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Bayesian inferential methods for the two parameter Weibull (and extreme-value distribution) are presented in a life-testing context. A practical method of calculating posterior distributions of the two parameters and a large class of functions of the parameters is presented. The emphasis is for the situation where the sample information is large relative to the prior information.

The relationship between the fiducial method and the Bayesian method is examined and certain properties which are desirable from the frequentist point of view are shown for the Bayesian method. The frequency properties of the Bayesian method are examined under type I and type II progressive censoring by simulation and by exact methods.

The probability of coverage of Bayesian confidence intervals, conditional on an ancillary event, is shown to be equal to the nominal probability of coverage, in the full sample case and under type II progressive censoring. Numerical examples of the Bayesian method were

given, comparing the Bayesian results with results given by other methods.

# Bayesian Inference for the Weibull Distribution

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The author wishes to dedicate this work to the memory of Abdul-Baha, who is his exemplar.

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# BAYESIAN INFERENCE FOR THE WEIBULL DISTRIBUTION

## I. INTRODUCTION

The purpose of this thesis is to present the Bayesian method of making inferences about parameters of the Weibull distribution. The setting is described in terms of life-testing, although the method is applicable to many other types of experimental situations. The method may be used to analyze censored and uncensored data. In many cases, it gives exact answers from a frequentist point of view and, in some situations, it is unique in this respect. It will be seen that inferential methods for the extreme-value distribution are applicable to the Weibull distribution and conversely, so the extreme-value distribution is of great interest also. An extensive bibliography for the Weibull and the extreme-value distributions can be found in Mann [18].

The Bayesian method presented here is for the construction of confidence intervals, although tests of hypothesis may be made from confidence intervals in the usual manner. The method presented here is intended primarily for situations where prior information is negligible relative to the sample information. Major attention is given to determining the frequentist properties of these confidence intervals, such as the frequency with which an $\alpha$-level Bayesian confidence interval actually covers a certain parameter.

The life testing problem is to infer about a population of life-lengths through a sample from the population. The term "life-length" should be interpreted in a general sense so as to include not only time to death or failure, but time to cure or any other event. These problems occur in industry, medicine, biology, and many other fields of study. Biological and medical applications usually fall under the heading of mortality studies; for examples, see Kimball [15], Harris, Meier, and Tukey [10], and Kaplan and Meier [14]. Applications in industry of the extreme-value distribution can be found in articles by Aziz [2], Eldredge [6], Gumbel [9], and Posner [21].

Because the two parameters of the extreme-value distribution are one to one functions of the two parameters of the Weibull, confidence intervals for the parameters of one distribution allow one to immediately calculate confidence intervals for corresponding parameters of the other. Since the extreme-value distribution is a location-scale parameter family, many of the methods which have been developed for location-scale parameter families apply to the Weibull distribution also. Mann's paper [18] gives a concise description of many of these general methods and some of these will be described in this dissertation also.

The density of the Weibull distribution may be written

$$f(t \mid \alpha, \beta) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} \exp\left(-\left(\frac{t}{\alpha}\right)^{\beta}\right), \quad t > 0, \ \alpha > 0, \ \beta > 0. \tag{1.1.1}$$

If a random variable $T$ obeys a Weibull distribution with parameters $(\alpha, \beta)$, then $X = \ln(T)$ obeys an extreme-value distribution with location and scale parameters $\theta = \ln(\alpha)$ and $\sigma = \beta^{-1}$. The extreme-value distribution has density

$$(1.1.2) \quad g(x|\theta, \sigma) = \frac{1}{\sigma} \exp\left(\frac{x-\theta}{\sigma}\right) \exp\left(-\exp\left(\frac{x-\theta}{\sigma}\right)\right), \ -\infty < x < \infty, \ \sigma > 0, \ -\infty < \theta < \infty.$$

In many practical situations, the experimenter is obliged to remove items from the test before they have failed. This is called censoring. There are two common types of censoring; the first is called Type I censoring. A typical example of type I censoring is to fix some time $t$ and censor all items at time $t$ which haven't failed by time $t$. A more complicated version of type I censoring is to fix times $t_j$ $(j=1, \cdots, J)$ and randomly censor $n_j$ $(j=1, \cdots, J)$ of the items which haven't failed by time $t_j$. If for some $k$, $1 \leq k \leq J$, there are less than $n_k$ items on test at time $t_k$, all the remaining items are censored at time $t_k$ and the test is terminated. In this type of censoring, the number of censored items is random.

Another common type of censoring is called type II censoring. It consists of starting the experiment with $n$ items, choosing an integer $k$ $(1 \leq k < n)$ and censoring all the surviving items at the time of the $k^{th}$ failure. The more complicated progressive type II censoring consists of choosing integers $k_j$ $(j=1, \cdots, J)$ and randomly censoring $n_j$ $(j=1, \cdots, J)$ items at the time of failure of the $k_j^{th}$ item.

There are cases in which the type of censoring rule is unspecified. For example, a doctor may test patients to determine the efficacy of a treatment. Sometimes he is able to treat them until a cure is achieved but in some cases he is forced to stop the treatment before a cure is reached. These latter patients are censored according to an unknown censoring rule. This is called unspecified censoring.

An important and useful feature of the Bayesian method is that it can be used with any type of censoring, specified or unspecified. This fact gives it a wide applicability which isn't available to most of the other published methods. Most of the other methods either are not applicable to the censored case at all or are applicable only to a specific type of censoring, such as type I or type II. An especially difficult problem from the frequentist point of view is unspecified censoring. In fact, with unspecified censoring, satisfactory inferences cannot be made from the frequentist point of view.

Another important aspect of the proposed Bayesian method is that it utilizes all of the information available in the order statistic. The other published methods reduce the minimal sufficient order statistic, and don't utilize all the available information. Some of the consequences of this failure to utilize all the available information will be explored in detail in Chapter 4.

In order to gain a better understanding of the problem and also put the Bayesian method in perspective, a selection of the more important existing methods is presented in the next section. Although this selection isn't exhaustive, it will serve the purpose of making clearer the advantages and disadvantages of the proposed Bayesian procedure.

## The Method of Thoman, Bain and Antle

The method of Thoman, Bain, and Antle [25] allows one to find confidence intervals for $\alpha$ and $\beta$ in the full sample case. It is based on the fact that $\hat{\alpha}/\alpha$ and $\hat{\beta}\ell n(\hat{\alpha}/\alpha)$ are statistics which have distributions independent of $(\alpha, \beta)$, where $\hat{\alpha}$ and $\hat{\beta}$ are the m. l. estimates of $\alpha$ and $\beta$, respectively. This is easy to see because $\hat{\beta}/\beta = \sigma/\hat{\sigma}$ and $\hat{\beta}\ell n(\hat{\alpha}/\alpha) = (\theta - \hat{\theta})/\hat{\sigma}$, where $\beta = \sigma^{-1}$ and $\theta = \ell n(\alpha)$ are the parameters of the associated extreme-value distribution. In a location-scale parameter family, such as the extreme-value distribution, with location parameter $\theta$ and scale parameter $\sigma$, $\sigma/\hat{\sigma}$ and $(\theta - \hat{\theta})/\hat{\sigma}$ always have distributions which are independent of $(\theta, \sigma)$, being invariant under changes of location and scale, and their method may be applied to any family of distributions which can be transformed to a location-scale parameter family. It also may be applied, at least in principle, to data with type II censoring, although this might be cumbersome in practice because it would require a separate simulation for each censoring pattern (this point is elaborated upon later).

This method may seem to be a completely satisfactory solution but it doesn't utilize all the information available in the order statistic, in view of the fact that the maximum likelihood estimators are not sufficient. There are further practical difficulties, such as the pres-sence of type I censoring and the simulated distribution being subject to errors of sampling. In Chapter IV, a further examination of this method is made.

## Large-sample Maximum Likelihood Estimation for the Extreme-Value Distribution

The method of m. l. estimation for parameters of the extreme-value distribution is described by Harter and Moore [11]. They consider samples which are sampled from above and below using a type II censoring scheme. An item is censored "from below" if all that is known about it is that it failed before a certain time. The authors simulated 2000 samples of sizes $n = 10$ and $n = 20$ and applied different type II censoring patterns to obtain samples with different amounts of censoring. They found the asymptotic covariance matrix of $(\hat{\theta}, \hat{\sigma})$, the m. l. estimates of $(\theta, \sigma)$ in closed form for different censoring patterns. Since $(\hat{\theta}, \hat{\sigma})$ are jointly asymptotically normal with known mean and covariance matrix, one can use this fact to find large-sample confidence limits on $\theta$ or $\sigma$ or other functions of $(\theta, \sigma)$. The confidence limits for $\theta$ and $\sigma$ (and other functions of $(\theta, \sigma)$) will converge asymptotically to the confidence limits given by the Bayesian method, as is known from the general theory of Bayesian procedures.

By simulation, Harter and Moore estimated the bias and the variance of $\theta$ and $\sigma$ for samples sizes $n = 10$ and $n = 20$ with different censoring patterns. Finally, using squared-error loss as the criterion, they found by simulation that the m. l. estimates compare favorably with best linear (in the order statistics) unbiased estimates and best linear (in the order statistics) invariant estimates. These two methods are often denoted by BLU and BLI, respectively.

Best Linear Unbiased and Best Linear Invariant Estimation

The methods of best linear unbiased and best linear invariant estimation have been investigated by Mann [17]. The purpose of these methods is to estimate $\theta = \ell n(\alpha)$ and $\sigma = \beta^{-1}$ or linear combinations of $\theta$ and $\sigma$. Certain linear combinations of $\theta$ and $\sigma$ correspond to quantiles of the extreme-value distribution, since it is a location-scale parameter family.

If $(T_{(1)}, \ldots, T_{(n)})$ are the order statistics from a Weibull distribution with parameters $(\alpha, \beta)$, then $X_{(i)} = \ell n(T_{(i)})$ $(i=1, \ldots, n)$ are the order statistics of an extreme-value distribution with parameters $\theta = \ell n(\alpha)$ and $\sigma = \beta^{-1}$. They may conveniently be represented by $X_{(i)} = \theta + \sigma Z_{(i)}$ $(i=1, \cdots, n)$ where $(Z_{(1)}, \ldots, Z_{(n)}$ are the order statistics from an extreme value distribution with parameters $\theta = 0$ and $\sigma = 1$. BLU estimation consists of finding constants $\{a_i\}$ and $\{b_i\}$ such that $\theta^* = \sum_{i=1}^{n} a_i X_{(i)}$ and $\sigma^* = \sum_{i=1}^{n} b_i X_{(i)}$ are unbiased estimates of $\theta$ and $\sigma$, respectively, and have minimum variance among all such unbiased estimators of $\theta$ and $\sigma$. All that is needed are tables of the expectation vector and covariance matrix of $(Z_{(1)}, \cdots, Z_{(n)})$, and these tables are referenced in [17]. BLI estimators of $\theta$ and $\sigma$ are minimum variance in the class of statistics with the property that the estimators thus obtained are transformed in the same way as the observations, under changes of scale and location. This class of estimators includes unbiased estimators. Further, BLI estimators have smallest expected loss among all linear estimators with invariant loss. Although BLI estimators are not unbiased, Mann has shown that they are easily obtained from BLU estimators, and have smaller expected loss than BLU estimators.

