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Title: A New Approach to Pricing Real Options on Swaps:

A New Solution Technique and Extension to the Non-a.s. Finite Stopping Realm.

Abstract approved: ______________________________

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This thesis consists of extensions of results on a perpetual American swaption problem. Companies routinely plan to swap uncertain benefits with uncertain costs in the future for their own benefits. Our work explores the choice of timing policies associated with the swap in the form of an optimal stopping problem. In this thesis, we have shown that Hu, Oksendal's (1998) condition given in their paper to guarantee that the optimal stopping time is a.s. finite is in fact both a necessary and sufficient condition. We have extended the solution to the problem from a region in the parameter space where optimal stopping times are a.s. finite to a region where optimal stopping times are non-a.s. finite, and have successfully calculated the probability of never stopping in this latter region. We have identified the joint distribution for stopping times and stopping locations in both the a.s. and non-a.s. finite stopping cases. We have also come up with an integral formula for the inner product of a generalized hyperbolic distribution with the Cauchy distribution.

Also, we have applied our results to a back-end forestry harvesting model where stochastic costs are assumed to exponentiate upwards to infinity through time.

A New Approach to Pricing Real Options on Swaps: A New Solution Technique and Extension to the Non-a.s. Finite Stopping Realm.

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I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Uran Chu, Author

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TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

TABLE OF CONTENTS (Continued)

A New Approach to Pricing Real Options on Swaps: A New Solution Technique and Extension to the Non-a.s. Finite Stopping Realm.

1 INTRODUCTION

1.1 Option to swap one risky asset for another

Finance and Economics professionals have been interested in the pricing of real options on swap contracts for some time. The holder of the option can choose to pick any time to swap the pay-offs of one asset for another. He/she gives up one asset, asset 2, for a different asset, asset 1. Upon request by the holder of the contract to exercise his/her option, the seller of the option, bound by contract, must consummate the swap. The seller gives the holder asset 1 in exchange for asset 2. When the swap option holder decides to swap, he/she exercises the option, and the time when he/she chooses to do so is called the exercise time. If the holder can exercise during any time before the expiry of the contract (if one exists), the option contract is of the American variety (instead of European). In a problem without an exercise time constraint (the contract time never goes out on expiry), the problem of contract valuation becomes one of valuation of a Perpetual American swap option contract. People call swap option contracts "swaptions" for short.

Within this area of study, researchers solve different problems based on different assumptions, such as whether markets exists for trading the two assets involved in the swap or not. If an external market exists for the traded assets, then valuators can bring valuation tools from Finance to help price them, as in Gerber and Shiu [12]. They posited a complete market model, and used risk-neutral valuation to come up with a swap option contract price. There, the martingale restrictions forced the required rate of return less the growth rate of the price processes to obey strict relationships with their respective volatilities.

Another approach for studying the valuation of these swap-options, or swaptions as we call them, is to utilize the expected-value for valuation, but with no market model assumed. Open markets usually do not trade stock certificates on companies' individual projects. They trade certificates on the right to claim companies' earnings as a whole. In these models, we assume stochastic models for revenues and costs, and the company valuates swaptions according to the expected values of revenue minus cost, subject to a random time at which the companies choose to exercise their options. The companies pick those random exercise times according to information on revenues and costs in real time. They do so to their advantage; they pick optimal times to exercise according to some predetermined optimal policies. When the valuators are big companies, or the government, which own well-diversified portfolios of projects, they do not expect to earn risk-premia and valuate projects using expected value of revenue minus cost. An example of this vein of research is Hu and Oksendal [13]. See also Brazee and Mendelsohn [5]. Here, average growth rate of revenue and cost processes do not necessarily follow martingale restrictions in the complete market risk-neutral valuation world, but are free parameters in the model. The growth rates of price processes here are not functions of the volatilities of the respective processes. Gerber and Shiu had more restrictions in the stochastic program in their models, and also came up with a different solution to the problem than Hu, Oksendal's [13]. Therefore, the two veins of research dealt with similar but different problems. In Hu, Oksendal [13], discounted price processes are strict super-martingales.

Motivation for our thesis problem stems from timber production in a world with carbon sequestration. As is well known, forests, and trees in particular, capture carbon and store them in their tissues until they die. After their deaths, woody debris and the soil (in the area of the tree root) release previously captured carbon back into the atmosphere in gradual decaying processes. See Jarnish and Harmon [14] for some selected decomposition rates. Thus, if scientists believe that carbon dioxide trapped in the atmosphere is warming up the entire earth, and excessive warming would eventually cause catastrophe, we as human beings living here have incentives to delay carbon releases into the atmosphere anyway we can. We would pay foresters for incremental capture of carbon. We would penalize foresters for cutting down forests and releasing carbon into the atmosphere.

Our thesis problem is in fact a first-stage problem to the overall carbon sequestration economic incentive problem described in the last paragraph. A National Science Foundation grant, under the I.G.E.R.T. Ecosystem Informatics program here at Oregon State University, motivated and funded our research. This program aimed at fostering interdisciplinary research on ecological problems. We started by looking at the distribution of time-to-harvest for forests, under both timber sales and carbon sequestration payments and penalties as economic incentives for foresters. Chladna's [9] paper gave us a lot of the framework in this thesis. As the growth of mature forests stagnates (becomes practically zero), foresters interested in profit maximization would choose an optimal policy so as to maximize their net-worth. They would pick the best time to receive sales revenue from timber sales, and simultaneously pay a penalty on carbon released back into the atmosphere. Both timber and carbon prices are stochastic, and the harvest decision is a one-shot decision. In actuality, there should be no restriction on the harvesting time that it should be finite: foresters, if it is in their best interest, can decide never to harvest and neither receive any revenue nor incur any penalty cost with the stand by just leaving it alone forever. We take up this last modeling aspect in this dissertation: it is different from Chladna's [9] model. By modeling in this difference, we could investigate whether, as a first approximation, our change would produce different qualitative results from Chladna's [9] solution. Our results could also serve as the back-end boundary condition to the first phase of the forestry harvesting problem, when forest wood volume growth follows a deterministic growth trajectory. Here, we have made an approximation that may or may not be a good one. Whether our approximation of zero timber-price growth is a good approximation to Chladna's [9] mean-reversion model is largely an empirical issue, which is outside the scope of this thesis.

We solve our simplified problem first. We let both stochastic price processes follow geometric Brownian motion. The use of this process in modelling timber prices has been called into question, especially by Plantinga [20]. Their issues with the model appear to be quite valid, but is outside the scope of this thesis. We eventually realized that, if we were to mimic Chladna's [9] set of parameters for her stochastic model, we would need to make a special extension of Hu, Oksendal's [13] results to regions beyond their parameter restrictions in the parameter space. We will report results in this aspect in Chapter 5 of this dissertation.

The above problem for the foresters is a swaption problem. Foresters have the option to pick the timing for harvests, based on observed prices in real time. They then formulate a harvesting policy that would on average, be optimal. During harvests, they incur benefits that are proportional to timber price and disproportional to carbon penalties. As the United States does not have a well-formed, universal carbon market as yet, but the damage to the environment can be substantial (damage to environment has not been internalized by foresters), a central planner looking to study the benefits and costs to society as a whole would look at optimizing the following objective function: $\sup_{\tau} E^{(x_2,x_1)}(X_{1,\tau}Q - X_{2,\tau}Q) = Q \sup_{\tau} E^{(x_2,x_1)}(X_{1,\tau} - X_{2,\tau})$. Here $X_{1,\tau}$ is the price of timber at harvesting; $X_{2,\tau}$ is the price of carbon penalty at the same time; τ is the random choice variable (the optimal harvesting time) for the harvester at his/her own choosing. Q is the quantity of timber harvested. By restricting our attention to no-growth, completely mature forests, Q is a constant quantity that is not dependent on time. The optimizing problem becomes one of trying to find an optimum policy τ^*, if one exists, so as to optimize the value of the objective function $\sup_{\tau} E^{(x_2,x_1)}(X_{1,\tau} - X_{2,\tau})$. This is the same problem that Hu and Oksendal studied in their 1998 paper.

The Hu, Oksendal [13] problem is interesting in the context of carbon sequestration in that carbon markets are not yet well formed everywhere around the world. As of this writing, it is unclear that the Kyoto Protocol will continue beyond year 2012.

The same situation also applies to a myriad of investment opportunities that are available to companies. As companies' claim to earnings from their individual projects do not trade as paper assets in external markets, valuations for such projects do not necessarily agree with valuation from any one of a variety of market models, such as the one used in Gerber and Shiu [12]. Thus, the Hu, Oksendal [13] model is more appropriate for valuation in our case. Chladna [9] and other papers, Brazee and Mendelsohn ([5], and Plantinga [20] echoed this same approach

Though the original motivation for this thesis was to solve an optimal stopping problem associated with a joint timber-carbon price model, the solution to the Hu, Oksendal [13] problem has wider applicability as well. In any investment timing problem, where a one-shot investment decision brings in a net profit of benefits minus costs, the results of this paper can apply. Other application areas include Research and Development spending; see Sadowsky [22], though they modelled their problem along the lines of Gerber and Shiu [12]. See also Williams [23] on a real estate development application. Since this is an optimal stopping problem, we can find its solution using techniques from Stochastic Optimization. We give a brief motivating example in the Appendix of this Chapter.

Hu and Oksendal [13] solved this swaption valuation problem, with restrictions on their solutions' applicability in the parameter space. In this thesis, we extend their work to other parts of the parameter space beyond their restrictions. In Chapter 2, we trace out the history of literature regarding our problem. In Chapter 3, we use a different approach to arrive at the same solution as Hu, Oksendal [13]'s method outlined in Section 1.6 below.

We also extend the results to a region beyond Hu, Oksendal's [13] restrictions. Chapter 4 will prove rigorously, based on mathematics, that the stopping boundary is of the same form as given in the Hu, Oksendal [5] paper, even for our region of the parameter space. Chapter 5 will report results in a region of the parameter space that mimics Chladna's [9] model. Chapter 6 will conclude.

1.2 From simple example to real problem

A simple example in Appendix A.1 illustrates some intricacies of the problem in this thesis. When we add uncertainties to the price processes, the problem becomes much harder: it becomes a stochastic optimization problem. We provide some basic material on stochastic integrals and the Itô Calculus in Appendix A.1 as well; readers unfamiliar with the subject of Stochastic Analysis should consult the Appendix for a brief introduction and references first before continuing reading; **the rest of this chapter will assume the notation and basic results as given in that Appendix**.

1.3 Hu, Oksendal's [13] Research Problem

We will review the problem and method of solution of Hu, Oksendal [13] in subsections 1.3-1.6.

A stopping time is a non-negative, possibly infinite random variable $\tau(\omega)$ such that for all $t \geq 0$, $\{\tau \leq t\}$ is $\mathcal{F}_t$-measurable. Please see Appendix A.1 for definitions. Let $\mathcal{Y}_0$ be the class of all a.s. finite stopping times.

The problem in Hu, Oksendal's [13] paper is to find a stopping time τ^*, if it exists, such that $\sup_{\tau \in \mathcal{Y}_0} E^{(x_2,x_1)}\{(X_{1,\tau} - X_{2,\tau})\} = E^{(x_2,x_1)}\{(X_{1,\tau^*} - X_{2,\tau^*})\}$, and evaluate the expectation as a function of (x_2, x_1), over the entire state-space $\mathbb{R}^2_+$ for the following stochastic optimization program:

$$\sup_{\tau \in \mathcal{Y}_0} E^{(x_2,x_1)}\{(X_{1,\tau} - X_{2,\tau})\}$$

$$\begin{aligned}
\text{s.t.} \qquad dX_{1,t} &= -p_1 X_{1.t} dt + q_{11} X_{1,t} dB_{1,t} + q_{12} X_{1,t} dB_{2,t}, \\
dX_{2,t} &= -p_2 X_{2.t} dt + q_{21} X_{2,t} dB_{1,t} + q_{22} X_{2,t} dB_{2,t}, \\
X_{1,0} &= x_1, \\
X_{2,0} &= x_2, \\
x_1, x_2, &> 0, \\
p_1, p_2 &> 0, \\
0 &\le q_{11}, q_{12}, q_{21}, q_{22} < \infty, \\
p_1 + \frac{1}{2}(q_{11}^2 + q_{12}^2) &\le p_2 + \frac{1}{2}(q_{21}^2 + q_{22}^2).
\end{aligned}$$

Further, we give other constant definitions here so as to make future dispositions more compact:

$a_{11} = q_{11}^2 + q_{12}^2,$

$a_{22} = q_{21}^2 + q_{22}^2,$

$a_{12} = q_{11}q_{21} + q_{21}q_{22},$

$\gamma = a_{11} + a_{22} - 2a_{12}.$

For example, we write the last constraint in the above program as $p_1 + \frac{1}{2}a_{11} \le p_2 + \frac{1}{2}a_{22}$, for the rest of this thesis. The $p_1, p_2, q_{11}, q_{12}, q_{21}, q_{22}$ are real constants. Note that the last condition is a reflection of the asymmetry in this problem: $X_{1,t}$ represents the stochastic benefit, while $X_{2,t}$ represents the stochastic cost, at time t. Here, $\{(X_{2,t}, X_{1,t}) : t \ge 0\}$ is a two-dimensional stochastic process with time index $t \in [0, \infty)$. $X_{2,t}, X_{1,t}$ share a common probability space $(\Omega, \mathcal{F}, P)$, and their joint stochastic components (B_{2t}, B_{1t}) are independent processes and generate a filtration $\{\mathcal{F}_t\}$ of increasing σ-fields, as defined in the Appendix. The two-tuple $(X_{2,t}, X_{1,t})$ jointly, at any particular time t, is the state of our problem.

We assume the matrix $\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$ as non-singular. This latter condition prevents a linearly dependent relationship between the two stochastic processes of the benefit and cost terms. This is what we usually see in practice between revenues and costs, or two different stochastic evolutions on two different assets. The parameters p_1, p_2 are positive, and $q_{11}, q_{12}, q_{21}, q_{22}$ are non-negative, but not all of them equal zero. Letting them be negative actually poses no obstruction. We can take them to be non-negative, not all of them zero, without losing any generality. We will change the conditions on p_1 and p_2 to $p_1 > 0$, $p_2 < 0$ in Chapter 5.

Under conditions for which τ^* is a.s. finite, Hu, Oksendal [13] obtained the solution to this problem. We name their parameter space Θ_{HO}. Parameter sets within this parameter space satisfy the following constraints:

(i) $p_1, p_2 > 0$,

(ii) $p_1 + \frac{1}{2}a_{11} \leq p_2 + \frac{1}{2}a_{22}$.

When the parameters violate the latter condition, we need to make changes to the above program.

This is the new wrinkle added to our thesis problem beyond Hu, Oksendal's [13]. If τ can take on the value ∞, $(X_{2,\infty}, X_{1,\infty})$ is undefined because it is a function of $(B_{2,t}, B_{1,t})$, which are undefined when $t \to \infty$. The Law of Iterated Logarithm (see Theorem 11.5 in Bhattacharya and Waymire)[2] gives $\lim_{t\to\infty} \sup B_{i,t}$ to be ∞, and $\lim_{t\to\infty} \inf B_{i,t}$ to be $-\infty$. Therefore, $\lim_{t\to\infty} B_{i,t}$ does not exist. It is fortunate that, as the swaption holder never exercises, he/she should incur an objective function of 0, so we understand the behavior of the objective function over the set $[\tau = \infty]$. In fact, though an a.s. limit to standard Brownian Motion fails to exist, the limit of an exponential dependence on Brownian Motion with a linear drift term does. For $p_i + \frac{1}{2}a_{ii} > 0$, one has, from the solution solved on page 138 and 139 of this thesis in the Appendix section,

$$\begin{aligned}
\lim_{t\to\infty} x_1 e^{-(p_i+\frac{1}{2}a_{ii})t+q_{11}B_{1,t}+q_{12}B_{2,t}} &= x_1 e^{-\lim\limits_{t\to\infty}\{t[(p_i+\frac{1}{2}a_{ii})+q_{11}\frac{B_{1,t}}{t}+q_{12}\frac{B_{2,t}}{t}]\}} \\
&= x_1 e^{-\lim\limits_{t\to\infty} t \lim\limits_{t\to\infty}\{[(p_i+\frac{1}{2}a_{ii})+q_{11}\frac{B_{1,t}}{t}+q_{12}\frac{B_{2,t}}{t}]\}} \\
&= x_1 e^{-\lim\limits_{t\to\infty} t(p_i+\frac{1}{2}a_{ii})} \quad \text{a.s.} \\
&= 0 \text{ a.s.}
\end{aligned}$$

The third equality is due to the Strong Law of Large Numbers. Combining with the fact that $(X_{2,t}, X_{1,t})$ is well defined over the set $[\tau < \infty]$, we can define our new objective function, with a slight <u>abuse of notation</u> as follows:

$$\sup_{\tau} E^{(x_2,x_1)}\{(X_{1,\tau} - X_{2,\tau})I_{[\tau<\infty]} + 0 \cdot I_{[\tau<\infty]}\} = \sup_{\tau}\{E^{(x_2,x_1)}(X_{1.\tau} - X_{2,\tau})I_{[\tau<\infty]}\},$$

where $I_{[\cdot]}$ is the indicator function for the event inside the brackets. The proper mathematical optimization is with respect to the following functional of the two paths X_1, X_2, and τ:

$$\Pi(X_2, X_1, \tau) = \begin{cases} (X_{1,\tau} - X_{2,\tau}) & \text{if } \tau < \infty \\ 0 & \text{if } \tau = \infty \end{cases} \tag{1.1}$$

Our objective function is then defined as $\sup\limits_{\tau\in\mathcal{Y}} E^{(x_2,x_1)}[(\Pi(X_2, X_1, \tau))I_{[\tau<\infty]}]$. In our thesis, we will prove that our method will obtain exactly the same solution as Hu, Oksendal's [13], but over a different region of the parameter space. We name our parameter space Θ_C. Parameters within our region satisfy

(i) $p_1, p_2 > 0$,

(ii) $0 \leq \left|\mu_2 + \dfrac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\right| < \sqrt{\mu_1^2+\mu_2^2}$.

We define constants μ_1, μ_2 as follows:

$$\mu_1 = \left(\frac{\left(p_1 + \frac{1}{2}a_{11}\right) - \left(p_2 + \frac{1}{2}a_{22}\right)}{\sqrt{\gamma}} \right);$$

$$\mu_2 = \frac{(\sigma_2\sigma_1\rho - \sigma_1^2)(p_2 + \frac{1}{2}a_{22}) + (\sigma_2\sigma_1\rho - \sigma_2^2)(p_1 + \frac{1}{2}a_{11})}{\sigma_1\sigma_2\sqrt{1-\rho^2}\sqrt{\gamma}},$$

where $\sigma_1 = \sqrt{a_{11}}$, $\sigma_2 = \sqrt{a_{22}}$, $\rho = \frac{a_{12}}{\sqrt{a_{11}a_{22}}}$, and $\gamma = a_{11} + a_{22} - 2a_{12}$.

With the above discussion, we can now formulate our version of the swaption problem in the next section. It is only a slight variant of Hu, Oksendal's [13] problem.

1.4 Our thesis problem under extension, and results obtained

Let $\mathcal{Y}$ be the class of stopping times including those τ's that can be ∞ with positive probability, the so called non-a.s. finite stopping times, as well as the class of stopping times that are a.s. finite. We solve a modified problem from that listed in 1.3. The modifications are as follows:

Find $\tau^* \in \mathcal{Y}$, if it exists, such that

$$\sup_{\tau \in \mathcal{Y}} E^{(x_2,x_1)}\{(X_{1,\tau} - X_{2,\tau})I_{[\tau<\infty]}\} = E^{(x_2,x_1)}\{(X_{1,\tau^*} - X_{2,\tau^*})I_{[\tau^*<\infty]}\}, \tag{1.2}$$

and evaluate the expectation as a function of (x_2, x_1), over the entire state space $\mathbb{R}_+^2$ for a stochastic program that is modified from Section 1.3's in the following manners: The objective function changes from $\sup_{\tau \in \mathcal{Y}_0} E^{(x_2,x_1)}\{(X_{1,\tau} - X_{2,\tau})\}$ to $\sup_{\tau \in \mathcal{Y}}\{E^{(x_2,x_1)}(X_{1.\tau} - X_{2,\tau})I_{[\tau<\infty]}\}$, and the constraint over the parameter space changes from $p_1 + \frac{1}{2}a_{11} \leq p_2 + \frac{1}{2}a_{22}$ to $p_1 + \frac{1}{2}a_{11} > p_2 + \frac{1}{2}a_{22}$. This change allows for τ^* to be non-a.s. finite. It was the author's original intention to make no changes to the rest of the constraints from the program on page 7. However, during the course of calculation, we needed to add a new constraint,

$0 \leq \left| \mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}} \right| < \sqrt{\mu_1^2+\mu_2^2}$, in order for our calculation to go through. In addition, our calculation went through whether $p_1 + \frac{1}{2}a_{11} \leq p_2 + \frac{1}{2}a_{22}$ is true or not. Thus, we added the constraint $0 \leq \left| \mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}} \right| < \sqrt{\mu_1^2+\mu_2^2}$, and dropped the constraint $p_1 + \frac{1}{2}a_{11} \leq p_2 + \frac{1}{2}a_{22}$ to arrive at our thesis problem in Chapter 4, and the result in Theorem 4.1.2.

Remark 1.4.1. *The new method is also applicable to part of Hu, Oksendal's [13] parameter region and provides a new derivation of their results as well.*

We will also identify Θ_{EXTHO} as the part of parameter space that satisfies the following constraints:

(i) $p_1 > 0, p_2 < 0$,

(ii) $p_1 + p_2 + \frac{1}{2}\gamma > 0$,

(iii) $p_1 + \frac{1}{2}a_{11} \leq p_2 + \frac{1}{2}a_{22}$, and

Θ_{EXTC} as part of the parameter space that satisfies the following constraints:

(i) $p_1 > 0, p_2 < 0$,

(ii) $0 \leq \left| \mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}} \right| < \sqrt{\mu_1^2+\mu_2^2}$,

(iii) $p_1 + p_2 + \frac{1}{2}\gamma > 0$, with

$$\gamma = (q_{11} - q_{21})^2 + (q_{12} - q_{22})^2 = a_{11} + a_{22} - 2a_{12}.$$

It is true that $\Theta_C \nsubseteq \Theta_{HO}$, and $\Theta_{HO} \nsubseteq \Theta_C$, and $\Theta_{EXTC} \nsubseteq \Theta_{EXTHO}$, and $\Theta_{EXTHO} \nsubseteq \Theta_{EXTC}$. Of course, $\Theta_C \cup \Theta_{HO}$ and $\Theta_{EXTC} \cup \Theta_{EXTHO}$ have empty intersections, because the latter parameter space is part of the $p_2 < 0$ sub-space, as opposed to the former ($p_2 > 0$).

The original contributions of this thesis are as follows:

i) Over Θ_{HO}, we provide a new derivation method on the solution of this problem over $\Theta_{HO} \cap \Theta_C$. We provide the marginal density of stopping times. We also provide the joint density of stopping times and stopping location in the transformed space. The joint density of stopping location in the original space is just one Jacobian transform away, and can be had using techniques from elementary calculus. From the joint density, we can calculate the various conditional and marginal densities. The marginal location density in the transformed space is also available.

ii) Over Θ_C, we provide the solution of this problem by our integral method. We also provide here a formula for the non-zero probability of never exercising the exchange option, i.e. $P(\tau^* = \infty)$, given our parameter values, in addition to the marginal, conditional, and joint densities as described in i). We also identify an interesting integral formula as a Corollary (Corollary 3.5.5): namely integral over a disintegrated measure results in an integral that is a product of a Cauchy density with a generalized hyperbolic density. It integrates out to a bi-variate power function. See Chang, Pollard [8].

iii) Over Θ_{EXTHO}, we extend Hu, Oksendal's [13] solution to this part of the parameter space, with an additional constraint that $p_1 + p_2 + \frac{1}{2}\gamma > 0$ to guarantee a positive objective function. We use the free boundary pde method for the extension. Over $\Theta_{EXTHO} \cap \Theta_{EXTC}$, we provide the marginal, conditional, and joint densities for hitting time and location, as described in i).

iv) Over Θ_{EXTC}, we extend Hu, Oksendal's [13] solution to this part of the parameter space, with an additional constraint that $p_1 + p_2 + \frac{1}{2}\gamma > 0$ to guarantee a positive objective function. We use our integral method for this extension. Once again, we provide a formula for the probability of never exercising the exchange option, i.e. $P(\tau^* = \infty)$, given our parameter values, in addition to the marginal, conditional, and joint densities as described in i).

1.5 Some preliminary important observations

We make important observations at this point.

1.5.1 Policy for stopping is static

Substituting the solutions for $X_{2,\tau}, X_{1,\tau}$ gives, for the objective function,

$$\sup_{\tau\in\mathcal{Y}} E^{(x_2,x_1)}\{(X_{1,\tau} - X_{2,\tau})I_{[\tau<\infty]}\}$$
$$= \sup_{\tau\in\mathcal{Y}} E^{(x_2,x_1)}\Big\{\big[x_1 e^{-(p_1+\frac{1}{2}a_{11})\tau+q_{11}B_{1,\tau}+q_{12}B_{2,\tau}} - x_2 e^{-(p_2+\frac{1}{2}a_{22})\tau+q_{21}B_{1,\tau}+q_{22}B_{2,\tau}}\big] I_{[\tau<\infty]}\Big\}.$$

A look inside the expectation makes us realize that the expectation integral involves the tri-variate joint density of $(B_{2,\tau}, B_{1,\tau}, \tau)$.

Note that $\sup_{\tau\in\mathcal{Y}} E^{(x_2,x_1)}\{(X_{1,\tau} - X_{2,\tau})I_{[\tau<\infty]}\}$ is not a function of t, nor is it a function of τ itself, for any stopping time τ. An optimal stopping time τ^*, if it exists, is likewise not a function of t. We can then view any specification of stopping time τ, a function of (x_2, x_1, ω), as a policy to prescribe what action (whether to stop and exercise the option, or to continue and not exercise the option) to take at each point (x_2, x_1) in the state space $\mathbb{R}^2_+$. It is a policy in the sense that once the joint process arrives at (x_2, x_1) in the state space, the action prescribed will not be dependent on that time of arrival, and the distribution of stopping times over Ω is fixed.

Given fixed $p_1, p_2, q_{11}, q_{12}, q_{21}, q_{22}$, we solve the problem for an optimal policy $\tau^*(x_2, x_1, \omega)$, if it exists, for every point (x_2, x_1) in our initial state space, which is the first quadrant $\mathbb{R}^2_+$. The objective function value is jointly determined with the optimal policy, as $X_{1,\tau} - X_{2,\tau}$ is integrated over the joint probability distribution of $(X_{1,\tau}, X_{2,\tau}, \tau)$, and leaves a real number per (x_2, x_1). It will be the highest value achievable for $E^{(x_2,x_1)}\{(X_{1,\tau} - X_{2,\tau})I_{[\tau<\infty]}\}$, given the optimal choice of τ^*, if such exists.

1.5.2 Supremum of objective function must be greater than or equal to the stopping pay-off

One possible policy for consideration is to stop immediately to take the current swap value, which is $X_{1,0} - X_{2,0} = x_1 - x_2$. This might or might not be the optimal policy. Since $\tau = 0$ with probability 1 is one of the policy choices, it is immediately clear that $\sup_{\tau \in \mathcal{Y}} E^{(x_2,x_1)}\{(X_{1,\tau} - X_{2,\tau})I_{[\tau<\infty]}\}$ must be $\geq x_1 - x_2$ at each (x_2, x_1) over the entire state space, as, besides the immediate stopping pay-off $x_1 - x_2$ that is available as the pay-off to a policy choice at each point, there might be another policy or other policies that could give higher objective function values.

As a consequence, we want to see if stopping immediately with probability 1 is optimal at any point in our initial state space. Either
$\sup_{\tau \in \mathcal{Y}} E^{(x_2,x_1)}\{(X_{1,\tau} - X_{2,\tau})I_{[\tau<\infty]}\} = x_1 - x_2$, or else,
$\sup_{\tau \in \mathcal{Y}} E^{(x_2,x_1)}\{(X_{1,\tau} - X_{2,\tau})I_{[\tau<\infty]}\} > x_1 - x_2$.

Thus, all points in $\mathbb{R}^2_+$ must belong to one of the following regions defined as follows:

Definition 1.5.1.

$$S = \big\{(x_2, x_1) : \sup_{\tau \in \mathcal{Y}} E^{(x_2,x_1)}\{(X_{1,\tau} - X_{2,\tau})I_{[\tau<\infty]}\} = x_1 - x_2\big\}$$

$$S^c = \big\{(x_2, x_1) : \sup_{\tau \in \mathcal{Y}} E^{(x_2,x_1)}\{(X_{1,\tau} - X_{2,\tau})I_{[\tau<\infty]}\} > x_1 - x_2\big\}.$$

We see that S, and hence S^c, are not functions of t. Therefore, ∂S must also be necessarily time-independent. S is often called the *stopping set* or *stopping region* of this problem; S^c, the *continuation set* or *region*; ∂S, the *optimal stopping* or *exercise boundary.*

Definition 1.5.2. *A stopping time (policy)* τ' *is* ϵ*-close-to-optimal if* $\sup_{\tau \in \mathcal{Y}} E^{(x_2,x_1)}\{(X_{1,\tau} - X_{2,\tau})I_{[\tau<\infty]}\} - E^{(x_2,x_1)}\{(X_{1,\tau'} - X_{2,\tau'})I_{[\tau'<\infty]}\} < \epsilon$*, where* $\epsilon > 0$*. A stopping time (policy) that prescribes* $P[\tau = 0] > 0$ *is called a* τ_s *policy.*

Lemma 1.5.3. *Assume $S^c \neq \varnothing$. Over S^c, $P[\tau^* = 0] = 0$ if τ^* exists. Moreover, regardless of whether τ^* exists, there exists an $\epsilon' > 0$ such that for all ϵ-close-to-optimal policies τ' s.t. $\epsilon' > \epsilon > 0$, $P[\tau' = 0] = 0$.*

Proof. From the class of stopping times, let any policy that is ϵ-close-to-optimal, such that

$$0 < \epsilon < \frac{\sup_{\tau \in \mathcal{Y}} E^{(x_2,x_1)}\{(X_{1,\tau} - X_{2,\tau})I_{[r<\infty]}\} - (x_1 - x_2)}{2} = \epsilon',$$

be called a τ' policy.

We observe that $[\tau^* = 0], [\tau_s = 0], [\tau' = 0] \in \mathcal{F}_0$, as Blumenthal's 0-1 Law applies to Brownian Motion in $\mathbb{R}^2$. Therefore, $P[\tau^* = 0], P[\tau' = 0], P[\tau_s = 0]$ are equal to 0 or 1. By definition, $P[\tau_s = 0] > 0$, and therefore, $P[\tau_s = 0] = 1$. But over S^c, stopping immediately with probability 1 gives a pay-off of

$$x_1 - x_2 < \sup_{\tau \in \mathcal{Y}} E^{(x_2,x_1)}\{(X_{1,\tau} - X_{2,\tau})I_{[\tau<\infty]}\},$$

and therefore any policy prescribing $P[\tau_s = 0] = 1$ cannot be the optimal policy. Thus, it also follows immediately, then, that, if τ^* exists, $P[\tau^* = 0] = 0$, because the only other choice, $P[\tau^* = 0] = 1$ would make τ^* sub-optimal. In addition, in order for any τ' policy to be ϵ-close-to-optimal for $0 < \epsilon < \epsilon'$, the policy cannot prescribe delivering $x_1 - x_2$ with probability 1. The only other choice for the value of $P[\tau' = 0]$ is 0. □

Remark 1.5.1. *We will show in Lemma 3.2.1 that $S^c \neq \varnothing$; therefore, the results of this Lemma will hold through out the parameter regions of our thesis.*

1.6 Solution to problem by free boundary P.D.E. method

One method of solution for this type of problem is to proceed with the solution of a corresponding free boundary value problem first. If this solution satisifes all the conditions from the Verification Theorem in Hu, Oksendal[13], then the free boundary pde solution is also the solution for the original optimal stopping, or stochastic optimization problem under Section 1.3; see Hu, Oksendal [13] on the use of the Verification Theorem 2.1 in their paper.

Let $V(x_2, x_1) \equiv \sup_{\tau} E^{(x_2,x_1)}\{(X_{1,\tau} - X_{2,\tau})\}$ for each $(x_2, x_1) \in S^c \cup \partial S$. We now show how to apply this method in our optimization problem. We have $\sup_{\tau} E^{(x_2,x_1)}\{(X_{1,\tau} - X_{2,\tau})\} = x_1 - x_2$ for each $(x_2, x_1) \in S$, by definition, and we are looking for solutions V that satisfy the following two conditions:

1. Value matching: $V(x_2, x_1) = x_1 - x_2$ over ∂S;

2. Smooth-pasting: $\nabla V(x_2, x_1) = \nabla(x_1 - x_2)$ over ∂S; see Dixit and Pindyck [9]. Hu, Oksendal [13] also required these conditions in their paper.

3. Additionally, given our stochastic differential equations specified in the program in Section 1.3, the infinitesimal generator $\mathcal{L}$ for the joint stochastic process $(X_{2,t}, X_{1,t})$ is a differential operator given by

$$
\begin{gathered}
-p_1 x_1 \frac{\partial}{\partial x_1} - p_2 x_2 \frac{\partial}{\partial x_2} + \frac{1}{2}(q_{11}^2 + q_{12}^2) x_1^2 \frac{\partial^2}{\partial x_1^2} \\
+(q_{11}q_{21} + q_{12}q_{22}) x_1 x_2 \frac{\partial^2}{\partial x_1 \partial x_2} + \frac{1}{2}(q_{21}^2 + q_{22}^2) x_2^2 \frac{\partial^2}{\partial x_2^2}.
\end{gathered}
$$

Also, $\mathcal{L}V(x_2, x_1) = 0$ for each $(x_2, x_1) \in S^c$. $\mathcal{L}V(x_2, x_1) = 0$ then implies

$$
\begin{aligned}
& - p_1 x_1 \frac{\partial V}{\partial x_1} - p_2 x_2 \frac{\partial V}{\partial x_2} + \frac{1}{2}(q_{11}^2 + q_{12}^2) x_1^2 \frac{\partial^2 V}{\partial x_1^2} \\
& + (q_{11}q_{21} + q_{12}q_{22}) x_1 x_2 \frac{\partial^2 V}{\partial x_1 \partial x_2} + \frac{1}{2}(q_{21}^2 + q_{22}^2) x_2^2 \frac{\partial^2 V}{\partial x_2^2} = 0, \\
\text{or} & - p_1 x_1 \frac{\partial V}{\partial x_1} - p_2 x_2 \frac{\partial V}{\partial x_2} + \frac{1}{2} a_{11} x_1^2 \frac{\partial^2 V}{\partial x_1^2} + a_{12} x_1 x_2 \frac{\partial^2 V}{\partial x_1 \partial x_2} + \frac{1}{2} a_{22} x_2^2 \frac{\partial^2 V}{\partial x_2^2} = 0.
\end{aligned}
$$

with $a_{11} \equiv q_{11}^2 + q_{12}^2$, $a_{12} \equiv q_{11}q_{21} + q_{12}q_{22}$, and $a_{22} \equiv q_{21}^2 + q_{22}^2$.

First, we guess a solution of the form $V(x_2, x_1) = Cx_1^a x_2^b$. We take first and second partial derivatives to substitute into the pde above to reduce that equation into an algebraic one, (1.3) below:

$$\frac{\partial V}{\partial x_1} = aCx_1^{a-1}x_2^b; \quad \frac{\partial V}{\partial x_2} = bCx_1^a x_2^{b-1}; \quad \frac{\partial^2 V}{\partial x_1^2} = a(a-1)Cx_1^{a-2}x_2^b;$$

$$\frac{\partial^2 V}{\partial x_2^2} = b(b-1)Cx_1^a x_2^{b-2}; \quad \frac{\partial^2 V}{\partial x_1 x_2} = abCx_1^{a-1}x_2^{b-1}.$$

Substituting all the terms back into the pde gives

$$[-p_1 a - p_2 b + \frac{1}{2}a_{11}a(a-1) + a_{12}ab + \frac{1}{2}a_{22}b(b-1)]V = 0. \tag{1.3}$$

In order for V not to be 0 throughout the entire region of S^c, we need

$$[-p_1 a - p_2 b + \frac{1}{2}a_{11}a(a-1) + a_{12}ab + \frac{1}{2}a_{22}b(b-1)] = 0. \tag{1.4}$$

We know, from McDonald and Siegel [16] that ∂S is of the form $\{(x_2, x_1) : x_1 = \mu x_2, \mu > 1\}$ for this problem (i.e. ∂S is a straight line emanating from the origin with positive slope that is greater than 1.)

We then utilize conditions 1 and 2 to derive three conditions at ∂S:

Condition 1:

$$Cx_1^a x_2^b = x_1 - x_2. \tag{1.5}$$

Condition 2:

$$\frac{\partial Cx_1^a x_2^b}{\partial x_1} = aCx_1^{a-1}x_2^b = \frac{\partial(x_1 - x_2)}{\partial x_1} = 1. \tag{1.6}$$

$$\frac{\partial Cx_1^a x_2^b}{\partial x_2} = bCx_1^a x_2^{b-1} = \frac{\partial(x_1 - x_2)}{\partial x_2} = -1. \tag{1.7}$$

Dividing (1.5) by (1.6) gives

$$\frac{x_1}{a} = x_1 - x_2.$$

Dividing (1.5) by (1.7) gives

$$-\frac{x_2}{b} = x_1 - x_2.$$

Equating the last two equations gives

$$x_2 = -\frac{b}{a}x_1.$$

Substituting back into (1.5) gives

$$Cx_1^a\Big(-\frac{b}{a}x_1\Big)^b = x_1 - x_2 = C(-1)^b\frac{b^b}{a^b}x_1^{a+b} = x_1 - \frac{x_1}{\mu} = x_1(1-\frac{1}{\mu}) = x_1(\frac{\mu-1}{\mu}).$$

Equating the exponent and the multiplying constants, we get $a+b=1$, and

$$C(-1)^b\frac{b^b}{a^b} = \frac{\mu-1}{\mu}. \tag{1.8}$$

Then letting $a=\lambda$, and $b=1-\lambda$, we can re-write (1.4) as

$$-p_1\lambda - p_2(1-\lambda) + \frac{1}{2}a_{11}\lambda(\lambda-1) + a_{12}\lambda(1-\lambda) + \frac{1}{2}a_{22}(1-\lambda)(-\lambda) = 0.$$

Grouping terms gives

$$(\frac{1}{2}a_{11} - a_{12} + \frac{1}{2}a_{22})\lambda^2 + (-p_1 + p_2 - \frac{1}{2}a_{11} + a_{12} - \frac{1}{2}a_{22})\lambda - p_2 = 0.$$

Set $\gamma = a_{11} - 2a_{12} + a_{22}$; we then have

$$(\frac{1}{2}\gamma)\lambda^2 + (p_2 - p_1 - \frac{1}{2}\gamma)\lambda - p_2 = 0.$$

The two roots for λ then can be solved by using the quadratic formula:

$$\begin{aligned}\lambda &= \frac{-(p_2 - p_1 - \frac{1}{2}\gamma) \pm \sqrt{(p_2 - p_1 - \frac{1}{2}\gamma)^2 + 4\frac{1}{2}\gamma p_2}}{2\frac{1}{2}\gamma} \\ &= \frac{\frac{1}{2}\gamma - (p_2 - p_1) \pm \sqrt{(p_2 - p_1 - \frac{1}{2}\gamma)^2 + 2\gamma p_2}}{\gamma}.\end{aligned}$$

Rehashing (1.5) through (1.7),

$$Cx_1^{\lambda} x_2^{1-\lambda} = x_1 - x_2. \tag{1.9}$$

$$aCx_1^{\lambda-1} x_2^{1-\lambda} = 1 = \lambda C(\frac{x_1}{x_2})^{\lambda-1}. \tag{1.10}$$

$$bCx_1^{\lambda} x_2^{-\lambda} = -1 = (1-\lambda) C(\frac{x_1}{x_2})^{\lambda}. \tag{1.11}$$

Dividing (1.9) by (1.10) gives

$$\frac{x_1}{\lambda} = x_1 - x_2. \tag{1.12}$$

Dividing (1.9) by (1.11) gives

$$\frac{x_2}{1-\lambda} = x_2 - x_1. \tag{1.13}$$

Combining (1.12) and (1.13) gives

$$\frac{x_1}{\lambda} = \frac{x_2}{\lambda - 1}, \text{ or } \frac{x_1}{x_2} = \frac{\lambda}{\lambda - 1} = \mu. \tag{1.14}$$

The stopped process is at the stopping boundary; this gives

$$Cx_1^{\lambda} x_2^{1-\lambda} = C(\frac{x_1}{x_2})^{\lambda} x_2 = C\mu^{\lambda} x_2 = x_1 - x_2 = (\mu - 1)x_2.$$

Solving for $C = \frac{\mu-1}{\mu^\lambda}$ determines the entire expression for the value of the objective function over S^c. Hu, Oksendal [13] showed on page 302 in their paper that $\lambda > 1$ and therefore $\mu > 1$; therefore, $C > 0$.

