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Title THE MATHEMATICAL STRUCTURE OF RECTANGULAR ARRANGEMENTS

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The author studies the class of rectangular arrangements in terms of two binary relations on the objects of the arrangement. He shows how a univalent matrix determines a unique rectangular arrangement, and how each rectangular arrangement is associated with one, two, or four distinct matrices, according to the number of corner elements of the arrangement. In this manner it is shown that the essence of rectangular arrangements can be captured in mathematical terms.

THE MATHEMATICAL STRUCTURE OF RECTANGULAR ARRANGEMENTS

by

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THE MATHEMATICAL STRUCTURE OF RECTANGULAR ARRANGEMENTS

CHAPTER I

INTRODUCTION

Mathematics is the study of objects of thought and relations on the objects. In this paper we will discuss a class of "simple mathematical structures" corresponding to rectangular arrangements of objects. We will abstract the study of the class of all such rectangular arrangements to the study of relations on sets of objects, independent of the nature of the objects. In this manner we will capture, in mathematical terms, the essence of the rectangular arrangement.

The following are examples of rectangular arrangements:

Example 1. The arrangement of the elements in a matrix. The rows and columns are specified by the definition of a matrix (see Definition 1.8 below).

Example 2. The arrangement of trees in rows and columns as in an orchard.

Example 3. The arrangement of people into rows and columns as in a football stadium.

Example 4. The arrangement of the Gaussian integers $m+ni$ where m and n are integers such that $0 \leq m \leq k$, $0 \leq n \leq \ell$.

As these examples show, the essential nature of a rectangular arrangement is quite independent of the particular objects which make up the arrangement. The essential thing is a set of objects of any kind, arrayed in rows and columns. To explain how this abstract notion can be captured in mathematical terms, we must first present some basic definitions leading up to the idea of a simple mathematical structure.

Definition 1.1. Let n be a positive integer, and X be a set. A sequence of length n in X is a function, defined on the set of positive integers less than or equal to n, into the set X. Such sequences are usually denoted by $(x_1, x_2, \cdots, x_n)$ where x_i denotes the value of the function at the integer i, for $i = 1, 2, \cdots, n$. The set of all such n-ary sequences is denoted by X^n.

Definition 1.2. Let n be a positive integer and X be a set. An n-ary relation on X is a subset of X^n. A 1-ary relation is called a unary relation, a 2-ary relation is called a binary relation, and a 3-ary relation is called a ternary relation.

The following are examples of relations on the set of all triangles.

Example 1. Unary relation: The set of all x such that x is an equilateral triangle.

Example 2. Binary relation: The set of pairs (x, y) such that x and y are congruent triangles.

Example 3. Ternary relation: The set of triples (x, y, z) such that the area of x plus the area of y is equal to the area of z.

The reader should be familiar with the following types of binary relations:

Definition 1.3. A binary relation A on a set X is symmetric if for all elements a, b of X, if $(a, b) \varepsilon A$ then $(b, a) \varepsilon A$.

Definition 1.4. A binary relation A on a set X is reflexive if for all elements a of X, $(a, a) \varepsilon A$.

Definition 1.5. A binary relation A on a set X is transitive if for all elements a, b, c of X, if (a, b) and (b, c) belong to A then $(a, c) \varepsilon A$.

Definition 1.6. A binary relation A on a set X is an equivalence relation if A is reflexive, symmetric and transitive.

In introducing the notion of a simple mathematical structure, we shall first present the abstract definition, followed by two familiar examples from algebra.

Definition 1.7. Let X be a set of objects and $\{A_1, A_2, \cdots, A_n\}$ be a finite set of relations on X. A simple mathematical structure on X is the set X together with the finite set $\{A_1, A_2, \cdots, A_n\}$ of relations on X. The mathematical structure is denoted by $(X, A_1, A_2, \cdots, A_n)$. The relations A_i are called the distinguished relations of the structure.

Example 1. A group is a mathematical structure (X, E, M, I) where X is the set of elements of the group, E is the unary relation consisting of the identity e:

$$E = \{e\},$$

M is a ternary relation, the group operation (multiplication):

$$M = \{(x, y, z) : z = xy\},$$

and I is a binary relation, the group inverse:

$$I = \{(x, y) : xy = e\}.$$

Example 2. The natural numbers form a simple mathematical structure (X, U, S) where X is the set of natural numbers, U is a unary relation consisting of the element 1:

$$U = \{1\},$$

and S is a binary relation, the successor:

$$S = \{(x, y) : y \text{ is the successor of } x\}.$$

Finally, we define the notion of a matrix in a set X.

Definition 1.8. Let X be a set, and m, n be positive integers. Let $m \times n$ denote the set of all ordered pairs of positive integers (i, j) such that $i \leq m$, $j \leq n$. An $m \times n$ matrix in X is a function defined on the set $m \times n$ into the set X. Such matrices are often represented by rectangular arrays as in Figure 1.1,

$$\begin{matrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{matrix}$$

Figure 1.1

where x_{ij} denotes the value of the function at the point (i, j). The notation (x_{ij}) is also used to represent the matrix. For each $i = 1, 2, \cdots, m$, the sequence

$$(x_{i1}, x_{i2}, \cdots, x_{in})$$

is called the i^{th} row of the matrix; for each $j = 1, 2, \cdots, n$, the sequence

$$(x_{1j}, x_{2j}, \cdots, x_{mj})$$

is called the j^{th} column. The matrix (x_{ij}) is univalent if and only if the function is one-to-one; i.e., $x_{ij} = x_{k\ell}$ implies $(i, j) = (k, \ell)$.

In the next chapter we shall define a rectangular arrangement as a simple mathematical structure consisting of a set X and two binary relations R, C (for "row" and "column"), satisfying certain conditions specified by the axioms of rectangular arrangements. We then show that every univalent matrix determines a unique rectangular arrangement of its elements. In Chapter III we shall study the properties of rectangular arrangements and establish the general relationship between these structures and matrices; namely, that each rectangular arrangement that is not a single row or column corresponds to four distinct matrices (depending on which corner element is chosen for x_{11}).

Note: The term "mathematical structure" will be used in place of "simple mathematical structure" throughout the remainder of this paper, for the sake of brevity.

CHAPTER II

THE MATHEMATICAL STRUCTURE (X, R, C)

To set up the mathematical structure of a rectangular arrangement, we take the set X consisting of the objects in the arrangement and two binary relations R, C on X. The interpretation of R is as follows: The pair (x, y) belongs to R if and only if $x = y$, or x is next to y in the same row. Similarly, (x, y) belongs to C if and only if $x = y$, or x is next to y in the same column. The mathematical structure for this arrangement is then denoted by (X, R, C).

Before writing down the axioms for this structure, it is convenient to introduce four new relations $R_\infty, C_\infty, E_R, E_C$. Here R_∞ is interpreted as the set of all pairs (x, y) such that x and y are in the same row, and E_R is a unary relation consisting of all end-of-row objects. The interpretation of C_∞ and E_C is similar, with "row" replaced by "column".

Definition 2. 1. The pair (x, y) belongs to $\underline{R_\infty}$ if there exists a sequence $(z_1, z_2, \cdots, z_n)$ such that $x = z_1$, $y = z_n$ and $(z_i, z_{i+1}) \varepsilon R$ for $i = 1, 2, \cdots, n-1$.

Definition 2. 2. The object x belongs to $\underline{E_R}$ if there exists at most one y such that $y \neq x$ and $(x, y) \varepsilon R$.

<u>Definition 2.3.</u> The pair (x, y) belongs to $\underline{C_\infty}$ if there exists a sequence $(z_1, z_2, \cdots, z_n)$ such that $x = z_1$, $y = z_n$ and $(z_i, z_{i+1}) \varepsilon C$ for $i = 1, 2, \cdots, n-1$.

<u>Definition 2.4.</u> The object x belongs to $\underline{E_C}$ if there exists at most one y such that $y \neq x$ and $(x, y) \varepsilon C$.

The mathematical structure (X, R, C) is called a rectangular arrangement if and only if the following axioms are satisfied for <u>all x, y, u and v belonging to X:</u>

The interpretation of Axioms 2.1 and 2.2 is obvious; they will be used without further explicit mention throughout the next chapter.

<u>Axiom 2.1.</u> The binary relations R, C on X are both reflexive.