The Method of Johns and Lieberman for the Reliability

The method of Johns and Lieberman [13] is used to find exact lower confidence bounds for $R(t)$, the reliability at time $t$. It depends on the fact that the natural logarithms of the observations of a Weibull have an extreme-value distribution, which is a location-scale parameter family; the method also can be used whenever the underlying family of distributions is a location-scale parameter family. Let $(X_{(1)}, \cdots, X_{(n)})$ be the order statistics from the Weibull distribution. Let $p$ be the percentage of observations which aren't censored (from above), and $r = [np]$, where $[x]$ means the largest integer not larger than $x$, and $Y_{(i)} = \ell n(X_{(i)} - \ell n(t))$ $(i=1, \cdots, n)$.

Since $R(t) = \exp(-\exp(\beta(\ell n(t) - \ell n(\alpha))))$, by letting $\mu = \ell n(\alpha) - \ell n(t)$, and $\sigma = \beta^{-1}$, it is sufficient to find a lower confidence bound on $\mu/\sigma$. By choosing weights $\{a_i\}$ and $\{b_i\}$, they define estimators of $\mu$ and $\sigma$ as $Z_a = \sum_1^n a_i Y_{(i)}$ and $Z_b = \sum_1^n b_i Y_{(i)}$, respectively. By choosing $\{a_i\}$ and $\{b_i\}$ appropriately, $Z_a$ and $Z_b$ can be made to be asymptotically jointly normal.

The statistics $V_a = \frac{1}{\sigma}(Z_a - \mu)$ and $V_b = \frac{1}{\sigma} Z_b$ have distributions independent of $(\mu, \sigma)$. A function, $L(t)$, can be defined by $P(L(t) \leq tV_b - V_a) = \gamma$, for all real $t$. Actual values of $L(t)$ are found by simulation. It follows directly that $P(L(Z_a/Z_b) < \frac{\mu}{\sigma}) = \gamma$. Thus, calculating $Z_a/Z_b$ and using the table of values of $L(t)$ provided by

Johns and Lieberman enables one to obtain an exact $\gamma$-level lower confidence bound for $\mu/\sigma$, provided there is no censoring or type II censoring.

## A Three Order Statistic Method for Quantiles

Another method, due to Mann [19], enables one to construct an exact .90-level lower confidence bound on the .05 quantile of a Weibull distribution based on three order statistics. The method is applicable to type II progressive censoring and a table has been constructed for all possible censoring patterns up to sample size six. If $X_\nu, X_p, X_q$ are the natural logarithms of the three order statistics in question, the bound is of the form $t_\nu = \exp(X_\nu + \underline{V}_{\nu,p,q}(\alpha)(X_q - X_p))$, where $\underline{V}_{\nu,p,q}(\alpha)$ is the $100\,\alpha\%$ point of a certain distribution which can be found analytically. For small samples sizes, Mann believes this three order statistic method to be efficient compared with the m. l. method, the BLI method, and the Johns-Lieberman method. She also believes it to be the only method, up to the time of publication, which is applicable to type II progressive censoring. In that case, the proposed Bayesian procedure is the only competitor at present.

## The Bayesian Method

The proposed Bayesian approach is described in detail in Chapter II; it will be seen that it has close ties with the fiducial method.

The association between the two methods has been discussed in Fraser [8], Hora and Buehler [12], Wallace [26], and Lindley [16]. However, the Bayesian approach also is applicable to censored cases of all types, while the fiducial method isn't.

The proposed Bayesian approach enjoys advantages over the other methods presented in the literature. One major advantage is that it utilizes all the available information, while other methods first reduce the data to something less than a sufficient statistic. Savage [23] has shown that the minimal sufficient statistic for the parameters of the extreme-value distribution, and therefore of the Weibull distribution, is the order statistic. The full distribution of the information in the order statistic by the Bayesian method is reflected in the fact that different order statistics give rise to different posterior distributions.

A second advantage is that the Bayesian method can be applied to all censoring situations, while other methods either cannot be applied to all censoring patterns or can be applied only with considerable difficulty.

A third advantage is that confidence regions can be calculated analytically, whereas some of the other approaches described here, such as the Johns and Lieberman method or the Thoman, Bain, and Antle approach, require simulation to obtain appropriate percentage points.

A fourth advantage of the Bayesian method is discussed in detail in Chapter IV, where it is shown that, in some circumstances, the conditional probability of coverage, given a function of certain ancillary statistics, of certain parameters is equal to the nominal level. Other methods do not possess this property. It is shown that any other location and scale invariant procedure cannot, conditional on an ancillary statistic, possess the nominal probability of coverage, even though it does unconditionally.

A fifth advantage is that the Bayesian approach allows one easily to make inferences about functions of parameters, as well as the parameters themselves. In this connection, the concept of an "invariantly estimable function," as defined by Hora and Buehler [12] is found to be very important.

The major disadvantage of the Bayesian procedure is that it requires at least two numerical integrations to obtain confidence bounds for $\alpha$ and $\beta$. However, in an experiment of real value, this doesn't seem to be too great a difficulty in order to obtain the other advantages the method possesses.

A good deal of attention is given to the frequency properties of the Bayesian method because we often find that it offers solutions to problems which are difficult, if not impossible, to solve from a frequentist point of view. In addition, these Bayesian solutions often have properties which are highly desirable from a frequentist point of view; specifically, the frequency with which an $\alpha$-level Bayesian confidence interval actually covers the parameter of interest is often either exactly equal to or close to the nominal level. This is shown by theoretical investigation and by simulation. Consequently, one doesn't have to be a "Bayesian" in order to find the proposed method of value. This is especially important since there is no frequentist solution at present to many of these problems.

# II. A BAYESIAN PROCEDURE FOR THE WEIBULL DISTRIBUTION

## Choice of a Prior Distribution and the Likelihood

The essential feature of any Bayesian approach is the assumption of a prior distribution; the matter of which prior distribution to assume can be a difficult one, particularly when no natural conjugate one exists, as in the present case. However, choosing the prior $\pi(\alpha, \beta) = \pi(\alpha)\pi(\beta) = \frac{1}{\alpha\beta}$ is the same as choosing the prior $\pi(\theta,\sigma) \propto \frac{1}{\sigma}$, where $\theta = \ell n(\alpha)$, $\sigma = \beta^{-1}$ are location and scale parameters. This choice of a prior for $(\theta,\sigma)$ is well-known to lead often to results which are the same as obtained from a classical or frequentist approach, so by choosing the indicated prior for $(\alpha, \beta)$, one might expect to get results which have desirable properties from a frequentist point of view. Other priors will be briefly considered.

In order to introduce standard notation, the idea of censoring needs to be mentioned. If n items are put on test, k may fail and n-k may be removed from the test before failure; all that is known about these n-k items is that at the time of removal, they hadn't failed. There are various types of rules for censoring which are discussed in Chapters I and III; since the likelihood function of the observations doesn't depend on the censoring rule, but only on the observations, the inferences made by the Bayesian procedure will not depend on the censoring rule. This fact gives the Bayesian procedure wider applicability than many frequentist procedures which are limited either to the full sample case (i. e. no censoring whatsoever) or a specific type of censoring, because the frequentist needs to specify the censoring rule.

Let $\omega$ be the (possibly vector-valued) parameter of a family of c. d. f. 's $F(x|\omega)$,

$n$ be the number of items put on test,

$k$ be the number of failures,

$t_1, \cdots, t_k$ be the times of the $k$ failures,

$t_{k+1}, \cdots, t_n$ be the times of the $n-k$ censorings.

Then, for any censoring rule, the likelihood function is proportional to

$$\prod_1^k f(t_i|\omega) \prod_{k+1}^n [1-F(t_i|\omega)]$$

where $f(t_i|\omega)$ is the density.

The two parameter Weibull distribution is

$$F(t|\alpha, \beta) = 1-\exp(-(\frac{t}{\alpha})^\beta) .$$

If $\pi(\alpha, \beta) \propto \frac{1}{\alpha\beta}$, we have for the joint posterior density of $(\alpha, \beta)$, given $\underline{T}$, the sample,

$$\pi(\alpha, \beta|\underline{T}) \propto \frac{\beta^{k+1}}{\alpha^{k\beta+1}} (\prod_1^k t_i)^\beta \exp(-\alpha^{-\beta} \sum_1^n t_i^\beta) .$$

By integrating out $\alpha$, we obtain the posterior marginal density of $\beta$

$$\Pi(\beta \mid \underline{T}) \propto \frac{\beta^{k-2} \left(\prod_{1}^{k} t_i\right)^{\beta}}{\left(\sum_{i=1}^{n} t_i^{\beta}\right)^{k}}.$$

The normalizing constant for $\Pi(\beta \mid \underline{T})$ is found by numerical integration

$$K(\underline{T}) = \int_0^{\infty} \frac{\beta^{k-2} \left(\prod_{1}^{k} t_i\right)^{\beta}}{\left(\sum_{1}^{n} t_i^{\beta}\right)^{k}} \, d\beta.$$

An upper $\gamma$-level Bayesian confidence bound, $\bar{\beta}(\underline{T}, \gamma)$, for $\beta$ is given by the solution to

$$\text{(2.1.1)} \qquad \frac{1}{K(\underline{T})} \int_0^{\bar{\beta}(\underline{T}, \gamma)} \frac{\beta^{k-2} \left(\prod_{1}^{k} t_i\right)^{\beta}}{\left(\sum_{1}^{n} t_i^{\beta}\right)^{k}} \, d\beta = \gamma .$$

The solution must be found by first finding an approximate solution, possibly by large sample theory, and then interpolating to find a more precise solution. The process is simplified because the c.d.f. is a monotone function.

One can easily obtain the posterior distribution of a function $g(\alpha, \beta)$ if it is a strictly monotone function of $\alpha$ for each fixed $\beta$.

(This includes, of course, the function $g(\alpha, \beta) \equiv \alpha$). In this case, $g(\alpha, \beta)$ can be solved (at least theoretically) for $\alpha$. Let this solution be $\alpha = h(g, \beta)$. Then, taking $g$ to be increasing in $\alpha$,

$$P[g(\tilde{\alpha}, \tilde{\beta}) \leq g \mid \underline{T}] = \int_0^\infty P[\tilde{\alpha} \leq h(g, \beta) \mid \beta, \underline{T}]\, \pi(\beta \mid \underline{T})\, d\beta \tag{2.1.2}$$

$$= \int_0^\infty G\left(\frac{\sum_1^n t_i^\beta}{(h(g,\beta))^\beta} \,\Big|\, k\right) \pi(\beta \mid \underline{T})\, d\beta$$

where $G(x|k) = \exp(-x)\left(\sum_{j=0}^{k-1} \frac{1}{j!} x^j\right)$, the compliment of the c. d. f. of a gamma distribution with $k$ degrees of freedom. Equation (2.1.2) follows from the fact that

$$\pi(\alpha \mid \beta, \underline{T}) = (\Gamma(k-1)\alpha^\beta)^{-1} \left(\sum_1^n t_i^\beta\right) \exp\left(-\alpha^{-\beta} \sum_1^n t_i^\beta\right).$$

This distribution is known as the inverted gamma distribution and is discussed by Bhattacharya [4].

It should be pointed out that even if $g(\alpha, \beta)$ isn't monotone in $\alpha$ for each fixed $\beta$, it would be possible, at least in principle, to find the posterior density of $g(\alpha, \beta)$ by dividing the interval $(0, \infty)$ into subintervals such that $g(\alpha, \beta)$ is monotone in $\alpha$ in each subinterval for each $\beta$. Since all functions of interest are monotone in $\alpha$ for each $\beta$, there is no point in investigating this aspect more deeply here.

