TRIHEDRAL CURVES

by

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1. INTRODUCTION

In the following we propose to study some space curves related to a given space curve. We shall employ vector analysis, using a right-handed rectangular cartesian coordinate system. Since a vector U will be considered as an ordered triple of numbers, we can, depending on convenience, let U define either a point or a direction. We shall, then, often speak of the point U, or the direction U, thus identifying the point or the direction with its vector representation. Capital letters will be consistently used to distinguish vectors from scalars.

The following notation will be used throughout the paper. When more than one curve occurs in a discussion, subscripts will be used to denote to which curve the symbols refer.

1. The vector U, having components u_1, u_2, u_3, will be written as $U = [u_1, u_2, u_3]$.
2. The scalar product of vectors U and V: $U \cdot V$
3. The vector product of vectors U and V: $|U, V|$
4. The determinant of three vectors U, V, W: $|U, V, W|$
5. Arc length: s
6. Torsion: t
7. Curvature: k (always taken positive)
8. Screw curvature: $m = (k^2 + t^2)^{\frac{1}{2}}$ (always taken positive)
9. Unit tangent: T

10. Unit principal normal: N

11. Unit binormal: B

12. Darboux vector: $D \equiv - tT + kB$

13. Primes will indicate differentiation with respect to arc length.

From the study of twisted curves one has the familiar result that, if T_1 is the unit tangent and s_1 the arc length of the curve $X_1 = X_1\ (s_1)$, then

$$X_1 = \int_0^{s_1} T_1\ ds_1.$$

This suggests two related curves given by

$$X_2 = \int_0^{s_1} N_1\ ds_1 \text{ and } X_3 = \int_0^{s_1} B_1\ ds_1,$$

where N_1 and B_1 are the unit principal normal and unit binormal of the original curve. The fact that all three curves are associated with the fundamental trihedral of the given curve suggests the name trihedral curves. Thus we shall have the T-trihedral, or given, curve, the associated N-trihedral curve, and the associated B-trihedral curve. Some of the geometry of these latter two curves, as well as another related curve to be discussed later, is embodied in this paper. We shall assume that our given curve is a regular analytic space curve (3, p. 18) and, moreover, in the regions under consideration, is free of singular points (3, p. 27).

2.01. DEFINITION. Given the curve $X_1 = X_1(s_1)$, then the curve

$$X_2 = \int_0^{s_1} N_1 \, ds_1$$

will be called the <u>N-trihedral curve</u>, or, more simply, the <u>N-trihedral</u> of the given curve.

2.02. THEOREM. <u>A curve and its N-trihedral have the same arc length</u>.

For

$$s_2 = \int_0^{s_1} \left(\frac{dX_2}{ds_1} \cdot \frac{dX_2}{ds_1} \right)^{\frac{1}{2}} ds_1 = \int_0^{s_1} (N_1 \cdot N_1)^{\frac{1}{2}} \, ds_1 = s_1 .$$

2.03. TABULAR DATA FOR N-TRIHEDRAL CURVE.

(1) $T_2 = N_1$

(2) $B_2 = | X_2', X_2'' | / (X_2'' \cdot X_2'')^{\frac{1}{2}}$

$= | N_1, -k_1 T_1 - t_1 B_1 | / (k_1^2 + t_1^2)^{\frac{1}{2}}$

$= (k_1 B_1 - t_1 T_1) / m_1$

$= D_1 / m_1$

(3) $N_2 = | B_2, T_2 | = | D_1, N_1 | / m_1 = N_1' / m_1$ (2, p. 93 or 1, p. 30)

(4) $D_2 = | N_2, N_2' |$ (2, p. 93)

$= | (-k_1 T_1 - t_1 B_1) / m_1, - m_1 N_1$

$+ (-k_1' T_1 - t_1' B_1) / m_1 + (1 / m_1)' (-k_1 T_1 - t_1 B_1) |$

$= k_1 B_1 - t_1 T_1 + (-k_1' t_1 + t_1' k_1) N_1 / m_1^2$

$= D_1 - (k_1 / m_1)^2 (t_1 / k_1)' N_1$

(5) $k_2 = m_1$

For

$$T_2' = k_2 N_2 = N_1'. \quad \text{Thus } k_2 N_1' / m_1 = N_1', \text{ or } k_2 = m_1.$$

$$\begin{aligned}(6)\ t_2 &= -|X_2', X_2'', X_2'''| / (X_2'' \cdot X_2'') \\ &= -|N_1, -k_1 T_1 - t_1 B_1, -k_1' T_1 - t_1' B_1 - m_1^2 N_1| / m_1^2 \\ &= -(-k_1 t_1' + k_1' t_1) / m_1^2 \\ &= (k_1 / m_1)^2 (t_1 / k_1)'\end{aligned}$$

2.04. THEOREM. <u>The N-trihedral is a plane curve if and only if the given curve is a helix.</u>

For, if the given curve is a helix, t_1 / k_1 is constant (3, p. 52). Then, from 2.03 (6), $t_2 \equiv 0$. Conversely, if $t_2 \equiv 0$. then $(t_1 / k_1)' \equiv 0$, or t_1 / k_1 is constant, and the given curve is a helix.

