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Title TANGENTIAL FAMILIES OF STUDY VECTORS

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This paper is devoted to the study of families of tangent lines to curves in Euclidean three dimensional space by the medium of a particular representation for lines.

First, a ring of elements called dual numbers is described, and a vector space over this ring, whose elements are called dual vectors, is defined.

Next, a subset of the dual vectors is singled out. The members of this subset are called Study vectors. The set of Study vectors is shown to be in one-to-one correspondence with the set of all directed lines in Euclidean three dimensional space. Study vectors are used to represent the lines to which they correspond.

A necessary and several sufficient conditions on a family of Study vectors are given in order that they form a family of tangents to a curve in Euclidean three dimensional space.

TANGENTIAL FAMILIES OF STUDY VECTORS

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This paper is dedicated to my wonderful, loving, and patient wife, Shirley.

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TANGENTIAL FAMILIES OF STUDY VECTORS

INTRODUCTION

The purpose of this paper is to determine conditions for a given one-parameter family of lines to be a family of tangent lines to a curve in space. To this end, a representation for lines is introduced which differs from the one usually given in Analytic Geometry.

Parts I and II are devoted to the introduction and study of a ring of elements called dual numbers, and a vector space over this ring, called the space of dual vectors.

In Part III, a subset of the space of dual vectors is singled out and the elements of this subset are shown to be in one-to-one correspondence with the set of all directed lines in space. These special vectors, called Study vectors, are used to represent the directed lines with which they correspond.

Part IV is devoted to the primary question of the paper. The general goal of this section is to determine necessary and sufficient conditions on a one-parameter family $\mathfrak{A}$ of Study vectors in order that the family of lines which $\mathfrak{A}$ represents be a family of tangents to a space curve.

The goal is not completely realized, however. We find conditions which are necessary and conditions which are sufficient, but not a set of necessary and sufficient conditions. The difficulties are of the same sort which occur in the representation of space curves. If $x = x(\tau)$ is a vector-valued function of the real variable τ and if x is differentiable, then as long as the derivative $x'(\tau)$ is not a zero vector, $x(\tau)$ is a representation of a curve. If $x'(\tau_0)$ is a zero vector and $x'(\tau) \neq 0$ in a deleted neighborhood of τ_0, then it may or may not happen that x is a representation of a curve.

PRELIMINARY REMARKS

Small Greek letters represent real numbers, or, when indicated, real valued functions of a real variable. The only exception is the letter ε which is given a special meaning in Part I which will hold throughout the paper.

R^3 designates the vector space of ordered triplets $w = (\omega_1, \omega_2, \omega_3)$ which we suppose to be metrized by the Euclidean norm $|w| = (\omega_1^2 + \omega_2^2 + \omega_3^2)^{1/2}$. Small Roman letters denote elements in R^3. The vectors $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0,0,1)$ form an orthonormal basis for R^3, and since $w = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3$, we call ω_1, ω_2, ω_3 Cartesian coordinates for w.

Any ordered triple of vectors (w, w', w'') is said to be positively oriented if

$$\begin{vmatrix} \omega_1 & \omega_2 & \omega_3 \\ \omega_1' & \omega_2' & \omega_3' \\ \omega_1'' & \omega_2'' & \omega_3'' \end{vmatrix} > 0 .$$

Thus, (e_1, e_2, e_3) is a positively oriented triple.

The inner (scalar) product of $w = (\omega_1, \omega_2, \omega_3)$ and $y = (\psi_1, \psi_2, \psi_3)$ is defined by

$$\langle w, y \rangle = |w| \cdot |y| \cdot \cos \phi$$

where ϕ is the angle between w and y. We take as known that

$$\langle w, y \rangle = \omega_1\psi_1 + \omega_2\psi_2 + \omega_3\psi_3$$

and that

$$\langle w, w \rangle = |w|^2 .$$

The outer (vector) product of w and y, designated by $[w, y]$, is zero if either w or y is zero, or w and y are proportional. Otherwise $[w, y]$ is the vector of magnitude $|w| \cdot |y| \cdot \sin \phi$ which is perpendicular to both w and y and is such that the ordered triple of vectors $(w, y, [w, y])$ is positively oriented. We take as known that :

$$[w, y] = \left(\begin{vmatrix} \omega_2 & \omega_3 \\ \psi_2 & \psi_3 \end{vmatrix}, \begin{vmatrix} \omega_3 & \omega_1 \\ \psi_3 & \psi_1 \end{vmatrix}, \begin{vmatrix} \omega_1 & \omega_2 \\ \psi_1 & \psi_2 \end{vmatrix} \right)$$

$$[w, y] = -[y, w]$$

$$[w, \alpha x + \beta y] = \alpha[w, x] + \beta[w, y] .$$

In particular,

$$[w, w] = 0 .$$

The symbols $< >$ and [ ] are reserved for special use so that only ( ) and { } are used as grouping symbols. Parentheses are also used to designate an ordered triple or an interval on the real line, and braces to designate a set of elements. The context makes the usage clear.

PART I: THE RING OF DUAL NUMBERS AND THE SPACE OF DUAL VECTORS

Let ε be an indeterminant. For α and $\bar{\alpha}$ real numbers, the combination $A = \alpha + \varepsilon\bar{\alpha}$ is called a dual number (1, p. 261). We define the addition and multiplication of dual numbers $A = \alpha + \varepsilon\bar{\alpha}$ and $B = \beta + \varepsilon\bar{\beta}$ by

(1.1) $$A + B = (\alpha + \beta) + \varepsilon(\bar{\alpha} + \bar{\beta})$$

(1.2) $$A \cdot B = \alpha \cdot \beta + \varepsilon(\alpha \cdot \bar{\beta} + \bar{\alpha} \cdot \beta).$$

Note that dual numbers are added and multiplied like polynomials in ε, subject only to the condition that $\varepsilon^2 = 0$. As has been done above, dual numbers will be designated by capital Roman letters.

For $A = \alpha + \varepsilon\bar{\alpha}$, $\alpha = R(A)$ is called the real part of A and $\bar{\alpha} = D(A)$ the dual part of A. The equality $A = B$ means $R(A) = R(B)$ and $D(A) = D(B)$.

The set of dual numbers $\{A\}$ forms a commutative ring with a unit element and with zero divisors. The zero element is $0 = 0 + \varepsilon(0)$, the unit element is $1 = 1 + \varepsilon(0)$ and the additive inverse of $A = \alpha + \varepsilon\bar{\alpha}$ is $-A = -\alpha + \varepsilon(-\bar{\alpha})$.

In a ring, if the product of two non-zero elements is zero then each of these elements is called a zero divisor. For dual numbers A and B for which $R(A) = R(B) = 0$, we have

$$A \cdot B = (0 + \varepsilon \overline{a})(0 + \varepsilon\overline{\beta}) = 0 + \varepsilon(0 \cdot \overline{\beta} + \overline{a} \cdot 0) = 0 .$$

Thus, dual numbers whose real parts are zero are zero divisors.

Let $A = a + \varepsilon \overline{a}$, with $R(A) = a \neq 0$. Then

$$A \cdot (1/a + \varepsilon(-\overline{a}/a^2)) = a \cdot 1/a + \varepsilon(a(-\overline{a}/a^2) + \overline{a} \cdot 1/a) = 1.$$

We see that a dual number whose real part is non-zero has a multiplicative inverse, and

$$A^{-1} = 1/a + \varepsilon(-\overline{a}/a^2).$$

If $R(A) = 0$, we write $A = \varepsilon \overline{a}$ and if $D(A) = 0$, $A = a$. The dual number $A = a + \varepsilon(-\overline{a})$ is written $A = a - \varepsilon\overline{a}$.

Suppose Ω is an ordered pair of ordered triplets of real numbers, or according to the notation introduced, $\Omega = (w, \overline{w}) = ((\omega_1, \omega_2, \omega_3), (\overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3))$. We say $\Omega = (w, \overline{w})$ and $\Psi = (y, \overline{y})$ are equal if and only if $w = y$ and $\overline{w} = \overline{y}$. The sum of Ω and Ψ is defined to be

(1.3) $$\Omega + \Psi = (w+y, \overline{w}+\overline{y}) = ((\omega_1+\psi_1, \omega_2+\psi_2, \omega_3+\psi_3), (\overline{\omega}_1+\overline{\psi}_1, \overline{\omega}_2+\overline{\psi}_2, \overline{\omega}_3+\overline{\psi}_3))$$

and the product of Ω by a dual number $A = a + \varepsilon\overline{a}$ is defined to be

(1.4) $$A \cdot \Omega = (aw, a\overline{w}+\overline{a}w) = ((a\omega_1, a\omega_2, a\omega_3), (a\overline{\omega}_1+\overline{a}\omega_1, a\overline{\omega}_2+\overline{a}\omega_2, a\overline{\omega}_3+\overline{a}\omega_3)).$$

Let ε be as before, simply a symbol which satisfies the equation $\varepsilon^2 = 0$. The element $\Omega = (w, \overline{w})$ will be represented hereafter by $\Omega = w + \varepsilon \overline{w}$. We see that the definitions of addition and multiplication as given by equations (1.3) and (1.4) can be rewritten in the form

(1.3') $$\Omega + \Psi = (w + y) + \varepsilon(\overline{w} + \overline{y})$$

(1.4') $$A\Omega = \alpha w + \varepsilon(\alpha\overline{w} + \overline{\alpha} w).$$

The set of all elements of the form $\Omega = w + \varepsilon\overline{w}$, with addition and multiplication defined according to equations (1.3') and (1.4') will be called the set of dual vectors. Dual vectors will be represented by capital Greek letters.

If $\Omega = w + \varepsilon\overline{w}$, then the vector $w = R(\Omega)$ is called the real part of Ω and the vector $\overline{w} = D(\Omega)$ the dual part of Ω. If $R(\Omega) = 0$, we write $\Omega = \varepsilon\overline{w}$ and if $D(\Omega) = 0$, $\Omega = w$.

Let $\theta_1 = (1, 0, 0) + \varepsilon(0, 0, 0) = (1, 0, 0)$, $\theta_2 = (0, 1, 0)$ and $\theta_3 = (0, 0, 1)$, and let $\Omega = w + \varepsilon\overline{w} = (\omega_1, \omega_2, \omega_3) + \varepsilon(\overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3)$ be an arbitrary dual vector. Then

$$\Omega = (\omega_1 + \varepsilon\overline{\omega}_1)\theta_1 + (\omega_2 + \varepsilon\overline{\omega}_2)\theta_2 + (\omega_3 + \varepsilon\overline{\omega}_3)\theta_3 .$$

Thus every dual vector is representable as a linear combination of the dual vectors θ_1, θ_2, θ_3, with scalars from the ring of dual numbers.

From the preceding paragraph and the definitions of addition and multiplication given by equations (1.3') and (1.4'), it is easily seen that the set of dual vectors is a three dimensional vector space over the ring of dual numbers, for which the set $\{\theta_1, \theta_2, \theta_3\}$ is a basis.