In applied work, it is usually advisable to calculate the posterior density of $g(\alpha, \beta)$. An expression for the density can be obtained if $h(g, \beta)$ is differentiable in $g$ for all positive $\beta$. The differentiation under the integral sign in expression (2.1.2) is justified by the general theory of Riemann integration. The expression for $f(g|\underline{T})$, the posterior density of $g(\alpha, \beta)$ evaluated at $g$, is

$$(2.1.3) \qquad f(g|\underline{T}) = \int_0^\infty \frac{1}{h(g,\beta)^{k\beta+1}} \frac{\beta(\sum_i^n t_i^\beta)^k h'(g,\beta)}{(k-1)!} \exp\left(-\frac{1}{(h(g,\beta))^\beta}\sum_i^n t_i^\beta\right)\pi(\beta|\underline{T})d\beta$$

where $h'(g, \beta) = \frac{\partial}{\partial g} h(g, \beta)$.

The posterior c.d.f. of $\alpha$ can be found from (2.1.2), and $\bar{\alpha}(\gamma, \underline{T})$, the $\gamma$-level upper Bayesian confidence bound for $\alpha$, can be calculated as the solution to the equation

$$(2.1.4) \qquad P[\tilde{\alpha} \leq \bar{\alpha}(\gamma, \underline{T})|\underline{T}] = \int_0^\infty G\left(\frac{1}{\bar{\alpha}^\beta}\sum_i^n t_i^\beta \Big| k\right)\pi(\beta|\underline{T})d\beta = \gamma.$$

This equation can be solved by first using an approximate solution, perhaps by large sample theory, and then interpolating to obtain a more precise solution. The process is simplified because the c.d.f. is a monotone increasing function. The posterior density of $\alpha$, evaluated at a, is obtained from (2.1.3)

(2.1.4)
$$\int_0^\infty \frac{\beta(\sum_1^n t_i^\beta)^k}{a^{\beta k+1}(k-1)!} \exp\left(-\frac{\sum_1^n t_i^\beta}{a^\beta}\right)\pi(\beta|\underline{T})d\beta .$$

Other functions which have the desired monotonicity property are the $\gamma^{th}$ quantile, $t_\gamma = \alpha(-\ell n(1-\gamma))^{1/\beta}$, the reliability function, $R(t) = \exp(-(t/\alpha)^\beta)$, and the mean, $\alpha\Gamma(1+\frac{1}{\beta})$. Thus, practically all of the parameters of interest have a posterior c. d. f. which can be expressed as a single integral. This fact is important in applied work because it is much easier to numerically integrated single integrals than double integrals.

In summary, Equation (2.1.2) enables one to easily find the posterior distributions of a class of functions which include most parameters of interest.

## Other Prior Distributions

When $\alpha$ and $\beta$ are both unknown in the Weibull distribution, there is no sufficient statistic of fixed dimensionality. This fact can be seen directly from the likelihood function

(2.2.1)
$$L(\underline{T}|\alpha,\beta) = \frac{\beta^k}{\alpha^{k\beta+1}}(\prod_1^k t_i)^\beta \exp[-\alpha^{-\beta}\sum_1^n t_i^\beta] .$$

Because of the expression $\sum_1^n t_i^\beta$, the likelihood cannot be reduced to the form

$$L(\underline{T}|\alpha, \beta) = K[y(\underline{T})|\alpha, \beta]P(s)$$

where $y(\underline{T})$ is of fixed dimensionality. The factorization criterion now asserts that there is no sufficient statistic of fixed dimensionality. (Savage [23] has shown this more rigorously). Consequently, there is no conjugate prior for $(\alpha, \beta)$ jointly. A standard reference is Raiffa and Schlaifer [22]. Since there is no conjugate prior, one may ask: what suitable families of priors exist for $(\alpha, \beta)$? The posterior distribution of $(\alpha, \beta)$ may be written:

(2.2.2) $$\pi(\alpha, \beta|\underline{T}) \propto \left[\frac{\pi(\beta)\beta^k(\prod_1^k t_i)^\beta}{(\sum_1^n t_i^\beta)^{k+1}}\right]\left[\frac{\pi(\alpha|\beta)(\sum_1^n t_i^\beta)^{k+1}\exp(-\alpha^{-\beta}\sum_1^n t_i^\beta)}{\alpha^{k\beta}}\right].$$

If $\pi(\alpha|\beta) = \pi(\alpha)$, the most mathematically convenient prior for $\alpha$ is the family of diffuse priors $\pi(\alpha) = \alpha^{-s}$ for $s > 0$. This family has the advantage that the second factor will have a known density, an inverted gamma distribution. The special case $s = 1$ will be the only one investigated since it often gives desirable answers from a frequentist point of view. It is an open question as to whether there is any advantage to allowing $s$ to be some other value.

By inspection of the form of the second factor, it can be seen that the most obvious other mathematically convenient prior distributions for $\alpha$ are of the form:

(2.2.3) $$\pi(\alpha|\beta) \propto \exp[-\frac{w}{\alpha^{\beta}}]/\alpha^{s}, \quad w > 0, \; s > 0 .$$

In this case, the prior on $\alpha$ depends on $\beta$. We have not pursued whether this is a useful form of prior dependence of $\alpha$ on $\beta$.

The first factor in (2.2.2) is $\pi(\beta|\underline{T})$; it corresponds to no standard probability distribution. In fact, if $\pi(\beta|\underline{T})$ is thought of as a member of a parameterized family there must be n parameters. Because $\beta$ is integrated numerically we can choose $\pi(\beta)$ in almost any manner that we wish; it won't make the calculations or theory substantially different except that for the prior $\pi(\beta) \propto \frac{1}{\beta}$, we get answers which are desirable from a frequentist viewpoint.

# III. FREQUENTIST PROPERTIES OF BAYESIAN METHODS

## The Censored Case

In order to investigate the frequentist properties of the proposed Bayesian method, the censoring pattern must be specified so that it is clear what a repetition of the experiment is. The work of various authors on different types of censoring is summarized in Mann [18].

The two most common models are called type I and type II censoring. In the most general setting, type I censoring requires starting with $n$ items on test. At a predetermined time, $n_i$ $(i=1, \cdots, k)$ of the items which haven't failed by time $T_i$ are randomly removed. If for some time $T_i$, there are less than $n_i$ items on test, all of the remaining items are censored at time $T_i$ and the test is terminated. The number of items which fail in this type of censoring is a random number.

Similarly, progressive type II censoring starts with $n$ items on test and at the failure of the $r_j^{th}$ items, exactly $n_j$ $(j=1, \cdots, k)$ items are randomly removed. It is easy to see that a large class of censoring patterns can be constructed using the basic ideas involved in these two types of censoring.

If the density governing time to failure is of the form $\frac{1}{\sigma}f(\frac{x-\theta}{\sigma})$, i.e. a location-scale parameter density, then the likelihood function of the observations under a type II censoring pattern is

$$L(\underline{x} \mid \theta, \sigma) \propto \prod_{j=1}^{k} \left\{ \frac{1}{\sigma} f\left(\frac{x_{r_j} - \theta}{\sigma}\right) \left[1 - F\left(\frac{x_{r_j} - \theta}{\sigma}\right)\right]^{n_j} \right\}.$$

Since the likelihood is a function of $\{\frac{x_{r_j} - \theta}{\sigma}, \ j=1, \cdots, k\}$, it is a location-scale parameter model. It will be seen that this is very important in studying the frequency properties of the Bayesian intervals.

When the censoring is of type I, we do not have a location-scale parameter problem. The frequentist properties of the Bayesian procedure for this case will be investigated in Chapter V using simulation.

In practice, one often doesn't known how the censoring came about. For example, a doctor wishes to investigate the curative powers of a treatment of some illness. Patients come to his office by some unknown rule of nature. He applies the treatment and continues applying it until a cure is effected or the treatment must be terminated before a cure for some non-medical reason, such as the patient leaving town. The data is censored, but the censoring rule isn't known, and it can't be classed either as a type I censoring pattern nor as a type II censoring pattern. Such a situation is called unspecified censoring and is very common. Since the Bayesian procedure depends only on the data and not on the censoring rule, it may still be applied.

Methods have been developed for analyzing censored data when the censoring pattern is known to be of type I or of type II. Most of the exact methods are for type II censoring. With unspecified censoring, the censoring pattern is unknown and so it is impossible to develop any method of inference from a frequentist point of view.

## The Fiducial Method and the Bayesian Method

An interesting and important fact about the proposed Bayesian method is that when we have essentially a location-scale parameter model (by transforming to the extreme-value distribution), it corresponds to the fiducial method for location and scale parameters. That is, when $\pi(\theta, \sigma) \propto \frac{1}{\sigma}$, the posterior densities of $\tilde{\theta} = \ell n(\tilde{\alpha})$, and $\tilde{\sigma} = \tilde{\beta}^{-1}$ are the same as their fiducial densities, both jointly and marginally. Thus, the two methods will give the same confidence intervals when there is no censoring or the censoring is of type II. We will also see that Bayesian confidence regions are closely related to Pitman confidence regions for $(\theta, \sigma)$, and this fact has bearing on the frequentist properties of the Bayesian method also. In this section, the fiducial method and its relation to the proposed Bayesian method will be discussed.

The fiducial method, as described by Fisher [7], for finding confidence intervals for $\theta$ and $\sigma$ would be applied as follows (assuming no censoring). First of all, a sufficient statistic of the same dimension as the unknown parameter is found. Then a function of the sufficient statistic and the parameter is found whose distribution is independent of the parameters is determined; any such function of a sufficient statistic and unknown parameters is called a pivotal quantity. The fiducial argument consists of formally changing the variables from the pivotal quantity to the density of the parameter, fixing the observed sufficient statistic. Thus, the parameter assumes the status of a random variable whose density is determined conditional on the sufficient statistic.

Many densities, including the extreme-value distribution, don't have a minimal sufficient statistic whose dimension is independent of the sample size. As Fraser [8] and Fisher [7] point out, it is sometimes possible to obtain a conditionally sufficient statistic whose dimensionality is the same as that of the unknown parameter by conditioning on an ancillary statistic. Consequently, a conditional fiducial distribution for the parameter is obtained. In the case of the extreme-value distribution this procedure may be applied as follows; the density is of the form $\frac{1}{\sigma} f(\frac{x-\theta}{\sigma})$. A sufficient statistic is the complete sample $\underline{X} = (X_1, \cdots, X_n)$ which has the density $\frac{1}{\sigma^n} \prod_1^n f(\frac{X_i - \theta}{\sigma})$. An ancillary statistic is $A(\underline{X}) = (\frac{X_3 - X_1}{X_2 - X_1}, \cdots, \frac{X_n - X_1}{X_2 - X_1}) = (A_3, \cdots, A_n)$. The statistic $(X_1, X_2)$ is conditionally sufficient, given $A(\underline{X})$, because it is possible to recover the complete sample $\underline{X}$ from $(X_1, X_2)$ and $A(\underline{X})$. The conditional density of $(X_1, X_2)$ given $A(\underline{X})$ is

$$h_{(\theta, \sigma)}(X_1, X_2 \mid A(\underline{X})) \propto \frac{|X_1 - X_2|^{n-2}}{\sigma^n} f(\frac{X_1 - \theta}{\sigma}) f(\frac{X_2 - \theta}{\sigma}) \prod_3^n f(\frac{A_j(X_2 - X_1) + X_1 - \theta}{\sigma}).$$

A pivotal statistic is $(T_1, T_2) = (\frac{X_1-\theta}{\sigma}, \frac{X_2-\theta}{\sigma})$. The distribution of $(T_1, T_2)$ is clearly independent of $(\theta, \sigma)$ and has conditional density

$$g(t_1, t_2) = h_{(0,1)}(t_1, t_2 \mid A(\underline{X})).$$

By a change of variable from $(T_1, T_2)$ to $(\theta, \sigma)$, the fiducial density of $(\theta, \sigma)$, conditional on $A(\underline{X})$, is found to be

$$K(\theta, \sigma \mid X_1, X_2, A(\underline{X})) \propto \frac{|X_1 - X_2|^{n-1}}{\sigma^{n+1}} f(\frac{X_1-\theta}{\sigma}) f(\frac{X_2-\theta}{\sigma}) \prod_3^n f(\frac{A_j(X_2-X_1) + X_1 - \theta}{\sigma})$$

$$\propto \frac{1}{\sigma^{n+1}} \prod_1^n f(\frac{X_i-\theta}{\sigma}).$$

This function is the same as the posterior distribution of $(\theta, \sigma)$ in the proposed Bayesian procedure. Since the joint densities are the same, the two methods will also give the same density for any function $g(\theta, \sigma)$ of the parameters. In particular, the marginal densities of $\theta$ and $\sigma$ will be the same.