For λ, we can take one of two possible roots, the one associated with taking the positive or negative sign. When we take the root with the negative sign, since the term under the radical is greater in magnitude than the term preceding it in the expression for λ, and the denominator $\gamma = (q_{11} - q_{21})^2 + (q_{21} - q_{22})^2 > 0$, $\lambda < 0$ results. This results in a $\mu = \frac{\lambda}{\lambda-1}$ that is between 0 and 1 in value. The number $\mu - 1$ and thus the entire objective function over the continuation region are then negative. Thus, we must take the term associated with the positive root so as to make λ and the objective function have the correct, positive, sign. Thus, the Hu, Oksendal [13] solution to this problem is

$$\lambda = \frac{\frac{1}{2}\gamma - (p_2 - p_1) + \sqrt{(p_2 - p_1 - \frac{1}{2}\gamma)^2 + 2\gamma p_2}}{\gamma}, \tag{1.15}$$

$$V(x_2, x_1) = \sup_\tau E^{(x_2,x_1)}\{(X_{1,\tau} - X_{2,\tau})\} = \frac{\mu - 1}{\mu^\lambda} x_1^\lambda x_2^{1-\lambda} \tag{1.16}$$

whenever $(x_2, x_1) \in S^c$,

$$V(x_2, x_1) = \sup_\tau E^{(x_2,x_1)}\{(X_{1,\tau} - X_{2,\tau})\} = x_1 - x_2 \tag{1.17}$$

whenever $(x_2, x_1) \in S$.

$$S = \{(x_2, x_1) : x_1 \geq \mu x_2 > 0\},\ S^c = \{(x_2, x_1) : 0 < x_1 < \mu x_2\},\ \mu = \frac{\lambda}{\lambda - 1}. \tag{1.18}$$

Hu, Oksendal [13] then checked against a verification theorem cited in their paper.

Let $\mathcal{Y}_0$ denote the class of all a.s. finite stopping times. We recite Theorem 2.1 on pages 298 to 299 in Hu, Oksendal [13].

(Sufficient variational inequalities)

(a) Suppose there exists a function $\phi : \mathbb{R}^2_+ \to \mathbb{R}$ such that $\phi \in C^1(\mathbb{R}^2_+)$ and $\phi(x_2, x_1) \geq g(x_2, x_1) = x_1 - x_2 \ \forall (x_2, x_1) \in \mathbb{R}^2_+$.
Define $D = \{(x_2, x_1) \in \mathbb{R}^2_+ : \phi(x_2, x_1) > g(x_2, x_1)\}$, (the continuation region, this is our S^c). Suppose

$$E^{(x_2,x_1)} \int_0^\infty I_{[\partial D]}(X_2(t), X_1(t))dt = 0 \quad \forall (x_2, x_1) \in \mathbb{R}^2_+.$$

Moreover, suppose the following:
$\phi \in C^2(\mathbb{R}^2_+/\partial D)$ and the second order derivatives of ϕ are locally bounded near ∂D, $\mathcal{L}\phi \leq 0$ for $(x_2, x_1) \in \mathbb{R}^2_+/\overline{D}$, the family $\{\phi(X_\tau)\}_{\tau \in \mathcal{F}_D}$ is uniformly integrable with respect to $Q^{(x_2,x_1)}$ for all $(x_2, x_1) \in D$, where $\mathcal{F}_D$ is the set of all bounded stopping times $\tau \leq \tau_D$, and ∂D is a Lipschitz surface. The symbol $\mathbb{R}^2_+/\partial D$, means, in accordance to standard mathematical notation, the set $\mathbb{R}^2_+$ without the boundary for set D.
Then $\phi(x_2, x_1) \geq E^{(x_2,x_1)}[g(X_{2,\tau}, X_{1,\tau})]$ for all stopping times τ.

(b) If, in addition, we have $\mathcal{L}\phi = 0$ for $(x_2, x_1) \in D$, and
$\tau_D = \{t > 0 : (X_2(t), X_1(t)) \notin D\} < \infty$ a.s. $Q^{(x_2,x_1)}$ for $(x_2, x_1) \in \mathbb{R}^2_+$, then $\phi(x_2, x_1) = \sup_{\tau \in \mathcal{Y}_0} E^{(x_2,x_1)}[g(X_{2,\tau}, X_{1,\tau})]$ and $\tau^* = \tau_D$ is optimal.

If the free boundary pde solution passes all the above conditions, then the solution is necessarily equal to the optimal stopping solution. If a solution does not pass all the conditions, then the free boundary value solution may or may not be equal to the optimal stopping solution. Throughout Hu, Oksendal's [13] paper, the authors had verified that the free boundary pde solution does indeed satisfy all of the conditions of their Theorem 2.1 by assuming two conditions on the parameter space:

(i) $p_1 > 0$, $p_2 > 0$, and

(ii) $p_1 + \frac{1}{2}a_{11} > p_2 + \frac{1}{2}a_{22}$.

These two conditions are necessary for the free boundary solution to be verified as the optimal stopping solution also. The latter condition guarantees a.s. finite stopping times, which suffices for the conclusion in part b of the verification theorem. Thus, (1.15) through (1.18) is indeed the solution to the original optimal stopping problem.

1.7 Our Thesis Problem

As we have seen, the solution in Hu, Oksendal [13] does not address what happens when $p_1 + \frac{1}{2}a_{11} > p_2 + \frac{1}{2}a_{22}$. Once again, this asymmetry is responsible for the different roles asset 1 and 2 play in the objective function: asset 1 is a benefit, asset 2 is a cost. In Chapter 3, we will be show that, over $\Theta_C \sim \Theta_{HO}$, where the sign $\sim$ denotes set subtraction, optimal stopping time will not be a.s. finite. The optimal stopping time distribution will have a positive point mass at infinity, and the density function over finite stopping times will therefore be a defective distribution that does not integrate out to 1. This could affect the valuation of the swap option (the objective function value) in that the holder of the swap option will, with positive probability, take a pay-off of 0 over the event $[\tau = \infty]$. He/she will only be able to get a positive pay-off when $[\tau < \infty]$.

This also has repercussions on the method of solution. The verification theorem guarantees that the free boundary value solution is also the solution to the optimal stopping problem, but only when optimal stopping time is a.s. fintie. The conditions listed within the verification theorem together give sufficient, but not necessary conditions for the solutions to be one and the same. Not having a.s. finite stopping times violates one of the conditions for the verification theorem (part b of Theorem 2.1 in Hu, Oksendal's [13] paper). Thus, the theorem cannot guarantee that the free-boundary solution is also the optimal stopping solution in Θ_c. Even if the free boundary pde solution were still the same as the optimal stopping solution, we would still need to prove that it is by some other means. We use an entirely new approach to the solution. This is the subject of Chapter 3.

2 LITERATURE REVIEW

Margrabe [15] wrote, as far as we are aware, the first paper on the option to swap two financial assets. The author utilized option pricing theory to give a risk neutral valuation in complete markets and came up with an option pricing formula. As so many others had, he drew upon work previously done by Black, Scholes [4]. Subsequently, two veins of literature developed on pricing these exchange or swap options.

One approach continued with the standard assumptions in Finance, complete markets and risk neutral valuation. These assumptions added restrictions to the basic stochastic programming problem outlined in Chapter 1. Representative works in this area include Gerber and Shiu [12], and a host of other papers that applied their important result. Carr [7] also used Margrabe's [15] framework to valuate sequential, or compound options. Williams [23] used it to model real estate development. Won [24] also used it to valuate coal development projects in Korea. Suitable applications for this model are for financial assets trading in existing, well-functioning markets.

McDonald and Siegel [16] started another vein of research: the vein that includes the work in this thesis. In their paper, they dealt with company projects not necessarily traded as stocks in external markets, and investment or abandonment timing can be on any date. Even though they assumed an equilibrium markets model for the latter half of that paper, Hu, Oksendal [13] subsequently dropped that assumption. McDonald and Siegel showed that, using option pricing theory, the optimal stopping boundary of this problem is a straight line emanating from the origin. Olsen and Stensland [19] then extended the exchange problem to one with a single source of benefits versus multiple sources of costs. Their method resembled breaking up the revenue term into a weighted sum split up across the various costs components. The authors identified a super-set of the stopping set, and went on to conjecture that the super set was indeed the actual stopping set.

Hu, Oksendal [13] then added to this vein of research. They realized that the last two groups of authors failed to apply a condition that guaranteed that the process would stop at finite time (i.e. the swaption holder would exercise at finite time), and previous authors' arguments were complete only under this critical assumption. Hu, Oksendal [13] added this condition for the single-revenue and single-cost case, and proceeded to use techniques from free boundary partial differential equations and weak variational inequalities to solve for the solution. They also derived conditions for the optimal stopping set to be identifiable as the super-set in Olsen, Stensland [19] in the multiple-cost problem. Olsen, Stensland [19] were right, but only when Hu, Oksendal [13] imposed additional restrictions on the parameter space. Hu, Oksendal [13] then went on to conjecture that the identification of the stopping region should hold in even more general circumstances. Nishide and Rogers [17] then studied the multiple-revenue, multiple-cost case (a more general than the one studied under Hu, Oksendal [13]).

Nishide and Rogers [17] made the definitive statement that over higher dimensions, when the objective function involves multiple sources of costs and revenues, the identification for the stopping set has not been done because the subset suggested by Hu, Oksendal [13] will always be only a proper subset of the stopping region, as long as the volatility matrix is regular. When there are at least two revenue and two cost sources, Hu, Oksendal's [13] Theorem 4.7 is no longer true. Thus, the optimal stopping region has been exactly identified only over certain parameter restrictions (with (4.27) and (4.29) holding in Hu, Oksendal's [13] paper) on the single-revenue/multiple-cost problem. Nishide and Rogers [17] also claimed that they had solved the single revenue-single cost problem over the entire parameter space where discounted price processes were strict supermartingales. At this point, it is unclear that they have done so. When we tried to impose an a.s. finite stopping condition on the Nishide and Rogers [17] problem so as to complete an argument, the resulting restriction of the parameter space appears to be a proper subset of the parameter space studied under Hu, Oksendal [13]. Thus, we can argue that our extension to the parameter space beyond Hu, Oksendal's [13] is a meaningful one.

As we applied our theoretical model to Chladna's [9] forestry problem, we were successful in coming up with a solution to the back-end problem that mimicked Chladna's [9] problem. Finding an optimal stopping time to swap timber revenues and harvesting costs, which may include carbon penalties, was similar to our thesis problem in this aspect. We made this point in Section 1.1 of Chapter 1. However, whether our model solution is a good approximation to their problem (whether 0-mean growth in timber-prices well approximates her mean-reverting process) remains an empirical issue. It depends on how well, given estimates of parameters on both models, our objective function values and optimal harvesting time distributions agree with one another. Empirical tests concerning this aspect are outside the scope of this thesis.

A classic reference concerning finding the optimum rotation length or time-to-harvest is Faustmann [11]. There, timber prices and costs were constant through time. So was the required rate of return for foresters. Foresters faced identical economic conditions through time so that the age (the rotation length) they selected for optimal harvest (in economic terms) was identical from harvest to harvest.

The assumption of constant prices from markets is obviously untrue. Brazee and Mendelsohn [5] allowed for fluctuating prices, but was self-inconsistent as their schedule of reservation prices lay completely above the mean market prices!

Plantinga [20] proved many of the results from other papers, except using a different stochastic model for timber prices. His prices were mean- reverting, which is more realistic since in actuality we do not see natural resource prices explode to infinity. He showed the existence of reservation prices that declined with time. Real options carry positive economic values. But there were some qualitative differences as well. Plantinga was able to find reservation prices below the long term mean, whereas this was not the case in Brazee and Mendelsohn's paper [5] when they used geometric Brownian motion pricing models.

Chladna's [9] paper addressed the issue of optimum forest rotation in a single-rotation framework while considering the joint stochastic model for timber and carbon prices. She solved the optimization problem under a dynamic programming framework, and came up with numerical solutions on various age groups of forests. In her model, timber prices followed a mean-reversion-diffusion model, while carbon prices followed geometric Brownian motion. Exponentiating average carbon price reflected the increasing marginal cost (damage) in emitting more and more carbon into the atmosphere.

Chladna found that the "receive-and-pay-as-you-go" approach to treating carbon payments perversely incentivized the foresters to harvest earlier, thus, arriving at a lower average carbon storage through time. However, in her computation, she had assumed that the terminal condition was always to exercise the option, after 200 plus years of simulation. Thus, it was not clear whether the perverse incentives were due to the terminal condition that she imposed or not.

This impelled us to try to solve the thesis problem in Chapter 5 as a first approximation. Given mean-reverting-like behavior when the price growth rate is about 0 (when $p_1 = r$), and exponentiating carbon penalties as in her paper such that the parameter p_2 is negative, what would the distribution for the optimal stopping times be, and what would the worth of the portfolio (value of the objective function) be? We answer these questions in Chapter 5.

In this thesis, we have accomplished the following original work: i) we have successfully used the de-correlational, integral approach to valuate the swaption; ii) via this approach, an integral identity involving the integral of an exponential against a disintegrated measure, a Cauchy density multiplied by a generalized hyperbolic density that is absolutely continuous with respect to the Lebesque measure, resulted in the bi-variate power function that appeared in all the solutions along the two veins of research for this problem; iii) this integral approach allowed us to verify independently the Hu, Oksendal [13] solution to part of their parameter space (using our different derivation), but more importantly extended their result to two regions outside their parameter space. The solution for one extension is given in Chapter 3; the other, Chapter 5. In fact, the region in the parameter space in Chapter 5 is such that, if someone were to try to take that model of our problem as a stock swaption, one of the company's stocks would be paying out a continuous stream of negative dividends. Chladna's [9] paper led us to examine this extension.

Also, though Broadie and DeTemple [6] had shown that the optimal stopping boundary for this one revenue-one cost problem was a straight line in R^2, they solved a problem that is different from ours. Their problem had a finite time-horizon before the swaption contract expired. This led to a time-varying optimal stopping boundary in their problem. Our stopping boundary is non-time-varying. Also, the authors had an extra constant cost term in the objective function; this made their optimal stopping boundary have a non-zero intercept in the state-space. As mentioned above, even though McDonald and Siegel [16] had shown that the optimal stopping boundary for the perpetual problem was a straight line, they had to use arguments from option pricing theory to establish it. The proof was in the appendix of their 1986 paper. This thesis contains a non-option-pricing-theory argument that the optimal stopping boundary for the perpetual problem is indeed a straight line emanating from the origin. It is a confirmation of McDonald and Siegel's [16] result.

3 AN ALTERNATIVE SOLUTION TECHNIQUE THAT ALSO HANDLES THE NON-A.S. FINITE STOPPING CASE

We first establish a theorem that highlights the difference between Hu, Oksendal's[13] parameterization and our parameterization. Hu, Oksendal [13], in their paper, solved the program under 1.3 under the following parameter restrictions:

(i) $p_1, p_2 > 0$,

(ii) $p_2 + \frac{1}{2}a_{22} \geq p_1 + \frac{1}{2}a_{11}$.

Under this set of parameter restrictions, or regime, knowing the form of the stopping boundary, and using the Law of the Iterated Logarithm, Hu, Oksendal [13] showed that the optimal stopping time is a.s. finite. See (3.34) to (3.37) and the surrounding discussions in that paper. Section 1.6 of this dissertation gives one of their major results. We will partially establish this claim with another approach that looks at the distribution of first hitting times of a standard Brownian motion. However, our approach requires an additional restriction in the parameter space for a density function used in the derivation to be well defined. Thus, our alternative method only covers part of the parameter space of Hu, Oksendal[13]. However, the same method does allow us to address what happens to our problem when ii) becomes $p_2 + \frac{1}{2}a_{22} < p_1 + \frac{1}{2}a_{11}$. With one added restriction necessitated by the density function, to be cited later in this chapter, we can obtain the solution to our stochastic programming problems under both Sections 1.3 and 1.4.

The goal of this chapter is to prove the following main theorem of this chapter:

3.1 Statement of the Theorem

Theorem 3.1.1. *Assume the following restrictions on parameters:*

(i) $p_1, p_2 > 0$,

(ii) $0 \leq \left| \mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}} \right| < \sqrt{\mu_1^2+\mu_2^2}$,

where

$$\mu_1 = \left(\frac{(p_1 + \frac{1}{2}a_{11}) - (p_2 + \frac{1}{2}a_{22})}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}} \right),$$

$$\mu_2 = \frac{(\sigma_2\sigma_1\rho - \sigma_1^2)(p_2 + \frac{1}{2}a_{22}) + (\sigma_2\sigma_1\rho - \sigma_2^2)(p_1 + \frac{1}{2}a_{11})}{\sigma_1\sigma_2\sqrt{1-\rho^2}\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}},$$

$$\sigma_1 = \sqrt{a_{11}}, \sigma_2 = \sqrt{a_{22}}, \text{ and } \rho\sigma_1\sigma_2 = a_{12}.$$

Let $S_\mu = \{(x_2, x_1) : x_1 \geq \mu x_2 > 0\}$, *and* $S_\mu^c = \{(x_2, x_1) : 0 < x_1 < \mu x_2\}$, *where* $\mu > 1$, *and define the following sub-class of stopping times* $\{\tau_\mu\}$: $\tau_\mu = \inf\{t \geq 0 : (X_{2,t}, X_{1,t}) \in S_\mu\}$. *Let* $\lambda = \frac{1}{2} + \frac{p_1 - p_2}{\gamma} + \frac{\sqrt{[\frac{\gamma}{2} + (p_1 - p_2)]^2 + 2p_2\gamma}}{\gamma}$, *with* $\gamma = a_{11} + a_{22} - 2a_{12}$. *Then,*

$$\sup_\mu E^{(x_2,x_1)}\Big\{(X_{1,\tau_\mu} - X_{2,\tau_\mu})\boldsymbol{I}_{[\tau_\mu<\infty]}\Big\} = \begin{cases} \frac{(\mu^*-1)}{\mu^{*\lambda}} x_1^\lambda x_2^{1-\lambda} & \text{whenever} \quad (x_2, x_1) \in S^c \\ x_1 - x_2 & \text{whenever} \quad (x_2, x_1) \in S, \end{cases} \tag{3.1}$$

with $\mu^* = \frac{\lambda}{\lambda-1}$.

We defer most of the proof of the Theorem to Section 3.7. In the rest of this section, we outline the basic steps. Our parameter space is neither a super-set nor a sub-set of Hu, Oksendal's [13]. Their parameter space contains a region of the parameter space to which our integral approach does not apply: when $p_2 + \frac{1}{2}a_{22} \geq p_1 + \frac{1}{2}a_{11}$, and when $0 \leq \left| \mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}} \right| < \sqrt{\mu_1^2+\mu_2^2}$ is false. Likewise, our integral approach provides a solution that is not covered by Hu, Oksendal's [13] work: when

$p_2 + \frac{1}{2}a_{22} < p_1 + \frac{1}{2}a_{11}$, and when $0 \leq \left|\mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\right| < \sqrt{\mu_1^2+\mu_2^2}$ is true.

3.1.1 Discussion on another paper: Nishide and Rogers [17]

In the course of developing the solution to the optimization problem described under Section 1.4 of this thesis, a recent paper by Nishide and Rogers [17] appeared addressing the same problem by entirely different methods. In both Hu, Oksendal [13] and in Nishide, Rogers [17], the methods imply $\tau^* < \infty$ a.s. In the former paper, the authors imposed these conditions to guarantee the a.s. finiteness explicitly, while in the latter the authors imposed conditions implicitly, since, contrary to their remarks under the proof of Proposition 2.1 in Nishide, Rogers [17], the argument that the bound by $v(Z_0)$ in the displayed equation following their (2.2) requires T $< \infty$. We can show this as follows:

For an optimal T, we will conclude in Theorem 4.1.2 that such optimal T can be non-a.s. finite over Θ_C. Thus, with $Z_0 = z$, $E^z(e^{-\rho T}(1-Z_T)) = E^z(e^{-\rho T}(1-Z_T)I_{[\tau<\infty]}) + E^z(e^{-\rho T}(1-Z_T)I_{[\tau=\infty]}) = E^z(v(Z_0)I_{[\tau<\infty]}) + 0 = v(Z_0)E^z(I_{[\tau<\infty]}) = v(Z_0)P^{Z_0}(\tau<\infty) < v(Z_0)$, because pay-off over the set $[T < \infty]$ is equal to $v(Z_0)$, while pay-off over the set $[T = \infty]$ is defined to be 0, on page 21 of the paper. Since when $p_1 + \frac{1}{2}a_{11} > p_2 + \frac{1}{2}a_{22}$, $P^{Z_0}(\tau < \infty) < 1$, the left hand side is strictly less than $v(Z_0)$ by an amount equal to $P^{Z_0}(T = \infty)v(Z_0)$, and the bound is not achieved.

Apart from not requiring Hu, Oksendal's [13] restriction, as will be noted later on page 40 in this thesis, the method developed in this thesis provides an explicit formula for $P(T = \infty) > 0$. Interestingly, in spite of the various parameter restrictions, either for finite T, or for the methods in this thesis, we would not be surprised if the formula derived in their paper does indeed hold for the entire parameter space where the only restrictions are $p_1 > 0$ and $p_2 > 0$, though its actual proof is outside the scope of this thesis, and a complete proof is still open as of the writing of this thesis.

3.1.2 Brief discussion on Nishide and Rogers' [17] solution

The following discussion uses notation from the Nishide and Rogers [17] paper.

Reduction from 2 dimensions to 1:

The authors first reduce the two dimensional problem by using Girsanov's Theorem; the arguments are on page 24 of their paper and we sketch them here:

$$\begin{aligned}
V(x_2, x_1) &= \sup_\tau E^{(x_2,x_1)}[e^{-\rho\tau}(X_{1,\tau} - X_{2,\tau})] \\
&= \sup_\tau E^{(x_2,x_1)}\left[e^{-\rho\tau}\left(1 - \frac{X_{2,\tau}}{X_{1,\tau}}\right)X_{1,\tau}\right] \\
&= \sup_\tau E^{(x_2,x_1)}\left[e^{-\rho\tau}\big(1 - Z_\tau\big)x_1 e^{(\mu_1 - \frac{1}{2}\sigma_1^2)\tau + \sigma_1 W_{1\tau}}\right] \\
&= x_1 \sup_\tau E^{(x_2,x_1)}\left[e^{-(\rho-\mu_1)\tau}\big(1 - Z_\tau\big)e^{(-\frac{1}{2}\sigma_1^2)\tau + \sigma_1 W_{1\tau}}\right],
\end{aligned}$$

where the supremum is taken over all a.s. finite or non-a.s. finite stopping times τ, and $Z_t = \dfrac{X_{2,t}}{X_{1,t}}$. The Radon-Nikodym derivative of the transformed measure $\tilde{P}$ with respect to the measure P is then $e^{-\frac{1}{2}\sigma_1^2 t + \sigma_1 W_{1,t}}$, which is an exponential martingale under the P-measure. Since this has finite expectation at $t = 0$, Girsanov's Theorem can be applied to any bounded stopping time $\tau \wedge k$. The authors left a gap in the argument by not taking the limit as $k \to \infty$, and it is unclear how to fill in the argument. See Oksendal [18] for a reference on Girsanov's Theorem.

If we could apply Girsanov's Theorem, even for non-a.s. finite stopping times τ,

$$x_1 \sup_\tau E_P^{(x_2,x_1)}\left[e^{-(\rho-\mu_i)\tau}\big(1 - Z_\tau\big)e^{(-\frac{1}{2}\sigma_1^2)\tau + \sigma_1 W_{1\tau}}\right] = x_1 \sup_\tau E_{\tilde{P}}^{(x_2,x_1)}\left[e^{-(\rho-\mu_i)\tau}\big(1 - Z_\tau\big)\right];$$

this results in equation (2.5) of Nishide and Rogers [17]. Notice now the problem has been transformed to a one-dimensional problem, under the measure $\tilde{P}$ for evaluation.

3.1.3 Nishide and Rogers' [17] solution in 1-dimension

The authors showed their solution method in 1-dimension on pg. 23. Briefly, they used the infinitesimal generator to come up with an operator on the function inside the continuation region. This gave them a quadratic equation to get at the coefficient for their power function in one dimension. They then solved for the optimal stopping boundary point as a function of the positive root for the quadratic equation. The authors could then quickly establish Properties (P1), (P2), and (P3) of their paper. They then used Itô's Lemma to get (2.2); they stated that the equation is a martingale because of boundedness in the pay-off function. The v function, as defined on p. 23 of Nishide and Rogers [17], belongs to C^1. The first derivative v' is not necessarily uniformly integrable when multiplied by Z_s, but we are fortunate that Z_s times v' just happens to be a constant times v for this problem, (See equation 2.1 in that paper). Boundedness on v does imply that the right side of their equation 2.2 is a proper martingale. Hence, the expectation of the stochastic integral (2nd term on the right of 2.2) does indeed equal 0.

Dynkin's formula then applies:

$E^z[e^{-(\rho-\mu_1)T}v(Z_T)] = v(Z_0) + E\int_0^T e^{-(\rho-\mu_1)s}(L-\rho+\mu_1)v(Z_s)ds.$

Then, if T is a finite stopping time,

$$\begin{aligned} E^z[e^{-(\rho-\mu_1)T}(1-Z_T)] &\leq E^z[e^{-(\rho-\mu_1)T}v(Z_T)] \\ &= v(Z_0) + E\int_0^T e^{-(\rho-\mu_1)s}(L-\rho+\mu_1)v(Z_s)ds \\ &= v(Z_0), \end{aligned}$$

because the ratio of prices Z_t has spent all its time within the continuation region between time 0 and T, where $(L-\rho+\mu_1)v(Z_s)=0$, and $1-Z_T \leq v(Z_T)$.

But as long as T is a finite stopping time,

$$E^z[e^{-(\rho-\mu_1)T}(1-Z_T)] = E^z[e^{-(\rho-\mu_1)T}v(Z_T)]$$

because at stopping the value matching condition holds, or $1 - Z_T = v(Z_T)$. The l.h.s. then can be identified as follows:

$$\begin{aligned} E^z[e^{-(\rho-\mu_1)T}(1-Z_T)] &= E^z[e^{-(\rho-\mu_1)T}v(Z_T)] \\ &= v(Z_0) + E\int_0^T e^{-(\rho-\mu_1)s}(L-\rho+\mu_1)v(Z_s)ds \\ &= v(Z_0) + 0 \\ &= v(Z_0), \end{aligned}$$

where the second equality uses Dynkin's formula (see Oksendal [18]) and equality holds. Since the inequality holds for all finite T, taking a sup over the l.h.s. over just the finite T's gives the following equality:

$$\sup_T E^z[e^{-(\rho-\mu_1)T}(1-Z_T)] = E^z[e^{-(\rho-\mu_1)T}v(Z_T)] = v(Z_0) = v(z).$$

The above claim is not necessarily true when $T = \infty$ with any positive probability: in this case,

$$\sup_T E^z[e^{-(\rho-\mu_1)T}(1-Z_T)] \leq v(Z_0),$$

because Z_T has stayed in the continuation space forever and $(1 - Z_t) \leq v(Z_t)$ for every t inside the continuation region, with equality not guaranteed there.

Since we need $\sup_T E^z[e^{-(\rho-\mu_1)T}(1-Z_T)]$ and $v(Z_0)$ to be equal over both sets $[\tau < \infty]$ and $[\tau = \infty]$, equality over one set only is not enough to establish the overall equality, which is the result they need for Proposition 2.1 in their paper.

Their proof might be salvaged by requiring $P[\tau < \infty] = 1$. Since their Z-process stochastic differential equation is given as $dZ_t = (\mu_2 - \mu_1)Z_t dt + (\sigma_2 - \sigma_1)Z_t d\tilde{W}_t$ after they change to the $\tilde{P}$ measure, the authors could ensure a.s.-finite stopping times when $\mu_2 - \mu_1 - \frac{(\sigma_2-\sigma_1)^2}{2} \le 0$. From Proposition 3.2.7 of this thesis, this then is a restriction both sufficient and necessary for part of the proof to go through. Nishide and Rogers' [17] claim that their solution is identical to Hu, Oksendal's [13] is not established by their method of proof without this restriction.

3.1.4 Nishide and Rogers' [17] restriction in Hu, Oksendal's [13] space

We examine the restriction $\mu_2 - \mu_1 - \frac{1}{2}(\sigma_2 - \sigma_1)^2 \le 0$ when translated to Hu, Oksendal's [13] parameterization:

$$
\begin{aligned}
Z_t = \frac{X_{2,t}}{X_{1,t}} &= \frac{x_2 e^{-(p_2+\frac{1}{2}a_{22})t+q_{21}B_{1,t}+q_{22}B_{2,t}}}{x_1 e^{-(p_1+\frac{1}{2}a_{11})t+q_{11}B_{1,t}+q_{12}B_{2,t}}} \\
&= \frac{x_2}{x_1} e^{(p_1+\frac{1}{2}a_{11}-p_2-\frac{1}{2}a_{22})t+(q_{21}-q_{11})B_{1,t}+(q_{22}-q_{12})B_{2,t}} \\
&= \frac{1}{h} e^{(p_1+\frac{1}{2}a_{11}-p_2-\frac{1}{2}a_{22})t+\sqrt{(q_{21}-q_{11})^2+(q_{22}-q_{12})^2}B_{3,t}} \\
&= \frac{1}{h} e^{(p_1+\frac{1}{2}a_{11}-p_2-\frac{1}{2}a_{22})t+\sqrt{\gamma}B_{3,t}}.
\end{aligned}
$$

Over the P measure,

$$
dZ_t = (p_1 + \frac{1}{2}a_{11} - p_2 - \frac{1}{2}a_{22} + \frac{1}{2}\gamma)Z_t dt + \sqrt{\gamma}Z_t dB_{3,t}.
$$

Over the the $\tilde{P}$ measure, as their $\tilde{B}_{3,t} = B_{3,t} - \sigma_1 t$,

$$
\begin{aligned}
dZ_t &= (p_1 + \frac{1}{2}a_{11} - p_2 - \frac{1}{2}a_{22} + \frac{1}{2}\gamma)Z_t dt + \sqrt{\gamma}Z_t dB_{3,t} \\
&= (p_1 + \frac{1}{2}a_{11} - p_2 - \frac{1}{2}a_{22} + \frac{1}{2}\gamma + \sigma_1\sqrt{\gamma})Z_t dt + \sqrt{\gamma}Z_t d\tilde{B}_{3,t} \\
&= (p_1 + \frac{1}{2}a_{11} - p_2 - \frac{1}{2}a_{22} + \frac{1}{2}\gamma + \sqrt{\gamma a_{11}})Z_t dt + \sqrt{\gamma}Z_t d\tilde{B}_{3,t}.
\end{aligned}
$$

Their restriction would be

$$p_1 + \frac{1}{2}a_{11} - p_2 - \frac{1}{2}a_{22} + \sqrt{\gamma a_{11}} \leq 0, \text{ or}$$

$$p_1 + \frac{1}{2}a_{11} + \sqrt{\gamma a_{11}} \leq p_2 + \frac{1}{2}a_{22} \implies p_1 + \frac{1}{2}a_{11} \leq p_2 + \frac{1}{2}a_{22}.$$

So, imposition of the right restriction in their proof results in a parameter space that is a subset of Hu, Oksendal's [13] parameter space.

Note that even when their parameter space satisfies the above condition, it is unclear that Nishide and Rogers'[17] argument will go through. The use of Girsanov's Theorem and the Optional Sampling Theorem without a.s. finite stopping times needs to be re-examined.

The following section discusses the restrictions we use in our thesis.

3.1.5 Discussion of joint restrictions in our part of the parameter space

Our restriction ii) is as follows: $0 \leq \left| \mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}} \right| < \sqrt{\mu_1^2+\mu_2^2}$. The square of the middle term is

$$\left[\mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\right]^2 = \mu_2^2 + \frac{2\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\mu_2 + \frac{\sigma_2^2\sigma_1^2(1-\rho^2)}{\gamma}.$$

The square of the right-hand side is

$$\mu_1^2 + \mu_2^2 = \left[\frac{(p_1+\frac{1}{2}a_{11})-(p_2+\frac{1}{2}a_{22})}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\right]^2 + \mu_2^2$$
$$= \frac{(p_1+\frac{1}{2}a_{11})^2-2(p_1+\frac{1}{2}a_{11})(p_2+\frac{1}{2}a_{22})+(p_2+\frac{1}{2}a_{22})^2}{\gamma} + \mu_2^2.$$

Since both the middle and r.h.s. terms are non-negative, the condition ii) is true iff

$$\frac{2\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\mu_2 + \frac{\sigma_2^2\sigma_1^2(1-\rho^2)}{\gamma} < \frac{(p_1+\frac{1}{2}a_{11})^2-2(p_1+\frac{1}{2}a_{11})(p_2+\frac{1}{2}a_{22})+(p_2+\frac{1}{2}a_{22})^2}{\gamma}. \tag{3.2}$$

The left hand side of (3.2) is modified to

$$
\begin{aligned}
&\frac{2\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\mu_2 + \frac{\sigma_2^2\sigma_1^2(1-\rho^2)}{\gamma}\\
&= \frac{2\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\frac{(\sigma_2\sigma_1\rho-\sigma_1^2)(p_2+\frac{1}{2}a_{22})+(\sigma_2\sigma_1\rho-\sigma_2^2)(p_1+\frac{1}{2}a_{11})}{\sigma_1\sigma_2\sqrt{1-\rho^2}\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}} + \frac{\sigma_2^2\sigma_1^2(1-\rho^2)}{\gamma}\\
&= \frac{2(\sigma_2\sigma_1\rho-\sigma_1^2)(p_2+\frac{1}{2}a_{22})+2(\sigma_2\sigma_1\rho-\sigma_2^2)(p_1+\frac{1}{2}a_{11})}{\gamma} + \frac{\sigma_2^2\sigma_1^2(1-\rho^2)}{\gamma}\\
&= \frac{2(\sigma_2\sigma_1\rho-\sigma_2^2)(p_2+\frac{1}{2}\sigma_2^2)+2(\sigma_2\sigma_1\rho-\sigma_2^2)(p_1+\frac{1}{2}\sigma_1^2)}{\gamma} + \frac{\sigma_2^2\sigma_1^2(1-\rho^2)}{\gamma}\\
&= \frac{2(\sigma_2\sigma_1\rho-\sigma_1^2)p_2+(\sigma_2\sigma_1\rho-\sigma_1^2)\sigma_2^2+2(\sigma_2\sigma_1\rho-\sigma_2^2)p_1+(\sigma_2\sigma_1\rho-\sigma_2^2)\sigma_1^2}{\gamma} + \frac{\sigma_2^2\sigma_1^2(1-\rho^2)}{\gamma}\\
&= \frac{2(\sigma_2\sigma_1\rho-\sigma_1^2)p_2+(\sigma_2\sigma_1\rho)\sigma_2^2+2(\sigma_2\sigma_1\rho-\sigma_2^2)p_1+(\sigma_2\sigma_1\rho)\sigma_1^2}{\gamma} + \frac{\sigma_2^2\sigma_1^2(-1-\rho^2)}{\gamma}\\
&= \frac{2(a_{12}-a_{11})p_2+a_{12}a_{22}+2(a_{12}-a_{22})p_1+a_{12}a_{11}-a_{11}a_{22}-a_{12}^2}{\gamma}.
\end{aligned}
$$

Since $\gamma > 0$, substituting the above in the original inequality in (3.2) and multiplying through by γ gives

$$
\begin{aligned}
&2(a_{12}-a_{11})p_2+a_{12}a_{22}+2(a_{12}-a_{22})p_1+a_{12}a_{11}-a_{11}a_{22}-a_{12}^2\\
&\qquad < (p_1+\frac{1}{2}a_{11})^2-2(p_1+\frac{1}{2}a_{11})(p_2+\frac{1}{2}a_{22})+(p_2+\frac{1}{2}a_{22})^2\\
&\qquad = p_1^2+p_1a_{11}+\frac{1}{4}a_{11}^2-2p_1p_2-a_{11}p_2-p_1a_{22}-\frac{1}{2}a_{11}a_{22}+p_2^2+p_2a_{22}+\frac{1}{4}a_{22}^2,
\end{aligned}
$$

$\Leftrightarrow$

$$
\begin{aligned}
&2a_{12}p_2+2a_{12}p_1-a_{11}p_2-a_{22}p_1+a_{12}a_{22}+a_{12}a_{11}-\frac{1}{2}a_{11}a_{22}-a_{12}^2\\
&\qquad < p_1^2+p_1a_{11}+\frac{1}{4}a_{11}^2-2p_1p_2+p_2^2+p_2a_{22}+\frac{1}{4}a_{22}^2,
\end{aligned}
$$

$\Leftrightarrow$

$$
2a_{12}(p_2+\frac{1}{2}a_{22}+p_1+\frac{1}{2}a_{11})-\frac{1}{2}a_{11}a_{22}-a_{12}^2-a_{11}p_2-a_{22}p_1
$$

$$< p_1^2 + p_1 a_{11} + \frac{1}{4}a_{11}^2 - 2p_1p_2 + p_2^2 + p_2 a_{22} + \frac{1}{4}a_{22}^2$$
$$= (p_1 - p_2)^2 + p_1 a_{11} + p_2 a_{22} + \frac{1}{4}a_{11}^2 + \frac{1}{4}a_{22}^2,$$

$\Leftrightarrow$

$$2a_{12}(p_2 + \frac{1}{2}a_{22} + p_1 + \frac{1}{2}a_{11}) - a_{12}^2$$
$$< (p_1 - p_2)^2 + p_1(a_{11} + a_{22}) + p_2(a_{11} + a_{22})$$
$$+ \left(\frac{1}{2}a_{11} + \frac{1}{2}a_{22}\right)^2$$
$$= (p_1 - p_2)^2 + (p_1 + p_2)(a_{11} + a_{22}) + \left(\frac{1}{2}a_{11} + \frac{1}{2}a_{22}\right)^2,$$

$\Leftrightarrow$

$$2a_{12}\Big(p_2 + \frac{1}{2}a_{22} + p_1 + \frac{1}{2}a_{11} - \frac{a_{12}}{2}) < (p_1 - p_2)^2 + (p_1 + p_2)(a_{11} + a_{22}) + \left(\frac{1}{2}a_{11} + \frac{1}{2}a_{22}\right)^2$$

This is possible as long as $a_{12} < 0$, and $p_2 + \frac{1}{2}a_{22} + p_1 + \frac{1}{2}a_{11} - \frac{a_{12}}{2} > 0$. Unequivocally, $\frac{1}{2}a_{22} + \frac{1}{2}a_{11} > \frac{a_{12}}{2}$ whether a_{12} is positive or not, because first, if $a_{12} \geq 0$, then $\frac{1}{2}a_{22} + \frac{1}{2}a_{11} > a_{12} > \frac{a_{12}}{2}$. The covariance is always less than half the sum of the variances. Second, if $a_{12} < 0$, then $\frac{1}{2}a_{22} + \frac{1}{2}a_{11} > 0 > \frac{a_{12}}{2}$. In either case, $\frac{1}{2}a_{22} + \frac{1}{2}a_{11} - \frac{a_{12}}{2} > 0$, and therefore, $p_2 + \frac{1}{2}a_{22} + p_1 + \frac{1}{2}a_{11} - \frac{a_{12}}{2}$ is unequivocally positive. The condition $a_{12} < 0$ guarantees that the original restriction, $0 \leq \left|\mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}}\right| < \sqrt{\mu_1^2 + \mu_2^2}$, holds, as long as $p_1, p_2 > 0$, and regardless of whether $p_1 + \frac{1}{2}a_{11} \leq p_2 + \frac{1}{2}a_{22}$, or $p_1 + \frac{1}{2}a_{11} > p_2 + \frac{1}{2}a_{22}$. Thus, in a non-trivial part of the parameter space, the following conditions can hold simultaneously:

(i) $p_1, p_2 > 0$,

(ii) $p_1 + \frac{1}{2}a_{11} \leq p_2 + \frac{1}{2}a_{22}$,

(iii) $0 \leq \left| \mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}} \right| < \sqrt{\mu_1^2+\mu_2^2}$,

and the following can also hold simultaneously:

(i) $p_1, p_2 > 0$,

(ii) $p_1 + \frac{1}{2}a_{11} > p_2 + \frac{1}{2}a_{22}$,

(iii) $0 \leq \left| \mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}} \right| < \sqrt{\mu_1^2+\mu_2^2}$.