<u>Axiom 2.2.</u> The binary relations R, C on X are both symmteric.

Axiom 2.3a ensures that for every object belonging to X, there are at most two objects next to it in the same row. Axiom 2.3b ensures the same property for columns.

<u>Axiom 2.3a.</u> There are at most two distinct z's such that $z \neq x$ and $(z, x) \varepsilon R$.

<u>Axiom 2.3b.</u> There are at most two distinct z's such that $z \neq x$ and $(z, x) \varepsilon C$.

Axiom 2.4a ensures that for every row of the arrangement there is at least one end-of-row object. Similarly, by Axiom 2.4b, for every column there is at least one end-of-column object.

<u>Axiom 2.4a.</u> There exists a $z \varepsilon E_R$ such that $(x, z) \varepsilon R_\infty$.

<u>Axiom 2.4b.</u> There exists a $z \varepsilon E_C$ such that $(x, z) \varepsilon C_\infty$.

We can interpret Axiom 2.5 as the assertion that each row intersects each column in exactly one point (see Figure 2.1).

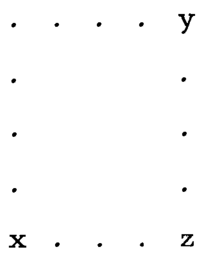

Figure 2.1

<u>Axiom 2.5.</u> There exists exactly one z such that $(x, z) \varepsilon R_\infty$ and $(y, z) \varepsilon C_\infty$.

If we have an arrangement as in Figure 2.2 where x and y are next to each other in the same row, x and u are next to each other in the same column, u and v are in the same row, and y and v are in the same column, then u is next to v in the same row and y is next to v in the same column. This is implied by Axiom 2. 6.

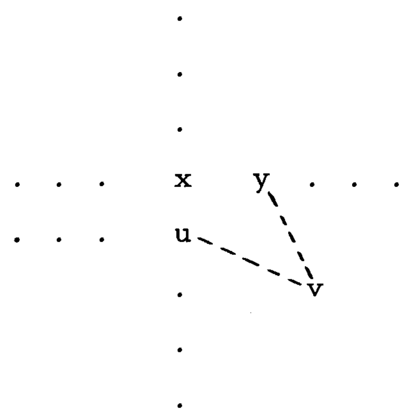

Figure 2. 2

<u>Axiom 2. 6.</u> If $(x, y)\varepsilon R$, $(x, u)\varepsilon C$, $(u, v)\varepsilon R_{\infty}$ and $(y, v)\varepsilon C_{\infty}$, then $(u, v)\varepsilon R$ and $(y, v)\varepsilon C$.

In order to exclude the example of an arrangement without objects, we require that the set X be non-empty. In this paper we shall be concerned with finite sets of objects.

<u>Axiom 2. 7.</u> The set X is finite and non-empty.

We pause to consider the independence of Axioms 2. 1 through 2. 7. We shall illustrate their independence by giving an example of an arrangement that satisfies all but one of Axioms 2. 1 through 2. 7. The proof that the example satisfies all but the specified axiom is left to the reader.

The mathematical structure with $X = \{a, b\}$, $R = \{(a, b), (b, a)\}$ and $C = \{(a, a), (b, b)\}$ satisfies all but Axiom 2. 1.

For the sake of brevity in presenting the remaining examples, we introduce a figure notation to simplify the representation of the arrangement. We shall use r as in Figure 2. 3a to denote that (a, b) and (b, a) belong to R, $\vec{r}$ as in Figure 2. 3b to denote that $(a, b) \varepsilon R$ and (b, a) does not belong to R, and c as in Figure 2. 4 to denote that (a, b) and (b, a) belong to C. It is understood that (a, a) and (b, b) belong to R and C.

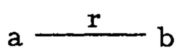

Figure 2. 3a

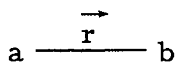

Figure 2. 3b

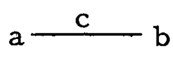

Figure 2. 4

The arrangement in Figure 2. 5 satisfies all but Axiom 2. 2.

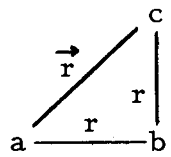

Figure 2.5

It is interesting to note that Axiom 2.2 can be omitted if Axiom 2.6 is replaced by Axiom 2.6* as follows:

<u>Axiom 2.6*.</u> If $(x, y)\varepsilon R$, $(x, u)\varepsilon C$, $(u, v)\varepsilon R_\infty$ and $(y, v)\varepsilon C_\infty$, then $(v, u)\varepsilon R$ and $(v, y)\varepsilon C$.

If $(x, y)\varepsilon R$, taking $x = u$ and $y = v$ it follows from this axiom that $(y, x)\varepsilon R$. If $(x, u)\varepsilon C$, taking $y = u$ and $v = u$ it follows that $(u, x)\varepsilon C$.

An arrangement as in Figure 2.6a satisfies all but Axiom 2.3a and the arrangement as in Figure 2.6b satisfies all but Axiom 2.3b.

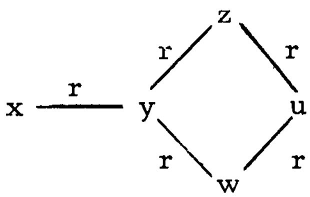

Figure 2.6a

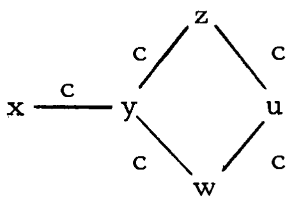

Figure 2.6b

A "circular arrangement" as in Figure 2.7a satisfies all but Axiom 2.4a and the "circular arrangement" as in Figure 2.7b satisfies all but Axiom 2.4b.

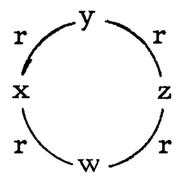

Figure 2.7a

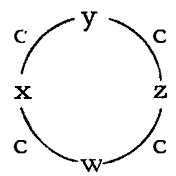

Figure 2.7b

Two separate rectangular arrangements, as in Figure 2.8, will satisfy all but Axiom 2.5.

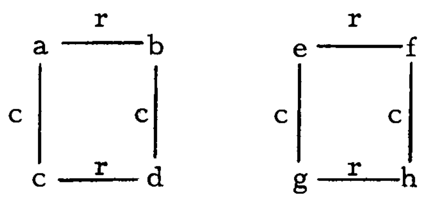

Figure 2.8

Figure 2.9 represents an arrangement that satisfies all but Axiom 2.6. (Take $x = a_{22}$, $y = a_{23}$, $u = a_{32}$, $v = a_{33}$ in Axiom 2.6)

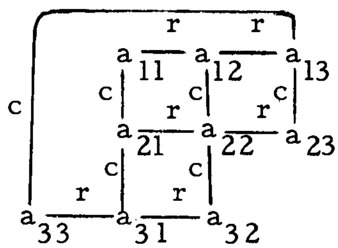

Figure 2.9

An "infinite arrangement" as represented in Figure 2.10 satisfies all but Axiom 2.7.

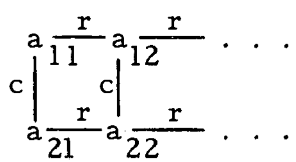

Figure 2.10

The proof that the following mathematical structures are rectangular arrangements is left to the reader.

<u>Example 2. 1.</u> Consider the row of objects a, b, c as in Figure 2. 11.

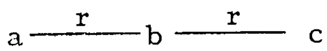

Figure 2. 11

The rectangular arrangement is given by the set

$$X = \{a, b, c\},$$

together with the binary relations R consisting of the pairs (a, a), (b, b), (c, c), (a, b), (b, a), (b, c) and (c, b), and the relation C consisting of the pairs (a, a), (b, b) and (c, c). From Definition 2. 1 it follows that R_∞ consists of all pairs of the set R and also the pairs (a, c) and (c, a). By Definition 2. 2,

$$E_R = \{a, c\}.$$

Similarly, by Definition 2. 3 it follows that $C_\infty = C$ and by Definition 2. 4 that

$$E_C = \{a, b, c\}.$$

<u>Example 2. 2.</u> Consider a univalent $m \times n$ matrix (x_{ij}) as given by Definition 1. 8 and illustrated in Figure 1. 1. A rectangular arrangement is given by the set

$$X = \{x_{ij} : 1 \leq i \leq m, \quad 1 \leq j \leq n\}$$

together with the binary relations R and C,

$$R = \{(x_{ij}, x_{i(j+1)}), (x_{i(j+1)}, x_{ij}), (x_{ij}, x_{ij}), (x_{in}, x_{in}) : 1 \leq i \leq m, \ 1 \leq j < n\},$$

and

$$C = \{(x_{ij}, x_{(i+1)j}), (x_{(i+1)j}, x_{ij}), (x_{ij}, x_{ij}), (x_{mj}, x_{mj}) : 1 \leq i < m, \ 1 \leq j \leq n\}.$$

This mathematical structure is called the rectangular arrangement (X, R, C) <u>associated with the univalent matrix (x_{ij})</u>.