The formal correspondence between the fiducial and the Bayesian method was shown for the full sample case, but it is important to understand that when there is censoring of a type that leaves the problem of a location-scale parameter type, the two methods still correspond and the frequentist properties of the Bayesian procedures are the same as those of the fiducial. We have already seen that with progressive type II censoring, the likelihood function of the observations with the extreme-value distribution is of location-scale parameter form, and so the fiducial and the Bayesian method will give the same answers and the properties of the Bayesian procedure are the same as those of the fiducial. We will see in the next section that these properties are the ones which are desirable from a frequency point of view.

## Frequentist Properties of the Fiducial Method

The frequentist properties of the fiducial method have been investigated by Hora and Buehler [12] and Bartholomew [3] and their findings are very relevant to the proposed Bayesian method. Although they are concerned mainly with point estimation, they prove a theorem about confidence intervals which throws light on properties of the Bayesian method. Their results are made especially useful because they apply to a large class of functions of $(\theta, \sigma)$ called invariantly estimable functions.

Let $\Omega$ represent the parameter space, $\Psi$ a group of transformations on $\Omega$, and $g$ a function on $\Omega$. A group $\Psi'$ of transformations on the range of $g$ is defined by $\psi'(g(\omega)) = g(\psi(\omega)), \forall \omega \epsilon \Omega$. A function $g$ is invariantly estimable if $g(\omega_1) = g(\omega_2) <=> g(\psi(\omega_1)) = g(\psi(\omega_2))$, $\forall \psi \epsilon \Psi$. We are interested in solutions $\overline{g}(\underline{x}, \gamma)$ to the equation

$$P_f\{g(\omega) \leq \overline{g}(\underline{x}, \gamma) | \underline{x}\} = \gamma \qquad (3.2.1)$$

where $f$ stands for fiducial probability (of $g(\omega)$ given $\underline{x}$). The following theorem is proved by Hora and Buehler:

If (1) $g$ is an invariantly estimable function,

(2) equation (3.2.1) has a unique solution,

(3) $\psi' g$ is strictly increasing in $g$ for all $\psi' \epsilon \Psi'$,

then $\overline{g}(\underline{x}, \gamma)$ has the confidence interval property.

The confidence interval property means that $\overline{g}(\underline{x}, \gamma)$ has the frequency interpretation associated with confidence regions. In our problem, either for the full sample or the type II censored situation,

$$\Omega = \{(\theta, \sigma) \mid -\infty < \theta < \infty, \quad \sigma > 0\}.$$

$$\Psi = \{\psi \mid \psi(\theta, \sigma) = (a + b\theta, \ b\sigma), -\infty < a < \infty, \ b > 0\}.$$

The important idea is that of an invariantly estimable function. However, an important example of a non-invariantly estimable function with the confidence interval property will be given.

Using the joint posterior density of $(\theta, \sigma)$, which corresponds to their fiducial distribution, it is possible to obtain the posterior density of $g(\theta, \sigma)$, a function of $(\theta, \sigma)$. The posterior density of $g(\theta, \sigma)$ will be the same as the fiducial density of $g(\theta, \sigma)$. Thus, it is possible to obtain confidence intervals for functions of the parameters, and these Bayesian intervals will be the same as the intervals obtained from the fiducial distribution, when we have no censoring or progressive type II censoring. Making an inference about $g(\theta, \sigma)$ is the same as making an inference about $g'(\alpha, \beta) = g(\theta, \sigma)$ where $\theta = \ell n(\alpha)$ and $\sigma = \beta^{-1}$.

As long as $g(\theta, \sigma)$ is invariantly estimable and satisfies the other conditions of the Hora-Buehler theorem, Bayesian upper confidence bounds for it will have the confidence interval property. Other types of intervals obtained by the posterior density of $g(\theta, \sigma)$ may not have the confidence interval property.

## Applications of the Theory of Invariantly Estimable Functions

A particularly useful application of this theory is to the function $t_\gamma = \alpha(-\ell n(1-\gamma))^{1/\beta}$, the $\gamma^{th}$ quantile. Since $\ell n(t_\gamma) = \ell n(\alpha) - \ell n(1-\gamma)/\beta = \theta - \sigma(\ell n(1-\gamma))$, and Hora and Buehler have shown functions of $(\theta, \sigma)$ of this type to be invariantly estimable and it meets the other conditions of the Hora-Buehler theorem, Bayesian upper confidence bounds for $t_\gamma$ have the confidence interval property. The associated problem of finding Bayesian tolerance regions has been studied by Aitchison [1].

An example of a function which isn't invariantly estimable is $R(t|\alpha, \beta) = \exp(-(t/\alpha)^\beta)$, the reliability at time $t$. Finding an upper Bayesian confidence bound for $R(t|\alpha, \beta)$ is the same as finding one for $(\ell n(t)-\theta)/\sigma$. This parameter is of great interest in practice and it also is an example of a non-invariantly estimable function for which a lower (or upper) Bayesian confidence bound has the confidence interval property. This fact is proved by the following argument. We know that functions of the type $\theta + c\sigma$ (with fixed c) are invariantly estimable and satisfy all the other conditions of the Hora-Buehler theorem. Therefore, the probability of coverage in the sample space is:

$$P_{x|\theta,\sigma}[P(\tilde{\theta} + c\tilde{\sigma} \le \theta + c\sigma | \underline{x}) \le \alpha] = \alpha .$$

This identity in $(c, \theta, \sigma)$ is an immediate consequence of the Hora-Buehler theorem and the relationship between the fiducial method and Bayes' method. If we chose a particular $c = (\ell n(t)-\theta)/\sigma$, this becomes

$$P_{x|\theta,\sigma}[P(\tilde{\theta} + \tilde{\sigma}(\ell n(t)-\theta)/\sigma \le \theta + \sigma(\ell n(t)-\theta)/\sigma | \underline{x}) \le \alpha]$$

$$= P_{x|\theta,\sigma}[P((\ell n(t)-\tilde{\theta})/\tilde{\sigma} \ge (\ell n(t) - \theta)/\sigma | \underline{x}) \le \alpha] = \alpha .$$

This shows that the Bayesian method gives lower confidence bounds for $(\ell n(t)-\theta)/\sigma$ which have the confidence interval property, and the same result holds for the reliability function of the Weibull distribution.

A general method of finding confidence intervals for functions $g(\alpha, \beta)$ which aren't invariantly estimable can be outlined as follows. (It should be pointed out, however, that these intervals do not necessarily have the confidence interval property.) First, find a $\gamma$-level joint confidence region $A(\underline{T}, \gamma)$ for $(\alpha, \beta)$, based on the joint posterior density $\pi(\alpha, \beta | \underline{T})$. Let

$$L(\underline{T}, \gamma) = \inf \{g(\alpha, \beta) | (\alpha, \beta) \in A(\underline{T}, \gamma)\} ,$$

$$U(\underline{T}, \gamma) = \sup \{g(\alpha, \beta) | (\alpha, \beta) \in A(\underline{T}, \gamma)\},$$

$$B(U, L) = \{(\alpha, \beta) | L \le g(\alpha, \beta) \le U\}.$$

Since $A(\underline{T},\gamma) \subseteq B(U(\underline{T},\gamma), L(\underline{T},\gamma))$, we have

$$P\{L(\underline{T},\gamma) \leq g(\tilde{\alpha},\tilde{\beta}) \leq U(\underline{T},\gamma) \mid \underline{T}\} \geq P\{(\tilde{\alpha},\tilde{\beta}) \epsilon A(\underline{T},\gamma) \mid \underline{T}\} = \gamma .$$

These probability statements signify Bayesian probability rather than frequentist probability. The frequency of the event $\{(\alpha, \beta) \epsilon A(\underline{T},\gamma)\}$ is of great interest, since it is a lower bound for the frequency of the event $\{L(\underline{T},\gamma) \leq g(\alpha,\beta) \leq U(\underline{T},\gamma)\}$. In the next section, a practical method of determining a region $A(\underline{T},\gamma)$ so that the event $\{(\alpha, \beta) \epsilon A(\underline{T},\gamma)\}$ occurs with frequency $\gamma$ will be presented.

## A Joint Confidence Region

In the last section, we saw the importance of determining a joint confidence region $A(\underline{T},\gamma)$ which covers $(\alpha, \beta)$ with frequency $\gamma$. Such a procedure is the following one. First, choose $\gamma_i (i=1,2)$ such that $0<\gamma_i<1$ $(i=1,2)$ and $\gamma_1\gamma_2 = \gamma$. Next, determine $\bar{\beta}(\underline{T},\gamma_1)$ as the solution to

$$\int_0^{\bar{\beta}(\underline{T},\gamma_1)} \pi(x \mid \underline{T}) dx = \gamma_1 . \tag{3.4.1}$$

If the sample, $\underline{T}$, contains exactly $k$ failures, then determine $k(\gamma_2)$ by $G(k(\gamma_2)\mid k) = \gamma_2$ where $G(x \mid k) = \exp(-x)(1 + \sum_{j=1}^{k-1} \frac{1}{j!} x^j)$. That is, the compliment of $G(x \mid k)$ is the c. d. f. of a gamma distribution with $k$ degrees of freedom. So $k(\gamma_2)$ can easily be determined. For each

$\beta$ satisfying $0 < \beta < \overline{\beta}$, define $\alpha(\beta, \gamma_2)$ by

(3.4.2) $$\alpha(\beta, \gamma_2) = \left[\frac{\sum_1^n t_i^\beta}{k(\gamma_2)}\right]^{1/\beta}.$$

Then

$$P\{\tilde{\alpha} \leq \alpha(\beta, \gamma_2) | \underline{T}, \beta\} = G\left(\frac{\sum_1^n t_i^\beta}{(\alpha(\beta, \gamma_2))^\beta} \Big| k\right) = G(k(\gamma_2)|k) = \gamma_2.$$

Define $A(\underline{T}, \gamma_1, \gamma_2) = \{(\alpha, \beta) | 0 < \beta < \overline{\beta}(\underline{T}, \gamma_1), \ 0 < \alpha < \alpha(\beta, \gamma_2)\}$. Then the Bayesian probability content of $A(\underline{T}, \gamma_1, \gamma_2)$ is $\gamma$.

