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The classical dimension theories of Menger-Urysohn and Lebesgue are equivalent on metric spaces. However, their infinite-dimensional analogues may differ, even on compact metric spaces. The three such infinite-dimensional dimension theories considered in this thesis are known as countable-dimensionality, property C, and weak infinite-dimensionality. The open questions regarding the relationships between these properties forms what is called "the Generalized Alexandroff Question".

In the second chapter of this thesis, results are obtained regarding the types of infinite-dimensional spaces involved with the use of various classes of maps. Inclusion maps, projection maps onto factors of product spaces, open mappings, refinable mappings, and approximately invertible maps such as hereditary shape equivalences are investigated.

In the final chapter of this thesis, a characterization of weak infinite-dimensionality in terms of binary open covers is generalized to give an infinite number of such characterizations. These will be used to better understand the differences between infinite-dimensional dimension theories.

Alternative Characterizations of Weak Infinite-Dimensionality and Their Relation to a Problem of Alexandroff's

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ALTERNATIVE CHARACTERIZATIONS OF WEAK INFINITE-DIMENSIONALITY AND THEIR RELATION TO A PROBLEM OF ALEXANDROFF'S

I. INTRODUCTION AND MATHEMATICAL PRELIMINARIES

The classical dimension theories of Menger-Urysohn and Lebesgue assign an integer, called the dimension, to topological spaces in a manner which extends the notions of dimension for manifolds and polyhedra. These classical dimension theories, large inductive dimension and covering dimension, may be thought of as inductively defined generalizations of the topological properties of normality and paracompactness. Thus, although quite difficult to show, it is not surprising that the two dimension theories are equivalent for the category of metric spaces.

A description of these theories is given in section 1.1, along with some elementary properties of those two theories. In addition, two important theorems are given. The first, called the decomposition theorem, gives a characterization of large inductive dimension in terms of unions of zero-dimensional subspaces. The second, due to Eilenberg and Otto, gives a characterization of large inductive dimension in terms of families of pairs of disjoint closed subsets of the space, now known as essential families. The section ends with a discussion of a result, due to Morita, concerning the equivalence of the two dimension theories when considered for the category of metric spaces.

On the other hand, the classification of infinite-dimensional spaces, first proposed for study by Hurewicz in 1928, is not as well understood. Although the classical dimension theories are equivalent on metric spaces, their infinite-dimensional analogues may differ, even for compact metric spaces. It appears to be very difficult to identify exactly what makes a given space have infinite dimension. As an example, there even exists an infinite-dimensional compact metric space which contains no positive-dimensional subspace [**Walsh** (1978)]. Since many important theorems of topology require finite-dimensionality as part of the hypothesis, infinite-dimensional dimension theories are of research interest.

In section 1.2, infinite-dimensional analogues of the finite-dimensional theories given in 1.1 are defined. The first such analogue, countable-dimensionality, was defined by Hurewicz as a generalization of the decomposition theorem. Variations of the original definitions are discussed, along with some relationships between those variations. Of particular interest is a theorem due to Nagata, which characterizes countable-dimensionality in terms of essential families.

Section 1.2 continues with the introduction of a covering property due to Haver known as property C. Property C was defined by Haver for metric spaces, and later generalized by Addis and Gresham for general topological spaces, in order to decide when certain infinite-dimensional locally contactible spaces were absolute neighborhood retracts. A survey of the results of Addis and Gresham which are pertinent to the rest of this thesis is presented. In particular, property C determines a dimension theory for paracompact strongly completely normal spaces.

The last infinite-dimensional dimension theory discussed in section 1.2 is known as weak infinite-dimensionality. Two versions, equivalent on compacta, are compared; the first due to Alexandroff, and the second due to Smirnov. Smirnov's version will prove to be deficient as a dimension theory, and will be discarded, for the most part, throughout the rest of this thesis.

The final section, section 1.3, of this chapter concerns itself, in detail, with a famous question posed by Alexandroff. Simply stated, this question asks whether or not the properties of countable-dimensionality and weak infinite-dimensionality are equivalent when considered on compact metric spaces. This question was recently negatively answered by Roman Pol [R. Pol (1981)]. However, as with most results in mathematics, this answer generates even more questions. A discussion of the construction of R. Pol's example is given, along with the remarks needed to show that this example is indeed an answer to the original Alexandroff question.

The section and this chapter is ended with a discussion of the relationships between the various infinite-dimensional dimension theories presented. Those relationships are combined with the questions raised by R. Pol's example to form what is called "The Generalized Alexandroff Question". The collective emphasis of this thesis is to research and better understand this problem.

1.1 Classical Dimension Theories

In this section, two classical dimension theories are given. The large inductive dimension was first published by Čech [Čech (1931)], although it might be related to earlier results of Brouwer, and is certainly related to the earlier theories of Menger and Urysohn [**Menger** (1923), Urysohn (1922)]. For a more complete exposition on the theory of large inductive dimension, the reader is referred to [**Engelking**, Ch. 2].

1.1.1 Definition. For every normal space X, the large inductive dimension of X is an integer $n \in \{-1, 0, 1, \cdots\}$, denoted by the $\operatorname{Ind} X = n$, or is said to be infinite, denoted by the $\operatorname{Ind} X = \infty$, which is assigned according to the following rules:

a) The $\operatorname{Ind} X = -1$ if and only if the space $X = \varnothing$.

b) The $\operatorname{Ind} X \leq n$, $n = \{0, 1, 2, \cdots\}$, if for each closed set $A \subset X$ and every open set $U \subset X$ with $A \subset U$ there exists an open set $V \subset X$ such that $A \subset V \subset \overline{V} \subset U$ with the $\operatorname{Ind} \operatorname{Fr} V \leq n-1$.

c) The $\operatorname{Ind} X = n$ if the $\operatorname{Ind} X \leq n$ and the $\operatorname{Ind} X > n-1$.

d) The $\operatorname{Ind} X = \infty$ if the $\operatorname{Ind} X > n$ for each $n \in \{-1, 0, 1, \cdots\}$.

As can be seen from the definition, large inductive dimension is basically an inductive version of nomality, and hence, is defined for any normal space. To obtain the standard theorems of a dimension theory, a stronger separation property, namely strong hereditary normality, must be assumed for the spaces involved. Since the emphasis of this thesis will be with metric spaces, and indeed since every metric space is strongly hereditarily normal, this distinction need be of little concern to the reader.

1.1.2 The Decomposition Theorem. [Engelking, p. 259] Let X be a non-empty metric space and let $n \in \{0, 1, 2, \cdots\}$ be fixed. The $\operatorname{Ind} X \leq n$ if and only if for each $k \in \{1, \cdots, n+1\}$ there exists a zero-dimensional subset $Z_k \subset X$ such that $X = \bigcup\{Z_k : k = 1, \cdots, n+1\}$.

Because of the important nature of the zero-dimensional subspace in 1.1.2, some elementary results of zero-dimensional spaces are combined to give the following statement.

1.1.3 Results for Zero-Dimensional Spaces. [**Engelking**, p. 33, 52, and 53]. Every normal space with the $\operatorname{Ind} X = 0$ is totally disconnected. Every compact totally disconnected normal space X has the $\operatorname{Ind} X = 0$.

However, there do exist totally disconnected separable metric spaces of all dimensions which can be constructed in a very axiomatic manner [**Rubin, Schori, and Walsh** (1979)]. The notation of the following theorem may also be found in [**Rubin, Schori, and Walsh** (1979)].

Although Eilenberg and Otto originally proved the following theorem only for the case where X was separable metric, the theorem remains true without the separability condition [**Engelking**, p. 230, 254].

1.1.4 [**Eilenberg and Otto** (1938)] A non-empty metric space X satisfies the inequality $\operatorname{Ind} X \leq n$ if and only if every $(n+1)$-family of pairs of disjoint closed subsets $\{(A_k, B_k) : k = 1, \cdots, n+1\}$ of X is inessential, that is for each $k \in \{1, \cdots, n+1\}$ there exists a closed subset $S_k \subset X$ which separates the pair (A_k, B_k) in X such that the $\bigcap\{S_k : k = 1, \cdots, n+1\} = \varnothing$.

The second classical dimension theory which will be presented, known as covering dimension, has its roots in a very early paper by Lebesgue [**Lebesgue** (1911)], and was formally defined by Čech [**Čech** (1933)]. To avoid an extra hypothesis, covering dimension will be defined for the category of paracompact spaces.

1.1.5 Definition. Let $\mathcal{U}$ be a collection of subsets from a set X. For any subset $A \subset X$ the order of A in $\mathcal{U}$ will be the largest number n of elements of $\mathcal{U}$ which contain some point $x \in A$, and will be denoted by the $\operatorname{ord}_A \mathcal{U} = n$. If $A = \{x\}$ for some $x \in X$, then the order of x in $\mathcal{U}$ will be denoted by the $\operatorname{ord}_x \mathcal{U} = n$. If no such largest integer exists, then A will be said to have infinite order in $\mathcal{U}$, and will be denoted by the $\operatorname{ord}_A \mathcal{U} = \infty$. The order of $\mathcal{U}$ will be defined and denoted by the $\operatorname{ord} \mathcal{U} = \sup\{\operatorname{ord}_x \mathcal{U} : x \in X\}$.

1.1.6 Definition. For every paracompact space X, the covering dimension of X is an integer $n \in \{-1, 0, 1, \cdots\}$, denoted by the $\dim X = n$ or is said to be infinite, denoted by the $\dim X = \infty$, which is assigned according to the following rules:

a) The $\dim X = -1$ if and only if the space $X = \varnothing$.

b) The $\dim X \leq n$, $n \in \{0, 1, 2, \cdots\}$, if every open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}$, with the $\operatorname{ord} \mathcal{V} \leq n+1$, which also covers X.

c) The $\dim X = n$ if the $\dim X \leq n$ and if the $\dim X > n-1$.

d) The $\dim X = \infty$ if the $\dim X > n$ for each $n \in \{-1, 0, 1, \cdots\}$.

In this form, the covering dimension is seen to be an inductive version of paracompactness. Since every metric space is paracompact, it is quite natural to suspect that the large inductive dimension and the covering dimension agree on metric spaces.

1.1.7 The Coincidence Theorem. [Katětov (1952), Morita (1954)]
For any metric space X, the $\operatorname{Ind} X = \dim X$.

Thus, the choice of category for this thesis will be that of metric spaces. For the remainder of this thesis, a space will always mean a metric space, and because of 1.1.7, all future references to dimension will refer to the covering dimension of Lebesgue with the notation $\dim X$.

1.2 Infinite-Dimensional Dimension Theories

Hurewicz was the first person to propose giving infinite-dimensional spaces their own dimension theory [Hurewicz (1928)]. In this section, three such infinite-dimensional dimension theories are presented. Although the classical dimension theories presented in the previous section are equivalent for metric spaces, their infinite-dimensional analogues may differ, even for compact metric spaces.

The first such theory, countable-dimensionality, was proposed by Hurewicz [Hurewicz (1928)] as a generalization of the decomposition theorem 1.1.2, and has been extensively discussed in the literature; [Nagata, chapter VI.] and [E. Pol (1983)] are two very good sources.

1.2.1 Definitions. A space X is said to be countable-dimensional, denoted by CD, if the space can be written as $X = \bigcup\{Z_k : k \in \mathbb{N}\}$ where each subspace $Z_k \subset X$ is finite-dimensional. If, in addition, each Z_k is a closed subset of X, then X is said to be strongly countable-dimensional. A space which is not countable-dimensional is said to be uncountable-dimensional.

It is clear from 1.1.2, that every finite-dimensional space is countable-dimensional, and thus, that every uncountable-dimensional space is infinite-dimensional. Many of the results in the literature concerning countable-dimensional spaces are much easier to prove, or indeed, can be shown to be true only for strongly countable-dimensional spaces. This is unfortunate since Smirnov has given the following result.

1.2.2 [**Smirnov** (1962)] There exists a CD compact metric space which is not strongly CD.

This discussion of countable-dimensional spaces is ended with the presentation of a theorem due to Nagata which has been rewritten in the language of essential families given in [**Rubin, Schori, and Walsh** (1979)].

1.2.3 [**Nagata** (1960)] A space X is countable-dimensional if and only if for any countable collection of pairs of disjoint closed subsets, henceforth called an ω-family, $\{(A_k, B_k): k \in \mathbb{N}\}$ of X, there exists for each $k \in \mathbb{N}$ a closed set $S_k \subset X$ which separates the pair (A_k, B_k) in X such that for each point $x \in X$ the $\mathrm{ord}_x \{S_k : k \in \mathbb{N}\} < \infty$.

The next infinite-dimensional dimension theory presented is a covering property first defined by Haver [**Haver** (1973)] for metric spaces, and later varied by Addis and Gresham [**Addis and Gresham** (1978)] for general topological spaces. The definition of property C given below in 1.2.4 is the one which was given by Addis and Gresham, and will be the only such definition used in this thesis. Haver's original definition is equivalent to 1.2.4 for compact metric spaces, however on non-compacta, the two properties obtained may differ.

1.2.4 Definitions. A subspace A of a space X is said to have property C in X, and is called a C-space, if for any sequence of covers $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of A by open subsets of X, there exists refinements $\mathcal{V}_n$ of $\mathcal{U}_n$, for $n \in \mathbb{N}$, which satisfies the following:

a) For each $n \in \mathbb{N}$ the elements of $\mathcal{V}_n$ are open in X.

b) For each $n \in \mathbb{N}$ the elements of each $\mathcal{V}_n$ are pairwise disjoint.

c) The $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ forms a cover of the subspace A.

Any refinement $\mathcal{V}_n$ of a cover $\mathcal{U}_n$ which satisfies the conditions a and b is said to be a C-refinement of the cover $\mathcal{U}_n$.

In [**Addis and Gresham** (1978)], results were presented to establish property C as an acceptable dimension theory. In order to establish the standard theorems of finite-dimensional dimension theories, Addis and Gresham had to assume that the spaces involved were paracompact strongly completely normal spaces, that is spaces where for each countable collection of separated subsets there exists a family of pairwise disjoint open subsets each of which contains exactly one of element of the collection of separated sets. Since this is much to ungainly for the purposes of this thesis, property C will only be considered for metric spaces. Of course, every metric space is paracompact and strongly completely normal. In particular, strong complete normality is needed to obtain the following basic result.

1.2.5 [**Addis and Gresham** (1978), p. 197] A subspace A of a metric space X has property C in X if and only if A has property C in itself.

In light of 1.2.5, and because all spaces considered in this thesis will be metric spaces, a subspace will simply be said to have property C with no mention of any ambient space. The results of [Addis and Gresham (1978)] which will frequently be used in the rest of this paper are stated next.

1.2.6 The Sum Theorem for Property C. [Addis and Gresham (1978), p. 197] If for each $n \in \mathbb{N}$ X_n is a subspace which has property C in a space X then the $\bigcup\{X_n : n \in \mathbb{N}\}$ also has property C.

1.2.7 The Subspace Theorem for Property C. [Addis and Gresham (1978), p. 197] If a space X has property C, then every closed subspace $A \subset X$ will also have property C. Thus, in light of 1.2.6, every $\mathfrak{F}_\sigma$ subspace of X will also have property C.

Haver's original definition of property C defined a property hereditary to all subspaces of a metric space. As will be shown in the next section, the definition of property C due to Addis and Gresham needs not be hereditary to all subspaces, even when the ambient space is a compact metric space.

Addis and Gresham also showed that property C was an infinite-dimensional analogue of a finite-dimensional dimension theory equivalent to covering dimension for paracompact strongly completely normal spaces, and hence also for metric spaces. Thus, it is quite correct to say that property C is an infinite-dimensional dimension theory.

1.2.8 [**Addis and Gresham** (1978), p. 197] A space X has the $\dim X \leq n$ if and only if for any sequence of covers $\{\mathcal{U}_k : k=1,\cdots,n+1\}$ by open subsets of X there exists for each $k \in \{1,\cdots,n+1\}$ a C-refinement $\mathcal{V}_k$ of $\mathcal{U}_k$ such that the $\bigcup\{\mathcal{V}_k : k=1,\cdots,n+1\}$ covers X.

The last infinite-dimensional dimension theory, weak infinite-dimensionality, is an infinite-dimensional analogue of the characterization of covering dimension which was given in 1.1.4. Two such analogues were proposed in the literature; the first by Alexandroff and the second by Smirnov. The best single source on weak infinite-dimensionality is found in [**Alexandroff and Pasynkov** (1973)], where the reader is referred to for general information about the following statements.

1.2.9 Definitions. A space X is said to be weakly infinite-dimensional in the sense of Alexandroff, which will be denoted by WID, if every given ω-family of pairs of disjoint closed subsets $\{(A_n, B_n) : n \in \mathbb{N}\}$ of X is inessential, that is for each $n \in \mathbb{N}$ there exists a closed set $S_k \subset X$ which separates the pair (A_n, B_n) in X such that the $\bigcap\{S_n : n \in \mathbb{N}\} = \varnothing$. If, in addition, the separators may be chosen such that for some finite integer N the $\bigcap\{S_n : n=1,\cdots,N\} = \varnothing$, then the space X is said to be weakly infinite-dimensional in the sense of Smirnov, which will be denoted by S-WID. If a space is not weakly infinite-dimensional, then the space will be referred to as being strongly infinite-dimensional, which will be denoted by SID.

It is clear that every S-WID space is also a WID space. Moreover, it is easy to show that the two versions of weak infinite-dimensionality are equivalent on compacta. Because of this, and unless stated otherwise, weak inifinite-dimensionality will always refer to weak infinite-dimensionality in the sense of Alexandroff.

The following theorems state results concerning WID spaces which will frequently be used in later sections of this thesis. The proofs follow the same pattern as the proofs of the corresponding theorems for property C, and are omitted.

1.2.10 The Sum Theorem for WID Spaces. If for each $n \in \mathbb{N}$ X_n is a WID subspace of a space X, then the $\bigcup\{X_n : n \in \mathbb{N}\}$ will also be a WID subspace of X.

The equivalent statement for S-WID spaces is false, even when each subspace is closed in X. A simple counter-example may be constructed by forming the nested union of n-cells where n ranges over all $n \in \mathbb{N}$. The resulting space, while clearly strongly countable-dimensional, is not S-WID.

1.2.11 The Subspace Theorem for WID Spaces. If a space X is WID (S-WID), then every closed subspace of X is also WID (S-WID). Thus, in light of 1.2.10, every $\mathcal{F}_\sigma$ subspace of a WID space is also WID.

