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Title PROXIMATE ORDERS AND GENERALIZED INDICATORS OF ENTIRE FUNCTIONS

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Frequently, in the theory of entire functions, one is confronted with the problem of comparing the growth of two functions \( f(z) \) and \( g(z) \) as the independent variable \( z \) tends to infinity.

A well-known procedure for measuring the growth of an entire function \( f(z) \) consists of defining the order \( \rho \) of \( f(z) \) by

\[
\rho = \lim_{r \to \infty} \frac{\log \log M(r)}{\log r}, \quad M(r) = \max \{|f(z)| : |z| = r\}
\]

and if \( \rho_f > \rho_g \), then the growth of \( f(z) \) is said to be greater than that of \( g(z) \). If \( \rho \) is finite, then another number \( \sigma \), called the type of \( f(z) \), is defined by

\[
\sigma = \lim_{r \to \infty} \frac{\log M(r)}{r^\rho},
\]

and if \( \rho_f = \rho_g \), but \( \sigma_f > \sigma_g \), then the growth of \( f(z) \) is said to be greater than that of \( g(z) \). If one is concerned with the
growth of $f(z)$ along a ray $\arg z = \theta$ in an angle $[\theta_1, \theta_2]$, then the indicator function $h(\theta)$ is defined by

$$h(\theta) = \lim_{r \to \infty} \log |f(re^{i\theta})| .$$

If $h_f(\theta) > h_g(\theta)$, then the growth of $f(z)$ is said to be greater than that of $g(z)$ along the ray $\arg z = \theta$.

A less well-known procedure for measuring growth consists of replacing the constant $\rho$ by a function $\rho(r)$, which is called a proximate order. The type of $f(z)$ with respect to $\rho(r)$ is then given by

$$\sigma = \lim_{r \to \infty} \log M(r) ,$$

and the generalized indicator is given by

$$h(\theta) = \lim_{r \to \infty} \log |f(re^{i\theta})| .$$

This thesis is concerned with the latter of these two procedures. It combines, organizes, and elaborates upon much of what has already appeared in the literature. Illustrative examples are given, and, in some cases, previous results have been extended. The aim of the author has been to present a concise but comprehensive dissertation on the properties of proximate orders and generalized indicators.
Chapter 1 introduces the reader to the concept of functional growth with a discussion of ordinary orders and indicators. Chapters 2 and 3 then extend these concepts to proximate orders and generalized indicators.
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CHAPTER 1

INTRODUCTION

We begin by presenting some of the basic elements of orders and indicators of entire functions, and their relationship to the general theory of growth of entire functions.

**Definition 1.1.** An entire function is a function of a complex variable which is holomorphic in the finite complex plane.

Since an entire function $f(z)$ is holomorphic in the finite complex plane, it has derivatives of all orders at every point of the plane and thus $f(z)$ can be represented by a MacLaurin Series,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n ,$$

which converges for all values of $z$. It is natural to think of entire functions as being a generalization of polynomials, and indeed many of their properties are quite similar to those of polynomials. When, in 1876, Weierstrass proved that every entire function can be expressed as an infinite product, he opened the door for the investigation of their properties and for their classification. At approximately
the same time, Laguerre, who was working on the relationships between entire functions and polynomials, introduced the important concept of the genus of an entire function. Later, around the turn of the century, Borel, Hadamard and Lindelöf provided connections between the growth of an entire function and the distribution of its zeros.

The rate of growth of a polynomial, as the independent variable goes to infinity is determined by its degree, which in turn, is equal to the number of zeros of the polynomial. Thus, the more zeros a polynomial has, the greater its growth. This relationship between the set of zeros of a polynomial and its growth can be generalized to arbitrary entire functions, and in fact many of the classical theorems of the theory of entire functions consist of establishing relationships between the distribution of the zeros of an entire function and its asymptotic behavior, that is, the behavior of the function as the independent variable goes to infinity.

To characterize the growth of an entire function \( f(z) \), we use the associated real-valued function

\[
M_f(r) = \max_{|z| = r} |f(z)|.
\]

The maximum modulus principle tells us that as \( r \) increases, \( M_f(r) \) grows monotonically. We shall see that this function plays an important role in determining the growth of \( f(z) \).
Theorem 1.1. Let $f(z)$ be an entire function. If there exists a positive integer $n$ such that

$$\lim_{r \to \infty} \frac{M_f(r)}{r^n} = K < \infty,$$

then $f(z)$ is a polynomial of degree at most $n$.

Proof. If

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad P(z) = \sum_{k=0}^{n} a_k z^k,$$

then the function $\phi(z)$, defined by

$$\phi(z) = [f(z) - P(z)] z^{-n-1}$$

is also entire, and since

$$\lim_{r \to \infty} \frac{M_f(r)}{r^n} = K < \infty$$

implies that for all $\epsilon > 0$ and all $R > 0$, there exists an $r > R$ such that

$$\frac{M_f(r)}{r^n} < K + \epsilon,$$

we have
\[
\lim_{r \to \infty} \phi(r) = \lim_{r \to \infty} \frac{f(r) - P(r)}{r^{n+1}} \\
\leq \lim_{r \to \infty} \frac{M_f(r)}{r^{n+1}} \\
< \lim_{r \to \infty} \frac{K + \epsilon}{r} = 0.
\]

It therefore follows from the maximum modulus principle that
\[
\phi(z) = 0, \quad \text{that is,} \quad f(z) = P(z).
\]

Thus, for an entire function \( f(z) \) that is not a polynomial, \( M_f(r) \) grows faster than any positive power of \( r \). The rate of growth of \( M_f(r) \) is an important characteristic of the behavior of the entire function \( f(z) \). We shall estimate the growth of \( f(z) \) by comparing the growth of \( M_f(r) \) with that of some other function. The preceding theorem has shown that comparison functions of the form \( r^n \) are not sufficient, since, for transcendental entire functions, \( M_f(r) \) grows much faster. Choosing our comparison functions to be of the form
\[
e^{r^k}
\]
provides the motivation for the following definition.

**Definition 1.** The entire function \( f(z) \) is said to be of order \( \rho \) if
By convention we shall say that a constant has order $\rho = 0$.

This definition tells us that given any $\epsilon > 0$ the following two properties hold:

1. $M_f(r) < e^{r^{\rho + \epsilon}}$ for all sufficiently large values of $r$;
2. $e^{r^{\rho - \epsilon}} < M_f(r)$ for some sequence $\{r_n\}$ of values of $r$ tending to infinity.

An alternative definition is the following.

**Definition 1.2'.** The entire function $f(z)$ is said to be of order $\rho$ if

$$f(z) = O(e^{r^{\rho + \epsilon}})$$

for all $\epsilon > 0$ and no $\epsilon < 0$ (6, p. 248).

With a little labor, one can show that these two definitions are equivalent.

**Theorem 1.2.** The order of the sum of two entire functions does not exceed the larger of the orders of the two summands.

**Proof.** Let $f(z)$ and $g(z)$ be entire functions with respective
orders \( \rho_f \) and \( \rho_g \). Then

\[
|f(z) + g(z)| \leq |f(z)| + |g(z)|
\]

\[= O(e^{r^{\rho_f + \epsilon}}) + O(e^{r^{\rho_g + \epsilon}})
\]

\[= O(e^{r^{\max(\rho_f, \rho_g) + \epsilon}}).
\]

Later in this chapter we shall consider the order of the product of two entire functions, and we shall find that a more precise statement can be made.

Now let us look at a few examples.

**Example 1.** The polynomial \( f(z) = \sum_{j=0}^{n} a_j z^j \) has order \( \rho = 0 \).

Proof. Since

\[
M_f(r) = \max_{|z|=r} |f(z)| \leq \sum_{j=0}^{n} |a_j| r^j \leq n \max |a_j| r^n = Kr^n,
\]

and

\[
\log M_f(r) \leq \log Kr^n = \log K + n \log r \leq 2n \log r
\]

for \( r \) sufficiently large, we have

\[
\rho = \lim_{r \to \infty} \frac{\log \log M_f(r)}{\log r} \leq \lim_{r \to \infty} \frac{\log 2n + \log \log r}{\log r} = 0,
\]

and since \( \rho \geq 0 \), we have \( \rho = 0 \).
Example 2. \( f(z) = \sin z \) has order \( p = 1 \).

Proof. Let \( |z| = |x+iy| = r \), then

\[
|\sin z| = \left| \frac{e^{iz} - e^{-iz}}{2i} \right| \leq \frac{1}{2} \left[ |e^{-y+ix}| + |e^{y-ix}| \right]
\]

\[
= \frac{1}{2} (e^{-y} + e^{y}) \leq e |y| \leq e |z| \leq e^{r^{1+\epsilon}}
\]

for all \( \epsilon > 0 \), that is

\[
\sin z = O\left( e^{r^{1+\epsilon}} \right)
\]

for all \( \epsilon > 0 \), and it remains to show that equality does not hold for any \( \epsilon < 0 \). Choose \( x = 0, \ y > 0 \), so that

\[
|\sin z| = \left| \frac{e^{-y} - e^{y}}{2i} \right| = \frac{1}{2} (e^{y} - e^{-y}) < \frac{1}{2} e^{y}.
\]

Now since, for any \( \epsilon > 0 \),

\[
y > y^{1-\epsilon} + \log 2K,
\]

for some \( K \), we have

\[
\frac{1}{2} e^{y} > Ke^{y^{1-\epsilon}},
\]

and therefore,

\[
|\sin z| > Ke^{y^{1-\epsilon}} = Ke^{r^{1-\epsilon}}.
\]
that is,

$$\sin z \neq O\left(e^{r^{1+\varepsilon}}\right)$$

for any $\varepsilon < 0$. From Definition 1.2', it follows that $\rho = 1$.

**Example 3.** $f(z) = e^{kz^a}$ has order $\rho = a$

**Proof.** By definition,

$$\rho = \lim_{r \to \infty} \frac{\log \log M_f(r)}{\log r} = \lim_{r \to \infty} \frac{\log \log kr^a}{\log r} = \lim_{r \to \infty} \frac{\log k + a \log r}{\log r} = a$$

**Example 4.** $f(z) = e^{e^z}$ has order $\rho = \infty$

**Proof.** By definition

$$\rho = \lim_{r \to \infty} \frac{\log \log e^r}{\log r} = \lim_{r \to \infty} \frac{r}{\log r} = \infty$$

We see from Example 3 that, for $0 < j < k$, the functions

$$e^{jz^\rho}, e^{kz^\rho}$$

are both of order $\rho$, and yet we know that the second grows faster than the first. Therefore, a more precise characterization of the growth of an entire function is desirable, so we can distinguish between the growths of functions of order $\rho$. 
Definition 1.3. The entire function $f(z)$ of finite order $\rho$ is of type $\sigma$ if

$$\lim_{r \to \infty} \frac{\log M_f(r)}{r^\rho} = \sigma.$$ 

$f(z)$ is said to be of maximal (or finite), normal (or mean), or minimal (or zero) type when $\sigma = \infty$, $0 < \sigma < \infty$, or $\sigma = 0$, respectively.

This definition tells us that given any $\epsilon > 0$, the following two properties hold:

1. $M_f(r) < e^{(\sigma + \epsilon) r^\rho}$ for all sufficiently large values of $r$;
2. $e^{(\sigma - \epsilon) r^\rho} < M_f(r)$ for some sequence $\{r_n\}$ of values of $r$ tending to infinity.

The definition also distinguishes between functions of the form of Example 3, since $\rho = a$ and $\sigma = k$.

We often consider a function of order not exceeding $\rho$, or a function of order $\rho$ but of type not exceeding $\sigma$. We shall use the following convention to classify such functions.

Definition 1.4. An entire function $f(z)$ of order not exceeding $\rho$, and if of order $\rho$, of type not exceeding $\sigma$, is said to be of growth $(\rho, \sigma)$.

Let $(\rho, \sigma)$ denote the set of functions of order $\rho^*$ and type
where either $p^* < p$ or $p^* = p$ and $\sigma^* \leq \sigma$. Thus for $i = 1, 2$, if $f_i(z)$ is of order $\rho_i$ and type $\sigma_i$ then $f_i(z) \in (\rho_i, \sigma_i)$. If $\rho_1 < \rho_2$, then $(\rho_1, \sigma_1)$ is a subset of $(\rho_2, \sigma_2)$ and we shall say that the growth of $f_1(z)$ is less than the growth of $f_2(z)$, regardless of the values of $\sigma_1$ and $\sigma_2$. If $\rho_1 = \rho_2$ and $\sigma_1 < \sigma_2$ then $(\rho_1, \sigma_1)$ is a subset of $(\rho_2, \sigma_2)$ and again we say that the growth of $f_1(z)$ is less than the growth of $f_2(z)$. If $\rho_1 = \rho_2$ and $\sigma_1 = \sigma_2$ then other methods must be used to distinguish between the "growths" of $f_1(z)$ and $f_2(z)$.

Since an entire function can be represented by an everywhere convergent MacLaurin Series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

the sequence $\{a_n\}$ determines the function completely, and (in theory) it should be possible to discover all the properties of the function by examining the sequence of coefficients. The following two theorems show us how to determine the order and type of a function from its MacLaurin Series coefficients.

**Theorem 1.** If the function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$
is entire, then \( f(z) \) is of finite order if and only if

\[
\mu = \lim_{n \to \infty} \left( \frac{n \log n}{\log |a_n|} \right)
\]

is finite, in which case \( \rho = \mu \). (In calculating \( \mu \), the quotient is to be taken as zero if \( a_n = 0 \).)

Proof. We shall make use of the following inequality:

\[
|a_n| = \left| \frac{f^{(n)}(0)}{n!} \right| \leq \frac{1}{2\pi} \int_{|z|=r} \frac{|f(z)|}{|z|^{n+1}} |dz| \leq \frac{1}{2\pi} \frac{M(r)}{r^{n+1}} \frac{2\pi r}{r} = \frac{M(r)}{r^n}
\]

The proof of this theorem is divided into two parts: the first showing that \( \rho \geq \mu \), and the second showing that \( \rho \leq \mu \).

Part I: \( \rho \geq \mu \). If \( \mu = 0 \), then \( \rho \geq \mu \) because \( \rho \geq 0 \).

Suppose that \( 0 < \mu \leq \infty \), and choose \( \epsilon \) such that \( 0 < \epsilon < \mu \). Then since

\[
\mu = \lim_{n \to \infty} \left( \frac{n \log n}{\log |a_n|} \right)
\]

we have, for infinitely many \( n \),

\[
n \log n \geq R \log \left| \frac{1}{a_n} \right|
\]

where
Thus
\[ R = \begin{cases} \mu - \epsilon & \text{if } 0 < \mu < \infty \\ \epsilon & \text{if } \mu = \infty. \end{cases} \]

\[
\log |a_n| \geq - R^{-1} n \log n.
\]

and, using our inequality \(|a_n| \leq r^{-n} M(r)|
we see that
\[
\log M(r) \geq n \log r + \log |a_n| \geq n \log r - R^{-1} n \log n
\]
\[
= n (\log r - R^{-1} \log n)
\]
\[
= n (\log (en)^R - R^{-1} \log n),
\]
where \( r = (en)^R \), i.e., \( n = \frac{1}{e} r^R \). Therefore
\[
\log M(r) \geq \frac{n}{R} = \frac{R}{eR} ,
\]
and
\[
\frac{\log \log M(r)}{\log r} \geq \frac{\log R - \log eR}{\log r} = R - \frac{\log eR}{\log r}.
\]

Taking \( \lim \sup \), we then obtain
\[ \rho = \lim_{r \to \infty} \frac{\log \log M(r)}{\log r} \geq \lim_{r \to \infty} \left( R - \frac{\log eR}{\log r} \right) = R = \begin{cases} \mu - \epsilon, & \text{if } 0 < \mu < \infty, \\ \epsilon, & \text{if } \mu = \infty, \end{cases} \]

which implies that \( \rho \geq \mu \).
Part II: $\rho \leq \mu$. If $\mu = \infty$ then $\rho \leq \mu$, because $\rho \leq \infty$.

Suppose that $0 \leq \mu < \infty$, then since

$$
\mu = \lim_{n \to \infty} \left( \frac{n \log n}{\log |a_n|} \right),
$$

we have, for all $\varepsilon > 0$ and for all $n$ sufficiently large,

$$
0 \leq \frac{n \log n}{\log |a_n|} \leq \mu + \varepsilon,
$$

which is equivalent to

$$
|a_n| \leq n^{-\frac{n}{\mu + \varepsilon}}.
$$

Adding a polynomial to an entire function does not change its order, so we may assume that this inequality holds for all values of $n$ (if $n = 0$, we shall interpret the right side as 1). Then we have, for $|z| \leq r$,

$$
f(z) = \sum_{n=0}^{\infty} a_n z^n \leq M(r) \leq \sum_{n=0}^{\infty} |a_n| r^n \leq \sum_{n=0}^{\infty} n^{-\frac{n}{\mu + \varepsilon}} r^n = S_1 + S_2,
$$

where

$$
S_1 = \sum_{n<(2r)^{\mu+\varepsilon}} n^{-\frac{n}{\mu+\varepsilon}} r^n, \quad S_2 = \sum_{n \geq (2r)^{\mu+\varepsilon}} n^{-\frac{n}{\mu+\varepsilon}} r^n.
$$
The function
\[ s(n) = n - \frac{n}{\mu + \epsilon} r^n \]
attains its maximum at \( n = e^{-1} r^{\mu + \epsilon} \), and
\[ \mu + \epsilon > 0 \implies 2^{\mu + \epsilon} > 1 > e^{-1} \implies e^{-1} r^{\mu + \epsilon} < (2r)^{\mu + \epsilon} \]

which implies that the maximum of \( s(n) \) is a term in \( S_1 \). We shall now obtain upper bounds for the quantities \( S_1 \) and \( S_2 \).

Considering \( S_1 \) first, we have
\[ S_1 = \sum_{n<(2r)^{\mu + \epsilon}} n - \frac{n}{\mu + \epsilon} r^n \leq r^{(2r)^{\mu + \epsilon}} \sum_{n=0}^{N} n - \frac{n}{\mu + \epsilon} , \]
and as \( N \to \infty \), the series converges (via the comparison test with the \( p \)-series) to some value \( K \), so that
\[ S_1 \leq Kr^{(2r)^{\mu + \epsilon}} \]
and
\[ S_1 = O(r^{(2r)^{\mu + \epsilon}}) = O(e^{(2r)^{\mu + \epsilon} \log r}) = O(e^{r^{\mu + 2 \epsilon}}) . \]

Before we obtain an upper bound for \( S_2 \), note that
\[ n \geq (2r)^{\mu + \epsilon} \implies n - \frac{1}{\mu + \epsilon} \leq \frac{1}{2r} \implies r^n - \frac{n}{\mu + \epsilon} \leq \frac{1}{2} \implies r^n - \frac{n}{\mu + \epsilon} \leq \left( \frac{1}{2} \right)^n . \]

Therefore,
\[ S_2 = \sum_{n \geq (2r)^{\mu+\epsilon}} r^n n^{-\frac{n}{n+\mu+\epsilon}} \leq \sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^n = \left( \frac{1}{1-\frac{1}{2}} \right) - 1 = 1, \]

and \( S_2 = O(1) \).

Combining our bounds, we have,

\[ M(r) \leq S_1 + S_2 = O(e^{r^{\mu+2\epsilon}}) + O(1) = O(e^{r^{\mu+3\epsilon}}), \]

for all \( \epsilon > 0 \), and finally

\[ \rho = \lim_{r \to \infty} \frac{\log \log M(r)}{\log r} \leq \mu + 3\epsilon, \]

that is, \( \rho \leq \mu \).

In conclusion, we have shown that for \( \mu \in [0, \infty) \), \( \rho \geq \mu \) and \( \rho \leq \mu \), which is equivalent to \( \rho = \mu \). Also, in Part II, for \( \mu \in [0, \infty) \),

\[ \mu = \lim_{n \to \infty} \left( \frac{n \log n}{\log \log a_n} \right) \iff |a_n| \leq n^{-\frac{n}{\mu+\epsilon}}. \]

Recalling from the theory of analytic functions, that the radius of convergence \( R \) of a power series

\[ \sum_{n=0}^{\infty} a_n z^n \]

is given by
\[
\frac{1}{R} = \lim_{n \to \infty} \sqrt[n]{|a_n|},
\]

we have that
\[
\frac{1}{R} \leq \lim_{n \to \infty} \left( \frac{1}{n} \right) = 0.
\]

Therefore, if \( \mu < \infty \), \( f(z) \) has radius of convergence \( R = \infty \), i.e. \( f(z) \) is entire. If \( \mu = \infty \), we have either \( \rho = \infty \) or \( f(z) \) is not entire, and this completes the proof.

**Theorem 1.4.** Let \( \beta \) be a number such that \( 0 < \beta < \infty \), and let the function
\[
f(z) = \sum_{n=0}^{\infty} a_n z^n
\]
be entire, and of finite order. Let
\[
\gamma = \lim_{n \to \infty} n|a_n|^{\beta/n}.
\]

If \( 0 < \gamma < \infty \), then \( f(z) \) is of order \( \rho = \beta \) and of type \( \sigma = \frac{\gamma}{\epsilon \rho} \).

**Proof.** If \( \gamma < \infty \) then for all \( \epsilon > 0 \) and \( n \) sufficiently large,
\[
n|a_n|^{\beta/n} \leq \gamma + \epsilon,
\]

and
\[
\log n + \frac{\beta}{n} \log |a_n| \leq \log (\gamma + \epsilon),
\]
which is equivalent to

\[ n \log n - n \log (\gamma + \epsilon) = n \log n \left(1 - \frac{\log(\gamma + \epsilon)}{\log n}\right) \leq -\beta \log |a_n| , \]

and finally, for \( n \) sufficiently large,

\[ \frac{n \log n}{\log |a_n|} \leq \frac{\beta}{1 - \frac{\log(\gamma + \epsilon)}{\log n}} . \]

By the previous theorem, since \( f(z) \) is of finite order, its order is given by

\[ \rho = \lim_{n \to \infty} \frac{n \log n}{\log |a_n|} \leq \lim_{n \to \infty} \frac{\beta}{1 - \frac{\log(\gamma + \epsilon)}{\log n}} = \beta , \]

and \( f(z) \) is of order at most \( \beta \).

If \( \gamma > 0 \) then for all \( \epsilon > 0 \) and for some sequence of values of \( n \) tending to infinity,

\[ n|a_n|^{\beta/n} \geq \gamma - \epsilon , \]

and, employing an argument similar to that above, we have

\[ \frac{n \log n}{\log |a_n|} \geq \frac{\beta}{1 - \frac{\log(\gamma - \epsilon)}{\log n}} , \]

which implies that \( \rho \geq \beta \), and thus if \( 0 < \gamma < \infty \), then \( \rho = \beta \).