PART II: INNER, OUTER AND TRIPLE PRODUCTS

The inner product of the dual vectors $\Omega = w + \varepsilon \overline{w}$ and $\Psi = y + \varepsilon \overline{y}$ is defined by

$$(2.1) \qquad (\Omega, \Psi) = \langle w, y\rangle + \varepsilon(\langle w, \overline{y}\rangle + \langle \overline{w}, y\rangle).$$

Because the inner product in R^3 is a symmetric, bilinear, real valued function, and the inner product of two dual vectors is a finite linear combination of inner products in R^3, the inner product defined by equation (2. 1) is a symmetric, bilinear function whose range lies in the ring of dual numbers.

The outer product of two dual vectors Ω and Ψ is defined by

$$(2.2) \qquad [\Omega, \Psi] = [w, y] + \varepsilon([w, \overline{y}] + [\overline{w}, y]).$$

The outer product of two dual vectors is again a dual vector.

Let $\Omega = w + \varepsilon \overline{w}$, $\Psi = y + \varepsilon \overline{y}$ and $\Sigma = s + \varepsilon \overline{s}$ be dual vectors. The following theorems will be of use.

<u>Theorem 2. 1</u>: $[\Omega, \Psi] = -[\Psi, \Omega]$.

Proof:

$$[\Omega, \Psi] = [w, y] + \varepsilon([w, \overline{y}] + [\overline{w}, y]) = -[y, w] + \varepsilon(-[y, \overline{w}] - [\overline{y}, w]) = -[\Psi, \Omega]. \blacksquare$$

Corollary 2.1.1: $[\Omega, \Omega] = 0.$

Theorem 2.2: For an arbitrary dual number $A = \alpha + \varepsilon \overline{\alpha}$,

$$[A\Omega, \Psi] = A[\Omega, \Psi] .$$

Proof:

$$[A\Omega, \Psi] = [\alpha w + \varepsilon(\alpha \overline{w} + \overline{\alpha} w), y + \varepsilon \overline{y}] = \alpha[w, y] + \varepsilon(\alpha[w, \overline{y}] + \alpha[\overline{w}, y] + \overline{\alpha}[w, y])$$

$$= \alpha[w, y] + \varepsilon(\alpha([w, \overline{y}] + [\overline{w}, y]) + \overline{\alpha}[w, y])$$

$$= A([w, y] + \varepsilon([w, \overline{y}] + [\overline{w}, y])$$

$$= A[\Omega, \Psi] . \blacksquare$$

Corollary 2.2.1: For arbitrary dual numbers $A = \alpha + \varepsilon \overline{\alpha}$ and $B = \beta + \varepsilon \overline{\beta}$,

$$[A\Omega, B\Omega] = 0.$$

Theorem 2.3: $[\Omega, \Psi + \Sigma] = [\Omega, \Psi] + [\Omega, \Sigma].$

Proof:

$$[\Omega, \Psi + \Sigma] = [w + \varepsilon \overline{w}, (y+s) + \varepsilon(\overline{y} + \overline{s})] = [w, y+s] + \varepsilon([w, \overline{y} + \overline{s}] + [\overline{w}, y+s])$$

$$= [w, y] + \varepsilon([w, \overline{y}] + [\overline{w}, y]) + [w, s] + \varepsilon([w, \overline{s}] + [\overline{w}, s])$$

$$= [\Omega, \Psi] + [\Omega, \Sigma] . \blacksquare$$

Theorem 2.4: $<\Omega, [\Psi, \Sigma]> = <[\Omega, \Psi], \Sigma>$.

Proof:

$$<\Omega, [\Psi, \Sigma]> = <w+\varepsilon\overline{w}, [y, s]+\varepsilon([y, \overline{s}]+[\overline{y}, s])>$$

$$= <w, [y, s]> + \varepsilon(<w, [y, \overline{s}]> + <w, [\overline{y}, s]> + <\overline{w}, [y, s]>)$$

$$= <[w, y], s> + \varepsilon(<[w, y], \overline{s}> + <[w, \overline{y}], s> + <[\overline{w}, y], s>)$$

$$= <[w,y] + \varepsilon([w, \overline{y}] + [\overline{w}, y]), s + \varepsilon\overline{s}>$$

$$= <[\Omega, \Psi], \Sigma> . \blacksquare$$

Theorem 2.5: $<\Omega, [\Psi, \Sigma]> = - <\Omega, [\Sigma, \Psi]>$.

Proof:

$$<\Omega, [\Psi, \Sigma]> = <\Omega, -[\Sigma, \Psi]> = - <\Omega, [\Sigma, \Psi]> . \blacksquare$$

Theorem 2.6: $[[\Omega, \Psi], \Sigma] = <\Omega, \Sigma>\Psi - <\Psi, \Sigma>\Omega$.

Proof:

$$[[\Omega, \Psi], \Sigma] = [[w, y] + \varepsilon([w, \overline{y}] + [\overline{w}, y]), s + \varepsilon\overline{s}]$$

$$= [[w, y], s] + \varepsilon([[w, y], \overline{s}] + [[w, \overline{y}], s] + [[\overline{w}, y], s])$$

$$= <w,s>y - <y,s>w + \varepsilon(<w, \overline{s}>y - <y, \overline{s}>w + <w, s>\overline{y}$$

$$- <\overline{y}, s>w + <\overline{w}, s>y - <y, s>\overline{w}),$$

using the vector identity in R^3,

$$[[u, v], w] = <u, w>v - <v, w>u.$$

Then

$$\begin{aligned}[[\Omega, \Psi], \Sigma] &= <w, s>y + \varepsilon(<w, \bar{s}>y + <\bar{w}, s>y + <w, s>\bar{y}) \\ &\qquad - <y, s>w - \varepsilon(<y, \bar{s}>w + <\bar{y}, s>w + <y, s>\bar{w}) \\ &= <\Omega, \Sigma>\Psi - <\Psi, \Sigma>\Omega . \quad \blacksquare\end{aligned}$$

<u>Corollary 2.6.1:</u> $[\Omega, [\Psi, \Sigma]] = <\Omega, \Sigma>\Psi - <\Omega, \Psi>\Sigma.$

The above theorems and corollaries show that the inner and outer products in the vector space of dual vectors have basic properties which closely resemble the corresponding properties of the inner and outer products in R^3. In theorems 3.4 and 3.5 a further resemblance is obtained.

PART III: STUDY VECTORS

Let $x = (\xi_1, \xi_2, \xi_3)$ and $y = (\psi_1, \psi_2, \psi_3)$ be distinct points in R^3. The set of points $z(\tau)$ of the form

$$(1-\tau)x + \tau(y), \quad -\infty < \tau < +\infty$$

forms a line through x and y. Note that as τ goes from 0 to 1, $z(\tau)$ goes from x to y. We call this oriented line the directed line from x to y. The directed line from y to x has the representation

$$w(\tau) = (1-\tau)y + \tau x, \quad -\infty < \tau < +\infty .$$

We say that a point

$$x_1 = (1-\tau_1)x + \tau_1 y$$

precedes a point

$$x_2 = (1-\tau_2)x + \tau_2 y$$

on the directed line from x to y if and only if $\tau_1 < \tau_2$. If x' and y' are points on the directed line from x to y, then this line has the representation

$$z'(\sigma) = (1-\sigma)x' + \sigma y', \quad -\infty < \sigma < +\infty$$

if and only if x' precedes y'.

For x and y distinct points in R^3, define

(3.1) $$a = (y-x)/|y-x| ,$$

(3.2) $$\bar{a} = [x,y]/|y-x| .$$

Then

$$<a,a> = <y-x, y-x>/|y-x|^2 = 1 ,$$

$$<a,\bar{a}> = <y-x,[x,y]>/|y-x|^2 = 0.$$

We record these formulas as

(3.3) $$<a,a> = 1,$$

(3.4) $$<a,\bar{a}> = 0 .$$

If the directed line from x to y is given and x' precedes y' on the line, then

$$x' = (1-\tau_1)x + \tau_1 y ,$$

$$y' = (1-\tau_2)x + \tau_2 y ,$$

with $\tau_1 < \tau_2$. Then

$$y'-x' = (\tau_2 - \tau_1)(y-x)$$

and

$$\frac{y'-x'}{|y'-x'|} = \frac{(\tau_2-\tau_1)(y-x)}{|\tau_2-\tau_1||y-x|} = a.$$

It follows that the vector a defined by equation (3.1) is independent of the choice of the points x and y on the directed line from x to y, as long as x precedes y. The unit vector a is called the direction of the directed line from x to y.

The vector $\overline{a}$ defined by equation (3.2) is also independent of the choice of points x and y as long as x precedes y. For if x' precedes y' on the directed line from x to y, then as before

$$x' = (1-\tau_1)x + \tau_1 y$$

$$y' = (1-\tau_2)x + \tau_2 y$$

with $\tau_1 < \tau_2$. Therefore,

$$[x', y'] / |y'-x'|$$

$$= \frac{(1-\tau_1)(1-\tau_2)[x,x] + \tau_1(1-\tau_2)[y,x] + (1-\tau_1)\tau_2[x,y] + \tau_1\tau_2[y,y])}{|\tau_2-\tau_1||y-x|}$$

$$= (\tau_2-\tau_1)[x,y] / |\tau_2-\tau_1| \cdot |y-x| = \overline{a} .$$

The directed line from x to y has associated with it a unique pair of vectors a and $\overline{a}$ in R^3, defined by equations (3.1) and (3.2) respectively. In the language of Part II, the directed line from x to y has a unique dual vector $a + \varepsilon\overline{a}$ associated with it.

Lemma: Let a directed line from x to y be given and let a be the direction of this line as determined by equation (3.1). The point z lies on this line if and only if there is a real number λ for which $z = x + \lambda a$.

Proof:

Suppose $z = x + \lambda a$. Then

$$z = x + \lambda(y-x)/|y-x| = (1-\lambda/|y-x|)x + (\lambda/|y-x|)y$$

and z lies on the line.

Suppose z lies on the line. Then

$$z = (1 - \tau_1)x + \tau_1 y = x + \tau_1(y-x) = x + \lambda a$$

for $\lambda = \tau_1|y-x|$. ∎

Theorem 3.1: Let a directed line from the point x to the point y be given and let $a + \varepsilon\overline{a}$ be its associated dual vector, where a and $\overline{a}$ are determined from equations

(3.1) and (3.2). A point z lies on this directed line if and only if $[z, a] = \bar{a}$.

Proof:

Suppose z lies on the line so that by the lemma, $z = x + \lambda a$. Then

$$[z, a] = [x + \lambda a, a] = [x, a]$$

$$= [x, y-x]/|y-x| = [x, y]/|y-x| = \bar{a}.$$

Conversely, suppose $[z, a] = \bar{a}$. From the above equations we see that $[x, a] = \bar{a}$, so that

$$[z-x, a] = \bar{a} - \bar{a} = 0.$$

Since $a \neq 0$, $z - x = \lambda a$ for some real λ and by the lemma, z lies on the line. ▮

An immediate consequence of this theorem is the following corollary.