This method of obtaining a joint confidence region for $(\alpha, \beta)$ has a direct connection with both the fiducial method and Pitman's method [20] for obtaining joint confidence regions for a location and scale parameter. This connection will become clear in the following paragraphs.

An analogous joint region for $\theta = \ell n(\alpha)$ and $\sigma = \beta^{-1}$ can be obtained as follows. First, transform the sample by taking natural logarithms of the Weibull data; call the transformed sample $\underline{X}$. Then, using the prior $\pi(\theta, \sigma) \propto \sigma^{-1}$, determine the marginal posterior distribution of $\sigma$. With $\gamma_1$ and $\gamma_2$ the same as those used in determining $A(\underline{T}, \gamma_1, \gamma_2)$, find $\overline{\sigma}(\underline{X}, \gamma_1)$, a $\gamma_1$-level upper confidence bound for $\sigma$, from the equation

$$P\{\tilde{\sigma} \leq \overline{\sigma}(\gamma_1, \underline{X}) | \underline{X}\} = \gamma_1.$$

This is equivalent to finding a $\gamma_1$-level upper confidence bound for $\sigma$ by the fiducial method or Pitman's method. It follows from the general theory of the fiducial method and Pitman confidence regions that the following equation holds

$$\bar{\sigma}(\sigma \underline{X} + \theta) = \sigma \bar{\sigma}(\underline{X}). \qquad (3.4.3)$$

Next, for each $\sigma$ with $0 < \sigma < \bar{\sigma}(\gamma_1, \underline{X})$, determine the conditional (on $\sigma$) posterior distribution of $\theta$ and solve for $\bar{\theta}(\underline{X}, \sigma, \gamma_2)$ from the equation

$$P\{\tilde{\theta} \leq \bar{\theta}(\underline{X}, \sigma, \gamma_2) \mid \underline{X}, \sigma\} = \gamma_2 .$$

The solution $\bar{\theta}(\underline{X}, \sigma, \gamma_2)$ is a conditional $\gamma_2$-level upper confidence bound for $\theta$. It follows from the general theory of the fiducial method and also Pitman's method that

$$\bar{\theta}(\sigma \underline{X} + \theta, \sigma, \gamma_2) = \theta + \bar{\theta}(\underline{X}, 1, \gamma_2). \qquad (3.4.4)$$

Let $B(\underline{X}, \gamma_1, \gamma_2) = \{(\theta, \sigma) \mid 0 < \sigma < \bar{\sigma}(\underline{X}, \gamma_1), \theta \leq \bar{\theta}(\underline{X}, \sigma, \gamma_2)\}$. It follows from (3.4.3) and (3.4.4) that

$$(\theta, \sigma) \in B(\sigma \underline{X} + \theta, \gamma_1, \gamma_2) <=> (0, 1) \in B(\underline{X}, \gamma_1, \gamma_2). \qquad (3.4.5)$$

Because the sample $\underline{X}$ is transformed from $\underline{T}$, it also follows that $(\alpha, \beta) \in A(\underline{T}, \gamma_1, \gamma_2) <=> (\theta, \sigma) \in B(\underline{X}, \gamma_1, \gamma_2)$ where $\theta = \ell n(\alpha)$, $\sigma = \beta^{-1}$.

So the probability of coverage of $(\theta, \sigma)$ is the same as the probability of coverage of $(\alpha, \beta)$. As Wallace points out, confidence regions for $(\theta, \sigma)$ which satisfy (3.4.5) are Pitman confidence regions, which are known to have exact probability of coverage. Therefore, the probability of coverage of $(\theta, \sigma)$ and of $(\alpha, \beta)$ by the regions $B(\underline{X}, \gamma_1, \gamma_2)$ and $A(\underline{T}, \gamma_1, \gamma_2)$, respectively, is exactly $\gamma$.

The method described here is useful because the region $A(\underline{T}, \gamma_1, \gamma_2)$ can be calculated analytically; it is especially easy to determine if a particular point $(\alpha, \beta)$ is in $A(\underline{T}, \gamma_1, \gamma_2)$, even without actually calculating the region explicitly. There are other Pitman confidence regions, all of which satisfy (3.4.5), such as regions of highest posterior density, but it is much more difficult to calculate these regions than the one already described. Another technique which may have some practical value is to calculate the posterior likelihood function $l(\alpha, \beta \mid \underline{T})$ in an area of interest. An example of such a grid is displayed in Figure 1. The technique requires obtaining $L$, the maximum of $l(\alpha, \beta \mid \underline{T})$, and a choice of $\alpha_0$, $\Delta_1$, and $\beta_0$ and $\Delta_2$. The values of $l(\alpha_0 + i\Delta_1, \beta_0 + j\Delta_2 \mid \underline{T}) = l(i, j)$ for $i=1$ to $I$ and $j=1$ to $J$, for suitably determined integers $I$ and $J$, are calculated. The ratios $L/l(i, j)$ are then found and the appropriate percentiles are displayed in the $(i, j)$ cell of the grid. Such a grid is relatively easy to calculate on a computer and quickly gives one an idea of what values of $(\alpha, \beta)$ have high posterior likelihood. Figure 1 displays such a grid using the data taken from Mann [19] which are (.38, .88, .96, 1.18, 1.78, 1.2). The first 5 observations are uncensored and the last is censored. The abscissa represents alpha and the ordinate represents beta.

Figure 1. Posterior likelihood function of alpha and beta.

The abscissa represents alpha. The ordinate represents beta.

| | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 3 | 8 | 13 | 17 | 19 | 19 | 16 | 13 | 10 | 8 | 6 | 4 | 3 | 2 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4.0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 5 | 10 | 15 | 20 | 22 | 21 | 19 | 15 | 12 | 9 | 6 | 5 | 3 | 2 | 2 | 1 | 1 | 1 | 1 | 1 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 6 | 12 | 19 | 24 | 26 | 25 | 22 | 18 | 14 | 10 | 7 | 5 | 4 | 3 | 2 | 1 | 1 | 1 | 1 | 1 |
| 3.8 | 1 | 1 | 1 | 1 | 1 | 1 | 3 | 8 | 15 | 22 | 28 | 29 | 28 | 24 | 20 | 16 | 12 | 9 | 6 | 4 | 3 | 2 | 2 | 1 | 1 | 1 | 1 |
| | 1 | 1 | 1 | 1 | 1 | 2 | 4 | 10 | 19 | 27 | 32 | 33 | 32 | 28 | 23 | 18 | 13 | 10 | 7 | 5 | 4 | 3 | 2 | 2 | 1 | 1 | 1 |
| 3.6 | 1 | 1 | 1 | 1 | 1 | 2 | 6 | 13 | 23 | 31 | 37 | 38 | 36 | 31 | 26 | 20 | 15 | 11 | 8 | 6 | 4 | 3 | 2 | 2 | 1 | 1 | 1 |
| | 1 | 1 | 1 | 1 | 1 | 3 | 8 | 17 | 27 | 36 | 42 | 43 | 40 | 35 | 29 | 23 | 17 | 13 | 10 | 7 | 5 | 4 | 3 | 2 | 2 | 1 | 1 |
| 3.4 | 1 | 1 | 1 | 1 | 1 | 4 | 11 | 21 | 32 | 42 | 47 | 48 | 44 | 39 | 32 | 25 | 20 | 15 | 11 | 8 | 6 | 4 | 3 | 2 | 2 | 1 | 1 |
| | 1 | 1 | 1 | 1 | 2 | 6 | 14 | 26 | 38 | 48 | 53 | 53 | 49 | 43 | 35 | 28 | 22 | 17 | 13 | 9 | 7 | 5 | 4 | 3 | 2 | 2 | 1 |
| 3.2 | 1 | 1 | 1 | 1 | 3 | 8 | 18 | 31 | 44 | 54 | 59 | 58 | 54 | 47 | 39 | 31 | 25 | 19 | 14 | 11 | 8 | 6 | 5 | 3 | 3 | 2 | 2 |
| | 1 | 1 | 1 | 1 | 4 | 11 | 22 | 37 | 50 | 60 | 65 | 64 | 58 | 51 | 43 | 35 | 27 | 21 | 16 | 12 | 9 | 7 | 5 | 4 | 3 | 2 | 2 |
| 3.0 | 1 | 1 | 1 | 2 | 6 | 14 | 28 | 43 | 57 | 67 | 70 | 69 | 63 | 55 | 47 | 38 | 30 | 24 | 18 | 14 | 11 | 8 | 6 | 5 | 4 | 3 | 2 |
| | 1 | 1 | 1 | 3 | 8 | 18 | 33 | 50 | 64 | 73 | 76 | 74 | 68 | 59 | 50 | 41 | 33 | 26 | 21 | 16 | 12 | 9 | 7 | 6 | 4 | 3 | 3 |
| 2.8 | 1 | 1 | 1 | 4 | 11 | 23 | 39 | 56 | 70 | 79 | 82 | 79 | 72 | 64 | 54 | 45 | 36 | 29 | 23 | 18 | 14 | 11 | 8 | 6 | 5 | 4 | 3 |
| | 1 | 1 | 2 | 6 | 15 | 29 | 46 | 63 | 77 | 85 | 87 | 83 | 76 | 67 | 58 | 48 | 39 | 32 | 25 | 20 | 16 | 12 | 10 | 8 | 6 | 5 | 4 |
| 2.6 | 1 | 1 | 3 | 8 | 19 | 35 | 53 | 70 | 83 | 90 | 91 | 87 | 80 | 71 | 61 | 51 | 42 | 34 | 28 | 22 | 18 | 14 | 11 | 9 | 7 | 5 | 4 |
| | 1 | 1 | 4 | 11 | 24 | 41 | 59 | 76 | 88 | 94 | 95 | 90 | 83 | 74 | 64 | 54 | 45 | 37 | 30 | 24 | 20 | 16 | 12 | 10 | 8 | 6 | 5 |
| 2.4 | 1 | 2 | 6 | 15 | 29 | 47 | 65 | 81 | 92 | 98 | 97 | 93 | 85 | 76 | 66 | 56 | 47 | 39 | 32 | 26 | 21 | 17 | 14 | 11 | 9 | 7 | 6 |
| | 1 | 3 | 9 | 19 | 35 | 53 | 71 | 86 | 96 | 99 | 99 | 94 | 86 | 77 | 68 | 58 | 49 | 41 | 34 | 28 | 23 | 19 | 16 | 13 | 10 | 9 | 7 |
| 2.2 | 2 | 5 | 12 | 24 | 40 | 58 | 75 | 89 | 97 | 99 | 99 | 94 | 87 | 78 | 69 | 60 | 51 | 43 | 36 | 30 | 25 | 21 | 17 | 14 | 12 | 10 | 8 |
| | 3 | 7 | 16 | 29 | 46 | 63 | 79 | 91 | 98 | 99 | 98 | 93 | 86 | 78 | 69 | 60 | 52 | 44 | 38 | 32 | 27 | 22 | 19 | 16 | 13 | 11 | 9 |
| 2.0 | 4 | 10 | 20 | 34 | 50 | 67 | 81 | 91 | 97 | 98 | 96 | 91 | 84 | 77 | 68 | 60 | 52 | 45 | 39 | 33 | 28 | 24 | 20 | 17 | 14 | 12 | 10 |
| | 6 | 13 | 24 | 38 | 54 | 69 | 81 | 90 | 94 | 95 | 93 | 88 | 82 | 74 | 67 | 59 | 52 | 45 | 39 | 34 | 29 | 25 | 21 | 18 | 15 | 13 | 11 |
| 1.8 | 8 | 17 | 28 | 42 | 57 | 69 | 80 | 87 | 90 | 90 | 88 | 83 | 78 | 71 | 64 | 57 | 51 | 45 | 39 | 34 | 30 | 26 | 22 | 19 | 16 | 14 | 12 |
| | 11 | 20 | 32 | 45 | 57 | 68 | 77 | 82 | 84 | 84 | 82 | 78 | 73 | 67 | 61 | 55 | 49 | 43 | 38 | 34 | 30 | 26 | 23 | 20 | 17 | 15 | 13 |
| 1.6 | 14 | 23 | 34 | 46 | 57 | 66 | 72 | 76 | 78 | 77 | 75 | 71 | 67 | 62 | 56 | 51 | 46 | 41 | 37 | 33 | 29 | 26 | 23 | 20 | 18 | 16 | 14 |
| | 17 | 26 | 36 | 46 | 54 | 61 | 66 | 69 | 70 | 69 | 67 | 64 | 60 | 56 | 51 | 47 | 43 | 39 | 35 | 31 | 28 | 25 | 22 | 20 | 18 | 16 | 14 |
| 1.4 | 19 | 27 | 36 | 44 | 51 | 56 | 59 | 61 | 61 | 60 | 58 | 56 | 53 | 49 | 46 | 42 | 39 | 35 | 32 | 29 | 26 | 24 | 21 | 19 | 17 | 16 | 14 |
| | 21 | 28 | 34 | 41 | 45 | 49 | 51 | 52 | 52 | 51 | 50 | 48 | 45 | 42 | 40 | 37 | 34 | 31 | 29 | 26 | 24 | 22 | 20 | 18 | 16 | 15 | 14 |
| 1.2 | 21 | 27 | 32 | 36 | 39 | 42 | 43 | 44 | 43 | 43 | 41 | 39 | 38 | 35 | 33 | 31 | 29 | 27 | 25 | 23 | 21 | 19 | 18 | 16 | 15 | 14 | 13 |
| | 20 | 24 | 28 | 31 | 33 | 34 | 35 | 35 | 35 | 34 | 33 | 32 | 30 | 29 | 27 | 25 | 24 | 22 | 21 | 19 | 18 | 17 | 15 | 14 | 13 | 12 | 11 |
| 1.0 | 18 | 21 | 23 | 25 | 26 | 27 | 27 | 27 | 27 | 26 | 25 | 24 | 23 | 22 | 21 | 20 | 19 | 18 | 17 | 16 | 15 | 14 | 13 | 12 | 11 | 11 | 10 |
| | 15 | 17 | 18 | 19 | 20 | 20 | 20 | 20 | 20 | 19 | 19 | 18 | 17 | 17 | 16 | 15 | 14 | 14 | 13 | 12 | 11 | 11 | 10 | 10 | 9 | 9 | 8 |
| .8 | 12 | 13 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | 13 | 13 | 13 | 12 | 12 | 11 | 11 | 10 | 10 | 9 | 9 | 8 | 8 | 8 | 7 | 7 | 7 | 6 |
| | | .8 | | .9 | | 1.0 | | 1.1 | | 1.2 | | 1.3 | | 1.4 | | 1.5 | | 1.6 | | 1.7 | | 1.8 | | 1.9 | | 2.0 | |