These results suggest that weak infinite-dimensionality in the sense of Smirnov is not an appropriate choice for an infinite-dimensional dimension theory. On the other hand, since weak infinite-dimensionality in the sense of Alexandroff does possess extensions of the finite-dimensional subspace and sum theorems, weak infinite-dimensionality in the sense of Alexandroff should be thought of as an infinite-dimensional dimension theory.

1.3 The Generalized Alexandroff Question

In the last section, three infinite-dimensional dimension theories were presented. Comparing theorem 1.2.3 with the definition of weak infinite-dimensionality given in 1.2.9 gives the following theorem.

1.3.1 Theorem. Every CD space is a WID space.

The converse of 1.3.1, first proposed by Alexandroff, remained an open question until only recently.

1.3.2 The Alexandroff Question. Is every WID space a CD space?

Until recently, the only examples of uncountable-dimensional spaces were strongly infinite-dimensional spaces such as the Hilbert cube. In 1981, Roman Pol combined two known theorems to negatively answer 1.3.2 [R. Pol (1981)]. Because of the importance of this example, and because the structure of this example will be used in later sections, a detailed discussion of its construction is given.

1.3.3 [R. Pol (1981)] There exists a WID compact metric space P which contains a SID subspace. Thus, P cannot be CD.

Proof. There exists a SID separable metric space X which is totally disconnected. Although this space was probably known previously, an explicit construction of such a space X as a subset of the Hilbert cube may be found in [**Rubin, Schori, and Walsh** (1978)] as a special case of a more general and powerful construction procedure. As R. Pol also mentions, the construction of this space may be modified to ensure that the constructed space X is topologically complete. A detailed proof may be found in [**Garity and Schori** (1986)]. Thus, the space X can be constructed to be a $\mathcal{G}_\delta$ subset of the Hilbert cube [**Dugundji**, p. 307].

A classical compactification theorem [**Lelek** (1965)] states that such a space admits a compactification P in the Hilbert cube with a CD remainder, i.e. $Z = P \setminus X$ is CD. Since the compactum X contains the SID subspace X, the theorem 1.3.1, together with the obvious hereditary nature of countable-dimensionality, implies that P cannot be CD. On the other hand, it is easy to show that P is WID. This will be obtained in a later section as a result of a more general theorem. Thus, the compactum P is seen to be a negative answer to the question of Alexandroff. This completes the proof.

It should be mentioned that, as noticed by E. Pol, R. Pol's compactum P also has property C. This will be obtained in a later section as a special case of a more general theorem.

<u>1.3.4</u> [**Addis and Gresham** (1978), p. 197 and p. 202] Every CD space has property C. Every space with property C is WID.

It has long been the opinion of many topologists working in this area, that property C captures the essential nature of countable-dimensionality. If this is indeed the case, then R. Pol's compactum cannot truly be considered to be an acceptable solution to the problem of Alexandroff, or perhaps it would be better to say that Alexandroff simply asked the wrong question. Therefore, in the remainder of this section, a generalized Alexandroff question will be posed. Contained in this generalized question are particular questions of importance, not only to dimension theory, but also to larger areas of topology such as manifold decompositions, shape theory and the study of dimension-raising maps.

It is obvious from the definition given in 1.2.1 that the property of being countable-dimensional is hereditary to all subspaces of a countable dimensional space. Haver's original definition of property C was hereditary to all subspaces of a metric C-space. On the other hand, R. Pol's compactum shows that property C, as defined by Addis and Gresham, and weak infinite-dimensionality, while hereditary to closed subspaces, need not be hereditary to all subspaces. Thus, the two versions of property C do differ. Because of this observation of the lack of an hereditary nature for property C and for weak infinite-dimensionality, it is quite natural to make the following definitions.

<u>1.3.5</u> <u>Definitions.</u> A space X is said to be <u>hereditarily weakly infinite-dimensional</u>, denoted by HWID, if every subspace $A \subset X$ is WID. A space X is said to have <u>property C hereditarily</u>, denoted by H-property C, if every subspace $A \subset X$ has property C.

1.3.6 Theorem. Metric spaces satisfy the following implications of properties.

$$\begin{array}{lllll} & & \overset{b}{\Longrightarrow} \ \text{HWID} \ \overset{d}{\Longrightarrow} & & \\ \text{CD} \ \overset{a}{\Longrightarrow} \ \text{H-property C} & & & & \text{WID} \\ & & \overset{c}{\Longrightarrow} \ \text{property C} \ \overset{e}{\Longrightarrow} & & \end{array}$$

Proof. These implication are obvious from the results in 1.3.4.

Since the compactum P constructed in 1.3.3 is a C-space with an HWID subspace, it is seen that the converses of the implications c and d are false. At the time of this writing, the converses to the remaining implications of 1.3.6 are open questions. These reverse implications, those of a, b, and e, together with the implication f of the following statement compose the generalized question of Alexandroff.

1.3.7 The Generalized Question of Alexandroff. Does every metric space always satisfy each of the following implications of properties?

$$\text{CD} \overset{a}{\Longleftarrow} \text{H-property C} \overset{b}{\Longleftarrow} \text{HWID} \overset{f}{\Longrightarrow} \text{property C} \overset{e}{\Longleftarrow} \text{WID}$$

Since the implication e of 1.3.7 implies the implications b and f of 1.3.7, it certainly seems that the question posed by e is of the most interest. Also of interest is the question posed by the implications a and b of 1.3.7 when taken together: Must every HWID space be CD?

Chapter two of this paper presents results which illustrate the importance of, and difficulties in, answering these questions. In 1.2.3, a characterization of countable-dimensionality in terms of essential families was given. However, at this time, no such characterization of property C in terms of essential families is known. Chapter three of this paper will explore the possibilities of such a characterization, as well as characterizing weak infinite-dimensionality in terms of sequences of open covers.

II. EXTENSIONS OF EXISTING RESULTS

In this chapter, extensions and generalizations of existing results will be given. Section 2.1 gives results concerning separations and decompositions of weakly infinite-dimensional spaces. A result, theorem 2.1.6, is given which generalizes E. Pol's proof of the fact that R. Pol's compactum has property C. The section ends with a question of importance in answering the generalized Alexandroff question.

In section 2.2, results concerning products of weakly infinite-dimensional spaces are presented. A brief review of known results is given, followed by some results about products of R. Pol's compactum with various WID spaces. A direct proof that the product of two compact C-spaces also has property C is given in 2.2.17. The section ends with some questions for further research.

The topic of section 2.3 is the preservation of weak infinite-dimensionality and property C by open maps with finite fibers. A very general theorem is given in 2.3.9, from which various results are obtained including a new result concerning property C under open maps with finite fibers. These results no longer follow if the condition on the fibers is relaxed. Some questions related to these topics are asked at the end of the section.

In section 2.4, a covering characterization of weak infinite-dimensionality is given. This characterization, given in 2.4.2, is exploited to generalize a result of Kato, by showing that refinable maps on compacta preserve property C. A slight variation in proof yields Kato's original result as well as a result due to Patten.

In the final section of this chapter, the major motivation for studying the relationships between different types of infinite-dimensional spaces is discussed; namely the cell-like dimension-raising map question. Results due to Kozlowski and Ancel related to this question are discussed. In particular, it is shown that approximately invertible maps, such as hereditary shape equivalences, preserve weak infinite-dimensionality. By a simple variation of proof, Kozlowski's result that hereditary shape equivalences preserve finite-dimensionality is obtained. The remainder of the section raises questions related to the cell-like dimension-raising map question.

2.1 Decompositions of WID Spaces

In this section some easily obtained results concerning decompositions of weakly infinite-dimensional spaces are presented. The emphasis is on the similarities between property C and weak infinite-dimensionality, and not on the results themselves.

2.1.1 [**Addis and Gresham** (1978), p. 200] A space X has property C if and only if for any pair of disjoint closed subsets (A, B) of X there exists a closed subset $S \subset X$ having property C in X such that S separates the pair (A, B) in X.

The following analogous result concerning the inductive nature of weak infinite-dimensionality is obtained.

2.1.2 Theorem. A space X is WID if and only if for any pair of disjoint closed subsets (A, B) of X there exists a closed WID subset $S \subset X$ such that S separates the pair (A, B) in X.

Proof. Suppose that X is WID and that S is any closed subset of X which separates the given pair of disjoint closed sets (A, B) in X. Since X is WID, and since S is closed in X, the separator S must be WID.

Suppose that the space X satisfies the hypothesis of the converse and let an ω-family of pairs of disjoint closed subsets $\{(A_n, B_n) : n \in \mathbb{N}\}$ of X be given. By the hypothesis, a closed subset $S_1 \subset X$ which separates the pair (A_1, B_1) in X may be chosen such that the separator S_1 is WID.

Since the collection $\{(A_n \cap S_1, B_n \cap S_1) : n \in \mathbb{N} \setminus \{1\}\}$ is an ω-family of pairs of disjoint closed subsets of S_1, for each $n \in \mathbb{N} \setminus \{1\}$ a closed set $T_n \subset S_1$ which separates the pair $(A_n \cap S_1, B_n \cap S_1)$ in S_1 may be chosen such that the $\bigcap\{T_n : n \in \mathbb{N} \setminus \{1\}\} = \varnothing$. Each T_n may then be enlarged to obtain a closed set $S_n \subset X$, with $S_n \cap S_1 \subset T_n$, which separates the pair (A_n, B_n) in X [**Engelking**, p. 13].

Thus, separators of the pairs comprising the given ω-family in X are obtained such that the

$$\begin{aligned}\bigcap\{S_n : n \in \mathbb{N}\} &= \bigcap\{S_n \cap S_1 : n \in \mathbb{N} \setminus \{1\}\} \\ &\subset \bigcap\{T_n : n \in \mathbb{N} \setminus \{1\}\} \\ &= \varnothing .\end{aligned}$$

Therefore the ω-family $\{(A_n, B_n) : n \in \mathbb{N}\}$ is inessential, and the space X is shown to be WID. This completes the proof.

The previous theorem motivates the following theorem for spaces which have property C.

<u>2.1.3</u> <u>Theorem.</u> A space X has property C if and only if for any given open cover $\mathcal{U}$ of X there exists a C-refinement $\mathcal{V}$ of $\mathcal{U}$ such that the complement $X \setminus \bigcup\{V : V \in \mathcal{V}\}$ has property C in X.

<u>Proof.</u> Suppose that the space X has property C and let $\mathcal{U}$ be a given open cover of X. If $\mathcal{V}$ is any C-refinement of $\mathcal{U}$, then the

complement $X\setminus\bigcup\{V:V\in\mathcal{V}\}$ is closed in X. Thus, since X has property C, $X\setminus\bigcup\{V:V\in\mathcal{V}\}$, being closed in X, must also have property C in X.

Suppose that the space X satisfies the hypothesis of the converse, and let $\{\mathcal{U}_n:n\in\mathbb{N}\}$ be a given sequence of open covers of X. By the hypothesis, a C-refinement $\mathcal{V}_1$ of $\mathcal{U}_1$ may be chosen such that the complement $X\setminus\bigcup\{V:V\in\mathcal{V}_1\}$ has property C in X.

Finally, for each $n\in\mathbb{N}\setminus\{1\}$ a C-refinement $\mathcal{V}_n$ of $\mathcal{U}_n$ may be chosen such that the $\bigcup\{\mathcal{V}_n:n\in\mathbb{N}\setminus\{1\}\}$ covers $X\setminus\bigcup\{V:V\in\mathcal{V}_1\}$. Thus, the $\bigcup\{\mathcal{V}_n:n\in\mathbb{N}\}$ covers all of X, which completes the proof.

Property C and weak infinite-dimensionality may also be related by the following theorem. The technique used in the proof of that theorem will also be used in a later section to characterize weak infinite-dimensionality in terms of open covers.

<u>2.1.4</u> <u>Theorem.</u> A space X is WID if and only if for any given open cover $\mathcal{U}$ of X there exists a C-refinement $\mathcal{V}$ of $\mathcal{U}$ such that the complement $X\setminus\bigcup\{V:V\in\mathcal{V}\}$ is WID.

<u>Proof.</u> Suppose that the space X is WID and that $\mathcal{U}$ is a given open cover of X. If $\mathcal{V}$ is any C-refinement of $\mathcal{U}$, then the complement $X\setminus\bigcup\{V:V\in\mathcal{V}\}$ is closed in X. Thus, since X is WID, $X\setminus\bigcup\{V:V\in\mathcal{V}\}$ being closed in X must also be WID.

Suppose that the space X satisfies the hypothesis of the converse.

Theorem 2.1.2 will be used to show that X is WID. Let (A, B) be a given pair of disjoint closed subsets of X, and use normality to choose open sets U_1 and U_2 such that

$$A \subset U_1 \subset \bar{U}_1 \subset X \setminus B \quad \text{and} \quad B \subset X \setminus U_1 \subset U_2 \subset \bar{U}_2 \subset X \setminus A.$$

Since the pair $\{U_1, U_2\}$ is an open cover of X, the hypothesis may be used to choose a C-refinement $\mathcal{V}$ of $\{U_1, U_2\}$ such that the complement $X \setminus \bigcup\{V : V \in \mathcal{V}\}$ is WID. Define

$$V_1 = (X \setminus \bar{U}_2) \cup (\bigcup\{V : V \in \mathcal{V},\ V \subset U_1\})$$

and

$$V_2 = (X \setminus \bar{U}_1) \cup (\bigcup\{V : V \in \mathcal{V},\ V \cap (X \setminus U_1) \neq \varnothing\}).$$

Clearly V_1 and V_2 are open subsets of X with

$$A \subset X \setminus \bar{U}_2 \subset V_1 \quad \text{and} \quad B \subset X \setminus \bar{U}_1 \subset V_2.$$

Furthermore, since the $U_1 \cup U_2 = X$, the sets V_1 and V_2 are disjoint.

Thus, if $S = X \setminus (V_1 \cup V_2)$ is defined, then S is seen to be a separator of the pair (A, B) in X. Since S is a closed subset of the WID subspace $X \setminus \bigcup\{V : V \in \mathcal{V}\}$, the separator S must also be WID, and then, by 2.1.2, the space X has been shown to be WID. This completes the proof.

The remainder of this section presents results on the decomposition of spaces into unions of subspaces of which at least one is WID.

2.1.5 [**Leibo** (1971)] If A is a WID subspace of a SID space X, then X contains a closed SID subspace Y such that $Y \cap A = \varnothing$.

<u>2.1.6</u> <u>Theorem.</u> If A is a subspace having property C in a space X which does not have property C, then X contains a closed subspace Y which does not have property C such that $Y \cap A = \varnothing$.

<u>Proof.</u> Since the space X is not a C-space, there exists a collection of open covers $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of X such that no C-refinements of the covers forms a cover of X. However, since A does have property C, for each $m \in \mathbb{N}$ C-refinements $\mathcal{V}_{2m}$ of each $\mathcal{U}_{2m}$ may be chosen such that the $\bigcup\{\mathcal{V}_{2m} : m \in \mathbb{N}\}$ forms an open cover of A in X.

Set $Y = X \setminus \bigcup\{V \in \mathcal{V}_{2m} : m \in \mathbb{N}\}$, then Y is a closed subset of X which is disjoint from A. No C-refinements of the remaining covers $\mathcal{U}_{2m-1}$ where $m \in \mathbb{N}$ can provide a cover of Y, lest it also complete a cover of C-refinements for X. Thus, Y cannot have property C, which completes the proof.

From 2.1.6 the following corollary, the first of which was observed by E. Pol, is obtained.

<u>2.1.7</u> <u>Corollary.</u> R. Pol's compactum P has property C, and thus is weakly infinite-dimensionality.

<u>Proof.</u> Recall that $P = X \cup Z$ is a subcompactum of the Hilbert cube where X is a totally disconnected SID space and Z is CD. If P did not have property C, then since Z, being CD, does have property C, the theorem 2.1.6 would imply the existence of a closed, hence compact, subspace $Y \subset P$ with $Y \cap Z = \varnothing$ which did not have property C. Thus, the

subspace $Y \subset X$ would be a compact totally disconnected space, and hence, by 1.1.3, would be zero-dimensional. But, this would contradict Y not having property C. Therefore, P must have property C which completes the proof of the corollary.

If property C is replaced by countable-dimensionality in 2.1.6, then the statement is no longer true. Pol's example is not countable-dimensional, and yet contains no closed uncountable-dimensional subspace missing Z.

<u>2.1.8</u> [**Skljarenko** (1959)] Let X be a S-WID space, then X contains a compact WID subspace K whose complement Z is CD.

Quite naturally, 2.1.8 raises questions of whether or not such decompositions exist for WID spaces, or for spaces which have property C.

<u>2.1.9</u> <u>Question.</u> If X is a WID space, then must X contain a compact WID subspace whose complement is CD?

<u>2.1.10</u> <u>Question.</u> If X is a space which has property C, then must X contain a compact subspace having property C whose complement is CD?

These questions will be answered negatively in the next section. It might be, assuming these answers, that 2.1.11 is the correct question to ask.

2.1.11 Question. If X is a WID space, then must X contain a compact WID subspace whose complement has property C?

An affirmative answer to 2.1.11, and a simple use of the sum theorem, would reduce the proof of implication e. of 1.3.7 to proving the implication for compacta. This would be a major step in solving the generalized Alexandroff question.

2.2 Products of WID Spaces

Results concerning when products of WID spaces are WID are very scarce amongst the literature of WID spaces. The answer to the following question is not even known [**R. Pol** (1982)].

2.2.1 Question. If X and Y are WID compacta, then must the product space $X \times Y$ be WID?

However, some results are known for non-compact spaces. As a guide to understanding the examples, some outline of proof is given.

2.2.2 [**R. Pol** (1982)] There exist two WID spaces whose product is not a WID space.

Proof. Pol's example of 2.1.7 can be written as $P = X \cup Z$ where $Z = P \setminus X$ is CD. Thus, it is possible to write $Z = B_1 \cup B_2$ where B_1 and B_2 are disjoint Bernstein sets, i.e. all compact subsets of B_1 or B_2 have countable cardinality [**Kuratowski**, p. 40].

It is then easy to show that the two spaces $X \cup B_1$ and $X \cup B_2$ are WID non-compacta. However, the product $(X \cup B_1) \times (X \cup B_2)$ contains the SID space $X \times X$ as a closed subspace. Thus, the product space $(X \cup B_1) \times (X \cup B_2)$ must be SID. This completes the proof.

This example also answers the related question about property C.

<u>2.2.3</u> [**Engelking and** E. **Pol** (1983)] The subspaces $X \cup B_1$ and $X \cup B_2$ have property C. Thus, the product of two non-compact spaces, both having property C, needs not have property C.