<u>Corollary 3.1.1:</u> If each of two directed lines has $a + \varepsilon\bar{a}$ as its associated dual vector, then the two lines coincide and have the same direction.

If $a + \varepsilon\bar{a}$ is the dual vector associated with the directed line from x to y, then by equations (3.3) and (3.4),

(3.5) $$\langle a+\varepsilon\bar{a}, a+\varepsilon\bar{a}\rangle = \langle a, a\rangle + 2\varepsilon\langle a, \bar{a}\rangle = 1.$$

Definition: A dual vector whose inner product with itself equals one will be called a Study vector, after E. Study (1862 - 1922), the German geometer who first studied the representation for lines which is given by theorem 3.2 (1, p. 263).

Study vectors are denoted by German capital script letters. Thus $\mathfrak{A} = a+\varepsilon\bar{a}$ is a Study vector if and only if $R(\mathfrak{A}) = a$ is a unit vector which is orthogonal to $D(\mathfrak{A}) = \bar{a}$. From equation (3.5) it follows that the dual vector associated with a directed line is a Study vector.

Theorem 3.2 (Study): The oriented lines in R^3 are in one to one correspondence with the Study vectors.

Proof:

It was shown above that the directed line from x to y has an associated Study vector $\mathfrak{A} = a+\varepsilon\bar{a}$ determined by equations (3.1) and (3.2), and that if two directed lines have the same associated Study vector, then the lines are coincident and have the same direction. Hence, with each directed line is associated a unique Study vector.

Let $\mathfrak{a} = a + \varepsilon\bar{a}$ be a Study vector and consider the two points $x = [a, \bar{a}]$ and $y = x + a$ in R^3. These determine a unique directed line from x to y. Let $\mathfrak{b} = b + \varepsilon\bar{b}$ be its associated Study vector. By equations (3.1) and (3.2),

$$b = (y - x)/|y - x| = a/|a| = a$$

and

$$\bar{b} = [x, y]/|y - x| = [x, x + a] = [x, a] = \bar{a}$$

by theorem 3.1. Thus each Study vector $\mathfrak{a}$ determines in this way a unique directed line which has $\mathfrak{a}$ as its associated Study vector. ∎

As a consequence of this theorem, each directed line in R^3 can, without ambiguity, be represented by its associated Study vector, and conversely each Study vector represents, without ambiguity, that line with which it is associated. In the remainder of Part III, the terms directed line and Study vector will be used interchangeably, with the choice of terminology governed by the context. The line $\mathfrak{a}$ will thus refer to the directed line which has $\mathfrak{a}$ as its associated Study vector.

Let $\mathfrak{a} = a + \varepsilon\bar{a}$ be a directed line. It has been noted in the discussion preceding theorem 3.1, that the vector $R(\mathfrak{a}) = \mathbf{a}$ is the unit vector in the direction of the line. The following shows the geometric relationship of the vector $D(\mathfrak{a}) = \bar{a}$ to the line.

It is clear from theorem 3.1 that $\bar{a} = 0$ if and only if the line $\mathfrak{a}$ passes through the origin. Now, suppose $\bar{a} \neq 0$ and let x be that unique point on $\mathfrak{a}$ which is nearest to the origin. The vector x is perpendicular to $\mathfrak{a}$, i.e., to its direction a, and hence to λa for all real $\lambda \neq 0$. Let y be any other point on $\mathfrak{a}$ such that x precedes y, so that

$$\bar{a} = [x, y] / |y-x| \quad .$$

Let ϕ be the angle between the vectors x and y. By the definition of outer product in R^3,

$$|\bar{a}| = |x| \cdot |y| \cdot \sin\phi / |y-x| .$$

According to the lemma preceding theorem 3.1,

$$y - x = \lambda a$$

for some real λ, so that x is perpendicular to $y-x$ and the sine of the angle between x and $y-x$ equals one. Therefore,

$$|x| \cdot |y| \cdot \sin\phi = |[x, y]| = |[x, y-x]| = |x| \cdot |y-x|$$

and it follows that $|\bar{a}| = |x|$. Also, by theorem 3.1, for each point

z on $\mathfrak{a}$ we have

$$< \bar{a}, z > = < [z, a], z > = 0 .$$

Thus $\bar{a}$ is perpendicular to the plane containing the line $\mathfrak{a}$ and the origin, and the triple $(z, a, \bar{a})$ of vectors is positively oriented for each z on the line $\mathfrak{a}$.

Theorem 3.3: Let $\Omega = w + \varepsilon \bar{w}$ be an arbitrary dual vector for which $R(\Omega) = w \neq 0$. Then the dual vector

$$\frac{< w, w > - \varepsilon < w, \bar{w} >}{|w|^3} \quad \Omega$$

is a Study vector.

Proof:

According to the definition, we must show that the inner product of this vector with itself equals one. Hence,

$$< \frac{(<w,w> - \varepsilon <w,\overline{w}>)}{|w|^3}\,\Omega, \quad \frac{(<w,w> - \varepsilon <w,\overline{w}>)}{|w|^3}\,\Omega >$$

$$= \frac{(<w,w>^2 - 2\varepsilon <w,w><w,\overline{w}>)}{|w|^6} <\Omega, \Omega>$$

$$= \frac{(<w,w>^2 - 2\varepsilon <w,w><w,\overline{w}>)}{|w|^6} (<w,w> + 2\varepsilon <w,\overline{w}>)$$

$$= \frac{<w,w>^3 + 2\varepsilon(<w,w>^2<w,\overline{w}> - <w,w>^2<w,\overline{w}>)}{|w|^6}$$

$$= \frac{<w,w>^3}{|w|^6} = 1 .$$

Thus,

$$\frac{<w,w> - \varepsilon <w,\overline{w}>}{|w|^3}\,\Omega$$

is a Study vector. ∎

Theorem 3.4: Let $\mathcal{A} = a + \varepsilon\overline{a}$ and $\mathcal{B} = b + \varepsilon\overline{b}$ be two directed lines which are not necessarily distinct. Then

$$<\mathcal{A}, \mathcal{B}> = \cos\phi - \varepsilon\overline{\phi}\sin\phi$$

where ϕ is the angle between $\mathcal{A}$ and $\mathcal{B}$ and $\overline{\phi}$ is the distance between them.

Proof:

By equation (2. 1),

$$<\mathfrak{A}, \mathfrak{B}> = <a, b> + \varepsilon(<a, \overline{b}> + <\overline{a}, b>).$$

Since a and b are unit vectors in the direction of the lines $\mathfrak{A}$ and $\mathfrak{B}$ respectively, the angle ϕ between $\mathfrak{A}$ and $\mathfrak{B}$ is the angle between a and b and by definition,

$$<a, b> = |a| \cdot |b| \cdot \cos\phi = \cos\phi .$$

Thus only the interpretation of $<a, \overline{b}> + <\overline{a}, b>$ remains.

The lines $\mathfrak{A}$ and $\mathfrak{B}$ are either skew, parallel or intersecting. If they are skew, they have a unique common perpendicular which intersects both. Let it intersect $\mathfrak{A}$ at x and $\mathfrak{B}$ at x^*. If the lines $\mathfrak{A}$ and $\mathfrak{B}$ are parallel, let x and x^* be points on $\mathfrak{A}$ and $\mathfrak{B}$ which lie on some common perpendicular. If $\mathfrak{A}$ and $\mathfrak{B}$ intersect, let $x = x^*$ be the point of intersection. In any case, the distance between $\mathfrak{A}$ and $\mathfrak{B}$ is the distance from x to x^*, which we call $\overline{\phi}$.

Since x is on $\mathfrak{A}$ and x^* on $\mathfrak{B}$, theorem 3. 1 implies

$$\overline{a} = [x, a]$$

and

$$\bar{b} = [x^*, b] \quad .$$

Therefore,

$$<a, \bar{b}> + <\bar{a}, b> = <a, [x^*, b]> + <[x, a], b>$$

$$= \; <[a, b], x> - <[a, b], x^*> \; = \; <[a, b], x - x^*> \quad .$$

If $\mathfrak{A}$ and $\mathfrak{B}$ intersect, then $x - x^* = \bar{\phi} = 0$. If $\mathfrak{A}$ and $\mathfrak{B}$ are parallel then $[a, b] = 0 = \sin\phi$. If $\mathfrak{A}$ and $\mathfrak{B}$ are not parallel and do not intersect, then $\bar{\phi} = |x - x^*| \neq 0$ and $\sin\phi \neq 0$. Further, the vector $x - x^*$ is perpendicular to a and b and, since $x - x^*$ goes from $\mathfrak{B}$ to $\mathfrak{A}$, it is seen that $x - x^*$ and $[a, b]$ have opposite directions. Thus,

$$<a, \bar{b}> + <\bar{a}, b> = 0 = -\bar{\phi} \sin \phi$$

if either $\mathfrak{A}$ and $\mathfrak{B}$ are parallel or intersecting. Otherwise

$$<a, \bar{b}> + <\bar{a}, b> \; = \; <[a, b], x - x^*> = (|a| \cdot |b| \cdot \sin\phi) \cdot |x - x^*| \cdot (-1)$$

$$= \; -\bar{\phi} \sin \phi .$$

Hence

$$\langle \mathfrak{A}, \mathfrak{B} \rangle = \cos \phi - \varepsilon \bar{\phi} \sin \phi .$$

In this paper, the importance of theorem 3.4 lies in the following corollaries.

Corollary 3.4.1: Two distinct lines $\mathfrak{A}$ and $\mathfrak{B}$ intersect if and only if

$$D(\langle \mathfrak{A}, \mathfrak{B} \rangle) = 0$$

and

$$| R(\langle \mathfrak{A}, \mathfrak{B} \rangle)| \neq 1 .$$

Corollary 3.4.2: Two lines $\mathfrak{A}$ and $\mathfrak{B}$ are perpendicular, and possibly skew, if and only if

$$R(\langle \mathfrak{A}, \mathfrak{B} \rangle) = 0.$$

Corollary 3.4.3: Two lines $\mathfrak{A}$ and $\mathfrak{B}$ intersect at right angles if and only if

$$\langle \mathfrak{A}, \mathfrak{B} \rangle = 0 .$$

Theorem 3.5: If two distinct lines intersect, then $[\mathfrak{A}, \mathfrak{B}] / | [a, b] |$ is the line which is perpendicular to and intersects each of them.