Since the integral method works whether $p_1 + \frac{1}{2}a_{11} \leq p_2 + \frac{1}{2}a_{22}$ holds or not, the restriction of the result by the integral method, as given in Theorem 3.1.1, is only given by conditions i) and iii) immediately above.

3.1.6 Comparison of our restriction versus Nishide and Rogers' [17], and Hu and Oksendal's [13]

As noted in Section 3.1.3, Nishide and Rogers [17] might have ensured a.s. finite stopping times by imposing the condition:

$$p_2 + \frac{1}{2}a_{22} \geq p_1 + \frac{1}{2}a_{11} + \sqrt{a_{11}}\gamma.$$

This is then a sub-space of Hu, Oksendal's [13]. For our parameter space,

$$2a_{12}\Big(p_2 + \frac{1}{2}a_{22} + p_1 + \frac{1}{2}a_{11} - \frac{a_{12}}{2}) < (p_1 - p_2)^2 + (p_1 + p_2)(a_{11} + a_{22}) + \Big(\frac{1}{2}a_{11} + \frac{1}{2}a_{22}\Big)^2$$

holds as long as $a_{12} < 0$. Thus, there is a non-trivial region of the parameter space whose solution is covered by our solution but not by Hu, Oksendal's [13] or Nishide, Rogers' [17]: our integral method works even when

(i) $p_1, p_2 > 0$,

(ii) $p_1 + \frac{1}{2}a_{11} > p_2 + \frac{1}{2}a_{22}$,

(iii) $0 < \left| \mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}} \right| < \sqrt{\mu_1^2+\mu_2^2}$.

Restriction iii) is less restrictive than the $a_{12} < 0$ condition. The pursuit of our solution is then a meaningful one. Note also that the Nishide and Rogers [17] solution is also the same as Hu, Oksendal's [13] in functional form. As we shall see, our solution is also identical to that of the other two sets of authors.

We now begin the proof of Theorem 3.1.1. First, we establish some preliminaries.

3.2 Brief discussion on form of stopping boundaries

As stated in Section 1.6, Hu, Oksendal [13] recognized the stopping boundary of this problem to be of the form $\{(x_2, x_1) : x_1 = \mu x_2\}$, a ray emanating from the origin and going off to infinity over the first quadrant. It is immediately clear that $\mu \in [0, \infty]$. But there are further range restrictions.

Lemma 3.2.1. *Over the entire region* $\{(x_2, x_1) : x_1 < x_2\}$, $(x_2, x_1) \in S^c$, *policies that prescribe* $P^{(x_2,x_1)}(\tau = 0) > 0$ *are sub-optimal.*

Proof. In this region, all immediate pay-offs $x_1 - x_2 < 0$. But

$$\sup_\tau E^{(x_2,x_1)}[(X_{1,\tau} - X_{2,\tau})I_{[\tau<\infty]}] \geq 0,$$

because not doing anything and letting $\tau = \infty$ would deliver a zero payoff. Thus,

$$\sup_\tau E^{(x_2,x_1)}[(X_{1,\tau} - X_{2,\tau})I_{[\tau<\infty]}] > x_1 - x_2, \text{ and } (x_2, x_1) \in S^c.$$

It is at this point that we establish that $S^c \neq \varnothing$. Now, we can invoke Lemma 1.5.3 if we wish. By Lemma 1.5.3, any optimal policy, if it exists, or any ϵ-close-to-optimal policy would prescribe $P^{(x_2,x_1)}(\tau = 0) = 0$. Any policy that prescribes $P^{(x_2,x_1)}(\tau = 0) > 0$ is then necessarily sub-optimal.

□

We first restrict the class of stopping regions to the optimal stopping regions found in Hu, Oksendal's [13] solution to forward the exposition:

Definition 3.2.2. *Let the following be jointly defined:*

$$\tau_\mu \equiv \inf\{t \geq 0 : X_{1,t} \geq \mu X_{2,t}\},$$

$$S_\mu \equiv \{(x_2, x_1) : E^{(x_2,x_1)}[(X_{1,\tau_\mu} - X_{2,\tau_\mu})I_{[\tau<\infty]}] = x_1 - x_2\},$$

$$S_\mu^c \equiv \{(x_2, x_1) : E^{(x_2,x_1)}[(X_{1,\tau_\mu} - X_{2,\tau_\mu})I_{[\tau<\infty]}] > x_1 - x_2\}$$

for any $\mu \in [1, \infty]$.

The boundary of S_μ is denoted by ∂S_μ.

Note: See Remark 3.2.1 to see why we require the condition that $\mu \in [1, \infty]$.

Definition 3.2.3. *Let $P_{(x_2,x_1)}(\mu) \equiv P^{(x_2,x_1)}(\tau < \infty)$.*

We will explore the values for $E^{(x_2,x_1)}[(X_{1,\tau_\mu} - X_{2,\tau_\mu})I_{[\tau_\mu<\infty]}]$ where we restrict the class of stopping times to those giving stopping regions of the form S_μ, where $\mu \in [1, \infty]$. First, we establish the following results.

Lemma 3.2.4. *Let $a_{ii} = q_{i1}^2 + q_{i2}^2$, $i = 1, 2$; then*

$$Z_{i,t} = e^{-(p_i+\frac{1}{2}a_{ii})t+q_{i1}B_{i,t}+q_{i2}B_{2,t}}$$

are super-martingales for $i = 1, 2$.

Proof. In the Appendix A.3. □

Lemma 3.2.5. *The boundary $\partial S \subseteq S$. S is therefore closed, and S^c is open.*

Proof. In the Appendix A.3. □

Lemma 3.2.6. *$\partial S \subseteq S$ implies the region $\{(x_2, x_1) : x_1 < x_2\}$ contain no points of S or ∂S.*

Proof. In the Appendix A.3. □

Remark 3.2.1. *When looking for a possible optimal policy from the class of policies τ_μ, μ can be restricted to the range $[1, \infty)$ because none of the points in S will belong to lines of the form $\{(x_2, x_1) : x_1 = \mu x_2\}$ when $\mu \in [0, 1)$.*

Lemma 3.2.6 says that $\{(x_2, x_1) : x_1 < x_2\}$ contain no points of S or ∂S. Therefore, when looking for ∂S_μ for the purpose that it might identify ∂S (or it might not), we can then exempt this set, or lines of the form, $\{(x_2, x_1) : x_1 = \mu x_2, 0 \leq \mu < 1\}$ from the set of possible stopping boundaries that are of this from.

Proposition 3.2.7. *Let our initial position $(x_2, x_1) \in \mathbb{R}^2_+$ be such that $\frac{x_1}{x_2} = h \leq \mu$.*

$$P(\tau_\mu < \infty) = \left\{ \begin{array}{ll} \left(\frac{\mu}{h}\right)^{\frac{2(p_2 + \frac{1}{2}a_{22} - p_1 - \frac{1}{2}a_{11})}{\gamma}} < 1, & \text{if } p_2 + \frac{1}{2}a_{22} - p_1 - \frac{1}{2}a_{11} < 0 \\ 1, & \text{if } p_2 + \frac{1}{2}a_{22} - p_1 - \frac{1}{2}a_{11} \geq 0, \end{array} \right\}, \tag{3.3}$$

where $\gamma = (q_{11} - q_{21})^2 + (q_{12} - q_{22})^2$.

Proof. In Section 3.7. □

Remark 3.2.2. *We have thus come up with a formula for $P(\tau_\mu = \infty)$. Also, we have shown that the sufficient condition for a.s. finite stopping, derived by Hu, Oksendal [13], is also a necessary one. Chapter 4 will then show that, since $\tau^* = \tau_{\mu^*}$, $P(\tau_{\mu^*} = \infty) = P(\tau^* = \infty)$, and we have an explicit formula for this, as claimed under Remark 1.4.1.*

The violation of Hu, Oksendal's[13] condition (ii) on pg. 20 of this dissertation means that the joint process, started within S_μ^c, might never hit the stopping boundary of the form $\{(x_2, x_1) \in \mathbb{R}_+^2 : x_1 = \mu x_2\}$ with positive probability for some ranges of μ (such as $\mu = \frac{\lambda}{\lambda - 1}$ or above, as larger μ's imply stopping boundaries that are farther and farther away from our initial position and more and more difficult to hit).

The stopping-time density function changes from one that has no point-probability mass at infinity to one that has a non-zero probability mass there, given by

$$1 - \left(\frac{\mu}{h}\right)^{\frac{2(p_2 + \frac{1}{2}a_{22} - p_1 - \frac{1}{2}a_{11})}{\gamma}}.$$

Given that the objective function returns 0 when the stopping time is infinite, we suspect that the value of the objective function might not be the same across the two parameter regimes. In any case, the verification theorem in Hu, Oksendal's [13] paper cannot guarantee that the solution of the free boundary pde problem is also the solution to the optimal stopping problem at hand.

Another approach, one that we take in this thesis, is to evaluate the integral $E^{(x_2,x_1)}[(X_{1,\tau} - X_{2,\tau})I_{[\tau<\infty]}]$ directly.

In Chapter 4, we will prove that the optimal τ^* must come from one of the policies τ_μ as defined in Definition 3.2.2 above. We will also evaluate

$$\sup_\tau E^{(x_2,x_1)}[(X_{1,\tau} - X_{2,\tau})I_{[\tau<\infty]}] = \sup_\mu E^{(x_2,x_1)}[(X_{1,\tau_\mu} - X_{2,\tau_\mu})I_{[\tau_\mu<\infty]}]$$

directly. This evaluation for the r.h.s. expression of last equation is then the subject for the rest of this Chapter.

We briefly remark here that Proposition 3.2.7 (as we have not yet proved Theorem 3.1.1) is the first original result of this thesis: the optimal stopping time for this problem is a.s. finite iff $p_1 + \frac{1}{2}a_{11} \le p_2 + \frac{1}{2}a_{22}$; Hu, Oksendal [13] only showed the if part.

3.3 (A thought experiment)

Under Hu, Oksendal's [13] conditions, the swap option holder can start off in $S_\mu^c = \{(x_2, x_1) \in \mathbb{R}_+^2 : 0 < x_1 < \mu x_2\}$, and wait an a.s. finite amount of time before the joint process hits the stopping boundary $\partial S_\mu^c = \{(x_2, x_1) : x_1 = \mu x_2\}$. Once there, the swap option holder exercises the option and swaps asset 2 for 1 to realize the most gain, on an expected value basis, that any policy could deliver him/her. When $\mu > 1$ gets bigger and bigger, the swap option holder will, on average, have to wait longer and longer for the joint price process to hit the stopping boundary, because the boundary is farther and farther away from his/her initial position. He/she can be more patient in waiting to exercise if once in a while he/she could get a really large $X_{1,t} - X_{2,t}$, because of the stochastic nature of both values. Thus, if the possibility of stopping at big pay-offs is large enough, then he/she might consider pushing the stopping boundary upwards (increase the stopping boundary slope μ) so as to increase $E^{(x_2,x_1)}[(X_{1,\tau} - X_{2,\tau})I_{[\tau<\infty]}]$. He/she, is then a finicky swap option holder. He/she prefers to wait for really big pay-offs before giving up his/her open swap option position by exercising.

Now, we conduct a thought experiment: suppose $p_2 + \frac{1}{2}a_{22} - p_1 - \frac{1}{2}a_{11} \to -\infty$, implying that $P_{(x_2,x_1)}(\mu) \to 0$. This implies $\tau_\mu \to \infty$ with probability 1. The stopping boundary for the swap option holder must approach $\{(x_2, x_1) \in \mathbb{R}_+^2 : x_1 = x_2\}$ because the holder recognizes that most of the time, he/she will be forced to take a 0 pay-off. The swap option holder is now forced to be less finicky as to what kind of possible pay-offs he/she will take to maximize the objective function (the swap option value). He/she would race to exercise at the first hint of positive profit because he/she might never see positive pay-offs out of the swap arrangement ever again!

In actuality, when $p_2 + \frac{1}{2}a_{22} - p_1 - \frac{1}{2}a_{11} \to -\infty$, caused by $p_1 \to \infty$, $X_{1,t}$'s mean trajectory is damping down to 0 so fast, relative to $X_{2,t}$'s mean trajectory, that $X_{1,t} - X_{2,t}$

is not expected to be positive over too many t's and sample paths (ω's). Therefore, the swap option holder will do well to exercise when a rare, positive profit opportunity does come along. He/she does not want to wait for a big, big payoff. The stopping boundary corresponding to this behavior calls for $\mu \to 1$. Thus, the optimal stopping boundary should tilt towards the $\mu = 1$ slope in X-Space, and the objective function value should approach the limiting value of 0, as the probability of non-finite stopping time increases. The Hu, Oksendal [13] solution does not apply to this range of parameters, when $p_2 + \frac{1}{2}a_{22} - p_1 - \frac{1}{2}a_{11} < 0$; but the same intuition applies when we compare their case with ours. Their case has a.s. finite stopping time; therefore, the swap option holder can afford, on average, to be more finicky. Our case has non-a.s. finite stopping; therefore, our holders are less finicky. The slope of our optimal stopping boundary, while still greater than 1, should be less than Hu, Oksendal's [13] counterpart. We will give a more definite statement regarding this phenomenon at the end of Chapter 4.

3.4 Computational Plan

The pay-off function of the objective function is a homothetic function of degree 1. Various authors, McDonald and Siegel [16], Olsen and Stensland [19], and Hu, Oksendal [13], have suggested that the stopping boundary of this problem should be a straight line. In addition, we need to verify that when the parameter values fall within Hu, Oksendal's [13] parameter regime, our integral solution would be the same as theirs. Therefore, it is natural at this point to consider only a sub-class of stopping times, the τ_μ's as defined in Definition 3.2.2. We will evaluate the objective function value by direct integration:

$$\begin{aligned} E^{(x_2,x_1)}[(X_{1,\tau_\mu} - X_{2,\tau_\mu})I_{[\tau_\mu<\infty]}] &= E^{(x_2,x_1)}[(\mu X_{2,\tau_\mu} - X_{2,\tau_\mu})I_{[\tau_\mu<\infty]}] \\ &= (\mu - 1)E^{(x_2,x_1)}[X_{2,\tau_\mu} I_{[\tau_\mu<\infty]}] \end{aligned}$$

$$= (\mu - 1)E^{x_2,x_1}\Big\{E\Big[X_{2,\tau_\mu} I_{[\tau_\mu<\infty]}\Big|\sigma(\tau_\mu)\Big]\Big\}.$$

However, a difficulty arises in using the iterated expectation. We cannot directly substitute the variable τ_μ into the integrand and treat it as a constant of integration when we perform the first stage integration over X_{2,τ_μ}, because X_{2,τ_μ} and τ_μ are not independent. We therefore resort to transforming the original $(X_{2,t}, X_{1,t})$, the X-Space, to a $(Z_{2,t}, Z_{1,t})$ Z-Space such that $(Z_{2,t}, Z_{1,t})$ are jointly normal and uncorrelated, at least over the set $[\tau_\mu < \infty]$. The two properties just described imply that $Z_{2,t}, Z_{1,t}$ are independent over that set. Therefore, if we can simultaneously transform the stopping boundary in the original problem, a ray emanating from the origin of the form $\{(x_2, x_1) : x_1 = \mu x_2\}$ in X-Space to the horizontal axis $\{(z_2, z_1) : z_1 = 0\}$ in Z-Space, and also make $Z_{2,t}, Z_{1,t}$ uncorrelated over $[\tau_\mu < \infty]$, then the stopping time τ_μ will only depend on $Z_{1,t}$, and will be conditionally independent of $Z_{2,t}$. Now, our transformed problem will look as follows:

$$\begin{aligned}
E^{(x_2,x_1)}[(X_{1,\tau_\mu} - X_{2,\tau_\mu})I_{[\tau_\mu<\infty]}] &= E^{(x_2,x_1)}[(\mu X_{2,\tau_\mu} - X_{2,\tau_\mu})I_{[\tau_\mu<\infty]}] \\
&= (\mu - 1)E^{(x_2,x_1)}[X_{2,\tau_\mu} I_{[\tau_\mu<\infty]}] \\
&= (\mu - 1)E^{(x_2,x_1)}\Big\{E\Big[h(Z_{2,\tau_\mu}, \tau_\mu)I_{[\tau_\mu<\infty]}\Big|\sigma(\tau_\mu)\Big]\Big\},
\end{aligned}$$

as $h(Z_{2,\tau_\mu}, \tau_\mu) = X_{2,\tau_\mu}$ when $\tau_\mu < \infty$, and $h(Z_{2,\tau_\mu}, \tau_\mu) = 0$ when $\tau_\mu = \infty$, which is consistent with the objective function value equalling zero when the stopping time is infinite. Because of the conditional independence between the two (conditioned upon the set $[\tau_\mu < \infty]$), we can now substitute τ_μ into the first partial integral of the double integral and treat it as a constant of integration with respect to the integration over Z_{2,τ_μ}, the random variable representing the location of the z_2 coordinate when the horizontal-axis of $z_1 = 0$ is hit by the joint process in Z-Space.

Following this approach, we can show by further derivation that the marginal density function of the location Z_{2,τ_μ}, as given by

$$g_\mu(z) = \int_0^\infty f(z|\tau)\{f(\tau)I_{[\tau<\infty]}\}d\tau,$$

resulted from the disintegration of the joint measure into a conditional and marginal probability measure, that, after a switch in order of integration via applying Tonelli's theorem, became a multiplication of a Cauchy and a generalized hyperbolic density function. This holds for both our parameter regime and part of Hu, Oksendal's [13]. Computationally, the generalized hyperbolic density is itself a generalized inner product between a Gaussian and a generalized inverse gaussian distribution. See Eberlein et. al. [10]. When the Hu, Oksendal [13] parameter constraints hold, this integral solution reduces to their free boundary solution; when their second constraint is modified to $p_2 + \frac{1}{2}a_{22} - p_1 - \frac{1}{2}a_{11} < 0$, the overall objective function would also yield the same solution.

A brief sketch of the derivation follows, and the actual proof is in the proof for Proposition 3.5.2, to be given in Section 3.7. This is the new derivation method that is different from the free boundary pde approach in Hu, Oksendal [13].

Let $f(\cdot)$ denote the probability density function of whatever variable that is inside. For instance, let $f(\tau)$ denote the probability density function for τ, and let $f(z|\tau)$ denote the conditional probability density function for Z_{2,τ_μ}, given τ_μ. Even though we use the same letter f, they are in general different densities. An exception to that rule is for $g_\mu(z) = \int_0^\infty f(z|\tau)\{f(\tau)I_{[\tau<\infty]}\}d\tau$ as will be defined later. We proceed with the valuation, where ν stands for the Lebesque measure in R^2.

$$\begin{aligned} &E^{(x_2,x_1)}[(X_{1,\tau} - X_{2,\tau})I_{[\tau<\infty]}] \\ &\quad = (\mu-1)E^{(x_2,x_1)}\left\{E\left(e^{\frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}Z_{2,\tau_\mu}} I_{[\tau_\mu<\infty]}\middle|\sigma(\tau_\mu)\right)\right\} \end{aligned}$$

$$= (\mu - 1) \int \int \left\{ e^{\frac{\sigma_2 \sigma_1 \sqrt{1-\rho^2}}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1 \sigma_2 \rho}} Z_{2,\tau_\mu}} I_{[\tau < \infty]} \right\} \{ f(z|\tau) I_{[\tau<\infty]} + f(z|\tau) I_{[\tau=\infty]} \}$$

$$\times \{ f(\tau) I_{[\tau<\infty]} + f(\tau) I_{[\tau=\infty]} \} (\nu(dz d\tau))$$

$$= (\mu - 1) \int_0^\infty \int_{-\infty}^\infty e^{\frac{\sigma_2 \sigma_1 \sqrt{1-\rho^2}}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1 \sigma_2 \rho}} z} I_{[\tau<\infty]} f(z|\tau) f(\tau) dz d\tau$$

$$+ (\mu - 1) \int \int 0 \cdot I_{[\tau<\infty]} I_{[\tau=\infty]} f(z|\tau) f(\tau) (\nu(dz d\tau)),$$

by the law of iterated expectation. Now, for events such as $[\tau = \infty]$, $Z_{2,\tau} = -\infty$, because $X_{2,\tau} \equiv 0$. But $f(z|\tau = \infty) f(\tau)$ is finite. Therefore, the second term after the third equality above must be 0, or using Tonelli's theorem,

$$E^{(x_2,x_1)}[(X_{1,\tau_\mu} - X_{2,\tau_\mu}) I_{[\tau_\mu<\infty]}] = (\mu - 1) \int_0^\infty \int_{-\infty}^\infty e^{\frac{\sigma_2 \sigma_1 \sqrt{1-\rho^2}}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1 \sigma_2 \rho}} z} I_{[\tau<\infty]} f(z|\tau) f(\tau) dz d\tau$$

$$= (\mu - 1) \int_{-\infty}^\infty \int_0^\infty e^{\frac{\sigma_2 \sigma_1 \sqrt{1-\rho^2}}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1 \sigma_2 \rho}} z} f(z|\tau) \{ f(\tau) I_{[\tau<\infty]} \} d\tau dz$$

$$= (\mu - 1) \int_{-\infty}^\infty e^{\frac{\sigma_2 \sigma_1 \sqrt{1-\rho^2}}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1 \sigma_2 \rho}} z} \int_0^\infty f(z|\tau) \{ f(\tau) I_{[\tau<\infty]} \} d\tau dz$$

$$= (\mu - 1) \int_{-\infty}^\infty e^{\frac{\sigma_2 \sigma_1 \sqrt{1-\rho^2}}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1 \sigma_2 \rho}} z} g_\mu(z) dz.$$

By evaluating $\int_0^\infty f(z|\tau) f(\tau) I_{[\tau<\infty]} d\tau \equiv g_\mu(z)$ we can then integrate against the exponential function to come up with the overall expectation on the first line of the equation immediately preceding. The function $f(\tau)$ is the density function of first hitting times, whether it is in the original state space or the transformed state space. The function $f(\tau) I_{[\tau<\infty]}$ is the part of the distribution that includes only finite τ's. It is defective for the problem of non-a.s. finite stopping time in that $0 \leq \int_0^\infty f(\tau_\mu) I_{[\tau_\mu<\infty]} d\tau_\mu <$

1. Proposition 3.5.2 will give the functional form of $\int_0^\infty f(z|\tau)f(\tau)I_{[\tau<\infty]}d\tau$. But first, we give the transformation from X to Z-Space.

3.5 Transformation from X-Space to Z-Space and proof of Theorem 3.1.1

In order to achieve the de-correlation and the transformation of the stopping boundary, we go through the following series of transformations:

(i) $\begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} = \begin{bmatrix} e^{S_{1,t}} \\ e^{S_{2,t}} \end{bmatrix}$ (This step turns our problem from geometric to arithmetic joint Brownian motion).

(ii) $\begin{bmatrix} V_{1,t} \\ V_{2,t} \end{bmatrix} = \begin{bmatrix} S_{1,t} - \ln \mu \\ S_{2,t} \end{bmatrix}$ (This step gets rid of the intercept associated with the new stopping boundary).

(iii) $\begin{bmatrix} Y_{1,t} \\ Y_{2,t} \end{bmatrix} = \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \begin{bmatrix} \sigma_2 & -\sigma_1\rho \\ 0 & \sigma_1\sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} V_{1,t} \\ V_{2,t} \end{bmatrix}$ (This is the de-correlation step). The coefficients on the last line bear relationships with parameters in the original model. They are

$\sigma_1 = \sqrt{q_{11}^2 + q_{12}^2}$,

$\sigma_2 = \sqrt{q_{21}^2 + q_{22}^2}$,

$a_{12} = q_{11}q_{21} + q_{12}q_{22} = \rho\sigma_1\sigma_2$ or $\rho = \dfrac{q_{11}q_{21} + q_{12}q_{22}}{\sigma_1\sigma_2} = \dfrac{q_{11}q_{21} + q_{12}q_{22}}{\sqrt{q_{11}^2 + q_{12}^2}\sqrt{q_{21}^2 + q_{22}^2}}$.

(iv) Let $\arctan\left[\frac{\sigma_2-\sigma_1\rho}{\sigma_1\sqrt{1-\rho^2}}\right] = \theta$, $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

$$\cos\theta = \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2(1-\rho^2) + (\sigma_2-\sigma_1\rho)^2}} = \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2 - \sigma_1^2\rho^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho + \sigma_1^2\rho^2}}$$

$$= \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}},$$

$$\sin\theta = \frac{\sigma_2-\sigma_1\rho}{\sqrt{\sigma_1^2(1-\rho^2)+(\sigma_2-\sigma_1\rho)^2}} = \frac{\sigma_2-\sigma_1\rho}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}.$$

The rotation matrix, rotating our coordinate system $(Y_{2,t}, Y_{1,t})$ by $-\theta$ degrees, is then

$$R = \begin{bmatrix} \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}} & -\frac{\sigma_2-\sigma_1\rho}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}} \\ \frac{\sigma_2-\sigma_1\rho}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}} & \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}} \end{bmatrix}.$$

Let $\begin{bmatrix} R_{1,t} \\ R_{2,t} \end{bmatrix} = R \begin{bmatrix} Y_{1,t} \\ Y_{2,t} \end{bmatrix}$; this step rotates the stopping boundary from making an angle of θ with the horizontal axis to the horizontal axis.

(v) Define $\begin{bmatrix} Z_{1,t} \\ Z_{2,t} \end{bmatrix} = \begin{bmatrix} -R_{1,t} \\ R_{2,t} \end{bmatrix}$. This reflects the state space from the lower half-plane to the upper half-plane.

The next four propositions, as well as Theorem 3.1.1, will be proved in Section 3.7.

Proposition 3.5.1. *After performing the transformations from i) to v) in this section, the following results:*

I.

$$\begin{aligned} Z_{1,t} = &-\Big(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\Big)\frac{\sigma_2\ln x_1-\sigma_1\rho\ln x_2-\sigma_2\ln\mu}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \\ &+\frac{\sigma_2-\sigma_1\rho}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\frac{\ln x_2}{\sigma_2} \\ &-\Big[\Big(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\Big)\frac{\sigma_1\rho(p_2+\frac{1}{2}a_{22})-\sigma_2(p_1+\frac{1}{2}a_{11})}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \\ &+\Big(\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\Big)\frac{p_2+\frac{1}{2}a_{22}}{\sigma_2}\Big]t \\ &-\Big[\Big(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\Big)\frac{q_{11}\sigma_2-\sigma_1\rho q_{21}}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \end{aligned}$$

$$
\begin{aligned}
&- \frac{\sigma_2 - \sigma_1\rho}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}} \frac{q_{21}}{\sigma_2}\Big] B_{1,t} \\
&- \Big[\Big(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\Big) \frac{q_{12}\sigma_2 - \sigma_1\rho q_{22}}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \\
&- \frac{\sigma_2 - \sigma_1\rho}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}} \frac{q_{22}}{\sigma_2}\Big] B_{2,t}; \\
Z_{2,t} =& \Big(\frac{\sigma_2 - \sigma_1\rho}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\Big) \frac{\sigma_2 \ln x_1 - \sigma_1\rho \ln x_2 - \sigma_2 \ln \mu}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \\
&+ \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}} \frac{\ln x_2}{\sigma_2} \\
&+ \Big[\Big(\frac{\sigma_2 - \sigma_1\rho}{\sqrt{\gamma}}\Big) \frac{\sigma_1\rho(p_2 + \frac{1}{2}a_{22}) - \sigma_2(p_1 + \frac{1}{2}a_{11})}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \\
&- \Big(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\Big) \frac{p_2 + \frac{1}{2}a_{22}}{\sigma_2}\Big] t \\
&+ \Big[\Big(\frac{\sigma_2 - \sigma_1\rho}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\Big) \frac{q_{11}\sigma_2 - \sigma_1\rho q_{21}}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \\
&+ \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}} \frac{q_{21}}{\sigma_2}\Big] B_{1,t} \\
&+ \Big[\Big(\frac{\sigma_2 - \sigma_1\rho}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\Big) \frac{q_{12}\sigma_2 - \sigma_1\rho q_{22}}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \\
&+ \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}} \frac{q_{22}}{\sigma_2}\Big] B_{2,t}.
\end{aligned}
$$

II. $Var(Z_{1,t}) = Var(Z_{2,t}) = t$. $Cov(Z_{1,t}, Z_{2,t}) = 0$.

III. We transform from the original space $\partial S_\mu = \{(x_2, x_1) : x_1 = \mu x_2\}$

$$\textit{to } \{(z_2, z_1) : z_1 = 0\}.$$

IV. The following identity holds:

$$E^{(x_2,x_1)}[(X_{1,\tau} - X_{2,\tau}) I_{[\tau<\infty]}] = (\mu - 1) E^{(x_2,x_1)}\Big(e^{\frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}} Z_{2,\tau_\mu}} I_{[\tau<\infty]}\Big).$$

Proof. Given in Section 3.7. □

Proposition 3.5.2. *As long as* $p_2 + \frac{1}{2}a_{22} \neq p_1 + \frac{1}{2}a_{11}$*, or* $(\sigma_2\sigma_1\rho - \sigma_1^2)(p_2 + \frac{1}{2}a_{22}) \neq (\sigma_2^2 - \sigma_1\sigma_2\rho)(p_1 + \frac{1}{2}a_{11})$*, for* $Z_{2,\tau_\mu} = u$*,*

$$\int_0^\infty f(u|\tau)f(\tau)I_{[\tau<\infty]}d\tau = \frac{\sqrt{\mu_1^2+\mu_2^2}e^{-[\mu_1 z_1+\mu_2 z_2]}}{\pi}\frac{z_1 e^{\mu_2 u}}{\sqrt{z_1^2+(u-z_2)^2}} \times K_1\sqrt{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]},$$

where

$$\begin{aligned}
z_1 &= -\Big(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\Big)\frac{\sigma_2\ln x_1 - \sigma_1\rho\ln x_2 - \sigma_2\ln\mu}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \\
&+ \frac{\sigma_2-\sigma_1\rho}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\frac{\ln x_2}{\sigma_2}, \\
z_2 &= \Big(\frac{\sigma_2-\sigma_1\rho}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\Big)\frac{\sigma_2\ln x_1 - \sigma_1\rho\ln x_2 - \sigma_2\ln\mu}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \\
&+ \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\frac{\ln x_2}{\sigma_2}, \\
\mu_1 &= \frac{(p_1+\frac{1}{2}a_{11})-(p_2+\frac{1}{2}a_{22})}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}, \\
\mu_2 &= \frac{(\sigma_2\sigma_1\rho-\sigma_1^2)(p_2+\frac{1}{2}a_{22})+(\sigma_2\sigma_1\rho-\sigma_2^2)(p_1+\frac{1}{2}a_{11})}{\sigma_1\sigma_2\sqrt{1-\rho^2}\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}},
\end{aligned}$$

and K_1 *is the modified Bessel function of the third kind with index 1.*

Proof. Given in Section 3.7. □

Remark 3.5.1. *We have just come up with the marginal density for the hitting location in Z-Space. The marginal density in X-Space is just a change of variable (with a Jacobian transform) away from the above function. We describe the exact form of the transform in Remark 3.7.1 below.*

With the $g_\mu(z)$ density, we can now evaluate the expectation in

$$E^{(x_2,x_1)}[(X_{1,\tau_\mu}-X_{2,\tau_\mu})I_{[\tau_\mu<\infty]}] = (\mu-1)E^{(x_2,x_1)}\Big\{E\Big(e^{\frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}Z_{2,\tau_\mu}}I_{[\tau_\mu<\infty]}\Big|\sigma(\tau_\mu)\Big)\Big\}.$$

Proposition 3.5.3. *If* $p_1>0, p_2>0,$ *and* $0<|\mu_2+\frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\Big|<\sqrt{\mu_1^2+\mu_2^2}$, *we have*

$$(\mu-1)E^{(x_2,x_1)}\Big\{E\Big(e^{\frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}Z_{2,\tau_\mu}}I_{[\tau_\mu<\infty]}\Big|\sigma(\tau_\mu)\Big)\Big\} =$$

$$e^{C_1C_4-(\mu_1C_2+\mu_2C_4)-C_2\sqrt{\mu_1^2+\mu_2^2-C_1^2}}\Big\{(\mu-1)\mu^{C_1C_5-(\mu_1C_3+\mu_2C_5)-C_3\sqrt{\mu_1^2+\mu_2^2-C_1^2}}\Big\},$$

where

$$\begin{aligned}
C_1 &= \frac{\rho\sigma_1\sigma_2(p_1+p_2+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_2^2)-\sigma_1^2p_2-\sigma_2^2p_1-\rho^2\sigma_1^2\sigma_2^2}{\sqrt{\gamma}\sigma_1\sigma_2\sqrt{1-\rho^2}}+\frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}} \\
&= \mu_2+\frac{\sigma_2\sigma_2\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}, && (3.4) \\
z_1 &= -\Big(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\Big)\frac{\sigma_2\ln x_1-\sigma_1\rho\ln x_2-\sigma_2\ln\mu}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \\
&\quad+\frac{\sigma_2-\sigma_1\rho}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\frac{\ln x_2}{\sigma_2}, \\
&= C_2+C_3\ln\mu. && (3.5)
\end{aligned}$$

Here,

$$\begin{aligned}
h &= \frac{x_1}{x_2}, \\
C_2 &= \frac{-\ln h}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}, && (3.6) \\
C_3 &= \frac{1}{\sqrt{\gamma}}; \quad \gamma=(q_{11}-q_{21})^2+(q_{12}-q_{22})^2; && (3.7) \\
z_2 &= \Big(\frac{\sigma_2-\sigma_1\rho}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\Big)\frac{\sigma_2\ln x_1-\sigma_1\rho\ln x_2-\sigma_2\ln\mu}{\sigma_1\sigma_2\sqrt{1-\rho^2}}+\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\frac{\ln x_2}{\sigma_2}, \\
&= C_4+C_5\ln\mu, && (3.8)
\end{aligned}$$

with

$$C_4 = \frac{\sigma_2^2 \ln x_1 + \sigma_1^2 \ln x_2 - \rho\sigma_1\sigma_2 \ln(x_1 x_2)}{\sigma_1\sigma_2\sqrt{1-\rho^2}\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}, \tag{3.9}$$

$$C_5 = -\frac{\sigma_2 - \sigma_1\rho}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\frac{1}{\sigma_1\sqrt{1-\rho^2}}, \tag{3.10}$$

and μ, μ_1, μ_2 as previously defined. The same result holds regardless of whether $p_2+\frac{1}{2}a_{22} \geq p_1 + \frac{1}{2}a_{11}$ or not.

Proof. Given in Section 3.7. □

Over both Θ_{HO} and Θ_C,

$$E^{(x_2,x_1)}[(X_{1,\tau_\mu} - X_{2,\tau_\mu})I_{[\tau<\infty]}]$$
$$= e^{C_1C_4-(\mu_1C_2+\mu_2C_4)-C_2\sqrt{\mu_1^2+\mu_2^2-C_1^2}}\left\{(\mu-1)\mu^{C_1C_5-(\mu_1C_3+\mu_2C_5)-C_3\sqrt{\mu_1^2+\mu_2^2-C_1^2}}\right\}.$$

Proposition 3.5.4. *The expression obtained from Proposition 3.5.3 is equal to the Hu, Oksendal [13] solution, or*

$e^{C_1C_4-(\mu_1C_2+\mu_2C_4)-C_2\sqrt{\mu_1^2+\mu_2^2-C_1^2}}\left\{(\mu-1)\mu^{C_1C_5-(\mu_1C_3+\mu_2C_5)-C_3\sqrt{\mu_1^2+\mu_2^2-C_1^2}}\right\}$
$= (\mu-1)\mu^{-\lambda}h^\lambda x_2$, *with* $\mu = \dfrac{\lambda}{\lambda-1}$, $h = \dfrac{x_1}{x_2}$, *and*

$$\lambda = \frac{\frac{1}{2}\gamma - (p_2 - p_1) + \sqrt{(p_1 - p_2 + \frac{1}{2}\gamma)^2 + 2\gamma p_2}}{\gamma}. \tag{3.11}$$

The same result applies whether $p_2 + \frac{1}{2}a_{22} \geq p_1 + \frac{1}{2}a_{11}$ holds or not.