It remains to show that \( f(z) \) is of type \( \sigma = \gamma/\epsilon \rho \).
Let $\gamma < \infty$, then for all $\epsilon > 0$ and $n$ sufficiently large,

$$|a_n| \leq \left( \frac{\gamma + \epsilon}{n} \right)^{n/\rho}.$$ 

Since we may add a polynomial to $f(z)$ without changing its type, we may assume that this inequality holds for all values of $n$, interpreting the right side to be 1 when $n = 0$. Then

$$M(r) \leq \sum_{n=0}^{\infty} |a_n| r^n \leq \sum_{n=0}^{\infty} \left( \frac{\gamma + \epsilon}{n} \right)^{n/\rho} r^n = \sum_{n=0}^{\infty} \left( r^\rho \frac{\gamma + \epsilon}{n} \right)^{n/\rho}.$$ 

Let $s(n)$ be defined by

$$s(n) = \left( r^\rho \frac{\gamma + \epsilon}{n} \right)^{n/\rho},$$

and by the usual methods of the calculus, we find that $s(n)$ attains a maximum value of $\exp \{ (\gamma + \epsilon) r^\rho / \epsilon \rho \}$ when $n = (\gamma + \epsilon) r^\rho / \epsilon$. Let $S_1$ denote that part of the series $\Sigma s(n)$ for which $n \leq (\gamma + 2 \epsilon) r^\rho$, then we have that

$$S_1 \leq (\gamma + 2 \epsilon) r^\rho \exp \{ -\frac{(\gamma + \epsilon) r^\rho}{\epsilon \rho} \}.$$ 

Let $S_2$ denote the rest of the series, then we have that

$$S_2 = \sum_{n>(\gamma + 2 \epsilon) r^\rho} \left( r^\rho \frac{\gamma + \epsilon}{n} \right)^{n/\rho} < \sum_{n>(\gamma + 2 \epsilon) r^\rho} \left( \frac{\gamma + \epsilon}{\gamma + 2 \epsilon} \right)^{n/\rho} < \sum_{n=0}^{\infty} \left( \frac{\gamma + \epsilon}{\gamma + 2 \epsilon} \right)^{n/\rho} = K,$$ 

where $K$ is the sum of the geometric series. Hence

$$M(r) \leq S_1 + S_2 \leq (\gamma + 2 \epsilon) r^\rho \exp \{ -\frac{(\gamma + \epsilon) r^\rho}{\epsilon \rho} \} + K,$$
and we have that
\[ \sigma = \lim_{r \to \infty} \frac{\log M(r)}{r^\rho} \leq \lim_{r \to \infty} \frac{(\gamma + \epsilon) r^\rho}{r^\rho e^\rho} = \frac{\gamma + \epsilon}{e^\rho} \]
for all \( \epsilon > 0 \), and therefore \( \sigma \leq \gamma/e^\rho \).

Now let us show that \( \sigma \geq \gamma/e^\rho \). Choose \( \epsilon \) such that \( 0 < \epsilon < \gamma \), then for a sequence of values of \( n \) tending to infinity,
\[ |a_n| \geq \left( \frac{\gamma - \epsilon}{n} \right)^{\rho/n}. \]
Recalling that \( |a_n| \leq r^{-n} M(r) \), we choose \( r \) such that
\[ r^\rho = \frac{ne}{\gamma - \epsilon} \]
for these values of \( n \). Then
\[ M(r) \geq |a_n| r^n \geq \left( \frac{\gamma - \epsilon}{n} \right)^{n/r^\rho} = e^{\eta^\rho} = e^{1/\rho} \left( \frac{\gamma - \epsilon}{\epsilon} r^\rho \right), \]
and
\[ \sigma = \lim_{r \to \infty} \frac{\log M(r)}{r^\rho} \geq \lim_{r \to \infty} \frac{1}{r^\rho} \left( \frac{1}{\rho} \frac{\gamma - \epsilon}{\epsilon} r^\rho \right) = \frac{\gamma - \epsilon}{e^\rho} \]
for all \( \epsilon \) such that \( 0 < \epsilon < \gamma \), and therefore \( \sigma \geq \gamma/e^\rho \), which completes the proof.

With the aid of these two theorems, we can easily construct entire functions of arbitrary order and type. Consider the entire function
\[ f(z) = \sum_{n=0}^{\infty} \frac{(Az)^n}{\Gamma(an+1)} \]
where \( A > 0 \) and \( a > 0 \). Applying Stirling's formula, we obtain

\[
\Gamma(an+1) = \left( \frac{an}{e} \right)^{an} \frac{1}{2} \delta/12an
\]

where \( 0 < \delta < 1 \). Thus, by our theorems,

\[
\rho = \lim_{n \to \infty} \frac{n \log n}{n \log \frac{\Gamma(an+1)}{A^n}}
\]

\[
= \lim_{n \to \infty} \frac{n \log n}{an[\log a + \log n - 1] + \frac{1}{2} \log n + \log 2\pi a + \delta/12an - n \log A}
\]

\[
= \lim_{n \to \infty} \frac{1}{a \left[ \log \frac{a}{\log n} + 1 - \frac{1}{\log n} \right] + \frac{1}{2} \left[ \frac{1}{n} + \frac{\log 2\pi a}{n \log n} + \frac{\delta}{12an} \right] - \frac{\log A}{\log n}}
\]

\[
= \frac{1}{a},
\]

and \( \sigma = \gamma / \epsilon \rho \), where

\[
\gamma = \lim_{n \to \infty} n \left| \left( \frac{\frac{A^n}{n \log \left( \frac{\Gamma(an+1)}{a^n} \right)}}{n \log \left( \frac{\Gamma(an+1)}{a^n} \right)} \right) \right|^{\gamma/n}
\]

\[
= \lim_{n \to \infty} n \left| \frac{\frac{A^n an}{(an)^{an} (2\pi an)^{1/2} \exp(-\frac{\delta}{12an})}}{\frac{A^n an}{(an)^{an} (2\pi an)^{1/2} \exp(-\frac{\delta}{12an})}} \right|^{\frac{1}{an}}
\]

\[
= \lim_{n \to \infty} \frac{A^{1/a} e^{\frac{-\delta}{12an}}}{(an)(2\pi an)^{1/2} \exp(-\frac{\delta}{12an})}
\]

\[
= \frac{A^{1/a} e}{a},
\]

That is, \( \sigma = A^{1/a} \).

Similarly, we see that
\[ f_1(z) = \sum_{n=1}^{\infty} \left( \frac{e^\rho}{n} \right)^{\eta/\rho} z^n \quad (0 < \rho < \infty, \quad 0 < \sigma < \infty) \]

is of order \( \rho \) and normal type \( \sigma \), and

\[ f_2(z) = \sum_{n=2}^{\infty} \left( \frac{\log n}{n} \right)^{\eta/\rho} z^n \quad (0 < \rho < \infty) \]

is of order \( \rho \) and maximal type, and finally

\[ f_3(z) = \sum_{n=2}^{\infty} \left( \frac{1}{n \log n} \right)^{\eta/\rho} z^n \quad (0 < \rho < \infty) \]

is of order \( \rho \) and minimal type.

Consider the order of the product of the two entire functions \( f(z) \) and \( g(z) \). In general it can be said that

\[ \rho_{fg} \leq \max(\rho_f, \rho_g) \]

However, in some cases a much more precise statement can be made. This is the subject of our next theorem, and in its proof we shall use Levin's theorem of the lower bound for the modulus of a holomorphic function (4, p. 21):

**Theorem 1.5.** Let \( f(z) \) be holomorphic in the circle \( |z| < 2eR (R > 0) \) with \( f(0) = 1 \), and let \( \eta \) be an arbitrary positive number not exceeding \( \frac{3e}{2} \). Then inside the circle \( |z| \leq R \), but outside a family of excluded circles the sum of whose radii is not greater than \( 4\eta R \),
we have

$$\log |f(z)| > - H(\eta) \log M(2eR)$$

for

$$H(\eta) = 2 + \log \frac{3e}{2\eta}.$$ 

**Theorem 1.6.** Let $f(z)$ and $g(z)$ be entire functions of order $\rho_1$, $\rho_2$ and type $\sigma_1, \sigma_2$ respectively. Let $h(z) = f(z)g(z)$. Then:

(A) If $\rho_1 > \rho_2$, then $h(z)$ is of order $\rho_1$ and type $\sigma_1$;

(B) If $\rho_1 = \rho_2$, $0 < \sigma_1 < \infty$, and $\sigma_2 = 0$, then $h(z)$ is of order $\rho_1$ and type $\sigma_1$.

(C) If $\rho_1 = \rho_2$, $\sigma_1 = \infty$, and $0 < \sigma_2 < \infty$, then $h(z)$ is of order $\rho_1$ and type $\sigma_1$.

Proof. We shall prove part (B); the proofs of (A) and (C) are similar. Let $f(z)$ be an entire function of order $\rho$ and normal type $\sigma$; let $g(z)$ be an entire function of order $\rho$ and minimal type; let $h(z) = f(z)g(z)$.

Since the inequalities

$$M_f(r) < e^{(\sigma + \frac{\epsilon}{2})r^\rho}, \quad M_g(r) < e^{\frac{\epsilon}{2}r^\rho}$$

hold asymptotically, we also have the asymptotic inequality

$$M_h(r) = M_f(r)M_g(r) < e^{(\sigma + \epsilon)r^\rho}.$$
It now remains to show that

$$M_h(r) > e^{(\sigma - \varepsilon)r^\rho}$$

for some sequence of values of $r$ tending to infinity. To this end we apply Theorem 1.5 (we may assume without loss of generality that $g(o) = 1$, since $g(z)$ and $cz^{-n}g(z)$ have the same order and type).

First we find a number $R_1$ arbitrarily large, such that

$$M_f(R_1) > e^{(\sigma - \varepsilon)R_1^\rho}$$

and for all $R \geq R_1$,

$$M_g(R) < e^{\delta R^\rho},$$

where $0 < \delta < 1$. Such an $R_1$ exists for arbitrarily given positive $\varepsilon$ and $\delta$ since the first inequality holds for some sequence tending to infinity, and the second holds asymptotically.

Now choose $\eta = \delta/\delta$, and inside the circle

$$|z| \leq R = R_1(1-\delta)^{-1}.$$  

We form the excluded circles described in Theorem 1.5. The sum of the diameters of all the excluded circles is less than $8\eta R = 5R$. Therefore, in the interval $(R_1, R)$, there is a number $r_1$ such that the circumference $|z| = r_1$ does not meet any of the excluded circles.
Applying Theorem 1.5 on this circumference, we have

\[ \log |g(z)| > - (2 + \log \frac{12e}{\delta}) \log M_g(2eR), \]

and since \( R_1 < r_1 < R_1 (1-\delta)^{-1} \), we have

\[ M_f(r_1) > M_f(R_1) > e^{(\sigma - \frac{\epsilon}{2})R_1^p} > e^{(\sigma - \frac{\epsilon}{2})(1-\delta)^p r_1^p}. \]

Therefore

\[
\log M_h(r_1) = \log M_f(r_1) + \log M_g(2eR) \\
> (\sigma - \frac{\epsilon}{2})(1-\delta)^p r_1^p + \max_{|z|=r_1} \log |g(z)| \\
> (\sigma - \frac{\epsilon}{2})(1-\delta)^p r_1^p - (2 + \log \frac{12e}{\delta}) \log M_g(2eR) \\
\]

and since

\[ \log M_g(2eR) < \delta(2eR)^p < \delta(2e)^p R_1^p (1-\delta)^{-p} < \delta(2e)^p r_1^p (1-\delta)^{-p}, \]

we have that

\[ \log M_h(r_1) > [(\sigma - \frac{\epsilon}{2})(1-\delta)^p - (2 + \log \frac{12e}{\delta})\delta(2e)^p (1-\delta)^{-p}] r_1^p. \]

Given \( \epsilon > 0 \), one can choose \( \delta \) so small that the expression in the brackets is not less than \( \sigma - \epsilon \). Therefore

\[ M_h(r_1) > e^{(\sigma - \epsilon)r_1^p} \]

for a sequence of values \( r_1 \) tending to infinity, and the theorem is proved.
In the opening remarks of this chapter, it was mentioned that many of the classical theorems of the theory of entire functions consist of establishing relationships between the growth of an entire function and the distribution of its zeros. The following theorem falls into this category.

**Theorem 1.7.** If \( f(z) \) is an entire function of order \( \rho \), and \( n(r) \) is the number of zeros of \( f(z) \) where \( |z| \leq r \), then

\[
\lim_{r \to \infty} \frac{\log n(r)}{\log r} \leq \rho.
\]

Before we prove this theorem, we recall Jensen's theorem (1,p.2):

**Theorem 1.8.** If \( f(z) \) is regular in \( 0 < r < R \), where \( |z| = r \), and \( f(0) \neq 0 \), then for \( 0 < r < R \) we have

\[
\int_0^r \frac{n(t)}{t} \, dt = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta - \log |f(0)|.
\]

We use this result in the proof of the following lemma.

**Lemma.** If \( f(z) \) is regular for \( |z| < R \) and \( |f(0)| = 1 \), then for \( 0 < r < R \),

\[
n(r) \leq \frac{\log M(r)}{\log \frac{R}{r}}.
\]

**Proof of Lemma.** We have
\[ n(r) \log \frac{R}{r} = n(r) \int_{0}^{R} \frac{u}{r} \, du \leq \int_{0}^{R} \frac{n(u)}{u} \, du, \]

because \( n(r) \) is increasing, and therefore

\[ n(r) \log \frac{R}{r} \leq \int_{0}^{R} \frac{n(u)}{u} \, du - \int_{0}^{r} \frac{n(u)}{u} \, du \leq \int_{0}^{R} \frac{n(u)}{u} \, du. \]

Applying Theorem 1.8, we obtain

\[
\begin{align*}
\frac{n(r)}{r} \log \frac{R}{r} & \leq \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(re^{i\theta})| \, d\theta - \log|f(0)| \\
& \leq \frac{1}{2\pi} \int_{0}^{2\pi} \log M(r) \, d\theta \\
& = \log M(r),
\end{align*}
\]

and the lemma is proved.

Proof of Theorem 1.7. Let \( f(z) \) be given by

\[ f(z) = \sum_{n=0}^{\infty} a_n z^n. \]

If \( f(z) \) has a zero of order \( h \) at \( z = 0 \), then we define \( F(z) \) to be

\[ F(z) = \frac{f(z)}{a_h z^h}. \]

If \( f(z) \) has no zero at \( z = 0 \), then we define \( F(z) \) by
\[ F(z) = \frac{f(z)}{a_0}. \]

In either case, \( F(z) \) has no zero at \( z = 0 \), and \( |F(0)| = 1 \). Also

\[ M_f(r) = \max_{|z|=r} |F(z)| = \max_{|z|=r} \frac{|f(z)|}{a_h z^h} = \frac{M_f(r)}{|a_h|^r}. \]

Now, applying our lemma, we obtain for \( 0 < r < R \)

\[
\begin{align*}
n(r) - h &= n_f(r) \leq \frac{\log M_f(R)}{\log \frac{R}{r}} \\
&= \frac{\log \left( \frac{M_f(R)}{|a_h|^R} \right)}{\log \frac{R}{r}} \\
&= \frac{\log M_f(R) - \log |a_h|^r - h \log R}{\log \frac{R}{r}},
\end{align*}
\]

and choosing \( R = 2r \) yields

\[ n(r) - h \leq \frac{\log M_f(2r) - \log |a_h|^r - h \log 2r}{\log 2} = \frac{\log M_f(2r) - \log |a_h|^r - h \log r}{\log 2}. \]

Therefore,

\[ n(r) \leq \frac{\log M_f(2r) - \log |a_h|^r - h \log r}{\log 2} \leq \frac{\log M_f(2r)}{\log 2}. \]

for \( r \) sufficiently large. Taking logarithms and dividing by \( \log 2r \), we have
\[ \frac{\log n(r)}{\log r + \log 2} < \frac{\log \log M_f(2r) - \log \log 2}{\log 2r}, \]

and finally,

\[ \lim_{r \to \infty} \left( \frac{\log n(r)}{\log r + \log 2} \right) < \lim_{r \to \infty} \frac{\log \log M_f(2r) - \log \log 2}{\log 2r}, \]

which is equivalent to

\[ \lim_{r \to \infty} \left( \frac{\log n(r)}{\log r} \right) < \lim_{r \to \infty} \frac{\log \log M_f(2r)}{\log 2r} = \rho, \]

and the proof is complete.

This theorem tells us that

\[ n(r) = O(r^{\rho+\epsilon}) \]

for all \( \epsilon > 0 \). In general we cannot make any lower estimate for \( n(r) \) since it is possible that \( f(z) \) may have large order but no zeros at all; for example, \( \rho = n \) when

\[ f(z) = e^z. \]

However, if \( f(z) \) has nonintegral order \( \rho \), it turns out that there are always zeros (in fact an infinite set of zeros), and \( n(r) \) can be closely approximated (2, p. 24-27).

Certain problems arise in which it is necessary to characterize the growth of an entire function in different directions. Actually,
before introducing such characterizations, one should understand clearly to what extent the growth of the function along a certain curve is independent of its growth along a nearby curve. Evgrafov discusses this in his book "Asymptotic Estimates and Entire Functions", even to the extent of showing how turns in the domain affect the growth (3, p. 94-125). In the remainder of this chapter, we shall be concerned with the growth of an entire function \( f(z) \) of positive finite order in an angle. To characterize the dependence of the growth of such a function in an angle on the direction in which \( z \) tends to infinity, E. Phragmén and E. Lindelöf, in 1908, introduced the concept of an indicator function. Many of their results are contained in these next few pages.

Since we are actually concerned with functions \( f(re^{i\theta}) \) which are holomorphic in the angle \( a < \arg z \leq \beta \), let us alter a few of our earlier definitions so that we may talk of the order and type of \( f(z) \) in the given angle.

**Definition 1.5.** \( M_f(r, \alpha, \beta) = \max_{\alpha < \theta < \beta} |f(re^{i\theta})| \).

**Definition 1.6.** If \( f(re^{i\theta}) \) is holomorphic in the angle \( a \leq \theta \leq \beta \), then the order \( \rho \) and the type \( \sigma \) of \( f(z) \) in the angle \([a, \beta]\) are defined by

\[
\rho = \lim_{r \to \infty} \frac{\log \log M_f(r, \alpha, \beta)}{\log r}, \quad \sigma = \lim_{r \to \infty} \frac{\log M(r, \alpha, \beta)}{r^\rho}
\]
Note that for $\beta = a + 2\pi$, $f(z)$ is entire and these definitions are identical to those presented earlier.

The indicator function of Phragmén and Lindelöf may now be defined as follows:

**Definition 1.7.** If $f(z)$ is holomorphic and of finite positive order $\rho$ in the angle $\theta_1 \leq \arg z \leq \theta_2$, then the indicator function of $f(z)$ is defined by

$$h(\theta) = \lim_{r \to \infty} \frac{\log |f(re^{i\theta})|}{r^\rho},$$

where $\theta_1 \leq \theta \leq \theta_2$.

The function

$$f(z) = e^{(a-ib)z^p}, \quad (a, b \text{ real})$$

which is holomorphic in any angle $\alpha \leq \arg z \leq \beta$, $\beta - \alpha \leq 2\pi$, has

$$|f(re^{i\theta})| = |e^{(a-ib)r^p(cos\theta+isin\theta)}| = e^{r^p(acos\theta+bsin\theta)},$$

and therefore has the indicator function

$$h(\theta) = \lim_{r \to \infty} \frac{1}{r^\rho} \frac{d}{dr}(f(re^{i\theta})| = a \cos \rho \theta + b \sin \rho \theta.$$

**Definition 1.8.** Let $H(\theta) = a \cos \rho \theta + b \sin \rho \theta$, where $a$ and $b$ are real. If $h(\theta) = H(\theta)$, then $h(\theta)$ is called a trigonometric indicator.
As we shall soon see, the sinusoid $H(\theta)$ plays an important role in the determination of many properties of $h(\theta)$. First, however, let us note that if $H(\theta_1) = h_1$ and $H(\theta_2) = h_2$, then

$$a \cos \rho \theta_1 + b \sin \rho \theta_1 = h_1,$$

$$a \cos \rho \theta_2 + b \sin \rho \theta_2 = h_2,$$

and solving for $a$ and $b$ we obtain

$$a = \frac{h_1 \sin \rho \theta_2 \cos \rho \theta_2 - h_2 \cos \rho \theta_1 \sin \rho \theta_1}{\cos \rho \theta_1 \sin \rho (\theta_2 - \theta_1)}, \quad b = \frac{h_2 \cos \rho \theta_1 - h_1 \cos \rho \theta_2}{\sin \rho (\theta_2 - \theta_1)}.$$

Substituting these quantities back into our original expression yields

$$H(\theta) = \frac{h_1 \sin \rho (\theta_2 - \theta) + h_2 \sin \rho (\theta - \theta_1)}{\sin \rho (\theta_2 - \theta_1)}.$$ 

**Theorem 1.9.** Let $h_f(\theta)$ be the indicator of $f(r e^{i \theta})$, a holomorphic function of order $\rho$ in the angle $\theta_1 \leq \theta \leq \theta_2$, $\theta - \theta_1 < \frac{\pi}{\rho}$, and $h_f(\theta_1) \leq h_1$, $h_f(\theta_2) \leq h_2$. Then for all values of $\theta$ in the angle $[\theta_1, \theta_2]$, $h_f(\theta) \leq H(\theta)$.

This property of the indicator function is called the trigonometric convexity property. The reason for this label can be justified by the following argument. If $h(\theta)$ is a trigonometric indicator
given by 

\[ h(\theta) = \frac{h(\theta_1)\sin(\theta_2 - \theta) + h(\theta_2)\sin(\theta - \theta_1)}{\sin(\theta_2 - \theta_1)} \]

where \( 0 < \theta_2 - \theta_1 < \frac{\pi}{\rho} \), then

\[ h\left(\frac{\theta_1 + \theta_2}{2}\right) = \frac{h(\theta_1)\sin(\frac{\theta_2 - \theta_1}{2}) + h(\theta_2)\sin(\frac{\theta_2 - \theta_1}{2})}{\sin(\theta_2 - \theta_1)} \]

\[ = \frac{h(\theta_1) + h(\theta_2)}{2 \cos(\frac{\theta_2 - \theta_1}{2})} \]

\[ \geq \frac{h(\theta_1) + h(\theta_2)}{2} , \]

that is, if \( h(\theta) \) is a trigonometric indicator then \( h(\theta) \) is a convex function.

In the proof of this theorem, we shall use the following form of the Phragmén-Lindelöf Theorem (4, p. 49).

**Theorem 1.10.** Let the function \( f(z) \) be holomorphic inside an angle of opening \( \pi/\alpha \) and continuous on the boundary. If \( |f(z)| \leq M \) on the sides of the angle and if \( \rho_f < \alpha \), then \( |f(z)| \leq M \) throughout the angle.

**Proof of theorem.** We begin by constructing the function

\[ H_\delta(\theta) = a_\delta \cos \rho \theta + b_\delta \sin \rho \theta \]

such that
\[ H_\delta(\theta_1) = h_1 + \delta, \quad H_\delta(\theta_2) = h_2 + \delta, \]

where \( \delta > 0 \). Next we note that \( g(z) \), which is defined by

\[ g(z) = f(z) e^{-(a_\delta - ib_\delta)z\rho} \]

has as its indicator function

\[
\begin{align*}
\text{h}_g(\theta) &= \lim_{r \to \infty} \frac{\log|g(re^{i\theta})|}{r^\rho} \\
&= \lim_{r \to \infty} \frac{\log|f(re^{i\theta})| + \log|e^{-(a_\delta - ib_\delta)z\rho}|}{r^\rho} \\
&= \lim_{r \to \infty} \frac{\log|f(re^{i\theta})|}{r^\rho} + \lim_{r \to \infty} \frac{\log|e^{-(a_\delta - ib_\delta)z\rho}|}{r^\rho} \\
&= h_f(\theta) - H_\delta(\theta). 
\end{align*}
\]

Therefore, for \( j = 1, 2 \),

\[ h_j(\theta_j) = h_j(\theta_j) - H_\delta(\theta_j) \leq h_j - (h_j + \delta) = -\delta, \]

and since

\[ -\delta > h_j(\theta_j) = \lim_{r \to \infty} \frac{\log|g(re^{i\theta})|}{r^\rho}, \]

we have that for all \( \epsilon > 0 \), and \( r \) sufficiently large,

\[ \log|g(re^{i\theta})| < r^\rho(\epsilon - \delta). \]

Choosing \( \delta > \epsilon \),
$$\log |g(re^{i\theta})| \rightarrow -\infty$$
as \(r \rightarrow \infty\), from which we deduce that \(g(z)\) tends to zero along the rays \(\arg z = \theta_1, \theta_2\).

