A TWO-DIMENSIONAL ANALOG OF THE DOUGALL METHOD

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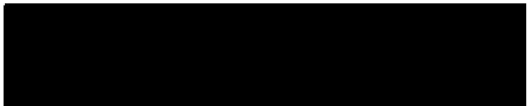the requirements for the
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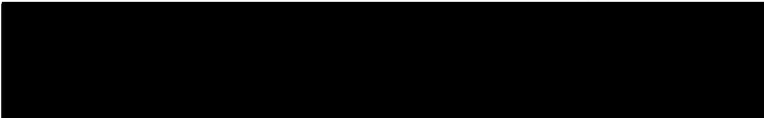Assistant Professor of Mathematics

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1. INTRODUCTION

The Green's function forms the basis for solutions
of a large number of equations in mathematics and physics.
Potential theory, heat conduction, elasticity, neutron
diffusion, and many other physical and mathematical con-
cepts employ the Green's function for solutions of prob-
lems. Physically, the Green's function represents the
reaction at a point P due to some kind of unit source at
another point, P'. In potential theory the source corre-
sponds to a charge and the reaction is the potential; in
heat conduction the source is a quantity of heat liberated
and the reaction is the temperature; in elasticity the
source is a force and the reaction a displacement; and so
on. The usual case however does not involve point sources,
but continuous sources. The total contribution to the
solution is obtained by integration over the source
strength values. As we shall see in Chapter 2, the mathe-
matical statement of the Green's function and the solution
of the Poisson equation makes this interpretation possible,
and with it, many solutions of problems can be written
down from a consideration of this meaning, provided the
Green's function is known.

This paper is devoted to the construction of the Green's function for the Laplacian, $\nabla^2 T$, using a method employed by John Dougall for the cylindrical and spherical coordinate systems (6, pp. 33-83). The analytical representations, many of which appear as bilinear forms, are easily adapted to calculations, and the method of construction is very direct once the singular form of the Green's function is represented. In his representations, the singular or fundamental form appears as an infinite or contour integral. To this form is added a solution of the Laplace equation which has arbitrary constants. The form of this solution is also an integral of similar structure to afford maximum opportunities for simplifications. In the usual way, the arbitrary parameters are used to satisfy the boundary conditions, and the resulting integrals are then evaluated by contour integration. The result is an expansion formula for the Green's function.

In view of the direct approach and easily manipulated results obtainable by this method, it seems desirable to extend this approach to other coordinate systems. The purpose of this thesis is to present an analog of the Dougall method in two dimensions, starting with the two dimensional logarithmic singularity. At the beginning, one may expect results similar to those obtained by

Dougall. Such is the case; however, the expansion
formula derived in this paper turns out to be a series
expansion for the classical solution for a circle.
Slight manipulations give equivalent forms in terms of
infinite integrals which are evaluated by both contour
integration and comparison with the canonical forms.

2. THE GREEN'S FUNCTION

Associated with the Laplacian, $\nabla^2 T$, and a region G is the Green's function K(P,P'), normally defined by the equation

$$(2.1) \qquad K(P, P') = \frac{1}{4\pi D(P, P')} + \beta(P, P')$$

where K(P,P') satisfies the Laplace equation

$$(2.2) \qquad \nabla^2 K = 0$$

everywhere in G except possibly at P = P', and D is the distance from P with coordinates (x,y,z) to P' with coordinates (x',y',z'). 1/D is called the fundamental solution, and β is called the complementary solution, since both 1/D and β satisfy equation (2.2). β, however, has no singular term and, with its first and second derivatives, is continuous in the region G. Thus, the Green's function is characterized by a singularity at P = P' since D = 0 at that point. Also associated with the Green's function is a homogeneous boundary condition, usually K = 0 or some linear combination of K and its normal derivative to the surface of G. This relationship between T and K can best be expressed by the following propositions (4, pp. 351-388 and 11, pp. 224-231):

PROPOSITION 1. If T is any function which satisfies a homogeneous boundary condition, say T = 0, is continuous, has continuous first and piecewise continuous second

derivatives in G, and satisfies the equation

(2.3) $$\nabla^2 T(P) = -\psi(P)$$

then T(P) is expressed by

(2.4) $$T(P) = \iiint\limits_{G} K(P,P')\,\psi(P')\,dV'$$

On the other hand, if ψ(P) is any function, which with its first derivatives is continuous in G, then the function expressed by

$$T(P) = \iiint\limits_{G} K(P,P')\,\psi(P')\,dV'$$

is continuous, has continuous first and second derivatives, and satisfies both the boundary condition and the differential equation

$$\nabla^2 T(P) = -\psi(P)$$

PROPOSITION 2. If T is a continuous function with continuous first derivatives, and satisfies the equation

(2.5) $$\nabla^2 T(P) = 0$$

then T(P) is expressed by

(2.6) $$T(P) = \iint\limits_{S} \left(K\frac{\partial T}{\partial n'} - T\frac{\partial K}{\partial n'}\right) dS'$$

for $K \neq 0$ on S and

(2.7) $$T(P) = -\iint\limits_{S} T\frac{\partial K}{\partial n'}\,dS'$$

for K = 0 on S, where S is the surface of G and the differentiation is taken along the external normal to S.

The restrictions on ψ in Proposition 1 can be relaxed somewhat to include a piecewise continuous ψ, and the proposition holds except at points of discontinuity of ψ where the second derivative is discontinuous (11, p. 64).

The existence proof for Proposition 2 is known as the first boundary value problem of potential theory or the Dirichlet problem. If the solution exists, it is given by equation (2.7), but no assurance is given from the statement or proof of the proposition that T(P) actually assumes the values assigned to it on the boundary when P approaches the boundary. The existence theorems for a large class of problems are known, and we make use of these results in expressing the solution for the interior of a circle in Chapter 4 (11, pp. 291-305).

Also, the physical interpretation becomes clear. From equation (2.4), we see that the total contribution to T is obtained by summation of $K\psi$ over the region in which ψ is defined. Thus, $K\psi$ must represent the reaction at P due to a source of strength ψ at P', or the total could not have been obtained by summation over G. When $\psi = 1$, K represents the reaction at P for a point source of unit strength at P'. Hence the physical interpretation usually accorded the Green's function.