Finally, there is the following result, recently discovered by E. Pol.

<u>2.2.4</u> [E. **Pol** (1986)] There exists a WID space X, having property C, whose product with some subspace B of the irrationals is SID, and therefore, cannot have property C.

Results concerning product spaces where at least one factor is compact are even more scarce in the literature. The following theorem is about all that is known.

<u>2.2.5</u> [**Addis and Gresham** (1978), p. 201] Let X and Y be two C-spaces where Y is compact. If Y has a basis $\mathfrak{B}$ of open sets such that for all $B \in \mathfrak{B}$ the product $X \times \mathrm{Bdy}(B)$ has property C, then the product $X \times Y$ also has property C.

<u>2.2.6</u> <u>Corollary.</u> If X has property C and Y is a σ-compact strongly-CD space, then $X \times Y$ has property C.

<u>Proof.</u> Since the space Y is σ-compact, Y can be written as $Y = \bigcup\{Y_n : n \in \mathbb{N}\}$ where each Y_n is a compact space. Moreover, since Y is

a strongly-CD space it is possible to write $Y = \bigcup\{Z_m : m \in \mathbb{N}\}$ where each Z_m is a closed finite-dimensional subspace of Y. Thus, since each Y_n can be written as $Y_n = \bigcup\{Z_m \cap Y_n : m \in \mathbb{N}\}$, it is seen that for each $n \in \mathbb{N}$ the compact subspace Y_n is also strongly-CD. An inductive application of 2.2.5 then gives that for each $n \in \mathbb{N}$ and every $m \in \mathbb{N}$ the product $X \times (Z_m \cap Y_n)$ has property C. Applying the sum theorem to

$$X \times Y_n = \bigcup\{X \times (Z_m \cap Y_n) : m \in \mathbb{N}\} \text{ and } X \times Y = \bigcup\{X \times Y_n : n \in \mathbb{N}\}$$

gives the desired result that the product space $X \times Y$ has property C.

It is now possible to answer the questions raised in 2.1.9 and 2.1.10. The following simple lemma is needed.

2.2.7 Lemma. Let P denote R. Pol's compactum and let $\mathbb{E}$ denote the Euclidian line. The complement of any compact subspace K of $P \times \mathbb{E}$ cannot be CD.

Proof. Let $K \subset P \times \mathbb{E}$ be a compact subset and let $\pi : P \times \mathbb{E} \to \mathbb{E}$ be the projection mapping. Since $\pi(K)$ is compact, the complement $\mathbb{E} \setminus \pi(K) \neq \emptyset$, and thus it is possible to choose $y \in \mathbb{E} \setminus \pi(K)$. But, then the product $P \times \{y\} = \pi^{-1}(y) \subset (P \times \mathbb{E}) \setminus K$, and since $P \times \{y\}$ is not CD, it is seen that the complement $(P \times \mathbb{E}) \setminus K$ cannot be CD. This completes the proof of the lemma.

2.2.8 Theorem. The product space $P \times \mathbb{E}$ provides negative answers to the questions raised in 2.1.9 and 2.1.10.

Proof. Applying 2.2.6, it is observed that the product $P \times \mathbb{E}$ has property C, and thus is WID. The lemma 2.2.7 then denies the existence of a decomposition of $P \times \mathbb{E}$ as called for in 2.1.9 or 2.1.10. This completes the proof.

Smirnov's compact CD space which is not strongly-CD, mentioned in 1.2.2, prevents the proof of 2.2.6 from generalizing to arbitrary compact CD spaces. Moreover, the proof of 2.2.5 depends heavily on the second factor Y having all of the structure. In the remainder of this section, results are presented which share this structure between the two factors. The case where one of the factors is R. Pol's compactum will be of particular interest.

The next theorem, while almost obvious, gives a necessary condition for the product space to have property C.

2.2.9 Theorem. If X and Y are spaces such that the product $X \times Y$ has property C, then both factors X and Y must also have the property C.

Proof. Fix a point $(x, y) \in X \times Y$. Since the fibers $X \times \{y\}$ and $\{x\} \times Y$ are closed subspaces of the product $X \times Y$ which has property C, the fibers $X \times \{y\}$ and $\{x\} \times Y$, as well as their homeomorphic copies X and Y, must both have property C. This completes the proof.

Next, results are presented which concern product spaces where one factor is R. Pol's compactum. The following lemmas are needed.

<u>2.2.10</u> **[Morita (1956)]** If $f:X \to Y$ is a closed mapping between spaces X and Y such that for each $y \in Y$ the fiber has $\dim f^{-1}(y) \leq 0$, then the $\dim X \leq \dim Y$.

<u>2.2.11</u> <u>Lemma.</u> If X is a space such that for any zero-dimensional space Z the product $X \times Z$ has property C, then for any CD space Y the product $X \times Y$ will also have property C.

<u>Proof.</u> Since the space Y can be written as $Y = \bigcup\{Y_n : n \in \mathbb{N}\}$, where for each $n \in \mathbb{N}$ the $\dim Y_n = 0$, the result follows from the hypothesis and the sum theorem.

<u>2.2.12</u> <u>Theorem.</u> If P is R. Pol's compactum and Y is any CD space, then the product $P \times Y$ will have property C.

<u>Proof.</u> By the lemma 2.2.11, it is sufficient to show that the product $P \times Y$ has property C where Y is any zero-dimensional space.

Suppose that Y is zero-dimensional and let $\pi : P \times Y \to Y$ denote the projection mapping. Recall that $P = X \cup Z$ where X is a totally disconnected SID space and Z is CD.

Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a given sequence of open covers of $P \times Y$. Since the product $Z \times Y$ is clearly CD, $Z \times Y$ has property C, and thus, for each $n \geq 2$ it is possible to choose a C-refinement $\mathcal{V}_n$ of $\mathcal{U}_n$ such that the $\bigcup\{\mathcal{V}_n : n \geq 2\}$ is an open cover of $Z \times Y$ in $P \times Y$.

By setting $K = (P \times Y) \setminus (\bigcup\{V \in \mathcal{V}_n : n \geq 2\})$ a closed, but not necessarily compact, subspace of $P \times Y$ which is contained in $X \times Y$ is obtained. Since P is compact, the projection π is a closed mapping, and thus the restriction $\pi : K \to Y$ to the closed subspace K remains a closed mapping. Fix a point $y \in Y$, then $\pi^{-1}(y) \cap K$ is a closed, hence compact, subspace of the compactum $P \times \{y\}$ contained in the totally disconnected subspace $X \times \{y\}$. Thus, by the remarks given in 1.1.3, it is seen that the $\dim(\pi^{-1}(y) \cap K) \leq 0$.

Using these results, 2.2.10 can be applied to obtain that the $\dim K \leq 0$, and thus by 1.2.8, a C-refinement $\mathcal{V}_1$ of the remaining cover $\mathcal{U}_1$ may be chosen such that $\mathcal{V}_1$ is an open cover of K in $P \times Y$.

The collection $\{\mathcal{V}_n : n \in \mathbb{N}\}$ is then a collection of C-refinements of the original sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ such that the $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ covers all of the product $P \times Y$. Thus, $P \times Y$ is shown to have property C which completes the proof.

<u>2.2.13</u> <u>Corollary.</u> The product of R. Pol's compactum and the space of irrationals has property C.

<u>Proof.</u> This is an obvious application of 2.2.12.

It should be noted that the result of 2.2.13 cannot be obtained from 2.2.5. Moreover, in light of E. Pol's result 2.2.4, the following question ought to be raised.

2.2.14 Question. Does there exist a compactum which has property C and yet whose product with the space of irrationals (or any CD space for that matter) does not have property C?

2.2.15 Theorem. If K is a compactum whose product with any zero-dimensional space Y has property C, then the product of R. Pol's compactum P with K also has property C.

Proof. As before write $P = X \cup Z$, then 2.2.11 implies that the product $Z \times K$ has property C. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a given sequence of open covers of the product $P \times K$, then for each $n \in \mathbb{N}$ a C-refinement $\mathcal{V}_{2n}$ of $\mathcal{U}_{2n}$ may be chosen such that the $\bigcup\{\mathcal{V}_{2n} : n \in \mathbb{N}\}$ forms an open cover of $Z \times K$ in the product $P \times K$.

Set $Y = (P \times K) \setminus (\bigcup\{V \in \mathcal{V}_{2n} : n \in \mathbb{N}\})$, then Y is a closed, hence compact, subspace of the compactum $P \times K$ contained in the subspace $X \times K$. Denote the projection mapping by $\pi : P \times K \rightarrow P$, then $\pi(Y)$ is a subcompactum of P contained entirely within the totally disconnected space X, and thus, by 1.1.3, the $\dim \pi(Y) \leq 0$.

Thus, Y is a closed subspace of the product $\pi(Y) \times K$ which, by the hypothesis, has property C. Thus, Y itself must also have property C. Therefore, using the remaining covers $\{\mathcal{U}_{2n-1} : n \in \mathbb{N}\}$, it is possible to choose C-refinements $\mathcal{V}_{2n-1}$ of each $\mathcal{U}_{2n-1}$ such that the $\bigcup\{\mathcal{V}_{2n-1} : n \in \mathbb{N}\}$ forms an open cover of Y in the product $P \times K$.

Therefore, the $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ covers all of $P \times K$, and thus the product $P \times K$ is shown to have property C. This completes the proof.

2.2.16 Corollary. Let P be R. Pol's compactum, then the product $P \times P$ has property C.

Proof. Let Z be any zero-dimensional space. By 2.2.12, the product $P \times Z$ has property C, and thus theorem 2.2.15 implies that the product $P \times P$ also has property C.

The theorem 2.2.15 motivates, and is made obsolete by, the following theorem. This will complete the results about product spaces where both factors are compact spaces which have property C.

2.2.17 Theorem. If X and Y are compact spaces, both of which have property C, then their product $X \times Y$ also has property C.

Proof. Let a sequence of open covers of the product $X \times Y$ be given. Rewrite the sequence as a countable collection of sequences of open covers, denoted by $\{\{\mathcal{U}_m^n : n \in \mathbb{N}\} : m \in \mathbb{N}\}$. Moreover, by compactness, it may be assumed that each $\mathcal{U}_m^n$ is a finite cover of the product $X \times Y$, where without loss of generality, each $\mathcal{U}_m^n$ is of the form $\mathcal{U}_m^n = \{A_k \times B_k : k = 1, \cdots, r_m^n\}$ with every A_k open in X and each B_k open in the space Y.

Fix $m \in \mathbb{N}$, $n \in \mathbb{N}$, and let $x \in X$ be fixed but arbitrary. Choose a finite subcover $\mathcal{U}_m^n(x)$ of the product $\{x\} \times Y$ from $\mathcal{U}_m^n$ such that for each element $A_k \times B_k \in \mathcal{U}_m^n(x)$ of the subcover the point, $x \in A_k$. By defining $A(x) = \bigcap\{A_k : A_k \times B_k \in \mathcal{U}_m^n(x)\}$, an open subset $A(x)$ of X with $x \in A(x)$ is obtained. Thus, by constructing $A(x)$ for each $x \in X$, and by defining $\mathcal{A}_m^n = \{A(x) : x \in X\}$, an open cover $\mathcal{A}_m^n$ of X is constructed.

Keep $m \in \mathbb{N}$ fixed, and construct such an open cover $\mathcal{A}_m^n$ of X for each $n \in \mathbb{N}$. Since X has property C, C-refinements $\mathcal{C}_m^n$ of each $\mathcal{A}_m^n$ may then be chosen such that the $\bigcup\{\mathcal{C}_m^n : n \in \mathbb{N}\}$ covers X. For each $n \in \mathbb{N}$ and every $C \in \mathcal{C}_m^n$ a single point $x \in X$ may be chosen such that $C \subset A(x) \in \mathcal{A}_m^n$. For each such C and x define

$$\mathcal{V}_m^n(C) = \{C \times B_k : A_k \times B_k \in \mathcal{U}_m^n(x)\} \text{ and } \mathcal{V}_m^n = \bigcup\{\mathcal{V}_m^n(C) : C \in \mathcal{C}_m^n\}.$$

Thus, for each $C \in \mathcal{C}_m^n$, $\mathcal{V}_m^n(C)$ is an open refinement of $\mathcal{U}_m^n(x)$, and hence also of $\mathcal{U}_m^n$, such that $\mathcal{V}_m^n(C)$ is an open cover of $C \times Y$. Similarly, since the elements of each $\mathcal{C}_m^n$ are pairwise disjoint, each $\mathcal{V}_m^n$ is an open refinement of $\mathcal{U}_m^n$ such that the collection $\{\bigcup\mathcal{V}_m^n(C) : C \in \mathcal{C}_m^n\}$ is a collection of pairwise disjoint "tubes" contained in the product $X \times Y$. Finally, since the $\bigcup\{\mathcal{C}_m^n : n \in \mathbb{N}\}$ covers X, define the set $\mathcal{V}_m = \bigcup\{\mathcal{V}_m^n : n \in \mathbb{N}\}$ to obtain an open cover of all of $X \times Y$.

Suppose that for each $m \in \mathbb{N}$ an open cover $\mathcal{V}_m$ of $X \times Y$ has been constructed in this manner. Fix $m \in \mathbb{N}$, and let $y \in Y$ be fixed but arbitrary. Since $X \times \{y\}$ is compact, a finite subcover $\mathcal{V}_m(y)$ of $X \times \{y\}$ may be chosen from $\mathcal{V}_m$ such that for each $C \times B_k \in \mathcal{V}_m(y)$ the point $y \in B_k$. As before, define $B(y) = \bigcap\{B_k : C \times B_k \in \mathcal{V}_m(y)\}$ to obtain an open subset B(y) of Y with $y \in B(y)$. Thus, by constructing B(y) for each $y \in Y$, and defining $\mathcal{B}_m = \{B(y) : y \in Y\}$, an open cover $\mathcal{B}_m$ of Y is obtained.

Suppose, in this manner, that for each $m \in \mathbb{N}$ such an open cover $\mathcal{B}_m$ has been constructed. Since Y has property C, for each $m \in \mathbb{N}$ a C-refinement $\mathfrak{D}_m$ of each $\mathfrak{B}_m$ may be chosen such that the $\bigcup\{\mathfrak{D}_m : m \in \mathbb{N}\}$ forms an open cover of Y. For each $m \in \mathbb{N}$ and every $D \in \mathfrak{D}_m$ choose a single point $y \in Y$ such that $D \subset B(y) \in \mathbb{B}_m$. For each such D and y define

$$\mathcal{W}_m(D) = \{C \times D : C \times B_k \in \mathcal{V}_m(y)\},$$

and then decompose $\mathcal{W}_m(D)$ into

$$\mathcal{W}_m(D) = \bigcup\{\mathcal{W}_m^n(D) : n \in \mathbb{N}\}$$

where for each $n \in \mathbb{N}$

$$\mathcal{W}_m^n(D) = \{C \times D : C \times B_k \in \mathcal{V}_m(y) \cap \mathcal{V}_m^n\}.$$

Thus, for each $D \in \mathfrak{D}_m$ the set $\mathcal{W}_m^n(D)$ is an open refinement of $\mathcal{V}_m^n$, and hence also of $\mathcal{U}_m^n$, such that $\mathcal{W}_m(D)$ covers the product $X \times D$.

Finally, set $\mathcal{W}_m^n = \bigcup\{\mathcal{W}_m^n(D) : D \in \mathfrak{D}_m\}$ to obtain an open refinement of $\mathcal{U}_m^n$. Furthermore, since each $\mathcal{W}_m(D)$ covers $X \times D$, where $D \in \mathfrak{D}_m$, and since the $\bigcup\{\mathfrak{D}_m : m \in \mathbb{N}\}$ covers all of the space Y, the union $\bigcup\{\mathcal{W}_m^n : m \in \mathbb{N},\, n \in \mathbb{N}\}$ is easily seen to cover all of the product $X \times Y$.

It only remains to show that for each $m \in \mathbb{N}$ and every $n \in \mathbb{N}$ the elements of $\mathcal{W}_m^n$ are pairwise disjoint. This is fairly immediate since any element of $\mathcal{W}_m^n$ has the form $C \times D$ where $C \in \mathcal{C}_m^n$ and $D \in \mathfrak{D}_m$. Since $\mathcal{C}_m$ and $\mathfrak{D}_m$ are collections of pairwise disjoint open sets of X and Y respectively, the elements of $\mathcal{W}_m^n$ are clearly pairwise disjoint.

Thus, for each $m \in \mathbb{N}$ and every $n \in \mathbb{N}$ it has been shown that $\mathcal{W}_m^n$ is a C-refinement of $\mathcal{U}_m^n$ such that the $\bigcup\{\mathcal{W}_m^n : m \in \mathbb{N},\, n \in \mathbb{N}\}$ covers all of the product $X \times Y$. Thus, the product $X \times Y$ has been shown to have property C which completes the proof of the theorem.

<u>2.2.18</u> <u>Corollary.</u> Let P denote R. Pol's compactum and let $n \in \mathbb{N}$ be fixed. The n-fold product P^n of P has property C.

<u>Proof.</u> This is obvious from the theorem 2.2.17.

It should be noted that the compactification procedure used to construct P involves adding non-degenerate polyhedra to the totally disconnected space X. Thus, P^{∞} is SID and cannot have property C.

This section is ended with some related questions for further research. The following is related to the question asked in 2.2.14.

<u>2.2.19</u> <u>Question.</u> If X and Y are spaces having property C where X is compact, then must their product $X \times Y$ always have property C?

<u>2.2.20</u> <u>Question.</u> Let $f: X \to Y$ be a closed (open and closed) mapping between spaces X and Y. If the space Y has property C, and if for every $y \in Y$ the fiber $f^{-1}(y)$ has property C, then must the space X also have property C?

An affirmative answer to 2.2.20, which was motivated by 2.2.10, would also give an affirmative answer to 2.2.19. It should also be noted that if property C in 2.2.20 is replaced by countable-dimensionality, then the answer is known to be no [R. Pol (1983)].

2.3 Property C and Open Mappings with Finite Fibers

A classic question of finite-dimensional dimension theory considers the existence of mappings which raise or lower dimension. One class of maps which has been extensively studied with regard to these questions is that of open mappings. In this section, results are obtained for the analogous questions concerning infinite-dimensional spaces.

2.3.1 Definition. A mapping $f:X\rightarrow Y$ is called an open mapping if f maps each open set $A\subset X$ to an open set $f(A)\subset Y$.

Standard examples of open mappings include the projections mappings of product spaces onto their factor spaces.