Applying Theorem 1.10, we see that \(g(z)\) is bounded throughout the angle, and

$$\frac{h_f(\theta)}{g} = \lim_{r \to \infty} \frac{\log|g(re^{i\theta})|}{r^p} \leq \lim_{r \to \infty} \frac{\log M}{r^p} = 0.$$ 

Thus for all \(\delta > 0\),

$$h_f(\theta) - H_\delta(\theta) \leq 0,$$

and letting \(\delta \to 0\), we have

$$h_f(\theta) \leq H(\theta)$$

for \(\theta_1 \leq \theta \leq \theta_2\), which completes the proof.

Let us look at three important corollaries of the trigonometric convexity property.

**Corollary 1.** If for \(i = 1, 2\), \(h_f(\theta_i) = h_i < \infty, \theta_2 - \theta_1 < \frac{\pi}{\rho}\), then \(h_f(\theta)\) is bounded above for all values of \(\theta\) in the interval \([\theta_1, \theta_2]\).

**Proof.** By Theorem 1.9, for \(\theta_1 \leq \theta \leq \theta_2\), we have

$$h_f(\theta) \leq H(\theta) = \frac{h_1 \sin \rho (\theta_2 - \theta) + h_2 \sin \rho (\theta - \theta_1)}{\sin \rho (\theta_2 - \theta_1)} \leq \frac{h_1 + h_2}{\sin \rho (\theta_2 - \theta_1)}.$$
**Corollary 2.** If $\theta_1 < \theta_2 < \theta_3$, $\theta_3 - \theta_1 < \frac{\pi}{\rho}$, $h(\theta_1) \leq H(\theta_1)$ and $h(\theta_2) \geq H(\theta_2)$. Then $h(\theta_3) \geq H(\theta_3)$.

**Proof.** Assume there exists a $\delta > 0$ such that $h(\theta_3) < H(\theta_3) - \delta$.

Let

$$H_\delta(\theta) = H(\theta) - \delta \sin \rho(\theta - \theta_1) \csc(\theta_3 - \theta_1),$$

then

$$H_\delta(\theta_1) = H(\theta_1),$$
$$H_\delta(\theta_2) = H(\theta_2) - \sin \rho(\theta_2 - \theta_1) \csc(\theta_3 - \theta_1) < H(\theta_2),$$
$$H_\delta(\theta_3) = H(\theta_3) - \delta.$$

Therefore

$$h(\theta_1) \leq H_\delta(\theta_1),$$
$$h(\theta_3) < H_\delta(\theta_3),$$

and applying Theorem 1.9, we have $h(\theta) \leq H_\delta(\theta)$ for $\theta_1 < \theta < \theta_3$.

In particular, for $\theta = \theta_2$,

$$h(\theta_2) \leq H_\delta(\theta_2) < H(\theta_2),$$

and this contradicts our hypothesis that $h(\theta_2) \geq H(\theta_2)$.

Our final corollary reformulates the trigonometric convexity property of the indicator in a symmetric form.

**Corollary 3.** The fundamental relation of the indicator: The indicator $h(\theta)$ of a function $f(z)$ holomorphic and of order $\rho$ inside the angle $\theta_1 \leq \arg z \leq \theta_3$, $\theta_3 - \theta_1 < \frac{\pi}{\rho}$, satisfies the relation...
\[ h(\theta_1) \sin \rho(\theta_2 - \theta_3) + h(\theta_2) \sin \rho(\theta_3 - \theta_1) + h(\theta_3) \sin \rho(\theta_1 - \theta_2) \leq 0 \]

for \( \theta_1 < \theta_2 < \theta_3 \).

**Proof.** Define \( H(\theta) = a \cos \rho + b \sin \rho \) such that \( H(\theta_1) = h(\theta_1) \) and \( H(\theta_2) = h(\theta_2) \). Then, by the preceding corollary, \( h(\theta_3) \geq H(\theta_3) \).

Now since
\[
H(\theta) = \frac{h(\theta_1) \sin \rho(\theta_2 - \theta) + h(\theta_2) \sin \rho(\theta - \theta_1)}{\sin \rho(\theta_2 - \theta_1)},
\]

we have that
\[-h(\theta_1) \sin \rho(\theta_2 - \theta) + h(\theta_2) \sin \rho(\theta - \theta_1) + H(\theta) \sin \rho(\theta_2 - \theta_1) = 0.\]

Setting \( \theta = \theta_3 \) and using \( h(\theta_3) \geq H(\theta_3) \), we obtain
\[-h(\theta_1) \sin \rho(\theta_2 - \theta_3) + h(\theta_2) \sin \rho(\theta_3 - \theta_1) + h(\theta_3) \sin \rho(\theta_2 - \theta_1) \geq 0,\]

which is equivalent to
\[ h(\theta_1) \sin \rho(\theta_2 - \theta_3) + h(\theta_2) \sin \rho(\theta_3 - \theta_1) + h(\theta_3) \sin \rho(\theta_1 - \theta_2) \leq 0.\]

Note that a cyclic permutation of the quantities \( \theta_1, \theta_2, \theta_3 \) leaves the fundamental relation unchanged, and interchanging any two of these quantities changes the direction of the inequality.

The fundamental relation opens the doorway to a number of analytical properties of the indicator function. A few of these properties are listed below (they shall be proved in Chapter 3 for the generalized indicator):
1). The function $h(\theta)$ is continuous.

2). If $h(\theta) = -\infty$ for even one value of $\theta$, then $h(\theta) \equiv -\infty$.

3). $h(\theta)$ has a derivative from the left and from the right at every point.

4). The right hand derivative is greater than or equal to the left hand derivative at every point.

5). The right hand derivative is continuous from the right and the left hand derivative is continuous from the left.

6). $h(\theta)$ has a derivative at all points except possibly a countable set.

7). If $\theta_0$ is a local maximum or minimum point for $h(\theta)$, then for $|\theta - \theta_0| \leq \frac{\pi}{\rho}$

$$h(\theta) \geq h(\theta_0) \cos \rho(\theta - \theta_0).$$
CHAPTER 2

PROXIMATE ORDERS

We shall now generalize our concept of the order of an entire function by replacing the constant \( \rho \) with a function \( \rho(r) \), which satisfies certain properties. The function \( \rho(r) \) shall be called a proximate order. Valiron is given credit for the initial development of proximate orders in his "Lectures on the General Theory of Entire Functions". He used the following definition (8, p. 64 and 67).

**Definition 2.1.** A function \( \rho(r) \) is a general proximate order of the entire function \( f(z) \) if:

1. \( \rho(r) \) is continuous and defined for \( r > r_0 \);
2. \( \lim_{r \to \infty} \rho(r) = \rho \), where \( 0 < \rho < \infty \);
3. \( \lim_{r \to \infty} \rho(r) > \beta \), where \( 0 < \beta < \rho \);
4. \( \lim_{r \to \infty} r \rho'(r) \log r = 0 \);
5. \( \lim_{r \to \infty} \frac{\log M_f(r)}{r^\rho(r)} = 1 \).

The theorems which Valiron subsequently proved are stated in terms of one of the following two special cases of general proximate orders (8, p. 67).
Definition 2.2. If \( p(r) \) is a general proximate order with the additional property

\[
\lim_{r \to \infty} p(r) = p, \\
\]

then \( p(r) \) is a proximate order \( L(\text{Lindelöf}) \), or simply an order \( L \).

Definition 2.3. If \( p(r) \) is a general proximate order with the additional properties that \( p \) is not an integer, and that

\[
\lim_{r \to \infty} p(r) > p, \\
\]

where \( p \) is the genus\(^1\) of \( f(z) \), then \( p(r) \) is a proximate order \( B(\text{Boutroux}) \), or simply an order \( B \).

In the early 1900's, Lindelöf and Boutroux experimented with functions \( p(r) \) having these more specific properties. Valiron, in 1923, initiated the terminology and generalized their results, thus giving birth to the theory of proximate orders.

Slight modifications of this original definition were made during the next forty years. In 1954, Boas gave the following (1,p.9):

Definition 2.4. \( p(r) \) is a proximate order of the entire

\[\text{Let } r_1, r_2, r_3, \ldots \text{ be the moduli of the nonzero zeros of } f(z). \]

The least integer \( p \) for which \( \sum_{n=1}^{\infty} r_n^{p-1} \) converges is the genus of \( f(z) \) (8, p. 53).
function \( f(z) \) if:

1. \( \rho(r) \) is monotone, nondecreasing and piecewise differentiable;
2. \( \lim_{r \to \infty} \rho(r) = \rho \); 
3. \( \lim_{r \to \infty} \rho'(r)r \log r = 0 \); 
4. \( \lim_{r \to \infty} \frac{\log M_f(r)}{r^\rho(r)} = 1 \).

And, in 1956, Cartwright gave still another definition (2, p. 54-55) which seems to be a compromise between definitions given by Valiron and that of Boas.

**Definition 2.5.** \( \rho(r) \) is a Lindelöf proximate order of the entire function \( f(z) \) if:

1. \( \rho(r) \) is real, continuous, and piecewise differentiable for \( r > \ell \);
2. \( \lim_{r \to \infty} \rho(r) = \rho \);
3. \( \lim_{r \to \infty} \rho'(r)r \log r = 0 \), where \( \rho'(r) \) is either the right or left hand derivative at points where they are different; 
4. \( \lim_{r \to \infty} \frac{\log M_f(r)}{r^\rho(r)} = 1 \).

And, \( \rho(r) \) is an ordinary proximate order if Condition 2 is replaced by

...
2'. \( \lim_{r \to \infty} \rho(r) = \rho, \lim_{r \to \infty} \rho(r) = \beta > 0 \).

The latest product in the evolution of definitions appears to be the one given by Levin (4, p. 32) in 1964. It is this definition which shall be used throughout this paper.

**Definition 2.6.** \( \rho(r) \) is a proximate order if it satisfies the following conditions:

1. \( \rho(r) \) is real, continuous, and piecewise differentiable for \( r > l \);
2. \( \lim_{r \to \infty} \rho(r) = \rho \);
3. \( \lim_{r \to \infty} \rho'(r) r \log r = 0 \).

Note that a proximate order is a real valued function of a real variable, and has been defined without any reference to an entire function.

**Definition 2.7.** If, for the entire function \( f(z) \), the quantity

\[
\sigma_f = \lim_{r \to \infty} \frac{\log M_f(r)}{r \rho(r)}
\]

is different from zero and infinity, then \( \rho(r) \) is called a proximate order of the entire function \( f(z) \). If \( \sigma_f \) is either zero or infinity, then \( \rho(r) \) is not a proximate order of \( f(z) \). In either case, \( \sigma_f \).
is called the type of \( f(z) \) with respect to the proximate order \( \rho(r) \). \( f(z) \) is said to be of maximal, normal, or minimal type with respect to \( \rho(r) \) if \( \sigma_f = \infty \), \( 0 < \sigma_f < \infty \), or \( \sigma_f = 0 \) respectively. Note that with respect to its own proximate orders, \( f(z) \) is always of normal type.

Before examining the properties of proximate orders of entire functions, we prove Shah's existence theorem (5, p. 326-328), which states that given any entire function \( f(z) \) of finite order, it is possible to construct a proximate order \( \rho(r) \) such that \( f(z) \) is of type 1 with respect to \( \rho(r) \).

**Theorem 2.1.** Let \( f(z) \) be entire and of finite order \( \rho \). It is possible to find a positive continuous function \( \rho(r) \) having the following properties:

1. \( \rho(r) \) is differentiable for \( r > r_0 \), except at isolated points where both left and right hand derivatives exist;
2. \( \lim_{r \to \infty} \rho(r) = \rho \)
3. \( \lim_{r \to \infty} r \rho'(r) \log r = 0 \);
4. \( \lim_{r \to \infty} \frac{\log M_f(r)}{r \rho(r)} = 1 \).

**Proof.** Let \( \sigma(r) \) be defined by

\[
\sigma(r) = \frac{\log \log M_f(r)}{\log r},
\]
so that
\[
\lim_{r \to \infty} \sigma(r) = \rho.
\]

We have two cases to consider: either (A) \( \sigma(r) > \rho \) for a sequence of values of \( r \) tending to infinity, or (B) \( \sigma(r) \leq \rho \) for all large \( r \).

Case (A): \( \sigma(r) > \rho \) for a sequence of values of \( r \) tending to infinity. First we define the step function \( \phi(r) \) by
\[
\phi(r) = \max \{\sigma(x)\}.
\]
\[x \geq r\]

We know that such a function \( \phi(r) \) exists and is a nonincreasing function of \( r \), since \( \sigma(r) \) is continuous, \( \lim_{r \to \infty} \sigma(r) = \rho \), and \( \sigma(r) > \rho \) for a sequence of values of \( r \) tending to infinity.

Let
\[r_1 > e^{e^e},\]
and let \( \phi(r_1) = \sigma(r_1) \) (such values will exist for a sequence of values of \( r \) tending to infinity). Let \( \rho(r_1) = \phi(r_1) \). Let \( t_1 \) be the smallest integer not less than \( 1 + r_1 \) such that \( \phi(r_1) > \phi(t_1) \), and let
\[\rho(r) = \rho(r_1) = \phi(r_1).\]

For \( r_1 \leq r \leq t_1 \). Now define \( u_1 \) as follows:
\( u_1 > t_1; \)
\[
\rho(r) = \rho(r) - \log \log \log r + \log \log \log t_1, \text{ for } t_1 \leq r < u_1;
\]
\[
\rho(r) = \phi(r) \text{ for } r = u_1 \text{ (that is, } u_1 \text{ is the value of } r \text{ for which } \rho(r) \text{ intersects } \phi(r)).
\]

Note that \( \rho(r) > \phi(r) \) for \( t_1 \leq r < u_1 \), since \( \rho(t_1) = \rho(u_1) = \phi(u_1) > \phi(t_1). \)

Let \( r_2 \) be the smallest value of \( r \) for which \( r_2 \geq u_1 \), and \( \phi(r_2) = \sigma(r_2) \). If \( r_2 > u_1 \) (that is, if \( u_1 \) is not an element of the sequence of values of \( r \) tending to infinity for which \( \sigma(r) > \rho \)) then let
\[
\rho(r) = \phi(r)
\]
for \( u_1 \leq r < r_2 \). Thus
\[
\rho(u_1) = \phi(u_1) = \max_{x \leq u_1} \{\sigma(x)\} = \sigma(r_2).
\]

Since \( \phi(r) \) is a constant for \( u_1 \leq r \leq r_2 \), we have that \( \rho(r) \) is continuous for \( u_1 \leq r \leq r_2 \). Define \( t_2 \) and \( u_2 \) analogous to \( t_1 \) and \( u_1 \).

Repeating this argument, we obtain the function \( \rho(r) \) which is differentiable in adjacent intervals. Further, either \( \rho'(r) = 0 \) or
\[
\rho'(r) = \frac{-1}{(r \log r)(\log \log r)}
\]
which implies that
\[
\lim_{r \to \infty} r \rho'(r) \log r = 0.
\]

Also: \(\rho(r) \geq \phi(r) \geq \sigma(r)\) for all \(r \geq r_1\); \(\rho(r) = \sigma(r)\) for an infinity of values \(r = r_1, r_2, \ldots\); \(\rho(r)\) is nonincreasing; and

\[
\lim_{r \to \infty} \phi(r) = \rho.
\]

Hence

\[
\lim_{r \to \infty} \rho(r) = \lim_{r \to \infty} \rho(r) = \rho,
\]

and since

\[
\log M_f(r) = r \sigma(r) = r \rho(r)
\]

for an infinity of \(r\),

\[
\log M_f(r) < r \rho(r)
\]

for the remaining \(r\), and

\[
\lim_{r \to \infty} \frac{\log M_f(r)}{r \rho(r)} = 1,
\]

which completes the proof of Case (A).
Case (B): \( \sigma(r) \leq \rho \) for all large \( r \). Here we have two possibilities:

Case B1: \( \sigma(r) = \rho \) for at least a sequence of values of \( r \) tending to infinity;

Case B2: \( \sigma(r) < \rho \) for all large \( r \).

In Case B1, take \( \rho(r) = \rho \) for all values of \( r \).

In Case B2, let

\[
\xi(r) = \max \{ \sigma(x) \}, \quad X \leq x \leq r
\]

where

\[
X > e^{e^e}
\]

is such that \( \sigma(x) < \rho \) whenever \( x > X \). Since

\[
\lim_{r \to \infty} \sigma(r) = \rho
\]

and \( \sigma(x) < \rho \) for \( x > X \), \( \xi(r) \) is nondecreasing. Take a suitably large value of \( r_1 > X \), and let:
\[ \rho(r_1) = \rho; \]

\[ \rho(r) = \rho + \log \log \log r - \log \log \log r_1, \quad \text{for} \quad s_1 \leq r \leq r_1, \]

where \( s_1 < r_1 \) is a number such that \( \xi(s_1) = \rho(s_1) \).

If \( \xi(s_1) \neq \sigma(s_1) \), then we take \( \rho(r) = \xi(r) \) for \( t_1 \leq r \leq s_1 \),

where \( t_1 < s_1 \) is the number nearest to \( s_1 \) such that \( \xi(t_1) = \sigma(t_1) \).

\( \rho(r) \) is then a constant for \( t_1 \leq r \leq s_1 \). If \( \xi(s_1) = \sigma(s_1) \), then

let \( t_1 = s_1 \).

Choose \( r_2 > r_1 \) suitably large, and let :

\[ \rho(r_2) = \rho; \]

\[ \rho(r) = \rho + \log \log \log r - \log \log \log r_2, \quad \text{for} \quad s_2 \leq r \leq r_2, \]

where \( s_2 < r_2 \) is a number such that \( \xi(s_2) = \rho(s_2) \).

Note: \( s_2 \) is given by the largest possible root of

\[ \xi(s_2) = \rho - \log \log \log r_2 + \log \log \log s_2. \]

If \( \xi(s_2) = \sigma(s_2) \) then let \( \rho(r) = \xi(r) \) for \( t_2 \leq r \leq s_2 \), where

\( t_2 < s_2 \) is the number nearest to \( s_2 \) such that \( \xi(t_2) = \sigma(t_2) \). If

\( \xi(s_2) = \sigma(s_2) \), then let \( t_2 = s_2 \).

For \( r < t_2 \), let

\[ \rho(r) = \rho(t_2) + \log \log \log t_2 - \log \log \log r \]

for \( u_1 \leq r \leq t_2 \), where \( u_1 < t_2 \) is the abscissa of the point of

intersection of the two curves.
\[ y = \rho, \]
\[ y = \rho(t_2) + \log \log \log t_2 - \log \log \log r. \]

Note: if \( u_1 < r_1 \), then pick \( r_2 \) larger so that \( u_1 \geq r_1 \). Let \( \rho(r) = \rho \) for \( r_1 \leq r < u_1 \).

We repeat this procedure, and note that \( \rho(r) \geq \xi(r) \geq \sigma(r) \) and \( \rho(r) = \sigma(r) \) for \( t = t_1, t_2, \ldots \). Hence

\[
\lim_{r \to \infty} \rho(r) = \rho,
\]

and since

\[
\log M_f(r) = r^\sigma(r) = r^\rho(r)
\]

for an infinity of \( r \), and

\[
\log M_f(r) < r^\rho(r)
\]

for the remaining \( r \), we have

\[
\lim_{r \to \infty} \frac{\log M_f(r)}{r^\rho(r)} = 1.
\]

Further, either \( \rho'(r) = 0 \), or

\[
\rho'(r) = -\frac{1}{(r \log r)(\log \log r)},
\]

which implies that

\[
\lim_{r \to \infty} r \rho'(r) \log r = 0.
\]
and the proof of Case (B) is complete.

Thus, given any entire function \( f(z) \) of finite order (and these are the only kind with which we shall be concerned), there exists at least one proximate order \( \rho(r) \) such that \( f(z) \) is of type 1 with respect to \( \rho(r) \). Our definition and the preceding theorem may lead us to believe that the proximate order and the corresponding type of an entire function are not uniquely determined. This is indeed the case, and is the subject of the following theorem.

**Theorem 2.2.** If \( \rho(r) \) is a proximate order of the entire function \( f(z) \), and \( \sigma_f \) is the type of \( f(z) \) with respect to \( \rho(r) \), and if \( c \) is any positive real number, then

\[
\rho^*(r) = \rho(r) + \frac{\log c}{\log r}
\]
is also a proximate order of $f(z)$, and

$$\sigma^*_f = \frac{\sigma_f}{c}$$

is the type of $f(z)$ with respect to $\rho^*(r)$.

**Proof.** First let us show that $\rho^*(r)$ is actually a proximate order. Obviously $\rho^*(r)$ is real, continuous and piecewise differentiable for $r > \ell$, since $\rho(r)$ has these properties. Secondly,

$$\lim_{r \to \infty} \rho^*(r) = \lim_{r \to \infty} \left[ \rho(r) + \frac{\log c}{\log r} \right] = \rho.$$

and finally,

$$\lim_{r \to \infty} \rho^*(r) r \log r = \lim_{r \to \infty} \left[ \left( \rho'(r) - \frac{\log c}{r(\log r)^2} \right) r \log r \right]$$

$$= \lim_{r \to \infty} \left[ \rho'(r) r \log r - \frac{\log c}{\log r} \right] = 0.$$

Completing the proof, we have

$$\sigma^*_f = \lim_{r \to \infty} \frac{\log M_f(r)}{r \rho^*(r)}$$

$$= \lim_{r \to \infty} \left[ \frac{\log M_f(r)}{r \rho(r)} \cdot \frac{1}{\frac{\log c}{r \log r}} \right]$$

$$= \frac{1}{c} \lim_{r \to \infty} \frac{\log M_f(r)}{r \rho(r)}$$

$$= \frac{\sigma_f}{c}.$$
Thus, by this and the previous theorem, we can construct a proximate order of any entire function, and once we have such a $\rho(r)$,

$$\rho^*(r) = \rho(r) + \frac{\log c}{\log r}$$

gives us a class of proximate orders of that entire function; one proximate order for every $c > 0$. Let us now turn our attention to the construction of proximate orders themselves, that is proximate orders without reference to $f(z)$. For this purpose, we give the following definition.

**Definition 2.8.** A positive function $L(r)$ is called a slowly increasing function (even if it is decreasing) if

$$\lim_{r \to \infty} \frac{r L'(r)}{L(r)} = 0.$$  

The following three properties of slowly increasing functions will be useful to us as we relate these functions to proximate orders.