In particular it follows that <u>if the given curve is a plane curve, then so also is its N-trihedral</u>.

2.05. THEOREM. <u>The N-trihedral is a circle if the given curve is a circular helix.</u>

By 2.04 the N-trihedral is a plane curve. It remains to show that its curvature is constant. But $k_2 = m_1 = (k_1^2 + t_1^2)^{\frac{1}{2}}$, which is constant if k_1 and t_1 are both constant.

2.06. THEOREM. <u>The Darboux vectors of the given curve and its N-trihedral are equal if and only if the given curve is a helix.</u>

By 2.03 (4), $D_2 = D_1 - (k_1 / m_1)^2 (t_1 / k_1)' N_1$. If the given curve is a helix, $(t_1 / k_1)' \equiv 0$, and $D_2 = D_1$. Conversely, if $D_2 = D_1$, then $(t_1 / k_1)' \equiv 0$, and the given curve is a helix.

2.07. THEOREM. <u>The N-trihedral of the N-trihedral is never the</u>

given curve.

For, if $X = \int_0^{s_2} N_2\, ds_2$, then $T = N_2 = N_1' / m_1 = -(k_1 / m_1)\, T_1$ $-(t_1 / m_1)\, B_1$. For T to be equal to T_1 we must have $t_1 \equiv 0$ and $k_1 \equiv -m_1$. The latter is impossible since $k_1 > 0$, $m_1 > 0$.

The next few theorems relate the N-trihedral of a given curve with the involutes of the given curve. It will be convenient to record here the tabular data for involutes (4, p. 352). Notation without subscripts will refer to an involute of the given curve.

2.08. TABULAR DATA FOR INVOLUTES.

(1) $X = X_1 + (c - s_1)\, T_1$

(2) $\epsilon = \pm 1$ according as $c - s_1 \gtrless 0$

(3) $ds = \epsilon k_1 (c - s_1)\, ds_1$

(4) $T = \epsilon N_1$

(5) $N = -\epsilon (k_1 T_1 + t_1 B_1) / m_1 = \epsilon N_1' / m_1$

(6) $B = (k_1 B_1 - t_1 T_1) / m_1 = D_1 / m_1$

(7) $k = \epsilon m_1 / k_1 (c - s_1)$

(8) $t = (k_1 / m_1)^2 (t_1 / k_1)' / k_1 (c - s_1)$

2.09. THEOREM. <u>The ratio of curvature to torsion for the N-trihedral and for an involute of the given curve are numerically equal.</u>

For, by 2.08 and 2.03, we have that

$$k / t = \epsilon m_1 / (k_1 / m_1)^2 (t_1 / k_1)' = \epsilon k_2 / t_2.$$

2.10. THEOREM. <u>The osculating, rectifying, and normal planes of the N-trihedral and of an involute of the given curve are parallel.</u>

2.11. THEOREM. <u>The Darboux vectors of the N-trihedral and of an involute of the given curve are parallel.</u>

For $N = \epsilon N_1' / m_1 = \epsilon N_2$, $dN / ds = \epsilon N_2' (ds_1 / ds)$. Therefore $D = |N, dN / ds| = (ds_1 / ds) |N_2, N_2'| = (ds_1 / ds) D_2$.

2.12. LEMMA. <u>A necessary and sufficient condition for a curve to be a helix is that its Darboux vector have a fixed direction, namely the direction of the axis of the helix.</u>

See (2, p. 106).

2.13. THEOREM. <u>If either the N-trihedral or an involute of the given curve is a helix, then so is the other, and their axes are parallel.</u>

This follows from 2.11 and 2.12.

We now conclude this section of the paper by illustrating some of the foregoing theory with an example.

2.14. EXAMPLE. For our given curve let us choose the helix

$$X_1 = \left[\tfrac{1}{4} \cos 2u, \cos u, \tfrac{1}{2}u - \tfrac{1}{4} \sin 2u\right], \quad 0 < u < \pi .$$

we find

$$ds_1 = \sqrt{2} \sin u \, du$$

$$s_1 = \sqrt{2} (1 - \cos u)$$

$$T_1 = \tfrac{1}{2} \sqrt{2} \left[-\cos u, -1, \sin u\right]$$

$$B_1 = \tfrac{1}{2} \sqrt{2} \left[-\cos u, 1, \sin u\right]$$

$$N_1 = \left[\sin u, 0, \cos u\right]$$

$$D_1 = \tfrac{1}{2}\sqrt{2}\ [0, \csc u, 0]$$

$$k_1 = \tfrac{1}{2} \csc u$$

$$t_1 = \tfrac{1}{2} \csc u$$

$$m_1 = \tfrac{1}{2}\sqrt{2} \csc u$$

We note that the helix makes a constant angle with the direction $[0, 1, 0]$.