PROOF OF PROPOSITIONS 1 AND 2. The first part of Proposition 1 follows immediately from an application of Green's formula,

$$(2.8) \quad \iiint_G (u \nabla^2 v - v \nabla^2 u) dv' = \iint_S \left(u \frac{\partial v}{\partial n'} - v \frac{\partial u}{\partial n'} \right) ds'$$

to the region G with u = K and v = T, accounting for the singular nature of K at P = P'. Thus,

$$(2.9) \quad \iint_S \left(k \frac{\partial T}{\partial n'} - T \frac{\partial k}{\partial n'} \right) ds' - \lim_{\epsilon \to 0} \iint_\alpha \left(T \frac{\partial k}{\partial n'} - k \frac{\partial T}{\partial n'} \right) ds' = - \iiint_G K(P,P') \psi(P') dv'$$

where α is a small sphere of radius ϵ enclosing the point P, and the differentiation is taken along the exterior normal. The integral over the surface of G vanishes since K and T both satisfy the same homogeneous boundary condition of the form

$$T = 0, \quad \frac{\partial T}{\partial n} = 0, \quad \text{or} \quad \frac{\partial T}{\partial n} + h T = 0$$

where h is a parameter. The use of spherical coordinates with P as the origin shows immediately that

$$(2.10) \quad \lim_{\epsilon \to 0} \iint_\alpha \left(T \frac{\partial k}{\partial n'} - k \frac{\partial T}{\partial n'} \right) ds' = T(P)$$

and

$$T(P) = \iiint_G K(P,P') \psi(P') dv'$$

This procedure of surrounding the point P by a sphere is convenient, although any region surrounding P produces the same results provided the volume of the region goes to zero through a sequence such that the maximum chord goes to zero (9, pp. 147-148).

The proof for the second part of Proposition 1 is made by noting that differentiation under the integral signs is permissible. Since β is a regular harmonic solution, with continuous first and second derivatives, differentiation under the integral sign is permissible for the derivatives of β (12, p. 59). The only term to be considered in detail is

$$(2.11) \qquad X = \frac{1}{4\pi} \iiint_G \frac{\psi(P')}{D(P,P')} \, dv'$$

and its derivatives. Consider first the following integrals written in spherical coordinates with P as the origin,

$$X = \frac{1}{4\pi} \iiint_G \psi \rho \sin\theta \, d\rho \, d\theta \, d\phi$$

$$\frac{1}{4\pi} \iiint_G \psi \, \frac{\partial\left(\frac{1}{D}\right)}{\partial x} \, dv' = \frac{1}{4\pi} \iiint_G \psi \, \frac{x'-x}{\rho} \sin\theta \, d\rho \, d\theta \, d\phi$$

In this form, X is absolutely convergent, and the integrand is continuous. Furthermore, the derivative of the integrand is also continuous since $\frac{|x'-x|}{\rho} \le 1$ always.

Hence, X may be differentiated under the integral sign to obtain the first derivative. Thus,

$$\frac{\partial X}{\partial x} = \frac{1}{4\pi} \iiint_G \psi \frac{\partial(\frac{1}{D})}{\partial x} dv' = -\frac{1}{4\pi} \iiint_G \psi \frac{\partial(\frac{1}{D})}{\partial x'} dv'$$

since

$$\frac{\partial(\frac{1}{D})}{\partial x} = -\frac{\partial(\frac{1}{D})}{\partial x'}$$

Alternatively, integrating by parts we have

$$(2.12) \quad \frac{\partial X}{\partial x} = -\frac{1}{4\pi} \iint_S \frac{\psi}{D} ds' + \frac{1}{4\pi} \iiint_G \frac{\partial \psi}{\partial x'} \cdot \frac{1}{D} dv'$$

Now, each of these integrands is continuous, and each derivative of these integrands is continuous by reasoning analogous to that for the first derivative. Hence, differentiation under each of these integral signs is again permitted. Thus,

$$\frac{\partial^2 X}{\partial x^2} = -\frac{1}{4\pi} \iint_S \psi \frac{\partial(\frac{1}{D})}{\partial x} ds' + \frac{1}{4\pi} \iiint_G \frac{\partial \psi}{\partial x'} \frac{\partial(\frac{1}{D})}{\partial x} dv'$$

$$(2.13)$$

$$= \frac{1}{4\pi} \iint_S \psi \frac{\partial(\frac{1}{D})}{\partial x'} - \frac{1}{4\pi} \iiint_G \frac{\partial \psi}{\partial x'} \frac{\partial(\frac{1}{D})}{\partial x'} dv'$$

with similar expressions for $\frac{\partial^2 X}{\partial y^2}$ and $\frac{\partial^2 X}{\partial z^2}$.

The Laplacian, $\nabla^2 T = \nabla^2 X$, becomes

$$(2.14) \quad \nabla^2 T = \frac{1}{4\pi} \iint_S \psi \frac{\partial(\frac{1}{D})}{\partial n'} ds' - \frac{1}{4\pi} \iiint_G \nabla \psi \cdot \nabla(\frac{1}{D}) dv'$$

using ∇ to denote the gradient operator in vector notation. Green's first identity,

$$(2.15) \quad \iiint_G u\nabla^2 v\,dv' + \iiint_G \nabla u \cdot \nabla v\,dv' = \iint_S u\frac{\partial v}{\partial n'}ds'$$

with $v = (1/D)$, applied to the region G, keeping in mind the singular nature at $P = P'$, gives

$$(2.16) \quad \iiint_G \nabla u \cdot \nabla\left(\tfrac{1}{D}\right)dv' = \iint_S u\frac{\partial\left(\tfrac{1}{D}\right)}{\partial n'}ds' - \lim_{\epsilon\to 0}\iint_\alpha u\frac{\partial\left(\tfrac{1}{D}\right)}{\partial\rho}\rho^2 d\theta d\phi$$

where α is a sphere with radius ϵ about the point P and the differentiation is taken along the external normal. Thus, with $u = \psi$ we have

$$(2.17) \quad \iiint_G \nabla\psi \cdot \nabla\left(\tfrac{1}{D}\right)dv' = \iint_S \psi\frac{\partial\left(\tfrac{1}{D}\right)}{\partial n'}ds' + 4\pi\,\psi(P)$$

and

$$(2.18) \quad \nabla^2 T(P) = -\psi(P)$$

This proves the relationship stated in Proposition 1.