Proof:

We must first show that $[\mathfrak{a}, \mathfrak{b}]/|[a,b]|$ is a line, i.e., that

$$<[\mathfrak{a}, \mathfrak{b}]/|[a,b]|, \quad [\mathfrak{a}, \mathfrak{b}]/|[a,b]| > = 1 .$$

By theorems 2.1 and 2.2, this is equivalent to showing that

$$<[\mathfrak{a}, \mathfrak{b}], [\mathfrak{a}, \mathfrak{b}]> = |[a,b]|^2 .$$

Now,

$<[\mathfrak{a}, \mathfrak{b}], [\mathfrak{a}, \mathfrak{b}]> = <\mathfrak{a}, [\mathfrak{b}, [\mathfrak{a}, \mathfrak{b}]]>$ by theorem 2.4

$= <\mathfrak{a}, <\mathfrak{b}, \mathfrak{b}>\mathfrak{a} - <\mathfrak{b}, \mathfrak{a}>\mathfrak{b}>$ by theorem 2.6

$= <\mathfrak{b}, \mathfrak{b}><\mathfrak{a}, \mathfrak{a}> - <\mathfrak{b}, \mathfrak{a}><\mathfrak{a}, \mathfrak{b}>$

$= 1 - <\mathfrak{a}, \mathfrak{b}>^2 = 1 - \cos^2\phi$ by theorem 3.4 and corollary 3.4.1

$= \sin^2\phi = |[a,b]|^2$

since $|b| = |a| = 1$. Thus $[\mathfrak{a}, \mathfrak{b}]/|[a,b]|$ is a line.

Now,

$$<\mathfrak{a}, [\mathfrak{a}, \mathfrak{b}]/|[a,b]|> = <[\mathfrak{a}, \mathfrak{a}], \mathfrak{b}>/|[a,b]| = 0$$

by theorems 2.1, 2.2, 2.3 and corollary 2.1.1. Similarly,

$$< \mathfrak{B}, [\mathfrak{A}, \mathfrak{B}] / |[a,b]| > = 0$$

by theorems 2.1, 2.2, 2.4 and corollary 2.1.1. Hence by corollary 3.4.3, $[\mathfrak{A}, \mathfrak{B}] / |[a,b]|$ intersects each of $\mathfrak{A}$ and $\mathfrak{B}$ at right angles. ∎

<u>Theorem 3.6</u>: Let $\mathfrak{A}$ and $\mathfrak{B}$ be distinct intersecting lines. A line $\mathfrak{L}$ lies in the plane of $\mathfrak{A}$ and $\mathfrak{B}$ if and only if

$$R(< [\mathfrak{A}, \mathfrak{B}], \mathfrak{L} >) = 0$$

and either:

$$D(< \mathfrak{A}, \mathfrak{L} >) = 0 \quad \text{and} \quad |R(< \mathfrak{A}, \mathfrak{L} >)| \neq 1,$$

or

$$D(< \mathfrak{B}, \mathfrak{L} >) = 0 \quad \text{and} \quad |R(< \mathfrak{B}, \mathfrak{L} >)| \neq 1.$$

Proof:

Suppose $\mathfrak{L}$ lies in the plane of $\mathfrak{A}$ and $\mathfrak{B}$. By theorem 3.5, $\mathfrak{L}$ is perpendicular to the line $[\mathfrak{A}, \mathfrak{B}] / |[a,b]|$ and so by corollary 3.4.2,

$$R(<[\mathfrak{A},\mathfrak{B}],\mathfrak{L}>) = 0.$$

Also, $\mathfrak{L}$ intersects either $\mathfrak{A}$ or $\mathfrak{B}$ and by corollary 3.4.1, either

$D(<\mathfrak{A},\mathfrak{L}>) = 0$ and $|R(<\mathfrak{A},\mathfrak{L}>)| \neq 1$

or

$D(<\mathfrak{B},\mathfrak{L}>) = 0$ and $|R(<\mathfrak{B},\mathfrak{L}>)| \neq 1$.

Next, suppose

$$R(<[\mathfrak{A},\mathfrak{B}],\mathfrak{L}>) = 0$$

and say

$$D(<\mathfrak{B},\mathfrak{L}>) = 0 \text{ and } |R(<\mathfrak{B},\mathfrak{L}>)| \neq 1 .$$

By corollary 3.4.2, $\mathfrak{L}$ is perpendicular to the line $[\mathfrak{A},\mathfrak{B}] / |[a,b]|$ and by corollary 3.4.1, $\mathfrak{L}$ intersects $\mathfrak{B}$. Thus $\mathfrak{L}$ lies in the plane of $\mathfrak{A}$ and $\mathfrak{B}$. The argument is similar if

$D(<\mathfrak{A},\mathfrak{L}>) = 0$ and $|R(<\mathfrak{A},\mathfrak{L}>)| \neq 1$. ∎

<u>Corollary 3.6.1:</u> Let $\mathfrak{A} = a + \varepsilon\bar{a}$ and $\mathfrak{B} = b + \varepsilon\bar{b}$

be distinct lines intersecting at the point x. Then $\mathcal{L} = c + \varepsilon \bar{c}$ lies in the plane of $\mathfrak{A}$ and $\mathfrak{B}$ and passes through the point x if and only if

$$< [a,b], c > = 0 \text{ and } [x,c] = \bar{c}\ .$$

We now let $\{\mathfrak{A}(\tau)\}$ be a one parameter family of lines in R^3. This means that $\mathfrak{A}(\tau) = a(\tau) + \varepsilon \bar{a}(\tau)$ where $a(\tau)$ and $\bar{a}(\tau)$ are functions of the real variable τ with ranges lying in R^3, and further that

$$< a(\tau), a(\tau) > = 1 \quad \text{and} \quad < a(\tau), \bar{a}(\tau) > = 0$$

for all τ. Let

$$\mathfrak{A}'(\tau) = a'(\tau) + \varepsilon \bar{a}'(\tau)$$

denote the derivative of $\mathfrak{A}$ with respect to τ, where

$$a'(\tau) = \lim_{h \to 0} \frac{a(\tau+h) - a(\tau)}{h}\ .$$

Higher derivatives are defined in a similar manner. The family $\{\mathfrak{A}(\tau)\}$ is called differentiable if $\mathfrak{A}'(\tau)$ exists for all τ.

<u>Theorem 3.7</u>: If $\{\mathfrak{A}(\tau)\}$ is a differentiable family of lines which all pass through the point x_0 and

$$< a(\tau),\ \bar{a}'(\tau) > \neq 0,$$

then

$$[\bar{a}'(\tau), \bar{a}(\tau)] \,/\, < a(\tau), \bar{a}'(\tau) > = x_0 .$$

Proof:

By theorem 3.1,

$$[x_0, a(\tau)] = \bar{a}(\tau)$$

for all τ. Hence

$$[x_0, a'(\tau)] = \bar{a}'(\tau) .$$

Therefore,

$$[\bar{a}'(\tau), \bar{a}(\tau)] \,/\, < a(\tau), \bar{a}'(\tau) >$$

$$= [[x_0, a'(\tau)] , [x_0, a(\tau)]] \,/\, < a(\tau), [x_0, a'(\tau)] >$$

$$= \frac{< [x_0, a'(\tau)], a(\tau) > x_0 - < [x_0, a'(\tau)], x_0 > a(\tau)}{< a(\tau), [x_0, a'(\tau)] >}$$

$$= x_0 ,$$

since

$$< [x_0, a'(\tau)] , x_0 > = 0. \blacksquare$$

Theorem 3.8: Suppose the family $\{\mathfrak{A}(\tau)\}$ of lines is differentiable and

$$[\bar{a}'(\tau), \bar{a}(\tau)] / \langle a(\tau), \bar{a}'(\tau) \rangle = x_0 ,$$

a non-zero constant vector. Then the vectors of the families $\{\bar{a}(\tau)\}$ and $\{\bar{a}'(\tau)\}$ all lie in a plane. Furthermore, each member of the family $\{\mathfrak{A}(\tau)\}$ intersects the unique line which is perpendicular to this plane and which passes through the origin.

Proof:

$$[\bar{a}'(\tau), \bar{a}(\tau)] / \langle a(\tau), \bar{a}'(\tau) \rangle = x_0$$

implies

$$[\bar{a}'(\tau), \bar{a}(\tau)] = \langle a(\tau), \bar{a}'(\tau) \rangle x_0$$

and hence $\bar{a}'(\tau)$ and $\bar{a}(\tau)$ are perpendicular to x_0 for all τ since $[\bar{a}'(\tau), \bar{a}(\tau)]$ is perpendicular to $\bar{a}'(\tau)$ and $\bar{a}(\tau)$. Thus the vectors of the families $\{\bar{a}(\tau)\}$ and $\{\bar{a}'(\tau)\}$ all lie in a plane.

Suppose $\mathfrak{L}$ is the unique line which is perpendicular to this plane and passes through the origin. Then $\bar{b} = 0$ and

$$\langle \bar{a}(\tau), b \rangle = 0$$

for all τ. Hence

$$<\mathfrak{A}(\tau), \mathfrak{B}> = <a(\tau), b> + \varepsilon(<a(\tau), \bar{b}> + <\bar{a}(\tau), b>) = <a(\tau), b>$$

and therefore,

$$(1) \qquad D(<\mathfrak{A}(\tau), \mathfrak{B}> = 0.$$

Since $x_0 \neq 0$, it follows that $\bar{a}(\tau) \neq 0$ and $\mathfrak{A}(\tau)$ does not pass through the origin. Hence $\mathfrak{A}(\tau)$ and $\mathfrak{B}$ are not parallel so that

$$(2) \qquad |R(<\mathfrak{A}(\tau), \mathfrak{B}>)| \neq 1 .$$

Equations (1) and (2) together with corollary 3.4.1 imply that $\mathfrak{A}(\tau)$ intersects $\mathfrak{B}$ for each τ. ∎

Theorem 3.9: Let $\{\mathfrak{A}(\tau)\}$ be a twice differentiable family of lines for which $a'(\tau)$ and $a''(\tau)$ do not vanish. Suppose each line in the family passes through the point x_0. The family $\{\mathfrak{A}(\tau)\}$ of lines lies in a plane if and only if

$$<\mathfrak{A}(\tau), [\mathfrak{A}'(\tau), \mathfrak{A}''(\tau)]> = 0$$

for all τ.

Proof:

Suppose the family $\{\mathfrak{A}(\tau)\}$ lies in a plane. There is a line $\mathcal{L}$ through the point x_0, common to all lines of $\{\mathfrak{A}(\tau)\}$, which is perpendicular to the plane of the family and hence to each member of the family. Since $\mathcal{L}$ intersects $\mathfrak{A}(\tau)$ at the point x_0 for each τ, corollary 3.4.3 gives

$$< \mathcal{L}, \mathfrak{A}(\tau) > = 0 .$$

Furthermore, theorem 3.1 gives

$$[x_0, a(\tau)] = \bar{a}(\tau)$$

since x_0 lies on each line in the family.