The N-trihedral is given by

$$X_2 = \int_0^{s_1} N_1\, ds_1 = \sqrt{2} \int_\pi^u N_1 \sin u\, du,$$

where

$$u = \cos^{-1} \tfrac{1}{2}\sqrt{2}\,(\sqrt{2} - s_1),\ 0 < u < \pi .$$

This turns out to be

$$X_2 = \tfrac{1}{2}\sqrt{2}\ \left[u - \tfrac{1}{2} \sin 2u - \pi,\ 0,\ \sin^2 u\right],$$

whence

$$dX_2 / du = \sqrt{2}\ \left[\sin^2 u,\ 0,\ \sin u \cos u\right],$$

and

$$ds_2 = \sqrt{2} \sin u\, (du / ds_1)\, ds_1 = ds_1 .$$

Thus $s_2 = s_1$, verifying theorem 2.02.

We also have

$$T_2 = [\sin u, 0, \cos u] = N_1$$

$$B_2 = [0, 1, 0] = D_1 / m_1$$

$$N_2 = [\cos u, 0, -\sin u] = N_1' / m_1$$

$$D_2 = [0, 2 \csc u, 0] = D_1$$

$$k_2 = \tfrac{1}{2}\sqrt{2} \csc u = m_1$$

$$t_2 = 0 = (k_1 / m_1)^2\, (t_1 / k_1)'$$

These results verify 2.03, 2.04, and 2.06.

3.01. DEFINITION. Given the curve $X_1 = X_1\ (s_1)$, then the curve

$$X_3 = \int_0^{s_1} B_1\ ds_1$$

will be called the <u>B-trihedral curve</u>, or, more simply, the <u>B-trihedral</u>, of the given curve.

3.02. THEOREM. <u>A curve and its B-trihedral have the same arc length.</u>

For

$$s_3 = \int_0^{s_1} \left(\frac{dX_3}{ds_1}\cdot\frac{dX_3}{ds_1}\right)^{\frac{1}{2}} ds_1 = \int_0^{s_1} (B_1\cdot B_1)^{\frac{1}{2}}\ ds_1 = s_1.$$

3.03. LEMMA. <u>If two curves,</u> $X_i = X_i\ (s_i)$ <u>and</u> $X_j = X_j\ (s_j)$, <u>are so related that</u> $s_i = s_j$ <u>and</u> $N_i = \pm N_j$, <u>then</u> (1) $D_i = D_j$ <u>and</u> (2) $m_i = m_j$.

If primes indicate differentiation with respect to s_i, then $N_i' = \pm N_j'$, whence

$$D_i = |N_i,\ N_i'| = |\pm N_j,\ \pm N_j'| = |N_j,\ N_j'| = D_j$$

and

$$m_i = (D_i\cdot D_i)^{\frac{1}{2}} = (D_j\cdot D_j)^{\frac{1}{2}} = m_j.$$

3.04. THEOREM. <u>If the given curve is a plane curve, the B-trihedral curve is a straight line parallel to</u> B_1.

For, if the given curve is a plane curve then B_1 is a constant vector, and we have

$$X_3 = \int_0^{s_1} B_1\ ds_1 = B_1 \int_0^{s_1} ds_1 = s_1\ B_1.$$

3.05. TABULAR DATA FOR B-TRIHEDRAL CURVE.

(1) $T_3 = B_1$.

(2) $\epsilon = \pm 1$ according as $t_1 \gtrless 0$.

(3) $B_3 = |X_3', X_3''| / (X_3'' \cdot X_3'')^{\frac{1}{2}}$

$= |B_1, t_1 N_1| / \epsilon t_1 = - \epsilon T_1$.

(4) $N_3 = |B_3, T_3| = \epsilon N_1$.

(5) $D_3 = D_1$. (by lemma 3.03)

(6) $k_3 = (T_3' \cdot T_3')^{\frac{1}{2}} = (B_1' \cdot B_1')^{\frac{1}{2}} = t_1$.

(7) $t_3 = - |X_3', X_3'', X_3'''| / (X_3'' \cdot X_3'')$

$= - |B_1, t_1 N_1, t_1' N_1 - t_1 k_1 T_1 - t_1^2 B_1| / t_1^2$

$= - k_1$.

(8) $m_3 = m_1$. (by lemma 3.03)

3.06. THEOREM. *The tangent, principal normal, and binormal of the B-trihedral are parallel, respectively, to the binormal, principal normal, and tangent of the given curve.*

3.07. THEOREM. *A curve and its B-trihedral have equal Darboux vectors.*

3.08. THEOREM. *If the given curve has constant curvature the B-trihedral has constant torsion; if the given curve has constant torsion, the B-trihedral has constant curvature.*

3.09. THEOREM. *The torsion of the B-trihedral is always negative.*

3.10. THEOREM. *If the given curve is a helix so also is the*

B-trihedral; if the given curve is a circular helix so also is the B-trihedral. In either case the two helices have parallel axes.

For $k_3 / t_3 = -\epsilon t_1 / k_1$. That the axes are parallel follows from theorem 3.07 and lemma 2.12.

3.11. THEOREM. *A curve and its B-trihedral have equal screw curvatures.*

3.12. THEOREM. *The B-trihedral of the B-trihedral is the given curve or its reflection in the origin according as the torsion of the given curve is negative or positive.*

For $Y = \int_0^{s_3} B_3 \, ds_3 = -\epsilon \int_0^{s_1} T_1 \, ds_1 = -\epsilon X_1$.

The final theorem of this section and several of those of the next section refer to Bertrand curves. For convenience of reference we here give a definition and a criterion for these curves.