However, if equation (2.18) becomes the Laplace equation, formula (2.4) is identically zero. On the other hand, the usual problem for the Laplace equation requires that T have boundary values other than zero.

It was also noted in the derivation that the surface integral in equation (2.9) vanished only if T and K satisfy the same homogeneous boundary condition. Thus, when K and T are different on the boundary, and $\psi = 0$, equation (2.9)

reduces to

(2.19)
$$T(P) = \iint_S \left(K \frac{\partial T}{\partial n'} - T \frac{\partial K}{\partial n'} \right) ds'$$

It is generally necessary to make K satisfy some homogeneous condition and apply equation (2.19). The simpliest and most direct is K = 0 on S, and we have

$$T(P) = - \iint_S T \frac{\partial K}{\partial n'} ds'$$

In general, a piecewise continuous ψ can also be treated by breaking the region G down into the parts in which ψ is continuous. Let $G = G_1 + G_2$ where $\psi = \psi_1$ in G_1 and $\psi = \psi_2$ in G_2. Let $T_1(P)$ denote the solution in G_1, $T_2(P)$ denote the solution in G_2, and γ the boundary between G_1 and G_2. Applying Green's formula to G_1 and G_2 respectively gives

$$T_1(P) + \iint_\gamma \left(K \frac{\partial T}{\partial n'} - T \frac{\partial K}{\partial n'} \right) ds' = \iiint_{G_1} K(P,P') \psi_1(P') dv'$$

$$- \iint_\gamma \left(K \frac{\partial T}{\partial n'} - T \frac{\partial K}{\partial n'} \right) ds' = \iiint_{G_2} K(P,P') \psi_2(P') dv'$$

and adding yields

$$T_1(P) = \iiint_G K(P,P') \psi(P') dv'$$

Also for the region G_2,

$$\iint_{\gamma} (K\frac{\partial T}{\partial n'} - T\frac{\partial K}{\partial n'})ds' = \iiint_{G_1} K(P,P')\psi_1(P')dv'$$

$$T_2'(P) - \iint_{\delta} (K\frac{\partial T}{\partial n'} - T\frac{\partial K}{\partial n'})ds' = \iiint_{G_2} K(P,P')\psi_2(P')dv'$$

and

$$T_2(P) = \iiint_{G} K(P,P')\psi(P')dv'$$

Since T_1 and T_2 are expressed by the same formula, we see that equation (2.4) also expresses the relationship for a piecewise continuous ψ.

It may also be noted that the Green's function is symmetric in its variables. Let P_1 and P_2 be points in G, and let α_1 and α_2 be spheres about P_1 and P_2 with radii ϵ_1 and ϵ_2 respectively. Green's formula, equation (2.8), applied to G with α_1 and α_2 excluded and with $u = K(P_1,P')$ and $v = K(P_2,P')$ gives (11, pp. 228-229)

$$\iint_{\alpha_1} \left[K(P_1,P')\frac{\partial K(P_2,P')}{\partial n'} - K(P_2,P')\frac{\partial K(P_1,P')}{\partial n'}\right]ds' + \iint_{\alpha_2}\left[K(P_1,P')\frac{\partial K(P_2,P')}{\partial n'} - K(P_2,P')\frac{\partial K(P_1,P')}{\partial n'}\right]ds' = 0$$

since $\nabla^2 K = 0$ everywhere in the modified region and the integrand of the surface integral is zero. Consequently, as ϵ_1 and ϵ_2 approach zero,

$$\lim_{\epsilon \to 0} \iint_{\alpha_1} \left[K(P_1,P') \frac{\partial K(P_2,P')}{\partial n'} - K(P_2,P') \frac{\partial K(P_1,P')}{\partial n'} \right] ds' = -K(P_2,P_1)$$

$$\lim_{\epsilon \to 0} \iint_{\alpha_2} \left[K(P_1,P') \frac{\partial K(P_2,P')}{\partial n'} - K(P_2,P') \frac{\partial K(P_1,P')}{\partial n'} \right] ds' = K(P_1,P_2)$$

and

$$K(P_1,P_2) = K(P_2,P_1)$$

Identical reasoning for the two-dimensional Poisson and Laplace equations with

$$K(P,P') = -\frac{1}{2\pi} \ln R + \beta(P,P')$$

produces equivalent results with volume integrals replaced by area integrals and surface integrals replaced by line integrals. R is the distance from P to P' in a plane.

3. THE DOUGALL METHOD

Bessel Functions

Before starting the discussion of the Dougall method, it is necessary to define some Bessel functions, especially for the second solutions since various authors add or delete first solutions and constants. Those used throughout this text are presented in reference (8). We define the following quantities by the series representations and note the asymptotic expansions needed in this development.

Functions

(3.1)
$$J_0(z) = \sum_{s=0}^{\infty} \frac{(-1)^s}{(s!)^2} \left(\frac{z}{2}\right)^{2s}$$

$$J_n(z) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} \left(\frac{z}{2}\right)^{n+2s}$$

(3.2)
$$I_0(z) = \sum_{s=0}^{\infty} \frac{1}{(s!)^2} \left(\frac{z}{2}\right)^{2s} = J_0(iz)$$

$$I_n(z) = \sum_{s=0}^{\infty} \frac{1}{s!(n+s)!} \left(\frac{z}{2}\right)^{n+2s} = i^{-n} J_n(iz)$$

(3.3)
$$K_0(z) = -I_0(z)\left[\gamma + \ln\frac{z}{2}\right] + \sum_{s=1}^{\infty} \frac{\phi(s)}{(s!)^2} \left(\frac{z}{2}\right)^{2s}$$

(3.4)
$$K_n(z) = \frac{1}{2} \sum_{s=0}^{n-1} \frac{(-1)^s(n-s-1)!}{s!} \left(\frac{2}{z}\right)^{n-2s} + (-1)^{n+1} I_n(z) \ln\frac{z}{2}$$

$$+ \frac{(-1)^n}{2} \sum_{s=1}^{\infty} \frac{\psi(s) + \psi(n+s)}{s!(n+s)!} \left(\frac{z}{2}\right)^{n+2s}$$

where

$$\psi(s) = \phi(s) + \gamma$$
$$\phi(s) = \left[1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{s} \right]$$
$$\gamma = 0.5772157\cdots$$