Differentiating $< \mathcal{L}, \mathfrak{A}(\tau) > = 0$ twice yields

$$< \mathcal{L}, \mathfrak{A}'(\tau) > = 0 \quad \text{and} \quad < \mathcal{L}, \mathfrak{A}''(\tau) > = 0 .$$

Since $a'(\tau) \neq 0$ and $a''(\tau) \neq 0$ by assumption, theorem 3.3 implies

$$\frac{(<a'(\tau), a'(\tau)> - \varepsilon <a'(\tau), \bar{a}'(\tau)>)}{|a'(\tau)|^3} \quad \mathfrak{A}'(\tau)$$

and

$$\frac{(<a''(\tau), a''(\tau)> - \varepsilon <a''(\tau), \bar{a}''(\tau)>)}{|a''(\tau)|^3} \quad \mathfrak{A}''(\tau)$$

are Study vectors, i.e., directed lines. From corollary 3.4.3 the relations

$$<\mathcal{L}, \mathfrak{a}'(\tau)> = 0 \quad \text{and} \quad <\mathcal{L}, \mathfrak{a}''(\tau)> = 0$$

imply that these two lines are perpendicular to $\mathcal{L}$.

Differentiating

$$[x_0, a(\tau)] = \bar{a}(\tau)$$

twice yields

$$[x_0, a'(\tau)] = \bar{a}'(\tau) \quad \text{and} \quad [x_0, a''(\tau)] = \bar{a}''(\tau).$$

Thus by theorem 3.1, x_0 lies on each of the lines

$$\frac{<a'(\tau), a'(\tau)> - \varepsilon <a'(\tau), \bar{a}'(\tau)>}{|a'(\tau)|^3} \; \mathfrak{a}'(\tau)$$

and

$$\frac{<a''(\tau), a''(\tau)> - \varepsilon <a''(\tau), \bar{a}''(\tau)>}{|a''(\tau)|^3} \; \mathfrak{a}''(\tau)$$

and these lines lie in the plane of the family $\{\mathfrak{a}(\tau)\}$. Since

$$<\mathfrak{a}, (\tau), \mathfrak{a}(\tau)> = 1$$

we have

$$< \mathfrak{a}(T), \mathfrak{a}'(T) > = 0$$

and by corollary 3.4.3, the lines $\mathfrak{a}(T)$ and

$$\frac{<a'(T), a'(T)> - \varepsilon <a'(T), a''(T)>}{|a'(T)|^3} \quad \mathfrak{a}'(T)$$

intersect at right angles. Hence theorem 3.5 implies

$$\mathfrak{b} = \frac{[\mathfrak{a}(T), \frac{(<a'(T), a'(T)> - \varepsilon <a'(T), \bar{a}'(T)>)}{|a'(T)|^3} \mathfrak{a}'(T)]}{[a,(T), \frac{(<a'(T), a'(T)> - \varepsilon <a'(T), \bar{a}'(T)>)}{|a'(T)|^3} a'(T)]}$$

$$= [\mathfrak{a}(T), \mathfrak{a}'(T)] / |[a(T), a'(T)] \; .$$

Thus,

$$< \mathfrak{b}, \mathfrak{a}''(T) > = 0$$

implies by theorems 2.2 and 2.4 that

$$< \mathfrak{a}(T), [\mathfrak{a}'(T), \mathfrak{a}''(T)] > = 0,$$

the desired result.

For the sufficiency, we first prove a

<u>Lemma:</u> Let $\{a(T)\}$ be a twice differentiable family of

unit vectors in R^3. The vectors of this family all lie in a plane if

$$<a(\tau), [a'(\tau), a''(\tau)]> = 0$$

for all τ.

Proof:

A non-zero constant vector $g = (\gamma_1, \gamma_2, \gamma_3)$ will be exhibited which will satisfy $<g, a(\tau)> = 0$ for all τ.

Suppose

$$a(\tau) = (a_1(\tau), a_2(\tau), a_3(\tau)).$$

Then

$$<a(\tau), [a'(\tau), a''(\tau)]> = 0$$

is true only if there are functions η_1, η_2, η_3 of τ which do not vanish simultaneously and for which

$$\eta_1(\tau)a(\tau) + \eta_2(\tau)a'(\tau) + \eta_3(\tau)a''(\tau) = 0$$

for all τ. This is equivalent to

$$\eta_1(\tau)a_i(\tau) + \eta_2(\tau)a_i'(\tau) + \eta_3(\tau)a_i''(\tau) = 0$$

for $i = 1, 2, 3$ and all τ.

Suppose $\eta_3(\tau)$ is never zero. The linear homogeneous differentiable equation

(1) $\eta_3(\tau)\theta''(\tau) + \eta_2(\tau)\theta'(\tau) + \eta_1(\tau)\theta(\tau) = 0$

has two linearly independent solutions $\xi(\tau)$ and $\psi(\tau)$, and every solution is a linear combination of these with constant coefficients. Since $a_i(\tau)$ satisfies equation (1), we have

$$a_i(\tau) = \lambda_i \xi(\tau) + \mu_i \psi(\tau),$$

$i = 1, 2, 3$, for constants $\lambda_1, \lambda_2, \lambda_3$ and μ_1, μ_2, μ_3.

Let $g = (\gamma_1, \gamma_2, \gamma_3)$ and consider the equation

$$\langle g, a(\tau)\rangle = \gamma_1 a_1(\tau) + \gamma_2 a_2(\tau) + \gamma_3 a_3(\tau) = 0.$$

This leads to the equation

$$(\gamma_1\lambda_1 + \gamma_2\lambda_2 + \gamma_3\lambda_3)\xi(\tau) + (\gamma_1\mu_1 + \gamma_2\mu_2 + \gamma_3\mu_3)\psi(\tau) = 0 .$$

Since $\xi(\tau)$ and $\psi(\tau)$ are linearly independent, this equation implies

$$\gamma_1\lambda_1 + \gamma_2\lambda_2 + \gamma_3\lambda_3 = 0 \quad \text{and} \quad \gamma_1\mu_1 + \gamma_2\mu_2 + \gamma_3\mu_3 = 0,$$

which has the solutions

$$\gamma_1 = \rho(\lambda_2\mu_3 - \lambda_3\mu_2), \gamma_2 = \rho(\lambda_3\mu_1 - \lambda_1\mu_3), \gamma_3 = \rho(\lambda_1\mu_2 - \lambda_2\mu_1)$$

for arbitrary ρ. Take $\rho = 1$. The vector $g = (\gamma_1, \gamma_2, \gamma_3)$ thus constructed is a non-zero constant vector such that

$$< g, a(\tau) > = 0$$

for all τ. Therefore each $a(\tau)$ is perpendicular to g and thus the vectors in the family $\{a(\tau)\}$ all lie in a plane.

Suppose $\eta_3(\tau) = 0$. Then

$$\eta_1(\tau)a_i(\tau) + \eta_2(\tau)a_i'(\tau) = 0$$

and, by differentiation,

$$\eta_1'(\tau)a_i(\tau) + (\eta_1(\tau) + \eta_2'(\tau))a_i'(\tau) + \eta_2(\tau)a_i''(\tau) = 0.$$

Hence, if $\eta_2(\tau) \neq 0$, we may reapply the preceding argument with η_1, η_2, η_3 replaced by $\eta_1', \eta_1 + \eta_2', \eta_2$ to construct the non-zero constant vector g. This completes the proof of the lemma.

We continue with the proof of the theorem.

$$< \mathfrak{a}(\tau), [\mathfrak{a}'(\tau), \mathfrak{a}''(\tau)] > = 0$$

implies

$$<a(\tau),[a'(\tau),a''(\tau)]> = 0$$

By the preceding lemma, the family $\{a(\tau)\}$ lies in a plane in R^3. Let τ_1 and τ_2 be such that $a(\tau_1) \neq a(\tau_2)$. Then for all τ,

$$<[a(\tau_1),a(\tau_2)], a(\tau)> = 0 .$$

Since the point x_0 lies on each line $\mathfrak{a}(\tau)$ by assumption, theorem 3.1 gives

$$[x_0, a(\tau)] = \bar{a}(\tau)$$

for all τ. Then corollary 3.6.1 implies that $\mathfrak{a}(\tau)$ lies in the plane determined by the lines $\mathfrak{a}(\tau_1)$ and $\mathfrak{a}(\tau_2)$ for all τ, i.e., the family $\{\mathfrak{a}(\tau)\}$ of lines lies in a plane. ∎

PART IV: FAMILIES OF TANGENT LINES TO CURVES IN R^3

Let C be a differentiable curve in R^3 with parametric representation $x = x(\tau)$ such that for all τ we have $x'(\tau) \neq 0$, i.e., x is a function from the real numbers to R^3 which is differentiable everywhere and whose derivative does not vanish. Then $x'(\tau)$ is a vector which is tangent to the curve C at the point $x(\tau)$. The tangent line to C at $x(\tau)$ contains the two distinct points $x(\tau)$ and $x(\tau) + x'(\tau)$ (2, p. 171).

In the remainder of this paper, the orientation of a line $\mathfrak{a} = a + \varepsilon\bar{a}$ will be immaterial. We will therefore consider coincident lines with opposite orientation as being equivalent, i.e., the lines $\mathfrak{a} = a + \varepsilon\bar{a}$ and $-\mathfrak{a} = -a - \varepsilon\bar{a}$ will be considered equivalent. Thus $\mathfrak{a} = a + \varepsilon\bar{a}$ will represent the (non-oriented) line, one of whose orientations would have $\mathfrak{a} = a + \varepsilon\bar{a}$ as its Study vector representation.

In the proofs of the following theorems, and in some of the discussion and examples, the real variable τ is omitted, except where clarity dictates, in order to simplify the reading.

Theorem 4.1: Let C be a differentiable curve with parametric representation $x = x(\tau)$ for which $x'(\tau) \neq 0$ for all τ. The family of tangents to C is

$T = \{ \mathfrak{A}(T): a(T) = x'(T)/|x'(T)|,\ \bar{a}(T) = [x(T), x'(T)] / |x'(T)| \}.$

Proof:

Let $\mathfrak{A} = a + \varepsilon\bar{a}$ be an element of T.

Because

$$[x, a] = [x, x'] / |x'| = \bar{a}$$

and

$$[x + x', a] = [x + x', x'] / |x'| = [x, x'] / |x'| = \bar{a},$$

theorem 3.1 implies that the two points x and $x + x'$ lie on the line $\mathfrak{A}$, and $\mathfrak{A}$ is tangent to the curve C.

Suppose $\mathfrak{L} = b + \varepsilon\bar{b}$ is tangent to C at x. The points x and $x + x'$ lie on the line $\mathfrak{L}$ and by the definitions of b and $\bar{b}$, i.e., equations (3.1) and (3.2),

$$b = \frac{(x + x') - x}{|(x+x') - x|} = \frac{x'}{|x'|}$$

and

$$\bar{b} = \frac{[x, x+x']}{|(x+x') - x|} = \frac{[x, x']}{|x'|}.$$

Thus $\mathcal{L}$ is an element of T. ∎

Consider now the problem of conditions which insure that a given family $\{\mathfrak{A}(\tau) = a(\tau) + \varepsilon \bar{a}(\tau)\}$ of lines is a family of tangent lines to a differentiable curve C in R^3.