# IV. CONDITIONAL PROBABILITIES OF COVERAGE OF NON-BAYESIAN PROCEDURES

We have already seen how inferences about parameters of a Weibull distribution are equivalent to inferences about parameters of the extreme-value distribution. The proposed Bayesian method was seen to give exact levels of coverage for invariantly estimable (and sometimes non-invariantly estimable) functions when there was no censoring or some type of type II censoring. When there is type I censoring or arbitrary censoring, the Bayesian method doesn't possess this property as can be seen from the results of simulation studies presented in Chapter V. On the other hand, most non-Bayesian methods do not possess the confidence interval property in the presence of type I censoring either. Moreover, when there is unspecified censoring, the notion of probability of coverage is not applicable.

In the introduction, it was pointed out that one of the advantages of the proposed Bayesian method was that it fully utilized the information contained in the minimal sufficient order statistic in the sense that different order statistics correspond to different posterior distributions. Other methods presented in the literature didn't fully utilize all the information. For example, the method of Thoman, Bain, and Antle utilizes only the m. l. estimates of $\alpha$ and $\beta$. Since different points in the sample space can give the same m. l. estimates $(\hat{\alpha}, \hat{\beta})$, the method can give the same $\gamma$-level confidence intervals for $\alpha$ and $\beta$ for different points in the sample space for all $0 < \gamma < 1$. However, this phenomenon could not happen with the Bayesian method since different sample points always give different posterior distributions. This is a consequence of the fuller utilization of the information made by the Bayesian method.

In this chapter some of the consequences of using less than all the available information in making inferences about $\alpha$ and $\beta$ will be investigated. Specifically, the conditional probabilities of coverage of $\alpha$ and $\beta$ by the Thoman, Bain, and Antle method will be found. The conditioning will be with respect to a certain function of ancillary statistic, which represents information in the order statistic which isn't contained in $(\hat{\alpha}, \hat{\beta})$. Although the full sample case is investigated in this chapter, the same methodology could fruitfully be extended to type II censoring situations and to other methods besides the Thoman, Bain and Antle.

The Thoman, Bain, and Antle method enables one to also find exact confidence bounds on $\theta = \ell n(\alpha)$ and $\sigma = \beta^{-1}$, the parameters of the associated extreme-value distribution. It will be simpler to present the relevant concepts by using a location-scale parameter family of distributions instead of the Weibull distribution. Let $\hat{\theta}$ and $\hat{\sigma}$ be the m. l. estimates of $\theta$ and $\sigma$ respectively. Let $X = \ell n(T_i)$ $(i=1, \cdots, n)$ be the natural logarithms of the original observations.

Define the following ancillary statistic:

$$R(\underline{X}) = \left(\frac{X_{(n)}-X_{(1)}}{X_{(2)}-X_{(1)}}, \cdots, \frac{X_{(3)}-X_{(1)}}{X_{(2)}-X_{(1)}}\right).$$

The statistic $(\hat{\theta}, \hat{\sigma}, R(\underline{X}))$ is in one to one correspondence with the order statistic and so it is minimal sufficient. The Thoman, Bain, and Antle method is based on $(\hat{\theta}, \hat{\sigma})$. The information which is ignored is contained in $R(\underline{X})$. So it is clear that the ancillary statistic $R(\underline{X})$ is of great interest and the methodology will be to examine conditional probabilities of coverage given a function of $R(\underline{X})$.

The following important fact should be noted about Bayesian confidence bounds. Suppose we are interested in a function $g(\theta, \sigma)$ and $\bar{g}(\underline{X}, \alpha)$ is the Bayesian $\alpha$-level upper confidence bound for $g(\theta,\sigma)$. If $g(\theta, \sigma)$ is invariantly estimable and $(\theta_x, \sigma_x)$ are any pair of possible values of $(\theta, \sigma)$ satisfying

(4.1.1) $$\bar{g}(\underline{X}, \alpha) = g(\theta_x, \sigma_x),$$

then

(4.1.2) $$\bar{g}(a\underline{X}+b, \alpha) = g(a\theta_x + b, a\sigma), \quad a > 0.$$

If $g(\theta, \sigma) = (t-\theta)/\sigma$, then

(4.1.3) $$\bar{g}(a\underline{X}+b, \alpha) = (t-a\theta_x - b)/a\sigma_x, \quad a > 0.$$

Let $g^*(\underline{X}, \gamma)$ be any other $\gamma$-level upper confidence bound which satisfies (4.1.2) when $g(\theta, \sigma)$ is invariantly estimable and (4.1.3) when $g(\theta, \sigma) = (t-\theta)/\sigma$. It is important for a method to possess this property so as not to give answers which depend on the units of measurement. The Thoman, Bain, and Antle method satisfies (4.1.2) when it is applied to $g(\theta, \sigma) = \theta$ and $g(\theta, \sigma) = \sigma$.

Thinking of $\gamma$ as fixed, the set $A_\alpha$ is defined by $A_\alpha = \{\underline{X} \mid \overline{g}(\underline{X}, \alpha) = g^*(\underline{X}, \gamma)\}$. The set $A_\alpha$ has the property that if $\underline{X} \epsilon A_\alpha$, then $a\underline{X}+b \epsilon A_\alpha$ for all $a > 0$. In the terminology of Buehler [5], $A_\alpha$ is a relevant subset. It follows that if $\underline{X}$ has a location and scale parameter distribution, the conditional distribution of $\underline{X}$ given $\underline{X} \epsilon A_\alpha$ is also of that type. Since the Bayesian upper confidence limit is exact in location-scale parameter families, we have

(4.1.4)
$$P_{x|\theta,\sigma}\{g^*(\underline{X}, \gamma) \geq g(\theta, \sigma) \mid g^*(\underline{X}, \gamma)\} = \overline{g}(\underline{X}, \alpha)\}$$
$$= P_{x|\theta,\sigma}\{\overline{g}(\underline{X}, \alpha) \geq g(\theta, \sigma) \mid g^*(\underline{X}, \gamma) = \overline{g}(\underline{X}, \alpha)\} = \alpha .$$

Thus, the conditional (on the event $g^*(\underline{X}, \gamma) = \overline{g}(\underline{X}, \alpha)$) probability of coverage is $\alpha$. The event $\{g^*(\underline{X}, \gamma) = \overline{g}(\underline{X}, \alpha)\}$ depends entirely on the ancillary statistic $A(\underline{X})$, so conditioning on it is the same as conditioning on a function of the ancillary statistic. In order to actually find the indicated $\alpha$, it is only necessary to calculate

(4.1.5)
$$\alpha = P\{g(\tilde{\theta}, \tilde{\sigma}) \leq g^*(\underline{X}, \gamma) \mid \underline{X}\}.$$

Thus, once $g^*(\gamma, \underline{X})$ is evaluated, $\alpha$ can be determined by numerical means from the posterior c. d. f. of $g(\theta, \sigma)$. This result can be applied to the function $g(\theta, \sigma) = \theta$ and $g(\theta, \sigma) = \sigma$, or equally well to $\alpha = \exp(\theta)$ and $\beta = \sigma^{-1}$. The posterior distributions of these functions can be calculated by numerical methods. If the Thoman, Bain, and Antle method gives $\alpha^*$ as a $\gamma$-level upper confidence bound for $\alpha$, then

$$(4.1.6) \qquad P[\tilde{\alpha} \leq \alpha^* | \underline{X}] = \int_0^\infty \pi(\beta | \underline{T}) G\left(\frac{\sum_1^n t_i^\beta}{\alpha^{*\beta}} \,\Big|\, k\right) d\beta$$

where $G(x|k) = \exp(-x)(\sum_0^{k-1} \frac{1}{j!} x^j)$, is the conditional probability of the event $\{\alpha^* \geq \alpha\}$. Similarly, if the Thoman, Bain, and Antle method gives $\beta^*$ as the upper bound for $\beta$, then

$$(4.1.7) \qquad P[\tilde{\beta} \leq \beta^* | \underline{T}] = \int_0^{\beta^*} \pi(\beta | \underline{T}) d\beta$$

is the conditional probability of the event $\{\beta^* \geq \beta\}$.

