AN ABSTRACT OF THE THESIS OF

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The focus of this study was junior-level mathematics students' perception of proof and its relationship to achievement. The following problems were investigated:

1) nature of perception of proof of undergraduate mathematics students who have enrolled in Advanced Calculus;

2) relationship between students' perception of selected aspects of proof and their achievement in Advanced Calculus; and

3) relationship between measures of perception of proof and achievement in Advanced Calculus.

Twenty versions of a questionnaire, each containing six items, were administered randomly to

47 students in Advanced Calculus. The questionnaire items measured selected aspects of students' perception of proof. Student responses to the questionnaire were evaluated and put into response categories by three judges.

An interview script was developed based on the results of the written questionnaires and a pilot study involving undergraduates with a similar backround as those in the study. The script assessed students'subjective perception of the nature of mathematical proof, degree to which students enjoy proof, and amount of confidence students have in their ability to construct proofs. Eight follow-up interviews were conducted. They were taped, analyzed, and categorized into an inductively developed category system.

Achievement data were obtained from student performance on tests and homework assignments. It was the total number of points accumulated by each student.

The following hypotheses were tested:

1) The correlation between total score obtained on the written questionnaire and achievement in Advanced Calculus for class A is not

significantly different than the same correlation for class B.

2) There is no significant proportion of variation in achievement that is associated with perception of proof.

3) There is no association between achievement in Advanced Calculus and perception of the aspects of proof addressed by each situation on the written questionnaire.

Data were analyzed using 2x2 contingency tables, correlation coefficient, and qualitative analysis of interviews. From these analyses the following conclusions were drawn:

1) the nature and role of hypothesis in mathematics is misunderstood by at least a large minority of junior-level mathematics students;

2) a significant proportion of variation in achievement is associated with perception of proof. Several recommendations for research and practice were discussed.

University Mathematics Students' Perception of Proof and its Relationship to Achievement.

by

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"For scholars and laymen alike it is not philosophy but active experience in mathematics itself that alone can answer the question: What is mathematics?"

-Courant and Robbins

"Who borrows the Medusa's eye,
Resigns to the empirical lie.
The knower petrifies the known:
The subtle dancer turns to stone."

-Theodore Roszak

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UNIVERSITY MATHEMATICS STUDENTS' PERCEPTION OF PROOF AND ITS RELATIONSHIP TO ACHIEVEMENT

I. INTRODUCTION

Undergraduate students of mathematics usually begin their university studies with calculus, what Eric Temple Bell calls "the chief instrument of applied mathematics" (Bell, 1951, p.320). It is characterized (for the most part) by relatively concrete algebraic manipulations and practical applications. In a standard modern calculus textbook, for example, the authors use the following words:

> "The power of calculus is impressive in its precise answers to realistic questions and problems. In the necessary development of the theory, we keep in mind the central question, how does one actually compute it? We hope that a robust numerical flavor enhances the concreteness of our exposition" (Edwards and Penney, 1982, p.ix).

Things change, however, as we see A. N. Whitehead's description of modern mathematics:

> "... the ideas, now in the minds of contemporary mathematicians, lie very

> remote from any notions which can be immediately derived by perception through the senses ... The point of mathematics is that in it we have always got rid of the particular instance, and even of any particular sorts of entities ... So long as you are dealing with pure mathematics, you are in the realm of complete and absolute abstraction" (Whitehead, 1956, pp.402-403).

It is this sort of mathematics that the undergraduate begins to study in his or her later undergraduate career. The "robust numerical flavor" is gone for many students who enroll in courses such as elementary analysis, abstract algebra, or general topology.

Textbook titles such as: A Bridge to Advanced Mathematics (Sentilles, 1975) and A Transition to Advanced Mathematics (Smith, et. al., 1983), indicate a felt need by mathematicians to

> "span the gap that exists between a practically oriented calculus sequence and the theoretically oriented courses in algebra, analysis and other areas which typically follow in {the student's} third and fourth years" (Sentilles, 1975, p.v).

The essence of modern mathematics is the axiomatic method and deductive reasoning, that is, mathematical proof.

> "... the axiomatic approach to a mathematical subject is the natural way to unravel the network of interconnections between the various facts and to exhibit the essential logical skeleton of the structure" (Courant and Robbins, 1941, p.216).

> "In general terms, the axiomatic point of view can be described as follows: To prove a theorem in a deductive system is to show that the theorem is a necessary logical consequence of some previously proved propositions; these in turn must themselves be proved; and so on. The process of mathematical proof would therefore be the impossible task of an infinite regression unless, in going back, one is permitted to stop at some point. Hence there must be a number of statements called postulates or axioms, which are accepted as true, and for which proof is not required. From these we may attempt to deduce all other theorems by purely logical argument" (Courant and Robbins, 1941, p.214).

From the "mythopoeic" thought of ancient man, where the gulf between subjective and objective was not so wide (Frankfort, et. al., 1946, p.11), to mathematicians' perception of proof and the role of proof in mathematics today we see a constantly evolving concept. Even after the rise of the deductive method the necessity of rigorous proof has not always gone unquestioned. For example, Morris Kline's description of the views of 18th century mathematicians:

> "The typical attitude of the century was: Why go to the trouble of proving by abstruse reasoning things which one never doubts in the first place, or of demonstrating what is more evident by what is less evident" (Kline, 1972, p.618)?

Clairaut states, for example, in Elements de Geometrie (1741): "All reasoning concerned with what common sense knows in advance, serves only to conceal the truth and to weary the reader and is today disregarded" (Kline, 1972, p.618).

Compare this with the findings of a recent study by Edgar Williams of Edmonton, Alberta (Williams, 1976). In his study, Williams developed a twelve-item questionnaire in an effort to better assess the extent to which high school students understand a number of selected aspects of proof in mathematics. Forty percent of the respondents found it unnecessary to prove a mathematical proposition that was found intuitively obvious (Williams, 1976, p.181).

Perceptual psychologists maintain that understanding is contained in perception inasmuch as that which is an element of understanding is also an element of perception.

> "... Differentiations in the perceptual field resulting in perceptions of seeing, hearing, smelling, or feeling are in our theoretical perspective fundamentally the same as those made in conceiving, knowing, or understanding" (Combs, et. al., 1976, p.17).

Further, "All behavior, without exception, is completely determined by and pertinent to the perceptual field of the behaving organism" (Combs, et. al., 1976, p.20).

From a perceptual point of view, behavior is a symptom of one's perceptual field. Consequently, it is likely that the way one perceives proof will affect one's behavior. In this framework it is natural to investigate students' perception of proof, including understanding.

Related to the problem of students' perception of proof is the work of P.M. van Hiele and the late Dina van Hiele-Geldof. Izaak Wirszup discusses five levels of thought development in geometry resulting from Russian post-experimental descriptions of the van Hiele levels (Wirszup, 1976).

Students at the point of transition previously described may be thought of as making a transition between what van Hiele calls level 2 and level 3. A student at level 2 is

> "... able to discern the possibility of one property following from another, and the role of definition is clarified ... However, at this level the student still does not grasp the meaning of deduction as a whole" (Wirszup, 1976, p.78).

A student at level 3 is able to grasp

> "the significance of deduction as a means of constructing and developing all geometric theory. The transition to this level is assisted by the pupils' understanding of the role and the essence of axioms, definitions and theorems; of the logical structure of a proof; and of the analysis of the logical relationships between concepts and statements" (Wirszup, 1976, p.78).

This research was developed in light of these definitions of perception and the problem of students perception of proof at the undergraduate level in mathematics. It is the plan of this study to use the items developed by Williams, and follow-up interviews to assess junior-level university mathematics students' perception of selected aspects of proof, identify variables related to proof in their perceptual field, and investigate the relationship between their perception of proof and achievement in one of the first courses involving proof at Oregon State University.

Need for the Study

There appears to be a transition point in the undergraduate curriculum for mathematics students occuring near the end of the sophomore and beginning of the junior year (Sentilles, 1975), (Smith, et.al., 1983). This is when students of mathematics generally finish a relatively concrete, practical calculus sequence and begin the study of more abstract mathematics, usually Advanced Calculus or Abstract Algebra. There has been a high attrition rate in MTH 311, Advanced Calculus, one of the first courses heavily involving proof at Oregon State University. This suggests that undergraduate mathematics students may not be ready for their first course involving the use of formal proof in undergraduate mathematics.

To construct proofs and understand the nature of mathematical proof and its techniques are essential objectives for the undergraduate mathematics major (CUPM, 1963). Yet, very little is known about the nature of students' perception of proof at this transition point in the undergraduate curriculum. Little is known about the relationship between a student's perception of proof and his or her growth and success in elementary theoretical courses such as Advanced Calculus.

In conversation, particularly with university mathematicians, a question commonly brought forth is: "Why should we investigate a student's perception of proof at this point at all? Proof is what success in Advanced Calculus is all about. Of course students with a good apprehension of proof will succeed more than those without apprehension of proof."

If students who will have problems in Advanced Calculus might be identified beforehand, then steps can be taken to alleviate these problems. Students might be advised and treated in order that they not fail in their first attempt at a course involving proof. This would save student, university, staff and taxpayers time, money and effort. Further, while it would seem that the more capable student would have a clearer understanding of proof as well as excel in Advanced Calculus- is there no room for doubt? It might be that the "practically oriented" calculus sequence weeds out the more "theoretically oriented" mathematics students, thus diminishing the correlation between success in Advanced Calculus and a clear understanding of proof at the outset of the course. If this stretches our credulity, then it may be best to recall that the role of research is often to confirm what is already suspected.

In the process of investigating the perception of proof of mathematics students who are beginning the study of abstract mathematics, useful information can be gathered concerning the validity of the questionnaire items used. There exists the potential to further investigate the construct validity of the items by correlating the results of their administration with a theoretical outcome of a proper understanding of selected topics related to proof; that is, success in Advanced Calculus. Further, the predictive ability of the items with respect to success in Advanced Calculus can be investigated since the population that is being sampled has had no formal introduction to techniques of proof in a systematic way at the university level. This measure of predictive ability is used to establish criterion-related validity (Gronlund, 1981), (Isaac and Michael, 1982).

In summary, the needs for the present study are as follows:

1) little is known about the nature of undergraduate mathematics students' perception of proof in general and, more specifically, their perception of selected aspects of proof when beginning the study of abstract mathematics;

2) research can yield information concerning the relationship between a student's perception of proof and his or her success in one of the first

courses involving proof in the undergraduate curriculum;

3) further research into the criterion-related and construct validity of the 12 items developed by Edgar Williams can provide useful information relating to its utility as a predictor of success in Advanced Calculus;

4) there has been a high attrition rate in beginning courses with an emphasis on proof, such as Advanced Calculus, at Oregon State University. This suggests that a better understanding of students' perception of proof may be needed for these courses to be more effective; and

5) there exists the potential for effective analysis, diagnosis, advisement, and treatment of mathematics students facing a difficult transition from concrete to abstract mathematics.

Statement of the Problem

The purpose of this research is to investigate undergraduate mathematics students' perception of a number of selected aspects of proof, the relationship between their perception of proof and achievement in first quarter Advanced Calculus, and to investigate the effectiveness of the items used as a predictor of success in Advanced Calculus.

The problems stated as questions follow:

1) What is the nature of the perception of proof of undergraduate mathematics students who have completed the prerequisites for and have enrolled in Advanced Calculus at Oregon State University?

2) Is there a relationship between a student's perception of selected aspects of proof upon entering the first quarter of Advanced Calculus and his or her achievement in Advanced Calculus?

3) What is the relationship between success on the 12-items developed by E. Williams (1976) and achievement in Advanced Calculus?

The selected aspects of students' perception of proof being investigated are as follows:

1) the need for proof in mathematics;

2) inductive argument and its inadequacy in supporting mathematical generalizations;

3) the role of definition and postulate;

4) indirect proof;

5) the fact that a single counter-example is sufficient to disprove a mathematical proposition;

6) the logical equivalence of a statement and its contrapositive; and

7) the fact that a statement and its converse are not logically equivalent (Williams, 1976).

Hypothesis Statements

The major hypothesis stated in the null form is:

There does not exist a relationship between the perception of proof of undergraduate mathematics students entering Advanced Calculus and their achievement in Advanced Calculus.

More detailed and specific hypotheses will be stated in chapter three.

Assumptions

The following assumptions are intrinsic to this study:

1) the items developed by Edgar Williams (1976) are valid and reliable measures of a student's understanding of the selected topics related to proof listed above;

2) the total points accumulated by students in Advanced Calculus in the fall quarter of 1984 in MTH 311 at Oregon State University are a valid and reliable measure of achievement; and

3) students will respond in an honest way to the questions in the interview.

Definition of Terms

1) Mathematical proof: The process of showing by means of an assumed logical process that what is to be proved follows from certain previously proven or axiomatically accepted propositions.

2) Perception: Any differentiation a person is capable of making in his or her perceptual field whether or not an objectively observable stimulus is present.

3) Perceptual Field: The entire universe, including himself, as it is experienced by an individual at the instant of action.

4) Differentiation: The rise of new characters into figure and the consequent lapse of other characters into ground in an individual's perceptual field.

5) Understanding of selected aspects of proof: In this study, the score received on each situation of the written questionnaire developed by E. Williams.

6) Advanced Calculus: The course offered at Oregon State University with the title MTH 311.

7) Achievement in Advanced Calculus: The total number of points accumulated by a student in MTH 311 at Oregon State University.

8) Proposition: A statement about mathematical entities.

9) Axiom/Postulate: A proposition that is to be accepted without proof.

10) Theorem: A proposition to be proved upon the basis of certain given hypotheses.

11) Counter-example: The method of disproving a theorem in which a single case is presented yielding a true hypothesis and a false conclusion.

12) Induction: Drawing conclusions from several known cases; reasoning from the particular to the general.

13) Converse: The theorem resulting from interchanging the hypothesis and conclusion.

14) Contrapositive: The theorem resulting from negating the hypothesis and conclusion and then interchanging them.

15) Indirect proof: The method of proof which supposes that the contrary to the fact to be proved is true and then shows that this supposition leads to an absurdity.

16) Hypothesis: An assumed proposition used as a premise in proving something else; a condition; that from which something follows.

17) Conclusion: The statement which follows as a consequence of the hypotheses of a theorem.

Limitations

This investigation is limited:

1) to mathematics students at Oregon State University who have successfully completed the prerequisites for and have enrolled in MTH 311 (first quarter Advanced Calculus) in fall quarter 1984;

2) by the degree of honesty that the students use when answering questions in the interviews; and

3) by the degree to which the questionnaire items are valid and reliable measures of students' perception of proof.

Delimitations

The following delimitations apply to the study:

1) the items of the questionnaire are not intended to test knowledge of mathematical concepts;

> "In selecting the content and wording of each item, it was hoped that most

> students could respond to the items in a meàningful way without having to rely on specific knowledge that may have been aquired in, for example, the study of plane geometry. In particular, it was considered important that students not perceive the items as a test of their knowledge but rather that they respond in a spontaneous manner using the first thoughts that come into their mind, thus maximizing the chances of obtaining student responses indicative of their subjective thought processes" (Williams, 1976, p.35).

2) the instructors and instruction at Oregon State University are not being evaluated;

3) the course MTH 311 is not being evaluated;

4) this study is not intended to solve the problem of teaching proof. Learning about mathematical proof is a lifelong endeavor.

Sample

The sample is composed of 47 students at Oregon State University who are enrolled in MTH 311 during the fall of 1984.

Instrumentation

To determine their perception of selected aspects of proof, each student in the study will be administered an instrument containing six items. These items were developed by E. Williams of The University of Alberta, Edmonton (1976). Williams developed twelve "mathematical situations" in the form of a dialogue between two people where some disagreement usually occurs about the nature of mathematical proof. The student is asked to respond to six of these situations in writing, usually by siding with one person or the other in the dispute and then defending his or her decision.

Their responses to each item will be put into response categories by three independently working judges from the Department of Mathematics at Oregon State University. From this procedure a response distribution for each item is obtained reflecting the students' perception of selected aspects of proof.

Follow-up interviews are also planned. These interviews will be conducted by the researcher and used to further clarify the students' perception of mathematical proof.

Procedure

The research will take place over an eleven week period during the fall 1984 quarter at Oregon State University, Corvallis, Oregon. Twenty versions of the questionnaire, each consisting of six items, will be administered randomly to the sample of students. This will take place during a regularly scheduled fifty-minute class period for each class during the first two weeks of the quarter.

Midterm scores and final grades will be collected for each student. The results of the 12-items and midterm exams will be analyzed. On the basis of this analysis and pilot student interviews, a format for follow-up interviews of selected students in the study will be developed and administered. Eight students will be selected randomly from the population. Each student will be interviewed by the researcher. The results of the interviews will then be analyzed.

Analysis of Results

The results of the written questionnaire will be reported, and a 2x2 contingency table will be given that compares the proportions of successful students that score highly on each item. The follow-up interviews will be analyzed with respect to specific variables to be listed in chapter three. Also, the interviews will be used in an attempt to define and refine further variables in students perception of proof. Exploratory analysis is also planned.

Organization of the Remainder of the Study

Chapter II presents a review of the literature related to the study of students' perception of proof and its role in mathematics. Chapter III defines the methodology of this study. The analysis and results are reported in Chapter IV.

The summary, conclusions, discussions, and recommendations are presented in Chapter V.

II. RELATED LITERATURE

This chapter is divided into three main sections. In the first section, a brief historical setting for phenomenology and phenomenological psychology is developed. While this is not the place for a detailed account of philosophy since Descartes, some basic statements regarding the motivation for the work of the phenomenologists and the relevance of phenomenology to psychology are necessary for a more complete understanding of the theoretical background of the research undertaken here.