Proof. Given in Section 3.7

□

The last Theorem just gave us our integral identity over the disintegrating measure, after a switch in order of integration has been applied:

Corollary 3.5.5. $\int_{-\infty}^{\infty} e^{\frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}u} \frac{\sqrt{\mu_1^2+\mu_2^2}e^{-[\mu_1 z_1+\mu_2 z_2]}}{\pi} \frac{z_1 e^{\mu_2 u}}{\sqrt{z_1^2+(u-z_2)^2}}$
$\times K_1\sqrt{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}du = (\mu-1)\mu^{-\lambda}h^{\lambda}x_2.$

The main result of this chapter, as given by Theorem 3.1.1 (p. 26) thus follows. The proof is the final section in Section 3.7.

3.6 Meaningful Extension

In this chapter, we have deliberately restricted the class of stopping boundaries, and thus the class of stopping times, to boundaries given by the form $\{(x_2, x_1) : x_1 = \mu x_2\}$ for $\mu > 1$. Under the Hu, Oksendal [13] parameter regime, the optimal stopping boundary does indeed come from one of these lines, and therefore, the solution using our approach is identical to the Hu, Oksendal [13] solution; though our additional restriction, ii) in Theorem 3.1.1 above, introduced because of the need to use the generalized hyperbolic distribution, showed that our proof applies to only a sub-space of Hu, Oksendal's [13].

Even with this limitation, the extension to the non-a.s. finite stopping setting by requiring the parameter conditions under Theorem 3.1.1 is still a meaningful one, as Hu, Oksendal [13] has not addressed the non-a.s. stopping problem at all, and Nishide and Rogers [17] have not completely addressed this same problem.

We have arrived at a solution for this case in Theorem 3.1.1, assuming that the stopping boundary is also of the form $\{(x_2, x_1) : x_1 = \mu x_2\}$ for some $\mu > 1$. In the next chapter, we will prove that the optimal stopping boundary for this problem, whether it is under the Hu, Oksendal [13] parameter regime or ours, will be given by boundaries of the same form as ∂S_μ in this chapter. Therefore, the optimal stopping time τ_{μ^*} found in this chapter is also optimal over the entire class of stopping times.

3.7 Proofs of results

Proof. (Proposition 3.2.7)

Let our initial position $(x_2, x_1) \in \mathbb{R}^2_+$ be such that $\frac{x_1}{x_2} = h \leq \mu$.
If $p_2 + \frac{1}{2}a_{22} - p_1 - \frac{1}{2}a_{11} < 0$, then $P_{(x_2,x_1)}(\mu) = \left(\frac{\mu}{h}\right)^{\frac{2(p_2+\frac{1}{2}a_{22}-p_1-\frac{1}{2}a_{11})}{\gamma}}$.
If $p_2 + \frac{1}{2}a_{22} - p_1 - \frac{1}{2}a_{11} \geq 0$, then $P_{(x_2,x_1)}(\mu) = 1$.

Take the ratio $\frac{X_1(t)}{X_2(t)}$ of the solutions of the stochastic differential equations on pg. 143, and 144. When the joint processes are stopped at time τ_μ, the ratio is

$$\frac{X_{1,\tau_\mu}}{X_{2,\tau_\mu}} = \frac{x_1}{x_2} \exp\left[\left(p_2 + \frac{1}{2}a_{22} - p_1 - \frac{1}{2}a_{11}\right)\tau_\mu + (q_{11} - q_{21})B_{1,\tau_\mu} + (q_{12} - q_{22})B_{2,\tau_\mu}\right] = \mu.$$

Let $k = p_2 + \frac{1}{2}a_{22} - p_1 - \frac{1}{2}a_{11}$, $h = \frac{x_1}{x_2}$ and let $k_1 = q_{11} - q_{21}$, $k_2 = q_{12} - q_{22}$. Then, the above ratio becomes

$$\frac{X_{1,\tau_\mu}}{X_{2,\tau_\mu}} = h \exp[k\tau_\mu + k_1 B_{1,\tau_\mu} + k_2 B_{2,\tau_\mu}] = \mu.$$

Taking logs on both sides, retaining the last equality, and doing some algebra, give

$$k_1 B_{1,\tau_\mu} + k_2 B_{2,\tau_\mu} = \ln\frac{\mu}{h} - k\tau_\mu.$$

Define $k_1 B_{1,t} + k_2 B_{2,t} = \sqrt{k_1^2 + k_2^2} B_{3,t}$, or $B_{3,t} = \dfrac{k_1 B_{1,t} + k_2 B_{2,t}}{\sqrt{k_1^2 + k_2^2}}$.

We know that $B_{3,0} = 0$; $B_{3,t}$ is a.s. continuous because $B_{1,t}$, $B_{2,t}$ are; $B_{3,t} \overset{d}{\sim} N(0,t)$, as $B_{1,t}$, $B_{2,t}$ are independent standard Brownian motions; $B_{3,t}$ has stationary and independent increments because independent processes $B_{1,t}$, $B_{2,t}$ do; thus, $\{B_{3,t} : t \geq 0\}$ is a standard Brownian process.

For rest of the chapter, let $\tau = \tau_\mu$. The stopping condition then becomes

$$B_{3,\tau} = \frac{\ln \frac{\mu}{h} - k\tau}{\sqrt{k_1^2 + k_2^2}},$$

as a function of τ.

Thus, the first hitting time τ for the joint process starting at (x_1, x_2), with $\frac{x_1}{x_2} = h < \mu$, and arriving at $(\mu X_{2,\tau}, X_{2,\tau})$ at time τ, is the same as the first hitting time for $B_{3,t}$ to hit the sloped line $\overline{h}(t) \equiv \dfrac{1}{\sqrt{k_1^2 + k_2^2}} \ln \dfrac{\mu}{h} - \dfrac{k}{\sqrt{k_1^2 + k_2^2}} t$. The distribution for this first hitting time is well known. We cite the following source to give the form of the density function: Bhattacharya and Waymire [3]. The density function is

$$f_\tau(t) = \frac{a}{t^{3/2}} \phi(\frac{a + bt}{\sqrt{t}}) = \frac{a}{\sqrt{2\pi} t^{3/2}} e^{-\frac{(\frac{a+bt}{\sqrt{t}} - 0)^2}{2}} = \frac{a}{\sqrt{2\pi} t^{3/2}} e^{-\frac{(a+bt)^2}{2t}},$$

(Note: $k < 0$, or $p_2 + \frac{1}{2} a_{22} < p_1 + \frac{1}{2} a_{11}$, is used here), where $a = \dfrac{\ln \frac{\mu}{h}}{\sqrt{k_1^2 + k_2^2}}$, and the coefficient $b = \dfrac{-k}{\sqrt{k_1^2 + k_2^2}}$. The probability for the event $[\tau < \infty]$ is then given by

$$\begin{aligned} P_{(x_2, x_1)}(\mu) &= \int_0^\infty f_\tau(t) dt = \int_0^\infty \frac{a}{t^{3/2}} \phi(\frac{a + bt}{\sqrt{t}}) dt \\ &= \int_0^\infty \frac{a}{\sqrt{2\pi} t^{3/2}} e^{-\frac{(\frac{a+bt}{\sqrt{t}} - 0)^2}{2}} dt \\ &= \int_0^\infty \frac{a}{t^{3/2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(a+bt)^2}{2t}} dt. \end{aligned}$$

Let $t = \dfrac{2}{b^2 x}; dt = -\dfrac{2}{b^2 x^2} dx; \dfrac{1}{t^{3/2}} = \Big(\dfrac{b^2 x}{2}\Big)^{3/2} = \dfrac{b^3 x^{3/2}}{2^{3/2}}$. Then

$$\int_0^\infty \frac{a}{t^{3/2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(a+bt)^2}{2t}} dt = \int_\infty^0 a \frac{b^3 x^{3/2}}{2^{3/2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(a + b\frac{2}{b^2 x})^2}{2 \frac{2}{b^2 x}}} \Big(- \frac{2}{b^2 x^2} \Big) dx$$

$$
\begin{aligned}
&= \int_0^\infty \frac{ab}{2}\frac{1}{\sqrt{\pi x}} e^{-\frac{(a+\frac{2}{bx})^2}{\frac{4}{b^2 x}}}\,dx \\
&= \int_0^\infty \frac{ab}{2}\frac{1}{\sqrt{\pi x}} e^{-\frac{(a^2+\frac{4a}{bx}+\frac{4}{b^2x^2})}{\frac{4}{b^2 x}}}\,dx \\
&= \frac{ab}{2}\int_0^\infty \frac{1}{\sqrt{\pi x}} e^{-(\frac{xa^2b^2}{4}+ab+\frac{1}{x})}dx \\
&= \frac{ab}{2}e^{-ab}\int_0^\infty \frac{e^{-\frac{1}{x}}}{\sqrt{\pi x}} e^{-(\frac{xa^2b^2}{4})}dx \\
&= \frac{ab}{2}e^{-ab}\mathcal{L}\left\{\frac{e^{-\frac{1}{t}}}{\sqrt{\pi t}}\right\}\Bigg|_{s=\frac{a^2b^2}{4}} \\
&= \frac{ab}{2}e^{-ab}\frac{1}{\sqrt{s}}e^{-2\sqrt{s}}\Big|_{s=\frac{a^2b^2}{4}} \\
&= \frac{ab}{2}e^{-ab}\frac{2}{ab}e^{-\frac{2ab}{2}} \\
&= e^{-2ab},
\end{aligned}
$$

where $\mathcal{L}$ stands for the Laplace transform of what is inside the curly brackets. The Laplace transform identity is on pg. 22 of Roberts and Kaufman [21], No. 6. As both a and b are positive, $Re(s) > 0$, and the Laplace identity holds for $a = 1$ (a here represents that a in Roberts, Kaufman [21], which is different from the $a = \frac{\ln\frac{\mu}{h}}{\sqrt{k_1^2+k_2^2}}$ defined as the intercept of the straight line that our standard Brownian motion is to hit above). Substituting for a and b in the formula gives

$$
\begin{aligned}
P_{(x_2,x_1)}(\mu) &= e^{-2ab} \\
&= e^{-\frac{2}{\sqrt{k_1^2+k_2^2}}\ln\frac{\mu}{h}\left(-\frac{k}{\sqrt{k_1^2+k_2^2}}\right)} \\
&= e^{\frac{2k}{k_1^2+k_2^2}\ln\frac{\mu}{h}} \\
&= e^{\frac{2k}{k_1^2+k_2^2}\ln\frac{\mu}{h}}
\end{aligned}
$$

$$= e^{\ln(\frac{\mu}{h})^{\frac{2k}{k_1^2+k_2^2}}}$$

$$= \left(\frac{\mu}{h}\right)^{\frac{2k}{k_1^2+k_2^2}}$$

$$= \left(\frac{\mu}{h}\right)^{\frac{2(p_2+\frac{1}{2}a_{22}-p_1+\frac{1}{2}a_{11})}{(q_{11}-q_{21})^2+(q_{12}+q_{22})^2}}$$

$$= \left(\frac{\mu}{h}\right)^{\frac{2(p_2+\frac{1}{2}a_{22}-p_1+\frac{1}{2}a_{11})}{\gamma}}$$

$$< 1,$$

as both a and b are positive.

We have proved the first half of the proposition.

If $k = p_2 + \frac{1}{2}a_{22} - p_1 - \frac{1}{2}a_{11} > 0$, this becomes the problem of a standard Brownian motion hitting a negatively-sloped line: $B_{3,\tau} = \overline{h}(\tau) = \dfrac{\ln\frac{\mu}{h} - k\tau}{\sqrt{k_1^2+k_2^2}}$. Note that $b < 0$ this time:

$$\begin{aligned} P_{(x_2,x_1)}(\mu) &= \int_0^\infty f_\tau(t)dt \\ &= \int_0^\infty \frac{a}{t^{3/2}}\phi(\frac{a+bt}{\sqrt{t}})dt \\ &= \int_0^\infty \frac{a}{\sqrt{2\pi}t^{3/2}}e^{-\frac{(\frac{a+bt}{\sqrt{t}}-0)^2}{2}}dt \\ &= \int_0^\infty \frac{a}{t^{3/2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{(a+bt)^2}{2t}}dt. \end{aligned}$$

Let $t = \dfrac{2}{b^2x}; \sqrt{t} = \sqrt{\dfrac{2}{b^2x}} = \dfrac{-1}{b}\sqrt{\dfrac{2}{x}}; dt = -\dfrac{2}{b^2x^2}dx; \dfrac{1}{t^{3/2}} = \left(\dfrac{b^2x}{2}\right)^{3/2} = -\dfrac{b^3x^{3/2}}{2^{3/2}}$.
Then,

$$\int_0^\infty \frac{a}{t^{3/2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{(a+bt)^2}{2t}}dt = \int_\infty^0 a\frac{-b^3x^{3/2}}{2^{3/2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{(a+b\frac{2}{b^2x})^2}{2\frac{2}{b^2x}}}\Big(-\frac{2}{b^2x^2}\Big)dx$$

$$= -\int_0^\infty \frac{ab}{2}\frac{1}{\sqrt{\pi x}} e^{-\frac{(a^2+\frac{4a}{bx}+\frac{4}{b^2x^2})}{\frac{4}{b^2x}}} dx$$

$$= -\frac{ab}{2}\int_0^\infty \frac{1}{\sqrt{\pi x}} e^{-(\frac{xa^2b^2}{4}+ab+\frac{1}{x})} dx$$

$$= -\frac{ab}{2}e^{-ab}\int_0^\infty \frac{e^{-\frac{1}{x}}}{\sqrt{\pi x}} e^{-(\frac{xa^2b^2}{4})} dx$$

$$= -\frac{ab}{2}e^{-ab}\mathcal{L}\left\{\frac{e^{-\frac{1}{t}}}{\sqrt{\pi t}}\right\}\Bigg|_{s=\frac{a^2b^2}{4}}$$

$$= -\frac{ab}{2}e^{-ab}\frac{1}{\sqrt{s}}e^{-2\sqrt{s}}\Big|_{s=\frac{a^2b^2}{4}}$$

$$= -\frac{ab}{2}e^{-ab}\frac{2}{-ab}e^{-\frac{2a(-b)}{2}}$$

$$= e^{ab}e^{-ab}$$

$$= 1.$$

Whenever a square-root is involved in the last calculation, since $t > 0$, $\sqrt{t} > 0$, $Re(s) > 0$, $Re(\sqrt{s}) > 0$, and $s = \frac{a^2b^2}{4}$, we take the negative root for b^2 in all the above cases. In other words, $\sqrt{b^2} = -b > 0$. The condition $Re(s) > 0$, of course, comes from the need for well-definedness of Laplace transforms. Now, for the remaining case $b = 0$. By the law of the iterated logarithm, a standard Brownian Process $B_{3,t}$ will hit any finite level a with probability 1. Thus, $P(\tau_\mu < \infty) = 1$ whenever $b \leq 0$, or whenever $p_2 + \frac{1}{2}a_{22} \leq p_1 + \frac{1}{2}a_{11}$. The second half of the Proposition is then also proved.

□

Proof. (Proposition 3.5.1) The solutions to the stochastic differential equations are

$$X_{1,t} = x_1 e^{-(p_1+\frac{1}{2}a_{11})t+q_{11}B_{1,t}+q_{12}B_{2,t}}, X_{2,t} = x_2 e^{-(p_2+\frac{1}{2}a_{22})t+q_{21}B_{1,t}+q_{22}B_{2,t}}.$$

In order to achieve the de-correlation, and also ease the difficulty in calculation, we go through the following series of transformations:

1. $\begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} = \begin{bmatrix} e^{S_{1,t}} \\ e^{S_{2,t}} \end{bmatrix}$; this changes the geometric Brownian motion into a problem with arithmetic Brownian motion.

$$S_{1,t} = \ln(x_1) - (p_1 + \frac{1}{2}a_{11})t + q_{11}B_{1,t} + q_{12}B_{2,t};$$

$$S_{2,t} = \ln(x_2) - (p_2 + \frac{1}{2}a_{22})t + q_{21}B_{1,t} + q_{22}B_{2,t}.$$

So it follows that

$$S_{1,0} = \ln(x_1); \tag{3.12}$$

$$S_{2,0} = \ln(x_2). \tag{3.13}$$

We transform the stopping boundary from $x_1 = \mu x_2$ to $e^{s_1} = \mu e^{s_2}$. Taking the log on both sides of the latter gives $s_1 = \ln\mu + s_2$.

2. $\begin{bmatrix} V_{1,t} \\ V_{2,t} \end{bmatrix} = \begin{bmatrix} S_{1,t} - \ln\mu \\ S_{2,t} \end{bmatrix}$;

This transformation gets rid of the intercept associated with the stopping boundary in (S_2, S_1) Space.

$$\begin{aligned} V_{1,t} &= \ln(x_1) - \ln\mu - (p_1 + \frac{1}{2}a_{11})t + q_{11}B_{1,t} + q_{12}B_{2,t} \\ &= \ln\frac{x_1}{\mu} - (p_1 + \frac{1}{2}a_{11})t + q_{11}B_{1,t} + q_{12}B_{2,t}, \\ V_{2,t} &= \ln x_2 - (p_1 + \frac{1}{2}a_{11})t + q_{21}B_{1,t} + q_{22}B_{2,t}. \end{aligned}$$

The stopping boundary is now $v_1 = v_2$ in (V_2, V_1) space.

3. $\begin{bmatrix} Y_{1,t} \\ Y_{2,t} \end{bmatrix} = \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \begin{bmatrix} \sigma_2 & -\sigma_1\rho \\ 0 & \sigma_1\sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} V_{1,t} \\ V_{2,t} \end{bmatrix}$; the coefficients on the last line bear the following relationships with parameters in the original model:

$$\sigma_1 = \sqrt{q_{11}^2 + q_{12}^2}$$
$$\sigma_2 = \sqrt{q_{21}^2 + q_{22}^2}$$
$$a_{12} = (q_{11}q_{21} + q_{12}q_{22}) = \rho\sigma_1\sigma_2 \text{ , or } \rho = \frac{q_{11}q_{21} + q_{12}q_{22}}{\sigma_1\sigma_2} = \frac{q_{11}q_{21} + q_{12}q_{22}}{\sqrt{q_{11}^2 + q_{12}^2}\sqrt{q_{21}^2 + q_{22}^2}}.$$

Solving for $V_{2,t}$ in terms of $Y_{2,t}$ gives

$$\begin{bmatrix} V_{1,t} \\ V_{2,t} \end{bmatrix} = \sigma_1\sigma_2\sqrt{1-\rho^2} \begin{bmatrix} \sigma_2 & -\sigma_1\rho \\ 0 & \sigma_1\sqrt{1-\rho^2} \end{bmatrix}^{-1} \begin{bmatrix} Y_{1,t} \\ Y_{2,t} \end{bmatrix} = \begin{bmatrix} \sigma_1\sqrt{1-\rho^2} & \sigma_1\rho \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} Y_{1,t} \\ Y_{2,t} \end{bmatrix}.$$

So,

$$V_{2,t} = \sigma_2 Y_{2,t}. \tag{3.14}$$

This transformation de-correlates the processes $(V_{2,t}, V_{1,t})$ and turns it into $(Y_{2,t}, Y_{1,t})$.

$$\begin{aligned}
\begin{bmatrix} Y_{1,t} \\ Y_{2,t} \end{bmatrix} &= \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \begin{bmatrix} \sigma_2 & -\sigma_1\rho \\ 0 & \sigma_1\sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} V_{1,t} \\ V_{2,t} \end{bmatrix} \\
&= \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \begin{bmatrix} \sigma_2 & -\sigma_1\rho \\ 0 & \sigma_1\sqrt{1-\rho^2} \end{bmatrix} \times \\
&\begin{bmatrix} S_{1,0} - \ln\mu - (p_1 + \frac{1}{2}a_{11})t + q_{11}B_{1,t} + q_{12}B_{2,t} \\ S_{2,0} - (p_2 + \frac{1}{2}a_{22})t + q_{21}B_{1,t} + q_{22}B_{2,t} \end{bmatrix} \\
&= \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \begin{bmatrix} \sigma_2\left[S_{1,0} - \ln\mu - (p_1 + \frac{1}{2}a_{11})t + q_{11}B_{1,t} + q_{12}B_{2,t}\right] \\ \sigma_1\sqrt{1-\rho^2}\left[S_{2,0} - (p_2 + \frac{1}{2}a_{22})t + q_{21}B_{1,t} + q_{22}B_{2,t}\right] \end{bmatrix}
\end{aligned}$$

$$
- \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \begin{bmatrix} \sigma_1\rho\left[S_{2,0} - (p_2 + \frac{1}{2}a_{22})t + q_{21}B_{1,t} + q_{22}B_{2,t}\right] \\ 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} \frac{\sigma_2(S_{1,0}-\ln\mu)-\sigma_1\rho S_{2,0}+\left[\sigma_1\rho\left(p_2+\frac{1}{2}a_{22}\right)-\sigma_2\left(p_1+\frac{1}{2}a_{11}\right)\right]t}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \\ \frac{1}{\sigma_2}\left[S_{2,0} - \left(p_2 + \frac{1}{2}a_{22}\right)t + q_{21}B_{1,t} + q_{22}B_{2,t}\right] \end{bmatrix}
$$

$$
+ \begin{bmatrix} \frac{(\sigma_2 q_{11}-\sigma_1\rho q_{21})B_{1,t}+(\sigma_2 q_{12}-\sigma_1\rho q_{22})B_{2,t}}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \\ 0 \end{bmatrix}. \tag{3.15}
$$

To check that the decorrelation has made $Cov(Y_{1,t}, Y_{2,t}) = 0$, we make the following calculation:

$$
\begin{aligned}
Cov(Y_{1,t}, Y_{2,t}) &= E\Big\{[Y_{1,t} - E(Y_{1,t})][Y_{2,t} - E(Y_{2,t})]\Big\} \\
&= E\Big\{\Big[\frac{(\sigma_2 q_{11} - \sigma_1\rho q_{21})B_{1,t} + (\sigma_2 q_{12} - \sigma_1\rho q_{22})B_{2,t}}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\Big]\Big(\frac{q_{21}B_{1,t} + q_{22}B_{2,t}}{\sigma_2}\Big)\Big\} \\
&= \frac{q_{21}(\sigma_2 q_{11} - \sigma_1\rho q_{21})t + q_{22}(\sigma_2 q_{12} - \sigma_1\rho q_{22})t}{\sigma_1\sigma_2^2\sqrt{1-\rho^2}} \\
&= \Big[\frac{q_{21}\sigma_2 q_{11} - \sigma_1\rho q_{21}^2 + \sigma_2 q_{12}q_{22} - \sigma_1\rho q_{22}^2}{\sigma_1\sigma_2^2\sqrt{1-\rho^2}}\Big]t \\
&= \Big[\frac{\sigma_2(q_{21}q_{11} + q_{12}q_{22}) - \sigma_1\rho(q_{21}^2 + q_{22}^2)}{\sigma_1\sigma_2^2\sqrt{1-\rho^2}}\Big]t \\
&= \Big[\frac{\sigma_2\rho\sigma_1\sigma_2 - \sigma_1\rho\sigma_2^2}{\sigma_1\sigma_2^2\sqrt{1-\rho^2}}\Big]t \\
&= 0.
\end{aligned} \tag{3.16}
$$

And thus, $Y_{1,t}, Y_{2,t}$ are uncorrelated. The stopping boundary has now become

$$
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \begin{bmatrix} \sigma_2 & -\sigma_1\rho \\ 0 & \sigma_1\sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}
$$

$$= \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \begin{bmatrix} \sigma_2 v_1 - \sigma_1\rho v_2 \\ \sigma_1\sqrt{1-\rho^2}v_2 \end{bmatrix}$$

$$= \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \begin{bmatrix} (\sigma_2 - \sigma_1\rho)v_1 \\ v_2\sigma_1\sqrt{1-\rho^2} \end{bmatrix},$$

giving $\frac{\sigma_1\sigma_2\sqrt{1-\rho^2}}{\sigma_2-\sigma_1\rho}y_1 = v_1$, and $\frac{\sigma_1\sigma_2\sqrt{1-\rho^2}}{\sigma_1\sqrt{1-\rho^2}}y_2 = v_2 = v_1 = \frac{\sigma_1\sigma_2\sqrt{1-\rho^2}}{\sigma_2-\sigma_1\rho}y_1$. Hence, $y_1 = \dfrac{\sigma_2 - \sigma_1\rho}{\sigma_1\sqrt{1-\rho^2}}y_2$, and the slope is not guaranteed to be zero. We want to transform the original problem into a problem with two uncorrelated processes $(Z_{2,t}, Z_{1,t})$ such that the first hitting time of the boundary only depends on $Z_{1,t}$, thus, making the first hitting time τ conditionally independent of the other process $Z_{2,t}$, as long as the hitting time is finite. This set-up will enable us to use the substitution Lemma in the transformed space to evaluate more easily the conditional expectation in the original problem in $(X_{2,t}, X_{1,t})$ Space. Thus, we still need to rotate the above stopping boundary in $(Y_{2,t}, Y_{1,t})$ to one that gives a horizontal stopping boundary $(z_1 = 0)$ so that only the $Z_{1,t}$ component determines τ.

4. We next make a linear transformation that rotates the stopping boundary to the horizontal axis. We start with the angle that the stopping boundary through the origin makes with the horizontal y_2-axis in $(Y_{2,t}, Y_{1,t})$: $\arctan\left[\frac{\sigma_2-\sigma_1\rho}{\sigma_1\sqrt{1-\rho^2}}\right] = \theta$. Our rotation to the horizontal axis is then given by the linear transformation through a rotation of $-\theta$ radians:

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix};$$

$$\cos\theta = \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2(1-\rho^2) + (\sigma_2 - \sigma_1\rho)^2}} = \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2 - \sigma_1^2\rho^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho + \sigma_1^2\rho^2}}$$

$$= \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}},$$

$$= \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}};$$

$$\sin\theta = \frac{\sigma_2 - \sigma_1\rho}{\sqrt{\sigma_1^2(1-\rho^2) + (\sigma_2 - \sigma_1\rho)^2}} = \frac{\sigma_2 - \sigma_1\rho}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}} = \frac{\sigma_2 - \sigma_1\rho}{\sqrt{\gamma}}.$$

The rotation matrix is then

$$R = \begin{bmatrix} \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}} & -\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}} \\ \frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}} & \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}} \end{bmatrix}.$$

Let

$$\begin{aligned}
\begin{bmatrix} R_{1,t} \\ R_{2,t} \end{bmatrix} &= R \begin{bmatrix} Y_{1,t} \\ Y_{2,t} \end{bmatrix} \\
&= R \begin{bmatrix} \frac{\sigma_2 \ln x_1 - \sigma_1\rho \ln x_2 - \sigma_2 \ln \mu}{\sigma_1\sigma_2\sqrt{1-\rho^2}} + \frac{[\sigma_1\rho(p_2 + \frac{1}{2}a_{22}) - \sigma_2(p_1 + \frac{1}{2}a_{11})]t}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \\ \frac{\ln x_2}{\sigma_2} - \frac{p_2 + \frac{1}{2}a_{22}}{\sigma_2}t + \frac{q_{21}}{\sigma_2}B_{1,t} + \frac{q_{22}}{\sigma_2}B_{2,t} \end{bmatrix} \\
&\quad + R \begin{bmatrix} \frac{(q_{11}\sigma_2 - \sigma_1\rho q_{21})B_{1,t}}{\sigma_1\sigma_2\sqrt{1-\rho^2}} + \frac{(q_{12}\sigma_2 - \sigma_1 q_{22})B_{2,t}}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \\ 0 \end{bmatrix}.
\end{aligned}$$

Grouping terms, $R_{1,t}$ becomes

$$\begin{aligned}
&\left(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\right)\left(\frac{\sigma_2 \ln x_1 - \sigma_1\rho \ln x_2 - \sigma_2 \ln \mu}{\sigma_1\sigma_2\sqrt{1-\rho^2}} + \frac{[\sigma_1\rho(p_2 + \frac{1}{2}a_{22}) - \sigma_2(p_1 + \frac{1}{2}a_{11})]t}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\right. \\
&\qquad \left. + \frac{(q_{11}\sigma_2 - \sigma_1\rho q_{21})B_{1,t}}{\sigma_1\sigma_2\sqrt{1-\rho^2}} + \frac{(q_{12}\sigma_2 - \sigma_1\rho q_{22})B_{2,t}}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\right) \\
&\qquad - \left(\frac{\sigma_2 - \sigma_1\rho}{\sqrt{\gamma}}\right)\left[\frac{\ln x_2}{\sigma_2} - \frac{p_2 + \frac{1}{2}a_{22}}{\sigma_2}t + \frac{q_{21}}{\sigma_2}B_{1,t} + \frac{q_{22}}{\sigma_2}B_{2,t}\right] \\
&\quad = \left(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\right)\left(\frac{\sigma_2 \ln x_1 - \sigma_1\rho \ln x_2 - \sigma_2 \ln \mu}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\right) - \frac{\sigma_2 - \sigma_1\rho}{\sqrt{\gamma}}\frac{\ln x_2}{\sigma_2}
\end{aligned}$$

$$
\begin{aligned}
&+\left[\left(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\right)\frac{\left[\sigma_1\rho\left(p_2+\frac{1}{2}a_{22}\right)-\sigma_2\left(p_1+\frac{1}{2}a_{11}\right)\right]}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\right.\\
&\left.+\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\frac{\left(p_2+\frac{1}{2}a_{22}\right)}{\sigma_2}\right]t\\
&+\left[\left(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\right)\frac{(q_{11}\sigma_2-\sigma_1\rho q_{21})}{\sigma_1\sigma_2\sqrt{1-\rho^2}}-\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\frac{q_{21}}{\sigma_2}\right]B_{1,t}\\
&+\left[\left(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\right)\frac{(q_{12}\sigma_2-\sigma_1\rho q_{22})}{\sigma_1\sigma_2\sqrt{1-\rho^2}}-\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\frac{q_{22}}{\sigma_2}\right]B_{2,t}.
\end{aligned}
$$

The second term $R_{2,t}$ becomes

$$
\begin{aligned}
&\left(\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\right)\left(\frac{\sigma_2\ln x_1-\sigma_1\rho\ln x_2-\sigma_2\ln\mu}{\sigma_1\sigma_2\sqrt{1-\rho^2}}+\frac{[\sigma_1\rho(p_2+\frac{1}{2}a_{22})-\sigma_2(p_1+\frac{1}{2}a_{11})]t}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\right.\\
&\qquad\left.+\frac{(q_{11}\sigma_2-\sigma_1\rho q_{21})B_{1,t}}{\sigma_1\sigma_2\sqrt{1-\rho^2}}+\frac{(q_{12}\sigma_2-\sigma_1\rho q_{22})B_{2,t}}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\right)\\
&\qquad+\left(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\right)\left[\frac{\ln x_2}{\sigma_2}-\frac{p_2+\frac{1}{2}a_{22}}{\sigma_2}t+\frac{q_{21}}{\sigma_2}B_{1,t}+\frac{q_{22}}{\sigma_2}B_{2,t}\right]\\
&=\left(\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\right)\left(\frac{\sigma_2\ln x_1-\sigma_1\rho\ln x_2-\sigma_2\ln\mu}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\right)+\left(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\right)\frac{\ln x_2}{\sigma_2}\\
&\qquad+\left[\left(\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\right)\frac{\left[\sigma_1\rho\left(p_2+\frac{1}{2}a_{22}\right)-\sigma_2\left(p_1+\frac{1}{2}a_{11}\right)\right]}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\right.\\
&\qquad\left.-\left(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\right)\frac{\left(p_2+\frac{1}{2}a_{22}\right)}{\sigma_2}\right]t\\
&\qquad+\left[\left(\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\right)\frac{(q_{11}\sigma_2-\sigma_1\rho q_{21})}{\sigma_1\sigma_2\sqrt{1-\rho^2}}+\left(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\right)\frac{q_{21}}{\sigma_2}\right]B_{1,t}\\
&\qquad+\left[\left(\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\right)\frac{(q_{12}\sigma_2-\sigma_1\rho q_{22})}{\sigma_1\sigma_2\sqrt{1-\rho^2}}+\left(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\right)\frac{q_{22}}{\sigma_2}\right]B_{2,t}.
\end{aligned}
$$

Also, the inverse relation is worked out so that we can use the result to figure out what the objective function will be transformed to:

$$\begin{aligned}
\begin{bmatrix} Y_{1,t} \\ Y_{2,t} \end{bmatrix} &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1} \begin{bmatrix} R_{1,t} \\ R_{2,t} \end{bmatrix} \\
&= \frac{1}{\cos^2\theta + \sin^2\theta} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} R_{1,t} \\ R_{2,t} \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} R_{1,t} \\ R_{2,t} \end{bmatrix}.
\end{aligned}$$

$$\begin{aligned}
Y_{1,t} = R_{1,t}\cos\theta + R_{2,t}\sin\theta &= \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}} R_{1,t} + \frac{\sigma_2 - \sigma_1\rho}{\sqrt{\gamma}} R_{2,t}; \\
Y_{2,t} = -R_{1,t}\sin\theta + R_{2,t}\cos\theta &= -\frac{\sigma_2 - \sigma_1\rho}{\sqrt{\gamma}} R_{1,t} + \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}} R_{2,t}. \qquad (3.17)
\end{aligned}$$

Because of non-correlation, $Cov\left(\begin{bmatrix} Y_{1,t} \\ Y_{2,t} \end{bmatrix}\right)$ must be of the form $\begin{bmatrix} y_{11} & 0 \\ 0 & y_{22} \end{bmatrix}$. Thus, after rotation,

$$\begin{aligned}
Cov\left(R\begin{bmatrix} Y_{1,t} \\ Y_{2,t} \end{bmatrix}, R\begin{bmatrix} Y_{1,t} \\ T_{2,t} \end{bmatrix}\right) &= R \cdot Var\left(\begin{bmatrix} Y_{1,t} \\ Y_{2,t} \end{bmatrix}\right) R^T \\
&= \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} y_{11} & 0 \\ 0 & y_{22} \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \end{bmatrix} \\
&= \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} r_{11}y_{11} & r_{21}y_{11} \\ r_{12}y_{22} & r_{22}y_{22} \end{bmatrix} \\
&= \begin{bmatrix} r_{11}^2 y_{11} + r_{12}^2 y_{22} & r_{11}r_{21}y_{11} + r_{12}r_{22}y_{22} \\ r_{21}r_{11}y_{11} + r_{22}r_{12}y_{22} & r_{21}^2 y_{11} + r_{22}^2 y_{22} \end{bmatrix}.
\end{aligned}$$

The off diagonal terms are in general not zero, unless $Var(Y_{1,t}) = Var(Y_{2,t})$, or $y_{11} = y_{22}$. In this case,

$$r_{11}r_{21}y_{11} + r_{12}r_{22}y_{22} = (r_{11}r_{21} + r_{12}r_{22})y_{22} = 0,$$

since the term inside the parentheses is $\cos\theta\sin\theta - \sin\theta\cos\theta = 0$. We check to see if variances are indeed equal:

$$\begin{aligned}
Var(Y_{1,t}) &= Var\left(\frac{(\sigma_2 q_{11} - \sigma_1\rho q_{21})B_{1,t} + (\sigma_2 q_{12} - \sigma_1\rho q_{22})B_{2,t}}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\right)\\
&= \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)}\left[t(\sigma_2 q_{11} - \sigma_1\rho q_{21})^2 + t(\sigma_2 q_{12} - \sigma_1\rho q_{22})^2\right]\\
&= \frac{t}{\sigma_1^2\sigma_2^2(1-\rho^2)}\left[\left(\sigma_2^2 q_{11}^2 - 2\sigma_2\sigma_1\rho q_{11}q_{21} + \sigma_1^2\rho^2 q_{21}^2\right)\right.\\
&\left.+\left(\sigma_2^2 q_{12}^2 - 2\sigma_2\sigma_1\rho q_{12}q_{22} + \sigma_1^2\rho^2 q_{22}^2\right)\right]\\
&= \frac{t}{\sigma_1^2\sigma_2^2(1-\rho^2)}\left[\sigma_2^2\left(q_{11}^2 + q_{12}^2\right) - 2\sigma_2\sigma_1\rho\left(q_{11}q_{21} + q_{12}q_{22}\right) + \sigma_1^2\rho^2\left(q_{21}^2 + q_{22}^2\right)\right]\\
&= \frac{t}{\sigma_1^2\sigma_2^2(1-\rho^2)}\left[\sigma_2^2\sigma_1^2 - 2\sigma_2\sigma_1\rho\rho\sigma_1\sigma_2 + \sigma_1^2\rho^2\sigma_2^2\right]\\
&= \frac{t}{\sigma_1^2\sigma_2^2(1-\rho^2)}\left[\sigma_2^2\sigma_1^2 - 2\sigma_1^2\rho^2\sigma_2^2 + \sigma_1^2\rho^2\sigma_2^2\right]\\
&= \frac{t}{\sigma_1^2\sigma_2^2(1-\rho^2)}\left[\sigma_2^2\sigma_1^2 - \sigma_1^2\rho^2\sigma_2^2\right] = t.
\end{aligned}$$

$$Var(Y_{2,t}) = Var\left(\frac{q_{21}B_{1,t} + q_{22}B_{2,t}}{\sigma_2}\right) = \frac{1}{\sigma_2^2}\left(q_{21}^2 + q_{22}^2\right)t = t.$$

Since both variances are equal to t, the rotation does not "re-correlate" the previous de-correlation, and $Var\left(\begin{bmatrix} R_{1,t} \\ R_{2,t} \end{bmatrix}\right)$ is still a diagonal matrix. The on-diagonal elements for $Var\left(\begin{bmatrix} R_{1,t} \\ R_{2,t} \end{bmatrix}\right)$ are as follows:

$$Var(R_{1,t}) = r_{11}^2 t + r_{12}^2 t = t\cos^2\theta + t\sin^2\theta = t.$$

$$Var(R_{2,t}) = r_{21}^2 t + r_{22}^2 t = t\sin^2\theta + t\cos^2\theta = t.$$

$$Var\left(\begin{bmatrix} R_{1,t} \\ R_{2,t} \end{bmatrix}\right) = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}.$$

The stopping boundary then becomes

$$\begin{aligned}
\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} &= \begin{bmatrix} \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}} & -\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}} \\ \frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}} & \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}} \frac{(\sigma_2-\sigma_1\rho)}{\sigma_1\sqrt{1-\rho^2}} y_2 - \frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}} y_2 \\ \frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}} \frac{(\sigma_2-\sigma_1\rho)}{\sigma_1\sqrt{1-\rho^2}} y_2 + \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}} y_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}} \frac{(\sigma_2-\sigma_1\rho)}{\sigma_1\sqrt{1-\rho^2}} - \frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}} \\ \frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}} \frac{(\sigma_2-\sigma_1\rho)}{\sigma_1\sqrt{1-\rho^2}} + \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}} \end{bmatrix} y_2 \\
&= \begin{bmatrix} \frac{\sigma_1\sigma_2\sqrt{1-\rho^2}-\sigma_1^2\rho\sqrt{1-\rho^2}-\sigma_1\sigma_2\sqrt{1-\rho^2}+\sigma_1^2\rho\sqrt{1-\rho^2}}{\left(\sigma_1\sqrt{1-\rho^2}\right)\sqrt{\gamma}} \\ \frac{(\sigma_2-\sigma_1\rho)^2+\sigma_1^2\left(1-\rho^2\right)}{\left(\sigma_1\sqrt{1-\rho^2}\right)\sqrt{\gamma}} \end{bmatrix} y_2 \\
&= \begin{bmatrix} 0 \\ \frac{\sqrt{\gamma}}{\left(\sigma_1\sqrt{1-\rho^2}\right)} \end{bmatrix} y_2.
\end{aligned}$$

The stopping boundary now rests on the horizontal axis $r_1 = 0$ in $(R_{2,t}, R_{1,t})$ space. Our original problem has the joint process' starting position below the stopping boundary; we wish to transform the problem to where (z_2, z_1) is above the stopping boundary. We need to perform an additional reflection, since z_1 needs to be positive for later calculations to be well defined:

5. $\begin{bmatrix} Z_{1,t} \\ Z_{2,t} \end{bmatrix} = \begin{bmatrix} -R_{1,t} \\ R_{2,t} \end{bmatrix}$. Thus,

$$\begin{aligned} Z_{1,t} = & -\left(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\right)\left(\frac{\sigma_2 \ln x_1 - \sigma_1\rho \ln x_2 - \sigma_2 \ln \mu}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\right) + \frac{\sigma_2 - \sigma_1\rho}{\sqrt{\gamma}} \frac{\ln x_2}{\sigma_2} \\ & - \left[\left(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\right) \frac{\left[\sigma_1\rho\left(p_2 + \frac{1}{2}a_{22}\right) - \sigma_2\left(p_1 + \frac{1}{2}a_{11}\right)\right]}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \right. \\ & \left. + \frac{\sigma_2 - \sigma_1\rho}{\sqrt{\gamma}} \frac{\left(p_2 + \frac{1}{2}a_{22}\right)}{\sigma_2}\right] t \\ & - \left[\left(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\right) \frac{(q_{11}\sigma_2 - \sigma_1\rho q_{21})}{\sigma_1\sigma_2\sqrt{1-\rho^2}} - \frac{\sigma_2 - \sigma_1\rho}{\sqrt{\gamma}} \frac{q_{21}}{\sigma_2}\right] B_{1,t} \\ & - \left[\left(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\right) \frac{(q_{12}\sigma_2 - \sigma_1\rho q_{22})}{\sigma_1\sigma_2\sqrt{1-\rho^2}} - \frac{\sigma_2 - \sigma_1\rho}{\sqrt{\gamma}} \frac{q_{22}}{\sigma_2}\right] B_{2,t}. \end{aligned}$$

$$\begin{aligned} Z_{2,t} = & \frac{\sigma_2 - \sigma_1\rho}{\sqrt{\gamma}} \frac{\sigma_2 \ln x_1 - \sigma_1\rho \ln x_2 - \sigma_2 \ln \mu}{\sigma_1\sigma_2\sqrt{1-\rho^2}} + \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}} \frac{\ln x_2}{\sigma_2} \\ & + \left\{ \frac{\sigma_2 - \sigma_1\rho}{\sqrt{\gamma}} \cdot \frac{\left[\sigma_1\rho\left(p_2 + \frac{1}{2}a_{22}\right) - \sigma_2\left(p_1 + \frac{1}{2}a_{11}\right)\right]}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \right. \\ & \left. - \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}} \cdot \frac{\left(p_2 + \frac{1}{2}a_{22}\right)}{\sigma_2} \right\} t \\ & + \left[\frac{\sigma_2 - \sigma_1\rho}{\sqrt{\gamma}} \cdot \frac{(q_{11}\sigma_2 - \sigma_1\rho q_{21})}{\sigma_1\sigma_2\sqrt{1-\rho^2}} + \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}} \frac{q_{21}}{\sigma_2}\right] B_{1,t} \\ & + \left[\frac{\sigma_2 - \sigma_1\rho}{\sqrt{\gamma}} \cdot \frac{(q_{12}\sigma_2 - \sigma_1\rho q_{22})}{\sigma_1\sigma_2\sqrt{1-\rho^2}} + \frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}} \cdot \frac{q_{22}}{\sigma_2}\right] B_{2,t}. \end{aligned}$$

The expressions for $Z_{1,t}$ and $Z_{2,t}$ above, yield result I.