3.13. DEFINITION. A *Bertrand curve* is a curve whose principal normals are the principal normals of another curve.

3.14. CRITERION. *A necessary and sufficient condition for a curve to be a twisted Bertrand curve is that there exist constants* $r \neq 0$, $w \neq n\pi$, $(n = 0, \pm 1, \pm 2, \ldots)$ *such that*

$$r\,k \sin w + r\,t \cos w = \sin w,$$

r being the constant distance between corresponding points of a Bertrand curve and its mate, and w being the constant angle between corresponding osculating planes of a Bertrand curve and its mate. (cf., 3, art. 23.)

3.15. THEOREM. <u>If a given twisted curve is a Bertrand curve whose osculating planes are not perpendicular to the corresponding osculating planes of its mate</u> ($w \neq (2n + 1)\,\pi/2$, $n = 0, \pm 1, \pm 2, \ldots$), <u>then the B-trihedral is also a twisted Bertrand curve</u>.

By 3.14 there exist constants r and w such that

$$rk_1 \sin w + rt_1 \cos w = \sin w,\quad r \neq 0,\quad w \neq n\pi .$$

By 3.05, (6) and (7), $k_1 = -t_3$, $t_1 = \epsilon k_3$. Therefore

$$-rt_3 \sin w + r\,\epsilon k_3 \cos w = \sin w.$$

Let

$$\Omega = -\epsilon(\pi/2 - w),\quad q = -r \tan\Omega .$$

Then

$$qk_3 \sin\Omega + qt_3 \cos\Omega = \sin\Omega,\quad q \neq 0,\quad \Omega \neq n\pi ,$$

and, again by 3.14, the B-trihedral is also a twisted Bertrand curve.

As in section 2, we shall conclude this section by illustrating some of the foregoing theory with an example.

3.16. EXAMPLE. For our given curve we choose the same curve used in 2.14. The B-trihedral is found to be

$$X_3 = \int_0^{s_1} B_1\, ds_1 = \sqrt{2} \int_\pi^u B_1 \sin u\, du,$$

where

$$u = \cos^{-1} \tfrac{1}{2} \sqrt{2}\ (\sqrt{2} - s_1),\quad 0 < u < \pi .$$

This turns out to be

$$X_3 = \left[\tfrac{1}{4}(\cos 2u - 1),\ -(1 + \cos u),\ \tfrac{1}{2}(u - \tfrac{1}{2} \sin 2u - \pi)\right],$$

whence

$$dX_3 / du = \left[-\sin u \cos u,\ \sin u,\ \sin^2 u\right],$$

and

$$ds_3 = \sqrt{2} \sin u \, (du / ds_1) \, ds_1 = ds_1.$$

Thus $s_3 = s_1$, verifying theorem 3.02.

We also have

$$T_3 = \tfrac{1}{2} \sqrt{2} [-\cos u, 1, \sin u] = B_1$$

$$B_3 = \tfrac{1}{2} \sqrt{2} [\cos u, 1, -\sin u] = -T_1$$

$$N_3 = [\sin u, 0, \cos u] = N_1$$

$$D_3 = \tfrac{1}{2} \sqrt{2} [0, \csc u, 0] = D_1$$

$$k_3 = \tfrac{1}{2} \csc u = t_1$$

$$t_3 = -\tfrac{1}{2} \csc u = -k_1$$

$$m_3 = \tfrac{1}{2} \sqrt{2} \csc u = m_1$$

These results verify theorems 3.06 through 3.11.

4.01. DEFINITION. The curve

$$X_{13} = X_1 \cos\Theta + X_3 \sin\Theta,$$

where Θ is any constant angle, will be called a <u>B-curve</u> of the given curve. If $\Theta = 0$ the B-curve is the given curve, and if $\Theta = \pi/2$ the B-curve is the B-trihedral of the given curve.

4.02. THEOREM. <u>A curve and its B-curves have the same arc length.</u>

For

$$\begin{aligned} s_{13} &= \int_0^{s_1}\left(\frac{dX_{13}}{ds_1}\cdot\frac{dX_{13}}{ds_1}\right)^{\frac{1}{2}} ds_1 \\ &= \int_0^{s_1}\left[(T_1\cos\Theta + B_1\sin\Theta)\cdot(T_1\cos\Theta + B_1\sin\Theta)\right]^{\frac{1}{2}} ds_1 \\ &= \int_0^{s_1}\left[(T_1\cdot T_1)\cos^2\Theta + 2(T_1\cdot B_1)\sin\Theta\cos\Theta + (B_1\cdot B_1)\right. \\ &\qquad \left.\sin^2\Theta\right]^{\frac{1}{2}} ds_1 \\ &= s_1. \end{aligned}$$

4.03. TABULAR DATA FOR B-CURVES.

(1) $T_{13} = T_1\cos\Theta + B_1\sin\Theta$

(2) $\epsilon = \pm 1$ according as $k_1\cos\Theta + t_1\sin\Theta \gtrless 0$

(3) $$\begin{aligned} B_{13} &= |X'_{13}, X''_{13}| \,/\, (X''_{13}\cdot X''_{13})^{\frac{1}{2}} \\ &= \epsilon|T_1\cos\Theta + B_1\sin\Theta, N_1(k_1\cos\Theta \\ &\qquad + t_1\sin\Theta)| \,/\, (k_1\cos\Theta + t_1\sin\Theta) \\ &= \epsilon|T_1\cos\Theta + B_1\sin\Theta, N_1| \\ &= \epsilon(B_1\cos\Theta - T_1\sin\Theta) \end{aligned}$$