2.3.2 [**Hausdorff** (1934)] Every separable metric space X is the image of some subset Z of the irrationals by an open mapping $f:Z\rightarrow X$.

Since 2.3.2 implies that the Hilbert cube can be written as the image of a zero-dimensional space by an open mapping, it is seen that arbitrary open mappings need not preserve countable-dimensionality, property C, or weak infinite-dimensionality. The next statements show that even the strong restriction to the class of open mappings which have countable fibers fails to prevent those mappings from arbitrarily raising dimension.

2.3.3 Definitions. A fiber of a mapping $f:X \rightarrow Y$ is the inverse image $f^{-1}(y)$ of a point $y \in Y$. If the cardinality of each fiber $|f^{-1}(y)|$ is countable, then f is said to have countable fibers. Similarly, if the cardinality of each fiber $|f^{-1}(y)|$ of f is finite, then f is said to have finite fibers.

2.3.4 [**Roberts** (1947)] If $f:Z \rightarrow X$ is an open mapping between separable metric spaces, then there exists a subset $A \subset Z$ such that the restriction $f|_A : A \rightarrow X$ is an open mapping with countable fibers onto X.

Thus, 2.3.2 and 2.3.4, when taken together, imply that the Hilbert cube may still be written as the image of a zero-dimensional space by an open mapping with countable fibers. Although open mappings with countable fibers defined on metric spaces may raise or lower dimension in an arbitrary manner, it might be possible to place enough structure on the spaces involved to obtain preservation of dimension. The following question is a generalization of a result due to Alexandroff [Alexandroff (1936)].

2.3.5 Question. If $f: \rightarrow Y$ is an open mapping with countable fibers between locally compact metric spaces, then must X have property C if and only if Y has property C?

Next, open mappings with finite fibers are considered. The following lemma will be needed in the proof of the main theorem of the section.

<u>2.3.6</u> <u>Lemma.</u> [**Nagami** (1960)] Let $f:X\to Y$ be an open mapping with finite fibers between spaces. For each $j\in\mathbb{N}$ define

a) $Y_j=\{y\in Y:|f^{-1}(y)|=j\}$,

b) $X_j=f^{-1}(Y_j)$, and

c) $f_j=f|_{X_j}:X_j\to Y_j$.

Then:

1. For each $n\in\mathbb{N}$ the $\bigcup\{Y_j:j=1,\cdots,n\}$ is closed in Y.
2. For each $j\in\mathbb{N}$ the mapping $f_j:X_j\to Y_j$ is a local homeomorphism.

<u>2.3.7</u> <u>Definition.</u> A property P of metric spaces will be called <u>similar to property C</u> if the property P satisfies:

1. The property P is hereditary to closed subspaces.
2. If $X=\bigcup\{X_j:j\in\mathbb{N}\}$, where each X_j is a closed subset of X that has the property P, then the space X also has the property P.
3. If each point of a space X has a neighborhood which has the property P, then the space X also has the property P.

This definition will be used to avoid redundancy of proofs. It should be mentioned that if the space X of condition 3 is separable, then condition 3 follows from the first two conditions in the obvious manner. Although condition 3 of 2.3.7 appears to be a very strong assumption, as the following lemma shows, it is satisfied by many topological properties.

<u>2.3.8</u> <u>Lemma.</u> If P is a property of metric spaces which satisfies the conditions 1 and 2 of 2.3.7, and which in addition, satisfies the condition

3′. For any discrete collection $\{X_\alpha : \alpha \in \Gamma\}$ of closed subspaces of a space X where each X_α has the property P, the $\bigcup\{X_\alpha : \alpha \in \Gamma\}$ also has the property P,

then the property P is similar to property C.

<u>Proof.</u> Let P be a property as per the hypothesis of 2.3.8. It is enough to show that the property P satisfies condition 3 of 2.3.7.

Suppose that $\mathcal{U}$ is an open cover of a space X by sets which have the property P. Since X is metric, $\mathcal{U}$ has a σ-discrete closed refinement $\mathcal{V} = \bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ where each $\mathcal{V}_n$ is a discrete collection of closed sets in X, which also covers X.

Fix $n \in \mathbb{N}$, and let $V \in \mathcal{V}_n$. Since there exists $U \in \mathcal{U}$ with $V \subset U$ as a closed subset, condition 1 of 2.3.7 implies that V has the property P. Condition 3′ then implies that the $\bigcup\{V : V \in \mathcal{V}_n\}$ has the property P. Finally, condition 2 of 2.3.7 implies that $X = \bigcup\{V : V \in \mathcal{V}_n, n \in \mathbb{N}\}$ also has the property P. This completes the proof.

<u>2.3.9</u> <u>Theorem.</u> Let P be any property of metric spaces which is similar to property C and let $f : X \rightarrow Y$ be an open mapping with finite fibers between metric spaces. The space X has the property P if and only if the space Y has the property P.

<u>Proof.</u> As in the lemma 2.3.6, for each $j \in \mathbb{N}$ define $Y_j = \{y \in Y : |f^{-1}(y)| = j\}$, $X_j = f^{-1}(Y_j)$, and $f_j = f|_{X_j} : X_j \rightarrow Y_j$. Thus,

the lemma 2.3.6 implies that for each $n \in \mathbb{N}$ the $\bigcup\{Y_j : j = 1, \cdots, n\}$ is closed in Y, and thus that each Y_j is an $\mathcal{F}_\sigma$ subset in Y. Therefore, each X_j is also an $\mathcal{F}_\sigma$ set in X.

Furthermore, from 2.3.6, for each $j \in \mathbb{N}$ the map $f_j : X_j \to Y_j$ is a local homeomorphism. Therefore, a cover $\{U_\alpha : \alpha \in \Gamma_j\}$ of X_j by open sets of X_j may be chosen such that for each $\alpha \in \Gamma_j$ the restriction $f_j : U_\alpha \to f_j(U_\alpha)$ is a homeomorphism. Moreover, since $X_j = f^{-1}(Y_j)$, each f_j remains an open mapping, and then for each $\alpha \in \Gamma_j$ the image $f_j(U_\alpha)$ will be an open subset in Y_j. Therefore, the collection $\{f_j(U_\alpha) : \alpha \in \Gamma_j\}$ is a cover of Y_j by open subsets of Y_j.

Suppose that the space X has the property P, then from conditions 1 and 2 of 2.3.7 every $\mathcal{F}_\sigma$ subset of X also has the property P. In particular, for each $j \in \mathbb{N}$ the subspace X_j has the property P. By the same argument, for each $\alpha \in \Gamma_j$ U_α, being open in X_j, is an $\mathcal{F}_\sigma$ subset of X_j, and thus has the property P.

Since for each $j \in \mathbb{N}$ and every $\alpha \in \Gamma_j$ $f(U_\alpha)$ is homeomorphic to U_α, the collection $\{f_j(U_\alpha) : \alpha \in \Gamma_j\}$ is a cover of Y_j by open subsets of Y_j, each of which has the property P. Thus, condition 3 of 2.3.7 implies that each Y_j has the property P. Finally, since $Y = \bigcup\{Y_j : j \in \mathbb{N}\}$, condition 2 of 2.3.7 implies that the space Y also has the property P.

The proof of the converse is similar. Suppose that the space Y has the property P, then since each Y_j is an $\mathcal{F}_\sigma$ subset of Y, each Y_j also has the property P. Thus, for each $j \in \mathbb{N}$ and every $\alpha \in \Gamma_j$ $f_j(U_\alpha)$, being an open, and hence $\mathcal{F}_\sigma$, subset of Y_j, also has the property P.

Since for each $j \in \mathbb{N}$ and every $\alpha \in \Gamma_j$ $f_j(U_\alpha)$ is homeomorphic to U_α, the collection $\{U_\alpha : \alpha \in \Gamma_j\}$ is a cover of X_j by open subsets of X_j, each of which has the property P, and thus condition 3 of 2.3.7 implies that each X_j also has the property P. Finally, since $X = \bigcup\{X_j : j \in \mathbb{N}\}$, condition 2 of 2.3.7 implies that the space Y must also have the property P. This completes the proof.

Corollaries. Let X and Y be as in 2.3.9 with $f : X \rightarrow Y$ being an open mapping with finite fibers between X and Y.

2.3.10 [**Nagami** (1960)] The $\dim X \leq n$ if and only if the $\dim Y \leq n$.

2.3.11 [**Arhangel'skiĭ** (1966)] The space X is CD if and only if the space Y is CD.

2.3.12 [**Polkowski** (1983)] The space X is WID if and only if the space Y is WID.

2.3.13 The space X has property C if and only if the space Y has property C.

Proofs. It is clear from the definitions involved and the results presented in the introductory sections that all four properties of the corollaries satisfy conditions 1 and 2 of 2.3.7. The local condition 3 is also easily seen to be satisfied for all four properties by an application of 2.3.8. Thus, all four properties are similar to property C, and the results follow from the theorem 2.3.9. This completes the proofs.

The following related question remains open.

2.3.14 Question. Let $f:X \to Y$ be an open mapping between spaces so that each fiber is separable with no fiber dense in itself. Is it the case that the domain X must have property C if and only if the range Y has property C?

This section is closed with a brief discussion concerning the preservation of weak infinite-dimensionality in the sense of Smirnov by open mappings with finite fibers.

2.3.15 Theorem. The property of weak infinite-dimensionality in the sense of Smirnov is not similar to property C.

Proof. It is easy to see that the free union of n-cells I^n where $n \in \mathbb{N}$ is not S-WID. Thus, weak infinite-dimensionality in the sense of Smirnov does not satisfy conditions 2 or 3 of 2.3.7, and thus is not similar to property C.

Theorem 2.3.15 is another example of the failure of weak infinite-dimensionality in the sense of Smirnov to be an acceptable infinite-dimensional dimension theory. Although the techniques of 2.3.9 do not apply, a partial result is still known.

2.3.16 [**Polkowski** (1983)] Let $f:X \to Y$ be an open mapping with finite fibers between spaces X and Y. If X is S-WID then Y is also S-WID.

However, as the following example shows, the converse to 2.3.16 is false. A similar example, in a different form, was also known to Polkowski [**Polkowski** (1983), Remark 3.6].

2.3.17 Example. Consider the product induced metric topology on the space

$$\mathbb{R}_0^\omega = \{(x_1, x_2, x_3, \cdots) : x_k \in \mathbb{R},\ x_k = 0 \text{ for all but finitely many } k \in \mathbb{N}\}.$$

For each $n \in \mathbb{N}$ define subsets of $\mathbb{R}_0^\omega$ by

$$A_n = \{x \in \mathbb{R}_0^\omega : (x_1 + 2n - 1)^2 + x_2^2 + \cdots + x_n^2 \leq 1,\ x_k = 0 \text{ if } k > n\}$$

and

$$B_n = \{x \in \mathbb{R}_0^\omega : (x_1 - 2 + 3(\tfrac{1}{2})^n)^2 + x_2^2 + \cdots + x_n^2 \leq (\tfrac{1}{4})^n,\ x_k = 0 \text{ if } k > n\}.$$

Set $X = \bigcup\{A_n : n \in \mathbb{N}\}$ and $Y = \bigcup\{B_n : n \in \mathbb{N}\} \cup \{(2, 0, 0, \cdots)\}$. Since Y is compact and CD, Y is S-WID. However, the space X, while also CD, is easily seen to not be S-WID. Thus, since X is a closed subspace of $X \cup Y$, the union $X \cup Y$ cannot be S-WID.

Define a function $f : X \cup Y \to Y$ by

$$f(x) = \begin{cases} e_n(x) & \text{if } x \in A_n,\ n \in \mathbb{N} \\ x & \text{if } x \in Y \end{cases}$$

where for each $n \in \mathbb{N}$ $e_n : A_n \to B_n$ is the homeomorphism given by

$$e_n(x) = \left(-\frac{x_1 - 2n + 2}{2^n}, \frac{x_2}{2^n}, \cdots, \frac{x_n}{2^n} \right).$$

It is easy to check that f is an open mapping with finite fibers. Indeed, f is two-to-one except at the point $(2, 0, 0, \cdots)$ where f is one-to-one. Thus, f is an open mapping with finite fibers onto a S-WID space whose domain fails to be S-WID.

2.4 Property C and Refinable Maps on Compacta

In this section, the dimension preserving nature of refinable maps is investigated. The main result of this section shows that the image of a refinable map defined on a compactum which has property C must also have property C. In addition, a covering characterization of weak infinite-dimensionality is given. Using this characterization, results of Patten and Kato are also obtained.

2.4.1 Definition. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Gamma\}$ be a collection of open subsets of a space X. A collection $\mathcal{V} = \{V_\alpha : \alpha \in \Gamma\}$ is a precise pairwise disjoint open shrinkage of $\mathcal{U}$ if $\mathcal{V}$ satisfies:

1. For each $V_\alpha \in \mathcal{V}$ the set V_α is open in X.
2. For any $\alpha \in \Gamma$ and $\beta \in \Gamma$ with $\alpha \neq \beta$, $V_\alpha \cap V_\beta = \varnothing$.
3. For each $\alpha \in \Gamma$, $V_\alpha \subset \overline{V}_\alpha \subset U_\alpha$.

2.4.2 Theorem. A space X is WID if and only if for any sequence of binary open covers $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of X there exists for each $n \in \mathbb{N}$ a precise pairwise disjoint open shrinkage $\mathcal{V}_n$ of $\mathcal{U}_n$ such that the $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ forms an open cover of X.

Proof. Suppose that X is WID, and for each $n \in \mathbb{N}$ let a binary open cover $\mathcal{U}_n = \{U_n^1, U_n^2\}$ of X be given. For each $n \in \mathbb{N}$ define $A_n = X \setminus U_n^2$ and $B_n = X \setminus U_n^1$. Thus, each (A_n, B_n) is a pair of disjoint closed sets of X. Since the space X is WID, for each $n \in \mathbb{N}$ a closed subset $S_n \subset X$ which separates the pair (A_n, B_n) in X may be chosen such that the $\bigcap\{S_n : n \in \mathbb{N}\} = \varnothing$.

Thus, since for each $n \in \mathbb{N}$ S_n separates the pair (A_n, B_n) in X, there exist open sets V_n^1 and V_n^2 of X such that $S_n = X \setminus (V_n^1 \cup V_n^2)$, $V_n^1 \cap V_n^2 = \varnothing$, $A_n \subset V_n^1$ and $B_n \subset V_n^2$. Then, for each $n \in \mathbb{N}$ the inclusions

$$V_n^1 \subset \overline{V}_n^1 \subset X \setminus V_n^2 \subset X \setminus B_n = U_n^1 \quad \text{and} \quad V_n^2 \subset \overline{V}_n^2 \subset X \setminus V_n^1 \subset X \setminus A_n = U_n^2$$

are obtained. Thus, $\mathcal{V}_n = \{V_n^1, V_n^2\}$ is seen to be a precise pairwise disjoint open shrinkage of $\mathcal{U}_n$.

Finally, the

$$\begin{aligned} \bigcup\{V_n^1 \cup V_n^2 : n \in \mathbb{N}\} &= X \setminus [\bigcap\{X \setminus (V_n^1 \cup V_n^2) : n \in \mathbb{N}\}] \\ &= X \setminus [\bigcap\{S_n : n \in \mathbb{N}\}] \\ &= X, \end{aligned}$$

and thus the $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ has been shown to be an open cover of X.

Assume the hypothesis of the converse and let $\{(A_n, B_n) : n \in \mathbb{N}\}$ be a given ω-family of pairs of disjoint closed subsets of X. For each $n \in \mathbb{N}$ apply normality to choose open sets U_n^1 and U_n^2 such that

$$A_n \subset U_n^1 \subset \overline{U}_n^1 \subset X \setminus B_n \quad \text{and} \quad B_n \subset X \setminus U_n^1 \subset U_n^2 \subset \overline{U}_n^2 \subset X \setminus A_n,$$

and thus, for each $n \in \mathbb{N}$ $\mathcal{U}_n = \{U_n^1, U_n^2\}$ is seen to be a binary open cover of X. Therefore, by the hypothesis, a precise pairwise disjoint open shrinkage $\mathcal{V}_n = \{V_n^1, V_n^2\}$ of each $\mathcal{U}_n$ may be chosen such that the $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ forms a cover of the space X.

For each $n \in \mathbb{N}$ define

$$S_n = X \setminus \{V_n^1 \cup (X \setminus \overline{U}_n^2) \cup V_n^2 \cup (X \setminus \overline{U}_n^1)\}.$$

It is clear that each S_n is a closed subset of X which separates the pair (A_n, B_n) in X. Finally, the

$$\bigcap\{S_n : n \in \mathbb{N}\} = \bigcap\{X \setminus (V_n^1 \cup (X \setminus \bar{U}_n^2) \cup V_n^2 \cup (X \setminus \bar{U}_n^1)) : n \in \mathbb{N}\}$$
$$\subset X \setminus \bigcup\{V_n^1 \cup V_n^2 : n \in \mathbb{N}\}$$
$$= \varnothing,$$

and thus the space X has been shown to be WID.

The following related results are obtained in a similar manner.

2.4.3 Theorem. Let X be a given space. The $\dim X \leq n$ if and only if for any given collection $\{\mathcal{U}_k : k = 1, \cdots, n+1\}$ of $n+1$ binary open covers of X there exists for each $k \in \{1, \cdots, n+1\}$ a precise pairwise disjoint open shrinkage $\mathcal{V}_k$ of $\mathcal{U}_k$ such that the $\bigcup\{\mathcal{U}_k : k - 1, \cdots, n+1\}$ forms a cover of X.

Proof. The proof is an obvious modification of the proof of 2.4.2 as applied to the characterization of dimension which was given in 1.1.4, and thus will be omitted.

2.4.4 Theorem. A space X is CD if and only if for any given sequence of binary open covers $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of X there exists for each $n \in \mathbb{N}$ a precise pairwise disjoint open shrinkage $\mathcal{V}_n = \{V_n^1, V_n^2\}$ of $\mathcal{U}_n = \{U_n^1, U_n^2\}$ such that for each point $x \in X$ the point $x \in V_n^1 \cup V_n^2$ for all but finitely many $n \in \mathbb{N}$.

Proof. The proof is an obvious modification of the proof of 2.4.2 as applied to the characterization of countable-dimensionality which was given in 1.2.3, and thus will be omitted.

Refinable maps were originally defined in [Ford and Rogers (1978)], and their dimension preserving nature has been investigated in [Patten (1982)] and [**Kato** (1983)]. The main result of this section states that refinable maps between compacta preserve property C. The results of Patten and Kato will also be obtained through a simple modification of proof using theorems 2.4.2 and 2.4.3.