Theorem 4. 2: If $\{\mathfrak{A}(\tau)\}$ is a differentiable family of tangent lines to a curve C and if $x = x(\tau)$ is a parametric representation of C such that $\mathfrak{A}(\tau)$ is tangent to C at $x(\tau)$ and $x'(\tau) \neq 0$, then

$$< a'(\tau), \bar{a}'(\tau) > = 0$$

for all τ.

Proof:

According to theorem 4. 1,

$$a = x' / |x'|$$

and

$$\bar{a} = [x, x'] / |x'| .$$

Hence

$$a' = x'' / |x'| + x' \cdot \frac{d}{d\tau}(1/|x'|)$$

and

$$\bar{a}' = [x' x'] / |x'| + [x, x''] / |x'| + [x, x'] \cdot \frac{d}{d\tau}(1/|x'|)$$

$$= [x, x''] / |x'| + [x, x'] \cdot \frac{d}{d\tau}(1/|x'|) .$$

It follows that

$$<a', \bar{a}'> = < \frac{x''}{|x'|} + x' \cdot \frac{d}{d\tau}(1/|x'|), \frac{[x,x'']}{|x'|} + [x,x'] \frac{d}{d\tau}(1/|x'|)>$$

$$= \frac{<x'', [x, x'']>}{|x'|^2} + \frac{<x', [x, x'']>}{|x'|} \cdot \frac{d}{d\tau}(1/|x'|)$$

$$+ \frac{<x'', [x, x']>}{|x'|} \frac{d}{d\tau}(1/|x'|) + <x', [x,x']> (\frac{d}{d\tau}(1/|x'|))^2$$

$$= 0$$

since

$$<x'', [x, x'']> = <x', [x, x']> = 0$$

and

$$<x', [x, x'']> = - <x'', [x, x']> . \blacksquare$$

<u>Theorem 4.3:</u> Let $T = \{\mathfrak{a}(\tau)\}$ be a twice differentiable

family of lines defined over a finite or infinite open interval (α, β) of values of τ. If

(i) $< a'(\tau), \bar{a}'(\tau) > = 0$ for $\tau \epsilon (\alpha, \beta)$

(ii) $< a(\tau), \bar{a}'(\tau) > \neq 0$ for $\tau \epsilon (\alpha, \beta)$

and (iii) $< \bar{a}(\tau), [\bar{a}'(\tau), \bar{a}''(\tau)] > \neq 0$ for $\tau \epsilon (\alpha, \beta)$,

then there is a differentiable curve C which has T as its family of tangent lines.

Proof:

Since

$$< a(\tau), \bar{a}'(\tau) > \neq 0 \quad \text{for} \quad \tau \epsilon (\alpha, \beta),$$

$$x(\tau) = [\bar{a}'(\tau), \bar{a}(\tau)] \, / < a(\tau), \bar{a}'(\tau) >$$

is defined for $\tau \epsilon (\alpha, \beta)$. Because $a(\tau)$ and $\bar{a}(\tau)$ are twice differentiable, $x'(\tau)$ exists for $\tau \epsilon (\alpha, \beta)$.

Now,

$$x' = \frac{\langle a,\bar{a}'\rangle([\bar{a}',\bar{a}'] + [\bar{a}'',\bar{a}]) - (\langle a,\bar{a}''\rangle + \langle a',\bar{a}'\rangle)[\bar{a}',\bar{a}]}{\langle a,\bar{a}'\rangle^2}$$

$$= \frac{\langle a,\bar{a}'\rangle[\bar{a}'',\bar{a}] - \langle a,\bar{a}''\rangle[\bar{a}',\bar{a}]}{\langle a,\bar{a}'\rangle^2}$$

$$= \frac{[\langle a,\bar{a}'\rangle\bar{a}'' - \langle a,\bar{a}''\rangle\bar{a}',\bar{a}]}{\langle a,\bar{a}'\rangle^2}$$

since

$$[\bar{a}',\bar{a}'] = 0 \quad \text{and} \quad \langle a',\bar{a}'\rangle = 0.$$

The vector identity

$$[[u,v],w] = \langle u,w\rangle v - \langle v,w\rangle u$$

is used to simplify this expression as follows.

$$x' = \frac{[[[\bar{a}',\bar{a}''],a],\bar{a}]}{\langle a,\bar{a}'\rangle^2} = \frac{\langle \bar{a},[\bar{a}',\bar{a}'']\rangle a - \langle a,\bar{a}\rangle[\bar{a}',\bar{a}'']}{\langle a,\bar{a}'\rangle^2}$$

$$= \frac{\langle \bar{a},[\bar{a}',\bar{a}'']\rangle}{\langle a,\bar{a}'\rangle^2} \cdot a$$

since $\langle a,\bar{a}\rangle = 0$.

We see by hypothesis (iii) that x' does not vanish anywhere in (α,β) so that x is not constant throughout any subinterval of (α, β). Therefore, $x = x(T)$ for $T \in (\alpha, \beta)$ is a parameterization of a differentiable curve C in R^3 for which $x'(T) \neq 0$ for $T \in (\alpha, \beta)$.

Since a is a unit vector and $x' \neq 0$, it follows from the above that

$$a = x' / |x'| .$$

Furthermore,

$$[x, x'] / |x'| = [[\bar{a}', \bar{a}], a] / < a, \bar{a}' >$$

$$= \frac{< \bar{a}', a > \bar{a} - < \bar{a}, a > \bar{a}'}{< a, \bar{a}' >} = \bar{a}$$

since $< \bar{a}, a > = 0$. The family T has the representation

$$T = \{\mathcal{O}\!\ell : a = x' / |x'|, \quad \bar{a} = [x, x'] / |x'|\}$$

and by theorem 4. 1, T is the family of tangent lines to the curve C. ∎

For a family T of lines, condition (ii) of theorem 4. 3 either is satisfied, or it fails in one of the following ways:

(1) at isolated points of (α, β) ,

(2) over subintervals of (α, β),

or (3) at points of (α, β) which are limit points of points at which (ii) holds and are also limit points of points at which (ii) fails to hold.

The next three theorems deal with case (2) above, and case (1) is discussed later.

For a differentiable family $\{\mathfrak{a}(T)\}$ of Study vectors,

$$< a(T), \bar{a}(T) > = 0$$

implies

$$< a, \bar{a}' > + < a', \bar{a} > = 0 \ .$$

Thus $< a, \bar{a}' > = 0$ if and only if $< a', \bar{a} > = 0$. Also, $< a', \bar{a} > = 0$ if and only if $a' = 0$, $\bar{a} = 0$, or a' is perpendicular to $\bar{a}$.

The following table shows the treatment of the problem under case (2) above, i.e., where $< a, \bar{a}' > = 0$ over a subinterval (γ, δ) of (α, β).

$a'(T)$ perpendicular to $\bar{a}(T)$ on (γ, δ) ----- theorem 4.4

$a'(T) = 0$ on (γ, δ) -------------------- theorem 4.5

$\bar{a}(T) = 0$ on (γ, δ) --------------------- theorem 4.6

<u>Theorem 4.4</u>: Let $T = \{\mathfrak{a}(T)\}$ be a twice differentiable

family of lines defined over a finite or infinite open interval (γ, δ) of values of τ. If, throughout (γ, δ),

(0) $a'(\tau) \neq 0$ and $\bar{a}(\tau) \neq 0$,

(i) $< a'(\tau), \bar{a}'(\tau) > = 0$,

(ii) $< a(\tau), \bar{a}'(\tau) > = 0$,

(iii) $|a'|^2 \cdot |\bar{a}| - (|a'|^2)' |\bar{a}|' / 2|a'|^2 + |\bar{a}|'' \neq 0$,

then there is a differentiable curve C which has T as its family of tangent lines.

Proof:

According to the discussion preceding the statement of the theorem, conditions (0) and (ii) imply that a' and $\bar{a}$ are perpendicular for all τ.

Equation (3. 3) states

$$< a(\tau), a(\tau) > = 1 \quad \text{for all} \quad \tau,$$

so that

$$< a(\tau), a'(\tau) > = 0 \quad \text{for all} \quad \tau.$$

Equation (3. 4) shows that

$$< a(\tau), \bar{a}(\tau) > = 0$$

and hence, the set $\{a, a', \bar{a}\}$ forms a mutually perpendicular set of vectors.

From

$$<a'(\tau), \bar{a}(\tau)> = 0,$$

it follows that

$$< a'', \bar{a} > + < a', \bar{a}' > = 0,$$

and by hypothesis (i),

$$< a'', \bar{a} > = 0.$$

Since $\bar{a} \neq 0$, either $a'' = 0$ or a'' is perpendicular to $\bar{a}$ and in either case,

$$< a, [a', a''] > = 0 \quad \text{for all} \quad \tau \epsilon (\gamma, \delta).$$

By the lemma used in the proof of theorem 3. 9, this implies that the vectors of the family $\{a(\tau) : \tau \epsilon (\gamma, \delta)\}$ all lie in a plane. Since a' is perpendicular to both a and $\bar{a}$, this plane is described by $\{w: < w, [a, a'] > = 0\}$.

The vectors of the family $\{\bar{a}(\tau)\}$ are all perpendicular to this plane.

Consequently,

$$\bar{a}(\tau) = \lambda(\tau) v_0$$

for some unit vector v_0 perpendicular to the plane and some non-vanishing, real valued function $\lambda(\tau)$ which is twice differentiable.

We prove the following two lemmas in order to separate the proof of the theorem into more comprehensible parts.

<u>Lemma 1:</u> Under the conditions of theorem 4.4,

$$a'' = -|a'|^2 a + ((|a'|^2)'/2|a'|^2) a' \; .$$

Proof:

Since a and a' are perpendicular,

$$a'' = \mu(\tau) a + \nu(\tau) a' ,$$

$$<a, a''> = \mu <a, a> = \mu ,$$

and

$$<a', a''> = \nu <a', a'> .$$

Further, $<a, a'> = 0$ implies

$$<a, a''> + <a', a'> = 0$$

so that

$$<a, a''> = -|a'|^2 .$$

Also,

$$(<a', a'>)' = 2<a', a''> .$$

Thus

$$\mu = <a, a''> = -|a'|^2,$$

$$\nu = <a', a''>/ <a', a'> = (|a'|^2)'/2|a'|^2$$

and

$$a'' = -|a'|^2 a + ((|a'|^2)'/2|a'|^2)a'$$

which proves the lemma.

Lemma 2: Under the conditions of theorem 4. 4,

$$(1) \quad <v_0, [a, a']><v_0, \overline{a}''> - <v_0, [a, a'']><v_0, \overline{a}'>$$

$$+ <v_0, [a', a'']> <v_0, \overline{a}>$$

is non-zero.