(4) $N_{13} = |B_{13}, T_{13}| = \epsilon N_1$

(5) $D_{13} = D_1$ (by lemma 3.03)

(6) $k_{13} = (T'_{13}\cdot T'_{13})^{\frac{1}{2}} = \epsilon(k_1\cos\Theta + t_1\sin\Theta)$

(7) $t_{13} = t_1 \cos\theta - k_1 \sin\theta$

For $t_{13} N_{13} = B'_{13} = \epsilon(t_1 N_1 \cos\theta - k_1 N_1 \sin\theta)$

$= N_{13} (t_1 \cos\theta - k_1 \sin\theta)$

(8) $m_{13} = m_1$ (by lemma 3.03)

The following six theorems are immediate consequences of the tabular data for B-curves.

4.04. THEOREM. _The principal normal of any B-curve is parallel to the principal normal of the given curve._

4.05. THEOREM. _The rectifying plane of any B-curve is parallel to the rectifying plane of the given curve._

4.06. THEOREM. _The Darboux vector of any B-curve is parallel to the Darboux vector of the given curve._

4.07. THEOREM. _If the given curve is a circular helix, so also is every B-curve a circular helix._

4.08. THEOREM. _If the given curve is a helix, so also is every B-curve a helix. The axes of the given curve and its B-curves are parallel._

This follows from 2.12 and (5) of 4.03.

4.09. THEOREM. _For any B-curve we have_

(1) $k_{13} \sin\theta + \epsilon t_{13} \cos\theta = \epsilon t_1$,

(2) $k_{13} \cos\theta - \epsilon t_{13} \sin\theta = \epsilon k_1$.

4.10. THEOREM. <u>If some B-curve is a circular helix, then every B-curve, including the given curve, is a circular helix.</u>

This follows from 4.07 and 4.09.

4.11. THEOREM. <u>Every twisted Bertrand curve is a B-curve of some curve of constant curvature.</u>

Let C be a given twisted Bertrand curve of curvature k and torsion t. By 3.14 there exist constants $r \neq 0$, $w \neq n\pi$, $(n = 0, \pm 1, \pm 2, \ldots)$ such that

$$k \sin w + t \cos w = (1 / r) \sin w. \tag{1}$$

Let $\epsilon = \pm 1$ according as $\sin w \gtrless 0$, and choose θ and k_1 such that

$$w = \pi/2 + \epsilon\theta, \quad (1 / r) \sin w = \epsilon k_1.$$

Then k_1 is a positive constant and (1) becomes

$$k \cos \theta - \epsilon t \sin \theta = \epsilon k_1.$$

Finally, choose t_1 such that

$$k \sin \theta + \epsilon t \cos \theta = \epsilon t_1.$$

By the fundamental theorem of space curves (3, p. 46) there exists a curve C_1 having k_1 and t_1 for curvature and torsion, and by 4.08 C is a B-curve of curve C_1. Since C_1 has constant curvature, the theorem is established.

4.12. THEOREM. <u>Every twisted Bertrand curve is a B-curve of some curve of constant torsion.</u>

Let C be a given twisted Bertrand curve of curvature k and torsion t. By 3.14 there exist constants $r \neq 0$, $w \neq n\pi$, $(n = 0, \pm 1, \pm 2, \ldots)$ such that

$$k \sin w + t \cos w = (1 / r) \sin w. \qquad (1)$$

Let

$$\epsilon = \pm 1 \text{ according as } k \cos w - t \sin w \gtrless 0,$$

and choose θ and t_1 such that

$$w = \epsilon \theta, \qquad (1 / r) \sin w = t_1.$$

Then t_1 is a constant and (1) becomes

$$\epsilon k \sin \theta + t \cos \theta = t_1,$$

or

$$k \sin \quad + \epsilon t \cos \quad = \epsilon t_1.$$

Finally, choose k_1 such that

$$k \cos \theta - \epsilon t \sin \theta = \epsilon k_1.$$

Observe that

$$k_1 = \epsilon k \cos \theta - t \sin \theta = \epsilon(k \cos w - t \sin w) > 0.$$

By the fundamental theorem of space curves (3, p. 46) there exists a curve C_1 having k_1 and t_1 for curvature and torsion, and by 4.08 C is a B-curve of curve C_1. Since C_1 has constant torsion, the theorem is established.

4.13. THEOREM. *Every B-curve of a curve of constant non-zero curvature, except those for which* $= (2n + 1) \pi / 2$, $(n = 0, \pm 1, \pm 2, ..)$ *is a twisted Bertrand curve.*

For we have, by (2) of 4.09,

$$k_{13} \cos \theta - \epsilon t_{13} \sin \theta = \epsilon k_1. \qquad (1)$$

Suppose k_1 is constant and choose w and r such that

$$\theta = \epsilon (w - \pi / 2), \qquad (1 / r) \sin w = \epsilon k_1.$$

Then (1) becomes

$$k_{13} \sin w + t_{13} \cos w = (1 / r) \sin w.$$

clearly $r \neq 0$ and $w \neq n\pi$, $(n = 0, \pm 1, \pm 2, \ldots)$. Therefore, by 3.14, the B-curve is a twisted Bertrand curve.