The stopping boundary is now $z_1 = 0$ after the reflection. This gives part III of Proposition 3.5.1. The state space is the upper half plane, including the horizontal axis in $(Z_{2,t}, Z_{1,t})$-space. Of course, the reflection will not make the variance-covariance matrix non-diagonal. Therefore, $Z_{2,t}, Z_{1,t}$ are still uncorrelated, and hence independent by their joint normality, as long as $t < \infty$.

Thus, $Var\left(\begin{bmatrix} Z_{1,t} \\ Z_{2,t} \end{bmatrix}\right) = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}$. Also, $Cov(Z_{1,t}, Z_{2,t}) = 0$, $Var(Z_{i,t}) = t$ for $i = 1, 2$, and this gives II.

Following are the stochastic differential equations, and parameter definitions identified with the first hitting time density distribution:

$$\begin{aligned} dZ_{1,t} = &-\Bigg[\left(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\right)\frac{\left[\sigma_1\rho\left(p_2+\frac{1}{2}a_{22}\right)-\sigma_2\left(p_1+\frac{1}{2}a_{11}\right)\right]}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \\ &+\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\frac{\left(p_2+\frac{1}{2}a_{22}\right)}{\sigma_2}\Bigg]dt \\ &-\left[\left(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\right)\frac{(q_{11}\sigma_2-\sigma_1\rho q_{21})}{\sigma_1\sigma_2\sqrt{1-\rho^2}}-\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\frac{q_{21}}{\sigma_2}\right]dB_{1,t} \\ &-\left[\left(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\right)\frac{(q_{12}\sigma_2-\sigma_1\rho q_{22})}{\sigma_1\sigma_2\sqrt{1-\rho^2}}-\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\frac{q_{22}}{\sigma_2}\right]dB_{2,t}. \end{aligned}$$

$$\begin{aligned} dZ_{2,t} = &\Bigg\{\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\cdot\frac{\left[\sigma_1\rho\left(p_2+\frac{1}{2}a_{22}\right)-\sigma_2\left(p_1+\frac{1}{2}a_{11}\right)\right]}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \\ &-\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\cdot\frac{\left(p_2+\frac{1}{2}a_{22}\right)}{\sigma_2}\Bigg\}dt \\ &+\left[\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\cdot\frac{(q_{11}\sigma_2-\sigma_1\rho q_{21})}{\sigma_1\sigma_2\sqrt{1-\rho^2}}+\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\frac{q_{21}}{\sigma_2}\right]dB_{1,t} \\ &+\left[\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\cdot\frac{(q_{12}\sigma_2-\sigma_1\rho q_{22})}{\sigma_1\sigma_2\sqrt{1-\rho^2}}+\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\frac{q_{22}}{\sigma_2}\right]dB_{2,t}, \end{aligned}$$

$$\begin{aligned} z_1 &= -\left(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\right)\left(\frac{\sigma_2\ln x_1-\sigma_1\rho\ln x_2-\sigma_2\ln\mu}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\right)+\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\frac{\ln x_2}{\sigma_2}, \\ z_2 &= \frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\cdot\frac{\sigma_2\ln x_1-\sigma_1\rho\ln x_2-\sigma_2\ln\mu}{\sigma_1\sigma_2\sqrt{1-\rho^2}}+\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\cdot\frac{\ln x_2}{\sigma_2}; \end{aligned}$$

$$\mu_1 = -\left[\left(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\right)\frac{\left[\sigma_1\rho\left(p_2+\frac{1}{2}a_{22}\right)-\sigma_2\left(p_1+\frac{1}{2}a_{11}\right)\right]}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\right.$$
$$\left.+\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\frac{\left(p_2+\frac{1}{2}a_{22}\right)}{\sigma_2}\right],$$
$$\mu_2 = \left\{\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\cdot\frac{\left[\sigma_1\rho\left(p_2+\frac{1}{2}a_{22}\right)-\sigma_2\left(p_1+\frac{1}{2}a_{11}\right)\right]}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\right.$$
$$\left.-\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\cdot\frac{\left(p_2+\frac{1}{2}a_{22}\right)}{\sigma_2}\right\},$$
$$\sigma_1'^2 = \left[\left(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\right)\frac{(q_{11}\sigma_2-\sigma_1\rho q_{21})}{\sigma_1\sigma_2\sqrt{1-\rho^2}}-\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\frac{q_{21}}{\sigma_2}\right]^2$$
$$+\left[\left(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\right)\frac{(q_{12}\sigma_2-\sigma_1\rho q_{22})}{\sigma_1\sigma_2\sqrt{1-\rho^2}}-\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\frac{q_{22}}{\sigma_2}\right]^2,$$
$$\sigma_2'^2 = \left[\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\frac{(q_{11}\sigma_2-\sigma_1\rho q_{21})}{\sigma_1\sigma_2\sqrt{1-\rho^2}}+\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\frac{q_{21}}{\sigma_2}\right]^2$$
$$+\left[\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\frac{(q_{12}\sigma_2-\sigma_1\rho q_{22})}{\sigma_1\sigma_2\sqrt{1-\rho^2}}+\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\frac{q_{22}}{\sigma_2}\right]^2.$$

Thus, the two Weiner processes $Z_{2,t}, Z_{1,t}$ are jointly normal, uncorrelated and therefore independent, as long as t is finite, and both have variances t identical to standard Brownian motion. Each is a sum of continuous or a.s. continuous functions of t when we fix ω. Thus, their sample paths are a.s. continuous.

We can put each in the following form:

$$Z_{1,t} = z_1 + \mu_1 t + W_{1,t}, \text{ and}$$
$$Z_{2,t} = z_2 + \mu_2 t + W_{2,t},$$

with $W_{1,t}, W_{2,t}$ independent standard Weiner processes given as follows:

$$W_{1,t} = -\left[\left(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\right)\frac{(q_{11}\sigma_2-\sigma_1\rho q_{21})}{\sigma_1\sigma_2\sqrt{1-\rho^2}}-\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\frac{q_{21}}{\sigma_2}\right]B_{1,t}$$

$$-\left[\left(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\right)\frac{(q_{12}\sigma_2-\sigma_1\rho q_{22})}{\sigma_1\sigma_2\sqrt{1-\rho^2}}-\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\frac{q_{22}}{\sigma_2}\right]B_{2,t},$$

$$\begin{aligned} W_{2,t} &= \left[\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\cdot\frac{(q_{11}\sigma_2-\sigma_1\rho q_{21})}{\sigma_1\sigma_2\sqrt{1-\rho^2}}+\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\frac{q_{21}}{\sigma_2}\right]B_{1,t} \\ &+ \left[\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\cdot\frac{(q_{12}\sigma_2-\sigma_1\rho q_{22})}{\sigma_1\sigma_2\sqrt{1-\rho^2}}+\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\frac{q_{22}}{\sigma_2}\right]B_{2,t}, \end{aligned}$$

with ρ the correlation of the original process components in X-space.

Following through the same steps of linear transformations 1 through 5 for the objective function, and utilizing equations (3.14) and (3.17) gives

$$\begin{aligned} &E^{(x_2,x_1)}\left[(X_{1,\tau}-X_{2,\tau})\,I_{[\tau<\infty]}\right] \\ &= E^{(x_2,x_1)}\left[\left(\frac{X_{1,\tau}-X_{2,\tau}}{X_{2,\tau}}\right)X_{2,\tau}I_{[\tau<\infty]}\right] \\ &= E^{(x_2,x_1)}\left[(\mu-1)\,X_{2,\tau}I_{[\tau<\infty]}\right] \\ &= (\mu-1)\,E^{(x_2,x_1)}\left(X_{2,\tau}I_{[\tau<\infty]}\right) \\ &= (\mu-1)\,E^{(x_2,x_1)}\left(e^{S_{2,\tau}}I_{[\tau<\infty]}\right) \\ &= (\mu-1)\,E^{(x_2,x_1)}\left(e^{V_{2,\tau}}I_{[\tau<\infty]}\right) \\ &= (\mu-1)\,E^{(x_2,x_1)}\left(e^{\sigma_2 Y_{2,\tau}}I_{[\tau<\infty]}\right) \\ &= (\mu-1)\,E^{(x_2,x_1)}\{\exp[\sigma_2(-\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}R_{1,\tau}+\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}R_{2,\tau})]I_{[\tau<\infty]}\} \\ &= (\mu-1)\,E^{(x_2,x_1)}\{\exp[\sigma_2(\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}Z_{1,\tau}+\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}Z_{2,\tau})]I_{[\tau<\infty]}\} \\ &= (\mu-1)\,E^{(x_2,x_1)}\{\exp[\sigma_2(\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\cdot 0+\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}Z_{2,\tau})]I_{[\tau<\infty]}\} \\ &= (\mu-1)\,E^{(x_2,x_1)}\{\exp[\sigma_2(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}Z_{2,\tau})]I_{[\tau<\infty]}\} \\ &= (\mu-1)\,E^{(x_2,x_1)}[\exp\left(\frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}Z_{2,\tau}\right)I_{[\tau<\infty]}]. \end{aligned}$$

This gives part IV of Proposition 3.5.1

□

Remark 3.7.1. *The last calculation identifies the aggregate transformation from X-Space to Z-Space:* $(X_{2,\tau}, \mu X_{2,\tau}) \mapsto \left(\exp\left[\frac{\sigma_1\sigma_2\sqrt{(1-\rho^2)}}{\sqrt{\gamma}} Z_{2,\tau}\right], 0\right)$
Knowing the joint density in hitting times and Z-locations, we can transform to the joint density of hitting times and X-locations by a simple Jacobian transform:

$f(x_2,\tau) = f(\frac{Ln(x_2)}{a}, \tau)\frac{1}{ax_2}$, *with* $a = \frac{\sigma_1\sigma_2\sqrt{(1-\rho^2)}}{\sqrt{\gamma}}$.

The f *on the right side is the joint density* $f(z_2,\tau)$.

The marginal density of hitting location also follows the same conversion calculation:

$g_\mu(x_2) = g_\mu(\frac{Ln(x_2)}{a})\frac{1}{ax_2}$, *with* $a = \frac{\sigma_1\sigma_2\sqrt{(1-\rho^2)}}{\sqrt{\gamma}}$.

The g_μ *on the right side is* $g_\mu(z_2)$. *The marginal density function for hitting times stays the same, since our de-correlation transform does not work on the time scale.*

With the above joint and marginal densities, even in the (x_2, x_1, τ)*-Space, we have complete knowledge for the conditional densities as well. These are some of the main contributions of this thesis: we know the joint, conditional, and marginal densities of hitting location and hitting times in both the Z and X-Spaces.*

Proof. (Proposition 3.5.2)

From Proposition 3.5.1, the joint process is of the form:

$$Z_{1,t} = z_1 + \mu_1 t + W_{1,t}, \text{and}$$

$$Z_{2,t} = z_2 + \mu_2 t + W_{2,t}.$$

Define τ to be the first hitting time of $Z_{1,t}$ on the z_2-axis: $\tau = \inf\{t > 0 : Z_{1,t} = 0\}$, with $\tau(\emptyset) \equiv \infty$. Now, because $\{Z_{1,t}\}_{t\geq 0}$, and $\{Z_{2,t}\}_{t\geq 0}$ are independent random variables for every finite time t; $Z_{2,t}$ and τ are also conditionally independent random variables over $[\tau < \infty]$.

We understand that Z_{2,τ_μ} is the location at $Z_{2,t}$ when $Z_{1,t} = 0$ for the first time, subject to the constraint that we are within the set $[\tau < \infty]$. Since $Z_{2,0} = z_2$ and the mean trajectory varies with $\mu_2 t$ as t increases, within the set $[\tau < \infty]$,

$$E^{z_2}\left(Z_{2,\tau}\,|\,\tau\right) = E\left[z_2 + \mu_2\tau + W_{2,\tau}\,|\,\tau\right] = z_2 + \mu_2\tau.$$

$$Var^{z_2}\left(Z_{2,\tau}\,|\,\tau\right) = Var\left[z_2 + \mu_2\tau + W_{2,\tau}\,|\,\tau\right] = \tau.$$

Thus, $f\left(Z_{2,\tau} = u\,|\,\tau\right) I_{[\tau<\infty]} = \frac{1}{\sqrt{2\pi\tau}} \exp\left[-\frac{1}{2\tau}(u - z_2 - \mu_2\tau)^2\right] I_{[\tau<\infty]}$.

Remark 3.7.2. *The above gives the conditional density of hitting location, given hitting time, in Z-Space. Multiplying the conditional density with the marginal density for the stopping time gives their joint density in Z-Space. As we remarked in Remark 3.7.1, the joint density of hitting location and hitting times in X-Space is just one simple Jacobian transform away from the joint density of hitting location and hitting times.*

Continuing,

$$\begin{aligned}\int_0^\infty f(u\,|\,\tau)f(\tau)I_{[\tau<\infty]}d\tau &= \int_0^\infty f(u\,|\,\tau)I_{[\tau<\infty]}f(\tau)I_{[\tau<\infty]}d\tau \\ &= \int_0^\infty \frac{1}{\sqrt{2\pi\tau}} \exp\left[-\frac{1}{2\tau}(u - z_2 - \mu_2\tau)^2\right] f(\tau)I_{[\tau<\infty]}d\tau,\end{aligned}$$

where the integral is over the distribution of (finite) τ.

The distribution for τ remains the same whether the process is in the original or transformed space, because we have only done a spatial transformation for the decorrelation without affecting the time scale. For $Z_{1,t} = z_1 + \mu_1 t + W_{1,t}$ to hit zero is the same as having $Z_{1,t} - z_1 - \mu_1 t = W_{1,t}$ hit a line that has time-dependence of $0 - z_1 - \mu_1 t = -z_1 - \mu_1 t$. Because of the perfect symmetry of paths of Brownian motion, the distribution of the first hitting time is the same as for $W_{1,t}$ hitting a sloped line $z_1 + \mu_1 t$. From

Bhattacharya and Waymire [3], the corresponding density function is then

$$f(\tau)I_{[\tau<\infty]} = \frac{z_1}{\sqrt{2\pi\tau^3}} \exp\left[-\frac{(z_1+\mu_1\tau)^2}{2\tau}\right]; \;\; \tau \geq 0.$$

Remark 3.7.3. *The last formula is then the marginal density for stopping times. It is the same density whether we are in X- or Z-Space.*

We have then

$$\begin{aligned}
&\int_0^\infty f(u\,|\,\tau)f(\tau)I_{[\tau<\infty]}d\tau \\
&\quad= \int_0^\infty f(u\,|\,\tau)I_{[\tau<\infty]}f(\tau)I_{[\tau<\infty]}d\tau \\
&\quad= \int_0^\infty \frac{1}{\sqrt{2\pi\tau}} \exp\left[-\frac{1}{2\tau}(u-z_2-\mu_2\tau)^2\right] \frac{z_1}{\sqrt{2\pi\tau^3}} \exp\left(-\frac{(z_1+\mu_1\tau)^2}{2\tau}\right) d\tau \\
&\quad= \int_0^\infty \frac{z_1}{2\pi\tau^2} \exp\left[-\frac{(u-z_2-\mu_2\tau)^2+(z_1+\mu_1\tau)^2}{2\tau}\right] d\tau \\
&\quad= \int_0^\infty \frac{z_1}{2\pi\tau^2} \exp\left[-\frac{(u-z_2)^2-2(u-z_2)\mu_2\tau+\mu_2^2\tau^2+(z_1^2+2\mu_1\tau z_1+\mu_1^2\tau^2)}{2\tau}\right] d\tau \\
&\quad= \int_0^\infty \frac{z_1}{2\pi\tau^2} \exp\left[-\frac{(u-z_2)^2+z_1^2+2\left[\mu_1 z_1-(u-z_2)\mu_2\right]\tau+(\mu_1^2+\mu_2^2)\tau^2}{2\tau}\right] d\tau \\
&\quad= \int_0^\infty \frac{z_1}{2\pi\tau^2} \exp\left[-\frac{\frac{(u-z_2)^2+z_1^2}{(\mu_1^2+\mu_2^2)}+2\frac{[\mu_1 z_1-(u-z_2)\mu_2]}{(\mu_1^2+\mu_2^2)}\tau+\tau^2}{2\tau}(\mu_1^2+\mu_2^2)\right] d\tau \\
&\quad= \int_0^\infty \frac{z_1}{2\pi\tau^2} \exp\left[-\frac{\frac{(u-z_2)^2+z_1^2}{(\mu_1^2+\mu_2^2)}+2\frac{[\mu_1 z_1-(u-z_2)\mu_2]}{(\mu_1^2+\mu_2^2)}\tau+\tau^2}{2\tau/(\mu_1^2+\mu_2^2)}\right] d\tau \\
&\quad= \int_0^\infty \frac{z_1}{2\pi\tau^2} \exp\left[-\frac{\left[\tau+\frac{[\mu_1 z_1-(u-z_2)\mu_2]}{(\mu_1^2+\mu_2^2)}\right]^2+\frac{(u-z_2)^2+z_1^2}{(\mu_1^2+\mu_2^2)}-\left[\frac{[\mu_1 z_1-(u-z_2)\mu_2]}{(\mu_1^2+\mu_2^2)}\right]^2}{2\tau/(\mu_1^2+\mu_2^2)}\right] d\tau
\end{aligned}$$

$$= \int_0^\infty \frac{z_1}{2\pi\tau^2} \exp\left[-\frac{\frac{(u-z_2)^2+z_1^2}{(\mu_1^2+\mu_2^2)} - \left[\frac{[\mu_1 z_1-(u-z_2)\mu_2]}{(\mu_1^2+\mu_2^2)}\right]^2}{2\tau/(\mu_1^2+\mu_2^2)}\right]$$

$$\times \exp\left\{-\frac{\left[\tau + \frac{[\mu_1 z_1-(u-z_2)\mu_2]}{(\mu_1^2+\mu_2^2)}\right]^2}{2\tau/(\mu_1^2+\mu_2^2)}\right\} d\tau.$$

Now,

$$\frac{\frac{(u-z_2)^2+z_1^2}{(\mu_1^2+\mu_2^2)} - \left[\frac{\mu_1 z_1-(u-z_2)\mu_2}{(\mu_1^2+\mu_2^2)}\right]^2}{2\tau/(\mu_1^2+\mu_2^2)} = \frac{(u-z_2)^2 + z_1^2 - \frac{[\mu_1 z_1-(u-z_2)\mu_2]^2}{(\mu_1^2+\mu_2^2)}}{2\tau}$$

$$= \frac{\left[(u-z_2)^2+z_1^2\right]\left(\mu_1^2+\mu_2^2\right) - [\mu_1 z_1 - (u-z_2)\mu_2]^2}{2\tau(\mu_1^2+\mu_2^2)}$$

$$= \frac{\mu_1^2(u-z_2)^2 + \mu_1^2 z_1^2 + \mu_2^2(u-z_2)^2 + \mu_2^2 z_1^2 - \mu_1^2 z_1^2 + 2\mu_1 z_1(u-z_2)\mu_2 - (u-z_2)^2\mu_2^2}{2\tau(\mu_1^2+\mu_2^2)}$$

$$= \frac{\mu_2^2 z_1^2 + \mu_1^2(u-z_2)^2 + 2\mu_1 z_1(u-z_2)\mu_2}{2\tau(\mu_1^2+\mu_2^2)}$$

$$= \frac{[\mu_2 z_1 + (u-z_2)\mu_1]^2}{2\tau(\mu_1^2+\mu_2^2)}.$$

Hence,

$$\int_0^\infty \frac{z_1}{2\pi\tau^2} \exp\left\{-\frac{\frac{(u-z_2)^2+z_1^2}{(\mu_1^2+\mu_2^2)} - \left[\frac{[\mu_1 z_1-(u-z_2)\mu_2]}{(\mu_1^2+\mu_2^2)}\right]^2}{2\tau/(\mu_1^2+\mu_2^2)}\right\} \exp\left\{-\frac{\left[\tau + \frac{[\mu_1 z_1-(u-z_2)\mu_2]}{(\mu_1^2+\mu_2^2)}\right]^2}{2\tau/(\mu_1^2+\mu_2^2)}\right\} d\tau$$

$$= \int_0^\infty \frac{z_1}{2\pi\tau^2} \exp\left\{-\frac{[\mu_2 z_1 + (u-z_2)\mu_1]^2}{2\tau(\mu_1^2+\mu_2^2)}\right\} \exp\left\{-\frac{\left[\tau + \frac{[\mu_1 z_1-(u-z_2)\mu_2]}{(\mu_1^2+\mu_2^2)}\right]^2}{2\tau/(\mu_1^2+\mu_2^2)}\right\} d\tau$$

$$= \int_0^\infty \frac{z_1}{2\pi\tau^2} \exp\left\{-\frac{[\mu_2 z_1 + (u-z_2)\mu_1]^2}{2\tau(\mu_1^2+\mu_2^2)}\right\}$$

$$\times \exp\left\{-\frac{(\mu_1^2+\mu_2^2)\left[\tau^2 + 2\frac{[\mu_1 z_1-(u-z_2)\mu_2]}{(\mu_1^2+\mu_2^2)}\tau + \frac{[\mu_1 z_1-(u-z_2)\mu_2]^2}{(\mu_1^2+\mu_2^2)^2}\right]}{2\tau}\right\} d\tau$$

$$= \int_0^\infty \frac{z_1}{2\pi\tau^2} \exp\left\{-\frac{[\mu_2 z_1 + (u - z_2)\mu_1]^2}{2\tau(\mu_1^2 + \mu_2^2)}\right\}$$

$$\times \exp\left(-\frac{\left\{(\mu_1^2 + \mu_2^2)\tau^2 + 2[\mu_1 z_1 - (u - z_2)\mu_2]\tau + \frac{[\mu_1 z_1 - (u-z_2)\mu_2]^2}{(\mu_1^2+\mu_2^2)}\right\}}{2\tau}\right) d\tau$$

$$= \int_0^\infty \frac{z_1}{2\pi\tau^2} \exp\left\{-\frac{[\mu_2 z_1 + (u - z_2)\mu_1]^2}{2\tau(\mu_1^2 + \mu_2^2)}\right\}$$

$$\times \exp\left\{-\frac{(\mu_1^2 + \mu_2^2)\tau}{2}\right\} \exp\left(-[\mu_1 z_1 - (u - z_2)\mu_2]\right)$$

$$\times \exp\left\{-\frac{[\mu_1 z_1 - (u - z_2)\mu_2]^2}{2(\mu_1^2 + \mu_2^2)\tau}\right\} d\tau$$

$$= \mathrm{e}^{(-[\mu_1 \mathrm{z}_1 - (\mathrm{u} - \mathrm{z}_2)\mu_2])} \int_0^\infty \frac{\mathrm{z}_1}{2\pi\tau^2} \exp\left[-\frac{(\mu_1^2 + \mu_2^2)\tau}{2}\right]$$

$$\times \exp\left\{-\frac{[\mu_2 z_1 + (u - z_2)\mu_1]^2 + [\mu_1 z_1 - (u - z_2)\mu_2]^2}{2\tau(\mu_1^2 + \mu_2^2)}\right\} d\tau.$$

Let

$$\frac{1}{w} = \frac{[\mu_2 z_1 + (u - z_2)\mu_1]^2 + [\mu_1 z_1 - (u - z_2)\mu_2]^2}{2\tau(\mu_1^2 + \mu_2^2)},$$

so that

$$\tau = \frac{[\mu_2 z_1 + (u - z_2)\mu_1]^2 + [\mu_1 z_1 - (u - z_2)\mu_2]^2}{2(\mu_1^2 + \mu_2^2)} w,$$

and

$$\tau^2 = \left\{\frac{[\mu_2 z_1 + (u - z_2)\mu_1]^2 + [\mu_1 z_1 - (u - z_2)\mu_2]^2}{2(\mu_1^2 + \mu_2^2)}\right\}^2 w^2,$$

$$d\tau = \left\{\frac{[\mu_2 z_1 + (u - z_2)\mu_1]^2 + [\mu_1 z_1 - (u - z_2)\mu_2]^2}{2(\mu_1^2 + \mu_2^2)}\right\} dw.$$

Then the integral above equals

$$\frac{z_1 e^{-[\mu_1 z_1 - (u-z_2)\mu_2]}}{\left(\frac{[\mu_2 z_1 + (u-z_2)\mu_1]^2 + [\mu_1 z_1 - (u-z_2)\mu_2]^2}{2(\mu_1^2+\mu_2^2)}\right)} \int_0^\infty \frac{1}{2\pi w^2} e^{-\frac{1}{w}}$$

$$\times e^{-\frac{[\mu_2 z_1 + (u-z_2)\mu_1]^2 + [\mu_1 z_1 - (u-z_2)\mu_2]^2}{4} w} dw.$$

Also, as an aside,

$$\begin{aligned}
&\frac{[\mu_2 z_1 + (u-z_2)\mu_1]^2 + [\mu_1 z_1 - (u-z_2)\mu_2]^2}{2(\mu_1^2+\mu_2^2)} \\
&= \frac{\mu_2^2 z_1^2 + 2\mu_2 z_1 (u-z_2)\mu_1 + (u-z_2)^2\mu_1^2 + \mu_1^2 z_1^2 - 2\mu_1 z_1 (u-z_2)\mu_2 + (u-z_2)^2\mu_2^2}{2(\mu_1^2+\mu_2^2)} \\
&= \frac{\mu_2^2 z_1^2 + (u-z_2)^2\mu_1^2 + \mu_1^2 z_1^2 + (u-z_2)^2\mu_2^2}{2(\mu_1^2+\mu_2^2)} \\
&= \frac{(\mu_2^2+\mu_1^2) z_1^2 + (\mu_2^2+\mu_1^2)(u-z_2)^2}{2(\mu_1^2+\mu_2^2)} \\
&= \frac{z_1^2 + (u-z_2)^2}{2},
\end{aligned}$$

or

$$[\mu_2 z_1 + (u-z_2)\mu_1]^2 + [\mu_1 z_1 - (u-z_2)\mu_2]^2 = (\mu_1^2+\mu_2^2)[z_1^2 + (u-z_2)^2].$$

Applying the above equality and looking at just the inside integral,

$$\begin{aligned}
&\int_0^\infty \frac{1}{2\pi w^2} e^{-\frac{1}{w}} e^{-\frac{[\mu_2 z_1 + (u-z_2)\mu_1]^2 + [\mu_1 z_1 - (u-z_2)\mu_2]^2}{4} w} dw \\
&\qquad = \int_0^\infty \frac{1}{2\pi w^2} e^{-\frac{1}{w}} e^{-\frac{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}{4} w} dw \\
&\qquad = \int_0^\infty \frac{1}{2\pi w^2} e^{-\frac{1}{w} - \frac{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}{4} w} dw.
\end{aligned}$$

Now,

$$\begin{aligned}
&\frac{d}{dw}\left[e^{-\frac{1}{w} - \frac{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}{4} w} \right] = \\
&\quad \left(\frac{1}{w^2} - \frac{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}{4} \right) e^{-\frac{1}{w} - \frac{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}{4} w}.
\end{aligned}$$

Thus, we can write the above integral as

$$\int_0^\infty \frac{1}{2\pi w^2} e^{-\frac{1}{w} - \frac{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}{4} w} dw$$

$$= \frac{1}{2\pi} \int_0^\infty \left\{ \frac{1}{w^2} - \frac{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}{4} \right\} e^{-\frac{1}{w} - \frac{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}{4} w} dw$$

$$+ \frac{1}{2\pi} \int_0^\infty \left\{ \frac{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}{4} \right\} e^{-\frac{1}{w} - \frac{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}{4} w} dw$$

$$= \frac{1}{2\pi} e^{-\frac{1}{w} - \frac{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}{4} w} \Bigg|_0^\infty$$

$$+ \frac{1}{2\pi} \int_0^\infty \left\{ \frac{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}{4} \right\} e^{-\frac{1}{w} - \frac{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}{4} w} dw$$

$$= \frac{1}{2\pi} \int_0^\infty \left\{ \frac{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}{4} \right\} e^{-\frac{1}{w} - \frac{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}{4} w} dw$$

$$= \frac{1}{2\pi} \left\{ \frac{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}{4} \right\} \int_0^\infty e^{-\frac{1}{w} - \frac{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}{4} w} dw.$$

Thus, the entire term, including the constant out front, becomes

$$\frac{z_1 e^{-[\mu_1 z_1 - (u-z_2)\mu_2]}}{\left(\frac{[\mu_2 z_1 + (u-z_2)\mu_1]^2 + [\mu_1 z_1 - (u-z_2)\mu_2]^2}{2(\mu_1^2+\mu_2^2)} \right)} \int_0^\infty \frac{1}{2\pi w^2} e^{-\frac{1}{w}} e^{-\frac{[\mu_2 z_1 + (u-z_2)\mu_1]^2 + [\mu_1 z_1 - (u-z_2)\mu_2]^2}{4} w} dw$$

$$= \frac{z_1 e^{-[\mu_1 z_1 - (u-z_2)\mu_2]}}{\left(\frac{(\mu_1^2+\mu_2^2)\left[z_1^2+(u-z_2)^2\right]}{2(\mu_1^2+\mu_2^2)} \right)} \frac{1}{2\pi} \left\{ \frac{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}{4} \right\} \int_0^\infty e^{-\frac{1}{w} - \frac{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}{4} w} dw$$

$$= \frac{(\mu_1^2+\mu_2^2) z_1 e^{-[\mu_1 z_1 - (u-z_2)\mu_2]}}{4\pi} \int_0^\infty e^{-\frac{1}{w} - \frac{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}{4} w} dw$$

$$= \frac{(\mu_1^2+\mu_2^2) z_1 e^{-[\mu_1 z_1 - (u-z_2)\mu_2]}}{2\pi} \mathcal{L} \left\{ \frac{1}{2} e^{-\frac{1}{w}} \right\} \Bigg|_{s=\frac{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}{4}}$$

$$= \frac{(\mu_1^2+\mu_2^2) z_1 e^{-[\mu_1 z_1 - (u-z_2)\mu_2]}}{2\pi} \frac{1}{\sqrt{s}} K_1\left(2\sqrt{s}\right) \Bigg|_{s=\frac{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}{4}}$$

$$= \frac{(\mu_1^2+\mu_2^2)z_1 e^{-[\mu_1 z_1-(u-z_2)\mu_2]}}{2\pi}\frac{1}{\sqrt{\frac{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}{4}}}K_1\left(2\sqrt{\frac{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}{4}}\right)$$

$$= \frac{\sqrt{(\mu_1^2+\mu_2^2)}e^{-[\mu_1 z_1-(u-z_2)\mu_2]}}{\pi}\frac{z_1}{\sqrt{[z_1^2+(u-z_2)^2]}}K_1\left(\sqrt{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}\right)$$

$$= \int_0^\infty f(u\,|\,\tau)f(\tau)I_{[\tau<\infty]}d\tau.$$

From Roberts and Kaufman [21], pg. 18, no. 13, the Laplace Transform identity, with $a = 1$, holds because $\mathrm{Re}(s) > 0$, as $z_1 > 0$, and $\mu_1^2 + \mu_2^2 > 0$ as well, as long as $\mu_1 \neq 0$, and $\mu_2 \neq 0$. The former non-zero constraint is equivalent to $p_2 + \frac{1}{2}a_{22} \neq p_1 + \frac{1}{2}a_{11}$. The latter is equivalent to $(\sigma_2\sigma_1\rho - \sigma_1^2)\left(p_2 + \frac{1}{2}a_{22}\right) + (\sigma_2\sigma_1\rho - \sigma_2^2)\left(p_1 + \frac{1}{2}a_{11}\right) \neq 0$, or $(\sigma_2\sigma_1\rho - \sigma_1^2)\left(p_2 + \frac{1}{2}a_{22}\right) \neq (\sigma_2^2 - \sigma_2\sigma_1\rho)\left(p_1 + \frac{1}{2}a_{11}\right)$. As long as one of these conditions is false, $\mu_1^2 + \mu_2^2 > 0$, and the above Laplace Transform identity holds for this derivation.