Asymptotic Expansions

($|z|$, large)

$$J_n(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4} - \frac{n\pi}{2}\right) \qquad -\frac{\pi}{2} < \arg z < \frac{\pi}{2}$$

(3.5)
$$I_n(z) \sim \frac{1}{\sqrt{2\pi z}} e^z \qquad 0 < \arg z < \pi$$

$$K_n(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \qquad -\pi < \arg z < \pi$$

K_n and I_n satisfy the relations

$$I_n(z) K_{n+1}(z) + I_{n+1}(z) K_n(z) = \frac{1}{z}$$

(3.6)
$$I_n'(z) K_n(z) - I_n(z) K_n'(z) = \frac{1}{z}$$

Also in the development we shall need the addition formulae

for $K_0(\lambda R)$ and $J_0(\lambda R)$, $R^2 = r^2 + r'^2 - 2rr' \cos(\theta - \theta')$,

which are expressed by

$$K_0(\lambda R) = I_0(\lambda r) K_0(\lambda r') + 2 \sum_{n=1}^{\infty} I_n(\lambda r) K_n(\lambda r') \cos n(\theta - \theta')$$

(3.7)
$$J_0(\lambda R) = J_0(\lambda r) J_0(\lambda r') + 2 \sum_{n=1}^{\infty} J_n(\lambda r) J_n(\lambda r') \cos n(\theta - \theta')$$

for $r \leq r'$ and

$$K_0(\lambda R) = I_0(\lambda r') K_0(\lambda r) + 2 \sum_{n=1}^{\infty} I_n(\lambda r') K_n(\lambda r) \cos n(\theta - \theta')$$

(3.8)
$$J_0(\lambda R) = J_0(\lambda r') J_0(\lambda r) + 2 \sum_{n=1}^{\infty} J_n(\lambda r') J_n(\lambda r) \cos n(\theta - \theta')$$

for $r > r'$ (4, p. 38 and p. 74).

The Green's Function for an Infinite Cylinder

The Green's function for an infinite cylinder is well known and can be expressed by (1, p. 341)

$$K(P,P') = \frac{1}{2\pi a^2} \sum_{n=-\infty}^{n=+\infty} \cos n(\theta-\theta') \sum_{m=1}^{\infty} \frac{J_n(\lambda_m r) J_n(\lambda_m r') e^{-\lambda_m |z-z'|}}{\lambda_m [J_n'(\lambda_m a)]^2}$$

$$J_n(\lambda_m a) = 0$$

The purpose of this section is to develop this function by the Dougall method, pointing out the distinquishing features and showing how other solutions for a semi-infinite or finite cylinder may be constructed. This section also provides the guide for Chapter 4 since many of the steps employed for the construction of the Green's function for a circle are directly analogous to those presented here.

The Dougall method consists of representing the singular term of the Green's function by an infinite or contour integral in such a way that the complementary solutions, β, are easily added to the integrand to affect simplifications. It is generally necessary to find an integral in terms of the complementary solutions. This procedure works very well in the cylindrical coordinate cases since two well known integrals in terms of the solutions of the Laplace equation represent 1/D. These integrals, commonly used as a starting point for this method, are given by (8, p. 65 and p. 75).

$$(3.9) \quad \frac{1}{D} = \frac{1}{\sqrt{R^2 + (z-z')^2}} = \int_0^\infty e^{-\lambda|z-z'|} J_0(\lambda R)\, d\lambda$$

$$(3.10) \quad \frac{1}{D} = \frac{2}{\pi} \int_0^\infty K_0(\lambda R) \cos\lambda(z-z')\, d\lambda + \beta(P,P')$$

and K is then expressed by either

$$(3.11) \quad K(P,P') = \frac{1}{4\pi} \int_0^\infty e^{-\lambda|z-z'|} J_0(\lambda R)\, d\lambda + \beta(P,P')$$

or

$$(3.12) \quad K(P,P') = \frac{1}{2\pi^2} \int_0^\infty K_0(\lambda R) \cos\lambda(z-z')\, d\lambda + \beta(P,P')$$

depending upon what boundary conditions are to be satis-
fied. In general, the first form is used when a boundary
condition on a z-plane is to be satisfied first, and the
result modified again by adding complementary functions
to satisfy the conditions on a cylindrical surface. The
second form is used for the reverse case. That is, the
boundary condition for the cylindrical surface is satis-
fied, and the result is the Green's function for an
infinite cylinder. Further modifications for finite
cylinders are obtained by adding other complementary
functions.

It is obvious that β must also be expressed in a form
similar to $1/4\pi D$ in order to affect any simplifications.

The solutions most common are found by separation of variables in $\nabla^2 \beta = 0$. With the aid of the addition formula for $K_0(\lambda R)$, the second form may be expressed by

$$K(P,P') = \frac{1}{2\pi^2} \int_0^\infty \cos\lambda(z-z') \sum_{n=-\infty}^{n=+\infty} \cos n(\theta-\theta') I_n(\lambda r') K_n(\lambda r) d\lambda + \beta$$

or

$$(3.13) \quad K(P,P') = \frac{1}{2\pi^2} \sum_{n=-\infty}^{n=+\infty} \cos n(\theta-\theta') \int_0^\infty I_n(\lambda r') K_n(\lambda r) \cos\lambda(z-z') d\lambda + \beta$$

for $r > r'$. The inversion of the summation and integration is justified by uniform convergence for $\theta \neq \theta'$ and any r and r'. Now it can be seen by separation of variables in $\nabla^2 \beta = 0$ that the function

$$\beta = A_\delta(\lambda) \cos\gamma(\theta-\theta') I_\delta(\lambda r) \cos\lambda(z-z')$$

is harmonic for arbitrary choices of A, γ and λ. However, in order to affect simplifications in K, these parameters are assigned roles similar to those in equation (3.13). Summation over these parameters does not affect the variables in r, θ and z as long as these sums with their first and second derivatives converge uniformly. Thus, with this condition on $A(\lambda)$, β may assume the form

$$\beta(P,P') = \frac{1}{2\pi^2} \sum_{n=-\infty}^{n=+\infty} \cos n(\theta-\theta') \int_0^\infty A_n(\lambda) I_n(\lambda r) \cos\lambda(z-z') d\lambda$$

for a solution of $\nabla^2 \beta = 0$ which is finite at r = 0. It also may take this form with $I_n(\lambda r)$ replaced by $K_n(\lambda r)$ for a solution which is finite at ∞, or the sum of the two for a hollow cylinder. Other modifications on the form of β can be made by letting either θ' or z' be zero. The boundary conditions generally dictate the form which is most compatible.