Proof:

We rewrite the given expression (1) as

$$<v_0, <v_0, [a,a'] > \bar{a}'' - <v_0, [a,a''] > \bar{a}' + <v_0, [a',a''] > \bar{a} >$$

which is zero if and only if

$$<v_0, [a,a'] > \bar{a}'' - <v_0, [a,a''] > \bar{a}' + <v_0, [a',a''] > \bar{a}$$

is zero, since v_0, $\bar{a}$, $\bar{a}'$ and $\bar{a}''$ all lie on a line.

Since $\bar{a}(T) = \lambda(T) v_0$ and $v_0 \neq 0$, this expression is zero if and only if

$$|[a,a']|\lambda'' - |[a,a'']|\lambda' + |[a',a'']|\lambda$$

is zero. We use lemma 1 to replace a'' in this last expression and get

$$|[a,a']|\lambda'' - \frac{|[a,(|a'|^2)'a']|}{2|a'|^2}\lambda' + |[a', -|a'|^2 a]|\lambda .$$

Since a and a' are perpendicular,

$$|[a,a']| = |a'|$$

which is non-zero by assumption. Thus, (1) is zero if and only if

$$\lambda'' - \frac{|(|a'|^2)'|}{2|a'|^2}\lambda' + |a'|^2\lambda$$

is zero. However, replacing λ by $|\bar{a}|$, we get

$$(|\bar{a}|)'' - \frac{|(|a'|^2)'|}{2|a'|^2}(|\bar{a}|)' + |a'|^2|\bar{a}|$$

which is non-zero by hypothesis (iii) of the theorem. This completes the proof of lemma 2.

Continuing the proof of the theorem, we define

$$x(T) = \frac{<v_0, \bar{a}'(T)> a(T) - <v_0, \bar{a}(T)> a'(T)}{<v_0, [a(T), a'(T)]>} .$$

Differentiating this expression we get,

$$x'(T) = \frac{(<v_0, \bar{a}''>a + <v_0, \bar{a}'>a' - <v_0, \bar{a}'>a' - <v_0, \bar{a}> a'')}{<v_0, [a, a']>}$$

$$- \frac{(<v_0, [a', a']> + <v_0, [a, a'']>)(<v_0, \bar{a}'>a - <v_0, \bar{a}>a')}{<v_0, [a, a']>^2}$$

$$= \frac{<v_0, [a, a']>(<v_0, \bar{a}''>a - <v_0, \bar{a}> a'')}{<v_0, [a, a']>^2}$$

$$- \frac{<v_0, [a, a'']>(<v_0, \bar{a}'>a - <v_0, \bar{a}> a')}{<v_0, [a, a']>^2}$$

$$= \frac{(<v_0, [a, a']> <v_0, \bar{a}''> - <v_0, [a, a'']><v_0, \bar{a}'>) a}{<v_0, [a, a']>^2}$$

$$+ \frac{<v_0, [a, a'']><v_0, \bar{a}>a' - <v_0, [a, a']><v_0, \bar{a}>a''}{<v_0, [a, a']>^2}$$

$$= \frac{\nu(T)a + <v_0, \bar{a}>(<[v_0, a], a''> a' - <[v_0, a], a'> a'')}{<v_0, [a, a']>^2}$$

where $\nu(T) = \langle v_0, [a, a'] \rangle \langle v_0, \bar{a}'' \rangle - \langle v_0, [a, a''] \rangle \langle v_0, \bar{a}' \rangle$.

Using the vector identity

$$[u, [v, w]] = \langle u, w \rangle v - \langle u, v \rangle w,$$

we have

$$x'(T) = \frac{\nu(T)a - \langle v_0, \bar{a} \rangle [[v_0, a], [a'', a']]}{\langle v_0, [a, a'] \rangle^2}$$

$$= \frac{\nu(T)a + \langle v_0, \bar{a} \rangle (\langle v_0, [a', a''] \rangle a - \langle a, [a', a''] \rangle v_0)}{\langle v_0, [a, a'] \rangle^2}$$

$$= \frac{\nu(T)a + \langle v_0, \bar{a} \rangle \langle v_0, [a', a''] \rangle a}{\langle v_0, [a, a'] \rangle^2}$$

since $\langle a, [a', a''] \rangle = 0$. Therefore,

$$x'(T) = \frac{(\langle v_0, [a, a'] \rangle \langle v_0, \bar{a}'' \rangle - \langle v_0, [a, a''] \rangle \langle v_0, \bar{a}' \rangle) a}{\langle v_0, [a, a'] \rangle^2}$$

$$+ \frac{\langle v_0, [a', a''] \rangle \langle v_0, \bar{a} \rangle a}{\langle v_0, [a, a'] \rangle^2}.$$

By lemma 2, $x'(T) \neq 0$ and since $a(T)$ is a unit vector,

$$a(T) = x'(T) / |x'(T)|.$$

Furthermore,

$$[x, a] = \frac{[<v_0, \bar{a}'> a - <v_0, \bar{a}> a', a]}{<v_0, [a, a']>} = \frac{<v_0, \bar{a}>[a, a']}{<v_0, [a, a']>}.$$

We recall that

$$[a, a'] = \mu(T)\bar{a}$$

so that

$$[x, a] = \frac{<v_0, \bar{a}> \mu(T)\bar{a}}{<v_0, \bar{a}> \mu(T)} = \bar{a}.$$

Theorem 4.1 shows that the family $\{\mathfrak{A}(T): T \epsilon (\gamma, \delta)\}$ is the family of tangent lines to the differentiable curve C with parametric representation $x = x(T)$. ∎

Suppose now that $a'(T) = 0$ for all T in a finite or infinite interval (γ, δ). Theorem 4.1 shows that if $\{\mathfrak{A}(T): T \epsilon (\gamma, \delta)\}$ is a family of tangent lines to a twice differentiable curve C with parametrization $x = x(T)$ for which $x'(T) \neq 0$, then

$$a(T) = x'(T)/|x'(T)|$$

and

$$\bar{a}(\tau) = [x(\tau), x'(\tau)/|x'(\tau)|] .$$

It follows that

$$a'(\tau) = x''/|x'| + x'(1/|x'|)' = 0$$

and

$$\bar{a}'(\tau) = [x', x'/|x'|] + [x, x''/|x'|] + [x, x'](1/|x'|)'$$

$$= [x, x'']/|x'| + [x, x'](1/|x'|)'$$

$$= -[x, x'](1/|x'|)' + [x, x'](1/|x'|)' = 0 .$$

Thus $\bar{a}'(\tau) = 0$ for all $\tau \epsilon (\gamma, \delta)$.

On the other hand, suppose a' and $\bar{a}'$ are identically zero on (γ, δ). The family $\{\mathfrak{a}(\tau): \tau \epsilon (\gamma, \delta)\}$ consists of a single line $\mathfrak{a}_0$ and this line is tangent to, and coincident with, the curve consisting of this line, or a segment thereof. This proves the following theorem.

<u>Theorem 4.5</u>: Let $T = \{\mathfrak{a}(\tau): \tau \epsilon (\gamma, \delta)\}$ be a twice differentiable family of lines for which $a'(\tau) = 0$ over (γ, δ). Then T is a family of tangent lines to a twice differentiable curve C if and only if $\bar{a}'(\tau) = 0$ over (γ, δ).

Finally, suppose $\bar{a}(\tau)$ is identically zero on a finite or infinite open interval (γ, δ). Then every line in the family $\{\mathfrak{a}(\tau): \tau \in (\gamma, \delta)\}$ passes through the origin and the family is a family of tangent lines to a differentiable curve C if and only if all of the lines coincide, i.e., $a'(\tau) = 0$ over (γ, δ). In this case, $\mathfrak{a}(\tau) = a_0$ for some constant vector $a_0 \in R^3$; we denote this line through the origin by $\mathfrak{a}_0$. The family $\mathfrak{a}(\tau) = \mathfrak{a}_0$ is tangent to and coincident with the curve consisting of the line $\mathfrak{a}_0$ or a segment thereof. This proves the final theorem.

<u>Theorem 4.6:</u> Let $T = \{\mathfrak{a}(\tau): \tau \in (\gamma, \delta)\}$ be a twice differentiable family of lines for which $\bar{a}(\tau)$ is identically zero. T is a family of tangent lines to a differentiable curve C if and only if $a'(\tau)$ is identically zero over (γ, δ). In this case C is a line through the origin or a segment thereof.

The discussion preceding theorem 4.3 shows that $\langle a, \bar{a}' \rangle = 0$ if and only if $\langle a', \bar{a} \rangle = 0$. Also, $\langle a', \bar{a} \rangle = 0$ if and only if a' is perpendicular to $\bar{a}$, or $a' = 0$, or $\bar{a} = 0$. Given a twice differentiable family of lines $T = \{\mathfrak{a}(\tau): \tau \in (\alpha, \beta)\}$ for which either $\langle a, \bar{a}' \rangle \neq 0$, or a' is perpendicular to $\bar{a}$, or $a' = 0$, or $\bar{a} = 0$ throughout the interval (α, β), theorems 4.2 through 4.6 show whether or not the family T is a family of tangent lines to a differentiable curve $C: x = x(\tau)$ for which $x'(\tau) \neq 0$.

In order to determine whether a given twice differentiable family of lines $T = \{\mathfrak{A}(\tau)\tau\epsilon(\alpha,\beta)\}$ is a family of tangent lines as described above, the following procedure may be used.

First check to see if theorem 4.2 is satisfied, i.e., if $< a'(\tau), \bar{a}'(\tau) > = 0$ over (α,β), since this is a necessary condition. If this is true, then check the conditions of theorems 4.3 through 4.6. If the conditions of one of these theorems are satisfied in a subinterval (γ,δ) of (α,β), we may apply the theorem to define a curve $C_{(\gamma,\delta)}: x = x(\tau)$ for $\tau\epsilon(\gamma,\delta)$. The subfamily $\{\mathfrak{A}(\tau): \tau\epsilon(\gamma,\delta)\}$ of the original family is a family of tangent lines to this curve. For example, if $<a,\bar{a}'> \neq 0$ over (γ,δ), we may apply theorem 4.3.

Suppose the conditions of theorem 4.3 or theorem 4.4 are satisfied in the subintervals (γ,δ) and (δ,ρ) of (α,β), but not at the point δ. The following examples show that the curve C defined over $(\gamma,\delta)\cup(\delta,\rho)$ to be $C_{(\gamma,\delta)}$ on (γ,δ) and $C_{(\delta,\rho)}$ on (δ,ρ) may or may not be definable at the point δ in such a way as to make it differentiable there.

In these examples, all of the families $\{\mathfrak{A}(\tau)\}$ are defined over the whole real axis, i.e., $(\alpha,\beta) = (-\infty, +\infty)$. It will be easy to see that the families satisfy condition (i) of theorems 4.3 and 4.4 everywhere, and it will be shown that they satisfy condition (ii) of either theorem 4.3 or 4.4 on the intervals $(-\infty,0)$ and $(0,\infty)$. The problem at the point $\tau = 0$, i.e., whether $a'(0) = 0$,

$\bar{a}(0) = 0$, etc., will be pointed out and the curves defined, as in the theorems, for all $\tau \neq 0$. Finally, whether or not the curves thus defined on $(-\infty, 0) \cup (0, +\infty)$ are differentiable at $\tau = 0$ will be examined.