2.4.5 Notation. For any metric space Y the distance between two points x and y of Y will be denoted by $d(x, y)$. If $A \subset Y$, then the diameter of A is defined and denoted by the

$$\operatorname{diam} A = \sup\{d(x, y) : x \in Y,\ y \in Y\}.$$

If $f: X \to Y$ and $g: X \to Y$ are two maps between compacta, then the distance between the two maps will mean the supremum metric which is defined and denoted by

$$\rho(f, g) = \sup\{d(f(x), g(x)) : x \in X\}.$$

2.4.6 Definition. Let $\epsilon > 0$ be given. A map $f: X \to Y$ between compacta is said to be an ϵ-map if for each $y \in Y$ the $\operatorname{diam} f^{-1}(y) < \epsilon$.

2.4.7 Definition. A map $r: X \to Y$ between compacta is refinable if for any $\epsilon > 0$ there exists a surjective ϵ-map $r_\epsilon: X \to Y$, called an ϵ-refinement of r, such that the $\rho(r, r_\epsilon) < \epsilon$.

Some preliminary results concerning the limit supremum of a sequence of closed sets in a compactum will be used in the proofs.

2.4.8 Definition. For each $j \in \mathbb{N}$ let A_j be a closed set in a compactum X. The limit supremum of the sequence $\{A_j : j \in \mathbb{N}\}$ is denoted by the $\limsup\{A_j : j \in \mathbb{N}\}$ and is defined to be the set of all $x \in X$ such that any open subset U of X with $x \in U \subset X$ has $U \cap A_j \neq \emptyset$ for infinitely many $j \in \mathbb{N}$. The limit supremum may also be characterized in terms of sequences by the

$$\limsup\{A_j : j \in \mathbb{N}\} = \{x \in X : \forall\, k \in \mathbb{N}\ \exists\, j_k \in \mathbb{N}\ \exists\, x_{j_k} \in A_{j_k} \text{ s.t. } x_{j_k} \to x\}.$$

2.4.9 Proposition. Let $\{A_j : j \in \mathbb{N}\}$ be a sequence of closed sets in a compactum X. If U is an open subset of X with the $\limsup\{A_j : j \in \mathbb{N}\} \subset U$, then there exists an $N \in \mathbb{N}$ such that for each $n \geq N$ the set $A_n \subset U$.

Proof. Suppose not, then for all $N \in \mathbb{N}$ there exists an integer $n \geq N$ and a point $x_n \in A_n \setminus U$. For each $N \in \mathbb{N}$, choose such a point x_n, and then, by compactness, extract a convergent subsequence $x_{n_k} \to x \in X$. From 2.4.8 and the hypothesis, the point $x \in \limsup\{A_j : j \in \mathbb{N}\} \subset U$, but then, by the convergence of the subsequence, for all sufficiently large k $x_{n_k} \in U$ which is a contradiction. Therefore, there must exist an $N \in \mathbb{N}$ such that $A_n \subset U$ for each $n \geq N$. This completes the proof.

The following technical lemma will also be needed in the proof of the main theorem of this section.

2.4.10 Lemma. [**Loncar and Mardešić** (1968)] Let $f:X\to A$ be a map from a compactum X to an ANR A. Given any $\epsilon>0$ there exists a $\delta>0$ such that for any surjective δ-map $g:X\to Y$ there exists a map $h:Y\to A$ with the $\rho(f, hg)<\epsilon$.

2.4.11 Theorem. Let $r:X\to Y$ be a refinable map between compacta. If the compactum X has property C, then Y must also have property C.

Proof. For each $n\in\mathbb{N}$ let $\mathcal{U}_n$ be a given open cover of Y. By the compactness of Y, it may be assumed without loss of generality that each $\mathcal{U}_n$ is of the form $\mathcal{U}_n=\{U_n^j:j=1,\cdots,m_n\}$ where each $m_n\in\mathbb{N}$. Thus, for each $n\in\mathbb{N}$ $r^{-1}(\mathcal{U}_n)\equiv\{r^{-1}(U_n^j):j=1,\cdots,m_n\}$ defines an open cover of X.

Since X has property C, and by the compactness of X, an integer N and for each $n\in\{1,\cdots,N\}$ a collection of open sets $\mathcal{V}_n$ of X may be chosen such that:

1. For each $n\in\{1,\cdots,N\}$ $\mathcal{V}_n=\{V_n^j:j=1,\cdots,m_n\}$ is a collection of open subsets of X with pairwise disjoint closures.
2. For each $n\in\{1,\cdots,N\}$ and every $j\in\{1,\cdots,m_n\}$ $V_n^j\subset\overline{V}_n^j\subset r^{-1}(U_n^j)$.
3. The $\bigcup\{\mathcal{V}_n:n=1,\cdots,N\}$ forms an open cover of X.

For each $n\in\{1,\cdots,N\}$ let A_n denote the wedge of m_n intervals, wedged at a common endpoint, and label the remaining endpoints $a_1, \cdots, a_{m_n}$. To avoid ambiguity, an endpoint a_j will always be referred to together with the A_n of which it is an endpoint. Each A_n is of course an ANR.

Indeed each A_n is an AR, and thus, by the compactness of X, for each $n \in \{1, \cdots, N\}$ maps $f_n : X \to A_n$ may be chosen such that for any $j \in \{1, \cdots, m_n\}$

$$f_n^{-1}(a_j) = r^{-1}(Y \setminus \bigcup\{U_n^k : k \neq j\}) \cup \overline{V}_n^j .$$

Since $f_n^{-1}(a_j) \subset r^{-1}(U_n^j)$, each $f_n(r^{-1}(Y \setminus U_n^j))$ is a closed subset of A_n which is disjoint from a_j.

For each $n \in \{1, \cdots, N\}$ complete normality may be used for each $j \in \{1, \cdots, m_n\}$ to choose open subsets M_n^j and N_n^j of A_n such that:

1. The point $a_j \in N_n^j$.
2. For each $n \in \{1, \cdots, N\}$, $\{N_n^j : j = 1, \cdots, m_n\}$ is a collection of pairwise disjoint sets.
3. The set $f_n(r^{-1}(Y \setminus U_n^j)) \subset M_n^j$.
4. The distance $d(\overline{M}_n^j, \overline{M}_n^j) \equiv D_n^j > 0$.

Fix $k \in \mathbb{N}$ and $n \in \{1, \cdots, N\}$. For each such n and k the lemma 2.4.10 guarantees the existence of a number $\delta_{n_k} > 0$ such that for any surjective δ_{n_k}-map $g : X \to Y$ there exists a map $h_{n_k} : Y \to A_n$ such that

$$\rho(f_n, h_{n_k} g) < \frac{1}{k} .$$

In particular, set $\delta_k = \min\{\frac{1}{k}, \delta_{n_k} : n = 1, \cdots, N\}$. By the refinability of r, a map $r_k : X \to Y$ may be chosen such that r_k is a δ_k-refinement of the map r. Since r_k is then also a δ_{n_k}-refinement of r for each $n \in \{1, \cdots, N\}$, the lemma 2.4.10 may be applied to obtain maps $h_{n_k} : Y \to A_n$

such that the distance

$$\rho(f_n, h_{n_k} r_k) < \frac{1}{k}.$$

Thus, by doing this procedure for each $k \in \mathbb{N}$, two sequences of maps are constructed. The sequence $(r_k : k \in \mathbb{N})$ is a sequence of $\frac{1}{k}$-refinements of the refinable map r, and for each $n \in \{1, \cdots, N\}$ the sequence $(h_{n_k} : k \in \mathbb{N})$ is a sequence of maps from Y to A_n where for each $k \in \mathbb{N}$ the distance

$$\rho(f_n, h_{n_k} r_k) < \frac{1}{k}.$$

<u>2.4.12</u> <u>Claim.</u> For each fixed $n \in \{1, \cdots, N\}$ and $j \in \{1, \cdots, m_n\}$ the

$$\limsup \{r_k^{-1}(Y \setminus U_n^j) : k \in \mathbb{N}\} \subset r^{-1}(Y \setminus U_n^j) \subset f_n^{-1}(M_n^j).$$

Indeed, given any $x \in \limsup \{r_k^{-1}(Y \setminus U_n^j) : k \in \mathbb{N}\}$, by 2.4.8 there exists a sequence of elements $x_{k_t} \in r_{k_t}^{-1}(Y \setminus U_t^j)$ where $t \in \mathbb{N}$ such that $x_{k_t} \to x$ as $t \to \infty$. Thus, the following inequalities are obtained;

$$\begin{aligned} d(r(x), Y \setminus U_n^j) &\leq d(r_{k_t}(x_{k_t}), Y \setminus U_n^j) + d(r_{k_t}(x_{k_t}), r(x_{k_t})) + d(r(x_{k_t}), r(x)) \\ &< 0 + \frac{1}{k_t} + d(r(x_{k_t}), r(x)) \\ &\to 0 \end{aligned}$$

as $t \to \infty$, $k_t \to \infty$, and $x_{k_t} \to x$.

Thus, since $Y \setminus U_n^j$ is closed in Y, the image $r(x) \in Y \setminus U_n^j$. This computation, together with condition 3 on the choice of M_n^j, gives the claimed inclusions and completes the proof of the claim.

It is now possible to choose $m \in \mathbb{N}$ sufficiently large enough that for each $n \in \{1, \cdots, N\}$ and every $j \in \{1, \cdots, m_n\}$

1. $\frac{1}{m} < d(a_j, A_n \setminus N_n^j)$,
2. $\frac{1}{m} < D_n^j$, and
3. $r_m^{-1}(Y \setminus U_n^j) \subset f_n^{-1}(M_n^j)$.

Condition 3 may be satisfied by applying 2.4.9 and 2.4.12 for all values of n and j.

Finally, for each $n \in \{1, \cdots, N\}$ and every $j \in \{1, \cdots, m_n\}$ define

$$W_n^j = h_{n_m}^{-1}(N_n^j) \quad \text{and} \quad \mathcal{W}_n = \{W_n^j : j = 1, \cdots, m_n\}.$$

Since for each $n \in \{1, \cdots, N\}$ the collection $\{N_n^j : j = 1, \cdots, m_n\}$ is a collection of pairwise disjoint open sets of A_n, it is seen that each $\mathcal{W}_n$ is also a collection of pairwise disjoint open sets of Y. It remains to be shown that for each $n \in \{1, \cdots, N\}$ the collection $\mathcal{W}_n$ refines $\mathcal{U}_n$, and that the $\bigcup\{\mathcal{W}_n : n = 1, \cdots, N\}$ covers Y.

Indeed, each $\mathcal{W}_n$ is actually a shrinkage of $\mathcal{U}_n$. For suppose that $y \in \overline{W}_n^j$, but that $y \in Y \setminus U_n^j$. Since r_m is surjective, a point $x \in r_m^{-1}(y)$ may be chosen. If $y \in Y \setminus U_n^j$ then condition 3 on the choice of m gives that $f_n(x) \in M_n^j$. On the other hand, since $W_n^j = h_{n_m}^{-1}(N_n^j)$ it is seen that $h_{n_m}(\overline{W}_n^j) \subset \overline{N}_n^j$. Thus,

$$h_{n_m}(y) = h_{n_m} r_m(x) \in \overline{N}_n^j.$$

But, then the

$$\begin{aligned} d(f_n(x), h_{n_m} r_m(x)) &\geq d(M_n, \overline{N}_n^j) \\ &\geq D_n^j \\ &> \tfrac{1}{m}, \end{aligned}$$

which is a contradiction. Therefore, it must be that $W_n^j \subset \overline{W}_n^j \subset U_n^j$, and hence, for each $n \in \{1, \cdots, N\}$ the collection $\mathcal{W}_n$ has been shown to be a shrinkage of $\mathcal{U}_n$.

Let $y \in Y$ be fixed but arbitrary, and choose a point $x \in r_m^{-1}(y)$. Then, since the $\bigcup\{\mathcal{V}_n : n = 1, \cdots, N\}$ covers the compactum X, integers $n \in \{1, \cdots, N\}$ and $j \in \{1, \cdots, m_n\}$ may be chosen such that the point $x \in V_n^j$. However, from the computation

$$\begin{aligned} d(f_n(x), h_{n_m} r_m(x) &= d(a_j, h_{n_m}(y)) \\ &< \tfrac{1}{m} < d(a_j, A_n \setminus N_n^j), \end{aligned}$$

it is seen that $h_{n_m}(y) \in N_n^j$. Thus, $y \in h_{n_m}^{-1}(N_n^j) = W_n^j$.

Since this shows that the $\bigcup\{\mathcal{W}_n : n = 1, \cdots, N\}$ forms an open cover of Y, the space Y is seen to have property C. This completes the proof.

Slight modifications of proof give the following theorems alluded to at the beginning of the section.

<u>2.4.13</u> <u>Theorem.</u> [**Kato** (1983)] Let $r: X \to Y$ be a refinable map between compacta. If X is WID then Y is also WID.

<u>Proof.</u> The result is obtained by using binary open covers in the proof of 2.4.11 as in the characterization of weak infinite-dimensionality given in 2.4.2.

<u>2.4.14</u> <u>Theorem.</u> [**Patten** (1982)] Let $r: X \to Y$ be a refinable map between compacta. If the $\dim X < n$ then the $\dim Y < n$.

Proof. The result is obtained by using the characterization of dimension given in 2.4.3 in the proof of 2.4.10.

This section is ended with a related open problem. In light of 2.4.11 and 2.4.12, the characterization of countable-dimensionality which was given in 2.4.4 might possibly be used to provide an answer.

2.4.13 Question. Let $r: X \rightarrow Y$ be a refinable map between compacta. If the domain X is CD, then must the range Y also be CD?

2.5 Hereditary Shape Equivalences on WID Spaces

In this section, the relationships between possible images of a cell-like map which raises dimension and the classification of infinite-dimensional spaces are investigated. The work of Kozlowski and Ancel is discussed. A result of Ancel is extended and related questions important to the subject are asked.

2.5.1 Definition. A map $f:X \to Y$ between spaces is called cell-like if it is proper, and if for every $y \in Y$, $f^{-1}(y)$ has the shape of the point $\{y\}$, i.e. for any ANR Z the induced function on the homotopy classes

$$f^{\#}:[\{y\}, Z] \to [f^{-1}(y), Z]$$

is a bijection.

The following is one of the major unsolved problems of topology. Answering it, either positively or negatively, would give important results in manifold decomposition theory, shape theory, ANR theory, as well as in dimension theory.

2.5.2 Question. Does there exist a cell-like map $f:X \to Y$ between spaces such that the $\dim Y > \dim X$?

A more complete discussion of this cell-like dimension-raising map question may be found in [Schori (1980)]. In particular, it is known that the image of a cell-like dimension-raising map must be infinite-dimensional.

Thus, it becomes important to determine what type of infinite-dimensional space could be such a possible image.

2.5.3 Definition. A map $f:X \rightarrow Y$ between spaces is called a shape equivalence if for every ANR Z, the induced function on homotopy classes

$$f^{\#}:[Y, Z] \rightarrow [X, Z]$$

is a bijection. A map $f:X \rightarrow Y$ between spaces is called an hereditary shape equivalence if for every closed set $A \subset Y$, the restriction $f|_{f^{-1}(A)}:f^{-1}(A) \rightarrow A$ is a shape equivalence.

It is obvious from the definitions that any proper hereditary shape equivalence is a cell-like map. Indeed, Kozlowski defined the hereditary shape equivalence as a generalization of the cell-like map to obtain the following theorem.

2.54 [**Kozlowski**, (to appear)] Let $f:X \rightarrow Y$ be a cell-like map between spaces where the $\dim X < \infty$. The $\dim Y > \dim X$ if and only if f is not an hereditary shape equivalence.

Ancel has extensively studied the dimension preserving properties of cell-like maps and hereditary shape equivalences, as well as those of related maps such as fine homotopy equivalences.

2.5.5 Definition. A map $f:X \to Y$ between spaces is said to be approximately invertible if for some closed embedding $e:X \to Z$ of X into a space Z (and thus, for any closed embedding into an ANR) the embedding satisfies the following condition. Given any collection $\mathcal{W}$ of open sets of Z which is refined by the collection $\{e(f^{-1}(y)):y \in Y\}$, there exists a map $g:Y \to Z$ such that the composition $gf:X \to Z$ is $\mathcal{W}$-close to the map e, i.e. for each $x \in X$ there exists a set $W \in \mathcal{W}$ such that $e(x) \cup gf(x) \subset W$.

2.5.6 [**Ancel** (1985a)] Every proper hereditary shape equivalence is approximately invertible.

Using this Ancel proved the following theorem.

2.5.7 [**Ancel** (1985b)] Let $f:X \to Y$ be an approximately invertible map between spaces such that for each $y \in Y$ the fiber $f^{-1}(y)$ is compact. If the domain X has property C, then the image Y also has property C.

Ancel combined 2.5.6 with 2.5.7 and used another theorem from [**Ancel** (1985a)] to obtain the following result.

2.5.8 [**Ancel** (1985b)] Let $f:X \to Y$ be a cell-like map between spaces where the domain X has property C. The image Y has property C if and only if f is an hereditary shape equivalence.

<u>2.5.9</u> <u>Corollary.</u> The image of a cell-like dimension-raising map cannot have property C.

<u>Proof.</u> This is immediate from 2.5.8 and 2.5.4.

The main result of this section is an application of 2.4.2 which extends 2.5.7.

<u>2.5.10</u> <u>Theorem.</u> Let $f:X \to Y$ be an approximately invertible map between spaces such that for each $y \in Y$ the fiber $f^{-1}(y)$ is compact in the space X. If the domain X is WID then the image Y is also WID.

<u>Proof.</u> For each $n \in \mathbb{N}$ let $\mathcal{U}_n = \{U_n^1, U_n^2\}$ be a binary open cover of the space Y. Define $f^{-1}(U_n) = \{f^{-1}(U_n^1), f^{-1}(U_n^2)\}$, then for each $n \in \mathbb{N}$ $f^{-1}(U_n)$ is a binary open cover of X. Since X is WID, 2.4.2 can be applied to obtain a precise pairwise disjoint open shrinkage $\mathcal{V}_n = \{V_n^1, V_n^2\}$ of each U_n where $n \in \mathbb{N}$ such that the $\mathcal{V} = \bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ is an open cover of X.