**Property 1.** If $L(r)$ is slowly increasing, then for each $k > 0$,

$$\lim_{r \to \infty} \frac{L(kr)}{L(r)} = 1.$$  

**Proof.** Since $L(r)$ is slowly increasing,

$$\lim_{r \to \infty} \frac{r L'(r)}{L(r)} = 0,$$

or in other words,
\[
\frac{L'(r)}{L(r)} = o\left(\frac{1}{r}\right)
\]
as \(r \to \infty\). Therefore,

\[
\log \frac{L(kr)}{L(r)} = \int_r^{kr} \frac{L'(t)}{L(t)} \, dt
\]

\[
= \int_r^{kr} o\left(\frac{1}{t}\right) dt
\]

\[
\leq \int_r^{kr} \frac{\epsilon}{t} dt = \log k^\epsilon
\]

for all \(\epsilon > 0\). Now since

\[
\int_r^{kr} o\left(\frac{1}{t}\right) dt \leq \log k^\epsilon
\]

implies that

\[
\int_r^{kr} o\left(\frac{1}{t}\right) dt = o(1),
\]

we have

\[
\log \frac{L(kr)}{L(r)} = o(1),
\]

which is equivalent to

\[
\frac{L(kr)}{L(r)} = e^{o(1)} = 1 + o(1),
\]

that is

\[
\lim_{r \to \infty} \frac{L(kr)}{L(r)} = 1.
\]
Property 2. If $L(r)$ is slowly increasing, and $a > 0$, then 

$[L(r)]^a$ is slowly increasing.

Proof. Applying our definition,

$$\lim_{r \to \infty} \frac{r([L(r)]^a)'}{[L(r)]^a} = \lim_{r \to \infty} \frac{ra[L(r)]^{a-1}L'(r)}{[L(r)]^a} = \lim_{r \to \infty} \frac{arL'(r)}{L(r)} = 0.$$ 

Property 3. If $L_1(r)$ and $L_2(r)$ are slowly increasing, then $L(r) = L_1(r)L_2(r)$ is slowly increasing.

Proof. Again, by our definition,

$$\lim_{r \to \infty} \frac{r[L_1(r)L_2(r)]'}{L_1(r)L_2(r)} = \lim_{r \to \infty} \frac{r[L_1(r)L_2'(r) + L_1'(r)L_2(r)]}{L_1(r)L_2(r)} = \lim_{r \to \infty} \frac{rL_2'(r)}{L_2(r)} + \lim_{r \to \infty} \frac{rL_1'(r)}{L_1(r)} = 0.$$ 

Now let us look at a few examples of slowly increasing functions.

Example 1. $L(r) = \ell^n(r)$ is slowly increasing, where $\ell^1(r) = \log r$ and $\ell^k(r) = \log (\ell^{k-1}(r))$. 
Proof. First let us note that

\[ L'(r) = [x^n(r)]' = \frac{1}{r} \prod_{j=1}^{n-1} \frac{1}{j^j(r)} \]

which can easily be proved by induction. Then

\[
\lim_{r \to \infty} \frac{rL'(r)}{L(r)} = \lim_{r \to \infty} \frac{r \cdot \prod_{j=1}^{n-1} \frac{1}{j^j(r)}}{\prod_{j=1}^{n} \frac{1}{j^j(r)}} = \lim_{r \to \infty} \frac{1}{\prod_{j=1}^{n} j^j(r)} = 0.
\]

Example 2. Let \( P(r) \) be a polynomial of order \( n \), where

\[
\lim_{r \to \infty} P(r) = +\infty.
\]

Then

\[ L(r) = \frac{K}{P(r)} + A, \]

where \( A > 0 \), and \( K \) is any real, is slowly increasing.

Proof. Applying the definition,

\[
\lim_{r \to \infty} \frac{rL'(r)}{L(r)} = \lim_{r \to \infty} \frac{r \cdot -KP'(r)}{P(r) + A} = \lim_{r \to \infty} \frac{-KrP'(r)}{P(r)[AP(r) - K]} = 0.
\]

Example 3. Let \( P(r) \) be a polynomial of order \( n \), where

\[
\lim_{r \to \infty} P(r) = +\infty.
\]

Then
\[ L(r) = Ke^{-P(r)} + A \]

where \( A < 0 \), and \( K \) is any real, is slowly increasing.

**Proof.** Applying the definition,

\[
\lim_{r \to \infty} \frac{rL'(r)}{L(r)} = \lim_{r \to \infty} \frac{-rKP'(r)e^{-P(r)}}{Ke^{-P(r)} + A}
\]

\[
= -\frac{K}{A} \lim_{r \to \infty} rP'(r)e^{-P(r)}
\]

\[
= 0 .
\]

**Example 4.** \( L(r) = \arctan(kr) \) is slowly increasing for \( r > 0, k > 0 \).

**Proof.** Applying the definition,

\[
\lim_{r \to \infty} \frac{rL'(r)}{L(r)} = \lim_{r \to \infty} \frac{r \left( \frac{k}{1+k^2r^2} \right)}{\arctan(kr)}
\]

\[
= k \lim_{r \to \infty} \frac{1}{\arctan(kr)} \cdot \frac{1}{\frac{1}{r} + k^2r}
\]

\[
= 0 .
\]

Note that Property 2 and Property 3 tell us that the functions

\[
\prod_{i=1}^{n} [l^i(r)]^{a_i} \cdot A^2 + \frac{AK}{e^P(r)} + \frac{AK}{P(r)} + \frac{K^2}{P(r)e^P(r)} \cdot (\log r^p) \arctan(kr)
\]

are also slowly increasing.

The following theorem shows the motivation for the preceding
discussion of slowly increasing functions.

**Theorem 2.3.** \( \rho(r) \) is a proximate order if and only if

\[
\rho(r) = \frac{\log L(r)}{\log r} + \rho,
\]

where \( L(r) \) is a slowly increasing function, and \( 0 < \rho < \infty \).

**Proof.** Assume \( \rho(r) \) is a proximate order. Then, since

\[
\rho(r) = \frac{\log L(r)}{\log r} + \rho \left( \rho(r) - \rho \right) \log r = L(r) \iff \rho(r) - \rho = L(r),
\]

we must show that

\[
L(r) = r^{\rho(r) - \rho}
\]

is slowly increasing. We see that

\[
L'(r) = r^{\rho(r) - \rho} \left[ (\rho(r) - \rho) \frac{1}{r} + \rho'(r) \log r \right],
\]

and therefore

\[
\lim_{r \to \infty} \frac{rL'(r)}{L(r)} = \lim_{r \to \infty} \frac{r^{\rho(r) - \rho} \left[ (\rho(r) - \rho) \frac{1}{r} + \rho'(r) \log r \right]}{r^{\rho(r) - \rho}}
\]

\[
= \lim_{r \to \infty} \left( \rho(r) - \rho \right) + \lim_{r \to \infty} \rho'(r) \log r
\]

\[
= 0,
\]

that is, \( L(r) \) is slowly increasing.

Now, assume \( L(r) \) is a slowly increasing function. Then
\[
\lim_{r \to \infty} \rho(r) = \lim_{r \to \infty} \left[ \frac{\log L(r)}{\log r} + \rho \right] = \rho .
\]

Also, since
\[
\rho'(r) = \frac{(\log r) \frac{L'(r)}{L(r)} - \frac{1}{r} \log L(r)}{(\log r)^2}
\]
we have that
\[
\lim_{r \to \infty} rp'(r) \log r = \lim_{r \to \infty} \left[ \frac{rL'(r)}{L(r)} - \frac{\log L(r)}{\log r} \right] = 0,
\]
and \( \rho(r) \) is a proximate order, which completes the proof.

Consider the entire function
\[
f(z) = e^{cz \frac{\text{Log} \, z}{\rho}} ,
\]
where \( 0 < c < \infty \), \( \rho \) is a positive integer, and \( \text{Log} \, z \) is the principal value of \( \log z \). Then since
\[
M_f(r) = e^{cr^\rho \log r}
\]
we have
\[
\log M_f(r) = cr^\rho \log r .
\]
We know that \( L(r) = \log r \) is slowly increasing, and that
\[
\rho(r) = \frac{\log L(r)}{\log r} + \rho \Leftrightarrow L(r) = r^\rho \log r^- \rho .
\]
Therefore, the type of \( f(z) \) with respect to the proximate order
\[ \rho(r) = \frac{\log \log r}{\log r} + \rho \]

is

\[ \sigma_f = \lim_{r \to \infty} \frac{\log M_f(r)}{r^\rho} = \lim_{r \to \infty} \frac{c r^\rho \log r}{r^\rho \log r} = c, \]

and \( \rho(r) \) is a proximate order of \( f(z) \).

We also know that \( L(r) = \log \log r \) is slowly increasing,

and that

\[ \rho^*(r) = \frac{\log \log \log r}{\log r} + \rho \]

is a proximate order. Since

\[ r^\rho^*(r) = r^\rho \log \log r, \]

the type of \( f(z) \) with respect to \( \rho^*(r) \) is

\[ \sigma_f = \lim_{r \to \infty} \frac{\log M_f(r)}{r^\rho^*} = \lim_{r \to \infty} \frac{c r^\rho \log r}{r^\rho \log r} = \infty, \]

and \( \rho^*(r) \) is not a proximate order of \( f(z) \).

The order of \( f(z) \) is given by

\[ \lim_{r \to \infty} \frac{\log \log M_f(r)}{\log r} = \lim_{r \to \infty} \frac{\log c + \rho \log r + \log \log r}{\log r} = \rho, \]

and the type of \( f(z) \) is

\[ \sigma = \lim_{r \to \infty} \frac{\log M_f(r)}{r^\rho} = \lim_{r \to \infty} \frac{c r^\rho \log r}{r^\rho} = \infty. \]

From these remarks, we see that by the methods of Chapter 1,
the functions
\[ f_1(z) = e^{c_1 z^\rho \log z}, \quad f_2(z) = e^{c_2 z^\rho \log z} \quad (0 < c_1 < c_2 < \infty), \]

are both considered to be of growth \((\rho, \infty)\), and although our intuition tells us that \( f_2(z) \) grows faster, we are unable to say that this is so. Therefore our growth scale of Chapter 1 is inadequate with regard to functions of the form
\[ f(z) = e^{cz^\rho \log z}. \]

On the other hand, using proximate orders and types, we are able to distinguish between their growths, since with respect to the proximate order
\[ \rho(r) = \frac{\log \log r}{\log r} + \rho, \]
we have
\[ \sigma_{f_1} = c_1 < c_2 = \sigma_{f_2}, \]

and our intuitive remark that the growth of \( f_2(z) \) is greater than the growth of \( f_1(z) \) is now a proven fact.

We saw in Chapter 1 that comparison functions of the form \( r^k, \quad k > 0, \) were not sufficient to measure the growth of nonpolynomial entire functions, and comparison functions which grow more rapidly than powers of \( r \) were desired. We chose functions of the
form
\[ e^{A r^k}, \]
\[ A > 0, \ k > 0, \] and though they seemed sufficient at the time, the preceding example has shown that there is at least one class of entire functions for which they are not sufficient. Therefore, we desire comparison functions which grow faster than
\[ e^{A r^k}, \]
and we now choose functions of the form
\[ e^{A r^k L(r)}, \]
where \( A > 0, \) and \( k(r) \) is a proximate order. These functions do grow faster since
\[ r^k(r) = r^k L(r), \]
where \( L(r) \) is a slowly increasing function, and
\[ \lim_{r \to \infty} L(r) = c \]
where \( 0 < c < \infty. \) Thus
\[ \frac{e^{A r^k}}{e^{A r^k L(r)}} = \frac{1}{e^{L(r)}} \to \frac{1}{e^c} \leq \frac{1}{e^0} = 1, \]
and for \( r > R, \) \( R \) sufficiently large,
\[ e^{A r^k} \leq e^{A r^k L(r)}. \]
Thus we see that proximate orders of entire functions and their types give us a "better" growth scale than that discussed in Chapter 1.

We shall now present some interesting and useful properties
Property 1. For \( \lim_{r \to \infty} \rho(r) = \rho > 0 \), the function

\[
f(r) = r^{\rho(r)}
\]

is an increasing function for \( r > r_0 \), \( r_0 \) sufficiently large.

Proof. Since

\[
f'(r) = (\rho(r) \log r)' r^{\rho(r)}
= [\rho'(r) \log r + \frac{1}{r} \rho(r)] r^{\rho(r)}
= [r \rho'(r) \log r + \rho(r)] r^{\rho(r)-1},
\]

we have that, for \( r > r_0 \) and for all \( \epsilon \) such that \( 0 < \epsilon < \rho \),

\[
f'(r) > (\rho - \epsilon) r^{\rho(r)-1} > 0.
\]

If \( \rho = 0 \), the inequality \( f'(r) > 0 \) still holds. To see this we replace the condition

\[
\lim_{r \to \infty} r \rho'(r) \log r = 0
\]

by the equivalent conditions

\[
\rho(r) > 0, \quad \lim_{r \to \infty} \frac{r \rho'(r) \log r}{\rho(r)} = 0.
\]

Before completing the proof, we justify this equivalence:
\[
\lim_{r \to \infty} \frac{r \rho'(r) \log r}{\rho(r)} = 0 \iff -\epsilon_1 < \frac{r \rho'(r) \log r}{\rho(r)} < \epsilon_1
\]

for all \( \epsilon_1 > 0 \) and \( r > r_1 \), and since \( \rho(r) > 0 \), we have

\[-\epsilon_1 \rho(r) < r \rho'(r) \log r < \epsilon_1 \rho_1 ,\]

and because \( \rho(r) \to 0 \), we obtain (for \( \epsilon_2 > 0 \), \( r \) sufficiently large)

\[-\epsilon = -\epsilon_1 \epsilon_2 < r \rho'(r) \log r < \epsilon_1 \epsilon_2 = \epsilon ,\]

which is equivalent to

\[
\lim_{r \to \infty} r \rho'(r) \log r = 0.
\]

Using this equivalence, we then have

\[
f'(r) = \rho(r) r^{\rho(r)-1} \left[ \frac{r \rho'(r) \log r}{\rho(r)} + 1 \right]
\]

\[
> \rho(r) r^{\rho(r)-1} (-\epsilon + 1)
\]

\[
> 0
\]

for all \( \epsilon > 0 \), and \( r > r_0 \), \( r_0 \) sufficiently large, and the proof is complete.

Since in the study of asymptotic properties, the behavior of the proximate order on a finite interval plays no role, we shall in the future assume that \( r^{\rho(r)} \) is a monotone function for \( r \geq 0 \), and we shall define it to be zero at \( r = 0 \). This may be done without
loss of generality.

Property 2. For $1 < k < \infty$

$$\frac{(kr)^\rho(kr)}{r^\rho(r)} \rightarrow k^\rho$$

uniformly as $r \rightarrow \infty$.

Proof. We know that

$$r^\rho(r) = r^\rho L(r),$$

where $L(r)$ is a slowly increasing function. Therefore

$$\frac{(kr)^\rho(kr)}{r^\rho(r)} = \frac{(kr)^\rho L(kr)}{r^\rho L(r)} = k^\rho \frac{L(kr)}{L(r)} \rightarrow k^\rho$$

uniformly as $r \rightarrow \infty$.

In the proof of our next property, we shall use the following theorem (2, p. 13).

Theorem 2.4. Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is regular for $|z| \leq R$, then $M_f(r)$ and

$$M_2(r) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta \right)^{\frac{1}{2}}$$

are increasing functions of $r$ for $0 < r < R$, and

$$M_2(r) \leq M_f(r) \leq \left( \frac{R+r}{R-r} \right)^{\frac{1}{2}} M_2(R)$$
for $0 < r < R$.

**Property 3.** \[ \lim_{r \to \infty} \frac{\log M_f(r)}{r^\rho(r)} = 1 \iff \lim_{r \to \infty} \frac{\log M_2(r)}{r^\rho(r)} = 1. \]

**Proof.** Let $R = kr$ in Theorem 2.4, then

\[ M_2(r) \leq M_f(r) \leq \left( \frac{k+1}{k-1} \right)^2 M_2(kr) \]

for $0 < r < kr$. Taking logarithms and dividing by $r^\rho(r)$, we obtain

\[ \frac{\log M_2(r)}{r^\rho(r)} \leq \frac{\log M_f(r)}{r^\rho(r)} \leq \frac{\log M_2(kr) + \frac{1}{2} \log \left( \frac{k+1}{k-1} \right)}{r^\rho(r)}. \]

Finally, we take lim sups and have

\[ \lim_{r \to \infty} \frac{\log M_2(r)}{r^\rho(r)} \leq \lim_{r \to \infty} \frac{\log M_f(r)}{r^\rho(r)} \leq \lim_{r \to \infty} \left[ \log M_2(kr) + \frac{1}{2} \log \left( \frac{k+1}{k-1} \right) \right]. \]

\[ \leq k^\rho \lim_{r \to \infty} \frac{\log M_2(kr)}{(kr)^\rho(kr)} \]

\[ = \lim_{r \to \infty} \frac{\log M_2(r)}{r^\rho(r)} , \]

letting $k \to 1$.

**Property 4.** For $\lambda < \rho + 1$, \[ \int_a^r t^\rho(t) - \lambda dt = \frac{1}{\rho+1-\lambda} r^\rho(r)+1-\lambda \]

\[ + o(r^\rho(r)+1-\lambda), \quad a \geq 0. \]
Proof. We have

\[ \int_a^x t^{\rho(t) - \lambda} \, dt = \int_a^x t^{\rho - \lambda} t^{\rho(t) - \rho} \, dt. \]

Integrating by parts, we let

\[ dv = t^{\rho - \lambda} \Rightarrow v = \frac{t^{\rho - \lambda + 1}}{\rho - \lambda + 1} \]

\[ u = t^{\rho(t) - \rho} \Rightarrow du = t^{\rho(t) - \rho} \left( (\rho(t) - \rho) \log t \right)' = t^{\rho(t) - \rho} \left( (\rho(t) - \rho) t^{-1} \rho'(t) \log t \right) \, dt \]

to obtain

\[ \int_a^x t^{\rho(t) - \lambda} \, dt = t^{\rho(t) - \rho} \left( \frac{t^{\rho - \lambda + 1}}{\rho - \lambda + 1} \right) \bigg|_a^x - \int_a^x \frac{t^{\rho - \lambda + 1}}{\rho - \lambda + 1} \left( t^{\rho(t) - \rho} \left( \rho(t) - \rho \right) t^{-1} \rho'(t) \log t \right) \, dt \]

\[ = \frac{t^{\rho(t) + 1 - \lambda}}{\rho - \lambda + 1} \bigg|_a^x - \frac{1}{\rho - \lambda + 1} \int_a^x \left[ t^{\rho(t) - \lambda} (\rho(t) - \rho) t^{\rho(t) + 1 - \lambda} \rho(t) \log t \right] \, dt. \]

From our definition of proximate order, we have asymptotically

\[ |\rho(t) - \rho| < \frac{\epsilon}{2} \quad \text{and} \quad |t \rho'(t) \log t| < \frac{\epsilon}{2} \]

for all \( \epsilon > 0 \). Substitution now yields

\[ \int_a^x t^{\rho(t) - \lambda} \, dt \leq \frac{1}{\rho - \lambda + 1} t^{\rho(t) + 1 - \lambda} + O(1) - \frac{\epsilon}{\rho + 1 - \lambda} \int_a^x t^{\rho(t) - \lambda} \, dt, \]

and transposing the integral gives us
\[
\int_{a}^{r} t^{\rho(t)-\lambda} dt \leq (1 + \frac{\epsilon}{\rho + 1 - \lambda})^{-1} \left[ \frac{1}{\rho + 1 - \lambda} r^{\rho(r)+1-\lambda} + O(1) \right]
\]

\[
= (1 + o(1))^{-1} \left[ \frac{1}{\rho + 1 - \lambda} r^{\rho(r)+1-\lambda} + O(1) \right]
\]

\[
= (1 + o(1)) \left[ \frac{1}{\rho + 1 - \lambda} r^{\rho(r)+1-\lambda} + O(1) \right], \text{ since }
\]

\[
\frac{1}{1 + o(1)} = 1 - \frac{o(1)}{1 + o(1)} = 1 + o(1),
\]

\[
= \frac{1}{\rho + 1 - \lambda} r^{\rho(r)+1-\lambda} + O(1) + \frac{o(1)r^{\rho(r)+1-\lambda}}{\rho + 1 - \lambda} + o(1)O(1)
\]

\[
= \frac{1}{\rho + 1 - \lambda} r^{\rho(r)+1-\lambda} + O(1) + o(r^{\rho(r)+1-\lambda}), \text{ since }
\]

\[
o(1)O(1) = O(1) \text{ and } O(1) + O(1) = O(1)
\]

\[
= \frac{1}{\rho + 1 - \lambda} r^{\rho(r)+1-\lambda} + o(r^{\rho(r)+1-\lambda}), \text{ since }
\]

\[
O(1) = o(r^{\rho(r)+1-\lambda}) \text{ and } 2o(f(r)) = o(f(r)).
\]

**Property 5.** For \( \lambda > \rho + 1 \),

\[
\int_{r}^{\infty} t^{\rho(t)-\lambda} dt = \frac{1}{\lambda - \rho + 1} r^{\rho(r)-\lambda + 1} + o(r^{\rho(r)-\lambda + 1}).
\]

**Proof.** Integrating by parts as in the proof of Property 4, we have,

for all \( \epsilon > 0 \),

\[
\int_{r}^{\infty} t^{\rho(t)-\lambda} dt = \frac{1}{\rho - \lambda + 1} t^{\rho(t)-\lambda + 1} \bigg|_{r}^{\infty} - \frac{\epsilon}{\rho + 1 - \lambda} \int_{r}^{\infty} t^{\rho(t)-\lambda} dt
\]

\[
= \frac{1}{\lambda - \rho - 1} r^{\rho(r)-\lambda + 1} + \frac{1}{\rho - \lambda + 1} \lim_{t \to \infty} (t^{\rho(t)+1-\lambda}) \frac{\epsilon}{\rho + 1 - \lambda} \int_{r}^{\infty} t^{\rho(t)-\lambda} dt.
\]
Since \( \lim_{t \to \infty} \rho(t) = \rho < \rho + 1 < \lambda, \ \lambda - \rho(t) - 1 > 0 \) for large \( t \), and

\[
\lim_{t \to \infty} t^{\rho(t)+1-\lambda} = \lim_{t \to \infty} \frac{1}{t^{\lambda-\rho(t)-1}} = 0.
\]

Transposing the integral on the right as in the proof of Property 4, we have

\[
\int_{r}^{\infty} t^{\rho(t)-\lambda} \, dt = (1 + \frac{\epsilon}{\rho+1-\lambda})^{-1} \frac{1}{\lambda - \rho - 1} t^{\rho(r)-\lambda+1}
\]

\[
= \left[1+o(1)\right] \frac{1}{\lambda - \rho - 1} r^{\rho(r)-\lambda+1}
\]

\[
= \frac{1}{\lambda - \rho - 1} r^{\rho(r)-\lambda+1} + o(1) \frac{1}{\lambda - \rho - 1} r^{\rho(r)-\lambda+1}
\]

\[
= \frac{1}{\lambda - \rho - 1} r^{\rho(r)-\lambda+1} + o(r^{\rho(r)-\lambda+1}).
\]

**Property 6.** If \( \phi(r) \) is a bounded function on each finite interval, then for \( \lambda < \rho + 1 \)

\[
\overline{\lim}_{r \to \infty} \left\{ r^{-\rho(r)+\lambda-1} \int_{x}^{r} \phi(t) \, dt \right\} \leq \frac{\overline{\Delta}}{\rho - \lambda + 1},
\]

where \( a > 0 \) and

\[
\overline{\Delta} = \overline{\lim}_{r \to \infty} \frac{\phi(r)}{r^{\rho(r)}}.
\]

