proof. Let $H: a(E) \to h(E)$ by $H(h+b) = h$ where $h \in h(E)$ and $b \in b(E)$. By Theorem 13, this projection is well-defined. Since $\mu | h(E)$ is localizable, there exists a compact $K \subset E$ and a Radon measure v on K such that

$$\mu(h) = \int_K h(x) dv(x)$$

for all h in $h(E)$. For each $x \in K$ let e_x be the conical measure evaluation-at-x, $e_x(h) = h(x)$ for all h in $h(E)$. We claim that $e_x \circ H = \mu(x; 1, 0)$ and that the map $p: k \to P(E)$ defined by $p(x) = e_x \circ H$ for x in K is continuous.

For let $f = h + b$. Then

$$\mu(x; 1, 0)(h+b) = \mu(1, 0)(h+b | Rx) = \mu(1, 0)(h | Rx)$$

since b is eventually constant along R^+x. But $\mu(1, 0)(h | Rx)$ equals the slope of h on the ray R^+x, which equals $h(x)$, by identification of Rx with R in the construction of $\mu(p, q)$. Therefore $e_x \circ H = \mu(x; 1, 0)$. To show that p is continuous, we need only show that for all f in $a(E)$ the map $f^* p : K \to R$ is continuous, where $f^* p(x) = h(x)$ for $f = h + b$. But this is clear since all of $h(E)$ are continuous functions.

But then

$$\int_{x \in K} p(x)(f) dv(x) = \int_{x \in K} H(f)(x) dv(x) = \mu(H(f)) = \mu(f)$$

so that

$$\mu = \int_{x \in K} p(x) dv(x)$$

Q. E. D.

This theorem clarifies Proposition 3 and its preceding remarks.

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APPENDIX

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