Note that condition (iii) of each of theorems 4.3 and 4.4 is checked implicitly when we define $x = x(\tau)$.

Example 1:

Let $$\alpha(\tau) = \frac{(1, 3\tau^2, 5\tau^4)}{(1+9\tau^4+25\tau^8)^{1/2}} + \frac{\varepsilon(2\tau^7, -4\tau^5, 2\tau^3)}{(1+9\tau^4+25\tau^8)^{1/2}} .$$

Then

$$a'(\tau) = \frac{(-18\tau^3-100\tau^7, 6\tau-15\tau^9, 20\tau^3+90\tau^7)}{(1+9\tau^4+25\tau^8)^{3/2}}$$

and

$$\bar{a}'(\tau) = \frac{(14\tau^6, -20\tau^4, 6\tau^2)}{(1+9\tau^4+25\tau^8)^{1/2}} - \frac{18\tau^3+100\tau^7}{(1+9\tau^4+25\tau^8)^{3/2}} (2\tau^7, -4\tau^5, 2\tau^3).$$

Thus

$$\langle a, \bar{a}'\rangle = -\langle a', \bar{a}\rangle = \frac{-16\tau^6-144\tau^{10}-400\tau^{14}}{(1+9\tau^4+25\tau^8)^2} = \frac{-16\tau^6}{1+9\tau^4+25\tau^8}$$

which is zero if and only if $\tau = 0$. For $\tau = 0$, we have both $a'(0) = 0$ and $\bar{a}(0) = 0$.

Next,

$$[\bar{a}'(\tau),\ \bar{a}(\tau)] = \frac{(-16\tau^7, -16\tau^9, -16\tau^{11})}{1+9\tau^4 + 25\,\tau^8}$$

$$= -16\tau^6(\tau, \tau^3, \tau^5)/(1 + 9\tau^4 + 25\tau^8)\ .$$

For $\tau \neq 0$, we define

$$x(\tau) = [\bar{a}'(\tau),\ \bar{a}(\tau)] \,/ < a(\tau), \bar{a}'(\tau) > = (\tau, \tau^3, \tau^5).$$

By defining $x(0) = 0$, we make the curve differentiable at the point $\tau = 0$ and hence the curve $C: x = x(\tau)$ is differentiable everywhere, with $x'(\tau) \neq 0$ for all τ.

Example 2:

Let $$\mathfrak{a}(\tau) = \frac{(-1, 0, \tau^4)}{(1+\tau^8)^{1/2}} + \frac{\varepsilon\,(\tau^4, -2\tau^2, 1)}{(1+\tau^8)^{1/2}}\ .$$

Then

$$a'(\tau) = (4\tau^7, 0, 4\tau^3)/(1+\tau^8)^{3/2}$$

and

$$\bar{a}'(\tau) = (4\tau^3, -4\tau+4\tau^9, -4\tau^7)/(1+\tau^8)^{3/2}\ .$$

Thus,

$$< a, \bar{a}' > = - < a', \bar{a} > = - \frac{4\tau^{11} + 4\tau^3}{(1+\tau^8)^2} = \frac{-4\tau^3}{1+\tau^8}$$

which is zero if and only if $\tau = 0$. We see that $a'(0) = 0$ and $\bar{a}(0) \neq 0$.

Next,

$$[\bar{a}'(\tau), \bar{a}(\tau)] = -4\tau(1, \tau^2, \tau^4)/(1+\tau^8) .$$

For $\tau \neq 0$, we define

$$x(\tau) = [\bar{a}', \bar{a}] / < a, \bar{a}' > = \left(\frac{1}{\tau^2}, 1, \tau^2 \right) .$$

We see that $\tau = 0$ is not a removable discontinuity of the curve $C : x = x(\tau)$, i. e., no definition of $x(0)$ will make the function $x = x(\tau)$ continuous at the point $\tau = 0$.

<u>Example 3:</u>

Let $$\mathfrak{a}(\tau) = \frac{(-1, \tau^2, 3\tau^4)}{(1+\tau^4+9\tau^8)^{1/2}} + \frac{\varepsilon (2\tau^5, -4\tau^3, 2\tau)}{(1+\tau^4+9\tau^8)^{1/2}} .$$

Then

$$a'(\tau) = \frac{(2\tau^3+36\tau^7, 2\tau-18\tau^9, 12\tau^3+6\tau^7)}{(1+\tau^4+9\tau^8)^{3/2}}$$

and

$$\bar{a}'(\tau) = \frac{(10\tau^4+6\tau^8+18\tau^{12}, -12\tau^2-4\tau^6+36\tau^{10}, 2-2\tau^4-54\tau^8)}{(1+\tau^4+9\tau^8)^{3/2}}.$$

Again,

$$< a, \bar{a}' > = - < a', \bar{a} > = \frac{-16\tau^4}{1+\tau^4+9\tau^8}$$

which is zero if and only if $\tau = 0$. We have both $a'(0) = 0$ and $\bar{a}(0) = 0$.

Now,

$$[\bar{a}', \bar{a}] = -16\tau^3(1, \tau^2, \tau^4)/(1+\tau^4+9\tau^8).$$

For $\tau \neq 0$, we define

$$x(\tau) = [\bar{a}', \bar{a}] / < a, \bar{a}' > = (\frac{1}{\tau}, \tau, \tau^3).$$

The point $\tau = 0$ is again not a removable discontinuity of the function $x = x(\tau)$ and the curve $C: x = x(\tau)$ for $\tau \neq 0$ cannot be made continuous at $\tau = 0$.

<u>Example 4:</u>

Let $\mathfrak{a}(\tau) = (\sin\tau, \cos\tau, 0) + \varepsilon(0, 0, \tau^2)$.

It follows that

$$a'(\tau) = (\cos \tau, \ -\sin \tau, \ 0) \neq 0$$

for all τ, and

$$\bar{a}'(\tau) = (0, 0, 2\tau) = 0$$

if and only if $\tau = 0$. Further,

$$< a', \bar{a} > = 0$$

for all τ, so that a' is perpendicular to $\bar{a}$ for all $\tau \neq 0$.

Let $v_0 = (0, 0, 1)$. Then

$$< v_0, \bar{a} > = \tau^2$$

and

$$< v_0, \bar{a}' > = 2\tau .$$

For $\tau \neq 0$, we define

$$x(\tau) = \frac{< v_0, \bar{a}' > a - < v_0, \bar{a} > a'}{< v_0, [a, a'] >}$$

$$= 2\tau(\sin \tau, \cos \tau, 0) - \tau^2(\cos \tau, \ -\sin \tau, \ 0).$$

By defining $x(0) = 0$ we make the curve $C : x = x(\tau)$ differentiable at the point $\tau = 0$ and hence, C is differentiable everywhere.

Example 5:

If $\mathfrak{a}(\tau) = (\tau^2, \tau^2, 1)/(2\tau^4+1)^{1/2} + \varepsilon(1,-1,0)$

then

$$a'(\tau) = (2\tau, 2\tau, -4\tau^3)/(2\tau^4+1)^{3/2}$$

and

$$\bar{a}'(\tau) = (0,0,0).$$

Note that $\bar{a}(\tau) \neq 0$ for all τ and $a'(\tau) = 0$ if and only if $\tau = 0$. Since $< a', \bar{a} > = 0$ for all τ, a' is perpendicular to $\bar{a}$ for $\tau \neq 0$.

Let $v_0 = (1,-1,0)/\sqrt{2}$. We have

$$< v_0, \bar{a}' > = 0,$$

$$< v_0, \bar{a} > = \sqrt{2},$$

and

$$[a, a'] = 2\tau(-1,1,0)/(2\tau^4+1).$$

We define, for $\tau \neq 0$,

$$x(\tau) = \frac{< v_0, \bar{a}' > a - < v_0, \bar{a} > a'}{< v_0, [a, a'] >} = \frac{(1, 1, -2\tau^2)}{(2\tau^4+1)^{1/2}}.$$

By defining $x(0) = (1, 1, 0)$ we make the curve $C : x = x(\tau)$ differentiable everywhere.

Example 6:

Let $$\mathfrak{a}(\tau) = \frac{(-1, -1, \tau^2 - 1)}{(\tau^4 - 2\tau^2 + 3)^{1/2}} + \frac{\varepsilon\,(\tau^2 + 2\tau - 1,\ -\tau^2 - 2\tau + 1, 0)}{(\tau^4 - 2\tau^2 + 3)^{1/2}}\ .$$

We have

$$a'(\tau) = \frac{(2\tau^3 - 2\tau,\ 2\tau^3 - 2\tau, 4\tau)}{(\tau^4 - 2\tau^2 + 3)^{3/2}}$$

and

$$\bar{a}'(\tau) = \frac{(-2\tau^4 + 4\tau + 6)\ (1, -1, 0)}{(\tau^4 - 2\tau^2 + 3)^{3/2}}\ .$$

Note that $a'(\tau) = 0$ if and only if $\tau = 0$ and $\bar{a}(\tau) = 0$ if and only if $\tau = -1 \pm \sqrt{2}$. For all other τ, $a'(\tau)$ is perpendicular to $\bar{a}(\tau)$.

Let $v_0 = (-1, 1, 0)/\sqrt{2}$. We have

$$< v_0, \bar{a}' > = \frac{4\tau^4 - 8\tau - 12}{\sqrt{2}(\tau^4 - 2\tau^2 + 3)^{3/2}}\ ,$$

$$< v_0, \bar{a} > = \frac{-2\tau^2 - 4\tau + 2}{\sqrt{2}(\tau^4 - 2\tau^2 + 3)^{1/2}}\ ,$$

and

$$[a, a'] = -2\tau(1, -1, 0)/(\tau^4 - 2\tau^2 + 3).$$

For $\tau \neq 0$, $\tau \neq -1 \pm \sqrt{2}$, we define

$$x(\tau) = \frac{\langle v_0, \bar{a}' \rangle a - \langle v_0, \bar{a} \rangle a'}{\langle v_0, [a, a'] \rangle} = \frac{(1 + \tau, 1 + \tau, 1 + \tau^2)}{\tau}.$$

By defining

$$x(-1 + \sqrt{2}) = \frac{(\sqrt{2}, \sqrt{2}, 2 - 2\sqrt{2})}{-1 + \sqrt{2}}$$

and

$$x(-1 - \sqrt{2}) = \frac{(-\sqrt{2}, -\sqrt{2}, 3 + 2\sqrt{2})}{-1 - \sqrt{2}},$$

we make the curve $C: x = x(\tau)$ differentiable at the points $\tau = -1 \pm \sqrt{2}$. The point $\tau = 0$ is not a removable discontinuity and the curve $C: x = x(\tau)$ cannot be made differentiable at the point $\tau = 0$.

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