For the purposes of continuing the discussion, the Green's function for an infinite cylinder will be constructed, satisfying the boundary condition K = 0 at r = a. If we combine the forms for $1/4\pi D$ and β, K(P,P')
becomes

$$(3.14) \quad K(P,P') = \frac{1}{2\pi^2} \sum_{n=-\infty}^{n=+\infty} \cos n(\theta - \theta') \int_0^\infty \left[I_n(\lambda r') K_n(\lambda r) + A_n(\lambda) I_n(\lambda r) \right] \cos \lambda(z - z') d\lambda$$

for r > r' with a similar expression for r ≤ r'. If K = 0 at r = a,

$$I_n(\lambda r') K_n(\lambda a) + A_n(\lambda) I_n(\lambda a) = 0$$

and

$$A_n(\lambda) = \frac{-I_n(\lambda r') K_n(\lambda a)}{I_n(\lambda a)}$$

Thus,

$$(3.15) \quad K(P,P') = \frac{1}{2\pi^2} \sum_{n=-\infty}^{n=+\infty} \cos n(\theta - \theta') \int_0^\infty \frac{I_n(\lambda r') \left[I_n(\lambda a) K_n(\lambda r) - K_n(\lambda a) I_n(\lambda r) \right] \cos \lambda(z - z') d\lambda}{I_n(\lambda a)}$$

If λ is now considered a complex variable, and the expression

$$\frac{1}{2} \int_{C'} \frac{I_n(\lambda\hbar')[K_n(\lambda\hbar)I_n(\lambda a) - K_n(\lambda a)I_n(\lambda\hbar)]e^{i\lambda|z-z'|}}{I_n(\lambda a)} d\lambda$$

is taken about a large semi-circular contour in the upper half λ plane and along the real axis indented at the origin, not passing through any poles, we find that the integral along the real axis reduces to

$$\int_0^\infty \frac{I_n(\lambda\hbar')[K_n(\lambda\hbar)I_n(\lambda a) - K_n(\lambda a)I_n(\lambda\hbar)]\cos\lambda(z-z')}{I_n(\lambda a)} d\lambda$$

In applying Cauchy's theorem, we also find that this integral is equal to $2\pi i$ times the sum of the residues at the poles inside the contour since the integral over the large semi-circle vanishes as the radius approaches infinity. This can be seen by substituting the asymptotic expansions for K_n and I_n into the integral. The function $I_n(\lambda a)$ has simple poles at the points $\lambda = i\lambda_m$ along the imaginary axis. Hence,

$$\int_0^\infty \frac{I_n(\lambda\hbar')[K_n(\lambda\hbar)I_n(\lambda a) - K_n(\lambda a)I_n(\lambda\hbar)]\cos\lambda(z-z')}{I_n(\lambda a)} d\lambda$$

$$= \pi i \sum_{m=1}^\infty \frac{-I_n(i\lambda_m\hbar')K_n(i\lambda_m a)I_n(i\lambda_m\hbar)e^{-\lambda_m|z-z'|}}{a I_n'(i\lambda_m a)}$$

with
$$I_n(i\lambda_m a) = 0 \quad \text{or} \quad J_n(\lambda_m a) = 0$$

and
$$I_n'(i\lambda_m a) = I_{|n|+1}(i\lambda_m a)$$

or
$$I_n'(i\lambda_m a) = i^{-n} e^{n\pi i} J_n'(\lambda_m a)$$

Also, from equation (3.6)

$$K_n(i\lambda_m a) = \frac{1}{i\lambda_m a\, I_n'(i\lambda_m a)}$$

and

$$\int_0^\infty \frac{I_n(\lambda r')[k_n(\lambda r)I_n(\lambda a) - k_n(\lambda a)I_n(\lambda r)]\cos\lambda(z-z')\,d\lambda}{I_n(\lambda a)}$$

$$= \frac{\pi}{a^2}\sum_{m=1}^\infty \frac{J_n(\lambda_m r')J_n(\lambda_m r)\,e^{-\lambda_m|z-z'|}}{\lambda_m[J_n'(\lambda_m a)]^2}$$

with $J_n(\lambda_m a) = 0$.

The residue at $\lambda = 0$ is zero for $n > 0$ and there is no pole at $\lambda = 0$ for $n = 0$. Consequently, the complete solution for the Green's function for an infinite cylinder is

$$(3.16)\quad K(P,P') = \frac{1}{2\pi a^2}\sum_{n=-\infty}^{n=+\infty}\cos n(\theta-\theta')\sum_{m=1}^\infty \frac{J_n(\lambda_m r)J_n(\lambda_m r')\,e^{-\lambda_m|z-z'|}}{\lambda_m[J_n'(\lambda_m a)]^2}$$

with $J_n(\lambda_m a) = 0$, λ_m taken only over the positive roots.

It was noted that only the expression for $r > r'$ was handled. Actually, due to the symmetric property of the Green's function, the second form for $r \le r'$ also reduces to the above equation.

If the function for a finite cylinder is needed, more complementary solutions with the form

$$(3.17)\quad \beta(P,P') = \frac{1}{2\pi a^2}\sum_{n=-\infty}^{n=+\infty}\cos n(\theta-\theta')\sum_{m=1}^\infty \frac{J_n(\lambda_m r)J_n(\lambda_m r')[A_m e^{\lambda_m z}+B_m e^{-\lambda_m z}]}{\lambda_m[J_n'(\lambda_m a)]^2}$$

may be added to equation (3.16) to satisfy the boundary conditions on the ends of the cylinder. An alternate expression for the finite cylinder may be derived also by starting with equation (3.10), satisfying the conditions for the z-plane surfaces first and then satisfying the boundary conditions on the cylinder (1, p. 341). The expansion in this case however is bilinear in trigonometric functions instead of Bessel functions.