Thus, the density function for $Z_{2,\tau_\mu} = u$ is then

$$g_\mu(u) = g_{CGH}(u) = \frac{\sqrt{(\mu_1^2+\mu_2^2)}e^{-[\mu_1 z_1+\mu_2 z_2]}}{\pi} \times \frac{z_1 e^{\mu_2 u}}{\sqrt{[z_1^2+(u-z_2)^2]}} \times K_1\left(\sqrt{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}\right).$$

We call this the Cauchy-generalized hyperbolic density, with an acronym of CGH, which is the subscript of the density function. □

Remark 3.7.4. *The joint density in Z-Space for hitting location and hitting time is given by:* $e^{(-[\mu_1 z_1-(u-z_2)\mu_2])}\frac{z_1}{2\pi\tau^2}\exp\left[-\frac{(\mu_1^2+\mu_2^2)\tau}{2}\right]\exp\left\{-\frac{[\mu_2 z_1+(u-z_2)\mu_1]^2+[\mu_1 z_1-(u-z_2)\mu_2]^2}{2\tau(\mu_1^2+\mu_2^2)}\right\}$, *where* u *is standing in as a realization of* $Z_{2,\tau}$*, the hitting location, and* τ *is the realization of the stopping time.*

Remark 3.7.5. *The marginal hitting location density is the* $g_\mu(u)$ *function as given above, where* u *is a realization of* $Z_{2,\tau}$*.*

Proof. (Proposition 3.5.3)

The condition that $0 < \left|\mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\right| < \sqrt{\mu_1^2+\mu_2^2}$ already guarantees that $\mu_1^2+\mu_2^2 > 0$, which means that either $p_2 + \frac{1}{2}a_{22} \neq p_1 + \frac{1}{2}a_{11}$, or $(\sigma_2\sigma_1\rho - \sigma_1^2)\left(p_2 + \frac{1}{2}a_{22}\right) \neq (\sigma_2^2 - \sigma_2\sigma_1\rho)\left(p_1 + \frac{1}{2}a_{11}\right)$, and the results in Proposition 3.5.2 holds. The quantity we are trying to solve for is then as follows:

$$\begin{aligned}(\mu-1)E^{(x_2,x_1)}&\left(e^{\frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}Z_{2,\tau}}I_{[\tau<\infty]}\right)\\ &= (\mu-1)\int_{-\infty}^{\infty}\frac{\sqrt{(\mu_1^2+\mu_2^2)}e^{-[\mu_1 z_1+\mu_2 z_2]}}{\pi}\frac{z_1 e^{\left(\mu_2+\frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\right)u}}{\sqrt{\left[z_1^2+(u-z_2)^2\right]}}\\ &\quad\times K_1\left(\sqrt{(\mu_1^2+\mu_2^2)[z_1^2+(u-z_2)^2]}\right)du. \qquad (3.18)\end{aligned}$$

The definitions for all constants, with further simplifications, are as follows:

$$\begin{aligned}\mu_1 &= -\left[\left(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\right)\frac{\left[\sigma_1\rho\left(p_2+\frac{1}{2}a_{22}\right)-\sigma_2\left(p_1+\frac{1}{2}a_{11}\right)\right]}{\sigma_1\sigma_2\sqrt{1-\rho^2}}+\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\frac{\left(p_2+\frac{1}{2}a_{22}\right)}{\sigma_2}\right]\\ &= -\left[\left(\frac{1}{\sqrt{\gamma}}\right)\frac{\left[\sigma_1\rho\left(p_2+\frac{1}{2}a_{22}\right)-\sigma_2\left(p_1+\frac{1}{2}a_{11}\right)\right]}{\sigma_2}+\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\frac{\left(p_2+\frac{1}{2}a_{22}\right)}{\sigma_2}\right]\\ &= -\left[\left(\frac{1}{\sqrt{\gamma}}\right)\frac{\left[-\sigma_2\left(p_1+\frac{1}{2}a_{11}\right)\right]}{\sigma_2}+\frac{\sigma_2}{\sqrt{\gamma}}\frac{\left(p_2+\frac{1}{2}a_{22}\right)}{\sigma_2}\right]\\ &= -\left[\left(\frac{1}{\sqrt{\gamma}}\right)\frac{\left[-\left(p_1+\frac{1}{2}a_{11}\right)\right]}{1}+\frac{1}{\sqrt{\gamma}}\frac{\left(p_2+\frac{1}{2}a_{22}\right)}{1}\right]\\ &= \left(\frac{\left(p_1+\frac{1}{2}a_{11}\right)-\left(p_2+\frac{1}{2}a_{22}\right)}{\sqrt{\gamma}}\right);\\ \mu_2 &= \left\{\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\cdot\frac{\left[\sigma_1\rho\left(p_2+\frac{1}{2}a_{22}\right)-\sigma_2\left(p_1+\frac{1}{2}a_{11}\right)\right]}{\sigma_1\sigma_2\sqrt{1-\rho^2}}-\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\cdot\frac{\left(p_2+\frac{1}{2}a_{22}\right)}{\sigma_2}\right\}\\ &= \left\{\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\cdot\frac{\left[\sigma_1\rho\left(p_2+\frac{1}{2}a_{22}\right)-\sigma_2\left(p_1+\frac{1}{2}a_{11}\right)\right]}{\sigma_1\sigma_2\sqrt{1-\rho^2}}-\frac{\sigma_1^2(1-\rho^2)}{\sqrt{\gamma}}\cdot\frac{\left(p_2+\frac{1}{2}a_{22}\right)}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\right\}\\ &= \frac{\left[\sigma_2\sigma_1\rho\left(p_2+\frac{1}{2}a_{22}\right)-\sigma_2^2\left(p_1+\frac{1}{2}a_{11}\right)\right]-\sigma_1^2\rho^2\left(p_2+\frac{1}{2}a_{22}\right)+\sigma_2\sigma_1\rho\left(p_1+\frac{1}{2}a_{11}\right)}{\sigma_1\sigma_2\sqrt{1-\rho^2}\sqrt{\gamma}}\\ &\quad-\frac{\sigma_1^2\left(p_2+\frac{1}{2}a_{22}\right)-\sigma_1^2\rho^2\left(p_2+\frac{1}{2}a_{22}\right)}{\sqrt{\gamma}}\cdot\frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sigma_2\sigma_1\rho\left(p_2+\frac{1}{2}a_{22}\right)+\sigma_2\sigma_1\rho\left(p_1+\frac{1}{2}a_{11}\right)-\sigma_2^2\left(p_1+\frac{1}{2}a_{11}\right)-\sigma_1^2\left(p_2+\frac{1}{2}a_{22}\right)}{\sigma_1\sigma_2\sqrt{1-\rho^2}\sqrt{\gamma}} \\
&= \frac{\left(\sigma_2\sigma_1\rho-\sigma_1^2\right)\left(p_2+\frac{1}{2}a_{22}\right)+\left(\sigma_2\sigma_1\rho-\sigma_2^2\right)\left(p_1+\frac{1}{2}a_{11}\right)}{\sigma_1\sigma_2\sqrt{1-\rho^2}\sqrt{\gamma}}. \\
z_1 &= -\left(\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\right)\left(\frac{\sigma_2\ln x_1-\sigma_1\rho\ln x_2-\sigma_2\ln\mu}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\right)+\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\frac{\ln x_2}{\sigma_2} \\
&= -\left(\frac{1}{\sqrt{\gamma}}\right)\left(\frac{\sigma_2\ln x_1-\sigma_1\rho\ln x_2-\sigma_2\ln\mu}{\sigma_2}\right)+\frac{\sigma_2\ln x_2-\sigma_1\rho\ln x_2}{\sqrt{\gamma}}\frac{1}{\sigma_2} \\
&= \frac{\sigma_2\ln x_2-\sigma_1\rho\ln x_2-\sigma_2\ln x_1+\sigma_1\rho\ln x_2+\sigma_2\ln\mu}{\sqrt{\gamma}\sigma_2} \\
&= \frac{\sigma_2\ln x_2-\sigma_2\ln x_1+\sigma_2\ln\mu}{\sqrt{\gamma}\sigma_2} \\
&= \frac{\ln x_2-\ln x_1+\ln\mu}{\sqrt{\gamma}} \\
&= \frac{\ln\frac{\mu}{h}}{\sqrt{\gamma}}; \\
z_2 &= \frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\frac{\sigma_2\ln x_1-\sigma_1\rho\ln x_2-\sigma_2\ln\mu}{\sigma_1\sigma_2\sqrt{1-\rho^2}}+\frac{\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\frac{\ln x_2}{\sigma_2} \\
&= \frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\frac{\sigma_2\ln x_1-\sigma_1\rho\ln x_2-\sigma_2\ln\mu}{\sigma_1\sigma_2\sqrt{1-\rho^2}}+\frac{\sigma_1^2(1-\rho^2)}{\sqrt{\gamma}}\frac{\ln x_2}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \\
&= \frac{\sigma_2^2\ln x_1-\sigma_1\sigma_2\rho\ln x_2-\sigma_2^2\ln\mu-\sigma_1\sigma_2\rho\ln x_1+\sigma_1^2\rho^2\ln x_2+\sigma_1\sigma_2\rho\ln\mu}{\sqrt{\gamma}\sigma_1\sigma_2\sqrt{1-\rho^2}} \\
&+ \frac{\sigma_1^2\ln x_2-\rho^2\sigma_1^2\ln x_2}{\sqrt{\gamma}\sigma_1\sigma_2\sqrt{1-\rho^2}} \\
&= \frac{\sigma_2^2\ln x_1+\sigma_1^2\ln x_2-\sigma_1\sigma_2\rho\ln x_2-\sigma_2^2\ln\mu-\sigma_1\sigma_2\rho\ln x_1+\sigma_1\sigma_2\rho\ln\mu}{\sqrt{\gamma}\sigma_1\sigma_2\sqrt{1-\rho^2}} \\
&= \frac{\sigma_2^2\ln x_1+\sigma_1^2\ln x_2-\sigma_1\sigma_2\rho\ln\left(x_1x_2\right)-\sigma_2^2\ln\mu+\sigma_1\sigma_2\rho\ln\mu}{\sqrt{\gamma}\sigma_1\sigma_2\sqrt{1-\rho^2}},
\end{aligned}
$$

where once again,

$$
\sigma_1 = \sqrt{q_{11}^2+q_{12}^2},
$$

$$
\sigma_2 = \sqrt{q_{21}^2+q_{22}^2};
$$

$$
\rho\sigma_1\sigma_2 = a_{12},
$$

$$\gamma = \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho.$$

Let us define

$$\begin{aligned} C_1 &= \frac{\rho\sigma_1\sigma_2\left(p_1 + p_2 + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2\right) - \sigma_1^2 p_2 - \sigma_2^2 p_1 - \sigma_1^2\sigma_2^2}{\sqrt{\gamma}\sigma_1\sigma_2\sqrt{1-\rho^2}} + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}} \\ &= \mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}. \end{aligned}$$

Only z_1, z_2 are functions of μ. We will separate out the part that is a function of μ from the part that is not in the above expression.

$$z_1 = -\frac{\ln h}{\sqrt{\gamma}} + \frac{\ln \mu}{\sqrt{\gamma}} = C_2 + C_3 \ln \mu,$$

where $C_2 = \frac{-\ln h}{\sqrt{\gamma}}$ and $C_3 = \frac{1}{\sqrt{\gamma}}$, and

$$z_2 = \frac{\sigma_2^2 \ln x_1 + \sigma_1^2 \ln x_2 - \sigma_1\sigma_2\rho \ln(x_1 x_2) - \sigma_2^2 \ln \mu + \sigma_1\sigma_2\rho \ln \mu}{\sqrt{\gamma}\sigma_1\sigma_2\sqrt{1-\rho^2}} = C_4 + C_5 \ln \mu,$$

where

$$C_4 = \frac{\sigma_2^2 \ln x_1 + \sigma_1^2 \ln x_2 - \rho\sigma_1\sigma_2 \ln(x_1 x_2)}{\sigma_1\sigma_2\sqrt{1-\rho^2}\sqrt{\gamma}}$$

and

$$C_5 = -\frac{\sigma_2 - \sigma_1\rho}{\sqrt{\gamma}} \frac{1}{\sigma_1\sqrt{1-\rho^2}}.$$

To make the dependence on μ explicit, we re-write the integral in (3.18) as a function of μ:

$$\psi(\mu) = (\mu - 1) \int_{-\infty}^{\infty} \frac{\sqrt{\left(\mu_1^2 + \mu_2^2\right)} e^{-[\mu_1 z_1 + z_2\mu_2]}}{\pi} \frac{z_1 e^{C_1 u}}{\sqrt{\left[z_1^2 + (u - z_2)^2\right]}}$$

$$\times K_1\left(\sqrt{\left(\mu_1^2+\mu_2^2\right)\left[z_1^2+(u-z_2)^2\right]}\right)du$$
$$=(\mu-1)\frac{z_1\sqrt{\left(\mu_1^2+\mu_2^2\right)}e^{-[\mu_1(C_2+C_3\ln\mu)+\mu_2(C_4+C_5\ln\mu)]}}{\pi}$$
$$\times\int_{-\infty}^{\infty}e^{C_1u}\frac{K_1\left(\sqrt{\left(\mu_1^2+\mu_2^2\right)\left[z_1^2+(u-z_2)^2\right]}\right)}{\sqrt{\left[z_1^2+(u-z_2)^2\right]}}du. \tag{3.19}$$

This is close in functional form with the Generalized Hyperbolic distribution. According to Eberlein, and von Hammerstein [10], page 3, the density function for the Generalized Hyperbolic distribution is

$$f_{GH(\lambda,\alpha,\beta,\delta,\eta)}(x)=\frac{\left(\alpha^2-\beta^2\right)^{\frac{\lambda}{2}}}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\delta^{\lambda}K_{\lambda}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}\left[\delta^2+(x-\eta)^2\right]^{\frac{\lambda-\frac{1}{2}}{2}}e^{\beta(x-\eta)}$$
$$\times K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\eta)^2}\right). \tag{3.20}$$

The Generalized Hyperbolic distribution is well defined with the following restrictions from (3.21) to (3.25):

$$\alpha=\sqrt{\mu_1^2+\mu_2^2}>0; \tag{3.21}$$
$$\beta=C_1;\ \ 0\le|\beta|<\alpha; \tag{3.22}$$
$$\lambda=\frac{3}{2}>0; \tag{3.23}$$
$$\delta=C_2+C_3\ln\mu=z_1>0; \tag{3.24}$$
$$\eta=C_4+C_5\ln\mu=z_2\in\mathbb{R}. \tag{3.25}$$

In particular, (3.21), (3.23), (3.24), (3.25) pose no additional restriction on our problem. However, (3.22) does.

$$0\le|\beta|<\alpha\iff 0\le|C_1|<\sqrt{\mu_1^2+\mu_2^2}$$

$$\iff 0 \leq \left| \mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}} \right| < \sqrt{\mu_1^2 + \mu_2^2}.$$

$$\mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}} =$$

$$\frac{\left(\sigma_2\sigma_1\rho - \sigma_1^2\right)\left(p_2 + \frac{1}{2}a_{22}\right) + \left(\sigma_2\sigma_1\rho - \sigma_2^2\right)\left(p_1 + \frac{1}{2}a_{11}\right)}{\sigma_1\sigma_2\sqrt{1-\rho^2}\sqrt{\gamma}} + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}$$

$$= \frac{\left(\sigma_2\sigma_1\rho - \sigma_1^2\right)\left(p_2 + \frac{1}{2}a_{22}\right) + \left(\sigma_2\sigma_1\rho - \sigma_2^2\right)\left(p_1 + \frac{1}{2}a_{11}\right) + \sigma_2^2\sigma_1^2(1-\rho^2)}{\sigma_1\sigma_2\sqrt{1-\rho^2}\sqrt{\gamma}}.$$

This is not a result of any prior restrictions, and has no obvious reduction to some other form. This restriction is thus added to our overall restrictions in the parameter space.

Under the same reference Eberlein, et. al. [10], we have an integral representation for the Generalized Hyperbolic density function as an infinite mixture of Gaussians weighted by a Generalized Inverse Gaussian distribution:

$$f_{GH(\lambda,\alpha,\beta,\delta,\eta)}(x) = \int_0^\infty f_{N(\eta+\beta y,y)}(x) f_{GIG\left(\lambda,\delta,\sqrt{\alpha^2-\beta^2}\right)}(y)dy. \tag{3.26}$$

Here, $f_{N(\eta+\beta y,y)}(x)$ is the Normal probability density function with mean $\eta + \beta y$, and variance y, and $f_{GIG\left(\lambda,\delta,\sqrt{\alpha^2-\beta^2}\right)}(y)$ is the probability density function of a Generalized Inverse Gaussian with corresponding parameters as listed within the parentheses. Furthermore,

$$f_{GIG\left(\lambda,\delta,\sqrt{\alpha^2-\beta^2}\right)}(y) = \left(\frac{\sqrt{\alpha^2-\beta^2}}{\delta}\right)^\lambda \frac{1}{2K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)} y^{\lambda-1} e^{-\frac{1}{2}\left[\frac{\delta^2}{y}+(\alpha^2-\beta^2)y\right]}$$

subject to the support constraint that $y > 0$. See Eberlein, et. al. (op. cit.).

The parameters for the G.I.G. distribution must obey one of the following sets of conditions in order for the density function to be well-defined:

$$\delta \geq 0,\ \sqrt{\alpha^2-\beta^2} > 0,\ \lambda > 0, \text{or} \tag{3.27}$$

$$\delta > 0,\ \sqrt{\alpha^2-\beta^2} > 0,\ \lambda = 0, \text{or} \tag{3.28}$$

$$\delta > 0,\ \sqrt{\alpha^2-\beta^2} > 0,\ \lambda < 0. \tag{3.29}$$

Since $\lambda = \frac{3}{2} > 0$, our set of parameters must obey restriction (3.27). The requirement $\delta \geq 0$ holds as $\delta > 0$ is already a requirement for the Generalized Hyperbolic distribution above. The condition $\gamma = \sqrt{\alpha^2-\beta^2} > 0$ also holds if $0 \leq |\beta| < \alpha$, per requirement (3.22) above as well. Thus, use of this integral identity poses no additional restrictions on our parameter space.

In what follows, we consider the integral in (3.19). The densities $f_{GH(\lambda,\alpha,\beta,\delta,\eta)}(u)$ and $f_{GIG\left(\lambda,\delta,\sqrt{\alpha^2-\beta^2}\right)}(y)$ are as defined above. Then, translating to notation of Eberlein et. al. [10],

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{C_1 u}\sqrt{\left[z_1^2+(u-z_2)^2\right]}\frac{K_1\left(\sqrt{\left(\mu_1^2+\mu_2^2\right)\left[z_1^2+(u-z_2)^2\right]}\right)}{\left[z_1^2+(u-z_2)^2\right]}du \\
&\qquad = \int_{-\infty}^{\infty} e^{\beta u}\sqrt{[\delta^2+(u-\eta)^2]}\frac{K_1\left(\alpha\sqrt{[\delta^2+(u-\eta)^2]}\right)}{[\delta^2+(u-\eta)^2]}du \\
&\qquad = e^{\beta\eta}\int_{-\infty}^{\infty} e^{\beta(u-\eta)}\sqrt{[\delta^2+(u-\eta)^2]}\frac{K_1\left(\alpha\sqrt{[\delta^2+(u-\eta)^2]}\right)}{[\delta^2+(u-\eta)^2]}du.
\end{aligned} \tag{3.30}$$

Now, from (3.20), with $\lambda = \frac{3}{2}$,

$$\begin{aligned}
&f_{GH(\lambda,\alpha,\beta,\delta,\eta)}(x) \\
&= \frac{\left(\alpha^2-\beta^2\right)^{\frac{3}{4}}}{\sqrt{2\pi}\alpha\delta^{\frac{3}{2}}K_{\frac{3}{2}}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}\left[\delta^2+(x-\eta)^2\right]^{\frac{1}{2}}e^{\beta(x-\eta)}K_1\left(\alpha\sqrt{\delta^2+(x-\eta)^2}\right),
\end{aligned}$$

or

$$\left[\delta^2+(x-\eta)^2\right]^{\frac{1}{2}} e^{\beta(x-\eta)} K_1\left(\alpha\sqrt{\delta^2+(x-\eta)^2}\right) \\ =\frac{\sqrt{2\pi}\alpha\delta^{\frac{3}{2}} K_{\frac{3}{2}}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{\left(\alpha^2-\beta^2\right)^{\frac{3}{4}}} f_{GH(\lambda,\alpha,\beta,\delta,\eta)}(x).$$

The integral in (3.30) then becomes equal to

$$\begin{aligned}
&e^{\beta\eta}\frac{\sqrt{2\pi}\alpha\delta^{\frac{3}{2}} K_{\frac{3}{2}}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{\left(\alpha^2-\beta^2\right)^{\frac{3}{4}}} \int_{-\infty}^{\infty} \frac{f_{GH(\lambda,\alpha,\beta,\delta,\eta)}(u)}{\left[\delta^2+(u-\eta)^2\right]} du \\
&= e^{\beta\eta}\frac{\sqrt{2\pi}\alpha\delta^{\frac{3}{2}} K_{\frac{3}{2}}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{\left(\alpha^2-\beta^2\right)^{\frac{3}{4}}} \int_{-\infty}^{\infty} \frac{f_{GH(\lambda,\alpha,\beta,\delta,\eta)}(x)}{\left[\delta^2+(x-\eta)^2\right]} dx \\
&= e^{\beta\eta}\frac{\sqrt{2\pi}\alpha\delta^{\frac{3}{2}} K_{\frac{3}{2}}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{\left(\alpha^2-\beta^2\right)^{\frac{3}{4}}} \int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{f_{N(\eta+\beta y,y)}(x)}{\left[\delta^2+(x-\eta)^2\right]} f_{GIG\left(\lambda,\delta,\sqrt{\alpha^2-\beta^2}\right)}(y)\,dy\,dx \\
&= e^{\beta\eta}\frac{\sqrt{2\pi}\alpha\delta^{\frac{3}{2}} K_{\frac{3}{2}}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{\left(\alpha^2-\beta^2\right)^{\frac{3}{4}}} \int_{0}^{\infty}\left\{\int_{-\infty}^{\infty} \frac{f_{N(\eta+\beta y,y)}(x)}{\left[\delta^2+(x-\eta)^2\right]} dx\right\} f_{GIG\left(\lambda,\delta,\sqrt{\alpha^2-\beta^2}\right)}(y)\,dy \\
&= e^{\beta\eta}\frac{\pi\sqrt{2\pi}\alpha\delta^{\frac{3}{2}} K_{\frac{3}{2}}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{\delta\left(\alpha^2-\beta^2\right)^{\frac{3}{4}}} \int_{0}^{\infty}\left\{\int_{-\infty}^{\infty} \frac{\delta f_{N(\eta+\beta y,y)}(x)}{\pi\left[\delta^2+(x-\eta)^2\right]} dx\right\} f_{GIG\left(\lambda,\delta,\sqrt{\alpha^2-\beta^2}\right)}(y)\,dy,
\end{aligned} \tag{3.31}$$

by Tonelli's theorem. The second equality utilizes (3.26).

Now, let us consider the inner integral first. By a change of variable,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\delta f_{N(\eta+\beta y,y)}(x)}{\pi\left[\delta^2+(x-\eta)^2\right]} dx &= \int_{-\infty}^{\infty} \frac{\delta e^{-\frac{(x-\eta-\beta y)^2}{2y}}}{\pi\left[\delta^2+(x-\eta)^2\right]} \frac{1}{\sqrt{2\pi y}} dx \\
&= \int_{-\infty}^{\infty} \frac{\delta e^{-\frac{(x-\beta y)^2}{2y}}}{\pi\left[\delta^2+x^2\right]} \frac{1}{\sqrt{2\pi y}} dx.
\end{aligned} \tag{3.32}$$

To evaluate the integral, we resort to using characteristic functions. We observe that

$\frac{\delta}{\pi[\delta^2+(x-\eta)^2]}$ and $\frac{e^{-\frac{(x-\eta-\beta y)^2}{2y}}}{\sqrt{2\pi y}}$ are both square-integrable, and therefore we can apply Parseval's Theorem (see pp. 154-155 of Yoshida [25]) to this generalized inner product. Let $i = \sqrt{-1}$. Then, because $e^{-\delta|\phi|}\cos(\beta y\phi)\, e^{-\frac{1}{2}y\phi^2}$ is an even function, and $e^{-\delta|\phi|}\sin(\beta y\phi)\, e^{-\frac{1}{2}y\phi^2}$ is an odd function,

$$\begin{aligned}
\int\limits_{-\infty}^{\infty} \frac{\delta e^{-\frac{(x-\beta y)^2}{2y}}}{\pi\left[\delta^2+x^2\right]}\frac{1}{\sqrt{2\pi y}}dx &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-\delta|\phi|}e^{i\beta y\phi}e^{-\frac{1}{2}y\phi^2}d\phi \\
&= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-\delta|\phi|}[cos(\beta y\phi)+isin(\beta y\phi)]e^{-\frac{1}{2}y\phi^2}d\phi \\
&= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-\delta|\phi|}cos(\beta y\phi)\ e^{-\frac{1}{2}y\phi^2}d\phi \\
&+ \frac{i}{2\pi}\int\limits_{-\infty}^{\infty} e^{-\delta|\phi|}sin(\beta y\phi)e^{-\frac{1}{2}y\phi^2}d\phi \\
&= \frac{1}{\pi}\int\limits_{0}^{\infty} e^{-\delta|\phi|}cos(\beta y\phi)\ e^{-\frac{1}{2}y\phi^2}d\phi + 0 \\
&= \frac{1}{\pi}\int\limits_{0}^{\infty} e^{-\delta|\phi|}cos(\beta y\phi)\ e^{-\frac{1}{2}y\phi^2}d\phi.
\end{aligned}$$

The last integral then equals

$$\begin{aligned}
\frac{1}{\pi}\int\limits_{0}^{\infty} e^{-\delta|\phi|}\cos(\beta y\phi)\, e^{-\frac{1}{2}y\phi^2}d\phi &= \frac{1}{\pi}\int\limits_{0}^{\infty}\cos(\beta y\phi)\, e^{-\delta\phi}e^{-\frac{1}{2}y\phi^2}d\phi \\
&= \frac{1}{\pi}\int\limits_{0}^{\infty}\cos(\beta y\phi)\, e^{-\frac{1}{2}y\left(\phi^2+\frac{2\delta\phi}{y}+\left(\frac{\delta}{y}\right)^2\right)+\frac{1}{2}y\left(\frac{\delta}{y}\right)^2}d\phi \\
&= \frac{1}{\pi}\int\limits_{0}^{\infty}\cos(\beta y\phi)\, e^{-\frac{1}{2}y\left(\phi+\frac{\delta}{y}\right)^2+\frac{1}{2}\left(\frac{\delta^2}{y}\right)}d\phi \\
&= e^{\frac{1}{2}\left(\frac{\delta^2}{y}\right)}\frac{1}{\pi}\int\limits_{0}^{\infty}\cos(\beta y\phi)\, e^{-\frac{1}{2}y\left(\phi+\frac{\delta}{y}\right)^2}d\phi
\end{aligned}$$

$$= e^{\frac{1}{2}\left(\frac{\delta^2}{y}\right)} \frac{1}{\pi} \int\limits_{\frac{\delta}{y}}^{\infty} \cos\left[\beta y \left(\phi' - \frac{\delta}{y}\right)\right] e^{-\frac{1}{2}y(\phi')^2} d\phi'$$

$$= e^{\frac{1}{2}\left(\frac{\delta^2}{y}\right)} \frac{1}{\pi} \int\limits_{\frac{\delta}{y}}^{\infty} \cos\left[\beta y \left(\phi - \frac{\delta}{y}\right)\right] e^{-\frac{1}{2}y\phi^2} d\phi$$

$$= e^{\frac{1}{2}\left(\frac{\delta^2}{y}\right)} \frac{1}{\pi} \left\{ \int\limits_{0}^{\infty} \cos\left[\beta y \left(\phi - \frac{\delta}{y}\right)\right] e^{-\frac{1}{2}y\phi^2} d\phi \right.$$

$$\left. - \int\limits_{0}^{\frac{\delta}{y}} \cos\left[\beta y \left(\phi - \frac{\delta}{y}\right)\right] e^{-\frac{1}{2}y\phi^2} d\phi \right\}. \quad (3.33)$$

To evaluate $\int_0^\infty \cos\left[\beta y \left(\phi - \frac{\delta}{y}\right)\right] e^{-\frac{1}{2}y\phi^2} d\phi$, let $t = \phi^2$; $dt = 2\phi d\phi$, or $d\phi = \frac{dt}{2\phi}$ $= \frac{dt}{2\sqrt{t}}$. The integral becomes $\int_0^\infty \frac{\cos\left[\beta y\left(\sqrt{t} - \frac{\delta}{y}\right)\right]}{2\sqrt{t}} e^{-\frac{1}{2}yt} dt$. Let $s = \frac{1}{2}y$, then we have

$$\int\limits_0^\infty \frac{\cos\left[\beta 2s\left(\sqrt{t} - \frac{\delta}{2s}\right)\right]}{2\sqrt{t}} e^{-st} dt = \int\limits_0^\infty \frac{\cos\left(\beta 2s\sqrt{t}\right)\cos\beta\delta + \sin\left(\beta 2s\sqrt{t}\right)\sin\beta\delta}{2\sqrt{t}} e^{-st} dt$$

$$= \cos\beta\delta \int\limits_0^\infty \frac{\cos\left(\beta 2s\sqrt{t}\right)}{2\sqrt{t}} e^{-st} dt$$

$$+ \sin\beta\delta \int\limits_0^\infty \frac{\sin\left(\beta 2s\sqrt{t}\right)}{2\sqrt{t}} e^{-st} dt. \quad (3.34)$$

Let $g_1(t), g_2(s)$ be functions of a real variable t and a complex variable s, respectively. We write $g_1(t) \doteq g_2(s)$, to denote that $g_2(s)$ is the Laplace transform of $g_1(t)$.

By noting that $\frac{1}{\sqrt{t}}\cos\left(a\sqrt{t}\right) \doteq \sqrt{\frac{\pi}{s}} e^{-\frac{a^2}{4s}}$, with $a = 2\beta s$, $Re(s) > 0$, the first integral in (3.34) can be evaluated (see pg. 34, no. 18, Roberts, G.E., and Kaufman, H.[21]). Also, $\frac{1}{\sqrt{t}}\sin\left(a\sqrt{t}\right) \doteq -\frac{i\sqrt{\pi}}{\sqrt{s}} e^{-\frac{a^2}{4s}} \operatorname{erf}\left(\frac{ia}{2\sqrt{s}}\right)$, with $a = 2\beta s$, $Re(s) > 0$, (from pg. 33, op. cit.), which enables us to evaluate the second integral in (3.34). The integral, on the l.h.s.

of (3.34) becomes,

$$
\begin{aligned}
&= \sqrt{\pi}\frac{\cos\beta\delta}{2}\frac{1}{\sqrt{s}}e^{-\beta^2 s^2/s} + \sqrt{\pi}\frac{\sin\beta\delta}{2}\int\limits_0^\infty \frac{\sin\left(2\beta s\sqrt{t}\right)}{\sqrt{\pi t}}e^{-st}dt \\
&= \sqrt{\pi}\frac{\cos\beta\delta}{2}\frac{1}{\sqrt{s}}e^{-\beta^2 s} + \frac{\sin\beta\delta}{2}\int\limits_0^\infty \frac{\sin\left(2\beta s\sqrt{t}\right)}{\sqrt{t}}e^{-st}dt \\
&= \sqrt{\pi}\frac{\cos\beta\delta}{2}\frac{1}{\sqrt{s}}e^{-\beta^2 s} - \frac{\sin\beta\delta}{2}\frac{i\sqrt{\pi}}{\sqrt{s}}e^{-\frac{4\beta^2 s^2}{4s}}\operatorname{erf}\left(\frac{i2\beta s}{2\sqrt{s}}\right) \\
&= \sqrt{\frac{\pi}{s}}\frac{e^{-\beta^2 s}}{2}\left(\cos\beta\delta - i\operatorname{erf}\left(i\beta\sqrt{s}\right)\sin\beta\delta\right) \\
&= \sqrt{\frac{\pi}{s}}\frac{e^{-\beta^2 s}}{2}\left(\cos\beta\delta - i\sin\beta\delta\int\limits_0^{i\beta\sqrt{s}} e^{-t^2}dt\right).
\end{aligned}
$$

(The Laplace transform over both the sine and cosine term requires $Re(s) = Re(y/2) > 0$. As the support for the G.I.G. density function is in $R^+, Re(y)$ will indeed be greater than 0).

Or,

$$
\begin{aligned}
\int\limits_0^\infty \frac{\cos\left[\beta 2s\left(\sqrt{t} - \frac{\delta}{2s}\right)\right]}{2\sqrt{t}}e^{-st}dt &= \sqrt{\frac{2\pi}{y}}\frac{e^{-\beta^2\frac{y}{2}}}{2}\left(\cos\beta\delta - i\sin\beta\delta\int\limits_0^{i\beta\sqrt{\frac{y}{2}}} e^{-t^2}dt\right) \\
&= \sqrt{\frac{\pi}{2y}}e^{-\beta^2\frac{y}{2}}\left[\cos\beta\delta - i\sin\beta\delta\operatorname{erf}\left(i\beta\sqrt{\frac{y}{2}}\right)\right].
\end{aligned} \tag{3.35}
$$

Next, consider the second integral in expression (3.33):

$$
\int\limits_0^{\frac{\delta}{y}} \cos\left[\beta y\left(\phi - \frac{\delta}{y}\right)\right]e^{-\frac{1}{2}y\phi^2}d\phi = \int\limits_0^{\frac{\delta}{y}} \cos\left[\beta y\left(\frac{\delta}{y} - \phi\right)\right]e^{-\frac{1}{2}y\phi^2}d\phi. \tag{3.36}
$$

This is a convolution integral. The Laplace Transform of this is equal to the Laplace Transform of $\cos(\beta y\phi)$ times the Laplace Transform of $e^{-\frac{1}{2}y\phi^2}$. Page 29 of Roberts and

Kaufman [21], No. 24 gives the following Laplace Transform pair: $\cos(\beta y\phi) \doteq \frac{s}{s^2+\beta^2y^2}$, with $Re(s) > Im(\beta y) = 0$, which does not impose any serious restriction or contradiction. Page 20 (op. cit.), 3.1, No. 2 gives the other Laplace Transform:

$$e^{-\frac{t^2}{a}} \doteq \frac{1}{2}\sqrt{\pi a}e^{\frac{as^2}{4}} \operatorname{erfc}\left(\frac{\sqrt{a}s}{2}\right),$$

or if $a = \frac{2}{y}$ and $t = \phi$, then

$$e^{-\frac{y\phi^2}{2}} \doteq \frac{1}{2}\sqrt{\frac{2\pi}{y}}e^{\frac{2s^2}{4y}} \operatorname{erfc}\left(\frac{\sqrt{\frac{2}{y}}s}{2}\right) = \sqrt{\frac{\pi}{2y}}e^{\frac{s^2}{2y}} \operatorname{erfc}\left(\sqrt{\frac{1}{2y}}s\right).$$

Here the restrictions are $Re(a) = Re(2/y) > 0$ and $Re(s) > -\infty$; both hold in this case.

The Laplace Transform of the convolution in (3.36) is then $\frac{s}{s^2+\beta^2y^2}\sqrt{\frac{\pi}{2y}}e^{\frac{s^2}{2y}} \operatorname{erfc}\left(s\sqrt{\frac{1}{2y}}\right)$. Now,

$$\frac{s}{s^2+\beta^2y^2} = \frac{A}{s-i\beta y} + \frac{B}{s+i\beta y},$$

where $A = \frac{s}{s+i\beta y}|_{s=i\beta y} = \frac{i\beta y}{2i\beta y} = \frac{1}{2}$, and $B = \frac{s}{s-i\beta y}|_{s=-i\beta y} = \frac{-i\beta y}{-2i\beta y} = \frac{1}{2}$. This gives

$$\frac{s}{s^2+\beta^2y^2} = \frac{1}{2}\frac{1}{s-i\beta y} + \frac{1}{2}\frac{1}{s+i\beta y},$$

and we have

$$\frac{s}{s^2+\beta^2y^2}\sqrt{\frac{\pi}{2y}}e^{\frac{s^2}{2y}} \operatorname{erfc}\left(s\sqrt{\frac{1}{2y}}\right) = \left(\frac{1}{2}\frac{1}{s+i\beta y} + \frac{1}{2}\frac{1}{s-i\beta y}\right)\sqrt{\frac{\pi}{2y}}e^{\frac{s^2}{2y}} \operatorname{erfc}\left(s\sqrt{\frac{1}{2y}}\right), \tag{3.37}$$

as long as $Re(s) > Re(i\beta y) = 0$. Now, $s = y/2 > 0$ because y is in the support of the G.I.G. distribution. From page 316, 16.3.1, No. 5 (op. cit.), the inverse Laplace Transform of $\frac{e^{s^2}\operatorname{erfc}(s)}{s-a}$ is $e^{a(t+a)}\left[\operatorname{erf}\left(\frac{t}{2}+a\right) - \operatorname{erf}(a)\right]$, so that

$$\frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{st}\frac{e^{s^2}\operatorname{erfc}(s)}{s-a}ds = e^{a(t+a)}\left[\operatorname{erf}\left(\frac{t}{2}+a\right)-\operatorname{erf}(a)\right] \tag{3.38}$$

for any $\gamma > 0$. In trying to find the inverse transform for both terms in (3.37), it is necessary that the poles associated with the integrand at a be such that $Re(a) < \gamma$. We have $Re\,(\pm i\beta y) = 0 < \gamma$ if γ is any positive real constant. Thus, the inverse Laplace transform is well defined as long as $\gamma > 0$. Now, we make the substitution $s' = \frac{s}{\sqrt{2y}}$, $ds = \sqrt{2y}ds'$. Then,

$$\frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{st}\sqrt{\frac{\pi}{2y}}\frac{1}{2}\frac{e^{\left(\frac{s}{\sqrt{2y}}\right)^2}\operatorname{erfc}\left(\frac{s}{\sqrt{2y}}\right)}{s+i\beta y}ds$$

$$= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\frac{\gamma-iT}{\sqrt{2y}}}^{\frac{\gamma+iT}{\sqrt{2y}}} e^{s'\sqrt{2y}t}\sqrt{\frac{\pi}{2y}}\frac{1}{2}\frac{e^{(s')^2}\operatorname{erfc}(s')}{\sqrt{2y}s'+i\beta y}\sqrt{2y}ds'$$

$$= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\frac{\gamma-iT}{\sqrt{2y}}}^{\frac{\gamma+iT}{\sqrt{2y}}} e^{s'\sqrt{2y}t}\sqrt{\frac{\pi}{2y}}\frac{1}{2}\frac{e^{(s')^2}\operatorname{erfc}(s')}{s'+i\frac{\beta y}{\sqrt{2y}}}ds'.$$

Utilizing (3.38), the above equals

$$\sqrt{\frac{\pi}{2y}}\frac{1}{2}e^{-i\beta\sqrt{\frac{y}{2}}\left(\sqrt{2y}t-i\beta\sqrt{\frac{y}{2}}\right)}\left[\operatorname{erf}\left(\frac{\sqrt{2y}t}{2}-i\beta\sqrt{\frac{y}{2}}\right)-\operatorname{erf}\left(-i\beta\sqrt{\frac{y}{2}}\right)\right]$$

$$= \sqrt{\frac{\pi}{2y}}\frac{1}{2}e^{-i\beta yt-\beta^2\frac{y}{2}}\left[\operatorname{erf}\left(\sqrt{\frac{y}{2}}t-i\beta\sqrt{\frac{y}{2}}\right)-\operatorname{erf}\left(-i\beta\sqrt{\frac{y}{2}}\right)\right],$$

as $\frac{\gamma}{\sqrt{2y}} > 0$, and $Re(s') = \frac{Re(s)}{\sqrt{2y}} > 0$ if $Re(s) > 0$.