These conditional probabilities of coverage of the Thoman, Bain, and Antle method have been calculated for $\alpha$ and $\beta$ for sample size $n = 5$ and 100 different samples. The level of confidence, $\gamma$, was set at .75, .90, .95, and .98. The confidence limits for $\alpha$ and $\beta$ were calculated from tables provided by Thoman, Bain, and Antle. The same 100 samples were used for all the results, so they aren't independent of one another. If these conditional probabilities are thought of as random variables, these 100 samples can be used to estimate the mean and standard deviation of the conditional probabilities.

| nominal level | .75 | .90 | .95 | .98 |
|---|---|---|---|---|
| mean | .747 | .901 | .950 | .974 |
| s.d. | .00148 | .00414 | .00374 | .0034 |

Figure 2. Conditional probabilities of coverage of alpha.

| nominal level | .75 | .90 | .95 | .98 |
|---|---|---|---|---|
| mean | .753 | .903 | .950 | .980 |
| s.d. | .00339 | .00741 | .00770 | .00595 |

Figure 3. Conditional probabilities of coverage of beta.

These figures are believed accurate to 3 significant digits. Since the Thoman, Bain, and Antle method is exact, the mean of the conditional probabilities should equal the nominal level. Of course, in practice, there must necessarily be some error due to sampling variation in the simulation, but the data presented here indicate that this error is quite small for most practical purposes.

The sample size n = 5 was chosen because the Thoman, Bain, Antle method and the Bayesian method are asymptotically equivalent. The tables of Thoman, Bain, and Antle do not present sample sizes smaller than 5.

The results of this simulation may also be summarized by a histogram-type table of the conditional probabilities of coverage for $\alpha$ and $\beta$ at various levels. The conditional probabilities are calculated from Equations (4.1.6) and (4.1.7) by Simpson's rule and are believed to be accurate to 3 significant digits.

It is believed that these data indicate that the information ignored by Thoman, Bain, and Antle method is negligible for most applications and can be used as well as the proposed Bayesian method. It should be remembered, however, that the Bayesian limits can be calculated for any type of censored data and have exact frequency properties for type II censoring. In theory, the Thoman, Bain, and Antle method could be applied to type II censoring but a simulation would have to be performed for each particular censoring pattern, i.e. each set of ancillary statistics.

| Probability | <.740 | .740-2 | .742-4 | .747-6 | .746-8 | .748-.75 | >.750 |
|---|---|---|---|---|---|---|---|
| No. Observ. | 0 | 1 | 4 | 12 | 66 | 17 | 0 |

Figure 4. Frequency distribution of conditional probabilities of coverage for alpha. Unconditional probability of coverage = .75.

| Probability | <.746 | .746-.748 | .748-.750 | .750-.752 | .752-.754 |
|---|---|---|---|---|---|
| No. Observ. | 0 | 1 | 19 | 22 | 30 |

| Probability | .754-.756 | .756-.758 | .758-.760 | .760-.762 | >.762 |
|---|---|---|---|---|---|
| No. Observ. | 10 | 8 | 6 | 2 | 2 |

Figure 5. Frequency distribution of conditional probabilities of coverage for beta. Unconditional probability of coverage = .75.

| Probability | <.890 | .890-.892 | .892-.894 | .894-.896 | .896-.898 |
|---|---|---|---|---|---|
| No. Observ. | 3 | 1 | 4 | 5 | 10 |

| Probability | .898-.900 | .900-.902 | .902-.904 | .904-.906 |
|---|---|---|---|---|
| No. Observ. | 19 | 15 | 21 | 22 |

Figure 6. Frequency distribution of conditional probabilities of coverage for alpha. Unconditional probability coverage = 90.

| Probability | <.890 | .890-2 | .892-4 | .894-6 | .896-8 | .898-.900 | .900-2 |
|---|---|---|---|---|---|---|---|
| No. Observ. | 2 | 7 | 6 | 6 | 9 | 11 | 4 |

| Probability | .902-4 | .904-6 | .906-8 | .908-.910 | .910-2 | >.912 |
|---|---|---|---|---|---|---|
| No. Observ. | 8 | 6 | 12 | 5 | 9 | 15 |

Figure 7. Frequency distribution of conditional probabilities of coverage for beta. Unconditional probability of coverage = .90.

| Probability | .940-2 | .942-4 | .944-6 | .946-8 | .948-.95 | .950-2 |
| --- | --- | --- | --- | --- | --- | --- |
| No. Observ. | 4 | 3 | 5 | 9 | 22 | 14 |

| Probability | .952-4 | .954-6 |
| --- | --- | --- |
| No. Observ. | 25 | 18 |

Figure 8. Frequency distribution of conditional probabilities of coverage for alpha. Unconditional probability of coverage = .95.

| Probability | <.940 | .940-2 | .942-4 | .944-6 | .946-8 | .948-.950 |
| --- | --- | --- | --- | --- | --- | --- |
| No. Observ. | 12 | 6 | 6 | 8 | 9 | 5 |

| Probability | .950-2 | .952-4 | .954-6 | .956-8 | .958-.960 | .960-2 | >.962 |
| --- | --- | --- | --- | --- | --- | --- | --- |
| No. Observ. | 8 | 11 | 9 | 4 | 9 | 8 | 5 |

Figure 9. Frequency distribution of conditional probabilities of coverage for beta. Unconditional probability of coverage = .95.

| Probability | <.970 | .970-.972 | .972-.974 | .974-.976 | .976-.978 | .978-.98 |
| --- | --- | --- | --- | --- | --- | --- |
| No. Observ. | 8 | 11 | 24 | 19 | 29 | 9 |

Figure 10. Frequency distribution of conditional probabilities of coverage for alpha. Unconditional probability of coverage = .98.

| Probability | <.970 | .970-2 | .972-4 | .974-6 | .976-8 | .978-.980 |
| --- | --- | --- | --- | --- | --- | --- |
| No. Observ. | 2 | 10 | 7 | 12 | 10 | 9 |

| Probability | .980-2 | .982-4 | .984-6 | .986-8 | .988-.990 | .990-2 |
| --- | --- | --- | --- | --- | --- | --- |
| No. Observ. | 11 | 13 | 13 | 10 | 1 | 2 |

Figure 11. Frequency distribution of conditional probabilities of coverage for beta. Unconditional probability of coverage = .98.

# V. PROBABILITIES OF COVERAGE WITH TYPE I CENSORING

The purpose of this chapter is to investigate, by simulation, the frequency properties of the Bayesian procedure under type I censoring.

The class of type I censoring patterns is, of course, large and there is no point in trying to do any sort of exhaustive analysis of the frequency properties of the Bayesian procedure under many different type I censoring patterns. But it is useful from an applied standpoint to gain whatever understanding is possible of the frequentist properties of the Bayesian procedure by looking at some simpler type I censoring patterns, especially the kind which may be found in practice.

Let us consider the extreme-value distribution and a type I censoring pattern consisting of censoring all items at time $X$ which haven't failed by time $X$. Let the observed sample be $(X_1, \cdots, X_n)$. If the location and scale parameters are $\theta$ and $\sigma$, respectively, the sample can be transformed to $((X_1-\theta)/\sigma, \cdots, (X_n-\theta)/\sigma)$ so that the transformed sample is from an extreme-value distribution with location and scale parameters 0 and 1 respectively. The censoring time is transformed to $(X-\theta)/\sigma$. The original and the transformed sample give analogous inferences and the frequency properties depend on the censoring time $(X-\theta)/\sigma$ as it is measured when the parameters are $(0, 1)$. This fact is a consequence of using invariant prior measures for $\theta$ and $\sigma$. So, for a particular sample size, the effect of censoring at a certain time X is a function of $(X-\theta)/\sigma$, only. Although it has been shown that under progressive type II censoring, the Bayesian procedure often has properties which are desirable from a frequentist standpoint, there is no reason to believe that this is the case with type I censoring. The simulation studies show this to be the case.

The preceeding discussion shows that for the Weibull distribution with parameters $\alpha$ and $\beta$, the effect of censoring at time T is a function of $(T/\alpha)^{\beta}$ only. In the simulation study, $\alpha = 1$ and $\beta = 1$, and T is varied. For sample size $n = 5$, we censor at the .20, .40, .60, and .80 quantiles. For each quantile at which we censor, 200 independent samples are generated. For each of these 200 samples, it is determined if the p-level lower Bayesian confidence bound for the parameters $\alpha, \beta$, and $R(t)$ actually cover the parameters. The nominal levels of p considered are .75, .90, .95, and .99 and the same 200 samples are used for each of these nominal values. The reliability time is $t = -\ln(.1) = 2.3$ which is the time corresponding to the .90 quantile. The percentage of coverages is then displayed. In a similar manner, upper confidence bounds for sample sizes $n = 5$ and $n = 20$ were calculated for the same parameters and the frequency of coverage displayed. In order to determine if, say, a one-sided upper p-level Bayesian confidence interval for $g(\alpha, \beta)$ actually covers $g(1, 1)$, it is only necessary to calculate

$$P[g(\tilde{\alpha}, \tilde{\beta}) \leq g(1,1) \mid \underline{T}] = \int_0^\infty \pi(\beta \mid \underline{T}) G\left( \frac{\sum_1^n t_i^\beta}{h^\beta(g(1,1), \beta)} \,\Big|\, k \right) d\beta$$

where $h(\alpha, \beta)$ is the unique solution to $g(h(\alpha, \beta), \beta) = \alpha$. (It is assumed that for fixed $\beta$, $g(\alpha, \beta)$ is monotone in $\alpha$. This is discussed in Chapter II.) The necessary numerical integration is accomplished by Simpson's rule. The upper p-level Bayesian confidence interval for $g(\alpha, \beta)$ actually covers the true parameter value $g(1, 1)$ if and only if

$$P[g(\tilde{\alpha}, \tilde{\beta}) \leq g(1,1) \mid \underline{T}] \leq p .$$

Table 1. Frequency of Coverage by One-Sided Bayesian Confidence Intervals.

Table 1a. Sample size = 5. Censoring time at the $20^{th}$ percentile. Lower confidence intervals. 200 samples.

| nominal level | .75 | .90 | .95 | .99 |
|---|---|---|---|---|
| alpha | .18 | .815 | .855 | .99 |
| beta | .87 | .925 | .955 | .99 |
| R(t) | .905 | .955 | .97 | .995 |
| standard error | .03 | .023 | .0158 | .0063 |

Table 1b. Sample size = 5. Censoring time at the 40th percentile. Lower confidence intervals. 200 samples.

| | | | | |
|---|---|---|---|---|
| nominal level | .75 | .90 | .95 | .99 |
| alpha | .445 | .79 | .865 | .92 |
| beta | .60 | .79 | .89 | .97 |
| R(t) | .73 | .835 | .895 | .96 |

Table 1c. Sample size = 5. Censoring time at the 60th percentile. Lower confidence intervals. 200 samples.

| | | | | |
|---|---|---|---|---|
| nominal level | .75 | .90 | .95 | .99 |
| alpha | .735 | .84 | .895 | .975 |
| beta | .375 | .61 | .76 | .945 |
| R(t) | .555 | .715 | .83 | .935 |

Table 1d. Sample size = 5. Censoring time at the 80th percentile. Lower confidence intervals. 200 samples.

| | | | | |
|---|---|---|---|---|
| nominal level | .75 | .90 | .95 | .99 |
| alpha | .84 | .94 | .97 | .995 |
| beta | .465 | .68 | .785 | .94 |
| R(t) | .475 | .665 | .80 | .935 |

Table 1e. Sample size = 5. Censoring time at the $20^{th}$ percentile. Upper confidence intervals. 200 samples.