Moreover, it may be assumed that $\mathcal{V}$ is locally finite since $\mathcal{V}$ may be replaced by a precise locally finite refinement which also covers X. Thus, given any $y \in Y$ and any $x \in f^{-1}(y)$ an open set $O_x \subset X$ may be chosen with $x \in O_x$ and with the

$$\operatorname{ord}_{O_x}\mathcal{V} = |\{V_n^j \in \mathcal{V} : V_n^j \cap O_x \neq \emptyset\} < \infty.$$

Fix $y \in Y$, then the collection $\{O_x : x \in f^{-1}(y)\}$ is an open cover of the non-empty compact fiber $f^{-1}(y)$. Thus, it is possible to extract a finite subcover $\{O_{x_j} : j = 1, \cdots, r_y\}$ of $f^{-1}(y)$ from $\{O_x : x \in f^{-1}(y)\}$. Define $O_y = \bigcup\{O_{x_j} : j = 1, \cdots, r_y\}$, then $O_y \subset X$ is open in X with $f^{-1}(y) \subset O_y$, and with the $\mathrm{ord}_{O_y} \mathcal{V} < \infty$.

<u>Claim.</u> Since the $\mathrm{ord}_{O_y} \mathcal{V} < \infty$ an open set $W_y \subset O_y$ may be chosen such that $f^{-1}(y) \subset W_y$, the $\mathrm{ord}_{W_y} \mathcal{V} < \infty$ and such that if $V_n^j \in \mathcal{V}_n$ with $V_n^j \cap W_y \neq \varnothing$ then $W_y \subset f^{-1}(U_n^j)$.

Indeed, define $N_n = \{n \in \mathbb{N} : \exists\, V_n^j \in \mathcal{V}_n,\ V_n^j \cap O_y \neq \varnothing\}$. If $n \in Ny$ is fixed, then there are two possibilities to consider.

If $y \in U_n^1 \cap U_n^2$, then $f^{-1}(y) \subset f^{-1}(U_n^1) \cap f^{-1}(U_n^2)$. In this case set

$$W_n = O_y \cap f^{-1}(U_n^1) \cap f^{-1}(U_n^2).$$

Then, W_n is an open subset of X with

$$f^{-1}(y) \subset W_n \quad \text{and} \quad W_n \subset f^{-1}(U_n^j)$$

for both $j = \{1, 2\}$.

Since $\mathcal{U}_n$ covers the space Y, the only other case is where the point y is in only one element of $\mathcal{U}_n$, say $y \in U_n^1 \setminus U_n^2$. Since $f^{-1}(y) \subset f^{-1}(U_n^2)$, and since $\overline{V}_n^2 \subset f^{-1}(U_n^2)$, the $f^{-1}(y) \cap \overline{V}_n^2 = \varnothing$. Thus, normality may be used to choose an open set $P_n \subset X$ with $f^{-1}(y) \subset P_n$ and $P_n \cap V_n^2 = \varnothing$. For this case, define

$$W_n = O_y \cap P_n \cap f^{-1}(U_n^1).$$

Then, W_n is open subset of X with

$$f^{-1}(y) \subset W_n, \quad V_n^2 \cap W_n = \varnothing, \quad \text{and } W_n \subset f^{-1}(U_n^1).$$

To finish the proof of the claim, define $W_y = \bigcap\{W_n : n \in N_y\}$. Since the $|N_y| < \infty$, the intersection Wy is an open subset of X, and clearly

$W_y \subset O_y$ with $f^{-1}(y) \subset W_y$. Since the $\mathrm{ord}_{O_y} \mathcal{V} < \infty$, this construction insures that the $\mathrm{ord}_{W_y} \mathcal{V} < \infty$. Suppose that $V_n^j \in \mathcal{V}_n$ with the $V_n^j \cap W_y \neq \emptyset$, then $n \in \mathbb{N}_y$ such that $W_y \subset W_n$. The choice of W_n assures that $W_n \subset f^{-1}(U_n^j)$, which then completes the proof of the claim.

The remainder of the proof follows Ancel's proof of 2.5.7. For each $y \in Y$ construct W_y as above. The collection $\{W_y : y \in Y\}$ can be used to form a collection of open sets $\mathcal{W}$ as in the definition of approximate invertibility such that $\mathcal{W}$ will be related to the original open covers $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of Y.

For simplicity consider X as a closed subspace of Z as per the approximate invertibility of f. Let $e : X \to Z$ denote inclusion. This inclusion map e will usually be omitted in statements if the context is clear. The metric on Z will be denoted by d.

For any subset $S \subset X$ let $\tilde{S} \subset Z$ be the subset defined by

$$\tilde{S} = \{z \in Z : d(z, S) < d(z, X \setminus S)\}.$$

The following propositions are clear from the definition, and thus given without proof.

1. Since the $d(z, \emptyset) = \infty$, $\tilde{\emptyset} = \emptyset$.
2. For any $S \subset X$, $\tilde{S}$ is open in Z.
3. If S is open in X, then $\tilde{S} \cap X = S$.
4. If $S \subset T \subset X$, then $\tilde{S} \subset \tilde{T} \subset Z$.
5. If $S \subset X$ and $T \subset X$, then $\tilde{S} \cap \tilde{T} = \widetilde{S \cap T}$.

Let $R = \bigcup\{\tilde{V}_n^j : V_n^j \in \mathcal{V}\}$, then R is an open subset of Z such that

$$\begin{aligned} R \cap X &= \bigcup\{\tilde{V}_n^j \cap X : V_n^j \in \mathcal{V}\} \\ &= \bigcup\{V_n^j : V_n^j \in \mathcal{V}\} \\ &= X. \end{aligned}$$

Let $\mathcal{W} = \{\tilde{W}_y \cap R : y \in Y\}$, then $\mathcal{W}$ is a collection of open sets of Z. Moreover, since the space $X \subset R$, it is also seen that for any given $y \in Y$ the image

$$\begin{aligned} ef^{-1}(y) &\subset e(W_y) \\ &= \tilde{W}_y \cap X \\ &\subset \tilde{W}_y \cap R \\ &\in \mathcal{W}. \end{aligned}$$

Thus, the set $\{ef^{-1}(y) : y \in Y\}$ refines $\mathcal{W}$.

Then, by the approximate invertibility of f, there exists a map $g : Y \to Z$ such that the composition $gf : X \to Z$ is $\mathcal{W}$-close to e. In particular for any $y \in Y$ and any $x \in f^{-1}(y)$ there exists a point $y' \in Y$ such that $e(x) \cup gf(x) \subset \tilde{W}_{y'} \cap R$. Thus, $gf(x) = g(y) \in R$ such that $g(Y) \subset R$.

For each $n \in \mathbb{N}$ the pair $\{V_n^1, V_n^2\}$ is a disjoint pair of open sets in X. The propositions then imply that the pair $\{\tilde{V}_n^1, \tilde{V}_n^2\}$ is a pair of disjoint open sets in Z, and thus $\{g^{-1}(\tilde{V}_n^1), g^{-1}(\tilde{V}_n^2)\}$ will also be a pair of disjoint open sets in Y. Moreover the

$$\begin{aligned} \bigcup\{g^{-1}(\tilde{V}_n^j) : V_n^j \in \mathcal{V}\} &= g^{-1}(\bigcup\{\bar{V}_n^j : V_n^j \in \mathcal{V}\}) \\ &= g^{-1}(R) \\ &= Y. \end{aligned}$$

Thus, the collection $\{g^{-1}(\tilde{V}_n^j) : V_n^j \in \mathcal{V}\}$ is an open cover of Y.

It only remains to be shown that for each $n \in \mathbb{N}$ the collection $\{g^{-1}(\tilde{V}_n^1), g^{-1}(\tilde{V}_n^2)\}$ is a shrinkage of $\mathcal{U}_n$. Using the normality of the space Y, it is enough to show that $g^{-1}(\tilde{V}_n^j) \subset U_n^j$ for each $n \in \mathbb{N}$ and both $j \in \{1, 2\}$ [**Nagata**, p. 2].

Let $y \in g^{-1}(\tilde{V}_n^j)$ be given, and choose any $x \in f^{-1}(y)$. Since gf is $\mathcal{W}$-close to the embedding e, there exists a point $y' \in Y$ such that $e(x)$ and $gf(x) = g(y)$ are both in $\tilde{W}_{y'} \cap R$. Thus,

$$g(y) \in \tilde{V}_n^j \cap \tilde{W}_{y'} = V_n^j \tilde{\cap} W_{y'},$$

and since $V_n^j \cap W_{y'} \neq \emptyset$ it is seen that $W_{y'} \subset f^{-1}(U_n^j)$. Finally, since the point $x \in \tilde{W}_{y'} \cap X = W_{y'}$, it is seen that $x \in f^{-1}(U_n^j)$. Thus, $f(x) = y \in U_n^j$, which shows that $g^{-1}(\tilde{V}_n^j) \subset U_n^j$. The characterization 2.4.2 then gives that Y is WID which completes the proof of the theorem.

As an easy application of 2.4.3, one implication of Kozlowski's theorem 2.5.4 is obtained.

<u>2.5.11</u> <u>Theorem.</u> Let $f: X \to Y$ be an hereditary shape equivalence between spaces X and Y. If the $\dim X < n$ then the $\dim Y < n$.

<u>Proof.</u> The result follows immediately by using the characterization of dimension given in 2.4.3 in the proof of 2.5.10 and thus is omitted.

Ancel's result 2.5.8 suggests the following important questions.

<u>2.5.12</u> <u>Question.</u> Let $f:X \to Y$ be a cell-like map from a WID space X onto Y. If the space Y is also WID, then must f be an hereditary shape equivalence?

<u>2.5.13</u> <u>Question.</u> Let $f:X \to Y$ be an approximately invertible map from a CD space X onto a space Y. Must the space Y be CD?

It is possible that the characterization of countable dimensionality which was given in 2.4.4 combined with the techniques used in the proof of 2.5.10 might provide a solution to 2.5.13. A positive answer to 2.5.12, or for that matter showing implication e of the generalized Alexandroff question 1.3.7, would then imply that the image of a cell-like dimension-raising map must be SID. A space which was a counter-example to the implication e of 1.3.7 would be a prime candidate for the image of a cell-like dimension-raising map.

III. ALTERNATIVE CHARACTERIZATIONS OF WEAK INFINITE-DIMENSIONALITY

In Chapter II., theorem 2.4.2, an alternative characterization of weak infinite-dimensionality was introduced. This characterization was exploited to obtain a number of results. In particular, this new characterization in terms of binary open covers bears a striking resemblance to the definition of property C. The ultimate goal of this chapter is to investigate possible solutions of the generalized Alexandroff question by determining a characterization of property C in terms of essential families.

A generalization of 2.4.2 is given in section 3.1, along with a fairly obvious new definition of essential family, each yielding new properties of infinite-dimensional spaces. These new properties are presented in 3.1, together with theorems relating them back to the original infinite-dimensional dimension theories.

Basic internal relationships of the new properties are presented in the next section. These results of 3.2 are, for the most part, immediate from the definitions of 3.1, however some technical work must be done if the spaces involved are not compact. With the additional assumption of compactness, most technical difficulties disappear. A brief discussion of the properties upon compacta is given in 3.1 which provides the initial hint to essential differences between property C and weak infinite-dimensionality. These differences will be discussed at greater length in later sections.

Sections 3.3 through 3.5 contain results relating the new properties to each other, and constitute the major work of this chapter. The results are given in order of discovery, and thus direct proofs of some earlier theorems will be given even though those same theorems may be proven indirectly by theorems presented in later sections. This author holds the opinion that the order of discovery, as well as the means of discovery, are often of greater importance than the discovery itself.

In the final section of this chapter, a summary of the essential differences between property C and weak infinite-dimensionality will be given. That section and the thesis concludes with a discussion of how a WID space which does not have property C might be constructed.

3.1 Definitions of the New Characterizations

In 2.4.2, a characterization of weak infinite-dimensionality was given in terms of open covers. This property is generalized by the following definitions.

3.1.1 Definitions. Let $r \in \{2, 3, 4, \cdots\}$. A space X will be said to have the property C_r if every countable sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X, where for each $n \in \mathbb{N}$ the $|\mathcal{U}_n| \leq r$, has for each $n \in \mathbb{N}$ a precise pairwise disjoint open shrinkage $\mathcal{V}_n$ of $\mathcal{U}_n$ such that the $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ forms an open cover of X. If a space X has property C_r for every $r \in \{2, 3, 4, \cdots\}$, then X will be said to have the property C_∞. A space X will be said to have the property C_ω if every countable sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X has for each $n \in \mathbb{N}$ a precise pairwise disjoint open shrinkage $\mathcal{V}_n$ of $\mathcal{U}_n$ such that the $\bigcup\{\mathcal{U}_n : n \in \mathbb{N}\}$ forms an open cover of X.

3.1.2 Proposition. A space X has property C_2 if and only if the space X is WID.

Proof. The property C_2 was precisely the characterization of weak infinite-dimensionality which was given in 2.4.2.

The following lemma will be used to avoid technical difficulties in later proofs of this section.

<u>3.1.3</u> <u>Lemma.</u> If an open cover $\mathcal{U} = \{U_\alpha : \alpha \in \Gamma\}$ of a space X is given, then there exists a precise open shrinkage $\mathcal{V} = \{V_\alpha : \alpha \in \Gamma\}$ of $\mathcal{U}$, that is for each $\alpha \in \Gamma$ the set $V_\alpha \subset \overline{V}_\alpha \subset U_\alpha$, which also covers X.

<u>Proof.</u> The cover $\mathcal{U}$ may be assumed to be locally finite, for if not $\mathcal{U}$ can be replaced with a precise locally finite open refinement which still covers the space X [**Dugundji**, p. 162]. If $\mathcal{U}$ is locally finite then normality can be used to choose the desired shrinkage [Nagata, p. 2].

<u>3.1.4</u> <u>Proposition.</u> The word "shrinkage" in the definitions of 3.1.1 may be replaced by the word "refinement" without altering the properties.

<u>Proof.</u> Apply 3.1.3 to the union of the refinements.

As in 2.4.2, the use of shrinkages yields true separators rather than the continuum-wise separators which would result from the use of refinements. Thus, the word "shrinkage" will be continued to be used in the statements of theorems, but in proofs often only refinements will be constructed, with the reader then referred to 3.1.3 or 3.1.4.

<u>3.1.5</u> <u>Theorem.</u> Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a given sequence of open covers of a space X. If for each $n \in \mathbb{N}$ $\mathcal{V}_n$ is a C-refinement of $\mathcal{U}_n$ such that the $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ covers X, then for each $n \in \mathbb{N}$ there exists a precise pairwise disjoint open shrinkage $\mathcal{W}_n$ of $\mathcal{U}_n$ such that the $\bigcup\{\mathcal{W}_n : n \in \mathbb{N}\}$ still covers X.

Proof. Let the open covers $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be given. Using ordinal numbers faithfully index each $\mathcal{U}_n$ by $\mathcal{U}_n = \{U_n^\alpha : \alpha \in \Gamma_n\}$ where each Γ_n is a well-ordered set of indices.

Similarly, given the C-refinements $\{\mathcal{V}_n : n \in \mathbb{N}\}$ such that the $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ covers the space X, use the ordinals to faithfully index each $\mathcal{V}_n$ by $\mathcal{V}_n = \{V_n^\beta : \beta \in \Lambda_n\}$ where each Λ_n is also a well-ordered set of indices.

For each fixed $n \in \mathbb{N}$ define a function $f_n : \Lambda_n \to \Gamma_n$ by setting $f_n(\beta) = \min\{\alpha \in \Gamma_n : V_n^\beta \subset U_n^\alpha\}$. Since for each $n \in \mathbb{N}$ $\mathcal{U}_n$ is refined by $\mathcal{V}_n$, it is clearly seen that each f_n is a well-defined function on Λ_n.

For each $\alpha \in \Gamma_n$ define $W_n^\alpha = \bigcup\{V_n^\beta \in \mathcal{V}_n : \beta \in f_n^{-1}(\alpha)\}$. For each $n \in \mathbb{N}$, by the definition of f_n and since $\mathcal{V}_n$ is a C-refinement of $\mathcal{U}_n$, each $\mathcal{W}_n = \{W_n^\alpha : \alpha \in \Gamma_n\}$ is then a precise pairwise disjoint open refinement of $\mathcal{U}_n$.

Suppose that for each $n \in \mathbb{N}$ such a $\mathcal{W}_n$ has been constructed. For any arbitrary $x \in X$ choose an $n \in \mathbb{N}$ and a $\beta \in \Lambda_n$ such that $x \in V_n^\beta$. But, then for $\alpha = f_n(\beta)$ the point $x \in W_n^\alpha$, and thus the $\bigcup\{\mathcal{W}_n : n \in \mathbb{N}\}$ covers X. Applying 3.1.3 to the $\bigcup\{\mathcal{W}_n : n \in \mathbb{N}\}$ completes the proof.

3.1.6 Corollary. A space X has property C if and only if the space X has property C_ω.

Proof. If the space X has property C, then the space X is seen to have property C_ω by a direct application of 3.1.5. Since any precise pairwise disjoint open shrinkage is clearly a C-refinement, it is obvious that if the space X has the property C_ω, then X will also have the property C.

The corollary 3.1.6 formalizes the resemblance between the definitions given in 3.11 and the definition of property C. The definitions in 3.1.1 were motivated by a characterization of weak infinite-dimensionality which had the same form as the definition of property C. In an attempt to answer the implication e. of 1.1.7, the problem can also be attacked from the reverse direction by finding a characterization of property C in terms of essential families.

Recall that a family of closed sets is discrete in a space if and only if the family is pairwise disjoint and locally finite in the space. In particular, the union of a discrete collection of closed sets is closed. Thus, every point of the space has a neighborhood which intersects at most one element of a discrete collection of closed sets.

A fairly obvious generalization of the definition of weak infinite-dimensionality is contained in the following definitions.