**Proof.** Since \( \overline{\Delta} = \overline{\lim}_{t \to \infty} \frac{\phi(t)}{t^{\rho(t)}} \) implies that for all \( \epsilon > 0 \) and \( t > t_0 \) (and thus \( r > r_0 \))
\[ \frac{\phi(t)}{t^\rho(t)} < \bar{\Delta} + \epsilon \Rightarrow \phi(t) < (\bar{\Delta} + \epsilon) t^{\rho(t)} \Rightarrow \frac{\phi(t)}{t^\lambda} < (\bar{\Delta} + \epsilon) t^{\rho(t)-\lambda}, \]

we have

\[ \int_a^r \frac{\phi(t)}{t^\lambda} \, dt < \int_a^r (\bar{\Delta} + \epsilon) t^{\rho(t)-\lambda} \, dt \]

\[ = \int_a^0 (\bar{\Delta} + \epsilon) t^{\rho(t)-\lambda} \, dt + \int_0^r (\bar{\Delta} + \epsilon) t^{\rho(t)-\lambda} \, dt \]

and

\[ \int_a^r \frac{\phi(t)}{t^\lambda} \, dt \leq O(1) + (\bar{\Delta} + \epsilon) \int_a^r t^{\rho(t)-\lambda} \]

\[ \leq O(1) + (\bar{\Delta} + \epsilon) \frac{r^{\rho(r)+1-\lambda}}{\rho+1-\lambda} + (\bar{\Delta} + \epsilon) o(r^{\rho(r)+1-\lambda}) \]

\[ = O(1) + (\bar{\Delta} + \epsilon) \frac{r^{\rho(r)+1-\lambda}}{\rho+1-\lambda} + o(r^{\rho(r)+1-\lambda}), \]

and finally

\[ \lim_{r \to \infty} \left[ r^{-\rho(r)+\lambda-1} \int_a^r \frac{\phi(t)}{t^\lambda} \, dt \right] \leq \lim_{r \to \infty} \left[ \frac{O(1)}{r^{\rho(r)-\lambda+1}} + \frac{\bar{\Delta} + \epsilon}{\rho-\lambda+1} + \frac{o(r^{\rho(r)+1-\lambda})}{r^{\rho(r)+1-\lambda}} \right] \]

\[ = \frac{\bar{\Delta}}{\rho-\lambda+1}. \]

**Property 7.** If \( \phi(r) \) is a bounded function on each finite interval, then for \( \lambda < \rho + 1 \)

\[ \lim_{r \to \infty} \left[ r^{-\rho(r)+\lambda-1} \int_a^r \frac{\phi(t)}{t^\lambda} \, dt \right] \geq \frac{\bar{\Delta}}{\rho+1-\lambda}, \]
where $a > 0$ and

$$
\Delta = \lim_{r \to \infty} \frac{\phi(r)}{r \rho(r)}.
$$

Proof. Since $\Delta = \lim_{t \to \infty} \frac{\phi(t)}{t \rho(t)}$ implies that for all $\epsilon > 0$ and $t > t_0$ (and thus $r > r_0$)

$$
\frac{\phi(t)}{t \rho(t)} > \Delta + \epsilon \Rightarrow \phi(t) > (\Delta + \epsilon) t \rho(t) \Rightarrow \frac{\phi(t)}{t^{\lambda}} > (\Delta + \epsilon) t^{\rho(t) - \lambda},
$$

we have

$$
\int_a^R \frac{\phi(t)}{t^\lambda} dt > \int_a^R (\Delta + \epsilon) t^{\rho(t) - \lambda} dt
$$

$$
= \int_a^0 (\Delta + \epsilon) t^{\rho(t) - \lambda} dt + \int_0^R (\Delta + \epsilon) t^{\rho(t) - \lambda} dt
$$

and

$$
\int_a^R \frac{\phi(t)}{t^\lambda} dt \geq O(1) + (\Delta + \epsilon) \int_0^R t^{\rho(r) - \lambda} dt
$$

$$
= O(1) + (\Delta + \epsilon) \frac{r^{\rho(r) - \lambda + 1}}{\rho - \lambda + 1} + (\Delta + \epsilon) o(r^{\rho(r) - \lambda + 1})
$$

$$
= O(1) + (\Delta + \epsilon) \frac{r^{\rho(r) - \lambda + 1}}{\rho - \lambda + 1} + o(r^{\rho(r) - \lambda + 1})
$$

and finally
\[
\lim_{r \to \infty} \left[ r^{-\rho(r)+\lambda-1} \int_{\alpha}^{r} \frac{\phi(t)}{t^\lambda} \, dt \right] \geq \lim_{r \to \infty} \left[ \frac{O(1)}{r^{\rho(r)-\lambda+1}} + \frac{\Delta + \epsilon}{\rho-\lambda+1} + \frac{o(r^{\rho(r)-\lambda+1})}{r^{\rho(r)-\lambda+1}} \right] \\
= \frac{\Delta + \epsilon}{\rho-\lambda+1} .
\]

**Property 8.** If \( \phi(t) \) is a bounded function on each finite interval, and the limit

\[
\Delta = \lim_{r \to \infty} \frac{\phi(r)}{r^{\rho(r)}}
\]

exists, then for \( \lambda < \rho+1 \), the limit

\[
\lim_{r \to \infty} \left[ r^{-\rho(r)+\lambda-1} \int_{\alpha}^{r} \frac{\phi(t)}{t^\lambda} \, dt \right] = \frac{\Delta}{\rho+1-\lambda}
\]

exists.

**Proof.** Since \( \Delta \) exists, we know that \( \Delta = \text{const} = \Delta \), and the result follows from Property 6 and Property 7.

Note that if \( f(z) \) is entire, then \( \phi(t) = \log M_f(t) \) satisfies the hypothesis of Property 6, and \( \Delta = \sigma_f \), the type of \( f(z) \) with respect to \( \rho(t) \). Therefore, Property 8 tells us that if \( \Delta \) exists,

\[
\lim_{r \to \infty} \left[ r^{\rho(r)+\lambda-1} \int_{\alpha}^{r} \frac{\log M_f(t)}{t^\lambda} \, dt \right] = \frac{\sigma_f}{\rho+1-\lambda} .
\]

Earlier it was shown that given any entire function \( f(z) \) there exists a proximate order \( \rho(r) \) such that the type of \( f(z) \) with respect to \( \rho(r) \) is \( \sigma_f = 1 \). We also know that any entire function has an infinite number of proximate orders. Consider the entire
function $f(z)$ where $f(r) > 0$ if $r > 0$. We shall now construct a proximate order which possesses two useful properties.

**Theorem 2.5.** If $f(r)$ is positive for $r > 0$ and satisfies the condition

$$\lim_{r \to \infty} \frac{\log f(r)}{\log r} = \rho < \infty,$$

then a proximate order $\rho(r)$ can be chosen so that for all $r > 0$

$$f(r) \leq r^{\rho(r)},$$

and for some sequence of values $r_n (n=1, 2, \cdots)$ tending to infinity

$$f(r_n) = r_n^{\rho(r_n)}.$$

**Proof.** Our proof shall be divided into three parts.

**Part 1.** In place of the given function $f(r)$, it will be more convenient for us to consider the function

$$\phi(r) = f(r)r^{-\rho}.$$ 

We note that the growth of $\phi(r)$ is less than any power of $r$, since

$$\lim_{r \to \infty} \frac{\log \phi(r)}{\log r} = \lim_{r \to \infty} \frac{\log f(r) + \log r^{-\rho}}{\log r} = \lim_{r \to \infty} \frac{\log f(r)}{\log r} + \lim_{r \to \infty} \frac{\log r^{-\rho}}{\log r} = \rho - \rho = 0,$$
and therefore, for all \( \epsilon > 0 \) and \( r > R \), \( R \) sufficiently large,

\[
\frac{\log \phi(r)}{\log r} < \epsilon \iff \log \phi(r) < \epsilon \log r \iff \phi(r) < r^\epsilon.
\]

Now let us pass to the logarithmic scale, setting \( x = \log r \) and \( y = \log \phi(r) \). We then have \( y = \phi_1(x) \) where \( \phi_1(x) = \log \phi(e^x) \) since

\[
y = \log \phi(r) = \log \phi(e^{\log r}) = \log \phi(e^x) = \phi_1(x).
\]

Also

\[
\lim_{x \to \infty} \frac{\phi_1(x)}{x} = \lim_{x \to \infty} \frac{y}{x} = \lim_{r \to \infty} \frac{\log \phi(r)}{\log r} = 0.
\]

Therefore, for all \( \epsilon > 0 \) and \( x > x_\epsilon \), the entire curve \( y = \phi_1(x) \) lies below the line \( y = \epsilon x \), because

\[
\lim_{x \to \infty} \frac{y}{x} = 0 \implies \frac{y}{x} < \epsilon \implies y = \phi_1(x) < \epsilon x
\]

for \( \epsilon > 0 \), \( x > x_\epsilon \). On the other hand, there are points of \( y = \phi_1(x) \) with arbitrarily large abscissas lying above the line \( y = -\epsilon x \), because

\[
\lim_{x \to \infty} \frac{y}{x} = 0 \implies -\epsilon < \frac{y}{x} \implies y = \phi_1(x) > -\epsilon x
\]

for \( \epsilon > 0 \) and a sequence \( \{x_n\} \) of values of \( x \) tending to infinity.

Note also that \( y = \phi_1(x) \) does not approach either of the lines \( y = \pm \epsilon x \) asymptotically since
Part 2. Suppose that \( \lim_{x \to \infty} \phi_1(x) = +\infty \), and form the smallest closed convex domain containing all the points of the curve \( y = \phi_1(x) \) and \( \{(x,0)/x \in \text{domain of } \phi_1(x)\} \), i.e. the positive ray of the x-axis "beneath" \( y = \phi_1(x) \). The part of the boundary of this domain that lies above the abscissa axis is a continuous function \( y = \psi(x) \), since, in a convex domain, any two points can be connected by a straight line which lies entirely within the domain.

Consider the point \((a, \psi(a))\) on this boundary. Given any length \( \epsilon_1 \), we construct an \( \epsilon_1 \)-neighborhood about \((a, \psi(a))\). Any point \((x_1, y_1)\) in this neighborhood has the property that

\[
\sqrt{(a-x_1)^2 + (\psi(a)-y_1)^2} < \epsilon_1.
\]

Since \((a, \psi(a))\) is a boundary point, the neighborhood contains both points inside and outside the convex hull, and in particular, it contains other boundary points \((x, \psi(x))\). Therefore,

\[
\sqrt{(a-x)^2 + (\psi(a)-\psi(x))^2} = \epsilon_2 + |\psi(a)-\psi(x)| < \epsilon_1, \epsilon_2 < \epsilon_1 \Rightarrow |\psi(a)-\psi(x)| < \epsilon = \epsilon_1 - \epsilon_2
\]

and \( y = \psi(x) \) is continuous.
The function \( y = \psi(x) \) has the following properties:

1. \( \phi_1(x) \leq \psi(x) \), since \( \psi(x) \) is the boundary of the convex domain containing \( \phi_1(x) \) and the positive x-axis "beneath" \( \phi_1(x) \), and \( \psi(x) \geq y \) when \( (x, y) \) is a point of the convex domain.

2. \( \lim_{x \to \infty} \frac{\psi(x)}{x} = 0 \), because
   \[
   \lim_{x \to \infty} \frac{\psi(x)}{x} = \lim_{x \to \infty} \frac{\phi_1(x)}{x} = 0.
   \]

3. \( \psi(x) \) is convex from above because \( \psi(x) \) is the "top half" of the boundary of a convex set (7, p. 105).

4. Each extreme point (i.e. each point not lying inside any line segment of the curve) of the curve \( y = \psi(x) \) also lies on the curve \( y = \phi_1(x) \), since the closed convex domain of \( S = \{(x, y)/0 \leq y \leq \phi_1(x), \ x \text{ in the domain of } \phi_1(x)\} \) is the closed domain of the set of extreme points of \( S \) (7, p. 138), and thus in forming the convex domain of \( S \), extreme points may only arise from extreme points of \( S \) (7, p. 13).

5. The curve \( y = \psi(x) \) has a sequence of extreme points that tend to infinity, since we know that
   \[
   \lim_{x \to \infty} \phi_1(x) = +\infty \Rightarrow \lim_{x \to \infty} \psi(x) = +\infty
   \]
(because of Property 1) and assuming \( \psi(x) \) doesn't have a sequence of extreme points implies that either \( \psi(x) \) has a finite number of extreme points or \( \psi(x) \) has a sequence of extreme points which does not tend to infinity, both of which are false as they imply that \( \lim_{x \to \infty} \psi(x) \neq +\infty \).

Now as it stands, \( y = \psi(x) \) is at worst piecewise differentiable (not being differentiable at the angular points). We can make changes in \( y = \psi(x) \) in a small neighborhood of each angular point in order to make \( y = \psi(x) \) everywhere differentiable. Therefore, since \( \lim_{x \to \infty} \psi(x) = +\infty \), we can apply L'Hopital's rule to \( \lim_{x \to \infty} \frac{\psi(x)}{x} = 0 \) and obtain

\[
\lim_{x \to \infty} \psi'(x) = 0 .
\]

Returning to our original variables,

\[
\phi_1(x) = \log \phi(e^x) = \log \phi(r) = \log f(r) r^{-\rho} \leq \psi(x) = \psi(\log r),
\]

which is equivalent to

\[
f(r) \leq r^\rho e^{\psi(\log r)} = r^\rho r^{\psi(\log r)} = r^{\rho + \frac{\psi(\log r)}{\log r}}.
\]

We now define \( \rho(r) \) by

\[
\rho(r) = \rho + \frac{\psi(\log r)}{\log r} , \quad r > 0 ,
\]

and we see immediately that
\[ f(r) \leq r^\rho(r). \]

Also, by properties 4 and 5 of \( \psi(x) \), there exists a sequence of values \( r_n \) tending to infinity for which

\[ f(r_n) = r_n^\rho(r_n) \]

(at these points we have \( \phi_1(x_n) = \psi(x_n) \) where \( x_n = \log r_n \)).

It now remains to show that the \( \rho(r) \) we have constructed

is a proximate order:

1. \( \rho(r) = \rho + \frac{\psi(\log r)}{\log r} \) is real valued, continuous and piece-wise differentiable for \( r > R, \ R \) sufficiently large;

2. \( \lim_{r \to \infty} \rho(r) = \rho + \lim_{r \to \infty} \frac{\psi(\log r)}{\log r} = \rho + \lim_{\log r \to \infty} \frac{\psi(\log r)}{\log r} = \rho + 0 = \rho; \)

3. \( \lim_{r \to \infty} r \rho'(r) \log r = \lim_{r \to \infty} r \log r \left[ \frac{(\log r)\psi'(\log r)\frac{1}{1-\frac{1}{r}} - \frac{1}{r} \psi(\log r)}{(\log r)^2} \right] \)

\[ = \lim_{r \to \infty} \left[ \psi'(\log r) - \frac{\psi(\log r)}{\log r} \right] \]

\[ = 0. \]

Part 3. In Part 2, we have proved the theorem under assumption that \( \lim_{x \to \infty} \phi_1(x) = + \infty \). Now, we shall show that the general case can be reduced to the case in Part 2.

Let us construct a concave function \( y = \psi_1(x) \) so that

\[ \lim_{x \to \infty} \frac{\psi_1(x)}{x} = 0, \lim_{x \to \infty} \psi'_1(x) = 0, \text{ and } \lim_{x \to \infty} \left[ \phi_1(x) + \psi_1(x) \right] = \infty, \]
where \( \phi_1(x) = \log \phi(e^x) = \log(f(x) e^{-xP}) \) as before. We construct \( \psi_1(x) \) in the following manner:

Pass a segment \( d_1 \) of the line \( y = -\varepsilon_1 x \) from the origin to a point \( (x_1, y_1) \) at which \( \phi_1 x > -\varepsilon_1 x_1 + 1 \). Such a point exists for all \( \varepsilon_1 > 0 \), because if it did not exist then \( \phi_1(x) \leq -\varepsilon_1 x_1 + 1 \) for all \( x \), which implies that \( \frac{\phi_1(x) - 1}{x} \leq -\varepsilon_1 \), and this is impossible since \( \lim_{x \to \infty} \frac{\phi_1(x) - 1}{x} = 0 \).

Now choose \( \varepsilon_2 \) such that \( 0 < \varepsilon_2 < \varepsilon_1 \), and from the point \( (x_1, -\varepsilon_1 x_1) \) pass a segment \( d_2 \) of the line \( y + \varepsilon_1 x_1 = -\varepsilon_2(x-x_1) \) to a point \( (x_2, y_2) \), \( x_2 > x_1 \), at which \( \phi_1(x_2) > -\varepsilon_1 x - \varepsilon_2(x_2-x_1) + 2 \). Again such a point exists, because if it did not exist then \( \frac{\phi_1(x) - (2-\varepsilon_1 x_1)}{x-x_1} < -\varepsilon_2 \) for all \( x \), which is impossible since \( \lim_{x \to \infty} \frac{\phi_1(x) - (2-\varepsilon_1 x_1)}{x-x_1} = 0 \).

Now choose \( \varepsilon_3 \) such that \( 0 < \varepsilon_3 < \varepsilon_2 \), and from the point \( (x_2, -\varepsilon_1 x_1 - \varepsilon_2(x_2-x_1)) \) pass a segment \( d_3 \) of the line \( y - [-\varepsilon_1 x_1 - \varepsilon_2(x_2-x_1)] = -\varepsilon_3(x-x_2) \) to a point \( (x_3, y_3) \), \( x_3 > x_2 \), at which \( \phi_1(x_3) > [-\varepsilon_1 x_1 - \varepsilon_2(x_2-x_1)] - \varepsilon_3(x_3-x_2) + 3 \). Again such a point exists.

Continuing our construction in this manner, we let \( y = \psi_1^*(x) \) be
the polygonal function whose graph consists of the segments \( d_i \).

Note that our positive numbers \( \epsilon_i \) have been chosen so that
\[
\epsilon_1 > \epsilon_2 > \cdots > \epsilon_n > \cdots \quad \text{and} \quad \epsilon_n \to 0,
\]
and the points \( x_i \) have been chosen so that \( x_1 < x_2 < \cdots < x_n < \cdots \) and \( x_n \to \infty \).

Clearly, \( y = \psi_1^*(x) \) satisfies the condition
\[
\lim_{x \to \infty} \frac{\psi_1^*(x)}{x} = 0.
\]

Now, \( y = \psi_1^*(x) \) is differentiable everywhere except at
\( x = x_1, x_2, \cdots \) (the angular points) but by changing \( \psi_1^*(x) \) in an
inessential manner in a small neighborhood of each angular point,
we can make \( \psi_1^*(x) \) everywhere differentiable.

Let \( \psi_1(x) \equiv -\psi_1^*(x) \) and we immediately see that \( \psi_1(x) \)
has the desired properties.

Now let us construct a convex majorant \( \psi_2(x) \) for the func-
tion \( \phi_1(x) + \psi_1(x) \) just as we did in Part 2.

Finally, we define \( \psi(x) \) by \( \psi(x) \equiv \psi_2(x) - \psi_1(x) \), noting that:

1. \( \psi(x) \geq \phi_1(x) \)
2. On some sequence \( \{x'_n\} \) of extreme points tending to infinity we have \( \phi_1(x'_n) = \psi(x'_n) \).

3. \( \lim_{x \to \infty} \psi'(x) = 0 \)

4. \( \lim_{x \to \infty} \frac{\psi(x)}{x} = 0 \).

As in Part 2, let \( \rho(r) = \rho + \frac{\psi(\log r)}{\log r} \), and by the same arguments, \( \rho(r) \) is a proximate order, \( f(r) \leq r^\rho(r) \), and for some sequence \( \{r'_n\} \) of extreme points tending to infinity, \( f(r'_n) = r^\rho(r'_n) \), and the proof of our theorem is complete.

In Chapter 1, we saw that the type of an entire function is related to the coefficients of the function's MacLaurin Series expansion by the equation

\[
\exp \rho = \lim_{n \to \infty} n! |a_n|^{p/n}.
\]

The following theorem generalizes this relationship by showing how the type of \( f(z) \), with respect to its proximate order \( \rho(r) \), is related to the coefficients of the MacLaurin Series expansion of \( f(z) \). Before proving this theorem, we shall prove the following lemma.

Lemma. If \( \phi(t) \) is defined to be the unique (for \( t > t_0 \)) solution of the equation

\[
t = r^{\rho(r)},
\]
then
\[ \lim_{t \to \infty} \frac{\phi(kt)}{\phi(t)} = k^{1/\rho} . \]

Proof. Since
\[ t = r \rho(r) \implies \log t = \rho(r) \log r \implies \frac{d \log t}{d \log r} = \rho(r) \frac{d \log r}{d \log r} \]
\[ + \log r \left[ \frac{d \rho(r)}{dr} \cdot \frac{dr}{d \log r} \right] = \rho(r) + \log r \rho'(r) \cdot r , \]
we have that
\[ \lim_{r \to \infty} \frac{d \log t}{d \log r} = \rho \implies \lim_{r \to \infty} \frac{d \log r}{d \log t} = \frac{1}{\rho} \implies \lim_{t \to \infty} \frac{d \log \phi(t)}{d \log t} = \frac{1}{\rho} . \]

Therefore, for all \( \epsilon > 0 \), and for all \( t > t_0 \), \( t_0 \) sufficiently large,
\[ \left| \frac{d \log \phi(t)}{d \log t} - \frac{1}{\rho} \right| < \epsilon , \]
which implies that
\[ \left( \frac{1}{\rho} - \epsilon \right) d \log t < d \log \phi(t) < \left( \frac{1}{\rho} + \epsilon \right) d \log t , \]
and
\[ \int_t^{kt} \left( \frac{1}{\rho} - \epsilon \right) d \log t < \int_t^{kt} d \log \phi(t) < \int_t^{kt} \left( \frac{1}{\rho} + \epsilon \right) d \log t , \]
which is equivalent to
\[ \left( \frac{1}{\rho} - \epsilon \right) \log k < \log \frac{\phi(kt)}{\phi(t)} < \left( \frac{1}{\rho} + \epsilon \right) \log k . \]
This, in turn, is equivalent to

\[ k \frac{1}{\rho} - \epsilon < \frac{\phi(kt)}{\phi(t)} < k \frac{1}{\rho} + \epsilon \quad \Rightarrow \lim_{t \to \infty} \frac{\phi(kt)}{\phi(t)} = k \frac{1}{\rho}. \]

**Theorem 2.5.** The type \( \sigma_f \) of the entire function

\[ f(z) = \sum_{n=0}^{\infty} c_n z^n \]

with respect to its proximate order \( \rho(r) \) is given by the equation

\[ \frac{\lim_{n \to \infty} (n|c_n|^n)}{\phi(n)} = (\sigma_f \rho e)^{1/\rho}, \]

where \( \phi(t) \) is defined to be the unique (for \( t > t_0 \)) solution of the equation

\[ t = r^{\rho(r)}. \]

**Proof.** From the inequality

\[ |c_n| \leq \frac{M_f(r)}{r^n}, \]

we derive that

\[ \log |c_n| = \log M_f(r) - n \log r. \]

Since the definition of \( \sigma_f \) tells us that for all \( \epsilon > 0 \) and \( r > R \), \( R \) sufficiently large,

\[ \log M_f(r) < \sigma_f r^{\rho(r)} + \epsilon, \]
we have
\[ \log |c_n| < \sigma r^\rho(r) - n \log r, \]
where \( \sigma > \sigma_f \). Choosing \( r \) to be the root of the equation
\[ n = \sigma r^\rho(r) \]
the equation
\[ r^\rho(r) = \frac{n}{\sigma \rho} \]
has the unique solution \( r = \phi \left( \frac{n}{\sigma \rho} \right) \), and our inequality becomes
\[ \log |c_n| < \frac{n}{\rho} - n \log \left( \frac{n}{\sigma \rho} \right) \]
which is equivalent to
\[ \log \left( \phi(n) \sqrt[n]{|c_n|} \right) < \frac{1}{\rho} \log \phi \left( \frac{n}{\sigma \rho} \right) + \log \phi(n) = \frac{1}{\rho} + \log \frac{\phi(n)}{\phi \left( \frac{n}{\sigma \rho} \right)} \]
and
\[ \phi(n) \sqrt[n]{|c_n|} < e^{\frac{1}{\rho} \log \phi \left( \frac{n}{\sigma \rho} \right)} \]
Taking the \( \lim \sup \) and applying our lemma, we obtain
\[ \lim_{n \to \infty} \phi(n) \sqrt[n]{|c_n|} \leq e^{\frac{1}{\rho} \lim_{n \to \infty} \log \left( \frac{\phi(n)}{\phi \left( \frac{n}{\sigma \rho} \right)} \right)} = e^{\frac{1}{\rho} \phi(r \rho) \frac{1}{\rho} \log (e \sigma \rho) \frac{1}{\rho}}, \]
and since \( \sigma > \sigma_f \) was arbitrary,
\[
\lim_{n \to \infty} \phi(n) \sqrt[n]{|c_n|} \leq (e^{\sigma_f \rho})^{1/\rho}.
\]

It remains to show that equality holds. To do this we define \(\sigma\) by the equation

\[
\lim_{n \to \infty} \phi(n) \sqrt[n]{|c_n|} = (e^{\sigma \rho})^{1/\rho}
\]

and show that the assumption \(\sigma < \sigma_f\) leads to a contradiction.