From Definition 2. 1 it follows that

$$R_{\infty} = \{(x_{ij}, x_{i\ell}) : 1 \leq i \leq m, \ 1 \leq j \leq n, \ 1 \leq \ell \leq n\}.$$

By Definition 2. 2,

$$E_R = \{x_{i1}, x_{in} : 1 \leq i \leq m\}.$$

From Definition 2. 3 it follows that

$$C_{\infty} = \{(x_{ij}, x_{\ell j}) : 1 \leq i \leq m, \ 1 \leq j \leq n, \ 1 \leq \ell \leq m\}.$$

By Definition 2.4,

$$E_C = \{x_{1j}, x_{mj}: 1 \leq j \leq \}.$$

The main result of the next chapter is that every rectangular arrangement is associated in this way with some univalent matrix. This result shows that our definition captures the essence of rectangular arrangements alone, by excluding arrangements of all other kinds.

CHAPTER III

SOME PROPERTIES OF RECTANGULAR ARRANGEMENTS (X, R, C)

In this chapter we shall derive several properties of an arbitrary rectangular arrangement (X, R, C).

We shall first show that each rectangular arrangement has a dual, obtained by interchanging R and C, which is again a rectangular arrangement.

Definition 3. 1. Let (X, R, C) be a rectangular arrangement. Then the structure (X, C, R) obtained by interchanging R and C, is called the dual of (X, R, C).

Theorem 3. 1. The dual, (X, C, R), of a rectangular arrangement (X, R, C) is itself a rectangular arrangement. Moreover, when R and C are interchanged to form the dual, R_∞ is interchanged with C_∞ and E_R is interchanged with E_C.

Proof: It follows from Definitions 2. 1-2. 4 that when R and C are interchanged, R_∞ is interchanged with C_∞, and E_R with E_C. Moreover, it is easy to check that Axioms 2. 1-2. 6 remain true if these interchanges are made in the axioms. Thus, Axiom 2. 1 and 2. 2 are essentially unchanged, Axioms 2. 3a and 2. 3b are interchanged, as are Axioms 2.4a and 2.4b; Axiom 2.5 is unchanged provided the "dummy variables" x and y are interchanged, and Axiom 2.6 is unchanged if the "dummy variables" y and u are interchanged. It follows that if (X, R, C) is a rectangular arrangement, so also is (X, C, R).

An important fact, closely related to Theorem 3.1, is that each proposition that can be proved in the theory of rectangular structures remains true if R, R_∞ and E_R are interchanged with C, C_∞ and E_C, respectively. The new proposition obtained in this way will be called the dual of the original proposition. To see that the dual of a proposition P is true if and only if P itself is true, notice that the dual of P says the same thing about the dual structure (X, C, R) as P says about (X, R, C).

Several lemmas and theorems of this chapter are arranged in dual pairs, marked "a" and "b", such as Theorem 3.3a and b. In each case only the "a" statement need be proved, since the "b" statement is its dual.

Theorem 3.2. The relations R_∞ and C_∞ are equivalence relations.

Proof: To prove that R_∞ is an equivalence relation, we must show that it is reflexive, symmetric and transitive (see Definitions 1.3 to 1.5).

For reflexivity, let a be any object in X, and let (z_1, z_2) be the sequence of length two such that $z_1 = z_2 = a$. By Axiom 2. 1, $(a,a)\varepsilon R$; hence by Definition 2. 1, $(a,a)\varepsilon R_\infty$, and R_∞ is reflexive.

For symmetry, let (a,b) be any pair belonging to R_∞. By Definition 2. 1 there is a sequence $(z_1, z_2, \cdots, z_n)$ such that $z_1 = a$, $z_n = b$ and $(z_i, z_{i+1})\varepsilon R$ for all $i = 1, 2, \cdots, n-1$. Consider the sequence $(w_1, w_2, \cdots, w_n)$ such that $w_i = z_{n-i+1}$, $i = 1, 2, \cdots, n$. Then $w_1 = b$, $w_n = a$ and by Axiom 2. 2, $(w_i, w_{i+1})\varepsilon R_\infty$ for $i = 1, 2, \cdots, n-1$. Hence $(b,a)\varepsilon R_\infty$, and R_∞ is reflexive.

For transitivity, suppose $(a,b)\varepsilon R_\infty$ and $(b,c)\varepsilon R_\infty$. By Definition 2. 1 there is a sequence $(v_1, v_2, \cdots, v_n)$ such that $v_1 = a$, $v_n = b$, $(v_i, v_{i+1})\varepsilon R$ for all $i = 1, 2, \cdots, n-1$, and a sequence $(u_1, u_2, \cdots, u_m)$ such that $u_1 = b$, $u_m = c$, $(u_j, u_{j+1})\varepsilon R$ for $j = 1, 2, \cdots, m-1$. Consider the sequence $(w_1, w_2, \cdots, w_{m+n-1})$ such that

$$w_i = \begin{cases} v_i & \text{for } i = 1, 2, \cdots, n-1 \\ u_{i-n+1} & \text{for } i = n, n+1, \cdots, m+n-1 \, . \end{cases}$$

Then $w_1 = a$, $w_{m+n-1} = c$ and $(w_i, w_{i+1})\varepsilon R$ for $i=1,2,\cdots,m+n-2$. Hence $(a,c)\varepsilon R_\infty$, and R_∞ is transitive.

We have shown that R_∞ is an equivalence relation. The dual assertion is that C_∞ is an equivalence relation.

It is clear from the interpretation of R_∞ and C_∞ (see Chapter II) that the equivalence classes [1] of these two equivalence relations are just the rows and columns of the rectangular arrangement. This is the motivation for

Definition 3. 2. The equivalence classes of R_∞ and C_∞ are called rows and columns, respectively.

If an object is not at the end of a row, then there must be two objects next to it in the same row. This, and the dual statement for columns, is the assertion of Theorem 3.3a-b.

Theorem 3.3a. If x does not belong to E_R, then there exist exactly two objects z such that $z \neq x$ and $(x, z)\varepsilon R$.

b. If x does not belong to E_C, then there exist exactly two objects z such that $z \neq x$ and $(x, z)\varepsilon C$.

Proof: If there are not at least two distinct z's such that $z \neq x$ and $(z, x)\varepsilon R$, by Definition 2. 2 we reach the contradiction that $x\varepsilon E_R$. Thus, there are at least two such z's and by Axiom 2. 3a there are at most two such z's. Hence there are exactly two z's and the theorem follows.

Out next theorem asserts that two objects which are in the same row, and also in the same column, are necessarily equal.

<u>Theorem 3.4.</u> If (x, y) belongs to both R_∞ and C_∞ then $x = y$.

<u>Proof:</u> By Theorem 3.2 $(x, x)\varepsilon R_\infty$ and $(y, y)\varepsilon C_\infty$. Hence the statement $(x, z)\varepsilon R_\infty$, $(y, z)\varepsilon C_\infty$ is true for $z = x$ and for $z = y$, but by Axiom 2.5 there is exactly one such z. Thus $x = z = y$.

Lemma 3.la asserts that the columns containing two different objects of a given row are disjoint. (See Figure 3.1.)

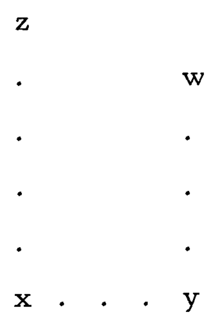

Figure 3.1

<u>Lemma 3.1a.</u> Let x, y, w, z be objects such that $x \neq y$, $(x, y)\varepsilon R_\infty$, and both (z, x) and (w, y) belong to C_∞. Then z and w are distinct.