The second section of this chapter concerns the basic tenents of phenomenological or perceptual psychology as presented in the work of Arthur Combs. Included here are implications for research within the context of perceptual psychology.

The third section surveys literature related to students' perception of mathematical proof.

Historical Backround

There is an inherent difficulty in defining the phenomenological movement.

> "Even after it had established itself as a movement conscious of its own identity, it kept reinterpreting its own meaning to an extent that makes it impossible to rely on a standard definition for the purpose of historical inclusion or exclusion" (Spiegelberg, 1982, p.1).

There are, however, several major themes of phenomenology that the various branches and off-shoots of the discipline have in common. There are also two philosophers whose work will be mentioned in this section. These themes and the work of these philosophers combine to define the spirit of this research. The particular interpretations and implications of these themes shall be discussed in the second part of this chapter; the section concerned with the perceptual psychology of Arthur Combs.

The two philosophers were both students of the German philosopher Franz Brentano (1838-1917). They are Carl Stumpf and Edmund Husserl. Carl

Stumpf was ten years Husserl's senior and not considered a phenomenologist in Husserl's tradition but his phenomenology influenced modern psychology to a large degree. Edmund Husserl is considered the motive force in the development of phenomenology (Stewart and Mickunas, 1974).

The importance of Carl Stumpf (1848-1936) lies in "the role he played in introducing phenomenological methods into psychology and transmitting them to some of its most active researchers" (Spiegelberg, 1982, p.52). Among these were the gestaltists "and, indirectly, the new 'phenomenological psychology' of Donald Snygg and Arthur W. Combs" (Spiegelberg, 1982, p.52).

Stumpf believed that

> "we have to carry out experiments in imagination. But even the experiment in reality proves helpful, if not indispensable ... he referred to a fundamental capacity of our consciousness to grasp the general in the particular and the necessary in the contingent, something for which the old expression 'intuitive knowledge' would be acceptable if it were not loaded with so many misleading associations. Specifically, Stumpf wanted to keep out of the idea of merely passive staring at the phenomena. What he wanted was active exploration by a whole set of

> mental operations" (Spiegelberg, 1982, p.53).

Edmund Husserl (1859-1938) developed phenomenology after working in the foundations of logic, and his first papers appeared while the branch of philosophy concerned with foundations was experiencing a shift "toward nihilism, most dramatically portrayed in the writings of Nietzsche, the poetry of Rainer Maria Rilke, and the novels of Franz Kafka" (Stewart and Mickunas, 1974, p.18). Husserl declared that even

> "the positive sciences, after three centuries of brilliant development, are now feeling themselves greatly hampered by obscurities in their foundations, in their fundamental concepts and methods" (Husserl, 1977, p.4).

It was these problems in the foundations of philosophy and science and the consequences of the work of Rene Descartes that moved Husserl to work in philosophy.

In his work Meditations on First Philosophy, Rene Descartes separated reality into two very different substances, substances of the mind (thinking substance), and material substances (extended substance). After having accepted this duality, philosophers have since argued about the nature of the relationship between the two. In one sense, the phenomenological movement grew as a result of this debate revolving around the dual nature of mind and body.

By the beginning of the twentieth century this debate had generated the schools of thought called empiricism and psychologism. One of the ideas of the phenomenologists that inspired the methodology of this study was that these two major schools of thought were in err. The work of these two schools of thought can be traced directly to the dualism of Rene Descartes. To the phenomenologists way of thinking these groups placed undue emphasis on, respectively, the extended substance and thinking substance of Descartes. The positivists wished to reduce reality to quantitative representations of sense data. The psychologists would explain that the foundations of logic (and therefore science) could be accounted for in terms of psychological laws.

To the positivists and empiriscists Spiegelberg states: "the question is whether there is any good reason to restrict data to sense data, thus refusing access to any other possible data without even looking at their credentials" (Spiegelberg, 1975, p. 681).

Husserl critiques psychologism in his work, Logical Investigations (Husserl, 1970), and rejects it as leading to absolute relativism. For, if logic is ultimately founded in the psychic process of individuals, then there exists as many valid logical (and ethical) systems as there are individuals. Thus, both the phenomenology of Stumpf and Husserl reject strictly empirical techniques and accept, at least somewhat, intuition as a path to knowledge.

Phenomenology as a philosophy does not reject empirical techniques out of hand. Kockelmans states:

> "While I admit the possibility and importance of the empirical sciences of man, my major concern is with the question of whether in addition to the empirical sciences there could not be developed a science of man that would some how 'compensate' for the losses which the limitations essentially connected with an empirical approach necessarily entail" (Kockelmans, 1973, p.257).

The above theme will be found to run through the perceptual psychology of Arthur Combs as well.

Husserl felt that the driving forces emanating from Descartes' work had lost their vitality because of the loss of the spirit of philisophical self-responsibility. In other words, a philisophical turn was needed away from "naive objectivism" and back to "transcendental subjectivism" and the methods of cartesian doubt (Husserl, 1977, p.4). Merleau-Ponty states:

> "Husserl tried to discover a way between logiscism and psychologism. By a truly radical reflection, which reveals the prejudices established in us by the external environment, he attempts to transform this automatic conditioning into a conscious conditioning" (Merleau-Ponty, 1973, p.53).

He continues:

> "He must find a way of knowing which is neither deductive nor purely empirical. This knowledge must not be purely conceptul in detaching itself from facts. Nevertheless it must be philosophical, or at least it must not make the existence of a philosophizing subject impossible. It is essential that our life should not be reduced exclusively to psychological events and that in and through these events there should be revealed a meaning which is irreducible to these particularities. This emergence of truth in and through

the psychological events is what Husserl called Wessensschau, the intuition of essences" (Merleau-Ponty, 1973, p.59).

In summary, phenomenology can be viewed as an attempt to alleviate some of the difficulties in the foundations of philosophy and science. It seeks to do this by a return to experiencing "the things in themselves" without prior prejudices. Husserl saw the phenomenological "intuition of essences" as what should be the first step for all ways of knowing. It is transcendental in that this "intuition" is a "peculiar, mystical operation that transports us beyond empirical facts" (Merleau-Ponty, 1973, p.59). It is subjective since the foundations of all fields are inadequate without tracing them back to their subjective roots.

Phenomenology can also be viewed as a reaction to the cartesian duality of mind and body. As seen in the words of Stumpf (Spiegelberg,1982), Kockelmans (Kockelmans, 1973), Merleau-Ponty (Merleau-Ponty, 1973), Spiegelberg (Spiegelberg, 1975), (Spiegelberg, 1982), and, later in this chapter, Combs (Combs, et. al., 1976) the methods of phenomenology are diverse. Phenomenologists reject a strictly empirical or intuitive approach to understanding phenomenon while accepting the validity, even necessity, of both.

Is it possible that philosophical phenomenology has relevance to psychology? Spiegelberg argues in the affirmative. Among the considerations offered by Spiegelberg are the following:

1) Unless, like the behaviorist, we abandon consciousness, psychology must describe intentional structures of consciousness in experience (whether matched by physical counterparts or not).

2) Phenomenology has relevance to the foundational problems of psychology. Its methods may make systematic the introduction and use of definitions. That is, psychological definitions may be derived

> "from what is called, perhaps a little pretentiously, essential insights, or a little more concretely, from grasping the essential types that can be intuited on the basis of a systematic variation of the observed phenomena" (Spiegelberg, 1975, p.255).

Further, Spiegelberg argues that phenomenology and cognitive field theory have a chance for cooperation inasmuch as the psychologists' cognitive field may be described in phenomenological terms. Consider the following definitions:

1) Kurt Lewin's life-space- "the totality of facts which determine the behavior of an individual at a certain moment" (Lewin, 1936, p.12).

2) Combs' perceptual field- "the entire universe, including himself, as it is experienced by an individual at the instant of action" (Combs, 1976, p.22).

Comparing these with Husserl's lifeworld (lebenswelt), an "individual with the ego at its center, as distinguished from the uncentered objective world of Galilean science, which, however, was supposed to have sprung from it" (Spiegelberg, 1975, p.6), one gets the feeling that they are describing the same thing, digging in the same tunnel.

Even in anthropology, the often defined word "culture" takes on a similar tone: behaving individuals in a material universe.

> "Culture consists of human ideas together with their derived behavioral and material manifestations learned by human beings and providing them the capability to persist as a species to organize for collective action, to communicate symbolically and to create new patterns for living" (Hogg, 1984).

Nevertheless,

> "as a basis for psychology, phenomenology is characterised by its epistemological radicalism, its opposition to a mere imitation of exact science in the study of man, and its determination to place experimental research in the realm of a completely new conceptual framework" (Thines, 1977).

In the next section, one such conceptual framework will be described along with some of its methodological implications.

Perceptual Psychology

In the United States, the original plea for the new frame of reference for psychology that we speak of here occurred in the early 1940's (Snygg, 1941). In this journal article Donald Snygg stated some basic postulates for phenomenological psychology. As the result of eight years of experimentation these postulates were expanded and explained more fully by Donald Snygg and Arthur W. Combs (Snygg and Combs, 1949). Their work went through a second edition (Combs and Snygg, 1959) and more recently a rewriting (Combs, Richards, and Richards, 1976).

Primary among the postulates of perceptual psychology is that at any given moment an individual's behavior is completely determined by the perceptual field of the behaving organism. The perceptual field is defined as the entire universe, including himself, as it is perceived by the individual at the moment of behaving (Combs, et. al., 1976), (Snygg and Combs, 1949), (Combs and Snygg, 1959), (Combs, Blume, Newman, and Wass, 1974), (Combs, 1982). Thus, an individual's actions depend upon no more or less than that individual's perception of the world around him at the instant that those actions take place. Perception, as it is used here, is intended to mean more than the collection of sense data by an individual. It is the result of an object or concept rising into figure or fading into ground. This rising and fading is known as differentiation and can occur whether or not an objectively observable stimulus is present. Hence, an individual can have perceptions of abstract entities such as hope, fear, or mathematical proof as well as physical objects.

> "Thus perceptions and the interrelationships among perceptions and behavior are the data with which the science of perceptual psychology is primarily concerned. Furthermore, since perceptions or meaning are not open to direct observation, the methods of perceptual research are often subjective or inferential, and the observer himself is often employed as an instrument of research. The problems of observer bias for perceptual psychology are not resolved by eliminating the observer, but by making the observer as reliable an instrument as possible" (Combs, et. al., 1976, p.368).

True to the tradition of phenomenology, the techniques and procedures of "non-perceptual" psychology are not meant to be overthrown, merely incorporated.

> "Perceptual psychology is not a denial of former psychologies ... it provides us with an additional explanation of particular value to practitioners and to those of us who are confronted with the practical problems of dealing with people, not as subjects in an experiment but, as striving, seeking human beings. It does not deny what we have known before. It extends beyond to give us a new string to our bow" (Combs, Avila, and Purkey, 1971b, p.118).

That new string is a frame of reference with the behaving individual at the center. If behavior is a function of the perceptual field of the behaver, then it stands to reason that the factors influencing the individual's perception should be the objects of investigation. Further, the influencing factors of perception should be understood from the point of view of the behaver.

One of the variables affecting perception that has a particularly strong effect on this research is that of self-concept or phenomenal self (Snygg and Combs, 1949), (Combs and Snygg, 1959), (Combs, Blume, Newman, and Wass, 1974), (Combs, Richards, and Richards, 1976), (Combs, 1982). The phenomenal self is that part of the perceptual field that refers to the individual. That is, "all those aspects of the perceptual field to which we refer when we say 'I' or 'me'" (Combs, Avila, and Purkey, 1971, p.120).

When describing characteristics of a particular self, several frames of reference may be taken, including from the individual himself (Combs, et.al., 1976, p.155). An individuals perception of his or her self is an important part of the perceptual field that Combs calls the self-concept (Combs, et.al., 1971b). Since an individual's perceptual field, and hence his or her self-concept, cannot be studied directly but only via perceptions, indirect methods of study must be utilized. One of these is known as the self-report. Care must be taken not to confuse the self-report with the self-concept. "The self-report is a behavior, the self-concept is a system of beliefs. Clearly these matters are not the same" (Combs, et.al., 1971b, p.52).

Some of the sources for error in the self-report are:

1) the degree of clarity of the subjects' awareness. The topic of interest will be reacted

to differently if it is figure rather than ground in the perceptual field of the subject;

2) the availability of adequate symbols to express oneself; and

3) social expectancy (Combs, et.al., 1976, p.373).

Thus, if the perceptual field of an individual is to be the object of study, indirect methods must be used. Among these methods may be included the direct observation of behavior in or out of a laboratory situation, or the evaluation of an individual's self-report in the form of a written report or an oral interview. In any case, inference plays an important role.

Related Research in Mathematics Education

Proof, the key element of the axiomatic method, has been the focus of much attention. The earliest example of a system relying strictly on the axiomatic method is The Elements of Euclid (See Heath, 1956).

As described in Chapter I, to prove a theorem is to show that it is a necessary consequence of previously proved theorems. Each theorem is based on those that have been proved before. To prevent this from being an infinite regression, certain propositions are assumed to be true without proof. These propositions are called axioms.

There are many accounts of the deductive method and the role of proof in mathematics. Some of these accounts are informal (Courant and Robbins, 1941), (Apostol, 1967), (Polya, 1973), some formal (Tarski, 1965), and others instructive (Greenberg, 1980), (Sentilles, 1975), (Solow, 1982), (Smith, et. al., 1983). Bell describes three roles of proof in mathematics.

> "The first is verification or justification, concerned with the truth of a proposition; the second is illumination, in that a good proof is expected to convey insight into why the proposition is true; this does not affect the validity of a proof, but its presence in a proof is aesthetically pleasing. The third sense of proof is the most characteristically mathematical, that of systematisation, i.e. the organisation of results into a deductive system ..." (Bell, 1976, p.24).

Seen from an international perspective, the presentation of proof in the classroom displays wide variations from country to country (Bell, 1976).

> "Underlying this divergence lies the tension between awareness that deduction is essential to mathematics, and the fact that generally only the ablest school pupils have achieved understanding of it" (Bell, 1976, p.23).

As varied as the approaches are to presenting proof in the classroom, so are the approaches to evaluating students' understanding of proof.

In his study, Bell provided 14 and 15-year old mathematics students with a series of numerical and geometrical problems. These problems required the students to provide an explanation and justification of a generalization. Their responses were then categorized and analyzed (Bell, 1976).

As has been mentioned, and as will be described in more detail in Chapter III, E. Williams developed a written questionnaire consisting of twelve "mathematical situations" (Williams, 1976). These situations were designed to elicit subjective responses about selected aspects of the proof domain. The students' responses were categorized by several judges and, based on the categories they were placed in, assigned a point score from 0 to 3. This score is to reflect no, low, medium, or a high level of understanding of the topic addressed by the situation.

Another type of questionnaire was developed by Vinner to assess high school and college students concept of definition (Vinner, 1976). Seven sentences were presented to the students and it was asked whether each sentence was an axiom, a postulate, a theorem, a fact, a definition, or other. The results were analyzed on the basis of the ability of the students to correctly identify the given statements as definitions.

Interviews were used by Galbraith as a tool for assessing 13-15 year-old students perception of the proof process. Rather than assigning numerical values to the students' responses, Galbraith's goal was to identify "clusters of mathematical reasoning characteristics within categories and items" (Galbraith, 1981).

Besides addressing aspects of proof, one theme of the above studies is the balance between the objective and the subjective; quantification and qualitative analysis (in fact, to the phenomenologist, all research is subject to this tension). The assumptions implicit in so-called "objective" research techniques have come under increasing fire both in the west, (Hoffmann, 1964), (Houts, 1977), (Gould, 1981) and in the east (Krutetski, 1967). Coupled with this is an increase in interest in clinical or naturalistic techniques in both mathematics education, (Easley, 1977) and science education (Welch, 1983).

Even a major opponent of clinical studies has reversed his position:

> "After all, man is, in his ordinary way, a very comptent knower, and qualitative common-sense knowing is not replaced by quantitative knowing. Rather, quantitative knowing has to trust and build on the qualitative, including ordinary perception" (Campbell, 1975).

That is a phenomenological statement.

Like the phenomenologists, educators advise that quantitative research not be abandoned, and that the dangers of qualitative study be recognized. In a recent article, Welch mentions a few problems with "naturalistic inquiry" (Welch, 1983). The first of these is the problem of objectivity. That is, can it be said that we have learned something when two competent researchers come to different conclusions based on the same evidence? Here Welch alludes to a story concerning a well-known anthropologist and his colleague coming to markedly different conclusions in two field studies of a primitive people (this is most likely the account Campbell gave of A. L. Kroeber and E. H. Erikson and their differences in describing the Yurok Indians of Northern California, see: Campbell, 1975, p.183).

Secondly, studies of a "naturalistic" nature are time-consuming, expensive and often carried out by poorly trained researchers.

Thirdly, these studies are coming into fashion. Welch argues that researchers "need to cautiously try this alternative approach, improve our skills and develop new procedures" (Welch, 1983, p.101). Merely jumping on the bandwagon will not do.

A mathematics educator echos these thoughts:

> "Research in mathematics education seems to be undergoing something of a revolution. When this journal began publication, the experiment was seen as the ideal research form for addressing questions in our field. Today naturalistic observation is becoming the ideal, and controlled experimentation is widely viewed as discredited. A major problem is that, whereas the canons for laboratory investigations are clear and well-developed, the canons for field investigations are just beginning to be laid down" (Kilpatrick, 1986).