Choose \(\sigma_1\) such that \(\sigma < \sigma_1 < \sigma_f\), then for all \(\epsilon > 0\) and \(n\) sufficiently large,

\[
\phi(n) \sqrt[n]{|c_n|} < (e^{\sigma \rho})^{1/\rho} + \epsilon < (e^{\sigma \rho})^{1/\rho} \iff |c_n| < \left[ \frac{(e^{\sigma_1 \rho})^{1/\rho}}{\phi(n)} \right]^n.
\]

Recalling our lemma again we have

\[
\lim_{n \to \infty} \frac{\phi(\frac{n}{\sigma_1 \rho})}{\phi(n)} = (\frac{1}{\sigma_1 \rho})^{1/\rho},
\]

which implies that for \(n > N,\ N\) sufficiently large

\[
\frac{\phi(\frac{n}{\sigma_1 \rho})}{\phi(n)} < (\frac{1}{\sigma_1 \rho})^{1/\rho} \iff \frac{1}{\phi(\frac{n}{\sigma_1 \rho})} > \frac{(\sigma_1 \rho)^{1/\rho}}{\phi(n)}.
\]

Using this, our original inequality becomes

\[
|c_n| < \left[ \frac{e^{1/\rho}}{\phi(\frac{n}{\sigma_1 \rho})} \right]^n.
\]
for \( n > N \), and thus

\[
|c_n r^n| < \left[ \frac{e^{1/\rho}}{\phi\left( \frac{n}{\sigma \rho} \right)} \right]^n r^n.
\]

The maximal term \( \mu_f(r) \) of the MacLaurin Series of \( f(r) \) then satisfies the inequality

\[
\mu_f(r) < \left[ \frac{e^{1/\rho}}{\phi\left( \frac{n}{\sigma \rho} \right)} \right]^n r^n
\]

for \( n > N \). Choosing

\[
n = \sigma \rho r \rho(r)
\]

in this inequality, we have

\[
\mu_f(r) < \left[ \frac{e^{1/\rho}}{\phi(r \rho(r))} \right]^\sigma \rho r \rho(r) r^\sigma \rho r \rho(r),
\]

and since

\[
r = \phi(t) \iff t = r^\rho(r) \implies \phi(r^\rho(r)) = \phi(t) = r
\]

our inequality becomes

\[
\mu_f(r) < \left[ \frac{e^{1/\rho}}{r} \right]^\sigma \rho r \rho(r) r^\sigma \rho r \rho(r) = e^{1/\rho} r^\sigma \rho r \rho(r).
\]

We shall use this fact shortly.

Since \( \rho(r) \) is a proximate order of \( f(z) \), \( \sigma_f \) is finite and
for all $\epsilon > 0$ and $r > R$, $R$ sufficiently large, we have seen that

$$M_f(r) < e^{(\sigma_f + \epsilon)r} \rho(r) r^k < e^{Ar^k},$$

where $\rho(r) = \rho < k$, and $A > \sigma_f$. Therefore

$$|c_n| < M_f(r) r^{-n} < e^{Ar^k} r^{-n}.$$

Now let us see where the function on the right takes its smallest value.

$$D(e^{Ar^k} r^{-n}) = e^{Ar^k} kAr^{-1} r^{-n} + e^{Ar^k} (-nr^{-1})$$

$$= e^{Ar^k} \left[ kAr^{-1} r^{-n} - \frac{n}{r^{n+1}} \right] = 0 \Rightarrow r = \left( \frac{n}{kA} \right)^{1/k}.$$

Checking the values of the derivative for

$$r > \left( \frac{n}{kA} \right)^{1/k}, \quad r < \left( \frac{n}{kA} \right)^{1/k}$$

we see that this is indeed the value of $r$ for which our function assumes its minimum value. Our asymptotic inequality now becomes

$$|c_n| < e^{A \left( \frac{n}{kA} \right)^{1/k}} \left[ \left( \frac{n}{kA} \right)^{1/k} \right]^{-n} = \left( \frac{eAk}{n} \right)^{n/k}.$$

We may assume that this inequality holds for all $n > N$, where $N$ is some function of $k$ and $A$.

Let us now obtain an upper bound for $M_f(r)$. We begin by choosing

$$N = 2^k eAk r^k.$$
for $r$ sufficiently large, so that

$$|c_n z^n| < \left[ \frac{e^{Ak}}{k! e^{Ak} r^k} \right]^{n/k} r^n = 2^{-n}$$

and

$$M_f(r) < \sum_{n=0}^{N} |c_n| r^n + \sum_{n=N+1}^{\infty} 2^{-n}.$$ 

This last series is geometric and has the value

$$\sum_{n=N+1}^{\infty} 2^{-n} = 2^{-N} - 1 \sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^{n-1} = 2^{-N-1} (1 - 1/2) = 2^{-N},$$

and thus

$$M_f(r) < \sum_{n=0}^{N} |c_n| r^n + 2^{-N}$$

$$\leq (N+1) \mu_f(r) + 2^{-N}$$

$$= (1 + 2^k e^{Ak} r^k) \mu_f(r) + 2^{-N}$$

$$< (1 + 2^k e^{Ak} r^k) \mu_f(r) + e$$

for $r$ sufficiently large. Now, using our upper bound on $\mu_f(r)$, we have

$$M_f(r) < (1 + 2^k e^{Ak} r^k) \sigma_1 r^\rho(r) + 2^{-N}$$

$$= (2 + 2^k e^{Ak} r^k) \sigma_1 r^\rho(r)$$
and
\[ \log M_f(r) < \log (2 + 2^k eAkr^k) + \sigma_1 r \rho(r) . \]

Finally, we divide by \( r^\rho(r) \) and take \( \limsup_s \), obtaining
\[ \sigma_f = \lim_{r \to \infty} \frac{\log M_f(r)}{r \rho(r)} < \lim_{r \to \infty} \frac{2 + 2^k eAkr^k}{r \rho(r)} + \sigma_1 \leq \sigma_1 \]

which is our contradiction.

In Chapter 1 we also saw that the order of \( f(z) \) and the distribution of the zeros of \( f(z) \) were related by the inequality
\[ \lim_{r \to \infty} \frac{\log n(r)}{\log r} \leq \rho, \]

where \( n(r) \) was the number of zeros of \( f(z) \) on the disc \( |z| < r \).

We shall conclude Chapter 2 with a similar theorem, which relates the distribution of the zeros of \( f(z) \) to the type of \( f(z) \) with respect to a given proximate order. In the proof of this theorem, we shall call upon the following results (7, p. 15):

**Theorem 2.7.** If \( f(z) \) is holomorphic in the circle \( |z| < e r \) and if \( |f(0)| = 1 \), then
\[ n(r) \leq \log M_f(er). \]

**Theorem 2.8.** Let \( f(z) \) be an entire function of nonintegral order \( \rho \) and let \( \rho(r) \) be a proximate order. Let
\[ \Delta_f = \lim_{r \to \infty} \frac{n(r)}{\rho(r)} , \]

where \( n(r) \) is the number of zeros of \( f(z) \) with \( |z| < r \). If:

- \( \Delta_f = 0 \), then \( f(z) \) is of minimal type with respect to \( \rho(r) \);
- \( 0 < \Delta_f < \infty \), then \( f(z) \) is of normal type with respect to \( \rho(r) \);
- \( \Delta_f = \infty \), then \( f(z) \) is of maximal type with respect to \( \rho(r) \).

Proof. Assume \( \rho(r) \) is a proximate order of \( f(z) \). Then \( \sigma_f \neq 0, \infty \), and for all \( \epsilon > 0 \) and \( r > r_0 \), \( r_0 \) sufficiently large, we have

\[ \log M_f(r) < (\sigma_f + \frac{\epsilon}{2})r^{\rho(r)} \rightarrow n(r) < (\sigma_f + \frac{\epsilon}{2})(\epsilon r)^{\rho(\epsilon r)} , \]

by Theorem 2.7. Property 2 of proximate orders tells us that

\[ \frac{\epsilon r^{\rho(\epsilon r)}}{r^{\rho(r)}} \rightarrow e^\rho \]

uniformly as \( r \to \infty \), and so for all \( \eta > 0 \), \( r > r_1 \), \( r_1 \) sufficiently large, we have

\[ \frac{\epsilon r^{\rho(\epsilon r)}}{r^{\rho(r)}} < (1+\eta)e^\rho \iff (\epsilon r)^{\rho(\epsilon r)} < (1+\eta)e^\rho r^{\rho(r)} . \]

Therefore

\[ \Delta_f = \lim_{r \to \infty} \frac{n(r)}{\rho(r)} \leq \lim_{r \to \infty} \frac{(\sigma_f + \frac{\epsilon}{2})(1+\eta)e^\rho r^{\rho(r)}}{r^{\rho(r)}} = e^\rho \sigma_f . \]
since $\epsilon$ and $\eta$ are arbitrary.

A similar argument holds for $\sigma_f = 0$ and $\sigma_f = \infty$, where $\rho(r)$ is any proximate order, and the same inequality is derived.

Now if $\rho$ is not an integer, we can obtain a converse estimate. By Hadamard's Theorem, if $f(z)$ is entire and of finite order, $f(z)$ can be written in the form

$$f(z) = z^m e^{P(z)} \prod_{n=1}^w G\left(\frac{z}{a_n}; \rho\right),$$

where $w \leq \infty$, $a_n$ are the nonzero roots of $f(z)$, $p \leq \rho$, $P(z)$ is a polynomial of degree $q \leq \rho$, $m$ is the multiplicity of the zero at the origin, and $G(u;p)$ is defined by

$$G(u;p) = (1-u)\exp\left[\sum_{j=1}^{p} \frac{u^j}{j}\right].$$

(4, p. 24).

Also, if the series

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}}$$

converges, then the infinite product

$$T(z) = \prod_{n=1}^{\infty} G\left(\frac{z}{a_n}; \rho\right)$$

satisfies the following inequality for all finite $z$:

$$\log|T(z)| < k \int_0^r t^p \left[ \int_0^t \frac{n(t)}{t^{p+1}} dt + r \int_t^{\infty} \frac{n(t)}{t^{p+2}} dt \right] dt.$$
where \( r = |z| \), \( k_0 = 1 \), and \( k_p = 3e(p+1)(2+\log p) \) for \( p > 0 \) (4, p. 11-12).

Let us apply these two theorems to our problem, and obtain:

\[
f(z) = e^{P(z)} \prod_{n=1}^{\infty} G\left(\frac{z}{a_n}; p\right);
\]

\[
\log |f(z)| = P(z) + \sum_{p} \frac{r^n(t)}{n!} \left(1 + \sum_{r=0}^{\infty} \frac{n(t)}{r!} \int_{0}^{r} \frac{dt}{t^{p+1}} + r \int_{r}^{\infty} \frac{dt}{t^{p+2}} \right);
\]

\[
\log M_f(r) \leq O(r^p) + \sum_{p} \frac{r^n(t)}{n!} \left(1 + \sum_{r=0}^{\infty} \frac{n(t)}{r!} \int_{0}^{r} \frac{dt}{t^{p+1}} + r \int_{r}^{\infty} \frac{dt}{t^{p+2}} \right),
\]

where \( p < \rho \). Now, since for all \( \epsilon > 0 \) and \( r > r_2 \), \( r_2 \) sufficiently large,

\[
\Delta_f = \lim_{r \to \infty} \frac{n(r)}{r^{\rho(r)}} \iff n(r) < (\Delta_f + \epsilon) r^{\rho(r)},
\]

we have

\[
\log M_f(r) \leq O(r^p) + (\Delta_f + \epsilon) \sum_{p} \frac{r^n(t)}{n!} \left(1 + \sum_{r=0}^{\infty} \frac{n(t)}{r!} \int_{0}^{r} \frac{dt}{t^{p+1}} + r \int_{r}^{\infty} \frac{dt}{t^{p+2}} \right).
\]

Properties 4 and 5 of proximate orders give us the values of these integrals:

\[
\int_{0}^{r} t^{\rho(t)-(p+1)} dt = \frac{1}{p-\rho} r^{\rho(r)-p} + o(r^{\rho(r)-p}) ;
\]

\[
r \int_{r}^{\infty} t^{\rho(t)-(p+2)} dt = r \left[ \frac{1}{p+1-\rho} r^{\rho(r)-p-1} + o(r^{\rho(r)-p}) \right] = \frac{1}{p+1-\rho} r^{\rho(r)-p} + o(r^{\rho(r)-p}).
\]
Substituting these values in our inequality yields

\[
\log M_f(r) \leq O(r^\rho) + (\Delta_f + \epsilon)k r^\rho \left[ \frac{1}{\rho-p} r^\rho(r)-p + o(r^\rho(r)-p) + \frac{1}{p+1-\rho} r^\rho(r)-p \right] + \right.
\]

\[
+ r o(r^\rho(r)-p-1) \left. \right] = O(r^\rho) + (\Delta_f + \epsilon)k r^\rho \left[ \frac{1}{\rho-p} + o(r^\rho(r)-p) + \frac{1}{p+1-\rho} + o(r^\rho(r)-p-1) \right].
\]

Asymptotically \(O(r^\rho) \leq Ar^\rho\) and therefore we have the asymptotic inequality

\[
\log M_f(r) \leq Ar^\rho + (\Delta_f + \epsilon)k r^\rho \left[ \frac{1}{\rho-p} + \frac{1}{p+1-\rho} \right],
\]

and

\[
\sigma_f = \lim_{r \to \infty} \frac{\log M_f(r)}{r^\rho} \leq \lim_{r \to \infty} \frac{Ar^\rho}{r^\rho} + \lim_{r \to \infty} \left[ (\Delta_f + \epsilon)k \left( \frac{1}{\rho-p} + \frac{1}{p+1-\rho} \right) \right]
\]

\[
= (\Delta_f + \epsilon)k \left( \frac{1}{\rho-p} + \frac{1}{p+1-\rho} \right) + K
\]

\[
\leq \frac{c}{p} \Delta_f
\]

where \(c\) is a constant independent of \(f\). We now have

\[
\Delta_f \leq e^\sigma_f \leq c \Delta_f,
\]

from which the conclusion of our theorem easily follows.
CHAPTER 3

THE GENERALIZED INDICATOR

In Chapter 2 we generalized the concept of order to that of proximate orders. Both of these give us a measure of the growth of $f(z)$ in the complex plane. In Chapter 1 we discussed the indicator function

$$h(\theta) = \lim_{r \to \infty} \frac{\log |f(re^{i\theta})|}{r^\rho}, \quad \theta_1 \leq \theta \leq \theta_2,$$

which measures the growth of $f(z)$ in the angle $[\theta_1, \theta_2]$. We shall now generalize the indicator function. We do this by replacing the order $\rho$ by the proximate order $\rho(r)$.

**Definition 3.1.** If $f(z)$ is holomorphic in the angle $\theta_1 \leq \arg z \leq \theta_2$, and $\rho(r)$ is a proximate order of $f(z)$, then the generalized indicator of $f(z)$ is defined by

$$h(\theta) = \lim_{r \to \infty} \frac{\log |f(re^{i\theta})|}{r^{\rho(r)}}, \quad \theta_1 \leq \theta \leq \theta_2.$$

Our first theorem shows that, like the indicator, the generalized indicator satisfies the fundamental relation (Corollary 3 of Theorem 1.9). We shall find it useful in deriving many properties of the generalized indicator.
Theorem 3.1. The Fundamental Relation of the Generalized Indicator:

The generalized indicator \( h(\theta) \) of a function \( f(z) \) which is holomorphic and of proximate order \( \rho(r) \), \( \rho(r) \to \rho > 0 \), inside the angle \( \theta_1 \leq \arg z \leq \theta_3 \), \( \theta_3 - \theta_1 < \frac{\pi}{\rho} \), satisfies the relation

\[
h(\theta_1) \sin \rho(\theta_2 - \theta_3) + h(\theta_2) \sin \rho(\theta_3 - \theta_1) + h(\theta_3) \sin \rho(\theta_1 - \theta_2) \leq 0
\]

for \( \theta_1 < \theta_2 < \theta_3 \).

Proof. Let \( \theta_1 < \theta < \theta_3 \), \( \theta_3 - \theta_1 < \frac{\pi}{\rho} \), and let

\[
H_\epsilon(\theta) = \frac{[h(\theta_1) + \epsilon] \sin \rho(\theta_2 - \theta) + [h(\theta_3) + \epsilon] \sin \rho(\theta - \theta_1)}{\sin \rho(\theta_3 - \theta_1)},
\]

for \( \epsilon > 0 \).

We call upon the following results from Levin (4, p. 69-70):

Let \( \rho(r) \to \rho > 0 \) be a given proximate order and let the opening of the angle \( \theta_1 \leq \arg z \leq \theta_2 \) be less than the smaller of the two numbers \( \frac{\pi}{\rho} \) and \( 2\pi \); then for arbitrary real numbers \( a \) and \( b \), a function \( w(z) \) can be constructed that is holomorphic and has no zeros in this angle, and such that for \( \theta_1 \leq \theta \leq \theta_2 \), we have uniformly

\[
\lim_{r \to \infty} \frac{\log |w(re^{i\theta})|}{r^{1/\rho(r)}} = a \cos \rho \theta + b \sin \rho \theta.
\]

Applying these results with
\[ a = \frac{[h(\theta_1) + \epsilon] \sin \rho \theta_3 - [h(\theta_3) + \epsilon] \sin \rho \theta_1}{\sin \rho (\theta_3 - \theta_1)} \]
\[ b = \frac{[h(\theta_3) + \epsilon] \cos \rho \theta_1 - [h(\theta_1) + \epsilon] \cos \rho \theta_3}{\sin \rho (\theta_3 - \theta_1)} \]

where \( \epsilon > 0 \), we obtain

\[
\lim_{r \to \infty} \frac{\log |w(re^{i\theta})|}{r^\rho(r)} = \frac{[h(\theta_1) + \epsilon] \sin \rho \theta_3 - [h(\theta_3) + \epsilon] \sin \rho \theta_1}{\sin \rho (\theta_3 - \theta_1)} \cos \rho \theta
\]
\[
+ \frac{[h(\theta_3) + \epsilon] \cos \rho \theta_1 - [h(\theta_1) + \epsilon] \cos \rho \theta_3}{\sin \rho (\theta_3 - \theta_1)} \sin \rho \theta
\]
\[
= \frac{[h(\theta_1) + \epsilon] \sin \rho (\theta_3 - \theta) + [h(\theta_3) + \epsilon] \sin \rho (\theta - \theta_1)}{\sin \rho (\theta_3 - \theta_1)}
\]
\[
= H_\epsilon (\theta).
\]

For all \( \epsilon_1 > 0 \) and \( r > r_0, \ r_0 \) sufficiently large, we have

\[
|w(re^{i\theta})| \geq e^{r^\rho(r)[H_\epsilon (\theta) - \epsilon_1]}, \ |f(re^{i\theta})| \leq e^{r^\rho(r)[h(\theta) + \epsilon_1]}
\]

and therefore

\[
\left| \frac{f(re^{i\theta})}{w(re^{i\theta})} \right| \leq e^{r^\rho(r)[h(\theta) - H_\epsilon (\theta) + 2\epsilon_1]}
\]

Since \( \epsilon > 0 \) was arbitrary, we choose \( \epsilon > 2\epsilon_1 \) so that

\( \epsilon_2 = \epsilon - 2\epsilon_1 > 0 \), and for \( j = 1, 3 \), we have
\[
\left| \frac{f(re^{i\theta})}{w(re^{i\theta})} \right| < e^{r \rho(r) \left[ h(\theta) - (h(\theta) + \epsilon) + 2 \epsilon \right]} = e^{-\epsilon r \rho(r)} \to 0
\]

as \( r \to \infty \). Thus

\[
f(z)w^{-1}(z) \to 0
\]

along the rays \( \theta = \theta_1, \theta_3 \) as \( r \to \infty \), and, by Theorem 1.10,

\( f(z)w^{-1}(z) \) is bounded in \( \theta_1 \leq \arg z \leq \theta_2 \). Thus, for some \( M > 0 \),

\[
|f(z)| \leq M |w(z)|
\]

and

\[
h(\theta) = \lim_{r \to \infty} \frac{\log|f(re^{i\theta})|}{r \rho(r)} \leq \lim_{r \to \infty} \frac{\log|w(re^{i\theta})| + \log M}{r \rho(r)} = H_\epsilon(\theta)
\]

for \( \theta_1 \leq \theta \leq \theta_3 \) and \( \epsilon > 0 \). Letting \( \epsilon \to 0 \) we have

\[
h(\theta) \leq \frac{h(\theta_1) \sin \rho(\theta_3 - \theta) + h(\theta_3) \sin \rho(\theta - \theta_1)}{\sin \rho(\theta_3 - \theta_1)},
\]

and finally we let \( \theta = \theta_2 \), \( \theta_1 < \theta_2 < \theta_3 \), to obtain the fundamental relation

\[
h(\theta_1) \sin \rho(\theta_2 - \theta_3) + h(\theta_2) \sin \rho(\theta_3 - \theta_1) + h(\theta_3) \sin \rho(\theta_1 - \theta_2) \leq 0
\]

for \( \theta_1 < \theta_2 < \theta_3 \).

Note that the inequality of Theorem 3.1 may also be written
We now exhibit a few properties of the indicator which follow directly from the fundamental relation.