4.14. THEOREM. <u>Every B-curve of a curve of constant non-zero torsion, except those for which $\Theta = n\pi$, $(n = 0, \pm 1, \pm 2, \ldots)$ is a twisted Bertrand curve.</u>

For we have, by (1) of 4.09,

$$k_{13} \sin \Theta + \epsilon t_{13} \cos \Theta = \epsilon t_1. \tag{1}$$

Suppose t_1 is constant and choose w and r such that

$$\Theta = \epsilon w, \quad (1 / r) \sin w = t_1.$$

Then (1) becomes

$$k_{13} = \sin w + t_{13} \cos w = (1 / r) \sin w.$$

clearly $r \neq 0$ and $w \neq n\pi$, $(n = 0, \pm 1, \pm 2, \ldots)$. Therefore, by 3.14, the B-curve is a twisted Bertrand curve.

4.15. THEOREM. <u>The normal planes of the given curve and its B-curve, corresponding to the angle Θ, intersect in the constant angle Θ.</u>

For

$$T_{13} \cdot T_1 = (T_1 \cos \Theta + B_1 \sin \Theta).\ T_1 = \cos \Theta .$$

4.16. THEOREM. <u>The osculating planes of the given curve and its B-curve, corresponding to the angle Θ, intersect in the constant angle Θ.</u>

For

$$B_{13} \cdot B_1 = \epsilon (B_1 \cos\Theta - T_1 \sin\Theta). \quad B_1 = \epsilon \cos\Theta.$$

4.17. THEOREM. The general equation of all curves of constant non-zero curvature c is

$$X = (1/c) \int V \left(\frac{dV}{dr} \cdot \frac{dV}{dr}\right)^{\frac{1}{2}} dr, \tag{1}$$

where V is an arbitrary unit vector function of the paremeter r.

Let us find the curvature of curve (1). We have

$$\frac{dX}{dr} = (1/c) \; V \left(\frac{dV}{dr} \cdot \frac{dV}{dr}\right)^{\frac{1}{2}},$$

whence

$$\frac{ds}{dr} = \left(\frac{dX}{dr} \cdot \frac{dX}{dr}\right)^{\frac{1}{2}} = (1/c) \left(\frac{dV}{dr} \cdot \frac{dV}{dr}\right)^{\frac{1}{2}}.$$

Therefore

$$T = V \quad \text{and} \quad kN = T' = \frac{dV}{dr} \left(\frac{dV}{dr} \cdot \frac{dV}{dr}\right)^{-\frac{1}{2}} c.$$

Thus

$$N = \frac{dV}{dr} \left(\frac{dV}{dr} \cdot \frac{dV}{dr}\right)^{-\frac{1}{2}} \quad \text{and} \quad k = c.$$

On the other hand (1) includes all curves having the constant non-zero curvature c. For suppose $k = c$. Let $T = V$, a unit vector function of a parameter r. Then

$$cN = T' = \frac{dV}{dr} \frac{dr}{ds} = \frac{dV}{dr} \left(\frac{dV}{dr} \cdot \frac{dV}{dr}\right)^{-\frac{1}{2}} \left(\frac{dV}{dr} \cdot \frac{dV}{dr}\right)^{\frac{1}{2}} \frac{dr}{ds},$$

whence

$$\frac{ds}{dr} = (1/c) \left(\frac{dV}{dr} \cdot \frac{dV}{dr}\right)^{\frac{1}{2}},$$

and

$$\frac{dX}{dr} = T \frac{ds}{dr} = (1/c) \; V \left(\frac{dV}{dr} \cdot \frac{dV}{dr}\right)^{\frac{1}{2}}.$$

4.18. THEOREM. <u>The general equation of all twisted Bertrand curves which are not circular helices is</u>

$$X = (1 / c) \cos\theta \int V \left(\frac{dV}{dr}\cdot\frac{dV}{dr}\right)^{\frac{1}{2}} dr$$
$$+ (1 / c) \sin\theta \int |V, dV / dr| \; dr, \qquad (1)$$

<u>where</u> V <u>is an arbitrary unit vector function of</u> r, <u>and</u> $c > 0$, $\theta \neq (2n + 1)\, \pi/2$, $(n = 0, \pm 1, \pm 2, \ldots)$ <u>are arbitrary constants</u>

By 4.11 every twisted Bertrand curve is a B-curve of some curve of constant curvature. With this in mind let us find the B-curves of (1), 4.17. By 4.13 these B-curves will all be twisted Bertrand curves if $\theta \neq (2n + 1)\, \pi/2$, $(n = 0, \pm 1, \pm 2, \ldots)$.