We, as investigators, must be sensitive to the problems that are built into the methods and techniques we use.

Summary

The main theme of this chapter has been the roles of quantitative versus qualitative techniques in scientific inquiry. A brief historical account of phenomenology has been given with the main focus on its justification of the use of qualitative as well as quantitative techniques in such inquiry.

In this chapter, Spiegelberg's arguments (Spiegelberg, 1975) in support of the relevance of philosophical phenomenology to psychology were stated and the basic tenents of the perceptual psychology of Donald Snygg and Arthur Combs were outlined. It should be noted that there exists a line of influence from Husserl and Stumpf through Lewin and Snygg to Combs.

Recent investigations into students' perception of topics related to proof were presented and their qualitative flavor noted. Further, the "myth" of objectivity has been discussed and possible benefits of "subjective" techniques listed. It must be realized that so-called "naturalistic investigations" that utilize the researcher as an instrument are filled with difficulties and must be approached with some awareness of the possible sources of error. Possible sources of error in "subjective" research have been listed in the section on perceptual psychology and some difficulties listed in the final section.

III. THE STUDY

Population

The sample was taken from the group of those students who took MTH 311, Advanced Calculus, during the fall quarter of 1984 at Oregon State University.

The Department of Mathematics offers a major of applied mathematics as well as pure mathematics. Along with students of both of these majors, MTH 311 is taken by engineering majors, pre-engineering students, some statistics majors, and an occasional science student.

The title of the course MTH 311 is Advanced Calculus. The textbook used was Elementary Analysis: The Theory of Calculus by Ross (Ross, 1984). The analysis sequence is described as follows in the Oregon State University General Catalog:

> "Foundations of one variable calculus including uniform convergence, uniform continuity and interchange of limits. An introduction to functions of two and three variables: differentiation, chain

rule, inverse and implicit function theorems, and Riemann integration. Examples and applications."

MTH 311 is the first quarter of the three quarter analysis sequence and has 16 hours of calculus for prerequisites including: differential calculus, integral calculus, vector calculus, sequences and series.

The approach taken is formal and rigorous in this junior level course in the university mathematics curiculum.

Sample

With the cooperation of the faculty at Oregon State University, two entire sections of MTH 311 were selected to participate in the study. Table 1 indicates the breakdown by class of the students that responded to the questionnaire.

Table 1 - Students Responding

CLASS	STUDENTS ENROLLED	WAITING LIST	POSSIBLE POOL	TOTAL
A	26	3	29	21
B	25	4	29	27
TOTAL			58	48

Of the 48 students that responded to the written questionnaire, scores in MTH 311 were unavailable for two. These students were not included in the study. One student in class B that did not respond to the written questionnaire was later interviewed. This set the total number of students that participated in the study at 47.

There were seventeen males and four females in class A, and twenty-four males and three females in class B. There were seventeen students from Oregon in class A, and thirteen students from Oregon in class B. There was one foreign student in each class. In class A, there was one student from southeast United States; in class B, there were four students from the western United States. Data were not available for two students from class A, and nine students from class B. Tables 1, 2, 3, 4, and 5 summarize some of the other characteristics of the students who responded to the written questionnaire:

Table 2 - Major

MAJOR	CLASS A	CLASS B
Mathematics	4	3
Mth. Sciences	5	4
Computer Science	4	3
Engineering	1	7
Science	4	1
Statistics	0	4
Physics	1	1
Geology	1	0
Mth. Education	0	1
Oceanography	0	1
General Science	1	0
TOTAL	21	25

Table 3 - Class Standing

YEAR	CLASS A	CLASS B
Graduate	0	7
Senior	13	9
Junior	6	8
Sophomore	2	2
TOTAL	21	26

Table 4 - Ages

AGE	CLASS A	CLASS B
30 and above	2	2
28	1	0
27	0	1
26	0	2
25	1	1
24	2	1
23	1	4
22	5	3
21	2	2
20	4	2
19	1	0
TOTAL	19	18

Table 5 - Grade Point Averages

AVERAGE	CLASS A	CLASS B
3.5-4.0	2	2
3.0-3.49	3	4
2.5-2.99	10	3
2.0-2.49	1	3
TOTAL	16	12

Grade-point averages were only available for those students in the college of science. Scores of students who were in engineering, computer science, and education were not available.

From these tables it can be seen that the two classes are similar with respect to major, class standing, ages, and grade-point average.

Instrumentation

The procedure described in this chapter was developed to describe mathematics students' perception of selected aspects of mathematical proof and to investigate the relationship between their perception of proof and success in MTH 311. There are three major steps that occur in the following chronological order. First, a written questionnaire was administered to the sample. Secondly, a pilot study was undertaken. In this stage, an interview script was developed and sample written questionnaires were administered. The sample written questionnaires were used in the training process for the judges. The third stage consisted of interviews of students from the main study.

From previous experience and a review of literature related to perception, the combined use of a written questionnaire and follow-up interviews was selected for this description and investigation. This "two-pronged" attack of the problem was chosen for the following reasons:

1) It was felt that using this approach presented the opportunity for the two techniques to be used in a complementary manner. That is, questions that arose during the administration and

analysis of the written questionnaire could be explored further in an interview setting.

2) The self-concept is an important factor in the theoretical framework of Perceptual Psychology. It was determined that the self-report in the form of a clinical interview and responses to a written questionnaire would provide data from which selected aspects of the nature of students' self-concept could be inferred.

Written Questionnaire

A written questionnaire (See Appendix A) developed by Edgar Williams of the University of Alberta, Edmonton was chosen to be used in this study (Williams, 1976). This questionnaire appeared to be developed in the spirit of perceptual psychology. It consists of twelve "mathematical situations" in the form of a dialog between two people where some disagreement usually occurs about the nature of mathematical argument. The student is asked to respond to each of these situations in writing, usually by siding with one person or the other in the dispute and then to defend his or her decision. Among the concepts the instrument addresses are: inductive/deductive reasoning, indirect proof, basic logic (converse, inverse and contrapositive), counter-example and the significance of hypotheses in mathematical argument.

During the process of validation Williams developed a number of items "which collectively were designed to assess student understanding of a wide variety of concepts in the proof domain" (Williams, 1976, p.15). Four judges examined the items with respect to content, wording and over-all appropriateness with the result of deletions and modifications. Williams then pilot-tested six instruments using the following guidelines:

1) Subjective thought-processes rather than rote knowledge was sought.

2) The investigator and judges were looking for the proper wording and format to elicit such responses.

3) The usability of each item was to be evaluated.

4) The time to be alotted for each item was to be determined.

After the pilot study, all items were then judged again by Williams, and the final twelve items were constructed. These items were then used to construct 8-item versions of an instrument that was administered to college-bound high school seniors in the Edmonton area.

The twelve items developed by Williams were chosen for the following reasons:

1) They were designed to maximize the possibility of obtaining responses indicative of the subjective thought-processes of the students that respond to it. This is concordant with the objectives of perceptual research as stated in chapter two.

2) They are easy to administer and the scoring, while not particularly easy, is explicitly stated.

3) They supply a numerical score allowing statistical analysis.

4) The extensive pilot-testing, judging, and testing of the items on the questionnaire by

Williams provide evidence that they are valid and reliable measures of the aspects of mathematical proof that they were designed to measure.

Since the items were developed for high school seniors who were taking college preparatory mathematics classes in the Edmonton area, the level of difficulty was deemed appropriate for university-level mathematics students in their first university course involving proof as well. Neither the students in the main study nor the students from the pilot study who responded to the written questionnaire expressed any difficulty in the reading level.

Interviews

The purpose of the follow-up interviews in this study was two-fold. First, the purpose was to investigate selected affective aspects of students' perception of mathematical proof and attempt to identify and refine other variables in the perceptual field of the interviewees that might have an effect on achievement in advanced calculus.

Specifically, the interviews were seeking to assess:

1) The students' subjective perception of the nature of mathematical proof and the role of proof in mathematics.

2) The degree to which students enjoy mathematical proof.

3) The degree of differentiation made by the student between proof and problem solving.

4) The amount of confidence the students have in their ability to construct proofs.

Secondly, the purpose of the interviews was to further probe responses to selected items from the questionnaire previously administered. Because of the varied responses received during the pilot-study, the finite geometry of situation five was of particular interest (Williams, 1976). Part of the interviews were intended to investigate the nature and breadth of the students' rejection of the four-point geometry.

Each interview was intended to go through two stages (Konold and Well, 1981). The first stage contained in-depth probes designed to address one or more of the four previously mentioned objectives. In the second stage, think-aloud type probes were designed to see if the interviewee will accept the four-point geometry of situation five of the written questionnaire.

The process of constructing and validating the interviews began with a pilot study conducted three weeks into the quarter. Six students from another class volunteered to respond to the written questionnaire and to be interviewed by the researcher. These students were enrolled in a mathematics class for prospective teachers which had an introduction to proof as one of its objectives. Since these students had a similar mathematics backround to those students in the main study, it was deemed appropriate to use these students in the development of the interview script.

Six questions were selected from the pool of twelve to be given to each of the volunteers. They were allowed to respond to this version of the questionnaire during a fifty-minute period of their choosing at the Mathematical Sciences Learning Center at Oregon State University.

The interview script was developed on the basis of a preliminary analysis of the responses to the written questionnaires, the pilot interviews and suggestions from a panel of judges representing mathematics, mathematics education and education (See Appendix D).

The pilot interviews were conducted by the researcher, taped, and approximately one half hour in length. The researcher had little experience in interview techniques and was therefore in contact with two faculty members of the Department of Mathematics who had such experience.

Procedure

The first part of the procedure involved the administration of questionnaires to the students. To determine their perception of selected aspects of proof in mathematics, each student in the study was administered six items to respond to.

Because of the shorter time period available (50 minutes compared to 80 minutes that Williams used for his 8-item version of the questionnaire) the number of situations to be presented in one questionnaire was set at six. Time did not appear to be a factor when the students from the pilot study responded to the six situations in 50 minutes; nor did it appear to be a factor in the main study. It was then decided to use all twelve of Williams' situations to obtain responses addressing a wider range of aspects of mathematical proof. Twenty versions of the questionnaire were constructed, each containing six situations for the students to respond to. The questionnaires were constructed by pairing off situations one and two, three and four, and so forth, concluding with eleven and twelve. This yielded six groups of two questions each. The twenty versions of the questionnaire resulted by taking all possible three-group combinations of the question pairs.

These questionnaires were then administered randomly to the two sections of advanced calculus on the same day, two weeks into the quarter. No warning was given to the students that anything out of the ordinary was to occur until that day.

The questionnaires were administered during the regularly scheduled class times, 8:30 AM for Class A and 2:30 PM for Class B (See Table-1).

The instructor introduced the researcher to the class and assured the students that what was about to follow would have no influence on their grades. The following words were then read to the students by the reseacher:

> "My purpose in being here today is to present you with some mathematical situations which you can read and think about. In general, each situation consists of a dialog between two hypothetical mathematics students. After reading the discussion in each situation, you will find several questions related to that situation. I wish to emphasize that these questions are not to be considered as some sort of test or examination. In fact, your responses to these items cannot be classified as right or wrong. The whole idea is to find out how you relate to the situations presented and to let you present your own ideas and thoughts not those of someone else. As a result you will be assisting myself and the mathematics department. Thank you."

The questionnaires were then passed out, and the directions read aloud to the students. They were further reassured that their responses could not be classified as right or wrong, asked to raise their hand if they had a question and told that they had approximately eight minutes to respond to each situation. Time did not appear to be a problem.

Over the course of the following four weeks, an interview script was developed on the basis of a preliminary analysis of the responses to the written questionnaires, the pilot interviews (described above), and suggestions from a panel of judges. For purposes of validity and reliability the researcher was to conduct all interviews and conduct those interviews using the same interview script. Thus, the final version of the interview script was constructed by the time the last interview of the pilot study was conducted.

The advanced calculus classes were stratified into three groups on the basis of their score on their first midterm. The stratifications were based on high, medium or low raw scores on that examination. Nine students were then selected at random so that each stratified group and each advanced calculus class were equally represented. These nine students were then contacted by a letter distributed by their respective advanced calculus professors (see Appendix C). Because of schedule problems one student was not interviewed, making the total number of interviewees eight.

Over a three-week period, a one-half hour interview was conducted with each of the eight students. Each of these interviews was conducted by the researcher. They were audio taped for later analysis and each was approximately one half hour in length.

To increase validity and reliability, the interviews began informally with the tape recorder off. This approach was used to ease tensions that sometimes arise in recorded interviews. The subjects were informed that their subjective thoughts on mathematical argument were of interest rather than their skill in constructing proofs. Further, they were provided earlier with an interview request form (see Appendix C) which states that the interviews had nothing to do with their grade, and the results were to remain confidential.

In the first phase of the interviews, questions from the interview script were read by the interviewer, and verbal responses were given. Care was taken that the interviewee had ample time to respond and understood the wording of the questions. For example, in one interview the student did not understand the word "postulate", but did understand the word "axiom". For the most part, response time and understanding did not seem to be a problem. Topics not explicitly forseen by the interview script were occasionally pursued.

In the second phase of the interviews, after the in-depth probes designed to address the previously mentioned objectives, a copy of situation five from the written questionnaire (See Appendix A) was produced and each interviewee was given some time to read and think about it. Paper and pencil were provided to the interviewee at this point for drawing figures if he or she wanted. Some students used the pencil and paper, others did not. Questioning by the interviewer then resumed.

The interviews were mainly conducted in a conference room at Oregon State University. The interviewer and interviewee were seated at a table across from one another with a tape recorder on the table in full view. As a result of a schedule problem, one interview was conducted, under similar conditions, in the office of a faculty member.

Analyses of Data

The responses to the written questionnaire were evaluated by three judges working independently. The judges were trained by the researcher using the response categories developed by Williams for each situation (Williams, 1976). The responses to the written questionnaire given to the students in the pilot study were then evaluated and scored by the judges. As a result of this training, most of the differences in interpreting Williams' response categories were resolved.

Each of the responses of the main study were then placed into response categories by the judges, each working independently. A numerical value of either 0,1,2, or 3 was given to each response on the basis of the response category it was placed in. This assignment of numerical values was also determined by Williams (see Appendix B). A meeting of the judges was held in order to resolve differences of opinion on the assignment of the responses to response categories. The number of differences between the judges in the assignment of responses to response categories that actually involved a difference in the assignment of a numerical value was relatively small. This was possible since more than one response category was often assigned the same numerical value. For example, in situation five, responses assigned to categories 0,1, and 2 received a numerical value of 0; category 3 received a numerical value of 1; categories 4 and 5 received a numerical value of 2; and categories 6 and 7 received a numerical value of 3.

The following hypotheses were tested:

1) The correlation between total score obtained on the written questionnaire and achievement in Advanced Calculus for class A is not

significantly different than the same correlation for class B.

2) There is no significant proportion of variation in achievement that is associated with perception of proof.

3) There is no association between achievement in advanced calculus and perception of the aspects of proof addressed by each situation on the written questionnaire.

For hypothesis 1, two independent estimates of the correlation between achievement in Advanced Calculus and total score were computed using the correlation coefficient, r (Snedecor and Cochran, 1980, p.477). The first estimate was for class A, the second for class B. These were then converted to z-scores and the significance of the difference between the two zs was tested (Snedecor and Cochran, 1980, p.186).

The correlation coefficient and its level of significance was calculated to test hypothesis 2 (Snedecor and Cochran, 1980, p.477).

For hypothesis 3, a 2x2 contingency table was constructed for each situation (Snedecor and

Cochran, 1980, p.200). The probabilities were then computed that the observed results would be obtained based on the null hypothesis (Fisher, 1973, p.96).

For the interviews, an interpretive method of analysis was primarily used (Konold and Well, 1981). This analysis was made with respect to the purposes of the interviews stated earlier in this chapter and involved two basic approaches.

The first approach was the elucidation and clarification of aspects of the perceptual field of the particular student. This involved making inferences about the student's perceptual field regarding selected affective aspects of his or her perception of proof. The student's reaction to situation five was analyzed in an effort to determine the extent to which he or she rejected the four-point geometry that was presented there.

The second aspect of the analysis of the interview data was an attempt to identify and refine other variables in the student's perceptual field that might have an impact on either their perception of proof, or their achievement in Advanced Calculus.

The analysis of the interview data was made by the researcher from transcripts and audio tapes of the interviews. The audio tapes were listened to by the researcher, and transcripts were made. The transcripts were read by the researcher and passages that related to the four areas the interviews were intended to address were marked. From this process a tally of interview quotes deemed pertinent by the researcher was constructed. Based on the tally, clusters of responses were identified, resulting in the definition of response categories associated with each aspect of proof addressed by the interviews.

Summary

This chapter presented the procedures used in the construction and administration of the written questionnaires as well as the development and conducting of follow-up interviews. Justification was given for the selection of Williams' questionnaire and for the use of follow-up interviews. The population was described in a series of tables. The interviews and questionnaires were described along with the administration and method of analysis for each.

IV. ANALYSIS AND RESULTS

The results obtained by the administration of the written questionnaires and the clinical interviews are presented in this chapter. Further, the chapter includes the analyses related to the stated objectives and hypotheses.

Written Questionnaire

The hypotheses that were tested follow:

1) The correlation between total score obtained on the written questionnaire and achievement in Advanced Calculus for class A is not significantly different than the same correlation for class B.

2) There is no significant proportion of variation in achievement that is associated with perception.

3) There is no association between achievement in MTH 311 and perception of the selected aspects of proof addressed by each situation.