<u>3.1.7</u> <u>Definitions.</u> A closed subset $S \subset X$ of a space X will be said to be a <u>separator</u> of a discrete collection of closed subsets $(A^\alpha : \alpha \in \Gamma)$ contained in the space X if $S \subset X$ separates the collection $(A^\alpha : \alpha \in \Gamma)$ in X; that is the complement $X \setminus S = \{U^\alpha : \alpha \in \Gamma\}$ where $\{U^\alpha : \alpha \in \Gamma\}$ is a collection of pairwise disjoint open subsets of X such that for each $\alpha \in \Gamma$ the closed set $A^\alpha \subset U^\alpha$. An ω-family of discrete collections of closed

subsets $\{(A_n^\alpha : \alpha \in \Gamma_n) : n \in \mathbb{N}\}$ of a space X will be called inessential if for each $n \in \mathbb{N}$ there exists a closed set $S_n \subset X$ which separates $(A_n^\alpha : \alpha \in \Gamma_n)$ in X such that the $\bigcap\{S_n : n \in \mathbb{N}\} = \emptyset$. Let $r \in \{2, 3, 4, \cdots\}$, then a space X will be said to be weakly infinite-dimensional with respect to r-tuples, and denoted by WID_r, if any ω-family of r-tuples of pairwise disjoint closed subsets $\{(A_n^j : j = 1, 2, \cdots, r) : n \in \mathbb{N}\}$ of X is inessential. If the space X is WID_r for every $r \in \{2, 3, 4, \cdots\}$, then the space X will be said to be WID_∞. If every ω-family of discrete collections of closed subsets $\{(A_n^\alpha : \alpha \in \Gamma_n) : n \in \mathbb{N}\}$ of a space X is inessential, then the space X will be said to be WID_ω.

3.1.8 Proposition. A space X is WID_2 if and only if X is WID.

Proof. This is obvious from the definitions.

The very general, but ungainly, definition of WID_ω simplifies when separability is assumed.

3.1.9 Proposition. A separable metric space X is WID_ω if and only if every ω-family of discrete sequences of closed subsets $\{(A_n^k : k \in \mathbb{N}) : n \in \mathbb{N}\}$ of X is inessential.

Proof. It is enough to realize that a discrete collection $\{A^\alpha : \alpha \in \Gamma\}$ in a separable space X has $A^\alpha = \emptyset$ for all but countably many A^α. Indeed, let X have a countable base of open sets and choose a point $x_\alpha \in A^\alpha$ from each $A^\alpha \neq \emptyset$. By the discreteness of the collection, there exists a basic open set $U_\alpha \subset X$ for each $\alpha \in \Gamma$ with $x_\alpha \in A^\alpha$ such that the point

$$x_\alpha \in U_\alpha \subset X \setminus \bigcup \{ A^\beta : \beta \in \Gamma \setminus \{\alpha\} \} .$$

Since the space has a countable base, there can be only a countable number of such U_α, and consequently only countably many non-empty A^α. With this restriction, the definition of WID_ω in 3.1.7 reduces to the statement of the proposition.

3.2 Basic Internal Relationships of the New Properties

In this section, basic relationships between the properties which were defined in the last section are given. Most of the relationships are immediate from the definitions.

3.2.1 Theorem. Let $r \in \{2, 3, 4, \cdots\}$. A space X satisfies the following four implications of properties:

1. If a space X has property C_{r+1}, then X has property C_r.
2. If a space X has property C_∞, then X has property C_r for each $r \in \{2, 3, 4, \cdots\}$.
3. If a space X has property C_ω, then X has property C_∞.
4. A space X has property C_ω if and only if X has property C.

Proofs.

1. This follows immediately from the definitions by regarding any cover $\mathcal{U}_n = \{U_n^j : j = 1, \cdots, r\}$ with cardinality r as a cover $\mathcal{U}_n = \{U_n^j : j = 1, \cdots, r+1\}$ with cardinality $r+1$ by setting $U_n^{r+1} = \varnothing$.
2. This is the definition of property C_∞ as given at the end of 3.1.1.
3. This follows in the same manner as the proof of implication 1.
4. This was proven in corollary 3.1.6 of the last section.

The following analogous results are obtained in a similar manner.

3.2.2 Theorem. Let $r \in \{2, 3, 4, \cdots\}$. A space X satisfies the following four implications of properties:

1. A space X is WID if and only if X is WID_2.
2. If a space X is WID_{r+1}, then X is WID_r.
3. If a space X is WID_∞, then X is WID_r for each $r \in \{2, 3, 4, \cdots\}$.
4. If a space X is WID_ω then X is WID_∞.

Proofs.

1. This was done in the proposition 3.1.8.
2. This follows immediately from the definitions by regarding any r-tuple $(A_n^1, \cdots, A_n^r)$ as an $(r+1)$-tuple $(A_n^1, \cdots, A_n^r, A_n^{r+1})$ by setting $A_n^{r+1} = \varnothing$.
3. This is the definition of WID_∞ as given at the end of 3.1.7.
4. This follows in the same manner as the proof of implication 2.

This section is ended with some superfluous, but illustrative, results concerning the properties C_ω and WID_ω for the category of compacta. The first such result is an obvious characterization of property C for compacta.

3.2.3 Theorem. A compactum X has property C if and only if every countable collection of finite open covers $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of X has precise pairwise disjoint open shrinkages $\mathcal{V}_n$ of U_n for each $n \in \mathbb{N}$ such that the $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ forms a cover of X.

Proof. The theorem follows immediately from the corollary 3.1.7 by choosing finite subcovers of each given countable cover.

Although of little importance, theorem 3.2.3 does suggest a similar theorem of more interest concerning compacta which are WID_ω.

3.2.4 Theorem. A compactum X is WID_ω if and only if every ω-family of discrete r_n-tuples of closed subsets $\{(A_n^k : k = 1, \cdots, r_n) : n \in \mathbb{N}\}$ of X, where each $r_n \in \mathbb{N}$, is inessential.

Proof. Since compacta are separable, 3.1.9 applies such that only ω-families of discrete sequences of closed sets $\{(A_n^k : k \in n) : n \in \mathbb{N}\}$ of X need be considered. Given such an ω-family, since X is compact, for each $n \in \mathbb{N}$ an $r_n \in \mathbb{N}$ may be chosen such that for each $k > r_n$ the set $A_n^k = \varnothing$.

Indeed, if $A_n^k \neq \varnothing$ for infinitely many $k \in \mathbb{N}$, then for each such k a point $x_k \in A_n^k$ may be chosen. Since X is compact, the sequence $(x_k : A_n^k \neq \varnothing)$ of such x_k has a convergent subsequence, say $x_{k_j} \to x \in X$. Thus, any open set with $x \in U$ has $x_k \in U$ for infinitely many x_k. But, then $A_n^k \cap U \neq \varnothing$ for infinitely many A_n^k, which contradicts the discreteness of $(A_n^k : k \in \mathbb{N})$.

Thus, the hypothesis implies that the ω-family $\{(A_n^k : k \in \mathbb{N}) : n \in \mathbb{N}\}$ is inessential, and X has been shown to be WID_ω which completes the proof.

Theorems 3.2.3 and 3.2.4 are contrasted with the following theorems which concern the properties C_∞ and WID_∞.

<u>3.2.5</u> <u>Theorem.</u> A space X has property C_∞ if and only if every countable collection of finite open covers $\{\mathcal{U}_n : n \in \mathbb{N}\}$, with the cardinality $|\mathcal{U}_n| = r_n$ and the $\sup\{r_n : n \in \mathbb{N}\} < \infty$, has for each $n \in \mathbb{N}$ a precise pairwise disjoint open shrinkage $\mathcal{V}_n$ of $\mathcal{U}_n$ such that the $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ forms a cover of X.

<u>Proof.</u> This follows immediately from the definition of C_∞ given at the end of 3.1.1.

Notice that the compactness assumption is not needed in 3.2.5, as well as in the following theorem.

<u>3.2.6</u> <u>Theorem.</u> A space X is WID_∞ if and only if every ω-family of r_n-tuples of closed subsets $\{A_n^k : k = 1, \cdots, r_n) : n \in \mathbb{N}\}$ of X, such that the $\sup\{r_n : n \in \mathbb{N}\} < \infty$, is inessential.

<u>Proof.</u> This follows immediately from the definition of WID_∞ given at the end of 3.1.7.

3.3 Relationships Between the Two Properties

In this section a result is given which relates the covering properties to the separation properties. Although the main result of this section is also proven indirectly in the next section a direct proof is included in this section. It is hoped that this will familiarize the reader with the dual nature of the new properties and that it will emphasize their connection to the motivating characterization in theorem 2.4.2. The following technical lemma will be needed.

3.3.1 Lemma. Let $\{A^\alpha : \alpha \in \Gamma\}$ be a discrete collection of closed sets in a space X. There exists a collection of open subsets $\{U^\alpha : \alpha \in \Gamma\}$ of X such that:

1. The collection $\{U^\alpha : \alpha \in \Gamma\}$ is locally finite in X.
2. For each $\alpha \in \Gamma$ $A^\alpha \subset U^\alpha \subset \overline{U}^\alpha \subset X \setminus \bigcup\{A^\beta : \beta \in \Gamma, \beta \neq \alpha\}$.
3. The collection $\{U^\alpha : \alpha \in \Gamma\}$ covers X.

Proof. Fix a single element $\gamma \in \Gamma$ and let $\alpha \in \Gamma \setminus \{\gamma\}$ be fixed but arbitrary. Since the collection $\{A^\beta : \beta \in \Gamma\}$ is discrete in X, the $\bigcup\{A^\beta : \beta \in \Gamma, \beta \neq \alpha\}$ is closed in X. Thus, by normality, an open set V^α may be chosen such that

$$A^\alpha \subset V^\alpha \subset \overline{V}^\alpha \subset X \setminus \bigcup\{A^\beta : \beta \in \Gamma, \beta \neq \gamma\}.$$

Thus, since the subset $V^\alpha \subset X \setminus A^\gamma$, it is also seen that $A^\gamma \subset X \setminus V^\alpha$.

Thus, if such a V^α is constructed for each $\alpha \in \Gamma \setminus \{\gamma\}$, then

$$\begin{aligned} A^\gamma &\subset \bigcap\{X \setminus V^\alpha : \alpha \in \Gamma, \alpha \neq \gamma\} \\ &= X \setminus \bigcup\{V^\alpha : \alpha \in \Gamma, \alpha \neq \gamma\}. \end{aligned}$$

On the other hand, since for each $\alpha \in \Gamma \setminus \{\gamma\}$ the set $A^\alpha \subset V^\alpha$, it is also seen that the

$$\bigcup\{A^\alpha : \alpha \in \Gamma,\ \alpha \neq \gamma\} \subset \bigcup\{V^\alpha : \alpha \in \Gamma,\ \alpha \neq \gamma\},$$

and thus, that the complement satisfies

$$X \setminus \bigcup\{V^\alpha : \alpha \in \Gamma,\ \alpha \neq \gamma\} \subset X \setminus \bigcup\{A^\alpha : \alpha \in \Gamma,\ \alpha \neq \gamma\}.$$

As before, normality then permits the choice of a final open set V^γ with

$$A^\gamma \subset X \setminus \bigcup\{V^\alpha : \alpha \in \Gamma,\ \alpha \neq \gamma\} \subset V^\gamma \subset \overline{V}^\gamma \subset X \setminus \bigcup\{A^\alpha : \alpha \in \Gamma,\ \alpha \neq \gamma\}.$$

At this point, an open cover $\{V^\alpha : \alpha \in \Gamma\}$ of X has been constructed such that for each $\alpha \in \Gamma$

$$A^\alpha \subset V^\alpha \subset \overline{V}^\alpha \subset X \setminus \bigcup\{A^\beta : \beta \in \Gamma,\ \beta \neq \alpha\}.$$

Then by paracompactness [**Dugundji**, p. 162], a precise locally finite open refinement $\{U^\alpha : \alpha \in \Gamma\}$ of $\{V^\alpha : \alpha \in \Gamma\}$ which also covers X may be chosen such that for each $\alpha \in \Gamma$ the set $U^\alpha \subset V^\alpha$.

Thus, for each $\alpha \in \Gamma$ it is seen that the set

$$U^\alpha \subset \overline{U}^\alpha \subset \overline{V}^\alpha \subset X \setminus \bigcup\{A^\beta : \beta \in \Gamma,\ \beta \neq \alpha\}.$$

Therefore, it only remains show that for each $\alpha \in \Gamma$ the set $A^\alpha \subset U^\alpha$.

Fix $\alpha \in \Gamma$, and let $x \in A^\alpha$ be fixed but arbitrary. Then, for some $\beta \in \Gamma$ the point $x \in U^\beta \subset V^\beta$. It must be shown that $\beta = \alpha$. If $\beta \neq \alpha$, then the set

$$V^\beta \subset X \setminus \bigcup\{A^\gamma : \gamma \in \Gamma,\ \gamma \neq \beta\} \subset X \setminus A^\alpha$$

which contradicts $x \in A^\alpha \cap V^\beta$. Thus, $\beta = \alpha$, $x \in U^\alpha$, and $A^\alpha \subset U^\alpha$.

Therefore, it has been shown that for each $\alpha \in \Gamma$

$$A^\alpha \subset U^\alpha \subset \overline{U}^\alpha \subset X \setminus \bigcup\{A^\beta : \beta \in \Gamma,\ \beta \neq \alpha\},$$

which completes the proof.

With the aid of 3.3.1, it now possible to prove the major result of this section.

<u>3.3.2</u> <u>Theorem.</u> Let $r \in \{2, 3, 4, \cdots\} \cup \{\omega\}$. If a space X has the property C_r then the space X is WID_r.

<u>Proof.</u> Suppose that the space X has the property C_r, and let $\{(A_n^\alpha : \alpha \in \Gamma_n) : n \in \mathbb{N}\}$ be a given ω-family of discrete collections, r-tuples if $r \in \{2, 3, 4, \cdots\}$, of closed sets of X. Fix $n \in \mathbb{N}$, and use the lemma 3.3.1 to choose a locally finite open cover $\mathcal{U}_n = \{U_n^\alpha : \alpha \in \Gamma_n\}$ of X such that for each $\alpha \in \Gamma_n$

$$A_n^\alpha \subset U_n^\alpha \subset \overline{U}_n^\alpha \subset X \setminus \bigcup\{A_n^\beta : \beta \in \Gamma_n,\ \beta \neq \alpha\}.$$

Suppose that for each $n \in \mathbb{N}$ such an open cover $\mathcal{U}_n$ has been constructed. Since the space X has property C_r, for each $n \in \mathbb{N}$ a precise pairwise disjoint open shrinkage $\mathcal{V}_n = \{V_n^\alpha : \alpha \in \Gamma_n\}$ of $\mathcal{U}_n$ may be chosen such that the $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ forms a cover of X.

Next, for any fixed $n \in \mathbb{N}$ and for each $\alpha \in \Gamma_n$ define

$$W_n^\alpha = V_n^\alpha \cup (X \setminus \bigcup\{\overline{U}_n^\beta : \beta \in \Gamma_n,\ \beta \neq \alpha\}).$$

Since the collection $\{U_n^\alpha : \alpha \in \Gamma_n\}$ is locally finite in the space X, each collection $\{\overline{U}_n^\beta : \beta \in \Gamma_n,\ \beta \neq \alpha\}$ is also locally finite in X. Therefore, the $\bigcup\{\overline{U}_n^\beta : \beta \in \Gamma_n,\ \beta \neq \alpha\}$ is closed in X, and thus, each $W_n^\alpha \subset X$ is open.

Fix $n \in \mathbb{N}$ and let $\alpha, \beta \in \Gamma_n$ with $\alpha \neq \beta$ be given. Since each $\mathcal{U}_n$ covers the space X, it is seen that

$$\begin{aligned} W_n^\alpha \cap W_n^\beta = & [V_n^\alpha \cup (X \setminus \bigcup\{\bar{U}_n^\gamma : \gamma \in \Gamma_n,\ \gamma \neq \alpha\})] \cap \\ & [V_n^\beta \cup (X \setminus \bigcup\{\bar{U}_n^\gamma : \gamma \in \Gamma_n,\ \gamma \neq \beta\})] \\ & \subset X \setminus \bigcup\{\bar{U}_n^\gamma : \gamma \in \Gamma_n\} \\ & = \varnothing . \end{aligned}$$

Fix an integer $n \in \mathbb{N}$, and let $\alpha \in \Gamma_n$ be fixed but arbitrary. For any $\beta \in \Gamma_n \setminus \{\alpha\}$ it is seen that

$$\bar{U}_n^\beta \subset X \setminus \bigcup\{A_n^\gamma : \gamma \in \Gamma_n,\ \gamma \neq \beta\},$$

such that the

$$\begin{aligned} \bigcup\{\bar{U}_n^\beta : \beta \in \Gamma_n,\ \beta \neq \alpha\} \subset & \bigcup\{X \setminus \bigcup\{A_n^\gamma : \gamma \in \Gamma_n,\ \gamma \neq \beta\} \\ & = X \setminus \bigcap\{\bigcup\{A_n^\gamma : \gamma \in \Gamma_n,\ \alpha \neq \beta\} : \beta \in \Gamma_n,\ \beta \neq \alpha\} \\ & = X \setminus A_n^\alpha . \end{aligned}$$

Thus, it is also seen that

$$A_n^\alpha \subset X \setminus \bigcup\{\bar{U}_n^\beta : \beta \in \Gamma_n,\ \beta \neq \alpha\} \subset W_n^\alpha .$$

Therefore, defining $S_n = X \setminus \bigcup\{W_n^\alpha : \alpha \in \Gamma_n\}$ gives a closed set $S_n \subset X$ which separates the collection $(A_n^\alpha : \alpha \in \Gamma_n)$ in X.

Finally, if for each $n \in \mathbb{N}$ the separator S_n has been defined, then since the $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ covers X, it is obtained that the

$$\begin{aligned} \bigcap\{S_n : n \in \mathbb{N}\} & = \bigcap\{X \setminus \bigcup\{W_n^\alpha : \alpha \in \Gamma_n\} : n \in \mathbb{N}\} \\ & = X \setminus \bigcup\{W_n^\alpha : n \in \mathbb{N},\ \alpha \in \Gamma_n\} \\ & \subset X \setminus \bigcup\{V_n^\alpha : n \in \mathbb{N},\ \alpha \in \Gamma_n\} \\ & = \varnothing . \end{aligned}$$

Thus, it has been shown that the ω-family $\{(A_n^\alpha : \alpha \in \Gamma_n) : n \in \mathbb{N}\}$ is inessential which completes the proof that the space X is WID_Γ.

<u>3.3.3</u> <u>Corollary.</u> If a space X has the property C_∞, then the space X is also WID_∞.

<u>Proof.</u> This is immediate from the theorem 3.3.2 and the definition of WID_∞ given at the end of 3.1.7

3.4 Equivalence of the Definitions of WID_r

Various results relating the definitions to each other have been given in the previous sections. This section begins with a brief summary of those results.