We have

$$X' = V,$$
$$X_1'' = c \frac{dV}{dr} \left(\frac{dV}{dr}\cdot\frac{dV}{dr}\right)^{-\frac{1}{2}},$$
$$B_1 = |X_1', X_1''| / (X_1''\cdot X_1'')^{\frac{1}{2}}$$
$$= (1 / c) \left| V, c \frac{dV}{dr} \left(\frac{dV}{dr}\cdot\frac{dV}{dr}\right)^{-\frac{1}{2}} \right|$$
$$= |V, dV / dr| \left(\frac{dV}{dr}\cdot\frac{dV}{dr}\right)^{-\frac{1}{2}},$$
$$X_3 = \int B_1 \, (ds_1 / dr)\, dr$$
$$= \int \left[|V, dV / dr| \left(\frac{dV}{dr}\cdot\frac{dV}{dr}\right)^{-\frac{1}{2}} \right] \left[(1 / c) \left(\frac{dV}{dr}\cdot\frac{dV}{dr}\right)^{\frac{1}{2}} \right] dr$$
$$= (1 / c) \int |V, dV / dr| \; dr. \qquad (2)$$

Curve (2), being the B-trihedral of a curve of constant curvature C, is a curve of constant torsion $-c$. We now obtain (1) as the general equation of the sought B-curves. When $\theta = (2n + 1)\, \pi/2$, $(n = 0, \pm 1, \pm 2, \ldots)$, the B-curve reduces to a curve of constant non-zero torsion. It will thus be a Bertrand curve only if it also has constant curvature, as is seen by referring to 3.14. But in this case it is a circular helix.

4.19. THE SIGNIFICANCE OF c AND θ IN (1) OF 4.18.

By 4.09, the linear relation between the curvature and torsion of (1) of 4.18 is

$$k \cos \theta - \epsilon t \sin \theta = \epsilon c \tag{1}$$

or

$$k - \epsilon t \tan \theta = \epsilon c / \cos \theta . \tag{2}$$

letting d be the constant distance between a Bertrand curve and its mate, and letting w be the constant angle between the osculating planes of the Bertrand curve and its mate, we have (3, p. 54)

$$d = (1 / c) \cos \theta , \tag{3}$$

$$\cot w = - \epsilon \tan \theta . \tag{4}$$

Notice that if we take $\theta = 0$, then, from (1), k is constant, and the Bertrand curve reduces to one having constant curvature. From (3) and (4) we then have $d = 1 / c$, $w = \pi / 2$.

4.20. THEOREM. <u>A twisted Bertrand curve is either a circular helix or is linearly dependent on two curves, one of constant curvature c and one of constant torsion -c, the second curve being the B-trihedral of the first. The Bertrand curve is a B-curve of the first.</u>

For, from (1) of 4.18, a twisted Bertrand curve, not a circular helix, is linearly dependent on the two curves

$$(1 / c) \int V \left(\frac{dV}{dr} \cdot \frac{dV}{dr} \right)^{\frac{1}{2}} dr,$$

a curve of constant curvature c, and

$$(1 / c) \int | V, dV / dr | \, dr,$$

a curve of constant torsion -c. In the proof of 4.18 it was shown that the second curve is the B-trihedral of the first, and that the Bertrand curve is a B-curve of the first.

4.21. THEOREM. The general equation of all curves of constant non-zero torsion c is

$$X = -(1 / c) \int | V, dV / dr | \, dr, \tag{1}$$

where V is an arbitrary unit vector function of the parameter r, and c is an arbitrary non-zero constant.

If $c < 0$ the curve X, being the B-trihedral of a curve of constant curvature -c, is a curve of constant torsion c. If $c > 0$, then -X is a curve of constant torsion. But the torsion of -X and X are the same except for sign since torsion, t, is given by

$$t = - | X', X'', X''' | / (X'' \cdot X''),$$

and replacing X by -X results only in an odd number of sign changes.

On the other hand (1) includes all curves having the constant non-zero torsion c. For suppose $t = c$. Set $B = V$, a unit vector function of r. Then

$$cN = B' = \frac{dV}{dr} \frac{dr}{ds} = \frac{dV}{dr} \left(\frac{dV}{dr} \cdot \frac{dV}{dr} \right)^{-\frac{1}{2}} \left(\frac{dV}{dr} \cdot \frac{dV}{dr} \right)^{\frac{1}{2}} \frac{dr}{ds},$$

whence

$$N = \pm \frac{dV}{dr} \left(\frac{dV}{dr} \cdot \frac{dV}{dr} \right)^{-\frac{1}{2}} \text{ and } \frac{ds}{dr} = \pm (1 / c) \left(\frac{dV}{dr} \cdot \frac{dV}{dr} \right)^{\frac{1}{2}}.$$

Therefore

$$T = \left| N, B \right| = \left| \pm\frac{dV}{dr} \left(\frac{dV}{dr}\cdot\frac{dV}{dr} \right)^{-\frac{1}{2}}, V \right|$$

$$= \mp \left| V, dV / dr \right| \left(\frac{dV}{dr}\cdot\frac{dV}{dr} \right)^{-\frac{1}{2}},$$

and

$$X = \int T \, (ds / dr) \, dr$$

$$= \int \left[\mp \left| V, dV / dr \right| \left(\frac{dV}{dr}\cdot\frac{dV}{dr} \right)^{-\frac{1}{2}} \right] \left[\pm (1 / c) \left(\frac{dV}{dr}\cdot\frac{dV}{dr} \right)^{\frac{1}{2}} \right] dr$$

$$= -(1 / c) \int \left| V, dV / dr \right| \, dr.$$

The following examples illustrate some of the theory of this section. We will use the unit vector function.