The responses to the written questionnaire were put into response categories by three judges working independently. Based on the response category a response was placed in (see Appendix B), a numerical score of 0, 1, 2, or 3 was given to the response indicating no, low, medium or high understanding of the concept addressed by the situation. Tables describing the results of this procedure are given for each of the twelve situations.

Following these tables, a 2x2 contingency table is provided for each situation. A grade of A, B, or C was recorded as success in MTH 311; D's, F's and withdrawals were recorded as failure in the course. Responses categorized as high or medium were recorded as success on the situation and responses categorized as low or zero were recorded as failure.

Summary of Responses for each Situation

Situation one is intended to assess the student's understanding of the "generalization principle". This principle asserts that if some mathematical proposition is proven for a fixed but arbitrary member of a class of elements, then that proposition is proven for all elements of said class. The results of situation one are given in Table 6.

Table 6 - Summary of Student Responses to Situation One

CLASS	0	LOW	MEDIUM	HIGH	TOTAL
A	2	1	1	7	11
B	8	1	1	3	13
TOTAL	10	2	2	10	24
%	42	8	8	42	

Forty-two percent of the respondents were found to have a high understanding of the generalization principle and convey that understanding on the written questionnaire. Of the others, a typical response to the question, Whose side would you be on in the above discussion? was "Tom's", because, "I agree with Tom that in general Joe's argument is not valid."

Sometimes it was not clear what areas the student was referring to. For example, a student sided with Tom "because if there (sic) not the same diagram, they may not be the same rectangle, and therefore have a different area."

The extent to which students accept empirical evidence as adequate for mathematical generalizations is to be assesed by situation two. The situation measures the degree to which students see the need for deductive proof rather than examples to support mathematical propositions. The results of situation two are given in Table 7.

Table 7 - Summary of Student Responses to Situation Two

CLASS	0	LOW	MEDIUM	HIGH	TOTAL
A	0	1	0	10	11
B	1	4	0	8	13
TOTAL	1	5	0	18	24
%	4	21	0	75	

Seventy-five percent of the students that responded to situation two saw the need for a deductive proof in this situation. When asked whose side she was on, one student wrote "Tom's", because, "you cannot come to a conclusion like Joe did just from repeated trials--he must find a way to prove his assumption for all whole #'s n>0."

Of the other twenty-five percent, some students chose Joe for reasons such as this: "It sounds reasonable. I would probably have to check it with a calculator, though."

Situation three is similar in nature to situation two except that the problem posed in this situation is considered elementary enough that the students might be expected to provide their own proof of the proposition. The responses to situation three are given in Table 8.

Table 8 - Summary of Student Responses to Situation Three

CLASS	0	LOW	MEDIUM	HIGH	TOTAL
A	0	0	1	9	10
B	0	0	1	13	14
TOTAL	0	0	2	22	24
%	0	0	8	92	

The respondents to situation three almost all recognized the proposition as true and attempted to provide a proof for it.

Similar to situations two and three, situation four seeks to assess the extent to which students accept examples as a proof for a mathematical proposition. In this situation, however, the proposition is false and students are given the oppurtunity to provide their own counter-example. Table 9 summarizes the responses to situation four.

Table 9 - Summary of Student Responses to Situation Four

CLASS	0	LOW	MEDIUM	HIGH	TOTAL
A	1	0	0	10	11
B	0	0	0	13	13
TOTAL	1	0	0	23	24
%	4	0	0	96	

As in situation three, the students who responded to situation four nearly all (ninety-six percent) responded with what was considered a high level of thinking. Some recognized that a simple algebraic rule applied. When asked if he thought that Joe's conclusion was true or false, one student wrote "false", because of a "counter example: $10^2 - 8^2 = 100 - 64 = 36$; $10 + 8 = 18$; $18 \neq 36$". This student went on to make the correct generalization "$a^2 - b^2 = (a+b)(a-b)$."

Situation five explores students notion of the role of hypothesis in mathematics. In an axiomatic system a mathematician is allowed to argue from elementary statements known as axioms or postulates. Often these statements are not only unfamiliar, but counter-intuitive. situation five is one such case. Hence, this situation seeks to determine whether students will reject an argument because it begins with assumptions that are counter-intuitive. Table 10 is a summary of the responses to situation five.

Table 10 - Summary of Student Responses to Situation Five

CLASS	0	LOW	MEDIUM	HIGH	TOTAL
A	2	6	0	2	10
B	5	5	2	2	14
TOTAL	7	11	2	4	24
%	29	46	8	17	

Eighty-three percent of the students that responded to situation five would not argue from a set of assumptions that were counter-intuitive regardless of what was accepted by Joe and Tom. For example, "a line contains an infinite number of points despite Joe's accertion (sic) that each of his lines only contains 2 points". One student simply stated that "the lines are not parallel".

Some of the students wanted clearer definitions of the concepts involved. One student, when asked whose side he was on, responded "neither", because, "no specification of definition of 'line' as being of infinite length".

An indirect argument is offered in situation six. This situation is meant to determine if students will accept an indirect argument even though it begins with a statement known to be false. The responses to situation six are summarized in Table 11.

Table 11 - Summary of Student Responses to Situation Six

CLASS	0	LOW	MEDIUM	HIGH	TOTAL
A	0	1	1	8	10
B	1	3	2	8	14
TOTAL	1	4	3	16	24
%	4	17	13	66	

Most of the students (sixty-six percent) accepted the indirect argument in situation six although it was unclear whether the logic used was understood by all, or whether the proposition was so obvious that any argument would have been accepted. For example, one student chose Joe's side and, when asked how he would show Tom why he disagreed with him, continued "either $1 = 0$ or $1 \neq 0$. Show each case using same 6 steps". Some students would not allow an argument from a false premise, choosing Tom's argument, because "by the definition of multiplication of a R by 1 which returns its value and 0 which returns zero -- does not permit supposing that $1=0$ (or vice versa)."

Situation seven presents another indirect argument to the student, but in this situation he or she is asked to explain the major steps of the argument. Thus, this situation is used to investigate the degree to which students understand these major steps. The responses to situation seven are summarized in Table 12.

Table 12 - Summary of Student Responses to Situation Seven

CLASS	0	LOW	MEDIUM	HIGH	TOTAL
A	3	2	1	5	11
B	2	2	5	5	14
TOTAL	5	4	6	10	25
%	20	16	24	40	

Sixty percent of the students that responded to situation seven could not explain the major steps in an indirect proof with a high degree of proficiency. For example, when asked to explain step 5 of Joe's argument to Tom (this is the step in the indirect argument where the contradiction is reached), a student responded "from the defn. of the problem, $y \neq 0$, therefore in order for $xy=0$ to be true, x cannot be non-zero". The same student virtually repeats this argument for the justification of the next step in the questionnaire (concluding that the negation of the induction hypothesis is true). While six of the students' responses were classified as medium due to the fact that they recognized the argument as an example of indirect reasoning, they still used an intuitive argument to justify the given conclusion. Thus, they were not able to describe the role of the major steps in the given example.

Situation eight measures whether the student deems a proof necessary when a proposition is intuitively obvious. Table 13 gives a summary of the responses to situation eight.

Table 13 - Summary of Student Responses to Situation Eight

CLASS	0	LOW	MEDIUM	HIGH	TOTAL
A	4	1	5	1	11
B	3	1	4	6	14
TOTAL	7	2	9	7	25
%	28	8	36	28	

Of the students' responses that were categorized as either low or zero, there appeared to be three types of responses. Some students felt that the statement in the situation was obvious and therefore needed no proof; for example: "Joe should have just looked this matter up in his book". Two students left the page blank. Three students' responses were unintelligible. One student seemed to grudgingly accept the necessity of proving intuitively obvious statements when he wrote "O.K.- proofs are necessary because intuition fails too often."

In situation nine, the students' ability to recognize the contrapositive of a statement is sought. Further, this situation is used to determine whether or not students consider a statement and its contrapositive logically equivalent. A summary of responses to situation nine is given in Table 14.

Table 14 - Summary of Student Responses to Situation Nine

CLASS	0	LOW	MEDIUM	HIGH	TOTAL
A	1	7	1	2	11
B	2	4	5	2	13
TOTAL	3	11	6	4	24
%	12	46	25	17	

Most students could not recognize the two statements in situation nine as the contrapositive of each other. Of the students that did successfully respond to situation nine, six out of ten failed to finish MTH 311 with success. Of the twelve situations, this situation showed the most pronounced tendency for students who failed to succeed in MTH 311 to succeed in the situation.

Situation ten presents the students with a propostion that is true for several cases along with one counter-example. Thus, this situation seeks to determine if students think one counter-example is sufficient to disprove a mathematical statement. A summary of student responses to situation ten is given by Table 15.

Table 15 - Summary of Student Responses to Situation Ten

CLASS	0	LOW	MEDIUM	HIGH	TOTAL
A	0	0	0	11	11
B	2	0	1	10	13
TOTAL	2	0	1	21	24
%	8	0	4	88	

Eighty-eight percent of the respondents to situation ten recognized that a single counter-example is sufficient to disprove a mathematical statement. The two students whose responses were categorized as zero did not give an indication of understanding the situation at all. One student left the page blank after apparently trying a few cases, the other student did not give an intelligible response.

Situation eleven was presented to determine if students prefer direct over indirect arguments. The summary of responses to situation eleven is given by Table 16.

Table 16 - Summary of Student Responses to Situation Eleven

CLASS	0	LOW	MEDIUM	HIGH	TOTAL
A	0	8	0	2	10
B	1	9	0	3	13
TOTAL	1	17	0	5	23
%	4	74	0	22	

Seventy-four percent of the respondents chose the direct argument. No student chose the indirect argument. Five students, 22 percent, argued that there was no reason to prefer one over the other.

Situation twelve is included to determine the extent to which students understand that a statement and its converse are not logically equivalent. Table 17 is a summary of the responses to situation twelve.

Table 17 - Summary of Student Responses to Situation Twelve

CLASS	0	LOW	MEDIUM	HIGH	TOTAL
A	2	0	0	8	10
B	4	0	0	9	13
TOTAL	6	0	0	17	23
%	26	0	0	74	

One response categorized as zero was, "all the lowest common factors may be prime, but in no way did you state, that so any multiplication of prime numbers assuming both are common factor to the odd number given is also a divisor". If the reader has an alternate interpretation of this passage, please contact me.

Tables 18, 19, 20, 21, 22, 23, 24, 25, 26, and 27 are 2x2 contingency tables for all but two of the twelve situations. The 2x2 contingency tables for Situation Three and Situation Four were left out since the large percentage of successful responses rendered the tables meaningless. For each table, a one-tailed test of the hypothesis of independence was made (Fisher, 1973, p.100). In other words, a test was made of the hypothesis that $a/(a+c) = b(b+d)$, where a,b,d, and c are the respective cell entries in the 2x2 contingency table (going counter-clockwise, starting with the upper left cell). For each 2x2 contingency table, the probability is given (as a note) that the observed table, or those tables more extreme, would occur assuming that differences in achievement are not associated with differences in perception of proof.

Table 18 - 2x2 Contingency Table for Situation One

	MTH 311		
	SUCCESS	FAILURE	TOTAL
QUESTIONNAIRE ITEM			
SUCCESS	6	6	12
FAILURE	4	8	12
TOTAL	10	14	24

Note. Situation one involves the generalization principle.

Note. p = .34.

Table 19 - 2x2 Contingency Table for Situation Two

	MTH 311 SUCCESS	MTH 311 FAILURE	TOTAL
QUESTIONNAIRE ITEM			
SUCCESS	9	9	18
FAILURE	1	5	6
TOTAL	10	14	24

Note. Situation two measures the degree to which students see the need for deductive proof rather than examples to support mathematical generalizations.

Note. $p = .17$.

Table 20 - 2x2 Contingency Table for Situation Five

	MTH 311		
	SUCCESS	FAILURE	TOTAL
QUESTIONNAIRE ITEM			
SUCCESS	3	3	6
FAILURE	6	12	18
TOTAL	9	15	24

Note. Situation five was designed to determine whether students will reject an argument because it begins with assumptions that are counter-intuitive.

Note. $p = .40$.

Table 21 - 2x2 Contingency Table for Situation Six

	MTH 311 SUCCESS	MTH 311 FAILURE	TOTAL
QUESTIONNAIRE ITEM			
SUCCESS	8	11	19
FAILURE	2	3	5
TOTAL	10	14	24

Note. Situation six is meant to determine if students will accept an indirect argument even though it begins with a statement known to be false.

Note. $p = .67$.

Table 22 - 2x2 Contingency Table for Situation Seven

	MTH 311		
	SUCCESS	FAILURE	TOTAL
QUESTIONNAIRE ITEM			
SUCCESS	7	9	16
FAILURE	2	7	9
TOTAL	9	16	25

Note. Situation seven is used to see if students understand the major steps in an indirect argument.

Note. $p = .26$.

Table 23 - 2x2 Contingency Table for Situation Eight

	MTH 311		
	SUCCESS	FAILURE	TOTAL
QUESTIONNAIRE ITEM			
SUCCESS	8	8	16
FAILURE	1	8	9
TOTAL	9	16	25

Note. Situation eight measures whether the student deems a proof necessary when a proposition is intuitively obvious.

Note. $p = .06$.

Table 24 - 2x2 Contingency Table for Situation Nine

	MTH 311		
	SUCCESS	FAILURE	TOTAL
QUESTIONNAIRE ITEM			
SUCCESS	4	6	10
FAILURE	10	4	14
TOTAL	14	10	24

Note. Situation nine measures the students' ability to recognize the contrapositive of a statement is sought.

Note. $p = .98$.

Table 25 - 2x2 Contingency Table for Situation Ten

	MTH 311		
	SUCCESS	FAILURE	TOTAL
QUESTIONNAIRE ITEM			
SUCCESS	14	8	22
FAILURE	0	2	2
TOTAL	14	10	24

Note. Situation ten seeks to determine if students think one counter-example is sufficient to disprove a mathematical statement.

Note. $p = .16$.

Table 26 - 2x2 Contingency Table for Situation Eleven

	MTH 311		
	SUCCESS	FAILURE	TOTAL
QUESTIONNAIRE ITEM			
SUCCESS	3	2	5
FAILURE	9	9	18
TOTAL	12	11	23

Note. Situation eleven was presented to determine if students prefer direct over indirect arguments.

Note. $p = .54$.

Table 27 - 2x2 Contingency Table for Situation Twelve

	MTH 311 SUCCESS	FAILURE	TOTAL
QUESTIONNAIRE ITEM			
SUCCESS	11	6	17
FAILURE	1	5	6
TOTAL	12	11	23

Note. Situation twelve is included to determine the extent to which students understand that a statement and its converse are not logically equivalent.

Note. $p = .06$.

Hypotheses

For hypothesis 1, two independent estimates of the correlation between achievement in Advanced Calculus and total score were computed using the correlation coefficient. The first estimate for class A, the second estimate for class B. These were then converted to z-scores, and the significance of the difference between the two z's was tested (Snedecor and Cochran, 1980, p.186). The results were at about the 63% level of significance in a two-tailed test. Thus, it wasconcluded that the r's are estimates of the same rho.

For hypothesis two, the correlation coefficient was calculated between achievement in advanced calculus and score on the written questionnaire. The resulting r=.442 was found to be significant at the 1% level (Snedecor and Cochran, 1980, p.477). The results of this procedure are summarized in Table 30.

Table 28 - Correlation Coefficient

CLASS	SAMPLE SIZE	r	r^2
A	21	.503	
B	25	.383	
TOGETHER	46	.442	.195

The proportion of variation in achievement that is associated with perception (r^2) is .195 or 19.5%. Thus, hypothesis two was rejected.

For hypothesis three, the probability was calculated that the observed 2x2 tables, or those tables more extreme, would occur assuming that differences in achievement are not associated with differences in perception of proof (Fisher, 1973, p.96). The following probabilities were obtained for each situation:

situation one (Table 18) - .34

situation two (Table 19) - .17

situation five (Table 20) - .40

situation six (Table 21) - .67

situation seven (Table 22) - .26

situation eight (Table 23) - .06

situation nine (Table 24) - .98

situation ten (Table 25) - .16

situation eleven (Table 26) - .54

situation twelve (Table 27) - .06

Based on a .2 level of probability, the results were deemed "significant" for situations two, eight, ten, and twelve. Based on a .4 level of probability, the results were deemed "significant" for situations one, five, and seven. The .98 probability for situation nine reveals that students who succeeded in MTH 311 had a tendency to give inappropriate responses to this situation. As a result of the very large percentage of students' responses classified as successful on situation six, the large probability was expected. Situation eleven was a preference question with a high percentage of the responses going in one direction. Situations three and four were not considered.

Interviews

The use of interviews is an example of the self-report as a tool to investigate aspects of students' perceptual fields (Combs, et. al., 1976). The primary interest of these interviews was in noting the similarities and differences in the interviewees' perceptual fields. These similarities and differences were inferred from their responses to particular questions. There were four general areas that the interviews were intended to address. The interview script is in Appendix D. The general areas addressed by the interviews are as follows:

1) the students' subjective perception of the nature of mathematical proof and the role of proof in mathematics;

2) the degree to which students enjoy mathematical proof;

3) the degree of differentiation made by the students between proof and problem-solving; and

4) the amount of confidence the students have in their ability to construct proofs.

The results are analyzed and reported with respect to these four areas. Transcripts were made from audio-tapes. Both the transcripts and the tapes were used in the analysis.

In each of the four areas, distinct classes of responses are identified and recorded. Further, other responses deemed interesting by the researcher are noted. The list of response "clusters" is not intended to be comprehensive, rather, they are intended to be representative.