**Property 1.** \( h(\theta) \) is continuous.

**Proof.** Choose an interval \( (\theta_1, \theta_3) \) such that \( \theta_3 - \theta_1 < \frac{\pi}{\rho} \).
Let \( \theta_2 \) be any point in the interval. Define the two functions \( h_1(\theta) \) and \( h_2(\theta) \) by

\[
\begin{align*}
    h_1(\theta) &= \frac{h(\theta_1) \sin \rho(\theta - \theta_1) + h(\theta_2) \sin \rho(\theta_2 - \theta_1)}{\sin \rho(\theta_2 - \theta_1)}, \\
    h_2(\theta) &= \frac{h(\theta_2) \sin \rho(\theta - \theta_2) + h(\theta_3) \sin \rho(\theta_3 - \theta_2)}{\sin \rho(\theta_3 - \theta_2)}.
\end{align*}
\]

For \( \theta_1 \leq \theta \leq \theta_2 \), the fundamental relation gives us

\[
h(\theta) \leq \frac{h(\theta_1) \sin \rho(\theta - \theta_1) + h(\theta_2) \sin \rho(\theta_2 - \theta_1)}{\sin \rho(\theta_2 - \theta_1)} = h_2(\theta).
\]

For \( \theta_2 \leq \theta < \theta_3 \), the fundamental relation yields

\[
h(\theta_2) \leq \frac{h(\theta) \sin \rho(\theta_2 - \theta) + h(\theta_3) \sin \rho(\theta_3 - \theta)}{\sin \rho(\theta_3 - \theta)}
\]
which is equivalent to
\[ h(\theta) \geq \frac{h(\theta_2)\sin \rho(\theta_2 - \theta_1) + h(\theta_3)\sin \rho(\theta_2 - \theta_3)}{\sin \rho(\theta_2 - \theta_1)} = h_1(\theta). \]

Similarly, consideration of \( \theta_2 \leq \theta \leq \theta_3 \) and \( \theta_1 < \theta_2 \leq \theta \) gives
\[ h_1(\theta) \leq h(\theta) \leq h_2(\theta). \]

We note that \( h_1(\theta_2) = h_2(\theta_2) = h(\theta_2) \), and therefore
\[ h_2(\theta) - h_2(\theta_2) \leq h(\theta) - h(\theta_2) \leq h_1(\theta) - h_1(\theta_2) \quad \text{for} \quad \theta_1 \leq \theta \leq \theta_2, \]
\[ h_1(\theta) - h_1(\theta_2) \leq h(\theta) - h(\theta_2) \leq h_2(\theta) - h_2(\theta_2) \quad \text{for} \quad \theta_2 \leq \theta \leq \theta_3. \]

Since \( h_1(\theta) \) and \( h_2(\theta) \) are continuous, we take limits as \( \theta \to \theta_2 \) and obtain
\[ 0 \leq \lim_{\theta \to \theta_2} [h(\theta) - h(\theta_2)] \leq 0 \]
for \( \theta_1 \leq \theta \leq \theta_2 \) and also for \( \theta_2 \leq \theta \leq \theta_3 \). Consequently, \( h(\theta) \)
is continuous in \( (\theta_1, \theta_3) \), and as this interval was chosen arbitrarily (such that \( \theta_3 - \theta_1 < \frac{\pi}{\rho} \)), \( h(\theta) \) is continuous for all values of \( \theta \).

**Property 2.** If \( h(\theta) = -\infty \) for even one value of \( \theta \), then \( h(\theta) \equiv -\infty \).

**Proof.** The fundamental relation does not exclude the possibility that \( h(\theta_0) = -\infty \) for some \( \theta_0 \) in \( (\theta_1, \theta_2) \). Suppose this to be
case and assume \( h(\theta_2) \) is finite. Take a value \( \theta_3 \) outside \([\theta_1, \theta_2]\), e.g. \( \theta_0 < \theta_3 < \theta_0 + \frac{\pi}{\rho} \). Let \( M \) be an arbitrarily large positive number, and choose \( a \) such that \( h(\theta_3) < a \). The fundamental relation now yields

\[
 h(\theta) \sin(\theta - \theta_0) \leq h(\theta) \sin(\theta_3 - \theta_0) + h(\theta_3) \sin(\theta - \theta_0) \\
\leq -M \sin(\theta_3 - \theta_0) + a \sin(\theta - \theta_0)
\]

for \( \theta_0 < \theta < \theta_3 \). Since \( M \) is arbitrary, we have \( h(\theta) = -\infty \) for \( \theta_0 < \theta < \theta_3 \), and in particular, \( h(\theta_2) = -\infty \), which contradicts our assumption.

A similar argument can be given if \( \theta_3 \) is chosen such that \( \theta_0 - \frac{\pi}{\rho} < \theta < \theta_0 \), and either of these cases can be analogously extended so that \( h(\theta) = -\infty \) for all \( \theta \).

Before continuing with the development of properties of the generalized indicator, we introduce the following two lemmas.

**Lemma 1.** For \( \theta_1 < \theta < \theta_3 \), \( \theta_3 - \theta_1 < \frac{\pi}{\rho} \), or for \( \theta < \theta_1 < \theta_3 \), \( \theta_3 - \theta < \frac{\pi}{\rho} \) we have the inequality

\[
\frac{h(\theta) - h(\theta_1)}{\sin(\theta - \theta_1)} \leq \frac{h(\theta_3) - h(\theta_1)}{\sin(\theta_3 - \theta_1)} + h(\theta_1) \frac{\theta_3 - \theta_1}{2} \sec(\frac{\theta_3 - \theta_1}{2}) \sec(-\theta_1).
\]

**Proof.** Let \( \theta_1 < \theta < \theta_3 \), \( \theta_3 - \theta_1 < \frac{\pi}{\rho} \), then the fundamental relation yields
\[ h(\theta) \sin \rho(\theta_3 - \theta_1) + h(\theta_3) \sin \rho(\theta_1 - \theta) \leq h(\theta_1) \sin \rho(\theta_3 - \theta) \]

which is equivalent to

\[ h(\theta) \sin \rho(\theta_3 - \theta_1) + h(\theta_3) \sin \rho(\theta_1 - \theta) + h(\theta_1)[ \sin \rho(\theta_1 - \theta_3) - \sin \rho(\theta_1 - \theta) ] \]

\[ \leq h(\theta_1)[ \sin \rho(\theta_3 - \theta) + \sin \rho(\theta_1 - \theta_3) - \sin \rho(\theta_1 - \theta) ] \]

\[ = h(\theta_1)[2 \sin \rho(\frac{\theta - \theta_1}{2}) \cos \rho(\frac{\theta_3 - \theta_1 - \theta}{2}) - 2 \sin \rho(\frac{\theta_1 - \theta}{2}) \cos \rho(\frac{\theta_1 - \theta}{2})] \]

\[ = -2h(\theta_1) \sin \rho(\frac{\theta - \theta_1}{2})[\cos \rho(\frac{\theta_3 - \theta_1 - \theta}{2}) - \cos \rho(\frac{\theta_1 - \theta}{2})] \]

\[ = -2h(\theta_1) \sin \rho(\frac{\theta - \theta_1}{2})[-2 \sin \frac{1}{2} \rho(\frac{\theta_3 - \theta_1}{2}) \sin \frac{1}{2} \rho(\frac{\theta_3 - \theta}{2})] \]

\[ = 4h(\theta_1) \sin \rho(\frac{\theta - \theta_1}{2}) \sin \rho(\frac{\theta_3 - \theta_1}{2}) \sin \rho(\frac{\theta_3 - \theta}{2}) \].

We now divide throughout by the positive quantity \( \sin \rho(\theta - \theta_1) \sin \rho(\theta_3 - \theta_1) \), and obtain

\[ \frac{h(\theta)}{\sin \rho(\theta - \theta_1)} - \frac{h(\theta_3)}{\sin \rho(\theta_3 - \theta_1)} + \frac{h(\theta_1)}{\sin \rho(\theta_1 - \theta)} + \frac{h(\theta_1)}{\sin \rho(\theta_3 - \theta_1)} \]

\[ \leq \frac{4h(\theta_1) \sin \rho(\frac{\theta - \theta_1}{2}) \sin \rho(\frac{\theta_3 - \theta_1}{2}) \sin \rho(\frac{\theta_3 - \theta}{2})}{2 \sin \rho(\frac{\theta - \theta_1}{2}) \cos \rho(\frac{\theta_1 - \theta}{2}) \cdot 2 \sin \rho(\frac{\theta_3 - \theta_1}{2}) \cos \rho(\frac{\theta_3 - \theta_1}{2})} \]

\[ = h(\theta_1) \sin \rho(\frac{\theta_3 - \theta}{2}) \sec \rho(\frac{\theta - \theta_1}{2}) \sec \rho(\frac{\theta_3 - \theta_1}{2}) \]

And finally
h(θ) - h(θ_1) \leq \frac{h(θ_3) - h(θ_1)}{\sin ρ(θ-θ_1)} + h(θ_1)\sin ρ}\left(\frac{θ_3-θ}{2}\right)\sec\left(\frac{θ-θ_1}{2}\right) - 2 \sin ρ\left(\frac{θ_1-2θ_3}{2}\right) - 2 \sin ρ\left(\frac{θ-θ_1}{2}\right)\cos ρ\left(\frac{θ_1}{2}\right)

Under the assumption θ < θ_1 < θ_3, θ_3 - θ < \frac{π}{ρ}, a similar argument yields the same inequality.

Lemma 2. For θ_3 < θ_1 < θ, θ - θ_3 < \frac{π}{ρ}, or for θ_3 < θ < θ_1, θ_1 - θ < \frac{π}{ρ}, we have the inequality

h(θ) - h(θ_1) \geq \frac{h(θ_3) - h(θ_1)}{\sin ρ(θ-θ_1)} + h(θ_1)\sin ρ\left(\frac{θ_3-θ}{2}\right)\sec\left(\frac{θ_1-θ_3}{2}\right)\sec\left(\frac{θ-θ_1}{2}\right).

Proof. Let θ_3 < θ_1 < θ, θ - θ_3 < \frac{π}{ρ}, then from the fundamental relation we have

h(θ)\sin ρ(θ_1 - θ) + h(θ_1)\sin ρ(θ_3 - θ_1) + h(θ)\sin ρ(θ_3 - θ_1) \leq 0

which is equivalent to

h(θ)\sin ρ(θ_1 - θ_1) + h(θ_3)\sin ρ(θ_3 - θ_1) + h(θ_1)\sin ρ(θ_3 - θ_1) - \sin ρ(θ_1 - θ_1)]

≥ h(θ_1) [\sin ρ(θ_3 - θ_1) + \sin ρ(θ_3 - θ_1) - \sin ρ(θ_1 - θ_1)]

= h(θ_1) [2 \sin ρ\left(\frac{θ_3-θ_1}{2}\right)\cos ρ\left(\frac{θ_3-θ_1}{2}\right) - 2 \sin ρ\left(\frac{θ_3-θ_1}{2}\right)\cos ρ\left(\frac{θ_1}{2}\right)

= 2h(θ_1)\sin ρ\left(\frac{θ_3-θ_1}{2}\right)[-2 \sin \left(\frac{θ_3-θ_1}{2}\right)\sin ρ\left(\frac{θ_3-θ_1}{2}\right)]

= -4h(θ_1)\sin ρ\left(\frac{θ_3-θ_1}{2}\right)\sin ρ\left(\frac{θ_3-θ_1}{2}\right)\sin ρ\left(\frac{θ_1}{2}\right).

We now divide throughout by the negative quantity \sin ρ(θ_1 - θ)sin(θ_1 - θ),
to obtain

\[ \frac{h(\theta) - h(\theta_1)}{\sin \rho(\theta_1 - \theta)} - \frac{h(\theta_3) - h(\theta_1)}{\sin \rho(\theta_1 - \theta_3)} < \frac{\theta - \theta_1}{2} \sin \rho(\frac{\theta_1 - \theta_3}{2}) \sin \rho(\frac{\theta_3 - \theta}{2}) \]

\[ \quad \times \frac{\theta - \theta_2}{2} \sin \rho(\frac{\theta - \theta_2}{2}) \cos \rho(\frac{\theta_1 - \theta_3}{2}) \sin \rho(\frac{\theta_1 - \theta_3}{2}) \]

\[ = \frac{h(\theta_1) \sin \rho(\frac{\theta_1 - \theta_3}{2}) \sec \rho(\frac{\theta_3 - \theta_1}{2}) \sec \rho(\frac{\theta_1 - \theta}{2})}{\sin \rho(\theta_1 - \theta_3)} \cdot \frac{\theta_3 - \theta_1}{\sin \rho(\theta_1 - \theta_3)} \cdot \frac{\theta - \theta_2}{2} \sin \rho(\frac{\theta - \theta_2}{2}) \cos \rho(\frac{\theta_1 - \theta_3}{2}) \sin \rho(\frac{\theta_1 - \theta_3}{2}) \]

and finally

\[ \frac{h(\theta) - h(\theta_1)}{\sin \rho(\theta_1 - \theta_1)} > \frac{h(\theta_3) - h(\theta_1)}{\sin \rho(\theta_3 - \theta_1)} + \frac{\theta_3 - \theta}{2} \sec \rho(\frac{\theta_3 - \theta_1}{2}) \sec \rho(\frac{\theta_1 - \theta}{2}) \]

Under the assumption \( \theta_3 < \theta < \theta_1, \quad \theta_1 - \theta_3 < \frac{\pi}{\rho} \), a similar argument yields the same inequality.

Now let us return to the properties of the generalized indicator.

**Property 3.** \( h(\theta) \) has a derivative from the left, \( h'_-(\theta) \), and a derivative from the right, \( h'_+(\theta) \), at every point.

**Proof.** First we observe that, for \( \theta_1 < \theta < \theta_3 \),

\[ \frac{\theta_3 - \theta}{2} \sec \rho(\frac{\theta_3 - \theta_1}{2}) \sec \rho(\frac{\theta_1 - \theta}{2}) = \frac{\sin \rho((\theta_3 - \theta_1) - (\theta - \theta_1))}{\cos \rho((\theta_3 - \theta_1) - (\theta - \theta_1)) \cos \rho(\theta - \theta_1)} \]

\[ = \tan \frac{1}{2} \rho(\theta_3 - \theta_1) - \tan \frac{1}{2} \rho(\theta_1 - \theta) \]

\[ = (\sec^2 \rho \phi) \frac{1}{2} \rho((\theta_3 - \theta_1) - (\theta - \theta_1)) \]

\[ = (\sec^2 \rho \phi) \frac{1}{2} \rho(\theta_3 - \theta_1) \]
where $\frac{1}{2} (\theta - \theta_1) < \phi < \frac{1}{2} (\theta_3 - \theta_1)$. Let $k$ be defined by

$$k = \max \left| \frac{1}{2} p h(\theta) \sec^2 \rho q \right|,$$

for $|\theta_3 - \theta_1| \leq q < \frac{\pi}{\rho}$, and $\theta_1 < \theta < \theta_3$, so that

$$h(\theta_1) \frac{1}{2} p \sec^2 \rho \phi \leq h(\theta_1) \frac{1}{2} p \sec^2 \rho q \leq k,$$

and Lemma 1 yields, for $\theta_1 < \theta < \theta_3$,

$$\frac{h(\theta) - h(\theta_1)}{\sin \rho (\theta - \theta_1)} \leq \frac{h(\theta_3) - h(\theta_1)}{\sin \rho (\theta_3 - \theta_1)} + k (\theta_3 - \theta).$$

Let the function $r(\theta, \theta_1)$ be defined by

$$r(\theta, \theta_1) = \frac{h(\theta) - h(\theta_1)}{\sin \rho (\theta - \theta_1)} + k (\theta - \theta_1),$$

and we note that for $\theta < \theta_3$

$$r(\theta, \theta_1) = \frac{h(\theta) - h(\theta_1)}{\sin \rho (\theta - \theta_1)} + k (\theta - \theta_1) \leq \frac{h(\theta_3) - h(\theta_1)}{\sin \rho (\theta_3 - \theta_1)} + k (\theta_3 - \theta_1) = r(\theta_3, \theta_1),$$

i.e. $r(\theta, \theta_1)$ is monotonically decreasing. Since, by Lemma 1 and Lemma 2,

$$\frac{h(\theta) - h(\theta_1)}{\sin \rho (\theta - \theta_1)}$$

is bounded below for $\theta_1 < \theta < \theta_3$ and bounded above for $\theta_3 < \theta_1 < \theta$, $r(\theta, \theta_1)$ is also bounded, and $\lim_{\theta \to \theta_1^+} r(\theta, \theta_1)$ exists.

We note that
\[ \rho r(\theta, \theta_1) = \frac{h(\theta) - h(\theta_1)}{\theta - \theta_1} \cdot \frac{\rho(\theta - \theta_1)}{\sin(\theta - \theta_1)} + k\rho(\theta - \theta_1), \]

and, therefore, \( h_+'(\theta_1) \) exists, because

\[ h_+'(\theta_1) = \lim_{\theta \to \theta_1^+} \frac{h(\theta) - h(\theta_1)}{\theta - \theta_1} = \lim_{\theta \to \theta_1^+} \frac{\rho r(\theta, \theta_1) - kp(\theta - \theta_1)}{\rho(\theta - \theta_1)} = \rho \lim_{\theta \to \theta_1^+} r(\theta, \theta_1). \]

Similarly, for \( \theta_3 < \theta < \theta_1 \), Lemma 2 yields

\[ \frac{h(\theta) - h(\theta_1)}{\sin(\theta - \theta_1)} \geq \frac{h(\theta_3) - h(\theta_1)}{\sin(\theta_3 - \theta_1)} + k(\theta_3 - \theta), \]

and in this interval,

\[ r(\theta, \theta_1) \geq \frac{h(\theta_3) - h(\theta_1)}{\sin(\theta_3 - \theta_1)} + k(\theta_3 - \theta) = r(\theta_3, \theta_1), \]

i.e. \( r(\theta, \theta_1) \) is monotonically increasing. Since

\[ \frac{h(\theta) - h(\theta_1)}{\sin(\theta - \theta_1)} \]

is bounded below for \( \theta_3 < \theta < \theta_1 \), and bounded above for \( \theta < \theta_1 < \theta_3 \),

\( r(\theta, \theta_1) \) is also bounded, and \( \lim_{\theta \to \theta_1^-} r(\theta, \theta_1) \) exists. Therefore

\[ h_-'(\theta_1) \] exists, because

\[ h_-'(\theta_1) = \lim_{\theta \to \theta_1^-} \frac{h(\theta) - h(\theta_1)}{\theta - \theta_1} = \lim_{\theta \to \theta_1^-} \frac{\rho r(\theta, \theta_1) - kp(\theta - \theta_1)}{\rho(\theta - \theta_1)} = \rho \lim_{\theta \to \theta_1^-} r(\theta, \theta_1). \]
Property 4. At each point $\theta = \theta_1$, $h_+'(\theta_1) \geq h'_-(\theta_1)$.

Proof. From Lemma 1 with $\theta < \theta_1 < \theta_3$, we have

$$\frac{h(\theta) - h(\theta_1)}{\sin \rho(\theta - \theta_1)} \leq \frac{h(\theta_3) - h(\theta_1)}{\sin \rho(\theta_3 - \theta_1)} + \frac{\theta_3 - \theta}{\frac{\theta_3 - \theta_1}{2}} \frac{\theta_3 - \theta_1}{2} \sec \frac{\theta - \theta_1}{2}.$$ 

Taking the limit as $\theta \to \theta_1^-$ and $\theta_3 \to \theta_1^+$, we obtain

$$\lim_{\theta \to \theta_1^-} \frac{h(\theta) - h(\theta_1)}{\sin \rho(\theta - \theta_1)} \leq \lim_{\theta_3 \to \theta_1^+} \frac{h(\theta_3) - h(\theta_1)}{\sin \rho(\theta_3 - \theta_1)} ,$$

which is equivalent to

$$\lim_{\theta \to \theta_1^-} \left[ \frac{h(\theta) - h(\theta_1)}{\theta - \theta_1} \cdot \frac{\rho(\theta - \theta_1)}{\sin \rho(\theta - \theta_1)} \right] \leq \lim_{\theta_3 \to \theta_1^+} \left[ \frac{h(\theta_3) - h(\theta_1)}{\theta_3 - \theta_1} \cdot \frac{\rho(\theta_3 - \theta_1)}{\sin \rho(\theta_3 - \theta_1)} \right] ,$$

i.e. $h_-'(\theta_1) \leq h_+'(\theta_1)$.

Property 5. $h_+'(\theta)$ is continuous from the right, and $h_-'(\theta)$ is continuous from the left.

Proof. Choose $\theta_1 < \theta < \theta_3$, $\theta_3 - \theta_1 < \frac{\pi}{\rho}$, from the fundamental relation, we have

$$h(\theta_1) \frac{\sin \rho(\theta - \theta_3) + h(\theta) \sin \rho(\theta_3 - \theta_1) + h(\theta_3) \sin \rho(\theta_1 - \theta)}{\sin \rho(\theta_1 - \theta_1)} \leq 0 ,$$

which is equivalent to
\[ h(0)\sin\theta(\theta_1 - \theta_3) + h(\theta_1)\sin\theta(\theta_3 - \theta) + h(\theta_3)[\sin\theta(\theta_3 - \theta_1) - \sin\theta(\theta_3 - \theta)] \]

\[ \geq h(\theta_3)[\sin\theta(\theta_1 - \theta) + \sin\theta(\theta_3 - \theta_1) - \sin\theta(\theta_3 - \theta)] \]

\[ = h(\theta_3)[2\sin\theta(\frac{\theta_3 - \theta}{2})\cos\theta(\frac{\theta_1 - \theta_3}{2}) - 2\sin\theta(\frac{\theta_3 - \theta_1}{2})\cos\theta(\frac{\theta_3 - \theta}{2})] \]

\[ = 2h(\theta_3)\sin\theta(\frac{\theta_3 - \theta}{2})[-2\sin\theta(\frac{\theta_1 - \theta}{2})\sin\theta(\frac{\theta_1 - \theta_3}{2})] \]

\[ = 4h(\theta_3)\sin\theta(\frac{\theta_3 - \theta}{2})\sin\theta(\frac{\theta_1 - \theta}{2})\sin\theta(\frac{\theta_1 - \theta_3}{2}) \cdot \]

We now divide throughout by the positive quantity \( \sin\theta(\theta_1 - \theta_3)\sin\theta(\theta_3 - \theta_1) \), and obtain

\[ \frac{h(\theta) - h(\theta_3)}{\sin\theta(\theta_3 - \theta)} \geq \frac{h(\theta_1) - h(\theta_3)}{\sin\theta(\theta_1 - \theta_3)} - h(\theta_3)\sin\theta(\frac{\theta_3 - \theta_1}{2})\sec\theta(\frac{\theta_3 - \theta_1}{2}) \sec\theta(\frac{\theta_1 - \theta_3}{2}) \]

\[ = \frac{h(\theta_1) - h(\theta_3)}{\sin\theta(\theta_1 - \theta_3)} - h(\theta_3)(\sec^2_2\rho\phi)\frac{1}{2}\rho[(\theta - \theta_3) - (\theta_1 - \theta_3)] \]

\[ \geq \frac{h(\theta_1) - h(\theta_3)}{\sin\theta(\theta_1 - \theta_3)} - k(\theta - \theta_1) \cdot \]

where \( \frac{1}{2}(\theta_1 - \theta_3) < \phi < \frac{1}{2}(\theta - \theta_3) \), and \( k = \max|\frac{1}{2}\rho h(\theta)\sec^2\rho\phi| \) for \( |\theta_3 - \theta_1| < \frac{\pi}{\rho}, \theta_1 < \theta < \theta_3 \). Since \( k(\theta - \theta_1) > 0 \),

\[ \rho \frac{h(\theta) - h(\theta_3)}{\sin\theta(\theta_3 - \theta)} \geq \rho \left[ \frac{h(\theta_1) - h(\theta_3)}{\sin\theta(\theta_1 - \theta_3)} - 2k(\theta - \theta_1) \right] \cdot \]

and taking the limit as \( \theta \to \theta_3 \) yields
\[
\begin{align*}
\frac{h(\theta) - h(\theta_1)}{\sin\theta - \sin\theta_1} &\leq \rho \left[ \frac{h(\theta_3) - h(\theta_1)}{\sin\theta - \sin\theta_1} + k(\theta_3 - \theta_1) \right] \\
&\leq \rho \left[ \frac{h(\theta_3) - h(\theta_1)}{\sin\theta - \sin\theta_1} + k(\theta_3 - \theta_1) \right] + 2k(\theta_3 - \theta_1) \\
&= \rho r(\theta_1, \theta_3) + 2k(\theta_3 - \theta_1).
\end{align*}
\]

From the proof of Property 3, we have

\[
\begin{align*}
\frac{h(\theta) - h(\theta_1)}{\sin\theta - \sin\theta_1} &\leq \rho \left[ \frac{h(\theta_3) - h(\theta_1)}{\sin\theta - \sin\theta_1} + k(\theta_3 - \theta_1) \right] \\
&\leq \rho \left[ \frac{h(\theta_3) - h(\theta_1)}{\sin\theta - \sin\theta_1} + k(\theta_3 - \theta_1) \right] + 2k(\theta_3 - \theta_1) \\
&= \rho r(\theta_1, \theta_3) + 2k(\theta_3 - \theta_1).
\end{align*}
\]

Subtracting we obtain

\[
h'_-(\theta_3) - h'_+(\theta_1) \geq -3k(\theta_3 - \theta_1),
\]

which is equivalent to

\[
[h'_-(\theta_3) + 3k(\theta_3)] - [h'_+(\theta_1) + 3k(\theta_1)] \geq 0.
\]

Now since \( h'_+(\theta) \geq h'_-(\theta) \), we have the two inequalities

\[
\begin{align*}
[h'_-(\theta_3) + 3k(\theta_3)] - [h'_+(\theta_1) + 3k(\theta_1)] &\geq 0, \\
[h'_+(\theta_3) + 3k(\theta_3)] - [h'_+(\theta_1) + 3k(\theta_1)] &\geq 0,
\end{align*}
\]

and the functions
\[ h_+'(\theta) + 3k\rho \theta, \quad h_-'(\theta) + 3k\rho \theta \]

are monotonically increasing on the interval \([\theta_1, \theta_1 + q]\), \(q < \frac{\pi}{\rho}\).