4. A TWO-DIMENSIONAL ANALOG

In the previous chapter, it was noted that a suitable representation of the singular form of the Green's function was necessary to apply the Dougall method. In two dimensions, K takes the form

$$(4.1) \qquad K(P,P') = -\frac{1}{2\pi} \ln R + \beta(P,P')$$

where $R^2 = r^2 + r'^2 - 2rr' \cos(\theta - \theta)$ in polar coordinates. Before going further, we need to develop the function ln R in the form of an integral, actually two integrals since it is both the real and the complex notation with which we shall be concerned.

The Green's Function Singularity

After obtaining the Green's function for an infinite cylinder, it is possible to specialize the result to two dimensions. If $\psi = \psi(r,\theta)$ only, then ψ is independent of z and the Green's function in two dimensions is obtainable by integrating z' from $-\infty$ to $+\infty$. The result is the Green's function for a circle. Thus,

$$K(P,P') = \frac{1}{\pi a^2} \sum_{n=-\infty}^{n=+\infty} \cos n(\theta - \theta') \sum_{m=1}^{\infty} \frac{J_n(\lambda_m r) J_n(\lambda_m r')}{\lambda_m^2 [J_n'(\lambda_m a)]^2}$$

with

$$J_n(\lambda_m a) = 0$$

is the bilinear form for the circle. One may also expect
this procedure to work for the singular form also. Such
is not the case however. The integral in equation (3.9),
when integrated with respect to z' from - to + , yields
a divergent integral. Other methods must be sought for
the representation of the singular form in two dimensions.

The representation found suitable for this analysis
is

$$(4.2) \qquad \ln R = -\int_0^\infty \frac{J_0(\lambda R) - J_0(\lambda)}{\lambda} d\lambda = -\frac{1}{\pi i} \int_C \frac{K_0(\lambda R e^{-\frac{\pi i}{2}}) - K_0(\lambda e^{-\frac{\pi i}{2}})}{\lambda} d\lambda$$

where C is a straight line contour along the real axis in
the upper half λ plane.

PROOF: The first integral is shown by considering a
function g'(x), integrable between 0 and ∞. Then,

$$\int_0^\infty g'(x)\, dx = g(\infty) - g(0)$$

If x is replaced by xy in this integral, we obtain

$$(4.3) \qquad \int_0^\infty g'(xy)\, dx = \frac{g(\infty) - g(0)}{y}$$

which converges uniformly for y > 0. We now integrate
both sides of this equation with respect to y between
u and v for u,v > 0. Thus,

$$\int_u^v dy \int_0^\infty g'(xy)\, dx = [g(\infty) - g(0)] \ln \frac{v}{u}$$

or

$$(4.4) \qquad \int_0^\infty dx \int_u^v g'(xy)\, dy = [g(\infty) - g(0)]\, \ln \frac{v}{u}$$

since the first integral on the left converges uniformly.

then,

$$(4.5) \qquad \int_0^\infty \frac{g(vx) - g(ux)}{x}\, dx = [g(\infty) - g(0)]\, \ln \frac{v}{u}$$

Now, $J_0(x)$ satisfies all the requirements for $g(x)$ and $g'(x)$. Hence

$$(4.6) \qquad \ln R = - \int_0^\infty \frac{J_0(\lambda R) - J_0(\lambda)}{\lambda}\, d\lambda$$

with $v = R$, $u = 1$, and $\lambda = x$.

It is this integral which gives the clue to the form of the contour integral to be considered. One familiar with Bessel functions would naturally assume the form

$$\frac{1}{\pi i} \int_{C'} \frac{K_0(\lambda R e^{-\frac{\pi i}{2}}) - K_0(\lambda e^{-\frac{\pi i}{2}})}{\lambda}\, d\lambda$$

around a semi-circular contour in the upper half λ plane and along the real axis indented at the origin (8, p. 23 and p. 103). By Cauchy's theorem, this integral is zero. Also, as the contour is made infinitely large, the integral over the semi-circle vanishes, and we have

$$(4.7) \quad \frac{1}{\pi i} \int_C \frac{K_0(\lambda R e^{-\frac{\pi i}{2}}) - K_0(\lambda e^{-\frac{\pi i}{2}})}{\lambda}\, d\lambda = \frac{1}{\pi i} \int_0^\infty \frac{\{[K_0(\lambda R e^{-\frac{\pi i}{2}}) - K_0(\lambda R e^{+\frac{\pi i}{2}})] - [K_0(\lambda e^{-\frac{\pi i}{2}}) - K_0(\lambda e^{\frac{\pi i}{2}})]\}}{\lambda}\, d\lambda$$

$$= \frac{1}{\pi} \int_0^\pi \lim_{\epsilon \to 0} \left[K_0(\epsilon R e^{i(\theta - \frac{\pi}{2})}) - K_0(\epsilon e^{i(\theta - \frac{\pi}{2})}) \right] d\theta$$

Also,

$$K_0(\epsilon R e^{i(\theta-\frac{\pi}{2})}) - K_0(\epsilon e^{i(\theta-\frac{\pi}{2})}) =$$

(4.8)
$$-I_0(\epsilon R e^{i(\theta-\frac{\pi}{2})})\left[\ln\frac{\epsilon R}{2}+\gamma+i(\theta-\frac{\pi}{2})\right] + I_0(\epsilon R e^{i(\theta-\frac{\pi}{2})})\left[\ln\frac{\epsilon}{2}+\gamma+i(\theta-\frac{\pi}{2})\right]$$

plus a power series starting in ϵ^2.