The first line of questions in the interview (after the preliminary questions like: "What is your name?" and so forth) sought to find out the students' perception of the nature of proof and the role of proof in mathematics. There were four main themes to this line of questioning. These themes were designed to determine the students' concept of the:

1) nature of mathematical proof;

2) role of proof in mathematics;

3) importance of proof (to the student, to mathematicians, and to the general public), and

4) understanding of certain particulars (e.g. axioms, theorems, deductive vs. inductive reasoning).

Before listing the major types of responses one note seems particularly important. It appeared that some students had previously thought about the type of questions that were asked in the interview. It appeared that others had not considered the questions prior to the interview. Why certain students had thought about these things before and what relationship there was between such thought and achievement is unclear at present. However, differences in responses were clear. The perceptual fields of those students who had not given much thought to proof were in a great state of flux. Their self-reports were changing even as they spoke. The following excerpt is an example of such a case.

Question: "What role does proof play in mathematics?"

Response: "No practical--well I don't know about that, it might--it's all so confus..."

Categories for student responses were developed in an inductive manner. After listening to the tapes and reviewing transcripts, the researcher identified clusters of responses that were related to each of the areas that the interviews sought to address. Responses were marked on the transcripts, tallied, and used as a guide to the researcher in formulating the response categories. Examples of each category are found in Appendix E. The students' perception of the nature of mathematical proof fell into three recognizable categories. These categories follow:

1) proof is a means of justification or verification of statements known or believed to be true;

2) proof is a way of getting to "fundamental" or basic ideas; working backward; and

3) proof is part of a building process from a certain groundwork up.

Probably the most revealing responses came from the part of the interviews that addressed the students' perception of the role of proof in mathematics. When asked directly what is mathematical proof, the interviewees generally had problems expressing themselves. When asked about the role of proof in mathematics, more specific responses were forthcoming.

Students' perception of the role of proof in mathematics fell into six categories. Students believe that proof:

1) is a means of justification in an abstract mathematical system,

2) is a means of justifying facts, truths about reality,

3) enhances understanding,

4) provides a foundation for mathematics,

5) has no role whatsoever, and

6) unclear.

There were four categories discovered that are related to students' perception of the importance of proof in mathematics. These categories are as follows:

1) proof is an activity that only theoretical mathematicians need be concerned with, to applied mathematicians and scientists it is not necessary at all;

2) proof is not necessary for "low-level" mathematics, but its importance increases as "you go higher";

3) proof is not necessary at all; and

4) unclear.

Responses about the perceived importance of proof to the interviewees ran along four different lines. They are as follows:

1) proof is necessary to provide rigor to mathematics, I wish to do mathematics, therefore proof is important to me;

2) proof is not a necessary ingredient of applied mathematics, I wish to do applied mathematics, therefore proof is not important to me;

3) I can do without it; and

4) unclear.

There was general agreement among the interviewees that the "general public" (non-mathematicians) had neither the urge nor the necessity to understand mathematical proof.

Certain particulars were addressed in the interviews. Some of these were in the interview script, others came up spontaneously during the course of some of the interviews. There were two particular components of an axiomatic system addressed in the interviews: axioms and theorems. The response categories relating to axioms fell into six categories. They are as follows:

1) axioms are arbitrary ground rules set by mathematicians so they can "play the game";

2) axioms are universal truths of nature;

3) axioms are statements that are hard to prove, hence they are assumed;

4) axioms are statements that are assumed because "everyone else does";

5) axioms are the foundation of mathematics; and

6) I don't know what an axiom is;

What is a theorem? How does it differ from an axiom? Six categories of responses to these questions were identified. The responses to these questions follow:

1) a theorem is a new result, derived from axioms;

2) a theorem is a statement that requires proof (as opposed to an axiom, which does not require proof);

3) a theorem is a postulate;

4) a theorem is a fact;

5) a theorem is a statement that may or may not be true; and

6) unclear.

Of the eight people interviewed, one person conveyed a sense of an arbitrary, axiomatic system, independent of nature or scientific reality. This student, it turns out, was at the top of his class and the top of the scores on the written questionnaire.

During some of the interviews the question of scientific knowledge and the difference between truth in science and truth in mathematics came up. Not all of the interviews got around to this question. It usually came up as a result of some comment or example given by the student from the physical sciences. There were four categories found relating to questions of this sort. The responses are summarized here:

1) scientific reasoning is inductive, generalizing from particular observations, mathematical reasoning is separate from reality, arbitrary;

2) science requires proof, otherwise "the data might be bad", mathematics proves what you can say about the world and science;

3) experiments or mathematics provide proof (it was unclear whether or not the student recognized a distinction between the two); and

4) unclear.

The students were asked about the degree of enjoyment they received from constructing mathematical proofs. Their responses were quite diverse. Ten response categories were identified. These categories follow:

1) it is satisfying to complete a proof;

2) some proofs are entertaining;

3) proofs are not enjoyable because it is hard to "visualize what's going on";

4) unsolved proofs are fun to ponder;

5) calculations are preferred to proofs;

6) calculations are disliked;

7) working proofs under pressure is particularly distasteful;

8) proofs are scary;

9) constructing proofs are a necessary hurdle on the way to a degree; and

10) proofs are not very stimulating.

In this line of questions there were two fruitful paths. First, students were asked to estimate there own prowess at constructing acceptable proofs. Secondly, they were asked about some of the techniques they employed when proving a theorem. A third line of questioning was not so illuminating. This was when students were asked what they would do if someone challenged one of there proofs. Amazingly enough, every one of the students would go back and look at it!

Four categories were found associated to students beliefs about there own ability to construct proofs. Students believed themselves to be:

1) not bad at proving things, but could use improvement;

2) not very good at proof;

3) not very good at proof, at least according to my professors; and

4) scared of proofs.

Techniques reported by the students fell into six categories. Some of the techniques that students say they use are:

1) starting with what is known and making associations;

2) utilizing short, powerful bursts of thinking;

3) memorizing of theorems;

4) using techniques from computer programming;

5) using direct and then indirect methods; and

6) "lying", putting statements in the proof just to fill up space when arriving at a difficult point in the proof.

Another part of the interviews involved problem-solving. The words "problem-solving" have become mathematics education jargon in the worst way. The gap between mathematics students' notion of problem-solving and mathematics educators' notion of problem-solving is illustrated by the two categories identified. Only one student fell into the first category. The response categories follow:

1) proving a theorem and problem-solving are, in some way, related or similar activities; and

2) solving problems is fundamentally an empirical task. Proof is a theoretical task.

The last phase of each interview was designed to investigate the extent to which students rejected the four-point geometry of situation five of the written questionnaire. The interviewee was given a copy of the situation and time to look at it. Then questions were asked from the interview script.

When presented with situation five, the students were unanimous in rejecting Joe's argument that there are three pairs of parallel lines in the situation. For every student, the basis for rejecting Joe's argument was that some property of the "real world" was being violated. The two points of contention were as folows:

1) Joe cannot state that there are only four points on the sheet of paper, because there are obviuosly more than that; and

2) lines must contain an infinite number of points, regardless of Joe's claim to the contrary.

Findings not Directly Related to the Hypotheses

Many thoughts came to mind during the course of planning, conducting, analyzing, and reporting on the interviews associated with this research. These "interview thoughts" have been compiled from notes taken during the entire process. Some of these thoughts were in the form of problems, some in the form of questions, others were procedures that supported the reliability and validity of the interview process. They are the result of reflective thinking based on two major activities: first, discussions with students, educators, mathematicians, and mathematics educators; second, thoughts arising during the interview process itself.

The interview process involves four major stages: planning, conducting, analyzing, and reporting. Two seemingly contradictory statements are worth considering:

1) the conducting, analyzing, and reporting phases should all be subheadings of the planning stage; and

2) no amount of planning is going to head off unexpected "disastors" in the process of conducting, analyzing, and reporting interview data.

In other words, while every step in the interview process should be planned and justified, unexpected things always come up. One of the major roles of the planning stage should be to minimize these unexpected occurences.

Other details came to mind during the interview process. Some of these are as follows:

1) use brand new, high quality tape;

2) for half-hour interviews, use 45-minute tapes;

3) develop the interview script with the aid of a pilot study;

4) do not work alone, groups of two or three are probably optimal;

5) do not rely solely on "expert" advice, student input can be much more valuble; and

6) use transcripts <u>and</u> audio tapes when analyzing the interviews, inflection and tone of voice convey meaning.

During the course of the interview process, it became apparent that the words "problem-solving" are, to use the words of N. R. Hanson, theory-laden (Hanson, 1960). That is, what problem-solving means to the average mathematics student and what problem-solving means to the average mathematics teacher are two different things. To the mathematics student problem-solving means solving problems; empirical problems. This means getting "the answer", whatever that may be. To mathematics teachers, especially university mathematics educators, problem-solving is a buzzword that contains within its scope an entire field of research and writing about generalized strategies employed to overcome "problems" or "difficulties", whatever they are.

It is made clear by this research that there exists a group of students who perceive mathematics to be a disconnected set of rules to be applied to meaningless symbols (occasionally these meaningless symbols take the form of clusters of intelligible sentences known as word problems). These students see little or no relationship between the process of constructing a proof and the process of solving a calculus problem of an empirical nature. It is arguable that this sort of attitude towards mathematics is fostered by a learning environment that emphasizes product over process. This is the problem that the advocates of problem-solving are trying to solve.

Advocators of problem-solving state that "the mathematics curriculum should become more strategy based and less content based" (Musser and Shaughnessy, 1980, p.136). This is correct as long as they do not mean to say "more strategy based <u>than</u> content based".

Mathematics, not problem-solving, is and should be the focus of mathematics education in the 80's. Although this statement is pithy, it leads to the first criticism of an over-emphasis on problem-solving. Mathematics means core mathematics and, core mathematics may mean many things. Core mathematics is history; it is content; and it is method. The interviews provide evidence that students do not understand the nature of mathematics. And problem-solving per se will not likely give them a better understanding of the nature of mathematics.

A second danger of too much faith in problem-solving as a focus for mathematics education can be viewed in the argument made by Theodore Roszak about dangers inherent in Logo (Roszak, 1986). In his book, Roszak warns that there exists a danger of depreciating the subject taught. In other words, according to Roszak's argument, if Logo cannot be raised to the level of the subject, then there is a temptation to lower the subject until "procedural thinking" (Pappert, 1980) works well. Roszak uses the examples of art and dance to illustrate his point: "So then: can Logo teach art? Only if art is defined as what Logo can do in the way of art, which is not much" (Roszak, p.78).

This criticism, he goes on, does not imply that we should reject Logo out of hand, merely that we should be aware of its pitfalls. So be it with problem-solving. "Problem-solving strategies", the key ingredient in the understanding-planning-acting-communicating process, are powerful tools to attack mathematical problems. But, mathematics is no more defined by problem-solving than art is defined by what Logo can do in the way of art or calculus is defined by the collection of facts in a first-year calculus text. Just as students may come to view mathematics as a collection of isolated facts if given only content, they may also see mathematics as a "bag of tricks" if problem-solving is allowed to be the focus of school mathematics and mathematics educators take the superficial view that mathematics is "merely problem-solving". A well-balanced curriculum should be based on content, method, and the place of mathematics in the context of the history of thought and the development of culture.

V. SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

Summary

The major focus of this study has been junior-level university mathematics students' perception of proof, where perception is defined as in the work of A. Combs, and its relationship to achievement. Students' perception have been measured and analyzed. The measure of perception represents a frozen image of a constantly changing phenomenon. Hence, the results should be viewed as such, i.e., a discrete, frozen image of a continuous process. The measurement represents the students' perception of proof at one particular point in time.

In the process of measuring and analyzing students' perception of proof and its relationship to achievement, the following problems were investigated:

1) the nature of perception of proof of undergraduate mathematics students who have completed the prerequisites for and have enrolled

in MTH 311, Advanced Calculus, at Oregon State University;

2) the relationship between students' perception of selected aspects of proof upon entering the first quarter of Advanced Calculus and his or her achievement in Advanced Calculus; and

3) the relationship between success on the 12 questionnaire items developed by E. Williams and achievement in Advanced Calculus.

The sample consisted of 47 students enrolled in MTH 311 (Advanced Calculus) at Oregon State University in the fall quarter of 1984. Twenty versions of a questionnaire, each containing six items, were administered randomly to two entire classes of MTH 311 in the fall of 1984 at Oregon State University. The 12 questionnaire items were previously developed by E. Williams of the University of Alberta for measuring selected aspects of students' perception of proof including: indirect proof, induction, deduction, elementary logic and counter-example. Student responses to the questionnaire were evaluated and put into response categories by three judges from the mathematics department at Oregon State University.

An interview script was developed based on the results of the administration of the written questionnaires and a pilot study involving undergraduates with a similar backround as those in the study. The interviews were developed with two main purposes in mind:

1) They were intended to assess students' subjective perception of the nature of mathematical proof, the degree to which students enjoy proof and the amount of confidence the students have in their ability to construct proofs.

2) They were also designed as a follow-up to the written questionnaires, allowing the researcher to further probe questions that arose from their evaluation, possibly identifying other variables related to students' perception of proof.

Eight follow-up interviews were conducted midway into fall quarter, 1984. The interviews were taped, conducted by the researcher, and one half-hour in length. Student responses to the interview questions were analyzed and categorized into an inductively developed category system.

Achievement data were obtained from student performance on tests and homework assignments in MTH 311. It was the total number of points accumulated by each student.

The data were analyzed and the following hypotheses were tested:

1) The correlation between total score obtained on the written questionnaire and achievement in Advanced Calculus for class A is not significantly different than the same correlation for class B.

2) There is no significant proportion of variation in achievement that is associated with perception of proof.

3) There is no association between achievement in Advanced Calculus and perception of the aspects of proof addressed by each situation on the written questionnaire.

The results of the evaluation of the interview transcripts, the responses to the written questionnaire, and the tests of the hypotheses are presented, in detail, in Chapter IV.

Conclusions

In his preface to Science and Hypothesis, Henri Poincare states:

> "To the superficial observer scientific truth is unassailable, the logic of science is infallible; and if scientific men sometimes make mistakes, it is because they have not understood the rules of the game. Mathematical truths are derived from a few self-evident propositions, by a chain of flawless reasonings; they are imposed not only on us, but on Nature itself ... From each experiment a number of consequences will follow by a series of mathematical deductions, and in this way each of them will reveal to us a corner of the universe. This, to the minds of most people, and to students who are getting their first ideas of physics, is the origin of certainty in science. This is what they take to be the role of science and mathematics" (Poincare, 1952, p. xxi).

This study supports Poincare's report of beginning mathematics students' perception of proof. This is indicated by their responses to the four-point geometry of situation five on the written questionnaire and the subsequent discussions in the follow-up interviews.

Statements deemed counter-intuitive by the students were generally not accepted as valid for use in a mathematical argument. Further, responses to interview questions about axioms and theorems indicated a feeling by most of the students that mathematical statements, particularly axioms, were "True", with a capital "T". That is, mathematical statements necessarily describe the nature of the universe.

While the students' responses to the written questionnaire, particularly situations two, three, and four, seemed to indicate a felt need for mathematical proof--further investigaton uncovers dissatisfaction with the process. Responses to situation eight, a situation involving a statement that was very obvious to the students, were less favorable. In the interviews this dissatisfaction is more pronounced. Consider the following: every student interviewed pointed out that the major difference between the calculus sequence and MTH 311 was in the treatment of proof. MTH 311 used proof extensively and the calculus sequence used proof not at all, or close to not at all but, only the most successful students argued that mathematical proof was important (a similar relationship between success in MTH 311 and acceptance of the importance of proof is seen in the 2x2 tables for situations two and eight). It would seem that there is a group of students who are willing to go along with the use of proof in mathematics if they have to, but have trouble seeing its worth.

To summarize and synthesize the results to this point, we state the following:

1) the nature and role of hypothesis in mathematics is misunderstood by at least a large number of Advanced Calculus students at Oregon State University;

2) there exists a group of students who express a superficial acceptance of the need for proof, but do not seem convinced when questioned further;

3) as evidenced by the interview responses and the 2x2 tables for situations two and eight,

the more successful students "hung on" to their acceptance of the need for proof longer than unsuccessful students.

The hypothesis, that the total score obtained on the written questionnaire and achievement in MTH 311 for class A is not significantly different from the same correlation for class B, on the basis of a 63% level of significance on a two-tailed test (see Chapter Four) is not rejected. Thus, it was concluded that the relationship between success on the written questionnaire and achievement in Advanced Calculus was similar for both classes.

The proportion of variation in achievement, both classes together, that is associated with perception was found to be 19.5%. Hypothesis two was rejected on the basis of these data. It was concluded that a significant proportion of variation in achievement is associated with perception.

For hypothesis three, 2x2 contingency tables were used. The hypothesis that differences in achievement are not associated with perception of proof for situations two, eight, ten, and twelve was rejected on the basis of a .2 probability. While situations three and four are similar to situation two, the high probabilities associated with these situations were expected because of the fact that a very large percentage of students' responses were catagorized as "high". Similarly, the high probability found associated with situation six was expected. From these data it was concluded that there is no association between achievement and perception of the aspects of proof addressed by these situations. The probabilities of .34, .4, and .26 found for situations one, five, and seven respectively, were not deemed "significant", but were low enough to be considered "suggestive". In other words, the results obtained from situations one, five, and seven were suggestive enough to warrant their possible inclusion on the modified version of the written questionnaire given in Appendix F. Situation eleven was an opinion question with a large number of responses going in one direction.