| | | | | |
|---|---|---|---|---|
| nominal level | .75 | .90 | .95 | .99 |
| alpha | .955 | .975 | .995 | 1.0 |
| beta | .215 | .225 | .790 | .805 |
| R(t) | .195 | .785 | .810 | .815 |

Table 1f. Sample size = 5. Censoring time at the $40^{th}$ percentile. Upper confidence intervals. 200 samples.

| | | | | |
|---|---|---|---|---|
| nominal level | .75 | .90 | .95 | .99 |
| alpha | .810 | .905 | .950 | .995 |
| beta | .680 | .840 | .845 | .940 |
| R(t) | .650 | .850 | .865 | .965 |

Table 1g. Sample size = 5. Censoring time at the $60^{th}$ percentile. Upper confidence intervals. 200 samples.

| | | | | |
|---|---|---|---|---|
| nominal level | .75 | .90 | .95 | .99 |
| alpha | .765 | .895 | .930 | .980 |
| beta | .90 | .925 | .930 | .930 |
| R(t) | .90 | .945 | .955 | .975 |

Table 1h. Sample size = 5. Censoring time at the 80$^{th}$ percentile. Upper confidence intervals. 200 samples.

| | | | | |
|---|---|---|---|---|
| nominal level | .75 | .90 | .95 | .99 |
| alpha | .65 | .855 | .92 | .985 |
| beta | .88 | .93 | .94 | .96 |
| R(t) | .915 | .97 | .97 | .985 |

Table 1k. Sample size = 20. Censoring at the 20$^{th}$ percentile. Upper confidence intervals. 100 samples.

| | | | | |
|---|---|---|---|---|
| nominal level | .75 | .90 | .95 | .99 |
| alpha | .92 | .98 | .99 | 1.0 |
| beta | .55 | .79 | .8 | .93 |
| R(t) | .52 | .74 | .8 | .95 |
| standard error | .0424 | .0328 | .0224 | .0089 |

Table 1l. Sample size = 20. Censoring at the 40$^{th}$ percentile. Upper confidence intervals. 100 samples.

| | | | | |
|---|---|---|---|---|
| nominal level | .75 | .90 | .95 | .99 |
| alpha | .85 | .94 | .97 | 1.0 |
| beta | .93 | .96 | .98 | .98 |
| R(t) | .82 | .89 | .94 | .97 |

Table 1m. Sample size = 20. Censoring at the $60^{th}$ percentile. Upper confidence intervals. 100 samples.

| | | | | |
|---|---|---|---|---|
| nominal level | .75 | .90 | .95 | .99 |
| alpha | .79 | .95 | .97 | .99 |
| beta | .93 | .96 | .96 | .96 |
| R(t) | .98 | .98 | 1.0 | 1.0 |

Table 1n. Sample size = 20. Censoring at the $80^{th}$ percentile. Upper confidence intervals. 100 samples.

| | | | | |
|---|---|---|---|---|
| nominal level | .75 | .90 | .95 | .99 |
| alpha | .74 | .88 | .98 | 1.0 |
| beta | .90 | .94 | .94 | .94 |
| R(t) | .96 | .99 | .99 | .99 |

Both small and wide differences can be observed between the estimated and nominal frequencies of coverage and it is difficult to draw a general conclusion about the effect of censoring. One observation that can be made is that when the level of coverage of a parameter by a p-level upper Bayesian confidence bound is greater than p, the lower p-level confidence bound will tend to have probability of coverage less than p , and conversely. This can be a result of skewness of the posterior distribution or possibly a shift.

# VI. NUMERICAL EXAMPLES

## A Posterior Distribution for Beta and $t_{.95}$

The first numerical example of the proposed Bayesian method is the posterior distribution of $\beta$, where the sample data are taken from Mann [19]. The data are (after division by 100) are (.38, .88, .96, 1.18, 1.78, 1.2). The first five observations were failures and the sixth was censored. The general formula for the posterior density (up to a constant of proportionality) is given by

$$\pi(\beta \mid \underline{T}) \propto \beta^{k-2} \left(\prod_{1}^{k} t_i\right)^{\beta} / \left(\sum_{1}^{n} t_i^{\beta}\right)^{k}$$

where $(t_1, \cdots, t_k)$ are the items that failed and $(t_{k+1}, \cdots, t_n)$ are the items that were censored. Simpson's rule was used to obtain the constant of proportionality and perform the numerical integration. The posterior density $\pi(\beta/\underline{T})$ was set equal to zero for $\beta > 5.5$ and the grid spacing for Simpson's rule was equal to 1/12. By comparing this spacing with finer spacings, it was found that spacings of 1/12 gave answers which were accurate to 4 significant places. The c.d.f. and density were calculated for various arguments.

Table 2. The posterior c. d. f. and density of beta.

| Argument | C. D. F. | Density |
|---|---|---|
| .5 | .006427 | .04817 |
| 1. | .07479 | .2427 |
| 1.5 | .2480 | .4318 |
| 2. | .4798 | .4669 |
| 2.5 | .6922 | .3685 |
| 3. | .8421 | .2318 |
| 3.5 | .9289 | .1223 |
| 4. | .9717 | .05588 |
| 4.5 | .9903 | .02268 |
| 5. | .9975 | .008332 |

Tables such as this could enable one to determine approximate confidence intervals and apply tests of hypothesis for $\beta$. One could also find a $\gamma$-level upper bound for $\beta$ by the method described in Chapter II. The .90-level upper bound for $\beta$ was found to be 3.30. Similarly, a .90 level upper bound for $\alpha$ was found to be 1.88.

The posterior c. d. f. of $t_{.95}$ is given in Table 3.

Table 3. The posterior c. d. f. of $t_{.95}$.

| Argument | C. D. F. |
|---|---|
| .1 | .118 |
| .2 | .283 |
| .3 | .478 |
| .4 | .668 |
| .5 | .821 |
| .6 | .921 |
| .7 | .972 |

A lower .90-level confidence bound for $t_{.95}$ was found to be .087. The comparable lower bound calculated by Mann was .0654.

A Comparison of Confidence Bounds Obtained by the Bayesian Method and the Maximum-Likelihood Method

Harter and Moore [11] investigated the properties of m. l. point estimation of $\theta$ and $\sigma$ of an extreme-value distribution and utilized the asymptotic normality properties of m. l. estimators to obtain large sample confidence bounds on $\alpha$ and $\beta$ from a randomly generated sample of size $n = 40$, with $\alpha = 1$ and $\beta = 2$. They censored the first 5 observations from below and the last five from above according to a type II rule. The upper 95% confidence bounds for $\beta$ and $\alpha$ were found to be 2.618 and 109.4, respectively. The corresponding values for the Bayesian proceedure were 2.271 and 111, respectively. However, the Bayesian calculations were done using the actual failure times of the first five observations, instead of censoring them from above at the failure time of the fifth observation. This was necessitated by the fact that the computer program wasn't designed to analyze data censored from above. We know that the two methods are asymptotically equivalent so there does not seem to be much purpose in attempting to make a more valid comparison between the two methods for these data.

A second set of data which will be analyzed to compare the two methods is provided by Sprott [24]. The data are (.2, 5.1, 3.3, 2.7, 1.4, 2.4, .4, 7.2, 6, 2.1). The first 7 observations are failures and the last 3 are censored by an unspecified rule. In applying the m. l. method to these data, the logarithms of the data were used to obtain the asymptotic covariance matrix of $\hat{\theta} = \ell n(\hat{\alpha})$ and $\hat{\vartheta} = \hat{\beta}^{-1}$. Harter and Moore apply asymptotic m. l. theory to the parameters of the extreme-value distribution, rather than the Weibull distribution. A .90-level upper confidence bound for $\alpha$ was 6.90 and a .90-level upper confidence bound for $\beta$ was 1.64. The corresponding upper .90-level Bayesian bounds for $\alpha$ and $\beta$ were 10.55 and 1.24, respectively. In this case, a rather wide difference in the bounds for $\alpha$ is observed. This may be due to the fact that asymptotic theory is applied to a small sample with censoring.

Summary and Conclusions

The main results of the investigation presented here are to demonstrate a practical method of applying Bayesian theory to the construction of confidence intervals for the parameters of a two-parameter Weibull distribution, and also for a large class of functions of these parameters. The introductory chapter presented the fundamental properties of the Weibull distribution and gave its relationship to the extreme-value distribution. It also presented a number of the more important point and interval techniques which have been developed for the Weibull and extreme-value distributions. The major advantages and disadvantages of the proposed Bayesian method are also discussed in this chapter.

In Chapter II, the Bayesian method was presented in detail. Specifically, it was shown how to calculate upper confidence bounds for $\alpha$, $\beta$, and a large class of functions of these parameters. In addition, a useful joint confidence region for $(\alpha, \beta)$ was presented. The prior $\pi(\alpha, \beta) = \pi(\alpha)\pi(\beta) = \frac{1}{\alpha\beta}$ was emphasized because it was found that with this prior, the Bayesian method often gave intervals which had desirable properties from a frequentist point of view. It was found that the Bayesian method had wide applicability because it could be applied to all types of censored and uncensored experimental situations and also because it could easily be applied to a large class of functions of the parameters of the Weibull distribution as well as the parameters themselves. The major disadvantage of the Bayesian method was that it required at least two numerical integrations to obtain a confidence bound for a parameter of interest.

The third chapter deals with the relationship between the fiducial method and the proposed Bayesian method. It was found that the two methods agree when there is no censoring or progressive type II censoring. Consequently, the Bayesian method possesses the confidence interval property for a class of functions which includes the class of invariantly estimable functions and specifically includes the parameters $\alpha, \beta$, and $R(t)$. The fact that the Bayesian lower confidence limit for $R(t)$, the reliability at time $t$, has the confidence interval property is important because it is an example of a non-invariantly estimable function with that property. An interesting function which is invariantly estimable is $t_\gamma$, the $\gamma^{th}$ quantile. Under progressive type II censoring, the Bayesian lower confidence limits have the confidence interval property. Mann has also developed a 3-order statistic method for obtaining exact lower confidence bounds for $t_{.90}$ under progressive type II censoring. The Bayesian method is the only competitor to the Mann method at present.

In Chapter IV a method is presented for finding conditional probabilities of coverage of $\alpha$ and $\beta$ by intervals constructed from the Thoman, Bain, and Antle method. The conditioning is done with respect to a function of the ancillary statistic for the full sample case. It is shown that the Bayesian method with the indicated priors is the only method whose conditional probability of coverage equals the nominal level. A simulation study of the conditional probabilities of coverage of the Thoman, Bain, and Antle method was adequate for most applied situations where there is no censoring, but it should be noted that the Bayesian method has wider applicability.

The fifth chapter was a simulation study of the frequency properties of Bayesian confidence intervals under type I censoring. Bayesian confidence intervals were constructed for the parameters $\alpha$, $\beta$, and $R(t)$ from data which had been censored by type I censoring patterns. The frequency of coverage was then presented.

The sixth and final chapter consists of a variety of numerical examples of the proposed method. The data are taken from the literature and the Bayesian results are compared with results obtained by other methods.

In conclusion, this author feels that the proposed method has practical value and that the results of this investigation justify a further study of the Bayesian method for the Weibull distribution. Particularly interesting areas of investigation include hypothesis testing for the Weibull, point estimation, and a decision theory approach to find optimal sample size in experimental design. Also the two-sample problem should be investigated with emphasis on parameters such as

$P[x < y]$ and $R_x(t)/R_y(t)$. Since this investigation was for the situation in which the sample information is large compared to the prior information, another important area still to be investigated is the converse case, where the prior information is appreciable relative to the sample information.

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