3.4.1 Summary. Let $r \in \{2, 3, 4, \cdots\}$. Metric spaces satisfy the following implications of properties.

$$\begin{array}{ccccccccccccc}
 & & C_2 & \Longleftarrow & C_r & \Longleftarrow & C_{r+1} & \Longleftarrow & C_\infty & \Longleftarrow & C_\omega & \Longleftrightarrow & \text{property } C \\
 & & \Updownarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\
WID & \Longleftrightarrow & WID_2 & \Longleftarrow & WID_r & \Longleftarrow & WID_{r+1} & \Longleftarrow & WID_\infty & \Longleftarrow & WID_\omega & &
\end{array}$$

The equivalence of the property C_2 with the property WID_2 was established in 2.4.2.

Since the ultimate goal is to answer the generalized Alexandroff question posed in 1.3.7, it is seen that the reverse implications of 3.4.1 must be shown. In this section, the reverse implications of the bottom row in 3.4.1 will be studied. The reverse implications of the top row will be considered in the next section.

The first subject of investigation is the relationship between elements and separators of ω-families. One such relationship is given by the following lemma, which will be used in the proof of the main theorem of this section.

<u>3.4.2</u> <u>Lemma.</u> Let $r \in \{2, 3, 4, \cdots\} \cup \{\omega\}$ and let $(A^\alpha : \alpha \in \Gamma)$ be a discrete collection, an r-tuple when $r \in \{2, 3, 4, \cdots\}$, of closed subsets of a space X. Let $S \subset X$ be a closed subset of X which separates $(A^\alpha : \alpha \in \Gamma)$ in X. If $T \subset X$ is a closed subset of X which separates the pair $(\bigcup\{A^\alpha : \alpha \in \Gamma\}, S)$ in X, then the closed subset T is also a separator of the original collection $(A^\alpha : \alpha \in \Gamma)$ in X.

<u>Proof.</u> Let $(A^\alpha : \alpha \in \Gamma)$ and S be given as stated in the hypothesis for a space X. Since $(A^\alpha : \alpha \in \Gamma)$ is discrete in X, the $\bigcup\{A^\alpha : \alpha \in \Gamma\}$ is closed in X. Moreover, since S separates $(A^\alpha : \alpha \in \Gamma)$ in X, there exists a pairwise disjoint collection of open subsets $\{U^\alpha : \alpha \in \Gamma\}$ of X such that $X \setminus S = \bigcup\{U^\alpha : \alpha \in \Gamma\}$ with the closed set $A^\alpha \subset U^\alpha$ for each $\alpha \in \Gamma$. Thus, the pair $(\bigcup\{A^\alpha : \alpha \in \Gamma\}, S)$ is a closed disjoint pair in X.

Let $T \subset X$ be a closed subset of X which separates the pair $(\bigcup\{A^\alpha : \alpha \in \Gamma\}, S)$ in X. Thus, there exist disjoint open subsets V_1 and V_2 of X such that

$$X \setminus T = V_1 \cup V_2 \quad \text{with} \quad \bigcup\{A^\alpha : \alpha \in \Gamma\} \subset V_1 \quad \text{and} \quad S \subset V_2.$$

Fix $\gamma \in \Gamma$ and define $W^\gamma = (U^\gamma \cap V_1) \cup V_2$. For each $\alpha \in \Gamma \setminus \{\gamma\}$ define $W^\alpha = U^\alpha \cap V_1$. Then, for each $\alpha \in \Gamma$ this gives

$$A^\alpha \subset U^2 \cap V_1 \subset W^\alpha.$$

Since $V_1 \cap V_2 = \varnothing$, for each $\alpha \in \Gamma \setminus \{\gamma\}$ the

$$\begin{aligned} W^\gamma \cap W^\alpha &= [(U^\gamma \cap V_1) \cup V_2] \cap (U^\alpha \cap V_1) \\ &= (U^\gamma \cap V_1) \cap (U^\alpha \cap V_1) \\ &= (U^\gamma \cap U^\alpha) \cap V_1 \\ &= \varnothing. \end{aligned}$$

Similarly, for any $\alpha, \beta \in \Gamma\setminus\{\gamma\}$ with $\alpha \neq \beta$ the

$$\begin{aligned} W^{\alpha} \cap W^{\beta} &= (U^{\alpha} \cap V_1) \cap (U^{\beta} \cap V_1) \\ &= (U^{\alpha} \cup U^{\beta}) \cap V_1 \\ &= \varnothing . \end{aligned}$$

Thus, the collection $\{W^{\alpha} : \alpha \in \Gamma\}$ is a collection of pairwise disjoint open sets of X such that the $X\setminus\bigcup\{W_n^{\alpha} : \alpha \in \Gamma\}$ is a closed subset of X which separates the collection $(A^{\alpha} : \alpha \in \Gamma)$ in X.

Finally, since $S \subset V_2$, and since $V_1 \subset X\setminus V_2 \subset X\setminus S$ such that

$$\begin{aligned} X\setminus T &= V_1 \cup V_2 \\ &= [(X\setminus S) \cap V_1] \cup V_2 \\ &= [(\bigcup\{U^{\alpha} : \alpha \in \Gamma\}) \cap V_1] \cup V_2 \\ &= [\bigcup\{U^{\alpha} \cap V_1 : \alpha \in \Gamma\}] \cup V_2 \\ &= [(U^{\gamma} \cap V_1) \cup V_2] \cup [\bigcup\{U^{\alpha} \cap V_1 : \alpha \in \Gamma,\ \alpha \neq \gamma\}] \\ &= W^{\gamma} \cup [\bigcup\{W^{\alpha} : \alpha \in \Gamma,\ \alpha \neq \gamma\}] \\ &= \bigcup\{W^{\alpha} : \alpha \in \Gamma\}, \end{aligned}$$

it is seen that $T = \bigcup\{W^{\alpha} : \alpha \in \Gamma\}$, which completes the proof.

This simple lemma is all that is needed to prove the main theorem of this section.

3.4.3 Theorem. Let $r \in \{2, 3, 4, \cdots\} \cup \{\omega\}$. If a space X is WID then the space X is also WID_r.

Proof. Suppose that X is WID and let $\{(A_n^{\alpha} : \alpha \in \Gamma_n) : n \in \mathbb{N}\}$ be a given ω-family of discrete collections, r-tuples when $r \in \{2, 3, 4, \cdots\}$, of closed subsets of X.

Since X, being metric, is collectionwise normal, and since each $(A_n^\alpha : \alpha \in \Gamma_n)$ is discrete in X, a closed subset $S_n \subset X$ may be chosen for each $n \in \mathbb{N}$ such that S_n is a separator of the collection $(A_n^\alpha : \alpha \in \Gamma_n)$ in the space X.

Construct an ω-family of pairs of disjoint closed subsets $\{(\bigcup\{A_n^\alpha : \alpha \in \Gamma_n\}, S_n) : n \in \mathbb{N}\}$ from X as in the proof of the lemma 3.4.2. Then, since X is WID, a closed subset $T_n \subset X$ may be chosen for each $n \in \mathbb{N}$ which separates the pair $(\bigcup\{A_n^\alpha : \alpha \in \Gamma_n\}, S_n)$ in X such that the $\bigcap\{T_n : n \in \mathbb{N}\} = \varnothing$. By applying the lemma 3.4.2, it is seen that for each $n \in \mathbb{N}$ T_n is also a separator of $(\bigcup\{A_n^\alpha : \alpha \in \Gamma_n\}, S_n)$. Thus, the ω-family $\{(A_n^\alpha : \alpha \in \Gamma_n) : n \in \mathbb{N}\}$ is inessential, and X has been shown to be WID_r which completes the proof.

<u>3.4.4</u> <u>Corollary.</u> If a space X is WID_r for one $r \in \{2, 3, 4, \cdots\}$, then the space X is WID_r for all $r \in \{2, 3, 4, \cdots\}$, that is the space X is WID_∞.

<u>Proof.</u> This is obvious from the definitions and results already presented.

The remarkable aspect of 3.4.3, especially in light of 3.2.4 and 3.2.6, is that the properties WID_∞ and WID_ω are seen to be equivalent. The boundedness of the number of elements in the discrete collections of the ω-families, or lack thereof, fails to make any difference, even when the spaces are not compact! The answer to implication e of the generalized Alexandroff question 1.3.7 has been reduced to a single implication which is formally stated in the following question.

3.4.5 Question. If a space X is WID_{ω} than must X also have the property C_{ω}?

3.5 Equivalence of the Definitions of Property C_r

After including the results of the last section into 3.4.1, the following summary is obtained.

3.5.1 Summary. Let $r \in \{2, 3, 4, \cdots\}$. Metric spaces satisfy the following implications of properties.

$$\begin{array}{ccccccccccccc}
 & & C_2 & \Leftarrow & C_r & \Leftarrow & C_{r+1} & \Leftarrow & C_\infty & \Leftarrow & C_\omega & \Longleftrightarrow & \text{property } C \\
 & & \Updownarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\
\text{WID} & \Longleftrightarrow & \text{WID}_2 & \Longleftrightarrow & \text{WID}_r & \Longleftrightarrow & \text{WID}_{r+1} & \Longleftrightarrow & \text{WID}_\infty & \Longleftrightarrow & \text{WID}_\omega & &
\end{array}$$

In this section, the reverse implications of the top row of 3.5.1 are investigated. The following simple lemma will be needed in the proof of the main theorem of this section.

3.5.2 Lemma. Let $r \in \{2, 3, 4, \cdots\} \cup \{\omega\}$. If a space X has the property C_r, then every closed ($\mathcal{F}_\sigma$) subspace $Y \subset X$ also has the property C_r in X.

Proof. The result follows in the same manner as the proof of the analogous statement about property C [**Addis and Gresham**, corollary 2.8].

3.5.3 Theorem. Let $r \in \{2, 3, 4, \cdots\}$. If a space X has the property C_r, then the space also has the property C_{r+1}.

<u>Proof.</u> Suppose that X has property C_r and let a countable sequence of open covers of X, each of cardinality $r+1$, be given. Rewrite the sequence of open covers as a countable collection of countable sequences

$$\{\{\mathcal{U}_{m,n} : n \in \mathbb{N}\} : m \in \mathbb{N}\}$$

where each cover has the form

$$\mathcal{U}_{m,n} = \{U^k_{m,n} : k = 1, 2, \cdots, r+1\}.$$

Fix $m \in \mathbb{N}$, and set $Y_m = X \setminus \bigcup\{U^{r+1}_{m,n} : n \in \mathbb{N}\}$. For each $m \in \mathbb{N}$ Y_m is a closed subspace of X which, by 3.5.2, has property C_r in X. Thus, since for each $n \in \mathbb{N}$ the collection $\{U^k_{m,n} : k = 1, \cdots, r\}$ is a cover of Y_m by open subsets of X, for each $n \in \mathbb{N}$ a precise pairwise disjoint open shrinkage $\{V^k_{m,n} : k = 1, \cdots, r\}$ of $\{U^k_{m,n} : k = 1, \cdots, r\}$ may be chosen such that the $\bigcup\{\{V^k_{m,n} : k = 1, \cdots, r\} : n \in \mathbb{N}\}$ forms an open cover of Y_m.

Keeping $m \in \mathbb{N}$ fixed, define the subsets $V^1_m \subset X$ and $V^2_m \subset X$ by

$$V^1_m = \bigcup\{U^{r+1}_{m,n} : n \in \mathbb{N}\} \text{ and } V^2_m = \bigcup\{V^k_{m,n} : k = 1, 2, \cdots, r,\ n \in \mathbb{N}\}.$$

Clearly, the pair $\{V^1_m, V^2_m\}$ is a binary open cover of the space X.

Doing this for each $m \in \mathbb{N}$ gives a countable sequence of such binary open covers of X. Since X has property C_r, by 3.2.1 the space X also has property C_2. Thus, for each $m \in \mathbb{N}$ a precise pairwise disjoint open refinement $\{W^1_m, W^2_m\}$ of each $\{V^1_m, V^2_m\}$ may be chosen such that the collection $\{W^j_m : j = 1, 2,\ m \in \mathbb{N}\}$ forms a cover of the space X.

Next, for each $m \in \mathbb{N}$, $n \in \mathbb{N}$, and for each $k \in \{1, 2, \cdots, r\}$ define

$$W^{r+1}_{m,n} = W^1_m \cap U^{r+1}_{m,n} \text{ and } W^k_{m,n} = W^2_m \cap V^k_{m,n},$$

and for each $m \in \mathbb{N}$ and $n \in \mathbb{N}$ set

$$\mathcal{W}_{m,n} = \{W^k_{m,n} : k = 1, \cdots, r+1\}.$$

Fix $m \in \mathbb{N}$ and $n \in \mathbb{N}$, then for any $k \in \{1, \cdots, r\}$ the

$$W_{m,n}^{r+1} \cap W_{m,n}^{k} \subset W_m^1 \cap W_m^2 = \varnothing .$$

Similarly, if $j, k \in \{1, \cdots, r\}$ with $j \neq k$, then the

$$W_{m,n}^{j} \cap W_{m,n}^{k} \subset V_{m,n}^{j} \cap V_{m,n}^{k} = \varnothing .$$

Clearly, the set $W_{m,n}^{r+1} \subset U_{m,n}^{r+1}$, and for any $k \in \{1, \cdots, r\}$ the set

$$W_{m,n}^{k} \subset V_{m,n}^{k} \subset U_{m,n}^{k} .$$

Thus, for each $m \in \mathbb{N}$ and $n \in \mathbb{N}$ the collection $\mathcal{W}_{m,n}$ is a precise pairwise disjoint open refinement of $\mathcal{U}_{m,n}$.

Let a point $x \in X$ be fixed but arbitrary, then since the collection $\{W_m^j : j = 1, 2, m \in \mathbb{N}\}$ covers X, integers $m \in \mathbb{N}$ and $j \in \{1, 2\}$ may be chosen such that $x \in W_m^j$. If the point $x \in W_m^1$, then since

$$W_m^1 \subset V_m^1 = \bigcup \{U_{m,n}^{r+1} : n \in \mathbb{N}\},$$

for some $n \in \mathbb{N}$ the point

$$x \in W_m^1 \cap U_{m,n}^{r+1} = W_{m,n}^{r+1} .$$

On the other hand, if the point $x \in W_m^2$, then since

$$x \in V_m^2 \subset \bigcup \{V_{m,n}^{k} : k = 1, \cdots, r,\ n \in \mathbb{N}\},$$

for some $k \in \{1, \cdots, r\}$ and $n \in \mathbb{N}$ the point $x \in V_{m,n}^{k}$. Thus, for this case, the point

$$x \in W_m^2 \cap V_{m,n}^{k} = W_{m,n}^{k} .$$

Since in either case the point x is in some $W_{m,n}^{k}$, it has been shown that the $\bigcup \{\mathcal{W}_{m,n} : m \in \mathbb{N}, n \in \mathbb{N}\}$ covers the space X.

Finally, apply theorem 3.1.4 to obtain that the space X has the property C_{r+1}. This completes the proof.

<u>3.5.4</u> <u>Corollary.</u> If for some $r \in \{2, 3, 4, \cdots\}$ a space X has property C_r, then X has property C_r for every $r \in \{2, 3, 4, \cdots\}$; that is the space X has property C_∞.

<u>Proof.</u> This is obvious from the theorems 3.5.3 and 3.2.1.

The theorem 3.5.3 gives further implications which are stated in the following theorem. This will complete the generalization of 2.4.2.

<u>3.5.5</u> <u>Theorem.</u> Let $r \in \{2, 3, 4, \cdots\}$. If a space X is WID_r, then the space X also has property C_r.

<u>Proof.</u> If the space X is WID_r, then by 3.2.2 the space X is WID_2. Since X is WID_2, the characterization 2.3.2 implies that X has property C_2. Thus, the corollary 3.5.4 then gives that the space X has property C_r.

<u>3.5.6</u> <u>Corollary.</u> If a space X is WID_∞, then the space X has property C_∞.

<u>Proof.</u> The proof is obvious from the definition of WID_∞, theorem 3.5.5 and the definition of C_∞.

3.6 Essential Differences Between the Properties

After including the results of the last section into 3.5.1, the following summary is obtained.

3.6.1 Summary. Let $r \in \{2, 3, 4, \cdots\}$. A space X satisfies the following implications of properties.

$$\begin{array}{ccccccccccc}
 & & C_2 & \Longleftrightarrow & C_r & \Longleftrightarrow & C_{r+1} & \Longleftrightarrow & C_\infty & \Longleftarrow & C_\omega & \Longleftrightarrow & \text{property } C \\
 & & \Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow & & \Downarrow & & \\
\text{WID} & \Longleftrightarrow & \text{WID}_2 & \Longleftrightarrow & \text{WID}_r & \Longleftrightarrow & \text{WID}_{r+1} & \Longleftrightarrow & \text{WID}_\infty & \Longleftrightarrow & \text{WID}_\omega & &
\end{array}$$

As can be seen for 3.6.1, only two reverse implications, each implying the other, remain unknown. Comparing theorems 3.2.3 and 3.2.5, it is also seen that for compacta the essential difference between C_∞ and C_ω seems to arise from the unboundedness of the cardinality of open covers involved. However, no such difficulty presented itself in the proof of the corresponding theorem 3.4.3, even without the assumption of compactness.

This suggests that theorem 3.5.3 might have a different type of proof. The proof of 3.5.3 given in section 3.5 was inductive in nature. The basic technique was to split a given cover into two subcollections, and then to use a previously proven theorem on the resulting covers of smaller cardinality. This technique no longer works if the cardinality of the covers is unbounded, for the resulting covers would then also have unbounded cardinality.

Thus, it would be very interesting to find a "generic proof" of 3.5.3, that is a proof which directly shows the equivalence of the properties C_2 and C_Γ without using any of the intermediate properties. Such a proof was given for theorem 3.4.3, and it seems likely that this is why that proof also showed the equivalence of the properties WID_∞ and WID_ω.

Of course, it might simply be the case that the properties C_∞ and C_ω are different. If that is the case, then the problem is to construct a counter-example, that is a space X with property C_∞ which does not have the property C_ω. From 3.2.3, it is seen that a compact counter example would have a sequence of open covers $\{\mathcal{U}_n : n \in \mathbb{N}\}$ with $|\mathcal{U}_n| \rightarrow \infty$ as $n \rightarrow \infty$. This observation might be used to give a construction procedure.

The other remaining unknown implication of 3.6.1 is whether or not every WID_ω space must also have the property C_ω. The major obstruction of this implication, and indeed the major obstruction to a direct non-inductive proof of the other related vertical implications of 3.6.1, is that a separator of a discrete collection of pairwise disjoint closed subsets $(A_n^\alpha : \alpha \in \Gamma)$ of a space X needs not form a precise pairwise disjoint open shrinkage of the related open cover. It is this obstruction which prevents a direct reversal of the argument used in lemma 3.3.1 and theorem 3.3.2.