To summarize the results regarding hypothesis three, there is evidence in support of the conclusion that there is an association between differences in perception of proof and achievement in Advanced Calculus. Specifically, it was found to be highly improbable that there is no association between achievement in Advanced Calculus and the perceptions of proof that were addressed by situations two, eight, ten, and twelve.

This leaves situation nine for consideration. It is puzzling to consider the results of situations nine and twelve side by side. Situation twelve seeks to determine if a student can tell that a statement and its converse are not equivalent. Situation nine seeks to determine whether or not a student can recognize the contrapositive of a given statement and that it is equivalent to the given statement. The results of the analysis of situation twelve were deemed significant. In fact, the probability of .06 was the lowest (tied with situation eight) found. In other words, the results can be construed as evidence in favor of the conclusion that there is an association between success in Advanced Calculus and an ability to recognize that a statement and its converse are logically equivalent. The results for situation nine, on the other hand, were the reverse of the results for situation twelve. There was a tendency for those students who succeeded in Advanced Calculus to fail in situation nine. There are several possibilities for this occurrence. One of these possibilities is that there is an inverse relationship between achievement in Advanced Calculus and understanding of the concepts addressed by stiuation nine. Another possibility is that the question was poorly posed, or considered ambiguous by the students.

In any event, sober reflection on the nature of the situation should lead us to the conclusion that no single aspect of perception of proof should be conclusive in determining the success or failure of a student in Advanced Calculus. Familiarity with the nature of the situations, and student responses to them, allow a specific complex of aspects of proof to be chosen. Then, their probabilities may be combined (Fischer, 1973, p.100).

In contrast to the object of discussion in the written questionnaires and the interviews, proof, an individual's perceptual field is not a logical system. Nor is an individuals perceptual field a stable system. A person may, at very close time intervals (say, within the span of one interview) report two logically contradictory beliefs that, at the instant of each report, are perfectly valid. One factor of this phenomenon is the amount of time and energy an individual spends thinking about the aspect of his perceptual field in question. How much time the nature and role of mathematical proof is spent in figure, rather than ground, is proportional to the stability of the individual's perception of proof. Not only this, but the students whose perceptual fields were most stable were the more successful students. This combination of circustances leads to interesting questions. Is there a relation between reflective thought and success in Advanced Calculus even if that thought is not directly related to the content of the course? Can this type of thought be promoted?

To summarize, the conclusions drawn from this study are as follows:

1) statements deemed counter-intuitive by the students were generally not accepted as valid for use in a mathematical argument;

2) it is held by many students that mathematical statements necessarily describe the nature of the universe;

3) the nature and role of hypothesis in mathematics is misunderstood by at least a large minority of junior-level university mathematics students;

4) results are mixed regarding students' felt need for mathematical proof;

5) successful students persisted in their acceptance of the need for proof longer than unsuccessful students;

6) the relationship between success on the written questionnaire and achievement in Advanced Calculus was similar for both classes;

7) a significant proportion of variation in achievement is associated with perception of proof;

8) it was found to be highly improbable that there is no association between achievement in Advanced Calculus and the perceptions of proof that were addressed by situations two, eight, ten, and twelve;

9) there was a tendency for those students who succeeded in Advanced Calculus to fail in situation nine;

10) an individual's perceptual field is neither a logical system nor a stable system; and

11) the students whose perceptual fields were most stable were the more successful students;

Recommendations

After describing students' perception of the role of proof in mathematics, Poincare goes on to state that:

> "upon more mature reflection the position held by hypothesis was seen; it was recognized that it is as necessary to the experimenter as it is to the mathematician. And then the doubt arose if all these constructions are built on solid foundations. The conclusion was drawn that a breath would bring them to the ground. This sceptical attitude

> does not escape the charge of superficiality. To doubt everything or to believe everything are two equally convenient solutions; both dispense with the necessity of reflection" (Poincare, 1952, p. xxii).

Reflection is necessary but not sufficient for success in making the transition to advanced mathematics. This statement of Poincare's is supported by the findings in this research. Students at Oregon State University whose responses seemed to indicate previous reflection on their part were more successful in MTH 311. Thus, the first recommendation supported by this research is to promote reflection on the nature and role of proof in mathematics and the difference between scientific and mathematical reasoning.

One possible method for promoting such thought may lie in the history of calculus. What is being suggested is not merely chronology and anecdote, but a study of the history of man's struggle with the fundamental ideas associated with calculus. Boyer writes:

> "The number L, thus abstractly defined as the derivative, is not to be regarded as an 'ultimate ratio,' nor may it be invoked as a means of 'visualizing' an

> instantaneous velocity or of explaining in a scientific or a metaphysical sense either motion or the generation of continuous magnitudes. It is such unclear considerations and unwarranted interpretations which, as we shall see, have embroiled mathematicians, since the time of Zeno and his paradoxes, in controversies which often misdirected their energy. On the other hand, however, it is precisely such suggestive notions which stimulated the investigations resulting in the formal elaboration of the calculus, even though this very elaboration was in the end to exclude them as logically irrelevant" (Boyer, 1949, p.8).

It is this very same sort of struggle with the relationship between mathematics and reality that the students in this study are encountering. Certainly the readiness of calculus students to grasp such subtleties may be in question. Seen in the context of the entire curriculum, however, it may be that such study would not be completely in vain. This would be interesting ground for future research.

Even if all of the data concerning this populations' perception of proof could be organized and ingested, the problem of what should be included in the curriculum would not be solved.

However, the results of this study tend to support the following statement:
the more important a faculty considers the actual content of Advanced Calculus to the undergraduate mathematics student, the more compelling the argument for a "transition course" that would introduce sudents to proof. The reason for this is if there is an association between perception of proof and achievement in Advanced Calculus, and if Advanced Calculus is the students' first encounter with formal mathematics, then it is reasonable to assume that there is some interaction between the course content and the techniques of proof that should be mastered. If the content is deemed important enough, a transition course could be developed. This could be a distinct course or it could be integrated into the already existing curriculum.

It is recommended that the nature of proof and its role in mathematics become an integral part of the undergraduate mathematics curriculum. Particular attention should be paid to proof during the first two years of the undergraduates' university experience. This attention to proof could take the form of a specific course, or proof could be integrated into the calculus sequence in a systematic way.

It is recommended that university and college mathematics faculty periodically discuss the nature and role of proof and the importance of students having an adequate perception of the nature of proof. Seminars, with first and second year mathematics students in mind, should be offered. These seminars could address topics both relevant to the topic of proof, and accessable to students in the calculus sequence. Such topics exist, in abundance, in the history of mathematics.

During the process of administering and analyzing the written questionnaire and the interviews, certain difficulties in their structures became apparant. These difficulties were not deemed serious enough to endanger their basic validity, but should be revised before being used in future research.

The proof given by Williams in situation six was found to be flawed. Recall that in his argument, Joe wished to show that 1 is not equal to 0. Step two of his proof assumes that it is possible to choose two numbers, a and b, that are not equal. This assumption is equivalent, however, to the stated object of the proof. Hence the reasoning is circular. The difficulty is subtle and students' responses did not appear to be affected.

In situation seven, it was determined that another, more typical, indirect proof would be more appropriate. This new proof would omit the use of any particular numbers (recall: situation seven utilizes the artifice of setting x=3 and then proceeding with the argument).

No other structural problems were found with the written questionnaire. There was occasional difficulty classifying the questionnaire responses into the categories stated by Williams. Appendix F contains a revised questionnaire which should alleviate these difficulties in future research. This revised, six-item, questionnaire should be suitable for administration during a 50-60 minute class period. The analysis of this instrument can be similar to that used in this study with the advantage that we now have a better idea what responses to expect from students at the junior-level in university mathematics.

Appendix G contains a revised interview script for use in future research. The revisions are based on a variety of concerns that surfaced as a result of the analysis of the interviews. The aspects of students' perception of proof that the revised interview script is intended to address are as follows:

1) the nature of mathematical proof (including what constitutes a valid proof; what are axioms and theorems; and what are the similarities and differences between mathematical reasoning and reasoning in the physical sciences);

2) the role of mathematical proof (including justification, illumination, systematization, and students' acceptance of proof, logic, and an axiomatic system);

3) students' enjoyment of mathematical reasoning in general and proof in particular; and

4) students' confidence in constructing valid proofs (including when do you know that you have finished).

It is suggested that the instruments developed and/or refined here (see Appendix F and Appendix G) be used to address a variety of research questions. Does the association between perception of proof and achievement in Advanced Calculus have long term implications for individual students? Can students' perception of proof be changed in one quarter? Can students' perception of proof be changed in one year? If so, does such change in perception of proof have an associated change in achievement?

Another line of questions arise from the notion of reflective thought. Is reflective thought a significant factor in formulating a students' perception of proof? Can reflective thinking be promoted? Does an historical approach affect such thought? Does an inadequate historical approach, based on anecdote and chronology, have an effect on students' perception of mathematical proof?

Since this study does not have the rigor of a true experimental design, it is recommended that the study be repeated with other subjects to further validate the findings.

Finally, there are a variety of sequences of mathematics courses for teachers in many universities and colleges throughout the country. The tools developed here might be useful in determining the effectivness of such sequences in conveying an adequate perception of proof to prospective teachers.

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Appendices

Appendix A

Written Questionnaire

The following is the written questionnaire developed by Williams and administered to the MTH 311 students in the fall of 1984.

Name______________________________

Date of Birth_______________________

Male_____________ Female_____________

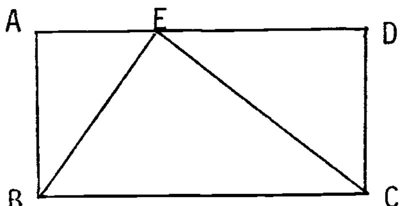
Class: Fr So Ju Sr Gr (circle one)

Instructor___________________________

Directions:

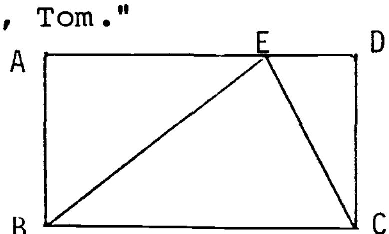
On the following pages you will be asked some questions related to mathematics. These questions are in no way intended as a test or examination. You are requested to read the discussion on each page carefully and answer each question in the space provided using the first thoughts or opinions that enter your mind.

Situation One



Joe's Diagram

Joe: "In my diagram, the altitude of triangle BEC is CD. Therefore, the area of triangle BEC is (BC)(CD)/2. But the area of rectangle ABCD is (BC)(CD). Therefore, the area of triangle BEC is 1/2 the area of rectangle ABCD. The same is true in your diagram, Tom."



Tom's Diagram

Tom: "I disagree with you Joe. My diagram is different from yours and I cannot say that something is true in my diagram just because it is true in yours."

Joe: "Of course you can. Because your diagram is not the same as mine doesn't matter."

<u>Tom</u>: "I don't agree. Our two diagrams are different and what's true in your digram, Joe, has nothing to do with what is true in my diagram."

Questions:

(a) Whose side would you be on in the above discussion?

Joe's__________ Tom's__________ Neither__________

(b) Why?

Situation Two

Joe has observed the following interesting pattern:

$4 - 1 = 3$ is divisible by three.

$4^2 - 1 = 15$ is divisible by three.

$4^3 - 1 = 63$ is divisible by three.

$4^4 - 1 = 255$ is divisible by three.

$4^5 - 1 = 1023$ is divisible by three.

$4^6 - 1 = 4095$ is divisible by three.

Joe borrowed a calculator and found out that $4^n - 1$ is divisible by 3 regardless of what value of n he tried. Therefore he came to the following conclusion:

$4^n - 1$ is divisible by 3 for all positive whole numbers n.

While Joe was working on this problem, Tom walked into the room. Tom looked at the conclusion and immediately stated that he was not convinced that Joe's conclusion was always true. Tom felt that while the conclusion was true for n = 1, 2, 3, 4, 5, 6 etc., this did not rule out the possibility of there being some number for which the conclusion was not true.

But Joe disagreed with Tom. Whatever value of n he had tried on the calculator confirmed the truth of his conclusion and therefore, as far as Joe was concerned, it was always true.

Questions:

(a) Whose side are you on?

Joe's__________ or Tom's__________

(b) Why?

Situation Three

Joe and Tom are discussing factors:

Joe: "Since 3 is a factor of both 24 and 33, therefore, 3 is a factor of 24 + 33."

Tom: "That's obvious, and you can even say more than that, Joe. If 3 is a factor of any two different numbers, then 3 is a factor of the sum of those two numbers."

Joe: "Well, if you put it that way, we can go even further and say that if any integer, n, is a factor of two different numbers, then n is a factor of the sum of those two numbers."

Tom: "Not too fast, Joe. If n is a factor of some number, p, and n is a factor of another number, q, can we always say that n is a factor of their sum, p+q?"

Joe: "Of course."

Tom: "I'm not convinced. In fact, I don't think that your conclusion is always true."

Questions:

(a) Who do you agree with?

Joe__________ or Tom__________

(b) If you agree with Joe, how would you convince Tom?

(c) If you agree with Tom, how would you convince Joe?

Situation Four

Joe has written some interesting equations, some of which are as follows:

$8^2 - 7^2 = 15$	$3^2 - 2^2 = 5$
$5^2 - 4^2 = 9$	$9^2 - 8^2 = 17$
$11^2 - 10^2 = 21$	$14^2 - 13^2 = 27$

From these equations, Joe concludes that for all integers, a and b,

$$a^2 - b^2 = a + b$$

Questions:

(a) Do you think that Joe's conclusion is true or false? __________

(b) If you think that Joe's conclusion is true, then state why.

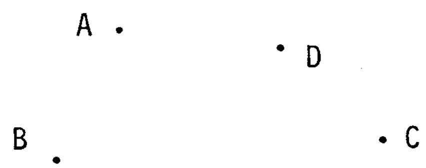
(c) If you think that Joe's conclusion is false, then state why.

(d) If you think that Joe's conclusion is false, then what do you think might be a correct conclusion?

Situation Five

Joe and Tom are discussing parallel lines.

Joe: "Suppose there are only four distinct points on this sheet of paper instead of an infinite number."



Tom: "So you are imagining that the four points A, B, C, and D above are the only points on this sheet of paper."

Joe: "Right. Now, since two points determine a unique line, these four points determine six distinct lines. Each of these six lines contains only two points. Do you see what the six lines are, Tom?"

Tom: "Yes."

Joe: "Now remember that if two lines have no point in common, then they are parallel."

Tom: "I agree."

<u>Joe</u>: "I claim that there are three pairs of parallel lines determined by the four given points."

<u>Tom</u>: "That's nonsense, Joe. None of the lines determined by the four given points can possibly be parallel."

<u>Joe</u>: "So you would say that the line determined by the points A and B (for example) is not parallel to the line determined by the points C and D."

<u>Tom</u>: "Of course these lines are not parallel."

<u>Joe</u>: "O.K., if these lines are not parallel, then they must intersect in some point. Since there are only four points on the sheet of paper, the lines must intersect in either A, B, C, or D. But clearly the line determined by A and B does not intersect the line determined by C and D in either C or D and vice versa. Therefore, these two lines must be parallel."

<u>Tom</u>: "I don't care what you say, the four given points do not determine any parallel lines."

<u>Joe</u>: "But I've shown otherwise Tom, and in fact I can show that there are three pairs of parallel lines determined by the four given points."

Questions:

(a) Whose side would you be on in the above discussion?

Joe's_________ Tom's_________ Neither_________

(b) Why?

Situation Six

Joe: "Tom, did you know that there is a way to show that $1 \neq 0$?"

Tom: "No, how?"

Joe: "Well, it goes like this:

1. Suppose that $1 = 0$.
2. Let a and b be any numbers such that $a \neq b$.
3. Since $1 = 0$, therefore $a = (a)(1) = (a)(0) = 0$.
4. Similarly, since $1 = 0$, $b = (b)(1) = (b)(0) = 0$.
5. Therefore $a = b$.
6. But $a = b$ is false and so $1 \neq 0$."

Tom: "But you started out by supposing $1 = 0$. How can you say something that isn't true? To me it doesn't make sense to suppose that $1 = 0$ in order to show just the opposite."

Joe: "I don't agree with you, Tom."

Questions:

(a) Whose side would you be on in the above discussion?

Joe's____________ or Tom's____________

(b) If you are on Joe's side, how would you show Tom why you disagree with him?

(c) If you are on Tom's side, how would you show Joe why you disagree with him?

<u>Situation Seven</u>

Joe has shown that the following statement is true for all real numbers x and y.

<u>Statement</u> Suppose $(x)(y) = 0$. If $y \neq 0$, then $x=0$.

Joe's argument is as follows:

(a) given $(x)(y) = 0$ and $y \neq 0$

(b) to show that $x = 0$

1. either $x = 0$ or $x \neq 0$;
2. for the sake of argument, suppose that $x = 3$;
3. since $(x)(y) = 0$, therefore $(3)(y) = 0$ and therefore $(1/3)(3y) = 0$;
4. but, since $(1/3)(3) = 1$, this means that $y = 0$;
5. but, $y = 0$ is false and so the supposition that $x \neq 0$ must be false;
6. therefore $x = 0$.

Although Joe's argument is correct, Tom does not understand it.

Questions:

(a) How would you explain step 2 of Joe's argument to Tom?

(b) How would you explain step 5 of Joe's argument to Tom?

(c) How would you explain step 6 of Joe's argument to Tom?

Situation Eight

Joe: "Tom, in situation seven on the previous page, I showed that for all real numbers, x and y, if $(x)(y) = 0$ and if $y \neq 0$, then $x = 0$."

Tom: "Yes, I see. However, I feel that your argument is completely unnecessary. Look, everybody knows that if $(x)(y) = 0$ and $y \neq 0$, then x must be equal to zero. There is no need to show it."