<u>b.</u> Let x, y, w, z be objects such that $x \neq y$, $(x, y)\varepsilon C_\infty$, and both (z, x) and (w, y) belong to R_∞. Then z and w are distinct.

<u>Proof:</u> Suppose that $z = w$. Then (w, x) and (w, y) belong to C_∞. It follows from Theorem 3.2 that $(x, y)\varepsilon C_\infty$. By Theorem 3.4

$x = y$, contradicting the hypothesis that $x \neq y$. Thus z and w must be distinct.

Theorem 3.5a asserts that an object next to and in the same column as an end-of-row object is also an end-of-row object (see Figure 3.2). Corollary 3.1a asserts the same property for <u>any</u> object in the same column as an end-of-row object.

$$\begin{array}{ccc} \cdot & \cdot & \\ x & x_1 & \cdots \\ y & y_1 & \cdots \\ \cdot & \cdot & \\ \cdot & \cdot & \end{array}$$

Figure 3.2

<u>Theorem 3.5a.</u> If $x \varepsilon E_R$ and $(x, y) \varepsilon C$, then $y \varepsilon E_R$.

<u>b.</u> If $x \varepsilon E_C$ and $(x, y) \varepsilon R$, then $y \varepsilon E_C$.

<u>Proof:</u> If $x = y$, then obviously $y \varepsilon E_R$. If $x \neq y$, suppose y does not belong to E_R. Then by Theorem 3.3a there are two distinct z's such that $z \neq y$ and $(z, y) \varepsilon R$. We shall call these objects z_1 and z_2. (See Figure 3.3.)

$$\begin{array}{ccc} u_1 & x & u_2 \\ z_1 & y & z_2 \end{array}$$

Figure 3.3

By Axiom 2.5 there exists a u_1 such that $(z_1, u_1)\epsilon C_\infty$ and $(u_1, x)\epsilon R_\infty$. It follows from Axiom 2.6 that $(u_1, x)\epsilon R$ and $(u_1, z_1)\epsilon C$.

If $u_1 = x$, then (x, z_1) and (x, y) belong to C, and by Theorem 3.2 $(z_1, y)\ \epsilon C_\infty$. It follows from Theorem 3.4 that $z_1 = y$, contradicting the assumption that $z_1 \neq y$. Thus $u_1 \neq x$. Similarly, there exists a u_2 such that $u_2 \neq x$, $(x, u_2)\epsilon R$, and $(u_2, z_2)\epsilon C$.

Since $x\epsilon E_R$, it follows from Definition 2.2 that $u_1 = u_2$. Thus by Theorem 3.2, $(z_1, z_2)\epsilon C_\infty$. Since by Theorem 3.2 $(z_1, z_2)\epsilon R_\infty$, it follows from Theorem 3.4 that $z_1 = z_2$, contradicting the fact that z_1 and z_2 are distinct. Hence $y\epsilon E_R$.

<u>Corollary 3.1a.</u> If $x\epsilon E_R$ and $(x, y)\epsilon C_\infty$ then $y\epsilon E_R$.

<u>b</u>. If $x\epsilon E_C$ and $(x, y)\epsilon R_\infty$ then $y\epsilon E_C$.

<u>Proof:</u> It follows directly from Theorem 3.5a by induction on the length of a sequence $(z_1, z_2, \cdots, z_n)$ such that $x = z_1$, $y = z_n$ and $(z_i, z_{i+1})\epsilon C$ for $i = 1, 2, \cdots, n-1$.

In order to establish that each rectangular arrangement is associated with some matrix, we introduce a special type of sequence.

<u>Definition 3.3.</u> A sequence $(w_1, w_2, \cdots, w_n)$ such that $w_i = w_j$ if and only if $i = j$ for $i, j = 1, 2, \cdots, n$ is called a <u>univalent sequence</u>.

The next theorem shows that we may always take the sequence

$(z_1, z_2, \cdots, z_n)$ in Definition 2.1 for R_∞, or in Definition 2.3 for C_∞, to be univalent.

<u>Theorem 3.6a.</u> If $(x, y) \varepsilon R_\infty$ and $x \neq y$, then there exists a unique univalent sequence $(u_1, u_2, \cdots, u_n)$ such that $x = u_1$, $y = u_n$ and $(u_i, u_{i+1}) \varepsilon R$ for $i = 1, 2, \cdots, n-1$.

<u>b</u>. If $(x, y) \varepsilon C_\infty$ and $x \neq y$, then there exists a unique univalent sequence $(u_1, u_2, \cdots, u_n)$ such that $x = u_1$, $y = u_n$ and $(u_i, u_{i+1}) \varepsilon C$ for $i = 1, 2, \cdots, n-1$.

<u>Proof:</u> Since $(x, y) \varepsilon R_\infty$, by Definition 2.1 there is at least one sequence $(u_1, u_2, \cdots, u_m)$ such that $x = u_1$, $y = u_m$ and $(u_i, u_{i+1}) \varepsilon R$ for $i = 1, 2, \cdots, m-1$. The set of lengths of all such sequences is a nonempty set of positive integers. Therefore it has a least element, which we shall denote by n. We shall show that every such sequence $(u_1, u_2, \cdots, u_n)$ of minimal length is a univalent sequence.

Suppose that $u_i = u_j$ for some $i < j$, $i, j = 1, 2, \cdots, n$. Then the sequence $(v_1, v_2, \cdots, v_{n-j+i})$ where $v_\ell = u_\ell$ for $\ell = 1, 2, \cdots, i$, and $v_\ell = u_{\ell - i + j}$ for $\ell = i+1, i+2, \cdots, n-j+i$ is a shorter sequence such that $x = v_1$, $y = v_{n-j+i}$ and $(v_\ell, v_{\ell+1}) \varepsilon R$ for $\ell = 1, 2, \cdots, n-j+i-1$. This contradicts the fact that $(u_1, u_2, \cdots, u_n)$ is the shortest sequence with this property; hence, $u_i \neq u_j$ for $i \neq j$.

The proof of uniqueness depends on the following:

Proposition 3.1. There is no univalent sequence $(u_1, u_2, \ldots, u_n)$ such that $n \geq 3$, $(u_i, u_{i+1}) \varepsilon R$ for $i = 1, 2, \ldots, n-1$ and $(u_1, u_n) \varepsilon R$.

Proof: Suppose such a sequence $(u_1, u_2, \ldots, u_n)$ exists. By Axiom 2.4a there is a $z \varepsilon E_R$ such that $(u_1, z) \varepsilon R_\infty$. By definition 2.2, no term of the sequence $(u_1, u_2, \ldots, u_n)$ belongs to E_R; hence, z is not a term of this sequence. As proved above, there is a univalent sequence $(v_1, v_2, \ldots, v_r)$ such that $v_1 = u_1$, $v_r = z$ and $(v_i, v_{i+1}) \varepsilon R$ for $i = 1, 2, \ldots, r-1$. Since $v_1 = u_1$, and $v_r = z$ is not a term of $(u_1, u_2, \ldots, u_n)$, there is a k, $1 \leq k \leq r$, such that $v_k = u_i$ for some $i = 1, 2, \ldots, n$ and $v_{k+1} \neq u_j$ for all $j = 1, 2, \ldots, n$.. Then there are three distinct values of w such that $w \neq v_k$ and $(v_k, w) \varepsilon R$; namely, $w = u_{i-1}$, u_{i+1} and v_{k+1}, where we take $u_0 = u_n$ if $i = 1$, and $u_{n+1} = u_1$ if $i = n$. Since this contradicts Axiom 2.4a, the proposition is proved.