Consequently, as $\epsilon \longrightarrow 0$

$$\ln R = -\frac{1}{\pi i}\int_C \frac{K_0(\lambda R e^{-\frac{\pi i}{2}}) - K_0(\lambda e^{-\frac{\pi i}{2}})}{\lambda}\,d\lambda$$

Also

$$K_0(\lambda R e^{-\frac{\pi i}{2}}) - K_0(\lambda R e^{\frac{\pi i}{2}}) =$$

(4.9)
$$-J_0(\lambda R)\left[\ln\frac{\lambda R}{2}+\gamma-\frac{\pi i}{2}\right] + J_0(\lambda R)\left[\ln\frac{\lambda R}{2}+\gamma+\frac{\pi i}{2}\right]$$

$$= \pi i\, J_0(\lambda R)$$

Similarly,

$$K_0(\lambda e^{-\frac{\pi i}{2}}) - K_0(\lambda e^{\frac{\pi i}{2}}) = \pi i\, J_0(\lambda)$$

and substitution into equation (4.7) gives

$$\ln R = -\int_0^\infty \frac{J_0(\lambda R) - J_0(\lambda)}{\lambda}\,d\lambda = -\frac{1}{\pi i}\int_C \frac{K_0(\lambda R e^{-\frac{\pi i}{2}}) - K_0(\lambda e^{-\frac{\pi i}{2}})}{\lambda}\,d\lambda$$

The Green's Function for a Circle

With the singular form represented by equation (4.2),
K becomes

(4.10) $$K(P,P') = \frac{1}{2\pi^2 i}\int_C \frac{K_0(\lambda R e^{-\frac{\pi i}{2}}) - K_0(\lambda e^{-\frac{\pi i}{2}})}{\lambda}\,d\lambda + \beta(P,P')$$

With the aid of the addition formula for $K_0(\lambda \text{Re}^{-\frac{\pi i}{2}})$,

K can also be written

$$K(P,P') = \frac{1}{2\pi^2 i} \int \frac{K_0(\lambda r e^{-\frac{\pi i}{2}}) I_0(\lambda r' e^{-\frac{\pi i}{2}}) - K_0(\lambda e^{-\frac{\pi i}{2}})}{\lambda} d\lambda$$

$$+ \frac{1}{\pi^2 i} \sum_{n=1}^{\infty} \cos n(\theta - \theta') \int_C \frac{K_n(\lambda r e^{-\frac{\pi i}{2}}) I_n(\lambda r' e^{-\frac{\pi i}{2}})}{\lambda} d\lambda + \beta$$

Now a suitable choice for β must be made. The function

$$\beta = \frac{1}{2\pi^2 i} \int_C \frac{A_0(\lambda) d\lambda}{\lambda} + \frac{1}{\pi^2 i} \sum_{n=1}^{\infty} \cos n(\theta - \theta') \int_C \frac{A_n(\lambda) \lambda^n}{\lambda} d\lambda$$

satisfies $\nabla^2 \beta = 0$ in two dimensions and K becomes

$$K(P,P') = \frac{1}{2\pi^2 i} \int_C \frac{[K_0(\lambda r e^{-\frac{\pi i}{2}}) I_0(\lambda r' e^{-\frac{\pi i}{2}}) - K_0(\lambda e^{-\frac{\pi i}{2}}) + A_0(\lambda)]}{\lambda} d\lambda$$

(4.11)

$$+ \frac{1}{\pi^2 i} \sum_{n=1}^{\infty} \cos n(\theta - \theta') \int_C \frac{K_n(\lambda r e^{-\frac{\pi i}{2}}) I_n(\lambda r' e^{-\frac{\pi i}{2}}) + \lambda^n A_n(\lambda)}{\lambda} d\lambda$$

provided $A_n(\lambda)$ is such that β with its first and second

derivatives is continuous. Now, K = 0 at r = a. Thus,

A_0 and A_n take the values

$$A_0 = -I_0(\lambda r' e^{-\frac{\pi i}{2}}) K_0(\lambda a e^{-\frac{\pi i}{2}}) + K_0(\lambda e^{-\frac{\pi i}{2}})$$

(4.12)

$$A_n = \frac{-I_n(\lambda r' e^{-\frac{\pi i}{2}}) K_n(\lambda a e^{-\frac{\pi i}{2}})}{a^n}$$

Consequently,

$$K(P,P') = \frac{1}{2\pi^2 i} \int \frac{I_0(\lambda r' e^{-\frac{\pi i}{2}})[K_0(\lambda r e^{-\frac{\pi i}{2}}) - K_0(\lambda a e^{-\frac{\pi i}{2}})]}{\lambda} d\lambda$$

(4.13)

$$+ \frac{1}{\pi^2 i} \sum_{n=1}^{\infty} \cos n(\theta - \theta') \int_C \frac{I_n(\lambda r' e^{-\frac{\pi i}{2}})[a^n K_n(\lambda r e^{-\frac{\pi i}{2}}) - \lambda^n K_n(\lambda a e^{-\frac{\pi i}{2}})]}{\lambda a^n} d\lambda$$

To evaluate these integrals, Cauchy's theorem is again applied to a contour, C', around a semi-circle and along the real axis indented at the origin in the upper half λ plane. Thus, following the pattern given in equations (4.7) to (4.9) we have

$$\frac{1}{2\pi^2 i} \int_{C} \frac{I_0(\lambda \imath' e^{-\frac{\pi i}{2}})[K_0(\lambda \imath e^{-\frac{\pi i}{2}}) - K_0(\lambda a e^{-\frac{\pi i}{2}})]}{\lambda} d\lambda = \frac{1}{2\pi} \int_{0}^{\infty} \frac{J_0(\lambda \imath')[J_0(\lambda \imath) - J_0(\lambda a)]}{\lambda} d\lambda$$

(4.14)

$$= \frac{1}{2\pi^2} \int_{0}^{\pi} \lim_{\epsilon \to 0} \left[K_0(\epsilon \imath e^{i(\theta - \frac{\pi}{2})}) - K_0(\epsilon a e^{i(\theta - \frac{\pi}{2})}) \right] d\theta = \frac{1}{2\pi} \ln \frac{a}{\imath}$$

Similarly,

$$\frac{1}{\pi^2 i} \int_{C} \frac{I_n(\lambda \imath' e^{-\frac{\pi i}{2}})[a^n K_n(\lambda \imath e^{-\frac{\pi i}{2}}) - \imath^n K_n(\lambda a e^{-\frac{\pi i}{2}})]}{\lambda} d\lambda$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \frac{J_n(\lambda \imath')[J_n(\lambda \imath) - J_n(\lambda a)]}{\lambda} d\lambda$$

$$= \frac{1}{\pi^2} \int_{0}^{\pi} \lim_{\epsilon \to 0} \left\{ \left[a^n K_n(\epsilon \imath e^{i(\theta - \frac{\pi}{2})}) - \imath^n K_n(\epsilon a e^{i(\theta - \frac{\pi}{2})}) \right] I_n(\epsilon \imath' e^{i(\theta - \frac{\pi}{2})}) \right\} d\theta$$