Joe: "I agree that everybody knows that this proposition is true, but I disagree that my argument is unnecessary."

Tom: "Look, if $3x = 0$, then $x = 0$; if $7x = 0$, then $x = 0$, and so on. You don't have to give me any argument to show me that the proposition is true."

Questions:

(a) Whose side would you be on in the above discussion?

Joe's____________ or Tom's____________

(b) Why?

Situation Nine

Statement A: Let f be any factor of some number, n. If n is an odd number, then f is an odd number.

Joe: "I think that Statenent A is true, Tom."

Tom: "Let me see. If n = 21, then the factors of n are 1, 3, 7, and 21. So, n is odd and all of its factors are odd. If n = 45, then the factors of n are 1, 3, 5, 9, 15, and 45. Again n is odd and all of its factors are odd. So Statement A seems to be true, Joe."

Joe: "I also think Statement B is true, Tom."

Statement B: Let f be any factor of some number, n. If f is an even number, then n is an even number.

Tom: "Why?"

Joe: "1. Since f is a factor of n, therefore n = (f)(m), where m is some integer.

2. If f is even, then f = 2k, where k is some integer.

3. Therefore, $n = (f)(m) = (2k)(m) = 2[(k)(m)]$.

4. Therefore, 2 is a factor of n and n is even."

<u>Tom</u>: "But Joe, this only shows that Statement B is true. It doesn't show that Statement A is true."

<u>Joe</u>: "Yes it does."

<u>Tom</u>: "I don't agree, Statement B has nothing to do with Statement A."

Questions:

(a) Whose side would you be on in the above discussion?

Joe's____________ or Tom's____________

(b) Why?

Situation Ten

Joe and Tom are discussing prime numbers. Recall that a prime number is a positive whole number, other than one, which is divisible only by one and itself.

Joe: "I've been trying to find a formula which will always give me a prime number and I've finally succeeded, Tom."

Tom: "What is your formula, Joe?"

Joe: "$n^2 - n + 17$.
When $n = 1$, $n^2 - n + 17 = 1^2 - 1 + 17 = 17$;
when $n = 2$, $n^2 - n + 17 = 2^2 - 2 + 17 = 19$;
when $n = 3$, $n^2 - n + 17 = 3^2 - 3 + 17 = 23$.
It just keeps giving me prime numbers."

Tom: "What about when $n = 17$? Then $n^2 - n + 17 = 17^2 - 17 + 17 = 17^2$."

Joe: "Well, that's only one exception and we can ignore that."

Questions:

(a) Whose side would you be on in the above discussion?

Joe's____________ or Tom's____________

(b) Why?

Situation Eleven

Suppose you are given as true the following facts:

1) Tokyo is larger than Los Angeles.

2) Toronto is smaller than Los Angeles.

3) Toronto is larger than Edmonton.

Joe and Tom wish to show that Edmonton is smaller than Tokyo using the above facts:

Joe argues as follows: Since Toronto is larger than Edmonton, therefore Edmonton is smaller than Toronto. Since Edmonton is smaller than Toronto and Toronto is smaller than Los Angeles, therefore Edmonton is smaller than Los Angeles. Since Tokyo is larger than Los Angeles, therefore Los Angeles is smaller than Tokyo. Since Edmonton is smaller than Los Angeles and since Los Angeles is smaller than Tokyo, therefore Edmonton is smaller than Tokyo.

Tom argues as follows: Either Edmonton is smaller than Tokyo or it is larger than Tokyo. Suppose Edmonton is larger than Tokyo. Then since Tokyo is larger than Los Angeles, therefore Edmonton would be larger than Los Angeles. Since Los Angeles is larger than Toronto, therefore Edmonton would be larger than Toronto. But this would contradict the fact that Toronto is larger than Edmonton. Therefore the original hypothesis that Edmonton is larger than Tokyo must be false. Therefore Edmonton is smaller than Tokyo.

Both Joe and Tom have shown that Edmonton is smaller than Tokyo.

Questions:

(a) Which argument would you have used?

Joe's____________ or Tom's____________

(b) Why would you have used this argument?

Situation Twelve

Joe: "All odd numbers greater than 627 are prime numbers."

Tom: "Show me."

Joe: "1. Suppose x is a prime number greater than 627.

2. It follows from the definition of a prime number that the only exact divisors of x are 1 and x itself.

3. Therefore 2 cannot be an exact divisor of x.

4. Therefore x cannot be even.

5. Therefore x is odd. So all odd numbers greater than 627 are prime numbers."

Question:

How would you reply if you were Tom?

Appendix B

Response Categories for Written Questionnaire

SITUATION ONE

Category:	Description of Category:
1.	(a) No response; (b) No meaningful response.
2.	Joe's conclusion that the area of triangle BEC is one half the area of rectangle ABCD is rejected and reasons given supporting the rejection.
3.	Joe's conclusion is accepted but it does not apply to both diagrams for reasons such as: (a) the two diagrams are not the same, or have different measurements or are not congruent; <u>or</u> (b) the conclusion is not generalizable to all other such diagrams. (All responses in this category indicate a lack of

understanding of the generalization principle.)

4. Joe's conclusion is accepted but proof is required in order to show that it applies to both diagrams. (Responses in this category include incomplete or incorrect attempts to provide such a proof.)

5. Joe's conclusion is true for both diagrams but the reasons given make it difficult to ascertain whether or not the student understands the generalization principle.

6. Joe's initial conclusion is true for both diagrams because the area of a triangle is always one half the base times the altitude. Included in this category are all responses which show some understanding of the generalization principle.

0. The response does not appear to fall

in any of the above categories.

SITUATION TWO

Category:	Description of Category:
1.	(a) No response; (b) No meaningful response.
2.	Joe's generalization is considered invalid on the basis of a "false" counter-example such as: 4^n-1 is not divisible by 3 when $n = 0$.
3.	Joe has made a correct generalization based on the examples given and this generalization is always true because: (a) Joe has given a sufficient number of examples to prove it, or (b), it is impossible for Joe to verify it for all positive integers individually, and since it is true in all the cases he tried, therefore it must be true always.

4. Joe has made a correct generalization, but it may not be true for all positive integers because Joe has not tested it enough or tried all of the possibilities. The responses in this (and the previous) category are characterized by thinking considered to be inductive in nature.

5. Joe has made a correct generalization, based on the evidence presented, which is always true because Tom has not provided a counter-example or any reason why it is not true. Included in this category are those responses which indicate that any proposition in mathematics is true unless or until it is proven false.

6. Joe has made a correct generalization based on the evidence presented but the generalization may not be true

for all positive integers because it is possible that a counter-example exists.

7. Joe has made a correct generaliztion based upon the evidence presented, but this evidence does not constitute proof of Joe's generalization. Responses in this category include complete, incomplete or incorrect attempts to prove Joe's generalization.

0. The response does not appear to fall in any of the above categories.

SITUATION THREE

Category:	Description of Category:
1.	(a) No response; (b) No meaningful response.
2.	Joe's proposition is untrue: (a) because 3 is "not" a factor of 57 as indicated in the first statement of

this item; (b) because a counter-example to the converse of Joe's proposition was provided; or (c) because some other "invalid" counter-example was given.

3. No definite position is taken with respect to the truth of Joe's propositon. Included in this category are those responses which consider and present arguments (sometimes contradictory in nature) supporting the points of view of both Joe and Tom.

4. Joe's proposition is always true and Tom can be convinced by providing him with a "sufficient" number of confirming instances of the proposition.

5. Joe's proposition may or may not be true because not enough confirming instances of the proposition have been provided.

6. Joe's proposition is always true because no counter-example to the proposition has been or can be provided. Tom can be convinced of this fact simply by asking him to produce at least one counter-example.

7. Joe's proposition may <u>not</u> be true (a) as can be demonstrated by providing a counter-example; or (b) because it is possible that a counter-example exists.

8. Joe's proposition is always true and Tom can be convinced by providing him with a proof of the proposition.

9. Joe's proposition is always true and a complete, incomplete or incorrect proof provided.

0. Responses which do not appear to fall into any of the above categories.

SITUATION FOUR

Category:	Description of Category:
1.	(a) No response.
	(b) The response is clearly based upon meaningless or irrelevant considerations.
2.	Joe has made a correct generalization based upon the evidence presented.
3.	The evidence given is insufficient to support any conclusion.
4.	Joe's conclusion is rejected but a valid reason is not given.
5.	Joe's conclusion is rejected (a) by providing an explicit counter-example; (b) by appealing to the identity $a^2 - b^2 = (a-b)(a+b)$, or (c) for other valid reasons.
6.	Joe's conclusion is rejected and a

correct generalization given based upon the evidence presented.

0. The response does not appear to fall in any of the above categories.

SITUATION FIVE

Category:	Description of Category:
1.	(a) No response; (b) No meaningful response.
2.	No definite position is taken with respect to the truth of Joe's conclusion. Included in this category are those responses which consider and present arguments (often contradictory in nature) supporting the points of view of both Joe and Tom.
3.	Joe's conclusion is rejected because of an apparent refusal to accept and use the given definition of a line

in this geometry. Normally, responses in this category refuse to accept any definition other than the Euclidean, thereby ignoring or refusing to accept the hypothesis of only four distinct points.

4. Joe's conclusion is partially accepted. Responses in this category, among others, accept the proposition that two pairs of parallel lines exist, but not three.

5. Joe's conclusion is accepted because Tom has not provided any justification for the position he takes.

6. Joe's conclusion that there exists three pairs of parallel lines in this geometry is accepted. Responses in this category justify the truth of Joe's conclusion by stating that the argument given follows from the definitions and assumptions given.

7. Joe's conclusion is accepted and justified by showing that one or both of the other two pairs of lines are parallel.

0. Responses which do not appear to fall into any of the above categories.

SITUATION SIX

Category:	Description of Category:
1.	(a) No response. (b) No meaningful response.
2.	Joe's argument is valid and yet it isn't. Responses in this category agree with both Joe and Tom and present arguments for both sides without realizing the contradictory nature of such a response.
3.	Joe's argument is rejected because it is "invalid" to "suppose that 1 = 0". Responses in this category do not

appear to accept the reasoning from a hypothesis which is considered to be "false".

4. Joe's argument is rejected because of an apparent lack of understanding of the role that the hypothesis in step one plays in the succeeding argument.

5. Joe's argument is rejected because the reasoning in steps 3, 4, and/or 5 of the argument is considered invalid.

6. Joe's argument is rejected because of an apparent lack of understanding of what the contradiction in step 5 really means.

7. Joe's argument is accepted but the reasons given indicate an acceptance of authority rather than an understanding of the methods used.

8. Joe's argument is accepted and the justification given is considered to

be based on a peripheral understanding of the methods used. Responses in this category include, where appropriate, those which suggest that one can show that a statement is valid by demonstrating that its negation is false or contradictory.

9. Responses which reflect a mature view of the method of indirect proof.

0. Responses which do not appear to fall in any of the above categories.

SITUATION SEVEN

Category:	Description of Category:
1.	No response.
2.	Response displays a complete lack of understanding of the methods used.
3.	The role of the hypothesis in step 2 of the argument is not adequately

explained.

4. The hypothesis in step 2 is adequately explained but the contradiction in step 5 is not. Responses in this category include those which argue in a circle by assuming the validity of the proposition being proven in order to explain step 5 of the argument.

5. The explanations given are judged to be based on an acceptance of authority rather than an understanding of the methods used.

6. A mature understanding of the argument given is indicated by adequate explanations of steps 2, 5, and 6.

0. Responses which do not appear to fall in any of the above categories.

SITUATION EIGHT

Category:	Description of Category:
1.	(a) No response; (b) No meaningful response.
2.	It is intuitively obvious that Joe's proposition is valid for all real numbers x and y and hence, no more convincing evidence (proof) is required.
3.	Joe's proposition is intuitively valid for all real numbers x and y, but a proof is required. Responses in this category are judged to be based upon an acceptance of authority rather than a mature understanding of the necessity for proof.
4.	Even though Joe's proposition may be considered to be intuitively valid for all real numbers x and y, a proof

is required to show its validity beyond all doubt. Responses in this category are judged to possess a mature understanding of the need for proof.

0. Responses which do not appear to fall in any of the above categories.

SITUATION NINE

Category:	Description of Category:
1.	(a) No response. (b) No meaningful response.
2.	Statement A or B is shown to be "invalid" by providing a counter-example to its converse.
3.	Response indicates that there is no relationship between statements A and B.
4.	Response indicates that there is a

relationship between A and B, but the nature of this relationship is not clearly explained or the explanation given is invalid.

5. Response indicates that statements A and B are equivalent because one is the contrapositive of the other.

0. Responses which do not fall in any of the above categories.

SITUATION TEN

Category:	Description of Category:
1.	(a) No response; (b) No meaningful response.
2.	Joe's claim is true because it is true in the majority of instances.
3.	Joe's claim is true because $n = 17$ might be the only exception.
4.	Joe's claim is untrue because $n = 17$

is a counter-example and this is sufficient reason for the claim to be rejected.

0. Responses which do not fall in any of the above categories.

SITUATION ELEVEN

Category:	Description of Category:
1.	(a) No response; (b) No meaningful response.
2.	The direct argument is selected because it is easier to follow, whereas Tom is arguing backwards.
3.	The indirect argument is selected because it is less confusing.
4.	Response which attempt to justify both arguments.
0.	None of the above.

SITUATION TWELVE

Category:	Description of Category:
1.	(a) No response; (b) No meaningful response.
2.	Joe's proposition is true because he has proved it.
3.	Joe's proposition is true only if he can prove it for any specific number greater than 627 that is selected.
4.	Joe's proposition is rejected and a counter-example given.
5.	Joe's proposition is rejected because his argument is that of the converse of the stated proposition.
0.	None of the above.

Appendix C

Interview Request

You have been selected to participate in follow-up interviews regarding the written questionnaire that you responded to at the beginning of the quarter.

The interview will take half an hour and can be arranged at your convenience any time during the remainder of this quarter except from 12 noon to 2:00 PM on monday, tuesday, wednesday, or friday.

I can be contacted through the secretary at the mathematics department on third floor Kidder Hall, extension 4686. Or you may leave your name, phone number and when I can reach you on this paper with your MTH 311 professor.

Thank you very much for your participation. The results of this study are intended to assist in defining the course MTH 311 and have nothing to do with your grade in MTH 311. Of course, the results are confidential.

NAME________________________

PHONE________________________

WHEN I CAN BE

REACHED____

POSSIBLE INTERVIEW

TIMES____

Appendix D

Interview Script

1) Without tape running, talk with interviewee in order to ease tensions. Explain that this interview is not intended as a test of their mathematical ability. Rather, it is an attempt to find out some of their subjective thoughts on mathematics in general and mathematical argument in particular.

2) Turn on tape and collect some demographic information:

(a) "What is your name?"

(b) "What is your major field of study?"

(c) "What mathematics courses are you taking now?"

(d) "Today's date is ----- ."

Turn the recorder off, rewind, and play to see if it is working.

3) Turn the recorder back on and ask the following questions:

(a) "What are the differences between calculus and MTH 311? ... explain."

(b) "How about proof, is there a significant difference in the treatment of proof between these courses? ... how?"

(c) "What is mathematical proof? ... explain."

(d) "What role does proof play in mathematics?"

(e) "What is the role of a postulate?" (axiom)

(f) "Do you need to understand proof in order to do mathematics?"

(g) "When do you think proof is really necessary? ... explain."

(h) "How important is proof to mathemeticians? ... to the general public?"

4) (a) "Do you enjoy proving things? ... why?"

(b) "Do you ever work mathematical proofs in your liesure time? ... why?"

(c) "Do you enjoy solving problems ... why?"

(d) "How about solving a mathematical problem? Do you ever work on that in your liesure time?"

(e) "What is the difference between proof and problem solving?"

(f) "How do you think other people feel about proof?"

5) (a) "Are you good at proving mathematical propositions?"

(b) "How do you begin when you are faced with the task of proving something?"

(c) "What would you do if someone challenged one of your proofs?"

6) Present the interviewee with your prepared material and ask:

(a) "What do you think about this (situation, argument, proof)? You may or may not have seen it before on the questionnaire you filled out at the beginning of the quarter. Take a little time to look it over."

Turn the recorder off and let the interviewee read Situation Five.

7) Turn the recorder back on and ask the following questions:

(a) "What exactly does Joe assume in this situation? ... can he do that? ... why? (or why not?)"

(b) "Given those assumptions, what is a line according to Joe? ... what do you think about that?"

(c) "Given those assumptions is Joe's conclusion that there exist three pair of parellel lines valid? ... why? (or why not?)"

(d) "Does a geometry like this have any purpose? ... explain."

8) End of interview. Thanks.

Appendix E

Interview Quotes

The following appendix contains examples of interview quotes that were used in determining the response categories outlined in Chapter Four.

The Nature of Mathematical Proof

1) proof is a means of justification or verification of statements known or believed to be true;

-"... it offers some kind of assurance to the mathematician that what he is doing isn't just merely hand-waving."

-"Mathematical proof, a precise statement of truths."

2) proof is a way of getting to "fundamental" or basic ideas; working backward;

-"... getting down to the very fundamental things that you can say are true."

3) proof is part of a building process from a certain groundwork up.

-"First you have to start with your basics ... then yoy can build them up to other things that are always true, uh, theorems..."

The Role of Proof in Mathematics

1) proof is a means of justification in an abstract mathematical system;

-"we all agree on certain rules that we're gonna play the game by and the proofs come out of those rules and if you can show that a new result is tied to those rules--that is a proof."