Returning to the proof of uniqueness in Theorem 3.6a, let $(w_1, w_2, \ldots, w_m)$ be a univalent sequence such that $x = w_1$, $y = w_m$ and $(w_i, w_{i+1}) \varepsilon R$ for $i = 1, 2, \ldots, m-1$. Since the sequence $(u_1, u_2, \ldots, u_n)$ is the shortest such sequence, $m \geq n$. Then there exists a greatest integer k, $1 \leq k \leq n$, such that $w_i = u_i$ for all $i \leq k$. If $k = n$, $u_n = w_n = w_m$, and since $(w_1, w_2, \ldots w_m)$ is univalent, $n = m$ and the two sequences are equal. If $k < n$, let ℓ be the least integer such that $k + 1 \leq \ell \leq n$ and $u_\ell = w_j$ for some

$j = k+1,\ k+2, \ldots, m$. Consider the sequence

$$(u_k, u_{k+1}, \ldots, u_{\ell-1}, w_{j-1}, \ldots, w_{k+2}, w_{k+1}).$$

By the definition of k, we have $u_k \neq u_{k+1} \neq w_{k+1} \neq u_k$; hence, the sequence is of length ≥ 3. Furthermore, by the definition of ℓ, the sequence is univalent. By Proposition 3.1 no such sequence exists. This contradiction rules out the case $k < n$, completing the proof.

We shall next show that a row with at least two objects has exactly two distinct end-of-row objects.

Theorem 3.7a. If there exist distinct x, y such that $(x, y) \varepsilon R_\infty$, then there exist exactly two distinct z's εE_R such that $(x, z) \varepsilon R_\infty$.

b. If there exist distinct x, y such that $(x, y) \varepsilon C_\infty$, then there exist exactly two distinct z's εE_C such that $(x, z) \varepsilon C_\infty$.

Proof: Since $x \neq y$ and $(x, y) \varepsilon R_\infty$, it follows from Theorem 3.6a that there exists at least one univalent sequence $(u_1, u_2, \ldots, u_m)$ such that $x = u_j$ for some $j = 1, 2, \ldots, m$ and $(u_i, u_{i+1}) \varepsilon R$ for $i = 1, 2, \ldots, m-1$. Since the set X is finite (Axiom 2.7), the set of lengths of all such sequences is a finite nonempty set of integers. Therefore it has a greatest element, which we shall denote by n. From Definition 3.3, $u_1 \neq u_n$. We shall show that u_1 and u_n belong to E_R.

Suppose u_1 does not belong to E_R. By Theorem 3.3a there exist exactly two z's such that $z \neq u_1$ and $(u_1, z) \varepsilon R$. Let these z's be denoted by z_1 and z_2. Since $u_2 \neq u_1$ and $(u_1, u_2) \varepsilon R$, it follows from Axiom 2. 3a that $u_2 = z_i$ for $i = 1$ or $i = 2$. We may assume without loss of generality that $z_2 = u_2$. We will show that $(z_1, u_1, u_2, \ldots u_n)$ is a univalent sequence, contradicting the fact that $(u_1, u_2, \ldots, u_n)$ is the longest such sequence.

Suppose that $z_1 = u_k$, and $z_1 \neq u_i$ for $i = 1, 2, \ldots, k-1$. Consider the univalent sequence $(z_1, u_1, u_2, \ldots, u_{k-1})$. Now $k \geq 3$ since $z_1 \neq u_i$ for $i = 1, 2$; and since $z_1 = u_k$ we have $(z_1, u_{k-1}) \varepsilon R$. By Proposition 3. 1 no such sequence exists. Thus no such k exists and $(z_1, u_1, \ldots, u_n)$ is a univalent sequence. This contradiction establishes that $u_1 \varepsilon E_R$. Using a similar argument, it follows that $u_n \varepsilon E_R$.

To prove that u_1 and u_n are the only two such objects, we need the following proposition:

<u>Proposition 3. 2.</u> Let $(u_1, u_2, \ldots, u_n)$ be a univalent sequence such that $(u_i, u_{i+1}) \varepsilon R$ for $i = 1, 2, \ldots, n-1$ and both u_1 and u_n belong to E_R. If z is any object such that $(z, u_1) \varepsilon R_\infty$, then $z = u_j$ for some $j = 1, 2, \ldots, n$.

<u>Proof:</u> Follows by an argument similar to that used in the first part of the proof of Proposition 3. 1.

Returning to the proof of the theorem, suppose E_R contains a third object z such that $u_1 \neq z \neq u_n$ and $(u_1, z) \varepsilon R_\infty$. By Proposition 3. 2 $z = u_i$ for some $i = 2, 3, \ldots, n-1$. By Definition 2. 2 u_i does not belong to E_R. This contradicts the assumption that $z \varepsilon E_R$, and completes the proof.

If an object is both an end-of-row and an end-of-column, we shall call it a "corner object". We show in Theorem 3.8 that every rectangular arrangement that is not the special case of a single row or column has four distinct corner objects. Before we show the existence of four such objects, we shall show that there is at least one corner object for each pair of objects, where one of the pair is an end-of-row and the other an end-of-column.

Lemma 3.2. If $x \varepsilon E_R$ and $y \varepsilon E_C$, there exists a $z \varepsilon E_R \cap E_C$ such that $(z, x) \varepsilon C_\infty$ and $(z, y) \varepsilon R_\infty$.

Proof: By Axiom 2.5 there is a z such that $(x, z) \varepsilon C_\infty$ and $(y, z) \varepsilon R_\infty$. By Corollary 3.1a $z \varepsilon E_R$, and by Corollary 3.1b $z \varepsilon E_C$. Therefore, $z \varepsilon E_R \cap E_C$, and the lemma follows.

Theorem 3.8. If there exists a pair (x, y) which does not belong to $R_\infty \cup C_\infty$, then $E_R \cap E_C$ contains exactly four objects.

Proof: We see that the hypothesis does not allow the arrangement to be a single row or column. As a guide in the proof we may visualize the arrangement as in Figure 3.4.

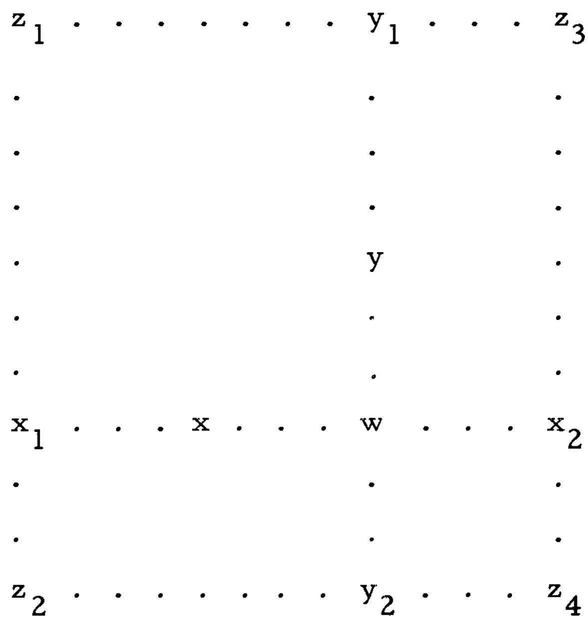

Figure 3.4

By Axiom 2.5 there exists a w such that $(x, w) \varepsilon R_\infty$ and $(w, y) \varepsilon C_\infty$. Since (x, y) does not belong to C_∞ or R_∞, it follows from Theorem 3.2 that $w \neq x$, and $w \neq y$. By Theorem 3.7a there are distinct x_1 and x_2 belonging to E_R such that (x_1, x) and (x_2, x) belong to R_∞. By Theorem 3.7b there are distinct y_1 and y_2 belonging to E_C such that (y_1, y) and (y_2, y) belong to C_∞. It follows from Lemma 3.2 that there exist the following z's:

(i) $z_1 \varepsilon E_R \cap E_C$ such that $(z_1, y_1) \varepsilon R_\infty$ and $(z_1, x_1) \varepsilon C_\infty$.

(ii) $z_2 \varepsilon E_R \cap E_C$ such that $(z_2, y_2) \varepsilon R_\infty$ and $(z_2, x_1) \varepsilon C_\infty$.

(iii) $z_3 \varepsilon E_R \cap E_C$ such that $(z_3, y_1) \varepsilon R_\infty$ and $(z_3, x_2) \varepsilon C_\infty$.

(iv) $z_4 \varepsilon E_R \cap E_C$ such that $(z_4, y_2) \varepsilon R_\infty$ and $(z_4, x_2) \varepsilon C_\infty$.

We see that $z_1 \neq z_2$; for if $z_1 = z_2$ then by Theorem 3. 2 $(y_1, y_2) \varepsilon R_\infty$. Since $(y_1, y_2) \varepsilon C_\infty$, by Theorem 3. 4 $y_1 = y_2$, contradicting the distinctness of y_1 and y_2. Similarly, it follows that $z_k \neq z_\ell$ for $k \neq \ell$, $k, \ell = 1, 2, 3, 4$.