Using the first terms of K_n and I_n in the expansions gives

$$\left[a^n K_n(\epsilon \imath e^{i(\theta - \frac{\pi}{2})}) - \imath^n K_n(\epsilon a e^{i(\theta - \frac{\pi}{2})}) \right] I_n(\epsilon \imath' e^{i(\theta - \frac{\pi}{2})}) =$$

(4.12)

$$= -\frac{\imath'^n}{2n} \left[\frac{\imath^n}{a^n} - \frac{a^n}{\imath^n} \right]$$

and, as $\epsilon \to 0$,

$$\frac{1}{\pi^2 i} \int_{C} \frac{I_n(\lambda \imath' e^{-\frac{\pi i}{2}})[a^n K_n(\lambda \imath e^{-\frac{\pi i}{2}}) - \imath^n K_n(\lambda a e^{-\frac{\pi i}{2}})]}{\lambda} d\lambda$$

$$= -\frac{\imath'^n}{2\pi n} \left[\frac{\imath^n}{a^n} - \frac{a^n}{\imath^n} \right]$$

Thus, the final form becomes

$$(4.13)\quad K(P,P') = -\frac{1}{2\pi}\ln\frac{r}{a} - \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{r'^n}{na^n}\left[\frac{r^n}{a^n}-\frac{a^n}{r^n}\right]\cos n(\theta-\theta')$$

for r > r' and treating the expression for r ≤ r' similarly yields

$$(4.14)\quad K(P,P') = -\frac{1}{2\pi}\ln\frac{r'}{a} - \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{r^n}{na^n}\left[\frac{r'^n}{a^n}-\frac{a^n}{r'^n}\right]\cos n(\theta-\theta')$$

We also have the results

$$(4.15)\quad \int_0^{\infty}\frac{J_0(\lambda r')[J_0(\lambda r)-J_0(\lambda a)]}{\lambda}d\lambda = \begin{cases}\ln\frac{a}{r} & \text{for } r'<r\leq a \\[2mm] \ln\frac{a}{r'} & \text{for } r\leq r'\leq a\end{cases}$$

$$(4.16)\quad \int_0^{\infty}\frac{J_n(\lambda r')[a^n J_n(\lambda r)-r^n J_n(\lambda a)]}{\lambda}d\lambda = \begin{cases}-\dfrac{r'^n}{2n}\left[\dfrac{r^n}{a^n}-\dfrac{a^n}{r^n}\right] & \text{for } r'<r\leq a \\[3mm] -\dfrac{r^n}{2n}\left[\dfrac{r'^n}{a^n}-\dfrac{a^n}{r'^n}\right] & \text{for } r\leq r'\leq a\end{cases}$$

These results may be regarded as generalizations of the Bateman integral (13, p. 406),

$$(4.17)\quad \int_0^{\infty}J_0(\lambda u)[1-J_0(\lambda v)]\frac{d\lambda}{\lambda} = \begin{cases}0 & \text{for } v<u \\[2mm] \ln\frac{v}{u} & \text{for } u\leq v\end{cases}$$

or expansions in terms of the Fourier-Bessel integral.

A direct approach through the addition theorem for $J_0(\lambda R)$ and equation (4.2) yields these integrals immediately, but does not provide their values in terms of elementary functions. Thus, with K expressed by

$$(4.18)\quad K(P,P') = \frac{1}{2\pi}\int_0^{\infty}\frac{J_0(\lambda r)J_0(\lambda r')-J_0(\lambda)}{\lambda}d\lambda + \frac{1}{\pi}\sum_{n=1}^{\infty}\cos n(\theta-\theta')\int_0^{\infty}\frac{J_n(\lambda r)J_n(\lambda r')}{\lambda}d\lambda + \beta$$

and β selected in the form of real integrals,

$$\beta(P,P') = \frac{1}{2\pi} \int_0^\infty \frac{A_0(\lambda)\,d\lambda}{\lambda} + \frac{1}{\pi} \sum_{n=1}^\infty \cos n(\theta-\theta') \int_0^\infty \frac{A_n(\lambda)\,r^n}{\lambda}\,d\lambda$$

we have

$$K(P,P') = \frac{1}{2\pi} \int_0^\infty \frac{J_0(\lambda r)J_0(\lambda r') - J_0(\lambda) + A_0(\lambda)}{\lambda}\,d\lambda$$

(4.19)

$$+ \frac{1}{\pi} \sum_{n=1}^\infty \cos n(\theta-\theta') \int_0^\infty \frac{J_n(\lambda r)J_n(\lambda r') + r^n A_n(\lambda)}{\lambda}\,d\lambda$$

and for $K = 0$ at $r = a$,

$$K(P,P') = \frac{1}{2\pi} \int_0^\infty \frac{J_0(\lambda r')\left[J_0(\lambda r) - J_0(\lambda a)\right]}{\lambda}\,d\lambda$$

(4.20)

$$+ \frac{1}{\pi} \sum_{n=1}^\infty \cos n(\theta-\theta') \int_0^\infty \frac{J_n(\lambda r')\left[a^n J_n(\lambda r) - r^n J_n(\lambda a)\right]}{\lambda a^n}\,d\lambda$$

The disadvantage of this approach lies in the fact that reducing these integrals to elementary functions is difficult. Hence the contour method is preferred in most cases.

If the expression for $r' > r$ is differentiated and substituted into the two dimensional equation corresponding to (2.7), we have the solution $T(P)$ for a circle in terms of the series expansion of the Poisson integral

formula. Thus,

$$(4.21) \qquad \frac{\partial K(P,P')}{\partial r'}\bigg|_{r'=a} = -\frac{1}{2\pi a} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{r^n}{a^{n+1}} \cos n(\theta-\theta').$$

and

$$T(P) = \frac{1}{2\pi} \int_0^{2\pi} a T(a,\theta') \left[\frac{1}{a} + 2\sum_{n=1}^{\infty} \frac{r^n}{a^{n+1}} \cos n(\theta-\theta') \right] d\theta'$$

or

$$T(P) = \frac{1}{2\pi} \int_0^{2\pi} \frac{a^2 - r^2}{a^2 + r^2 - 2ar \cos(\theta-\theta')} T(a,\theta') d\theta'$$

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