-"we need to do proofs ... to give some semblance of rigor to the calculus."

-"... proving a statement is true means ... you're sure that the statement exists within the specified conditions."

-"Taking premises and drawing conclusions; there's gotta be a systematic way of doing it."

2) proof is a means of justifying facts, truths about reality;

-"He could prove that only if he could assert it to something that's real."

-"... a way of describing a phenomenon precisely, maybe?"

-"We're just describing things the way we want them to work, and the way we think, physically, they should."

3) proof enhances understanding;

-"... when your answer looks like it doesn't make any sense."

4) proof provides a foundation for mathematics;

-"... to provide a foundation, something meaningful."

-"I have never thought about it ... I think the role of it is basically a setting up of a foundation for ideas and to make those ideas firm."

-"... everything has to be, um, founded upon something."

5) proof has no role whatsoever;

-"Nothing in particular that I can see."

6) unclear.

-"... what it's good for, I'm not sure."

-"I don't know why, but ... you have to go through some proofs -- I think to come up with an equation."

-"... proof, oh, is, shows you a relationship between two systems of equations."

The Importance of Proof in Mathematics

1) proof is an activity that only theoretical mathematicians need be concerned with, to applied mathematicians and scientists it is not necessary at all;

-"If you are working pure mathematics, I think that is very essential but as for applied mathematics, I don't think it is necessary at all."

-"You don't really need to understand the proofs to do the manipulations."

2) proof is not necessary for "low-level" mathematics, but its importance increases as "you go higher";

-"I don't see a real practical purpose in being able to do a proof unless you're going to be really involved in mathematics."

-"It depends on how high up you go, eventually I think you do."

3) proof is not necessary at all;

-"only when things aren't obvious."

4) unclear.

-"I guess I don't exactly know what a mathematicians gotta do."

-"No practical, well, I don't know about that. It's all so confus ..."

<u>The Importance of Proof to the Interviewees</u>

1) proof is necessary to provide rigor to mathematics, I wish to do mathematics, therefore proof is important to me;

-"I plan on doing theoretical mathematics when I get out of college, and when I hit that level, I'm gonna have to be able to write up my work."

2) proof is not a necessary ingredient of applied mathematics, I wish to do applied mathematics, therefore proof is not important to me;

-"*I* don't think so. I just want to do the numerical analysis ... reduce errors and things like that."

3) I can do without it;

-"I think I could survive without the proofs."

4) unclear.

-"Well, I don't know, it doesn't, it's helped me in some numerical analysis ..."

The Nature of Axioms

1) axioms are arbitrary ground rules set by mathematicians so they can "play the game";

-"Well, an axiom is a ground rule that can't be proved ... that's what I mean by the rules of the game."

2) axioms are universal truths of nature;

-"It's beyond the scope of what we know ... we can see that it's true, but ... proving, we're not quite sure how to do that."

-"... axiom is just the way it is."

-"I'm not really sure, the derivation of axioms ... sort of universal truth, you know?"

3) axioms are statements that are hard to prove, hence they are assumed;

-"... it's something that seems obvious, but it's hard to get the proof to."

-"... something that you can't really prove it's true, but you can't disprove it either."

4) axioms are statements that are assumed because "everyone else does";

-"... fundamental things that can't be proved, but we're just gonna assume because everyone else does."

5) axioms are the foundation of mathematics;

-"Everything else works because that is true."

-"It has to be your foundation."

6) I don't know what an axiom is;

-"I really don't know the difference between the theorems and the postulates."

<u>The Nature of Theorems</u>

1) a theorem is a new result, derived from axioms;

-"... a theorem is a new result derived from that axiom."

-"A theorem is built out of postulates."

-"All the theorems are based on axioms."

-"The theorem, you can go back ... it may take a long time but you can trace that right back to field properties..."

2) a theorem is a statement that requires proof (as opposed to an axiom, which does not require proof);

-"... not necessarily fundamental truth because you have to prove that ... once we prove that it just makes it easier."

3) a theorem is a postulate;

4) a theorem is a fact;

-"... sort of a mathematical fact that has been found over time not to be not true, or something like that."

5) a theorem is a statement that may or may not be true;

-"... the theorem is a statement that may or may not be true."

6) unclear.

-"... um, well, I may be 100% incorrect but my general feeling is that, uh, I don't really know why, why this is so but my general feeling is that I, uh, I guess I'm not really sure."

The Difference Between Scientific and Mathematical Reasoning

1) scientific reasoning is inductive, generalizing from particular observations, mathematical reasoning is separate from reality, arbitrary;

-"the mathematician can be much more sure within his realm of thinking that what he says is true ... given that these axioms are correct. The experiment is basically saying this is the way it goes based on -- we see it occurring alot."

-"(mathematics) doesn't have to answer to any reality, or anything like that. It's its own separate reality. Given a certain set of axioms you can do anything you want."

2) science requires proof, otherwise "the data might be bad", mathematics proves what you can say about the world and science;

-"If you're a scientist, you can run an experiment and get your data and you may think it means something, but if you don't have any proof ... maybe you've lost all significance."

-"... mathematicians prove ... what you can say about the world and what you can say about science and data."

-"Proof ... lays the whole foundation of mathematics, or any kind of science for that matter."

3) experiments or mathematics provide proof (it was unclear whether or not the student recognized a distinction between the two);

-"... proving a statement is true, whether experimentally or mathematically ... maybe the method that you go about are one and the same, but the procedures, but, uh ..."

4) unclear.

Students' Enjoyment of Mathematical Proof

1) it is satisfying to complete a proof;

-"I guess I enjoy it in the same perverse way I enjoy doing mathematical problems ... it's a really nice feeling to establish it."

2) some proofs are entertaining;

-"Sometimes proofs are -- certain proofs are -- very entertaining."

3) proofs are not enjoyable because it is hard to "visualize what's going on";

-"I like doing that with the computer ... more than with math because I can see what's going on."

-"I have trouble visualizing..."

-"Sometimes it's frustrating, I think I don't know how; I just miss it."

4) unsolved proofs are fun to ponder;

5) calculations are preferred to proofs;

-"I liked going through the derivatives ... solving the problems ... long and tedious."

-"I do number crunching in my spare time."

-"I enjoy applications more than I enjoy theory."

6) calculations are disliked;

-"Calculate this. That to me ... doesn't hold alot of meaning ... Alot of times -- here's an integral, you gotta do it right now -- there's no real thinking process involved."

7) working proofs under pressure is particularly distasteful;

-"... not under pressure."

8) proofs are scary;

-"I guess I'm kind of scared of proofs."

-"... scary."

9) constructing proofs are a necessary hurdle on the way to a degree;

-"I look at it as something I gotta get through to get my goal."

10) proofs are not very stimulating.

-"I don't like to do the proofs ... it was hard for me to take something that looked obvious."

-"I don't want to spend the rest of my life proving things."

-"I get my fill of it from my math classes."

-"... I don't hate it."

Students' Estimates of Their Ability to Construct Proofs

1) not bad at proving things, but could use improvement;

-"I wouldn't say I'm bad, but I think I could be alot more efficient."

2) not very good at proof;

-"I can't really say that (I'm good at constructing proofs)."

-"No, not really."

-"Well, you know, I'm giving it my best ... I still have trouble."

-"... somehow I seldom have what I need to show ... that specific proof."

-"I haven't developed a self-confidence about doing proofs."

3) not very good at proof, at least according to my professors;

-"Not according to my professor."

4) scared of proofs.

Techniques Reported by Students

1) starting with what is known and making associations;

-"... by looking at the givens and making some kind of association."

-"I try to find anything that I can relate to it that I know is true."

2) utilizing short, powerful bursts of thinking;

-"... my mind is mulling over these proofs all the time ... suddenly, I'll get the idea ... it's just that short burst of very powerful thinking that will give you the proof."

3) memorizing of theorems;

4) using techniques from computer programming;

-"I found that the biggest advantage I have in 311 is that I have a good programming backround and that allows me to break the problem up in a number of pieces."

5) using direct and then indirect methods;

-"The first thing I do is direct proof, if I can't do it I go to indirect proof by contraindication."

6) "lying", putting statements in the proof just to fill up space when arriving at a difficult point in the proof.

-"... like one thing I've noticed, I catch myself lying ... get to a point and you're stuck and you knom that you can skip over some steps ... I haven't proved this one little part."

Proof and Problem Solving

1) proving a theorem and problem-solving are, in some way, related or similar activities;

-"Well, problem-solving is just that -- well, o.k., problem-solving is in a way, can be, a proof. I mean it has the same flavor to it."

-"A problem and a proof are the same because we are trying to find some relationship usually."

2) solving problems is a fundamentally empirical task. Proof is a theoretical task.

-"With problem-solving you know what to do. Doing a proof you don't know where exactly to begin, and where to go from where you started."

-"... solving problems is just like a cookbook."

-"It's easier to solve problems than do proofs."

<u>The Four-Point Geometry of Situation Five</u>

1) Joe cannot state that there are only four points on the sheet of paper, because there are obviuosly more than that; and

-"No, space is filled with points for one thing, and around those points you can find other points."

-"... four distinct points on this sheet of paper ... you can't say that."

-"... out there, somewhere in the distance, there's another point, E, where AB intersects CD."

2) lines must contain an infinite number of points, regardless of Joe's claim to the contrary.

-"How could you even draw a line? There are only four distinct points on the sheet of paper."

-"... if you only have four points, you have four points and you don't have a line."

-"... there's just no line there."

-"In the real universe, these lines would intersect."

Appendix F

Revised Written Questionnaire

Name________________________________

Date of Birth________________________

Male______________ Female______________

Class: Fr So Ju Sr Gr (circle one)

Instructor_____________________________

Directions:

On the following pages you will be asked some questions related to mathematics. These questions are in no way intended as a test or examination. You are requested to read the discussion on each page carefully and answer each question in the space provided using the first thoughts or opinions that enter your mind.

Situation One

Joe has observed the following interesting pattern:

$4 - 1 = 3$ is divisible by three.

$4^2 - 1 = 15$ is divisible by three.

$4^3 - 1 = 63$ is divisible by three.

$4^4 - 1 = 255$ is divisible by three.

$4^5 - 1 = 1023$ is divisible by three.

$4^6 - 1 = 4095$ is divisible by three.

Joe borrowed a calculator and found out that $4^n - 1$ is divisible by 3 regardless of what value of n he tried. Therefore he came to the following conclusion:

$4^n - 1$ is divisible by 3 for all positive whole numbers n.

While Joe was working on this problem, Tom walked into the room. Tom looked at the conclusion and immediately stated that he was not convinced that Joe's conclusion was always true. Tom felt that while the conclusion was true for n = 1, 2, 3, 4, 5, 6 etc., this did not rule out the possibility of there being some number for which the conclusion was not true.

But Joe disagreed with Tom. Whatever value of n he had tried on the calculator confirmed the truth of his conclusion and therefore, as far as Joe was concerned, it was always true.

Questions:

(a) Whose side are you on?

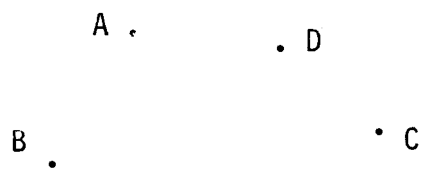
Joe's__________ or Tom's__________

(b) Why?

Situation Two

Joe and Tom are discussing parallel lines.

Joe: "Suppose there are only four distinct points on this sheet of paper instead of an infinite number."



Tom: "So you are imagining that the four points A, B, C, and D above are the only points on this sheet of paper."

Joe: "Right. Now, since two points determine a unique line, these four points determine six distinct lines. Each of these six lines contains only two points. Do you see what the six lines are, Tom?"

Tom: "Yes."

Joe: "Now remember that if two lines have no point in common, then they are parallel."

Tom: "I agree."

Joe: "I claim that there are three pairs of parallel lines determined by the four given points."

Tom: "That's nonsense, Joe. None of the lines determined by the four given points can possibly be parallel."

Joe: "So you would say that the line determined by the points A and B (for example) is not parallel to the line determined by the points C and D."

Tom: "Of course these lines are not parallel."

Joe: "O.K., if these lines are not parallel, then they must intersect in some point. Since there are only four points on the sheet of paper, the lines must intersect in either A, B, C, or D. But clearly the line determined by A and B does not intersect the line determined by C and D in either C or D and vice versa. Therefore, these two lines must be parallel."

Tom: "I don't care what you say, the four given points do not determine any parallel lines."

Joe: "But I've shown otherwise Tom, and in fact I can show that there are three pairs of parallel lines determined by the four given points."

Questions:

(a) Whose side would you be on in the above discussion?

Joe's_________ Tom's_________ Neither_________

(b) Why?

Situation Three

Joe has shown that the following statement is true for all real numbers x and y.

Statement Suppose $(x)(y) = 0$. If $y \neq 0$, then $x=0$.

Joe's argument is as follows:

(a) given $(x)(y) = 0$ and $y \neq 0$

(b) to show that $x = 0$

1. either $x = 0$ or $x \neq 0$;
2. for the sake of argument, suppose that $x \neq 0$;
3. since $(x)(y) = 0$, therefore $(1/x)(x)(y) = 0$ ($1/x$ is allowed since $x \neq 0$);
4. but, since $(1/x)(x) = 1$, this means that $y = 0$;
5. but, $y = 0$ is false and so the supposition that $x \neq 0$ must be false;
6. therefore $x = 0$.

Although Joe's argument is correct, Tom does not understand it.

Questions:

(a) How would you explain step 2 of Joe's argument to Tom?

(b) How would you explain step 5 of Joe's argument to Tom?

(c) How would you explain step 6 of Joe's argument to Tom?

Situation Four

Joe: "Tom, in situation seven on the previous page, I showed that for all real numbers, x and y, if $(x)(y) = 0$ and if $y \neq 0$, then $x = 0$."

Tom: "Yes, I see. However, I feel that your argument is completely unnecessary. Look, everybody knows that if $(x)(y) = 0$ and $y \neq 0$, then x must be equal to zero. There is no need to show it."

Joe: "I agree that everybody knows that this proposition is true, but I disagree that my argument is unnecessary."

Tom: "Look, if $3x = 0$, then $x = 0$; if $7x = 0$, then $x = 0$, and so on. You don't have to give me any argument to show me that the proposition is true."

Questions:

(a) Whose side would you be on in the above discussion?

Joe's____________ or Tom's____________

(b) Why?

Situation Five

Joe and Tom are discussing prime numbers. Recall that a prime number is a positive whole number, other than one, which is divisible only by one and itself.

Joe: "I've been trying to find a formula which will always give me a prime number and I've finally succeeded, Tom."

Tom: "What is your formula, Joe?"

Joe: "$n^2 - n + 17$.
When $n = 1$, $n^2 - n + 17 = 1^2 - 1 + 17 = 17$;
when $n = 2$, $n^2 - n + 17 = 2^2 - 2 + 17 = 19$;
when $n = 3$, $n^2 - n + 17 = 3^2 - 3 + 17 = 23$.
It just keeps giving me prime numbers."

Tom: "What about when $n = 17$? Then $n^2 - n + 17 = 17^2 - 17 + 17 = 17^2$."

Joe: "Well, that's only one exception and we can ignore that."

Questions:

(a) Whose side would you be on in the above discussion?

Joe's____________ or Tom's____________

(b) Why?

Situation Six

Joe: "All odd numbers greater than 627 are prime numbers."

Tom: "Show me."

Joe: "1. Suppose x is a prime number greater than 627.

2. It follows from the definition of a prime number that the only exact divisors of x are 1 and x itself.

3. Therefore 2 cannot be an exact divisor of x.

4. Therefore x cannot be even.

5. Therefore x is odd. So all odd numbers greater than 627 are prime numbers."

Question:

How would you reply if you were Tom?

Appendix G

Revised Interview Script

1) Without tape running, talk with interviewee in order to ease tensions. Explain that this interview is not intended as a test of their mathematical ability. Rather, it is an attempt to find out some of their subjective thoughts on mathematics in general and mathematical argument in particular.

2) Turn on tape and collect some demographic information:

(a) "What is your name?"

(b) "What is your major field of study?"

(c) "What mathematics courses are you taking now?"

(d) "Today's date is ----- ."

Turn the recorder off, rewind, and play to see if it is working.

3) Turn the recorder back on and ask the following questions:

(a) "What is mathematical proof? ... explain."

(b) "What function does it perform -- what is the role of proof?"

(c) "How do you know if a proof is valid or not?"

(d) "How do axioms and theorems differ?

(e) "What is the role, or function, of axioms and threorems?"

(f) "Where does logic fit in this picture? For example, what is the use of truth tables?"

(g) "In mathematics, do you think proof is always necessary? ... explain."

(h) "Do you need to understand proof in order to do mathematics?"

(i) "Is there a difference between the kind of reasoning that a mathematician uses and the kind of reasoning that a scientist uses? ... explain.

(j) "What does it mean when a mathematician says that a proposition is 'true'?"

(k) "Is this kind of 'truth' different from the kind of 'truth' a scientist considers? ... explain."

4) (a) "Do you like working mathematics problems?"

(b) "How about proofs, do you enjoy constructing mathematical proofs? ... why?"

(c) "Do you ever work mathematical proofs in your liesure time? ... why?"

(d) "How do you think other people feel about proof?"

5) (a) "Are you good at proving mathematical propositions?"

(b) "How do you begin when you are faced with the task of proving something?"

(c) "How do you know when you are done?"

6) (a) "Have you ever thought before about the kind of questions we have been talking about?"

(b) "Have you ever discussed these kinds of questions with any body before? ... Who?"

End of interview. Thanks.