Now suppose that $E_R \cap E_C$ contains a fifth point u, distinct from z_1, z_2, z_3, z_4. By Axiom 2. 5 there exists a u_1 such that $(z_1, u_1) \varepsilon C_\infty$ and $(u, u_1) \varepsilon R_\infty$, and a u_2 such that $(z_1, u_2) \varepsilon R_\infty$ and $(u, u_2) \varepsilon C_\infty$. (See Figure 3. 5.)

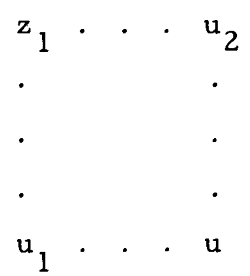

Figure 3. 5

By Corollary 3. 1a, u_1 and u_2 belong to E_R and by Corollary 3. 1b, u_1 and u_2 belong to E_C. Thus by Theorem 3. 7b, $u_1 = z_1$ or $u_1 = z_2$, and by Theorem 3. 7a $u_2 = z_1$ or $u_2 = z_3$. If $u_1 = z_1$, then (z_1, u_2) and (z_1, u) belong to R_∞ and by Theorem 3. 2, $(u_2, u) \varepsilon R_\infty$. It follows from Theorem 3. 4 that $u_2 = u$.

This contradicts the supposition that $u \neq z_i$ for $i = 1, 2, 3, 4$. Thus $u_1 \neq z_1$, so that $u_1 = z_2$. Similarly, $u_2 = z_3$ and by Axiom 2. 5 $u = z_4$, contradicting the assumption that u is distinct from z_1, z_2, z_3, z_4. Thus the theorem follows.

Theorem 3.9 implies that a rectangular arrangement with one object has one corner object, and a rectangular arrangement having one row or one column and at least two objects has exactly two corner objects.

Theorem 3. 9. If the conditions for Theorem 3. 8 are not satisfied; i. e., every pair $(x, y) \varepsilon R_\infty \cup C_\infty$, then $E_R \cap E_C$ contains one or two objects.

Proof: If the set X contains exactly one object z, then by Definition 2. 2 $z \varepsilon E_R$ and by Definition 2. 4 $z \varepsilon E_C$. Thus one object belongs to $E_R \cap E_C$.

If the set X contains at least two objects, let x, y be any two distinct objects. Then $(x, y) \varepsilon R_\infty \cup C_\infty$. It follows from Theorem 3. 4 that (x, y) does not belong to $R_\infty \cap C_\infty$. Thus $(x, y) \varepsilon R_\infty$ or $(x, y) \varepsilon C_\infty$ but not both. Suppose that $(x, y) \varepsilon R_\infty$ and (x, y) does not belong to C_∞. Let w be any object belonging to X. If (x, w) does not belong to R_∞, it follows from Theorem 3. 2 that (w, y) does not belong to R_∞. Thus both (w, x) and (w, y) belong to C_∞. By Theorem 3. 2 $(x, y) \varepsilon C_\infty$, contradicting the supposition that

(x, y) does not belong to C_∞. Thus $(x, w)\varepsilon R_\infty$. Hence, there is one row. It follows from Theorem 3.7a that there exist exactly two u's belonging to E_R. If there exists a y such that $(y, u)\varepsilon C$, it follows from Theorem 3.4 that $u = y$. Thus by Definition 2.4 $u\varepsilon E_C$. Thus $u\varepsilon E_R \cap E_C$. Since only two objects belong to E_R, there are exactly two corner objects.

Similarly, if $(x, y)\varepsilon C_\infty$ and (x, y) does not belong to R_∞, there is one column with two corner objects.

Figure 3.6 illustrates Theorem 3.10a.

```
z   w
.   .
.   .
.   .
x   y
```

Figure 3.6

<u>Theorem 3.10a.</u> If $(x, y)\varepsilon R$, (x, z) and (y, w) belong to C_∞, and $(z, w)\varepsilon R_\infty$, then $(z, w)\varepsilon R$.

<u>b</u>. If $(x, y)\varepsilon C$, (x, z) and (y, w) belong to R_∞, and $(z, w)\varepsilon C_\infty$, then $(z, w)\varepsilon C$.

<u>Proof:</u> By Definition 2.1 there exists a sequence $(u_1, u_2, \cdots, u_n)$ such that $x = u_1$, $z = u_n$ and $(u_i, u_{i+1})\varepsilon C$ for $i = 1, 2, \cdots, n-1$. By Axiom 2.5 there exists a v_2 such that $(v_2, u_2)\varepsilon R_\infty$ and

$(v_2, y)\varepsilon C_\infty$, and by Axiom 2.6 $(v_2, u_2)\varepsilon R$ and $(v_2, y)\varepsilon C$. By continuing in this way, we obtain a sequence $(v_2, v_3, \cdots, v_n)$ such that $(v_n, u_n)\varepsilon R$ and $(v_i, v_{i+1})\varepsilon C$ for $i = 2, 3, \cdots, n-1$. It follows from Definition 2.3 and Theorem 3.2 that $(y, v_n)\varepsilon C_\infty$. Hence the statement $(z, u)\varepsilon R_\infty$ and $(y, u)\varepsilon C_\infty$ is true for $u = v_n$ and for $u = w$. By Axiom 2.5 there is exactly one such u. Thus $u = v_n = w$. Hence $(z, w)\varepsilon R$.

If a row has at least two distinct objects, and one object in the row is a corner object, then there exists another object in the same row that is a corner object. This is the assertion of Lemma 3.3a.

Lemma 3.3a. Let x, y be objects such that $x \neq y$, $(x, y)\varepsilon R_\infty$ and $x\varepsilon E_R \cap E_C$. Then there exists a unique z such that $z \neq x$, $(x, z)\varepsilon R_\infty$, and $z\varepsilon E_R \cap E_C$.

b. Let x, y be objects such that $x \neq y$, $(x, y)\varepsilon C_\infty$ and $x\varepsilon E_R \cap E_C$. Then there exists a unique z such that $z \neq x$, $(x, z)\varepsilon C_\infty$, and $z\varepsilon E_R \cap E_C$.

Proof: By Theorem 3.7a there exists a z such that $z \neq x$, $(x, z)\varepsilon R_\infty$ and $z\varepsilon E_R$. It follows from Corollary 3.1b that $z\varepsilon E_C$. Therefore, $z\varepsilon E_R \cap E_C$. Furthermore, by Theorem 3.7a, there is only one z satisfying the lemma.

Lemma 3.4a asserts that any row with at least two objects can be arranged in a univalent sequence in exactly two ways.

Lemma 3.4a. If there exist distinct x, y such that $(x, y)\varepsilon R_\infty$, then there exist exactly two distinct univalent sequences $(u_1, u_2, \cdots, u_n)$ such that $x = u_j$ for some $j = 1, 2, \cdots, n$, both u_1 and u_n belong to E_R, and $(u_i, u_{i+1})\varepsilon R$ for $i = 1, 2, \cdots, n-1$; moreover, if $(z, x)\varepsilon R_\infty$ then $z = u_j$ for some j.

b. If there exist distinct x, y such that $(x, y)\varepsilon C_\infty$, then there exist exactly two distinct univalent sequences $(u_1, u_2, \cdots, u_n)$ such that $x = u_j$ for some $j = 1, 2, \cdots, n$, both u_1 and u_n belong to E_C and $(u_i, u_{i+1})\varepsilon C$ for $i = 1, 2, \cdots, n-1$; moreover, if $(z, x)\varepsilon C_\infty$ then $z = u_j$ for some j.

Proof: By Theorem 3.7a there are exactly two distinct z'sεE_R such that $(x, z)\varepsilon R_\infty$. Let the z's be denoted by z_1 and z_2. By Theorem 3.6a there exists a univalent sequence $(u_1, u_2, \cdots, u_n)$ such that $u_1 = z_1$, $u_n = z_2$ and $(u_i, u_{i+1})\varepsilon R$ for $i = 1, 2, \cdots, n-1$. It follows by Proposition 3.2 that $x = u_j$ for some j. Consider the sequence $(w_1, w_2, \cdots, w_n)$ such that $w_i = u_{n-i+1}$ for $i = 1, 2, \cdots, n$. Then $w_1 = z_2$, $w_n = z_1$, $(w_i, w_{i+1})\varepsilon R$ for $i = 1, 2, \cdots, n-1$, and the sequence is univalent. Since $w_1 \neq u_1$ the two univalent sequences are different.