$$V = \tfrac{1}{2}\sqrt{2} \left[\cos r, \sin r, 1 \right].$$

4.22. EXAMPLE OF A CURVE OF CONSTANT CURVATURE.

Let

$$c = 2$$

then

$$X = 2 \int V \left(\frac{dV}{dr}\cdot\frac{dV}{dr} \right)^{\frac{1}{2}} dr$$

$$V = \tfrac{1}{2}\sqrt{2} \left[\cos r, \sin r, 1 \right]$$

$$\frac{dV}{dr} = \tfrac{1}{2}\sqrt{2} \left[-\sin r, \cos r, 0 \right]$$

$$\left(\frac{dV}{dr}\cdot\frac{dV}{dr} \right)^{\frac{1}{2}} = \tfrac{1}{2}\sqrt{2}$$

$$X = 2 \left(\tfrac{1}{2}\sqrt{2} \right) \left(\tfrac{1}{2}\sqrt{2} \right) \int \left[\cos r, \sin r, 1 \right] dr$$

$$= \left[\sin r, -\cos r, r \right].$$

To verify that the curvature of X is constant and equal to $1 / c = \frac{1}{2}$,

$$ds = \left(\frac{dX}{dr}\cdot\frac{dX}{dr}\right)^{\frac{1}{2}} dr = \tfrac{1}{2}\ \sqrt{2}\ dr$$

$$T = \frac{dX}{dr}\frac{dr}{ds} = \tfrac{1}{2}\ \sqrt{2}\ \left[\cos r, \sin r, 1\right]$$

$$kN = T' = \tfrac{1}{2}\ \left[-\sin r, \cos r, 0\right]$$

thus

$$k = \tfrac{1}{2} = 1 / c.$$

4.23. EXAMPLE OF A CURVE OF CONSTANT TORSION.

Let

$$1 / c = -2$$

$$X = 2 \int \left| V, \frac{dV}{dr} \right| dr$$

$$\left| V, \frac{dV}{dr} \right| = \tfrac{1}{2}\ \left[-\cos r, -\sin r, 1\right]$$

$$X = 2\cdot\tfrac{1}{2} \int \left[-\cos r, -\sin r, 1\right]\ dr$$

$$= \left[-\sin r, \cos r, r\right].$$

To verify that the torsion is constant and equal to $c = -\frac{1}{2}$,

$$X' = \tfrac{1}{2}\ \sqrt{2}\ \left[-\cos r, -\sin r, 1\right]$$

$$X'' = \left[\sin r, -\cos r, 0\right]$$

$$X''' = \tfrac{1}{2}\ \sqrt{2}\ \left[\cos r, \sin r, 0\right]$$

$$\left|X'', X'''\right| = \tfrac{1}{2}\ \sqrt{2}\ \left[0, 0, 1\right]$$

$$t = -\left|X', X'', X'''\right| / (X''\cdot X'')$$

$$= -\tfrac{1}{2}.$$

4.24. EXAMPLE OF A FAMILY OF BERTRAND CURVES.

Let X_1 be the curve of constant curvature of 4.22 and X_3 be the curve of constant torsion of 4.23. Then

$$X_{13} = X_1 \cos\theta + X_3 \sin\theta, \quad 0 < \theta < \pi/4$$

$$= [\sin r, -\cos r, r] \cos\theta + [-\sin r, \cos r, r] \sin\theta$$

$$= [\sin r (\cos\theta - \sin\theta), -\cos r (\cos\theta - \sin\theta),$$

$$r(\cos\theta + \sin\theta)]$$

$$\frac{dX_{13}}{dr} = [\cos r (\cos\theta - \sin\theta), \sin r (\cos\theta - \sin\theta),$$

$$(\cos\theta + \sin\theta)]$$

$$\frac{dX_{13}}{dr} \cdot \frac{dX_{13}}{dr} = (\cos\theta - \sin\theta)^2 + (\cos\theta + \sin\theta)^2 = 2$$

$$ds_{13} = \left(\frac{dX_{13}}{dr} \cdot \frac{dX_{13}}{dr}\right)^{\frac{1}{2}} dr = \sqrt{2}\; dr$$

$$T_{13} = \tfrac{1}{2}\sqrt{2}\, [\cos r (\cos\theta - \sin\theta),$$

$$\sin r (\cos\theta - \sin\theta), (\cos\theta + \sin\theta)]$$

$$k_{13} N_{13} = T'_{13} = \tfrac{1}{2} [-\sin r (\cos\theta - \sin\theta),$$

$$\cos r (\cos\theta - \sin\theta), 0]$$

$$= \tfrac{1}{2}(\cos\theta - \sin\theta) \; [-\sin r, \cos r, 0]$$

thus

$k_{13} = \frac{1}{2}(\cos\theta - \sin\theta)$, a constant, so X_{13} is a Bertrand curve.

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