For the second term in (3.37), making the same substitutions as for the first term,

$$\frac{1}{2\pi i}\lim_{T\to\infty}\int\limits_{\gamma-iT}^{\gamma+iT} e^{st}\sqrt{\frac{\pi}{2y}}\frac{1}{2}\frac{e^{\left(\frac{s}{\sqrt{2y}}\right)^2}\operatorname{erfc}\left(\frac{s}{\sqrt{2y}}\right)}{s-i\beta y}ds$$

$$=\frac{1}{2\pi i}\lim_{T\to\infty}\int\limits_{\frac{\gamma-iT}{\sqrt{2y}}}^{\frac{\gamma+iT}{\sqrt{2y}}} e^{s'\sqrt{2y}t}\sqrt{\frac{\pi}{2y}}\frac{1}{2}\frac{e^{(s')^2}\operatorname{erfc}(s')}{\sqrt{2y}s'-i\beta y}\sqrt{2y}ds'$$

$$=\frac{1}{2\pi i}\lim_{T\to\infty}\int\limits_{\frac{\gamma-iT}{\sqrt{2y}}}^{\frac{\gamma+iT}{\sqrt{2y}}} e^{s'\sqrt{2y}t}\sqrt{\frac{\pi}{2y}}\frac{1}{2}\frac{e^{(s')^2}\operatorname{erfc}(s')}{s'-i\frac{\beta y}{\sqrt{2y}}}ds'$$

$$=\sqrt{\frac{\pi}{2y}}\frac{1}{2}e^{i\beta\sqrt{\frac{y}{2}}\left(\sqrt{2y}t+i\beta\sqrt{\frac{y}{2}}\right)}\left[\operatorname{erf}\left(\frac{\sqrt{2y}t}{2}+i\beta\sqrt{\frac{y}{2}}\right)-\operatorname{erf}\left(i\beta\sqrt{\frac{y}{2}}\right)\right]$$

$$=\sqrt{\frac{\pi}{2y}}\frac{1}{2}e^{i\beta yt-\beta^2\frac{y}{2}}\left[\operatorname{erf}\left(\sqrt{\frac{y}{2}}t+i\beta\sqrt{\frac{y}{2}}\right)-\operatorname{erf}\left(i\beta\sqrt{\frac{y}{2}}\right)\right].$$

The convolution integral in (3.36) was, of course, a function of $\frac{\delta}{y}$. Thus, we substitute $\frac{\delta}{y}$ into the above, for every occurrence of t. We get

$$\int\limits_0^{\frac{\delta}{y}}\cos\left[\beta y\left(\phi-\frac{\delta}{y}\right)\right]e^{-\frac{1}{2}y\phi^2}d\phi$$

$$=\sqrt{\frac{\pi}{2y}}\frac{1}{2}e^{-i\beta\delta-\beta^2\frac{y}{2}}\left[\operatorname{erf}\left(\sqrt{\frac{1}{2y}}\delta-i\beta\sqrt{\frac{y}{2}}\right)-\operatorname{erf}\left(-i\beta\sqrt{\frac{y}{2}}\right)\right]$$

$$+\sqrt{\frac{\pi}{2y}}\frac{1}{2}e^{i\beta\delta-\beta^2\frac{y}{2}}\left[\operatorname{erf}\left(\sqrt{\frac{1}{2y}}\delta+i\beta\sqrt{\frac{y}{2}}\right)-\operatorname{erf}\left(i\beta\sqrt{\frac{y}{2}}\right)\right]$$

$$=\sqrt{\frac{\pi}{2y}}\frac{1}{2}e^{-\beta^2\frac{y}{2}}\times$$

$$\left\{e^{-i\beta\delta}\left[\operatorname{erf}\left(\sqrt{\frac{1}{2y}}\delta-i\beta\sqrt{\frac{y}{2}}\right)-\operatorname{erf}\left(-i\beta\sqrt{\frac{y}{2}}\right)\right]\right.$$

$$\left.+e^{i\beta\delta}\left[\operatorname{erf}\left(\sqrt{\frac{1}{2y}}\delta+i\beta\sqrt{\frac{y}{2}}\right)-\operatorname{erf}\left(i\beta\sqrt{\frac{y}{2}}\right)\right]\right\}$$

$$=\sqrt{\frac{\pi}{2y}}\frac{1}{2}e^{-\beta^2\frac{y}{2}}\times$$

$$\left\{e^{-i\beta\delta}\left[\operatorname{erf}\left(\sqrt{\frac{1}{2y}}\delta - i\beta\sqrt{\frac{y}{2}}\right) + \operatorname{erf}\left(i\beta\sqrt{\frac{y}{2}}\right)\right]\right.$$
$$\left. + e^{i\beta\delta}\left[\operatorname{erf}\left(\sqrt{\frac{1}{2y}}\delta + i\beta\sqrt{\frac{y}{2}}\right) - \operatorname{erf}\left(i\beta\sqrt{\frac{y}{2}}\right)\right]\right\}$$
$$= \sqrt{\frac{\pi}{2y}}\frac{1}{2}e^{-\beta^2\frac{y}{2}}\left\{e^{-i\beta\delta}\operatorname{erf}\left(\sqrt{\frac{1}{2y}}\delta - i\beta\sqrt{\frac{y}{2}}\right) + e^{i\beta\delta}\operatorname{erf}\left(\sqrt{\frac{1}{2y}}\delta + i\beta\sqrt{\frac{y}{2}}\right)\right\}$$
$$+ \sqrt{\frac{\pi}{2y}}\frac{1}{2}e^{-\beta^2\frac{y}{2}}\left\{e^{-i\beta\delta} - e^{i\beta\delta}\right\}\operatorname{erf}\left(i\beta\sqrt{\frac{y}{2}}\right)$$
$$= \sqrt{\frac{\pi}{2y}}\frac{1}{2}e^{-\beta^2\frac{y}{2}}\left\{2\cdot\operatorname{Re}\left[e^{i\beta\delta}\operatorname{erf}\left(\sqrt{\frac{1}{2y}}\delta + i\beta\sqrt{\frac{y}{2}}\right)\right] - 2i\sin(\beta\delta)\operatorname{erf}\left(i\beta\sqrt{\frac{y}{2}}\right)\right\}$$
$$= \sqrt{\frac{\pi}{2y}}e^{-\beta^2\frac{y}{2}}\left\{\operatorname{Re}\left[e^{i\beta\delta}\operatorname{erf}\left(\sqrt{\frac{1}{2y}}\delta + i\beta\sqrt{\frac{y}{2}}\right)\right] - i\sin(\beta\delta)\operatorname{erf}\left(i\beta\sqrt{\frac{y}{2}}\right)\right\}.$$

Thus, the original (inner) integral in (3.31) integrates out to

$$\int_{-\infty}^{\infty}\frac{\delta}{\pi\left[\delta^2+x^2\right]}\frac{e^{-\frac{(x-\beta y)^2}{2y}}}{\sqrt{2\pi y}}dx$$
$$= \frac{e^{\frac{1}{2}\left(\frac{\delta^2}{y}\right)}}{\pi} \times \sqrt{\frac{\pi}{2y}}e^{-\frac{\beta^2 y}{2}}$$
$$\times\left[\left(\cos\beta\delta - i\sin(\beta\delta)\operatorname{erf}\left(i\beta\sqrt{\frac{y}{2}}\right)\right)\right.$$
$$\left. - \left\{\operatorname{Re}\left[e^{i\beta\delta}\operatorname{erf}\left(\sqrt{\frac{1}{2y}}\delta + i\beta\sqrt{\frac{y}{2}}\right)\right] - i\sin(\beta\delta)\operatorname{erf}\left(i\beta\sqrt{\frac{y}{2}}\right)\right\}\right]$$
$$= \frac{e^{\frac{1}{2}\left(\frac{\delta^2}{y}\right)}}{\pi}\sqrt{\frac{\pi}{2y}}e^{-\frac{\beta^2 y}{2}}\left(\cos\beta\delta - \operatorname{Re}\left[e^{i\beta\delta}\operatorname{erf}\left(\sqrt{\frac{1}{2y}}\delta + i\beta\sqrt{\frac{y}{2}}\right)\right]\right)$$
$$= \sqrt{\frac{1}{2\pi y}}e^{\frac{1}{2}\left(\frac{\delta^2}{y}\right)}e^{-\frac{\beta^2 y}{2}}$$
$$\times\left\{\cos\beta\delta - \frac{\left[e^{i\beta\delta}\operatorname{erf}\left(\sqrt{\frac{1}{2y}}\delta + i\beta\sqrt{\frac{y}{2}}\right)\right] + \left[e^{-i\beta\delta}\operatorname{erf}\left(\sqrt{\frac{1}{2y}}\delta - i\beta\sqrt{\frac{y}{2}}\right)\right]}{2}\right\}. \quad (3.39)$$

Before proceeding further, we recall the definition of the Generalized inverse Gaussian density function, from Eberlein, and Hammerstein, V., [10],

$$f_{GIG\left(\lambda,\delta,\sqrt{\alpha^2-\beta^2}\right)}(y) = \left(\frac{\sqrt{\alpha^2-\beta^2}}{\delta}\right)^{\lambda} \frac{1}{2K_{\lambda}\left(\delta\sqrt{\alpha^2-\beta^2}\right)} y^{\lambda-1} e^{-\frac{1}{2}\left(\frac{\delta^2}{y}+\left(\alpha^2-\beta^2\right)y\right)} \mathrm{I}_{(y>0)}.$$

Thus,

$$\begin{aligned} f_{GIG\left(\frac{3}{2},\delta,\sqrt{\alpha^2-\beta^2}\right)}(y) &= \left(\frac{\sqrt{\alpha^2-\beta^2}}{\delta}\right)^{\frac{3}{2}} \frac{1}{2K_{\frac{3}{2}}\left(\delta\sqrt{\alpha^2-\beta^2}\right)} y^{\frac{3}{2}-1} e^{-\frac{1}{2}\left(\frac{\delta^2}{y}+\ \left(\alpha^2-\beta^2\right)y\right)} \mathrm{I}_{(y>0)} \\ &= \left(\frac{\sqrt{\alpha^2-\beta^2}}{\delta}\right)^{\frac{3}{2}} \frac{1}{2K_{\frac{3}{2}}\left(\delta\sqrt{\alpha^2-\beta^2}\right)} y^{\frac{1}{2}} e^{-\frac{1}{2}\left(\frac{\delta^2}{y}+\left(\alpha^2-\beta^2\right)y\right)} \mathrm{I}_{(y>0)} \\ &= \frac{\left(\alpha^2-\beta^2\right)^{\frac{3}{4}}}{\delta^{\frac{3}{2}}} \frac{1}{2K_{\frac{3}{2}}\left(\delta\sqrt{\alpha^2-\beta^2}\right)} \sqrt{y} e^{-\frac{1}{2}\left(\frac{\delta^2}{y}+\left(\alpha^2-\beta^2\right)y\right)} \mathrm{I}_{(y>0)}. \end{aligned} \tag{3.40}$$

Substituting (3.39) into the bracketed term in (3.31) and (3.40) into the $f_{GIG\left(\lambda,\delta,\sqrt{\alpha^2-\beta^2}\right)}$ term in (3.31) turns (3.31) into

$$\begin{aligned} & e^{\beta\eta} \frac{\pi\sqrt{2\pi}\alpha\delta^{\frac{3}{2}} K_{\frac{3}{2}}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{\delta\left(\alpha^2-\beta^2\right)^{\frac{3}{4}}} \int_0^{\infty} \left\{ \int_{-\infty}^{\infty} \frac{f_{N(\eta+\beta y,y)}(x)}{\delta^2+(x-\eta)^2} dx \right\} f_{GIG\left(\lambda,\delta,\sqrt{\alpha^2-\beta^2}\right)}(y)\, dy \\ &= e^{\beta\eta} \frac{\pi\sqrt{2\pi}\alpha\delta^{\frac{3}{2}} K_{\frac{3}{2}}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{\delta\left(\alpha^2-\beta^2\right)^{\frac{3}{4}}} \int_0^{\infty} \left\{ \sqrt{\frac{1}{2\pi y}} e^{\frac{1}{2}\left(\frac{\delta^2}{y}\right)} e^{-\frac{\beta^2 y}{2}} \right. \\ & \quad \times \left[\cos\beta\delta - \frac{\left[e^{i\beta\delta}\operatorname{erf}\left(\sqrt{\frac{1}{2y}}\delta + i\beta\sqrt{\frac{y}{2}}\right)\right] + \left[e^{-i\beta\delta}\operatorname{erf}\left(\sqrt{\frac{1}{2y}}\delta - i\beta\sqrt{\frac{y}{2}}\right)\right]}{2} \right] \\ & \quad \left. \times f_{GIG\left(\lambda,\delta,\sqrt{\alpha^2-\beta^2}\right)}(y) \right\} dy \\ &= \frac{e^{\beta\eta}\pi\alpha}{2\delta} \left[\int_0^{\infty} \cos(\beta\delta) e^{-\frac{\alpha^2 y}{2}} dy \right. \end{aligned}$$

$$- \int_0^\infty \frac{\left\{ e^{-i\beta\delta} \operatorname{erf}\left(\sqrt{\frac{1}{2y}}\delta - i\beta\sqrt{\frac{y}{2}}\right) + e^{i\beta\delta} \operatorname{erf}\left(\sqrt{\frac{1}{2y}}\delta + i\beta\sqrt{\frac{y}{2}}\right) \right\}}{2} e^{-\frac{\alpha^2}{2}y} dy \Bigg]$$

$$= \frac{e^{\beta\eta}\pi\alpha}{2\delta} \cos(\beta\delta) \left(\frac{2}{\alpha^2}\right) - \frac{e^{\beta\eta}\pi\alpha}{2\delta} \left[\int_0^\infty \frac{\left\{ e^{-i\beta\delta} \operatorname{erf}\left(\sqrt{\frac{1}{2y}}\delta - i\beta\sqrt{\frac{y}{2}}\right) + e^{i\beta\delta} \operatorname{erf}\left(\sqrt{\frac{1}{2y}}\delta + i\beta\sqrt{\frac{y}{2}}\right) \right\}}{2} e^{-\frac{\alpha^2}{2}y} dy \right]. \tag{3.41}$$

We now concentrate on the integral in (3.41), i.e.

$$\frac{1}{2} \int_0^\infty \left\{ e^{-i\beta\delta} \operatorname{erf}\left(\sqrt{\frac{1}{2y}}\delta - i\beta\sqrt{\frac{y}{2}}\right) + e^{i\beta\delta} \operatorname{erf}\left(\sqrt{\frac{1}{2y}}\delta + i\beta\sqrt{\frac{y}{2}}\right) \right\} e^{-\frac{\alpha^2}{2}y} dy. \tag{3.42}$$

Let $t = \frac{y}{2}$; $y = 2t$; $dy = 2dt$. Then (3.42) becomes

$$\frac{1}{2} \int_0^\infty \left\{ e^{-i\beta\delta} \operatorname{erf}\left(\sqrt{\frac{1}{t}}\frac{\delta}{2} - i\beta\sqrt{t}\right) + e^{i\beta\delta} \operatorname{erf}\left(\sqrt{\frac{1}{t}}\frac{\delta}{2} + i\beta\sqrt{t}\right) \right\} e^{-\frac{\alpha^2}{2}2t} 2dt$$

$$= \int_0^\infty e^{-\beta^2 t} \left\{ e^{-i\delta\beta} \operatorname{erf}\left(\sqrt{\frac{1}{t}}\frac{\delta}{2} - i\beta\sqrt{t}\right) + e^{i\delta\beta} \operatorname{erf}\left(\sqrt{\frac{1}{t}}\frac{\delta}{2} + i\beta\sqrt{t}\right) \right\} e^{-(\alpha^2-\beta^2)t} dt$$

$$= \int_0^\infty e^{-\beta^2 t} \left\{ e^{-i\delta\beta} \left[1 - \operatorname{erfc}\left(\sqrt{\frac{1}{t}}\frac{\delta}{2} - i\beta\sqrt{t}\right)\right] + e^{i\delta\beta} \left[1 - \operatorname{erfc}\left(\sqrt{\frac{1}{t}}\frac{\delta}{2} + i\beta\sqrt{t}\right)\right] \right\} e^{-(\alpha^2-\beta^2)t} dt$$

$$= \int_0^\infty \left\{ e^{-i\delta\beta} + e^{i\delta\beta} \right\} e^{-\alpha^2 t} dt$$

$$-\int_0^\infty e^{-\beta^2 t}\left\{e^{-i\delta\beta}\operatorname{erfc}\left(\sqrt{\frac{1}{t}}\frac{\delta}{2}-i\beta\sqrt{t}\right)+e^{i\delta\beta}\operatorname{erfc}\left(\sqrt{\frac{1}{t}}\frac{\delta}{2}+i\beta\sqrt{t}\right)\right\}e^{-(\alpha^2-\beta^2)t}dt. \tag{3.43}$$

The first term above in (3.43) is an integral that we can evaluate by elementary methods, after the conversion of $\left\{e^{-i\delta\beta}+e^{i\delta\beta}\right\}$ into $2\cos\delta\beta$. The second integral term hinges upon another Laplace Transform pair from Roberts and Kaufman [21], pg. 87, 16.3.2, No. 13:

$$e^{-b^2 t}\left[e^{-iab}\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}-ib\sqrt{t}\right)+e^{iab}\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}+ib\sqrt{t}\right)\right]\doteq\frac{2e^{-a\sqrt{s}}}{s+b^2},$$

with the identification $b=\beta$, $a=\delta$, evaluated at $s=\alpha^2-\beta^2$. If the following conditions hold: $\operatorname{Re}\delta^2=\delta^2>0$, $\operatorname{Re}\left(\alpha^2-\beta^2\right)>-\operatorname{Re}\beta^2=-\beta^2$, the Laplace Transform identity holds. Since $\delta=z_1, z_1>0$, the first condition holds. Also,

$$\alpha^2>\beta^2 \iff \alpha^2-\beta^2>0\geq-\operatorname{Re}\left(\beta^2\right)=-\beta^2.$$

Thus, both conditions hold.

Thus, the integral, under the guise of a Laplace Transform, with $s=\alpha^2-\beta^2$, evaluates to

$$\frac{2e^{-\delta\sqrt{\alpha^2-\beta^2}}}{\alpha^2-\beta^2+\beta^2}=\frac{2e^{-\delta\sqrt{\alpha^2-\beta^2}}}{\alpha^2}.$$

Hence (3.43) equals

$$\frac{2\cos\delta\beta}{\alpha^2}-\left.\frac{2e^{-\delta\sqrt{s}}}{s+\beta^2}\right|_{s=\alpha^2-\beta^2}=\frac{2\cos\delta\beta}{\alpha^2}-\frac{2e^{-\delta\sqrt{\alpha^2-\beta^2}}}{\alpha^2-\beta^2+\beta^2}$$

$$=\frac{2\cos\delta\beta}{\alpha^2}-\frac{2e^{-\delta\sqrt{\alpha^2-\beta^2}}}{\alpha^2}$$

$$= \frac{2\cos\delta\beta - 2e^{-\delta\sqrt{\alpha^2-\beta^2}}}{\alpha^2}.$$

Substituting it back into (3.41) gives

$$\frac{e^{\beta\eta}\pi\alpha}{2\delta}\left[\cos(\beta\delta)\left(\frac{2}{\alpha^2}\right) - \frac{2\cos\delta\beta - 2e^{-\delta\sqrt{\alpha^2-\beta^2}}}{\alpha^2}\right] = \frac{e^{\beta\eta}\pi\alpha}{2\delta}\left(\frac{2e^{-\delta\sqrt{\alpha^2-\beta^2}}}{\alpha^2}\right)$$
$$= \frac{e^{\beta\eta}\pi}{\delta\alpha}\left(e^{-\delta\sqrt{\alpha^2-\beta^2}}\right).$$

Finally, we return to the integral in (3.19). Using the equalities in (3.21) through (3.25), and rearranging terms,

$$\begin{aligned}\psi(\mu) &= (\mu-1)\frac{(C_2+C_3\ln\mu)\sqrt{(\mu_1^2+\mu_2^2)}e^{-(\mu_1C_2+\mu_2C_4)}\mu^{-(\mu_1C_3+\mu_2C_5)}}{\pi}\cdot\frac{e^{\beta\eta}\pi}{\delta\alpha}\left(e^{-\delta\sqrt{\alpha^2-\beta^2}}\right)\\ &= (\mu-1)e^{-(\mu_1C_2+\mu_2C_4)}\mu^{-(\mu_1C_3+\mu_2C_5)}e^{\beta\eta}\left(e^{-\delta\sqrt{\alpha^2-\beta^2}}\right)\\ &= (\mu-1)e^{-(\mu_1C_2+\mu_2C_4)}\mu^{-(\mu_1C_3+\mu_2C_5)}e^{C_1(C_4+C_5\ln\mu)}\left[e^{-\sqrt{\mu_1^2+\mu_2^2-C_1^2}(C_2+C_3\ln\mu)}\right]\\ &= (\mu-1)e^{C_1C_4-(\mu_1C_2+\mu_2C_4)}\mu^{C_1C_5-(\mu_1C_3+\mu_2C_5)}\left[e^{-C_2\sqrt{\mu_1^2+\mu_2^2-C_1^2}}\mu^{-C_3\sqrt{\mu_1^2+\mu_2^2-C_1^2}}\right]\\ &= e^{C_1C_4-(\mu_1C_2+\mu_2C_4)-C_2\sqrt{\mu_1^2+\mu_2^2-C_1^2}}\left\{(\mu-1)\mu^{C_1C_5-(\mu_1C_3+\mu_2C_5)-C_3\sqrt{\mu_1^2+\mu_2^2-C_1^2}}\right\},\end{aligned} \tag{3.44}$$

as $\delta = C_2 + C_3\ln\mu$ and $\alpha = \sqrt{\mu_1^2+\mu_2^2}$, as given by (3.24) and (3.21), respectively.

This result holds true whether $p_2 + \frac{1}{2}a_{22} \geq p_1 + \frac{1}{2}a_{11}$ or not. This proves Proposition 3.5.3. □

Proof. (Proposition 3.5.4)

We highlight the dependence on μ by the later terms with parentheses {} in (3.44). None of the other terms preceding the brackets depend on μ . By the definitions for all constants under Proposition 3.5.3, we seek to equate the solution from Proposition 3.5.3 to the Hu, Oksendal [13] solution. We want (3.44) to be exactly identical to the

Hu, Oksendal [13] solution, which is $(\mu-1)\mu^{-\lambda}h^{\lambda}x_2$. We immediately can recognize the $(\mu-1)$ terms on both. If we can show that

$$\lambda=\frac{\frac{1}{2}\gamma-(p_2-p_1)+\sqrt{\left(p_1-p_2+\frac{1}{2}\gamma\right)^2+2\gamma p_2}}{\gamma}=C_3\sqrt{\mu_1^2+\mu_2^2-C_1^2}+\mu_1C_3+\mu_2C_5-C_1C_5,$$

and $h^{\lambda}x_2=e^{C_1C_4-(\mu_1C_2+\mu_2C_4)-C_2\sqrt{\mu_1^2+\mu_2^2-C_1^2}}$, then the two solutions are indeed identical. To achieve the former, we try to equate what is inside the square root on both expressions; we then try to equate what is outside the square root on both expressions as well. We have

$$\begin{aligned}
&\left(\mu_1^2+\mu_2^2-C_1^2\right)\\
&\quad=\mu_1^2+\mu_2^2-\left(\mu_2+\frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\right)^2\\
&\quad=\mu_1^2+\mu_2^2-\mu_2^2-2\frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\mu_2-\frac{\sigma_2^2\sigma_1^2\left(1-\rho^2\right)}{\gamma}\\
&\quad=\frac{\left(p_1+\frac{1}{2}a_{11}\right)^2-2\left(p_1+\frac{1}{2}a_{11}\right)\left(p_2+\frac{1}{2}a_{22}\right)+\left(p_2+\frac{1}{2}a_{22}\right)^2}{\gamma}\\
&\qquad-2\left\{\frac{\left[\sigma_2\sigma_1\rho-\sigma_1^2\right]\left(p_2+\frac{1}{2}a_{22}\right)+\left[\sigma_1\sigma_2\rho-\sigma_2^2\right]\left(p_1+\frac{1}{2}a_{11}\right)}{\gamma}\right\}-\frac{\sigma_2^2\sigma_1^2\left(1-\rho^2\right)}{\gamma}\\
&\quad=\frac{\left(p_1+\frac{1}{2}a_{11}\right)^2-2\left(p_1+\frac{1}{2}a_{11}\right)\left(p_2+\frac{1}{2}a_{22}\right)+\left(p_2+\frac{1}{2}a_{22}\right)^2-a_{11}a_{22}+a_{12}^2}{\gamma}\\
&\qquad-2\left\{\frac{\left[\sigma_2\sigma_1\rho-\sigma_1^2\right]\left(p_2+\frac{1}{2}a_{22}\right)+\left[\sigma_1\sigma_2\rho-\sigma_2^2\right]\left(p_1+\frac{1}{2}a_{11}\right)}{\gamma}\right\}\\
&\quad=\frac{p_1^2+p_1a_{11}+\frac{1}{4}a_{11}^2-2p_1p_2-p_1a_{22}-p_2a_{11}-\frac{1}{2}a_{11}a_{22}+p_2^2+p_2a_{22}}{\gamma}\\
&\qquad+\frac{\frac{1}{4}a_{22}^2-a_{11}a_{22}+a_{12}^2}{\gamma}-2\left\{\frac{\left(a_{12}-a_{11}\right)\left(p_2+\frac{1}{2}a_{22}\right)+\left(a_{12}-a_{22}\right)\left(p_1+\frac{1}{2}a_{11}\right)}{\gamma}\right\}\\
&\quad=\frac{p_1^2+p_1a_{11}+\frac{1}{4}a_{11}^2-2p_1p_2-p_1a_{22}-p_2a_{11}}{\gamma}\\
&\qquad+\frac{-\frac{1}{2}a_{11}a_{22}+p_2^2+p_2a_{22}+\frac{1}{4}a_{22}^2-a_{11}a_{22}+a_{12}^2}{\gamma}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{2p_2a_{12} - 2p_2a_{11} + a_{12}a_{22} - a_{11}a_{22} + 2a_{12}p_1 - 2a_{22}p_1 + a_{12}a_{11} - a_{11}a_{22}}{\gamma} \quad (3.45)\\
&= \frac{p_1^2 - 2p_1p_2 + p_2^2 + p_1a_{11} - p_1a_{22} - 2a_{12}p_1 + 2a_{22}p_1 - p_2a_{11} + p_2a_{22} - 2p_2a_{12} + 2p_2a_{11}}{\gamma}\\
&+ \frac{\frac{1}{4}a_{11}^2 - \frac{1}{2}a_{11}a_{22} + \frac{1}{4}a_{22}^2 - a_{11}a_{22} + a_{12}^2 - a_{12}a_{22} + a_{11}a_{22} - a_{12}a_{11} + a_{11}a_{22}}{\gamma}\\
&= \frac{(p_1-p_2)^2 + p_1\gamma + p_2\gamma}{\gamma} + \frac{\frac{1}{4}a_{11}^2 - \frac{1}{2}a_{11}a_{22} + \frac{1}{4}a_{22}^2 + a_{12}^2 - a_{12}a_{22} - a_{12}a_{11} + a_{11}a_{22}}{\gamma}\\
&= \frac{(p_1-p_2)^2 + p_1\gamma + p_2\gamma}{\gamma} + \frac{a_{11}^2 + 2a_{11}a_{22} + a_{22}^2 + 4a_{12}^2 - 4a_{12}a_{22} - 4a_{12}a_{11}}{4\gamma}\\
&= \frac{(p_1-p_2)^2 + p_1\gamma + p_2\gamma}{\gamma} + \frac{(a_{11} + a_{22} - 2a_{12})^2}{4\gamma}\\
&= p_1 + p_2 + \frac{(p_1-p_2)^2}{\gamma} + \frac{\gamma}{4}\\
&= \frac{\left\{\frac{\gamma^2}{4} + (p_1-p_2)^2 + (p_1+p_2)\gamma\right\}}{\gamma}\\
&= \frac{\left\{\frac{\gamma^2}{4} + (p_1-p_2)^2 + (p_1-p_2)\gamma + 2p_2\gamma\right\}}{\gamma}\\
&= \frac{\left[\frac{\gamma}{2} + (p_1-p_2)\right]^2 + 2p_2\gamma}{\gamma}. \quad (3.46)
\end{aligned}
$$

The final expression in (3.46) is non-negative, and taking the sqare root of it would not introduce any problems. Therefore,

$$
C_3\sqrt{\mu_1^2 + \mu_2^2 - C_1^2} = \frac{1}{\sqrt{\gamma}}\sqrt{\frac{\left[\frac{\gamma}{2} + (p_1-p_2)\right]^2 + 2p_2\gamma}{\gamma}} = \frac{\sqrt{\left[\frac{\gamma}{2} + (p_1-p_2)\right]^2 + 2p_2\gamma}}{\gamma}. \quad (3.47)
$$

For the λ expression, the square root term does indeed agree with the square root term in our expression.

Now, for the term outside the square root for the λ expression:

$$
C_1C_5 - \mu_1C_3 - \mu_2C_5 = (C_1 - \mu_2)C_5 - \mu_1C_3
$$

$$
\begin{aligned}
&= -\frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\frac{\sigma_2-\sigma_1\rho}{\sqrt{\gamma}}\frac{1}{\sigma_1\sqrt{1-\rho^2}} \\
&\quad - \left[\frac{\left(p_1+\frac{1}{2}a_{11}\right)-\left(p_2+\frac{1}{2}a_{22}\right)}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\right]\frac{1}{\sqrt{\gamma}} \\
&= \frac{\rho\sigma_1\sigma_2-\sigma_2^2}{\gamma} - \frac{\left(p_1+\frac{1}{2}a_{11}\right)-\left(p_2+\frac{1}{2}a_{22}\right)}{\gamma} \\
&= \frac{a_{12}-a_{22}-(p_1-p_2)-\frac{1}{2}a_{11}+\frac{1}{2}a_{22}}{\gamma} \\
&= -\frac{(p_1-p_2)}{\gamma} + \frac{-\frac{1}{2}a_{11}-\frac{1}{2}a_{22}+a_{12}}{\gamma} \\
&= -\frac{(p_1-p_2)}{\gamma} - \frac{1}{2}.
\end{aligned}
$$

Multiplying through by -1 gives the identity for the λ expression, outside its square root:

$$
-C_1C_5+\mu_1C_3+\mu_2C_5 = \frac{(p_1-p_2)}{\gamma}+\frac{1}{2} = \mu_1C_3+\mu_2C_5-C_1C_5.
$$

What remains is to show that $e^{C_1C_4-(\mu_1C_2+\mu_2C_4)-C_2\sqrt{\mu_1^2+\mu_2^2-C_1^2}}$ is equal to $h^\lambda x_2$.

We know that, from (3.46), and the definition for C_3,

$$
\sqrt{\mu_1^2+\mu_2^2-C_1^2} = \sqrt{\frac{\left[\frac{\gamma}{2}+(p_1-p_2)\right]^2+2p_2\gamma}{\gamma}}.
$$

Second,

$$
\begin{aligned}
C_1C_4-(\mu_1C_2+\mu_2C_4) &= (C_1-\mu_2)C_4-\mu_1C_2 \\
&= \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\gamma}}\frac{\sigma_2^2\ln x_1+\sigma_1^2\ln x_2-\rho\sigma_1\sigma_2\ln(x_1x_2)}{\sigma_1\sigma_2\sqrt{1-\rho^2}\sqrt{\gamma}} \\
&\quad + \left[\frac{\left(p_1+\frac{1}{2}a_{11}\right)-\left(p_2+\frac{1}{2}a_{22}\right)}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\right]\frac{\ln h}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}} \\
&= \frac{\sigma_2^2\ln x_1+\sigma_1^2\ln x_2-\rho\sigma_1\sigma_2\ln(x_1x_2)}{\gamma} \\
&\quad + \frac{\left[\left(p_1+\frac{1}{2}a_{11}\right)-\left(p_2+\frac{1}{2}a_{22}\right)\right]\ln h}{\gamma}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(a_{22}-a_{12})\ln x_1 + (a_{11}-a_{12})\ln x_2 + (p_1-p_2)\ln h}{\gamma} \\
&\qquad + \frac{\frac{1}{2}a_{11}\ln x_1 - \frac{1}{2}a_{11}\ln x_2 - \frac{1}{2}a_{22}\ln x_1 + \frac{1}{2}a_{22}\ln x_2}{\gamma} \\
&= \frac{\left(\frac{1}{2}a_{11} + \frac{1}{2}a_{22} - a_{12}\right)\ln x_1 + \left(\frac{1}{2}a_{11} + \frac{1}{2}a_{22} - a_{12}\right)\ln x_2}{\gamma} \\
&\qquad + \frac{(p_1-p_2)\ln h}{\gamma} \\
&= \frac{\frac{1}{2}\gamma\ln x_1 + \frac{1}{2}\gamma\ln x_2}{\gamma} + \frac{(p_1-p_2)\ln h}{\gamma} \\
&= \frac{1}{2}\ln x_1 + \frac{1}{2}\ln x_2 + \frac{(p_1-p_2)\ln h}{\gamma} \\
&= \frac{1}{2}\ln x_1 - \frac{1}{2}\ln x_2 + \frac{(p_1-p_2)\ln h}{\gamma} + \ln x_2 \\
&= \left(\frac{1}{2} + \frac{p_1-p_2}{\gamma}\right)\ln h + \ln x_2.
\end{aligned}
$$

Also, from the definition of C_2 on page 50, and the above identity gives

$$
\sqrt{\mu_1^2 + \mu_2^2 - C_1^2} = \sqrt{\frac{\left[\frac{\gamma}{2} + (p_1-p_2)\right]^2 + 2p_2\gamma}{\gamma}},
$$

we have

$$
\begin{aligned}
-C_2\sqrt{\mu_1^2 + \mu_2^2 - C_1^2} &= -\frac{-\ln h}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}}\sqrt{\frac{\left[\frac{\gamma}{2} + (p_1-p_2)\right]^2 + 2p_2\gamma}{\gamma}} \\
&= \frac{\ln h}{\gamma}\sqrt{\left[\frac{\gamma}{2} + (p_1-p_2)\right]^2 + 2p_2\gamma}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
C_1C_4 - &(\mu_1 C_2 + \mu_2 C_4) - C_2\sqrt{\mu_1^2 + \mu_2^2 - C_1^2} \\
&= \left(\frac{1}{2} + \frac{p_1-p_2}{\gamma}\right)\ln h + \ln x_2 + \frac{\ln h}{\gamma}\sqrt{\left[\frac{\gamma}{2} + (p_1-p_2)\right]^2 + 2p_2\gamma}
\end{aligned}
$$

$$= \left(\frac{1}{2} + \frac{p_1 - p_2}{\gamma} + \frac{\sqrt{\left[\frac{\gamma}{2} + (p_1 - p_2)\right]^2 + 2p_2\gamma}}{\gamma} \right) \ln h + \ln x_2$$

$$= \lambda \ln h + \ln x_2,$$

and $e^{C_1C_4-(\mu_1C_2+\mu_2C_4)-C_2\sqrt{\mu_1^2+\mu_2^2-C_1^2}} = e^{\lambda \ln h + \ln x_2} = h^\lambda x_2$, as desired.

Putting all the pieces back together gives

$$e^{C_1C_4-(\mu_1C_2+\mu_2C_4)-C_2\sqrt{\mu_1^2+\mu_2^2-C_1^2}} \left\{ (\mu-1)\mu^{C_1C_5-(\mu_1C_3+\mu_2C_5)-C_3\sqrt{\mu_1^2+\mu_2^2-C_1^2}} \right\}$$

$= (\mu - 1)\mu^{-\lambda} h^\lambda x_2$, which is the Hu, Oksendal [13] solution. The proof also works for our region of the parameter space, as the expression is unmodified whether $p_2 + \frac{1}{2}a_{22} \geq p_1 + \frac{1}{2}a_{11}$ or not. □

Proof. (Theorem 3.1.1)

First of all, our function $(\mu - 1)\mu^{-\lambda} h^\lambda x_2$ is well defined only when $\mu \geq 1$, since the stopping boundary must be "above" all points within the set $\{(x_2, x_1) : x_1 < x_2\}$.

Regardless of whether $p_2 + \frac{1}{2}a_{22} < p_1 + \frac{1}{2}a_{11}$ is true or not, when $0 \leq x_1 \leq \mu x_2$ from Proposition 3.5.4, we have

$$E^{(x_2,x_1)}\left[\left(X_{1,\tau_\mu} - X_{2,\tau_\mu} \right) I_{[\tau<\infty]} \right] = (\mu - 1)\mu^{-\lambda} h^\lambda x_2 = (\mu - 1) \left(\frac{\mu}{h} \right)^{-\lambda} x_2,$$

where $\lambda = \frac{1}{2} + \frac{p_1 - p_2}{\gamma} + \frac{\sqrt{\left[\frac{\gamma}{2} + (p_1 - p_2)\right]^2 + 2p_2\gamma}}{\gamma}$. Hu, Oksendal [13] showed that $\lambda > 1$. The person owning the swap option can decide on choosing any μ as the slope of the stopping boundary. He/she will choose a $\mu \geq 1$ if he/she is rational (does not want to take a loss), when a 0 pay-off is readily available. Naturally, he/she would choose a μ that maximizes the worth of the swap option, or the value of the objective function; otherwise, he/she would not be taking the supremum over all possible stopping times.

We then proceed to maximize the entire function with respect to μ. Let

$$\psi_\mu(x_2, x_1) = E^{(x_2,x_1)}\left[\left(X_{1,\tau_\mu} - X_{2,\tau_\mu}\right) I_{[\tau<\infty]}\right] = (\mu - 1)\left(\frac{\mu}{h}\right)^{-\lambda} x_2,$$

defined only when $0 < x_1 < \mu x_2$. Setting

$$\frac{d\psi_\mu}{d\mu} = (\mu - 1)\left[-\lambda\left(\frac{\mu}{h}\right)^{-\lambda-1}\right]\frac{1}{h}x_2 + \left[\left(\frac{\mu}{h}\right)^{-\lambda}\right]x_2 = 0,$$

we get $(\mu - 1)\left[-\lambda(\mu)^{-1}\right] + 1 = 0$, or that $1 = (\mu - 1)\left[\lambda(\mu)^{-1}\right]$. This gives $\mu = \frac{\lambda}{\lambda-1}$ as an interior stationary point, since $\lambda > 1$, $1 < \mu < \infty$.

To show that it is a maximum, take the second derivative:

$$\frac{d^2\psi_\mu}{d\mu^2} = -\lambda\left(\frac{\mu}{h}\right)^{-\lambda-1}\frac{1}{h}x_2 + \lambda(\lambda+1)(\mu-1)\left(\frac{\mu}{h}\right)^{-\lambda-2}\frac{1}{h^2}x_2 - \lambda\left(\frac{\mu}{h}\right)^{-\lambda-1}\frac{1}{h}x_2,$$

and evaluate it at $\mu = \frac{\lambda}{\lambda-1}$. In the above expression, each term has an h^λ, λ, and x_2 in it. Thus, we get rid of those terms and leave the sign of the above terms unchanged, since h^λ, λ, and x_2 are all positive.

After dividing the above expression by $\lambda h^\lambda x_2$, the above becomes

$$-(\mu)^{-\lambda-1} + (\lambda+1)(\mu-1)(\mu)^{-\lambda-2} - (\mu)^{-\lambda-1}.$$

Further manipulation leads to

$$\begin{aligned}
-(\mu)^{-\lambda-1} - (\mu)^{-\lambda-1} + (\lambda+1)(\mu-1)(\mu)^{-\lambda-2} &= -2(\mu)^{-\lambda-1} + (\lambda+1)(\mu-1)(\mu)^{-\lambda-2} \\
&= (\mu)^{-\lambda-2}\left[-2\mu + (\lambda+1)(\mu-1)\right] \\
&= (\mu)^{-\lambda-2}\left[-2\mu + (\lambda+1)\mu - (\lambda+1)\right] \\
&= (\mu)^{-\lambda-2}\left[-1\mu + (\lambda)\mu - (\lambda+1)\right]
\end{aligned}$$

$$= (\mu)^{-\lambda-2}\left[(\lambda-1)\mu-(\lambda+1)\right]$$
$$= (\mu)^{-\lambda-2}\left[\lambda-(\lambda+1)\right]$$
$$= -(\mu)^{-\lambda-2} < 0.$$

This indicates our stationary point above is at a maximum. In fact, since there is only one stationary point, it is an unique global interior maximum. A quick check on the boundary conditions verifies the last statement. At $\mu = 0$,

$$\psi_\mu(x_2, x_1) = (0-1)\left(\frac{0}{h}\right)^{-\lambda} x_2 = -\infty,$$

as $\lambda > 1$. At $\mu = \infty$,

$$\psi_\mu(x_2, x_1) = \lim_{\mu\to\infty}(\mu-1)\left(\frac{\mu}{h}\right)^{-\lambda} x_2 = 0,$$

as $\lambda > 1$. Additionally, $\psi_\mu(x_2, x_1) = 0$ at $\mu = 1$, whereas, $\psi_\mu(x_2, x_1) = (\mu-1)\left(\frac{\mu}{h}\right)^{-\lambda} x_2 > 0$, as $\mu, \lambda > 1$, and $\mu \geq h$ over $S^c \cup \partial S$, where this function is defined. Thus, as long as λ is greater than 1, $\mu^* = \frac{\lambda}{\lambda-1}$. This holds true whether $p_2 + \frac{1}{2}a_{22} < p_1 + \frac{1}{2}a_{11}$ holds or not, as the above evaluation does not depend on this condition. Notice also that this maximum does not depend on the location of (x_2, x_1). □

The ψ_μ function is maximized at the stationary point $\mu^* = \frac{\lambda}{\lambda-1}$. Thus, the Hu, Oksendal [13] solution in their parameter regime also carries this property that the slope for their stopping boundary is "optimal" in the above sense, (i.e., that it maximizes the objective function inside the region $\{(x_2, x_1) : 0 < x_1 < x_2\}$. Our proof is also valid for their case, under the additional restriction that

$$0 \leq \left|\mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\right| < \sqrt{\mu_1^2+\mu_2^2}.$$

Thus, since our integral approach works as long as restrictions (i) and (ii) of Theorem 3.1.1 is satisfied, there is no need to treat the parameter restrictions $p_1+\frac{1}{2}a_{11} \leq p_2+\frac{1}{2}a_{22}$, and $p_1 + \frac{1}{2}a_{11} > p_2 + \frac{1}{2}a_{22}$, as two separate cases, and we obtain Theorem 3.1.1.