Now suppose that there exists a third univalent sequence $(v_1, v_2, \cdots, v_m)$ such that $x = v_j$ for some $j = 1, 2, \cdots, m$, both v_1 and v_m belong to E_R and $(v_i, v_{i+1}) \varepsilon R$ for $i = 1, 2, \cdots, m-1$. It follows from Theorem 3.7a that $z_1 = v_1$ and $z_2 = v_m$, or $z_1 = v_m$ and $z_2 = v_1$. If $v_1 = z_1 = u_1$, it follows from Definition 2.2 that $v_2 = u_2$ and from Axiom 2.3a that $v_i = u_i$ for $i = 1, 2, \cdots, k$ where k is the smaller of m and n. Since the sequences $(u_1, u_2, \cdots, u_n)$ and $(v_1, v_2, \cdots, v_m)$ are univalent, $n = m$. This implies that the two sequences are the same, contradicting the assumption that they are different. Thus $z_2 = v_1$ and $z_1 = v_m$. Using a similar argument, it follows that $m = n$ and $w_i = v_i$ for $i = 1, 2, \cdots, n$, contradicting the assumption that they are different. Hence there exist exactly two such sequences.

If a rectangular arrangement is associated with a univalent matrix (x_{ij}), we shall show that x_{11} <u>has to be</u> one of the corner objects. Once this corner object has been picked, this matrix is uniquely determined. This is the assertion of the following lemma.

<u>Lemma 3.5.</u> If a rectangular arrangement (X, R, C) is associated with a univalent matrix (x_{ij}), then $x_{11} \varepsilon E_R \cap E_C$. If a rectangular arrangement is associated with two univalent matrices $(x_{ij}), (w_{ij})$ with $x_{11} = w_{11}$ then $(x_{ij}) = (w_{ij})$.

Proof: It is easy to verify by Example 2. 2 that $x_{11} \varepsilon E_R \cap E_C$. Suppose the matrices (x_{ij}) and (w_{ij}) are of dimension $m_1 \times n_1$ and $m_2 \times n_2$, respectively. Let n be the lesser of n_1 and n_2. It follows from Definition 2. 2 and Axiom 2. 3a by induction that $w_{1j} = x_{1j}$ for $j = 1, 2, \cdots, n$. We assume without loss of generality that $n_2 \geq n_1$. If there exists a $w_{i(n+1)}$, then $(x_{1n}, x_{1(n-1)})$ and $(x_{1n}, w_{1(n+1)})$ belong to R, contradicting the fact that $x_{1n} \varepsilon E_R$ (see Example 2. 2). Thus $n_1 = n_2 = n$. Similarly, letting m be the lesser of m_1 and m_2, it follows that the matrices (x_{ij}) and (w_{ij}) are both $m \times n$. For each fixed i, $1 \leq i \leq m$, by induction on j, using Definition 2. 2 and Axiom 2. 3a, we have $x_{ij} = w_{ij}$ for all j, $1 \leq j \leq n$. Thus $(x_{ij}) = (w_{ij})$ and the lemma follows.

In Chapter II (Example 2. 2) we saw that a rectangular arrangement is associated with every univalent matrix. The question we now consider is: For every rectangular arrangement (X, R, C) is there a $m \times n$ univalent matrix (x_{ij}) such that the rectangular arrangement (X, R, C) is associated with this matrix (x_{ij})? The answer to this question is yes; in fact for every rectangular arrangement there is one, two or four distinct matrices, depending on the number of corner objects. This will be proved in the next theorem.

Theorem 3. 11. Let (X, R, C) be a rectangular arrangement with n corner objects. Then (X, R, C) is associated with exactly n univalent matrices.

Proof: From Theorem 3.8 and 3.9 it is obviously sufficient to consider the following four cases. Case 1 will consider the rectangular arrangement with one object, Case 2a will consider the rectangular arrangement having one row such that the set contains at least two objects, Case 2b will consider the rectangular arrangement having one column such that the set contains at least two objects and Case 3 will consider the rectangular arrangement having at least two rows and two columns.

Case 1. Let (X, R, C) be a rectangular arrangement such that the set X contains exactly one object. This rectangular arrangement is associated with exactly one matrix; namely, the one-by-one matrix (x_{11}), where x_{11} is the single member of X.

Case 2a. Let (X, R, C) be a rectangular arrangement having only one row, and such that X contains at least two objects. By Lemma 3.4a this row is the range of either of two distinct univalent sequences. Let one of these be denoted $(u_1, u_2, \cdots, u_n)$. Define the $1 \times n$ matrix (x_{1j}) with $x_{1j} = u_j$ for $j = 1, 2, \cdots, n$. Using Example 2.2, the rectangular arrangement associated with this $1 \times n$ matrix (x_{1j}) is the set

$$X = \{x_{1j} : 1 \leq j \leq n\},$$

and the two binary relations R and C,

$$R = \{(x_{1j}, x_{1(j+1)}), (x_{1(j+1)}, x_{1j}), (x_{1j}, x_{1j}), (x_{1n}, x_{1n}) : 1 \leq j < n\},$$

and

$$C = \{(x_{1j}, x_{1j}) : 1 \leq j \leq n\} .$$

Thus we see that (X, R, C) is the rectangular arrangement associated with this $1 \times n$ matrix (x_{1j}).

Similarly, define the $1 \times n$ matrix (x_{1j}) with $x_{1j} = u_{n-j+1}$ for $j = 1, 2, \cdots, n$. It follows that (X, R, C) is also the rectangular arrangement associated with this $1 \times n$ matrix (x_{1j}).

The two matrices are different since their one-one elements are different. By Lemma 3.5 there exist at most two such matrices. Thus each row is associated with two distinct univalent matrices.

Case 2b. Let (X, R, C) be a rectangular arrangement having only one column and such that X contains at least two objects. Using an argument similar to that used in Case 2a, it follows that this rectangular arrangement is associated with two distinct matrices coinciding with the two univalent sequences of Lemma 3.4b.

Case 3. Let (X, R, C) be a rectangular arrangement having at least two rows and two columns. By Theorems 3.8 and 3.9 there exist four distinct elements belonging to $E_R \cap E_C$. Let one of these be denoted by z. Let $w, y, (u_1, u_2, \cdots, u_n)$ and $(v_1, v_2, \cdots, v_m)$ be defined as follows:

1) w is the object given by Lemma 3.3a such that $w \neq z$, $w \varepsilon E_R \cap E_C$ and $(w, z) \varepsilon R_\infty$.

2) y is the object given by Lemma 3.3b such that $y \neq z$, $y \varepsilon E_R \cap E_C$ and $(y, z) \varepsilon C_\infty$.

3) $(u_1, u_2, \cdots, u_n)$ is the univalent sequence given by Theorem 3.6a such that $z = u_1$, $w = u_n$ and $(u_j, u_{j+1}) \varepsilon R$ for $j = 1, 2, \cdots, n-1$.

4) $(v_1, v_2, \cdots, v_m)$ is the univalent sequence given by Theorem 3.6b such that $z = v_1$, $y = v_m$ and $(v_i, v_{i+1}) \varepsilon C$ for $i = 1, 2, \cdots, m-1$.

For fixed i, j, $1 \leq i \leq m$, $1 \leq j \leq n$, let x_{ij} be the unique object given by Axiom 2.5 such that $(x_{ij}, u_j) \varepsilon C_\infty$ and $(x_{ij}, v_i) \varepsilon R_\infty$.

By Theorem 3.2, $(x_{ij}, x_{i\ell}) \varepsilon R_\infty$ and $(x_{ij}, x_{kj}) \varepsilon C_\infty$ for $1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq \ell \leq n$, $1 \leq k \leq m$. Thus it follows from Theorem 3.10a that $(x_{ij}, x_{i(j+1)})$ and $(x_{i(j+1)}, x_{ij})$ belong to R for $1 \leq i \leq m$, $1 \leq j < n$. By Theorem 3.10b, $(x_{ij}, x_{(i+1)j})$ and $(x_{(i+1)j}, x_{ij})$ belong to C for $1 \leq i < m$, $1 \leq j \leq n$. By Axiom

2.1, $(x_{ij}, x_{ij}) \varepsilon R$ and $(x_{ij}, x_{ij}) \varepsilon C$ for $1 \leq i \leq m$, $1 \leq j \leq n$. By Corollary 3.1a, x_{i1} and x_{in} belong to E_R for $1 \leq i \leq m$ and by Corollary 3.1b, x_{1j} and x_{mj} belong to E_C for $1 \leq j \leq n$.