Thus because

\[ h_+'(\theta_1) + 3k\rho \theta_1 \leq \rho r(\theta_1, \theta_3) + \rho k(2\theta_3 + \theta_1) \leq h_-'(\theta_3) + 3k\rho \theta_3, \]

the limits

\[
\lim_{\theta_3 \to \theta^+} h_+'(\theta_3), \quad \lim_{\theta_1 \to \theta^-} h_-'(\theta_1)
\]

exist. Recalling that \(\theta_1, \theta_3\) were arbitrarily chosen such that \(\theta_3 - \theta_1 < \frac{\pi}{\rho}\), we have shown the limits

\[
\lim_{a \to \theta^+} h_+'(a), \quad \lim_{\theta \to \theta^-} h_-'(\theta)
\]

exist for all values of \(\theta\).

Let us find the value of each of the above limits. For \(s > 0\), we have

\[
\frac{h(\theta + s) - h(\theta)}{s} = \frac{1}{s} \int_{\theta}^{\theta + s} h_+'(a) da,
\]

and letting \(s \to 0^+\) yields

\[
h_+'(\theta) = \lim_{s \to 0^+} \left[ \frac{1}{s} \int_{\theta}^{\theta + s} h_+'(a) da \right] = \lim_{s \to 0^+} \frac{h_+'(\theta + s)}{s} = \lim_{a \to \theta^+} h_+'(a).
\]

Similarly, for \(s < 0\),
\[
\frac{h(\theta+s) - h(\theta)}{s} = \frac{1}{s} \int_{\theta}^{\theta+s} h'(a) da ,
\]

and letting \( s \to 0^- \) yields

\[
h'(\theta) = \lim_{s \to 0^-} \left[ \frac{\int_{\theta}^{\theta+s} h'(a) da}{s} \right] = \lim_{s \to 0^-} h'(\theta+s) = \lim_{a \to \theta^-} h'(a) ,
\]

which completes the proof.

**Property 6.** The generalized indicator \( h(\theta) \) has a derivative at all points except possibly on a countable set.

**Proof.** In the proof of Property 5 we saw that

\[
h'_-(\theta_3) - h'_+(\theta_1) \geq -3k\rho(\theta_3 - \theta_1) ,
\]

which we now rewrite as

\[
h'_+(\theta_1) - h'_-(\theta_1) - 3k\rho(\theta_3 - \theta_1) \leq h'_-(\theta_3) - h'_-(\theta_1) .
\]

If \( \theta = \theta_1 \) is a point of continuity of \( h'_-(\theta) \), then letting \( \theta_3 \to \theta_1^+ \) we obtain

\[
h'_+(\theta_1) \leq h'_-(\theta_1) ,
\]

and \( h'(\theta) = h'_+(\theta) = h'_-(\theta) \) exists for all points \( \theta = \theta_1, \theta_1 \) a point of continuity of \( h'_-(\theta) \).

Also from the proof of Property 5, the function
is a bound monotone function and therefore has at most a countable number of discontinuities, which implies that \( h'_0(\theta) \) has at most a countable number of discontinuities, and consequently \( h'(\theta) = h'_0(\theta) = h_-(\theta) \) at all points except possibly on a countable set.

**Property 7.** If \( \theta_0 \) is a local extreme point for the generalized indicator \( h(\theta) \), then for

\[
|h(\theta) - h(\theta_0)| \leq \frac{\pi}{\rho}
\]

\[
h(\theta) \geq h(\theta_0) \cos \rho(\theta - \theta_0).
\]

**Proof.** If \( \theta = \theta_0 \) is a local maximum point, \( h'_0(\theta_0) = 0 \), while if \( \theta = \theta_0 \) is a local minimum point, \( h'_0(\theta_0) > 0 \geq h'_0(\theta_0) \). Substituting \( \theta_1 = \theta_0 \) in Lemma 1 gives

\[
\frac{h(\theta)-h(\theta_0)}{\sin \rho(\theta-\theta_0)} \leq \frac{h(\theta_3)-h(\theta_0)}{\sin \rho(\theta_3-\theta_0)} + h(\theta_0) \sin \rho(\theta_3-\theta_0) \sec \left( \frac{\theta_3-\theta_0}{2} \right) \sec \left( \frac{\theta-\theta_0}{2} \right),
\]

for \( \theta < \theta < \theta_3 \), and passing to the limit as \( \theta \to \theta_0^+ \) yields

\[
0 \leq \frac{1}{\rho} h'_0(\theta_0) \leq \frac{h(\theta_3)-h(\theta_0)}{\sin \rho(\theta_3-\theta_0)} + h(\theta_0) \sin \rho(\theta_3-\theta_0) \sec \left( \frac{\theta_3-\theta_0}{2} \right) \sec \left( \frac{\theta_3-\theta_0}{2} \right).
\]

Transposing, we have

\[
\frac{h(\theta_3)-h(\theta_1)}{\sin \rho(\theta_3-\theta_1)} \geq -h(\theta_0) \tan \rho(\theta_3-\theta_0) = -h(\theta_0) \left[ \frac{1-\cos \rho(\theta_3-\theta_0)}{\sin \rho(\theta_3-\theta_0)} \right].
\]
which is equivalent to

\[ h(\theta_3) \geq h(\theta_0) \cos \rho (\theta_3 - \theta_0) \]

for \( 0 < \theta_3 - \theta_0 < \frac{\pi}{\rho} \).

An analogous argument using Lemma 2 for \( \theta_3 < \theta < \theta_1 \), yields the same inequality for \( 0 < \theta - \theta_3 < \frac{\pi}{\rho} \).

It remains to show that the inequality holds for \( |\theta - \theta_0| = \frac{\pi}{\rho} \), i.e. for \( \theta = \theta_0 \pm \frac{\pi}{\rho} \). Consider \( \theta_0 < \theta_2 < \theta_3 < \theta_0 + \frac{\pi}{\rho} \). The fundamental relation gives

\[ h(\theta_0) \geq \frac{h(\theta_2) \sin \rho (\theta_3 - \theta_0) + h(\theta_3) \sin \rho (\theta_0 - \theta_2)}{\sin \rho (\theta_3 - \theta_2)} \]

for \( \theta_0 < \theta_2 < \theta_3 \), and also

\[ h(\theta_0 + \frac{\pi}{\rho}) \geq \frac{h(\theta_2) \sin [\theta_3 - (\theta_0 + \frac{\pi}{\rho})] + h(\theta_3) \sin [\theta_0 + \frac{\pi}{\rho} - \theta_2]}{\sin \rho (\theta_3 - \theta_2)} \]

\[ = \frac{-h(\theta_2) \sin (\theta_3 - \theta_0) - h(\theta_3) \sin (\theta_0 - \theta_2)}{\sin \rho (\theta_3 - \theta_2)} \]

for \( \theta_2 < \theta_3 < \theta_0 + \frac{\pi}{\rho} \). Thus

\[ h(\theta_0 + \frac{\pi}{\rho}) + h(\theta_0) \geq 0, \]

and

\[ h(\theta) = h(\theta_0 + \frac{\pi}{\rho}) \geq -h(\theta_0) = h(\theta_0) \cos \rho (\theta_0 + \frac{\pi}{\rho} - \theta_0), \]

for \( \theta = \theta_0 + \frac{\pi}{\rho} \). A similar proof for \( \theta = \theta_0 - \frac{\pi}{\rho} \) is given by
considering the interval \( \theta_0 - \frac{\pi}{\rho} < \theta_2 < \theta_3 < \theta_0 \).

Now let us depart from these properties of the generalized indicator and derive a few general properties of the growth of entire functions of proximate order \( \rho(r) \) along rays. We note that these results will also be true for functions of normal type with respect to the usual order \( \rho \), that is, in the special case \( \rho(r) = \rho \).

**Theorem 3.2.** If \( f(z) \) is holomorphic and has the proximate order \( \rho(r) \), \( \rho > 0 \), inside the angle \( a \leq \arg z \leq \beta \), then to each \( \epsilon > 0 \) there corresponds a number \( r_\epsilon \) such that the inequality

\[
\log |f(re^{i\theta})| < [h(\theta) + \epsilon] r^{\rho(r)}
\]

is valid for \( r > r_\epsilon \) for all values of \( \theta \) in the interval \( [a, \beta] \).

**Proof.** We begin by dividing the interval \( [a, \beta] \) into subintervals with points of subdivision \( a = \theta_0, \theta_1, \theta_2, \ldots, \theta_{n-1}, \theta_n = \beta \), and for each subinterval \( (\theta_j, \theta_{j+1}) \) we construct the trigonometric function

\[
H_j(\theta) = \frac{[h(\theta_j) + \frac{\epsilon}{3}] \sin \rho(\theta_{j+1} - \theta) + [h(\theta_j) + \frac{\epsilon}{3}] \sin \rho(\theta - \theta_j)}{\sin \rho(\theta_{j+1} - \theta_j)}.
\]

Noting that

\[
H_j(\theta_j) = h(\theta_j) + \frac{\epsilon}{3}, \quad H_j(\theta_{j+1}) = h(\theta_{j+1}) + \frac{\epsilon}{3}.
\]

The intervals \( (\theta_j, \theta_{j+1}) \) can be chosen so small that the oscillation
of the functions \( h(\theta) \) and \( H_j(\theta) \) on each interval is less than \( \frac{\epsilon}{3} \).

Thus in each interval

\[
H_j(\theta) < h(\theta) + \frac{2\epsilon}{3}.
\]

In each of these intervals we construct the function \( w_j(z) \) with indicator \( H_j(\theta) \) as was done in the proof of Theorem 3.1. Again we see that the modulus of the function

\[
f(z)w_j^{-1}(z)
\]

is bounded on the sides of the angle \( \theta_j \leq \text{arg} z \leq \theta_{j+1} \), and by Theorem 1.10, it is bounded inside this angle. Therefore, for \( R_j(\epsilon) \) sufficiently large and \( r > R_j(\epsilon) \), we have

\[
\log |f(re^{i\theta})| < \log |w_j(re^{i\theta})| < \left[H_j(\theta) + \frac{\epsilon}{3}\right] r^\rho(r),
\]

for \( \theta_j < \theta < \theta_{j+1} \), and if \( r_\epsilon = \max R_j(\epsilon) \), then for \( r > r_\epsilon \)

\[
\log |f(re^{i\theta})| < \left[h(\theta) + \frac{2\epsilon}{3} + \frac{\epsilon}{3}\right] r^\rho(r) = \left[h(\theta) + \epsilon\right] r^\rho(r)
\]

for \( a \leq \theta \leq \beta \).

We shall put this useful inequality to work in the proof of our next theorem, which shows the relationship between the type \( \sigma_f \) and the generalized indicator \( h_f(\theta) \).

**Theorem 3.3.** The maximum value of the indicator \( h(\theta) \) of the
function \( f(z) \) on the interval \( a \leq \theta \leq \beta \) is equal to the type \( \sigma \) of this function inside the angle \( a \leq \arg z \leq \beta \).

Proof. For all \( \epsilon > 0 \) and \( r > r_0 \), we have, by the preceding theorem

\[
\log |f(re^{i\theta})| < [h(\theta) + \epsilon] r^\rho(r),
\]

which is equivalent to

\[
h(\theta) > \frac{\log |f(re^{i\theta})|}{r^\rho(r)} - \epsilon,
\]

and thus

\[
\max_{a \leq \theta \leq \beta} h(\theta) \geq \lim_{r \to \infty} \frac{\log M_f(r)}{r^\rho(r)} = \sigma.
\]

Using the definition of the generalized indicator, we reverse the inequality in the following manner:

\[
\max_{a \leq \theta \leq \beta} h(\theta) = \max_{a \leq \theta \leq \beta} \lim_{r \to \infty} \frac{\log |f(re^{i\theta})|}{r^\rho(r)} \leq \lim_{r \to \infty} \frac{\log M_f(r)}{r^\rho(r)} = \sigma.
\]

**Theorem 3.4 (Wiman).** If \( \rho \) is the order of the entire function \( f(z) \), and \( \rho < 1 \), then there exists a sequence

\[
r_1 < r_2 < r_3 < \cdots \quad (r_n \to \infty)
\]

such that for arbitrary \( \epsilon > 0 \) and \( n > n_\epsilon \)

\[
m_f(r_n) > [M_f(r_n)]^{\cos \pi \rho - \epsilon},
\]

where
Proof. By Hadamard's Theorem, let

\[ f(z) = cz^m \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right) \]

be an entire function of order \( \rho < 1 \). Then

\[ cr^m \prod_{k=1}^{\infty} \left(1 - \frac{r}{a_k}\right) \leq |f(z)| \leq c r^m \prod_{k=1}^{\infty} \left(1 + \frac{r}{a_k}\right), \]

where \( |z| = r \). Now let

\[ f_1(z) = cz^m \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right), \]

and let \( h_1(\theta) \) be the generalized indicator of \( f_1(z) \). Since

\[ h_1(\theta) = \lim_{r \to \infty} \frac{\log |f_1(re^{i\theta})|}{\rho(r)} = \lim_{r \to \infty} \frac{\log c + m \log z + \sum_{k=1}^{\infty} \log \left|1 - \frac{re^{i\theta}}{a_k}\right|}{r \rho(r)} \]

\( h_1(\theta) \) assumes its maximum when

\[ \left|1 - \frac{re^{i\theta}}{a_k}\right| \]

assumes its maximum. We note that
\[\left| 1 - \frac{re^{i\theta}}{a_k} \right| = \frac{1}{|a_k|} \left| a_k - r \cos \theta - ir \sin \theta \right|\]
\[= \frac{1}{|a_k|} \sqrt{(|a_k| - r \cos \theta)^2 + (r \sin \theta)^2}\]
\[= \frac{1}{|a_k|} \sqrt{|a_k|^2 - 2|a_k|r \cos \theta + r^2}\]

which is maximum when \( \theta = \pi \). Therefore

\[\max h_1(\theta) = h_1(\pi),\]

and Property 7 of the generalized indicator yields

\[h_1(\theta) \geq h_1(\pi) \cos \rho(\theta - \pi)\]

for \( |\theta - \pi| < \frac{\pi}{\rho} \), and in particular, since \( \rho < 1 \),

\[h_1(0) \geq h_1(\pi) \cos \rho \pi.\]

We use this fact in the following sequence of inequalities:

\[\lim_{r \to \infty} \frac{\log m_f(r)}{\log M_f(r)} \geq \lim_{r \to \infty} \frac{\log |f_1(r)|}{\log |f_1(-r)|}\]
\[\geq \lim_{r \to \infty} \frac{\log |f_1(r)|}{h_1(\pi) r \rho(r)}\]
\[\geq \frac{h_1(0)}{h_1(\pi)}\]
\[\geq \cos \pi \rho.\]
Finally, by the definition of $\lim \, \sup$, we have that for all $\epsilon > 0$ and $n > 0$, there exists an $n > n_0$ such that

$$\frac{\log m_f(r_n)}{\log M_f(r_n)} > \cos \pi \rho - \epsilon$$

which is equivalent to

$$m_f(r_n) > \lfloor M_f(r_n) \rfloor$$

As a consequence of this theorem, we see that if $\rho < \frac{1}{2}$ and if $\epsilon$ is chosen such that $\epsilon > 1$, then there is a sequence of circles $|z| = r_n$ on which $m(r_n) \to \infty$, and that the order of growth of $m(r_n)$ is the same as the order of growth of $M(r_n)$, since

$$\lim_{n \to \infty} \frac{\log \log M(r_n)}{\log r_n} = \lim_{n \to \infty} \frac{\log \log M_m(r_n)}{\log r_n} = \lim_{n \to \infty} \frac{\log \log M_f(r_n) - \log \log m_f(r_n)}{\log r_n}$$

$$= 0.$$ 

Also, the type of $m(r_n)$ is greater than or equal to $\sigma_f \cos \pi \rho$, because

$$\sigma_m = \lim_{r_n \to \infty} \frac{\log M_m(r_n)}{r_n^\rho} = \lim_{r_n \to \infty} \frac{\log m_f(r_n)}{r_n^\rho} = \lim_{r_n \to \infty} \frac{(\cos \pi \rho - \epsilon) \log M_f(r_n)}{r_n^\rho} = \left[\cos \rho \pi - \epsilon\right] \sigma_f.$$
for all $\epsilon > 0$.

We conclude Chapter 3 with Bernstein's Theorem. This theorem gives us an estimate of the density of the set of those $r$ on which, for fixed $\theta$ and arbitrary $\epsilon > 0$, the inequality

$$\log|f(re^{i\theta})| > [h(\theta)-\epsilon] r^\rho(r)$$

is valid.

**Theorem 3.5 (Bernstein).** Let the function $f(z)$ be holomorphic and of proximate order $\rho(r)$ in the angle $\alpha < \arg z < \beta$. To each choice of the positive numbers $\delta > 0$, $\epsilon > 0$ and $0 < w < 1$, there corresponds, on each fixed ray $\arg z = \theta$, a sequence of intervals

$$r_n < r < r_n (1+\delta), \quad r_n \to \infty$$

on each of which the inequality

$$\log|f(re^{i\theta})| > [h(\theta)-\epsilon] r^\rho(r)$$

is satisfied except perhaps on a set of measure not exceeding $w\delta r_n$.

**Proof.** Without loss of generality we may assume that $\theta = 0$.

From the definition of the generalized indicator, for each $\gamma > 0$, there exists a sequence $\{r_n\}$, $r_n \to \infty$, such that

$$\log|f(r_n)| > [h(\theta)-\gamma] r^\rho(r_n).$$
Also, if we choose \( \delta > 0 \) sufficiently small, then for

\[ |\theta| < \arcsin(2e\delta) \quad \text{and} \quad r > r(\gamma, \delta), \]

we have

\[
\log|f(re^{i\theta})| < \left[ (h(\theta) + \gamma) r^\rho(r) \right] |f(r)| < \left[ (h(0) + 2\gamma) r^\rho(r) \right].
\]

Now let

\[
\phi_n(z) = \frac{f(r_n + z)}{f(r_n)},
\]

so that \( \phi_n(0) = 1 \), and for \( |z| < 2e\delta r \),

\[
\log|\phi_n(z)| = \log|f(r_n + z)| - \log|f(r_n)|
\]

\[
< \left[ (h(\theta') + \gamma) (r_n + |z|)^\rho(r_n + |z|) \right] - \left[ (h(0) - \gamma) r_n^\rho(r_n) \right]
\]

\[
< \left[ (h(0) + 2\gamma) (r_n + |z|)^\rho(r_n + |z|) \right] - \left[ (h(0) - \gamma) (r_n + |z|)^\rho(r_n + |z|) \right]
\]

\[
= 3\gamma(r_n + |z|)^\rho(r_n + |z|),
\]

where \( \theta' = \arg(r_n + z) \). Applying Theorem 1.5, with \( R = \delta r_n \), \( \eta = \frac{w}{4} \), to \( \phi_n(z) \) gives us the inequality

\[
\log|\phi_n(z)| > -3\gamma H\left(\frac{w}{4}\right)(r_n + 2e\delta r_n)^\rho(r_n + 2e\delta r_n)
\]

which is satisfied inside the circle \( |z| < \delta r_n \), but outside of a family of circles the sum of whose radii is not greater than \( w\delta r_n \).

Now, returning to the function \( f(z) \), we see that
\[ \log |f(r_n + z)| = \log |f(r_n)| + \log |\phi(z)| \]

\[ > [h(0) - \gamma] r^\rho(r_n) - 3 \gamma H(\frac{w}{4})(r_n + 2e \delta r_n) \]

\[ = \frac{\rho(r_{n+2e \delta r_n})}{\rho(r_{n+2e \delta r_n})} \]

and thus the asymptotic inequality

\[ \log |f(r)| > [h(0) - \gamma] r^\rho(r_n) - 3 \gamma H(\frac{w}{4})(1 + 2e \delta) r_n \]

\[ = [h(0) - \gamma - 3 \gamma H(\frac{w}{4})(1 + 2e \delta)] r_n \]

is satisfied on the whole interval \((1 - \delta)r_n < r < (1 + \delta)r_n\), except perhaps for intervals the sum of whose lengths is less than \(2w \delta r_n\).

Since the function \(r^\rho(r)\) is increasing, also have

\[ r^\rho(r) < [(1 + \delta)r_n]^\rho((1 + \delta)r_n) \]

and the asymptotic inequality

\[ \frac{\rho(r_n)}{r_n} > (1 + \delta)^{-\rho} r^\rho(r) \]

for all values of \(r\) on the interval \((1 - \delta)r_n < r < (1 + \delta)r_n\). Combining our asymptotic inequalities yields

\[ \log |f(r)| > [h(0) - \gamma - 3 \gamma H(\frac{w}{4})(1 + 2e \delta)] (1 + \delta)^{-\rho} r^\rho(r) \]

and choosing \(\gamma\) and \(\delta\) so small that
\[ h(0) - \gamma - 3\gamma H \left( \frac{w}{4} \right) (1 + 2ae\delta)^\rho > [h(0) - \varepsilon] (1 + \delta)^\rho \]

we have

\[ \log |f(r)| > [h(0) - \varepsilon] r^{\rho(r)} \]

is satisfied on each interval of the sequence \( \{(1 - \delta)_n, (1 + \delta)_n \} \)

except possibly on a set of measure not exceeding \( w_\delta n \).
BIBLIOGRAPHY