Now suppose that $x_{ij} = x_{rs}$. (See Figure 3.7.)

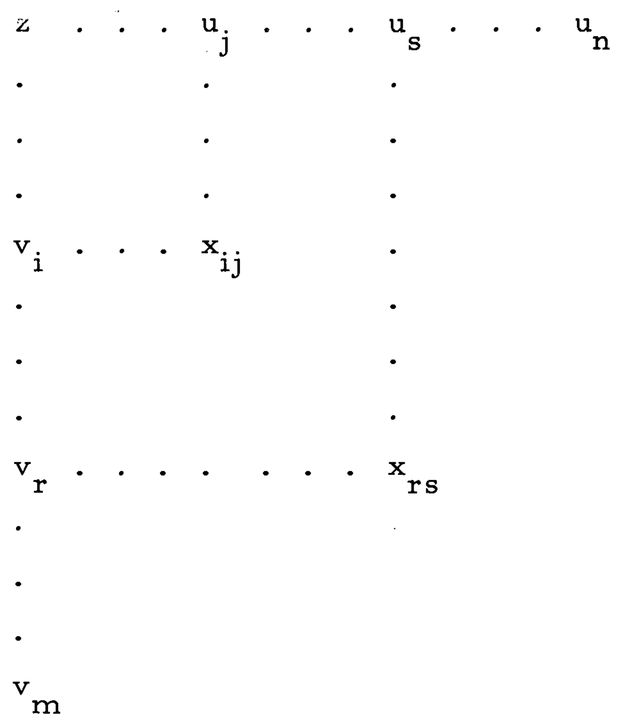

Figure 3.7

By Axiom 2.2, $(x_{ij}, x_{rs}) \varepsilon R$. It follows from Theorem 3.2 that $(v_i, v_r) \varepsilon R_\infty$. By Theorem 3.4, $v_i = v_r$; hence, $i = r$ since the sequence $(v_1, v_2, \cdots, v_m)$ is univalent. Using a similar argument, it follows that $u_j = u_s$, which is true if and only if $j = s$. Thus $x_{ij} = x_{rs}$ if and only if $(r, s) = (i, j)$.

If $(x_{i1}, x_{rs}) \varepsilon R$ for $(r,s) \neq (i,1)$, $(r,s) \neq (i,2)$, then by Definition 2. 2 x_{i1} does not belong to E_R, contradicting the fact that $x_{i1} \varepsilon E_R$. If $(x_{in}, x_{rs}) \varepsilon R$ for $(r,s) \neq (i,n)$, $(r,s) \neq (i,n-1)$, then by Definition 2. 2 x_{in} does not belong to E_R, contradicting the fact that $x_{in} \varepsilon E_R$. If $(x_{ij}, x_{rs}) \varepsilon R$ for $1 < i < m$, $1 < j < n$, $(r,s) \neq (i,j)$, $(r,s) \neq (i,j+1)$, $(r,s) \neq (i,j-1)$, Axiom 2. 3a is contradicted. Thus it follows that

$$R = \{(x_{ij}, x_{i(j+1)}), (x_{i(j+1)}, x_{ij}), (x_{ij}, x_{ij}), (x_{in}, x_{in}): 1 \leq i \leq m, \quad 1 \leq j < n \}.$$

Using a similar argument, it follows that

$$C = \{(x_{ij}, x_{(i+1)j}), (x_{(i+1)j}, x_{ij}), (x_{ij}, x_{ij}), (x_{mj}, x_{mj}): 1 \leq i < m, \quad 1 \leq j \leq n \}.$$

Thus we see that a rectangular arrangement (X, R, C) is associated with at least one $m \times n$ univalent matrix. Let z_1, z_2, z_3 and z_4 be the four corner objects. Take $x_{11} = z_i$ for $i = 1, 2, 3, 4$. Then the rectangular arrangement (X, R, C) is associated with these four $m \times n$ univalent matrices. The four matrices are distinct since their one-one elements are different. By Lemma 3. 5 there exist at most four such matrices. Thus the theorem follows.

With this theorem we have completed the association of rectangular arrangements with univalent matrices.

CHAPTER IV

EXTENSIONS OF THE CONCEPT OF RECTANGULAR ARRANGEMENTS

Two dimensional rectangular arrangements can be generalized further by allowing the set X to be infinite; i.e., deleting the finite property of Axiom 2.7. In this case rectangular arrangements can still be associated with univalent matrices except that the matrices may be infinite. If there is only one corner object, then the set has exactly one element or the set is infinite. If the set is infinite, then either the rectangular arrangement is a row as in Figure 4.1, or a column, or an arrangement as in Figure 4.2.

$$a_{11} \quad a_{12} \quad a_{13} \quad \cdots$$

Figure 4.1

$$\begin{matrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{matrix}$$

Figure 4.2

If there are two corner objects and the set is finite, we have a single row or single column. If the set is infinite, then we have an arrangement as in Figure 4.3a or Figure 4.3b.

$$\begin{array}{cccc} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdots \\ a_{m1} & a_{m2} & \cdots \end{array}$$

Figure 4.3a

$$\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \end{array}$$

Figure 4.3b

If there are more than two corner objects, then there exist four corner objects and Axiom 2.7 follows.

Figures 4.4 and 4.5 typify infinite arrangements that are not included in this generalization.

$$\begin{array}{cccccc} \cdots & a_{1-1} & a_{10} & a_{11} & a_{12} & \cdots \\ \cdots & a_{2-1} & a_{20} & a_{21} & a_{22} & \cdots \\ \cdots & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdots & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdots & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdots & a_{m-1} & a_{m0} & a_{m1} & a_{m2} & \cdots \end{array}$$

Figure 4.4

$$
\begin{array}{ccccccccc}
\cdot & & \cdot & \cdot & \cdot & \cdot & & \cdot \\
& \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
& & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot\;\cdot\;\cdot & & a_{-1-1} & a_{-10} & a_{-11} & a_{-12} & & \cdot\;\cdot\;\cdot \\
\cdot\;\cdot\;\cdot & & a_{0-1} & a_{00} & a_{01} & a_{02} & & \cdot\;\cdot\;\cdot \\
\cdot\;\cdot\;\cdot & & a_{1-1} & a_{10} & a_{11} & a_{12} & & \cdot\;\cdot\;\cdot \\
& & \cdot & \cdot & \cdot & \cdot & \cdot & \\
& \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & & \cdot & \cdot & \cdot & \cdot & & \cdot
\end{array}
$$

Figure 4. 5

We have considered only rectangular arrangements of two dimension. It seems that with appropriate relations it would be possible to capture in mathematical terms rectangular arrangements of dimension n. As an example it seems that a three dimensional rectangular arrangement as in Figure 4. 6 could be captured with three binary relations, R and C as given and a third binary relation D for "next to in depth".

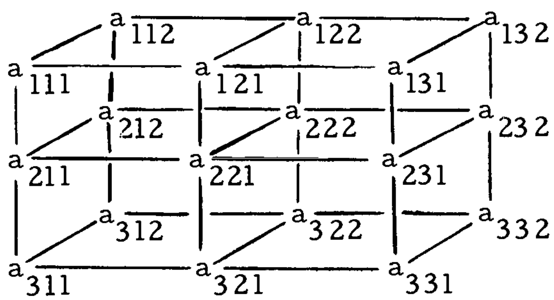

Figure 4. 6

We have seen that it is possible to capture with two binary relations the essence of a rectangular arrangement. Can a rectangular arrangement be captured with one binary relation? It would seem that a more abstract kind of rectangular arrangement could be presented with a single binary relation N, for "next to". In this abstraction the distinction between rows and columns would be lost. It seems that this rectangular arrangement would be associated with one, four or eight matrices, depending on the number of corner objects. Further work is being done.

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