4 THE OPTIMAL STOPPING BOUNDARY FOR BOTH THE HU, OKSENDAL [13] AND OUR EXTENDED PROBLEMS

4.1 The optimal stopping boundary in the extended problem

4.1.1 Goal of Chapter 4:

In the previous chapter, we found the value of μ^* of μ, such that stopping the process at time τ_{μ^*}, when $\frac{X_{1,t}}{X_{2,t}}$ first equals μ^*, gives the maximal value of the objective function over all stopping times of the form τ_μ. In this chapter, we will show that the optimal such τ_μ maximizes the objective function over all possible stopping times, so that the optimal policy in general is to stop when a certain straight-line boundary is hit. (As noted in Chapter 3, the hitting time of this boundary may not be a.s. finite.)

4.1.2 Definitions and the Main Theorem for Chapter 4:

We make an additional definition that will streamline our exposition:

Definition 4.1.1. *For a stopping time τ, let $V_\tau \equiv (X_{1,\tau} - X_{2,\tau}) I_{[\tau<\infty]}$, so that the objective function is $\sup_\tau E^{(x_2,x_1)}(V_\tau) \equiv \sup_\tau E^{(x_2,x_1)}[(X_{1,\tau} - X_{2,\tau}) I_{[\tau<\infty]}]$. We extend the permissible class of stopping times to allow for the value ∞ to be taken with positive probability.*

Condition 4.1.2

We assume that we are in the following parameter regime for the entire proof of Theorem 4.1.1 below:

i) $p_1 > 0, p_2 > 0$,

ii) $p_2 + \frac{1}{2}a_{22} < p_1 + \frac{1}{2}a_{11}$,

iii) $0 \leq \left| \mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_!\sigma_2\rho}} \right| < \sqrt{\mu_1^2 + \mu_2^2}$.

The objective of this chapter is to prove the following:

Theorem 4.1.1. *If we change condition (ii) under Section 3.1 to* $p_1 + \frac{1}{2}a_{11} > p_2 + \frac{1}{2}a_{22}$, *and we make the additional restriction* $0 \leq \left|\mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\right| < \sqrt{\mu_1^2+\mu_2^2}$, *the solution to the program in our region of the parameter space is also*
$\sup_\tau E^{(x_2,x_1)}\{(X_{1,\tau} - X_{2,\tau})I_{[\tau<\infty]}\} = \frac{(\mu^*-1)}{u^{*\lambda}}x_1^\lambda x_2^{1-\lambda}$, *whenever* $(x_2, x_1) \in S^C$,
$\sup_\tau E^{(x_2,x_1)}\{(X_{1,\tau} - X_{2,\tau})I_{[I<\infty]}\} = x_1 - x_2$, *whenever* $(x_2, x_1) \in S$,
with $S = \{(x_2, x_1) : x_1 \geq \mu^* x_2\}$, *and* $S^C = \{(x_2, x_1) : x_1 < \mu^* x_2\}, \mu^* = \frac{\lambda}{\lambda-1}$.

The constants involved in the above expressions are as in Chapter 3:

$$\lambda = \left(\frac{1}{2} + \frac{p_1 - p_2}{\gamma} + \frac{\sqrt{[\frac{\gamma}{2} + (p_1 - p_2)]^2 + 2p_2\gamma}}{\gamma}\right)$$

where $\gamma = (q_{11} - q_{21})^2 + (q_{12} - q_{22})^2$, *as defined in the Hu, Oksendal [13] paper. In addition,* $\sigma_1 = \sqrt{a_{11}}, \sigma_2 = \sqrt{a_{22}}, \rho = \frac{a_{12}}{\sqrt{a_{11}a_{22}}}$,

$$\begin{aligned}
\mu_1 &= \left[\frac{(p_1 + \frac{1}{2}a_{11}) - (p_2 + \frac{1}{2}a_{22})}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}}\right] = \frac{(p_1 + \frac{1}{2}a_{11}) - (p_2 + \frac{1}{2}a_{22})}{\sqrt{\gamma}}, \text{and} \\
\mu_2 &= \frac{(\sigma_2\sigma_1\rho - \sigma_1^2)(p_2 + \frac{1}{2}a_{22}) + (\sigma_2\sigma_1\rho - \sigma_2^2)(p_1 + \frac{1}{2}a_{11})}{\sigma_1\sigma_2\sqrt{1-\rho^2}\sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}} \\
&= \frac{(a_{12} - a_{11})(p_2 + \frac{1}{2}a_{22}) + (a_{12} - a_{22})(p_1 + \frac{1}{2}a_{11})}{\sqrt{a_{11}a_{22}}\sqrt{1 - \frac{a_{12}^2}{a_{11}a_{22}}}\sqrt{\gamma}} \\
&= \frac{(a_{12} - a_{11})(p_2 + \frac{1}{2}a_{22}) + (a_{12} - a_{22})(p_1 + \frac{1}{2}a_{11})}{\sqrt{a_{11}a_{22} - a_{12}^2}\sqrt{\gamma}}.
\end{aligned} \tag{4.1}$$

Proof of Theorem 4.1.1.

The following Propositions and Lemmas together prove Theorem 4.1.1.

Proposition 4.1.2. . *Let* $(x_2, x_1) \in \mathbb{R}_+^2$. *Then* $(x_2, x_1) \in S$ *iff* $(\eta x_2, \eta x_1) \in S \ \forall \eta > 0$.

Proof. If a point $(x_2, x_1) \in \mathbb{R}^2_+$ belongs to S, then $\sup_\tau E^{(x_2,x_1)} V_\tau = x_1 - x_2$. Any point on the ray emanating from the origin going through (x_2, x_1) has the coordinate description of $(\eta x_2, \eta x_1)$, for some $\eta > 0$.

Thus, for any $\eta > 0$, with I_E denoting the indicator function for event E,

$$\begin{aligned}
\sup_\tau E^{(\eta x_2,\eta x_1)} V_\tau &= \sup_\tau E^{(\eta x_2,\eta x_1)} \{(X_{1,\tau} - X_{2,\tau}) I_{[\tau<\infty]}\} \\
&= \sup_\tau E\{[\eta x_1 e^{-(p_1+\frac{1}{2}a_{11})\tau + q_{11}B_{1,\tau} + q_{12}B_{2,\tau}} \\
&\qquad - \eta x_2 e^{-(p_2+\frac{1}{2}a_{22})\tau + q_{21}B_{1,\tau} + q_{22}B_{2,\tau}}] I_{[\tau<\infty]}\} \\
&= \eta \sup_\tau E^{(x_2,x_1)} \{[x_1 e^{-(p_1+\frac{1}{2}a_{11})\tau + q_{11}B_{1,\tau} + q_{12}B_{2,\tau}} \\
&\qquad - x_2 e^{-(p_2+\frac{1}{2}a_{22})\tau + q_{21}B_{1,\tau} + q_{22}B_{2,\tau}}] I_{[\tau<\infty]}\} \\
&= \eta \sup_\tau E^{(x_2,x_1)} \{(X_{1,\tau} - X_{2,\tau}) I_{[\tau<\infty]}\} \\
&= \eta \sup_\tau E^{(x_2,x_1)} V_\tau \\
&= \eta(x_1 - x_2) \\
&= \eta x_1 - \eta x_2.
\end{aligned}$$

Hence, $(\eta x_2, \eta x_1)$ also belongs to S. The converse is immediate. Take $\eta = 1$. □

Note: Observe that whenever a point (x_2, x_1) belongs in S, $\sup_\tau E^{(x_2,x_1)} V_\tau = x_1 - x_2 < \infty$.

Corollary 4.1.3. *A ray emanating from the origin, as described by the equation* $x_1 = \mu x_2, \mu \in [0, \infty]$ *is either contained entirely in* S *or* S^C.

Remark 4.1.1. *Lemma 3.2.1 states that, if* $0 < x_1 < x_2$*, then the policy that specifies* $P(\tau = 0) > 0$ *cannot be the optimal policy. Thus, the optimal policy or* ϵ*-close-to-optimal policy (one with small enough* ϵ*) must specify* $P(\tau = 0) = 0$ *when the above condition holds. Therefore, the set* $\{(x_2, x_1) : 0 < x_1 < x_2\}$ *belongs entirely in* S^c*, and the region will not contain any point of* S*, or* ∂S*. While looking for* ∂S *then, possible stopping* *boundaries from the class of stopping boundary of the form $\{(x_2, x_1) : x_1 = \mu x_2\}$ must exclude μ's in the range of $0 < \mu < 1$, because these lines contain points in the $0 < x_1 < x_2$ region.*

Definition 4.1.4. *Let $E^{(x_2,x_1)}V_{\tau_\mu} = \psi_\mu(x_2, x_1)$ when $0 < x_1 \leq \mu x_2$. Let us call this function the $\mu-$policy choice function.*

Lemma 4.1.5. *Both λ and the optimal value μ^* are greater than 1.*

Proof. Hu, Oksendal [13] proved that $\lambda > 1$ (See p 302 in Hu, Oksendal[13]). The corresponding $\mu^* > 1$ follows immediately because $\mu^* = \frac{\lambda}{\lambda-1}$. Even under our parameter regime with a different set of parameter constraints, the same result holds because $p_2 > 0$ still.

□

Proposition 4.1.6. *For all points such that $0 < x_1 < \mu x_2$, $E^{(x_1,x_2)}V_{\tau_\mu} = \psi_\mu(x_1, x_2) > x_1 - x_2$, for any $1 \leq \mu \leq \mu^*$. In particular, the above holds when $\mu = \mu^*$.*

Proof. The following proof contains essentially the same idea as in Hu, Oksendal[13], only it does not rely on the smooth-pasting condition. So, it requires a slightly weaker set of conditions to get the same result.

For every x_2, there exists a point $(x_2, \mu x_2) \in \mathbb{R}^2_+$ s.t. the point is on the line described by the equation $x_1 = \mu x_2$. Let $h = \frac{x_1}{x_2}$.

First, let $1 < \mu \leq \mu^*$. At $x_1 = \mu x_2$, $h = \mu$. Therefore,
$\psi_\mu(x_2, x_1) = (\mu - 1)(\frac{\mu}{h})^{-\lambda} x_2 = (\mu - 1)(\frac{\mu}{\mu})^{-\lambda} x_2 = (\mu - 1)x_2 = x_1 - x_2$. Now, look at the region $0 < x_1 < \mu x_2$, where $\frac{x_1}{x_2} = h < \mu$. Fix x_2, and let $x_1 = 0^+$. Then,

$$\begin{aligned}\lim_{x_1 \downarrow 0} \psi_\mu(x_2, x_1) &= \lim_{x_1 \downarrow 0} (\mu - 1) \left(\frac{\mu}{h}\right)^{-\lambda} x_2 \\ &= \lim_{x_1 \downarrow 0} (\mu - 1) \left(\frac{x_2 \mu}{x_1}\right)^{-\lambda} x_2\end{aligned}$$

$$= \lim_{x_1 \downarrow 0} (\mu - 1) \left(\frac{x_1}{x_2 \mu} \right)^{\lambda} x_2 = 0,$$

since $\lambda > 1$. The continuity for ψ_μ then implies that, at $x_1 = 0^+, \psi_\mu > 0$ is arbitrarily close to 0. Define $\phi(x_2, x_1) = \psi_\mu(x_2, x_1) - (x_1 - x_2)$. ϕ is a continuous function and $\lim_{x_1 \to 0} \phi(x_2, x_1) = \lim_{x_1 \to 0} \psi_\mu(x_2, x_1) - \lim_{x_1 \to 0} (x_1 - x_2) = 0 - (-x_2) = x_2$.

At $x_1 = \mu x_2$, $\phi(x_2, \mu x_2) = \psi_\mu(x_2, \mu x_2) - (\mu x_2 - x_2) = x_1 - x_2 - (x_1 - x_2) = 0$;

$$\begin{aligned}
\frac{\partial \phi}{\partial x_1} &= \frac{\partial \psi_\mu}{\partial x_1} - 1 \\
&= \frac{\partial (\mu - 1) \left(\frac{\mu}{h}\right)^{-\lambda} x_2}{\partial x_1} - 1 \\
&= \frac{\partial (\mu - 1) \left(\frac{x_2 \mu}{x_1}\right)^{-\lambda} x_2}{\partial x_1} - 1 \\
&= \frac{\partial}{\partial x_1} \frac{(\mu - 1)}{\mu^\lambda} x_1^\lambda x_2^{1-\lambda} - 1 \\
&= \lambda \frac{(\mu - 1)}{\mu^\lambda} x_2^{1-\lambda} x_1^{\lambda - 1} - 1 \\
&= \frac{\mu^*}{\mu^* - 1} \frac{(\mu - 1)}{\mu^\lambda} h^{\lambda - 1} - 1 \\
&= \frac{\frac{\mu^*}{\mu^* - 1}}{\frac{\mu}{\mu - 1}} \left(\frac{h}{\mu}\right)^{\lambda - 1} - 1.
\end{aligned}$$

For $1 < \mu \leq \mu^*$, $\frac{\frac{\mu^*}{\mu^*-1}}{\frac{\mu}{\mu-1}} < 1$, and $\frac{\frac{\mu^*}{\mu^*-1}}{\frac{\mu}{\mu-1}} \left(\frac{h}{\mu}\right)^{\lambda-1} - 1 < 1 - 1 = 0$, because $\lambda > 1$ also.

Thus, $\frac{\partial \phi}{\partial x_1} < 0$ when $h < \mu$ and $1 < \mu \leq \mu^*$.

As long as (x_2, x_1) is such that $0 < x_1 < \mu x_2$, ϕ is a strictly monotonically decreasing function of x_1, with x_2 fixed. The ϕ function starts out positive with a value arbitrarily close to x_2 at $(x_2, 0^+)$, then monotonically decreases to 0 as x_1 increases to μx_2. The continuity and monotonicity of ϕ then establish that ϕ must be > 0 for all $x_1 \in (0, \mu x_2)$, and therefore $\psi_\mu(x_2, x_1) > (x_1 - x_2)$ for all $x_1 \in (0, \mu x_2)$. Since $x_2 \in \mathbb{R}_+^2$ was arbitrary, $\psi_\mu(x_2, x_1) > (x_1 - x_2)$ for all (x_2, x_1) such that $0 < x_1 < \mu x_2$.

The above argument applies to all $1 < \mu \leq \mu^*$, and the Proposition applies to $\mu = \mu^*$

in particular. For $\mu = 1, \forall(x_2, x_1) : x_1 < \mu x_2 = x_2, \psi_\mu(x_2, x_1) = (1-1)h^\lambda x_2 = 0 > x_1 - x_2$. Thus, the proposition applies to values of $\mu \in [1, \mu^*]$.

□

Now, we can claim that if ∂S is of the form $\{(x_2, x_1) : x_1 = \mu x_2\}$, then μ must be $\geq \mu^*$.

Corollary 4.1.7. $\{(x_2, x_1) : 0 < x_1 < \mu^* x_2\} \subseteq S^C$.

Proof. If $0 < x_1 < \mu^* x_2$, by Proposition 4.1.6,

$$\sup_\tau E^{(x_2,x_1)} V_\tau \geq \sup_\mu E^{(x_2,x_1)} V_\tau = \psi_{\mu^*}(x_2, x_1) > x_1 - x_2.$$

For all points s.t $0 < x_1 < \mu^* x_2$, we have $(x_2, x_1) \in S^C$, by Definition 1.4.3.

□

Proposition 4.1.8. *Consider the collection of stopping times $C(\tau_\infty)$ specifying stopping boundaries that are any subset of the union of the following lines: $x_1 = 0, x_2 = 0, x_1 = \infty, x_2 = \infty$. None of the policies in $C(\tau_\infty)$ can be optimal. In fact, any policy that gives $\tau = \infty$ a.s cannot be optimal.*

Proof. In order for the process to hit the boundaries $x_1 = 0$, or $x_2 = 0$, $(p_i + \frac{1}{2}a_{ii})\tau - q_{i1}B_{1,\tau} - q_{i2}B_{1,\tau} = \infty$ a.s. for either $i = 1, 2$. The joint process will not hit these sets in finite time a.s. The same can be said for $x_1 = \infty, x_2 = \infty$, only that the condition is now $(p_i + \frac{1}{2}a_{ii})\tau - q_{i1}B_{1,\tau} - q_{i2}B_{1,\tau} = -\infty$, but the condition can only possibly hold over a $P^{(x_2,x_1)}$-null set. Thus, the joint process will not be able to hit any part of the four boundaries in the first orthant, $x_1 = 0, x_2 = 0, x_1 = \infty, x_2 = \infty$ with positive $P^{(x_2,x_1)}$-measure, unless $\tau = \infty$.

Thus, if we let $\Gamma(\tau_\infty)$ denote the class of policies that prescribe $\tau = \infty$ a.s, then $C(\tau_\infty) \subseteq \Gamma(\tau_\infty)$. Properties proved for $\Gamma(\tau_\infty)$ also apply to $C(\tau_\infty)$. For $\tau \in \Gamma(\tau_\infty)$, $\tau = \infty$ leads to an objective function value that is equal to 0 a.s. But the τ_{μ^*}-policy has objective function, with $\mu^* > 1$,

$$E^{(x_2,x_1)}[V_{\tau_{\mu^*}}] = (\mu^* - 1)(\frac{h}{\mu^*})^\lambda x_2, \text{ whenever } 0 < x_1 < \mu^* x_2,$$

$$E^{(x_2,x_1)}[V_{\tau_{\mu^*}}] = x_1 - x_2, \text{ whenever } 0 < \mu^* x_2 \leq x_1.$$

Both are > 0 everywhere within their respective ranges inside $\mathbb{R}^2_+$. In fact, any τ_μ policy, with $\mu > 1$, will deliver strictly positive objective function values over the entire state-space. Thus, policies that give $\tau = \infty$ a.s. will not be optimal, as the τ_{μ^*}-policy gives an objective function that delivers strictly higher values everywhere.

□

Lemma 4.1.9. *The continuation set* $S^C \neq \mathbb{R}^2_+$.

Proof. Suppose the contrary, $S^C = \mathbb{R}^2_+$. Let τ be the optimal or ϵ-close-to-optimal policy for (x_2, x_1). The probability $P^{(x_2,x_1)}$ [stop at τ] $= E^{(x_2,x_1)}[P^{(x_2,x_1)}$ (stop at $\tau)|\mathcal{F}_\tau] = E^{(X_{2,\tau},X_{1,\tau})}[P^{(x_2,x_1)}$ (stop at 0)] $= E^{(X_{2,\tau},X_{1,\tau})}[0] = 0$, because $S^C = \mathbb{R}^2_+$, where we have invoked the strong Markov Property in the second equality above. The penultimate equality is due to Lemma 1.5.3, because no optimal or ϵ-close-to-optimal policy would stop at time 0 with any positive probability. If our optimal stopping region is empty, then the process would not stop at any finite τ with positive probability. This means that $\Gamma(\tau_\infty)$ would then contain the optimal policy or policies, because they are the only policies that would be undertaken with any positive probability. They all deliver an a.s. 0 pay-off. As once again, the τ_{μ^*} policy delivers strictly positive pay-offs everywhere, these $\Gamma(\tau_\infty)$ policies are not optimal. Thus, a contradiction has been arrived at. The policy or policies that specify $S^C = \mathbb{R}^2_+$ as the stopping set are then necessarily sub-optimal. Any optimal or ϵ-close-to-optimal policy with a small enough ϵ would necessarily prescribe an optimal stopping region s.t. $S^C \neq \mathbb{R}_+^2$.

$\square$

Remark 4.1.2. *Thus, the stopping set of this problem contains at least one point that belongs to* $\mathbb{R}_+^2$.

From the last lemma, there exists at least one point within $\mathbb{R}_+^2$ *that belongs to* S. *By Proposition 4.1.2, the entire ray emanating from the origin, through the interior point that belongs to* S, *also belongs to* S. *Supposing a stopping boundary between the sets* S *and* S^c *that is not of this form will quickly lead to a contradiction. This leads to the following Lemma.*

Lemma 4.1.10. ∂S *is either a straight line of the form* $\{(x_2, x_1) : x_1 = \mu x_2\}$, *or it is a union of such lines.*

Let $L_\mu = \{(x_2, x_1) : x_1 = \mu x_2\}$. Call a point (x_2, x_1) in $\mathbb{R}_+^2$, "above L_μ", when $x_1 > \mu x_2$. Write $\partial S = \cup_{\{\mu : L_\mu \subseteq \partial S\}} L_\mu$.

Proposition 4.1.11. *If* $L_\mu \subseteq \partial S$, *then for all* $(x_2, x_1) \in \{(x_2, x_1) : x_1 > \mu x_2\}, (x_2, x_1) \in S$.

Proof. Let η be an arbitrary real number greater than 1; suppose $(x_2, x_1) \in \partial S$.

$$\sup_\tau E^{(x_2,\eta x_1)}[(X_{1,\tau} - X_{2,\tau})I_{[\tau<\infty]}] = \sup_\tau E^{(x_2,x_1)}[(\eta\mu - 1)X_{2,\tau}I_{[\tau<\infty]}]$$
$$= \sup_\tau E^{(x_2,x_1)}[(\eta\mu - \mu + \mu - 1)X_{2,\tau}I_{[\tau<\infty]}]$$
$$= \sup_\tau E^{(x_2,x_1)}\{[(\eta - 1)\mu + (\mu - 1)]X_{2,\tau}I_{[\tau<\infty]}\}$$
$$\leq \sup_\tau E^{(x_2,x_1)}\{[\eta - 1)\mu]X_{2,\tau}I_{[\tau<\infty]}\} + \sup_\tau E^{(x_2,x_1)}\{[(\mu - 1)]X_{2,\tau}I_{[\tau<\infty]}\}$$
$$= \sup_\tau E^{(x_2,x_1)}\{(X_{1,\tau} - X_{2,\tau})I_{[\tau<\infty]}\} + \sup_\tau(\eta - 1)E^{(x_2,x_1)}\{\mu X_{2,\tau}I_{[\tau<\infty]}\}$$
$$= \sup_\tau E^{(x_2,x_1)}\{(X_{1,\tau} - X_{2,\tau})I_{[\tau<\infty]}\} + \sup_\tau(\eta - 1)E^{(x_2,x_1)}\{X_{1,\tau}I_{[\tau<\infty]}\}$$
$$= \sup_\tau E^{(x_2,x_1)}\{(X_{1,\tau} - X_{2,\tau})I_{[\tau<\infty]}\} + (\eta - 1)x_1$$

$= x_1 - x_2 + (\eta - 1)x_1 = \eta x_1 - x_2.$

Since the objective function is greater than or equal to the stopping pay-off everywhere, $\sup_{\tau} E^{(x_2,\eta x_1)}[(X_{1,\tau} - X_{2,\tau})I_{[\tau<\infty]}] = \eta x_1 - x_2$. Since (x_2, x_1) is an arbitrary point on ∂S, and η is an arbitrary real number > 1, all points in $\mathbb{R}^2_+$ above L_μ belong to S.

□

Proposition 4.1.12. *∂S consists of a single line emanating from the origin.*

Proof. Lemma 4.1.9 says that there exists at least one interior point that is in S. Therefore, by Corollary 4.1.3, the entire ray emanating from the origin through that interior point is in S. Let us call the slope of this line h. In this manner, perhaps there might be more than one line in S that is of this form. What matters is that there is at least one. This collection must come from lines with slope in $[\mu^*, \infty]$, by Corollary 4.1.7, since lines with slope in $[0, \mu^*)$ belong within the region $\{(x_2, x_1) : 0 \le x_1 < \mu^* x_2\} \subseteq S^C$. In this region, $\sup_{\tau} E^{(x_2,x_1)}[V_\tau] > x_1 - x_2$ so lines with slope in $[0, \mu^*)$ cannot belong to either S or ∂S.

Let L_μ denote the line given by the equation $x_1 = \mu x_2$ in $\mathbb{R}^2_+$.

Let SL be the set of $\mu's$ such that $L_\mu \subseteq S$. Since $h = \frac{x_1}{x_2}$, over $L_h, \sup_{\tau} E^{(x_2,x_1)}[V_\tau] = x_1 - x_2$.

Consider $\inf_{\mu} SL$. Since $L_h \subseteq S$, $\inf_{\mu} SL \le h$. At the same time, since none of the lines with slope $\mu \in [0, \mu^*)$ belong in S, $\inf_{\mu}$ SL $\ge \mu^*$. Thus, $h \ge \mu^*$. If $h > \mu^*$, $\inf_{\mu} SL$ $\in [\mu^*, h]$, and $\mu_{min} = \inf_{\mu} SL$ exists within the interval, because S is closed. The line $L_{\mu_{min}}$ is a boundary for S. Since S is closed, $L_{\mu_{min}}$ belongs to S, which means that μ_{min} belongs to SL, and it exists within the said interval. If $h = \mu^*$, then since none of the L_μ with $\mu \in [0, \mu^*)$ can belong in S, $\mu_{min} = \mu^* = h$. Either way, μ_{min} exists, and $L_{\mu_{min}}$ belongs to S. It is clear that $h < \mu^*$ cannot hold. $\sup_{\tau} E^{(x_2,x_1)}[V_\tau] = x_1 - x_2$ over the line $L_{\mu_{min}}$, and Proposition 4.1.11 says that the region $\{(x_2, x_1) : x_1 > \mu_{min} x_2)\}$ belongs entirely in S.

Take any $L_{\mu'}$ s.t. $0 < \mu' < \mu_{min}$. Then $L_{\mu'}$ by definition is not contained in S, and therefore must be contained in S^C. The region $\{(x_2, x_1) : 0 < x_1 < \mu_{min} x_2\}$ thus is contained in S^C. Then clearly, $\{(x_2, x_1) : x_1 = \mu_{min} x_2\}$ is the unique line that is the unique stopping boundary ∂S to this problem.

□

Note: At this point, we can conclude that $\sup_\tau E^{(x_2,x_1)}[(X_{1,\tau} - X_{2,\tau}) I_{[\tau<\infty]}] < \infty$. If (x_2, x_1) is in S, then the note under Proposition 4.1.2 says that $\sup_\tau E^{(x_2,x_1)}[(X_{1,\tau} - X_{2,\tau}) I_{[\tau<\infty]}] < \infty$. If the point belongs to S^c instead, then $\sup_\tau E^{(x_2,x_1)}[(X_{1,\tau} - X_{2,\tau}) I_{[\tau<\infty]}] = \psi_{\mu_{min}}(x_2, x_1) < \infty$, for any (x_2, x_1) in the initial state space, since $1 \le \mu^* \le \mu_{min} < \infty$.

Proposition 4.1.13. $\mu_{min} = \mu^*$.

Proof. It is clear from Proposition 4.1.12 that the stopping boundary in this problem is of the form $\{(x_2, x_1) : x_1 = \mu_{min} x_2\}$. Thus, $\sup_\tau E^{(x_2,x_1)}[V_\tau] = \sup_\mu E^{(x_2,x_1)}[V_{\tau_\mu}] = E^{(x_2,x_1)}[V_{\tau_{\mu^*}}]$.

At any point within the region $\{(x_2, x_1) : 0 < x_1 < x_2\} \subseteq S^C$, we can choose an optimal stopping policy that maximizes the objective function, based on maximizing the objective function with ∂S of this form. Over that region, all τ_μ-policies s.t. $\mu \ge 1$ are eligible for consideration as the optimal policy. By Theorem 3.1.1, τ_{μ^*} is the unique optimal policy. By the uniqueness of ∂S shown in the proof of Proposition 4.1.12, $\{(x_2, x_1) : x_1 = \mu^* x_2\}$ is therefore the ∂S in this problem. □

Lemma 4.1.14. $S = \{(x_2, x_1) : x_1 \ge \mu^* x_2\}$; $S^C = \{(x_2, x_1) : x_1 < \mu^* x_2\}$.

Proof. For all the points in the set $\{(x_2, x_1) : x_1 = \mu^* x_2\}$, $\sup_\tau E^{(x_2,x_1)} V_\tau = x_1 - x_2$, as shown in the proof of Proposition 4.1.6. Stopping immediately is optimal. In fact,

$$\partial S = \{(x_2, x_1) : x_1 = \mu^* x_2\},$$

by Proposition 4.1.13. Proposition 4.1.11 says that all points above the line are in S. The proof for Proposition 4.1.6 says that all points below the line are in S^C. Thus, we can readily identify $\{(x_2, x_1) : x_1 \geq \mu^* x_2\} = S$, because each point within the set satisfies $\sup_\tau E^{(x_2,x_1)} V_\tau = x_1 - x_2$, but each point outside the set does not.

For all points in the set $\{(x_2, x_1) : 0 < x_1 < \mu^* x_2\}$,

$$\sup_\tau E^{(x_2,x_1)}[V_\tau] = (\mu^* - 1)\left(\frac{\mu^*}{h}\right)^{-\lambda} x_2 > x_1 - x_2$$

by Theorem 3.1.1, Proposition 4.1.6, and Proposition 4.1.13. This set is S^c. All points within this set satisfy the inequality condition over the last statement. All points outside do not. $\square$

Lemma 4.1.14 identifies S and S^c as given in the statement of Theorem 4.1.1. Over S, $\sup_\tau E^{(x_2,x_1)} V_\tau = x_1 - x_2$. This is just a re-statement of its definition. Over S^c, $\sup_\tau E^{(x_2,x_1)}[V_\tau] = (\mu^* - 1)\left(\frac{\mu^*}{h}\right)^{-\lambda} x_2$, as given by Theorem 3.1.1 and Proposition 4.1.13, as the theorem also gives $\mu^* = \frac{\lambda}{\lambda - 1}$.

However, we never used condition ii) under Condition 4.1.2 in this proof. We have only used condition iii) when we proved Theorem 3.1.1. Therefore, we can cite the next theorem with identical results, but different conditions on the parameter space as the main result of this Chapter:

Theorem 4.1.2. *When the parameter space satisfies the following conditions:*

i) $p_1 > 0, p_2 > 0$,

ii) $0 < \left|\mu_2 + \frac{\sigma_2 \sigma_1 \sqrt{1-\rho^2}}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_! \sigma_2 \rho}}\right| < \sqrt{\mu_1^2 + \mu_2^2}$,

the unique solution to the program in our region of the parameter space is

$$\sup_\tau E^{(x_2,x_1)}\left\{(X_{1,\tau} - X_{2,\tau})\boldsymbol{I}_{[\tau<\infty]}\right\} = \begin{cases} \frac{(\mu_1 - 1)}{\mu^{\lambda_1}} x_1^{\lambda_1} x_2^{1-\lambda_1} & \textit{whenever} \quad (x_2, x_1) \in S^c \\ x_1 - x_2 & \textit{whenever} \quad (x_2, x_1) \in S. \end{cases}$$

4.1.3 Comparison between Hu, Oksendal's [13] solution and the solution for the extended problem

The problem investigated in this thesis is the following: find the optimal τ, if it exists, for the following problem:

$\sup_{\tau\in\mathcal{Y}_0} E^{(x_2,x_1)}\{(X_{1,\tau}-X_{2,\tau})I_{[\tau<\infty]}+0\cdot I_{[\tau=\infty]}\} = \sup_{\tau\in\mathcal{Y}_0} E^{(x_2,x_1)}\{(X_{1,\tau}-X_{2,\tau})I_{[\tau<\infty]}\}$

$$
\begin{aligned}
\text{s.t.}\qquad dX_{1,t} &= -p_1X_{1.t}dt + q_{11}X_{1,t}dB_{1,t} + q_{12}X_{1,t}dB_{2,t},\\
dX_{2,t} &= -p_2X_{2.t}dt + q_{21}X_{2,t}dB_{1,t} + q_{22}X_{2,t}dB_{2,t},\\
X_{1,0} &= x_1,\\
X_{2,0} &= x_2,\\
x_1, x_2, &> 0,\\
p_1, p_2 &> 0,\\
0 \le q_{11}, q_{12}&, q_{21}, q_{22} < \infty,\\
p_1+\frac{1}{2}a_{11} \le p_2+\frac{1}{2}a_{22} &\text{ or } p_1+\frac{1}{2}a_{11} > p_2+\frac{1}{2}a_{22},
\end{aligned}
$$

and $\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$ is non-singular, over the following restrictions on the parameters:

i) $p_1 > 0, p_2 > 0,$

ii) $0 < \left|\mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_!\sigma_2\rho}}\right| < \sqrt{\mu_1^2+\mu_2^2},$

with $a_{11} = q_{11}^2 + q_{12}^2$, $a_{22} = q_{21}^2 + q_{22}^2$, $a_{12} = q_{11}q_{21} + q_{12}q_{22}$, $\gamma = a_{11} + a_{22} - 2a_{12}$. Give also the objective function value $\sup\limits_{\tau\in\mathcal{Y}_0} E^{(x_2,x_1)}\{(X_{1,\tau}-X_{2,\tau})I_{[\tau<\infty]}\}$.

We have successfully obtained Hu, Oksendal's[13] solution using a different method, under the additional restriction ii) over and above Hu, Oksendal's [13].

In addition, we also obtained the same solution when the corresponding condition under Section 1.3 was modified to $p_1 + \frac{1}{2}a_{11} > p_2 + \frac{1}{2}a_{22}$.

In as much as $g(x) = \frac{x}{x-1}$ is a monotonically decreasing function over $x \in (1, \infty]$, we see immediately that when $p_1 + \frac{1}{2}a_{11} > p_2 + \frac{1}{2}a_{22}$, and when p_1's value is elevated relative to p_2's while there is no change in $q_{11}, q_{12}, q_{21}, q_{22}, \lambda = \left(\frac{1}{2} + \frac{p_1 - p_2}{\gamma} + \frac{\sqrt{[\frac{\gamma}{2} + (p_1 - p_2)]^2 + 2p_2\gamma}}{\gamma} \right)$ would tend to increase, but only when $p_1 \geq p_2$. This gives a decrease in μ. The swaption holder would rather exercise over a shallower slope with lower sets of net profits than when p_1's value is not as elevated relative to p_2. This is consistent with the intuition that as the revenue term's mean trajectory damps down faster than the cost's mean trajectory, the swaption holder would not want to bypass too many positive-profit opportunities once they present themselves. Waiting around longer in hopes of bigger pay-offs might not be the optimal strategy because revenues, on average, are decreasing at a more rapid rate than costs.

When a_{11} increases relative to a_{22}, γ increases. This makes λ decrease, as it can be re-written as $\lambda = \left(\frac{1}{2} + \frac{p_1 - p_2}{\gamma} + \sqrt{\left[\frac{1}{2} + \frac{(p_1 - p_2)}{\gamma}\right]^2 + \frac{2p_2}{\gamma}} \right)$, but, once again, only when $p_1 \geq p_2$. This in turn, makes μ increase. Having higher volatility in revenue affords the swaption holder higher probabilities of realizing large revenue at the tails of the revenue distribution, and it therefore pays to be more patient in waiting for these large revenue opportunities, because there is a higher probability that they would occur. The same can be said with increases in a_{22}, without any change in a_{11} or a_{12}. The swaption holder can afford to be more patient in waiting for more extreme low costs to occur because their probability of occurrence has just increased. Also, when processes are less positively correlated, high revenues occur more often with low costs and low revenues occur more often with high costs. Swaption holders would gladly wait longer for the former's joint occurrence and therefore would choose to exercise later for a larger set of pay-offs.

However, the result is ambiguous when $p_1 < p_2$. λ can increase or decrease as p_1 or γ increases.

In this chapter, we concern ourselves primarily with the situation when $p_1 + \frac{1}{2}a_{11} >$

$p_2 + \frac{1}{2}a_{22}$. If our elevated $p_1 + \frac{1}{2}a_{11}$ is due to increases in p_1, we as swaption holders would tend to exercise sooner and the realized set of stopping payoffs would tend to be lower. This conforms to the intuition outlined in Section 3.3. but only when $p_1 \geq p_2$. If the increase in value for the term on the right side of the inequality $p_2 + \frac{1}{2}a_{22} < p_1 + \frac{1}{2}a_{11}$ is due to increases in revenue volatility a_{11}, then when $p_1 \geq p_2$, our swaption holders would tend to exercise later and the set of realized stopping pay-offs would tend to be higher. However, when $p_1 < p_2$, the result is indeterminate without knowing further what the other parameter values are. Thus, the intuition outlined in Section 3.3 is only partially true, only when $p_1 \geq p_2$ and when p_1's value is elevated with respect to p_2.

5 WHAT HAPPENS WHEN WE MIMIC CHLADNA'S [9] PARAMETERS IN OUR PROBLEM?

5.1 Parameter restriction modification

Here, we explore the same questions regarding the stochastic problem mentioned in **1.3**, but with different restrictions. We want to let $p_1 = r > 0$, but $p_2 < 0$ as we want to model Chladna's [9] severe damage costs scenario due to carbon emissions. Damage caused by accelerating rates of carbon emissions to society could exponentiate upwards. A central planner looking at our objective function would also internalize these costs. The condition $p_1 = r > 0$ here mimics mean-reverting revenue behavior. In other words, we want the price growth to be 0 so as to mimic the behavior of a timber revenue pattern that always fluctuates about a long term mean, even as damage costs' mean trajectory exponentiates out to infinity in time. See Chladna [9].

5.1.1 Examination of Terminal Condition for New Parameter Restrictions

We investigate what happens to the following limits as $t \to \infty$:

$T = \lim_{t\to\infty} \left\{ x_1 e^{-\left(p_1+\frac{1}{2}a_{11}\right)t+q_{11}B_{1,t}+q_{12}B_{2,t}} - x_2 e^{-\left(p_2+\frac{1}{2}a_{22}\right)t+q_{21}B_{1,t}+q_{22}B_{2,t}} \right\} = 0 \quad a.s.$ if $a_{22} > -2p_2$, and

$T = \lim_{t\to\infty} \left\{ x_1 e^{-\left(p_1+\frac{1}{2}a_{11}\right)t+q_{11}B_{1,t}+q_{12}B_{2,t}} - x_2 e^{-\left(p_2+\frac{1}{2}a_{22}\right)t+q_{21}B_{1,t}+q_{22}B_{2,t}} \right\} = -\infty \quad a.s.$ if $a_{22} < -2p_2$.

The limit does not exist when $a_{22} = -2p_2$. Thus, it is natural to divide this problem into sub-regions as to whether $a_{22} > -2p_2$ or not.

However, as we consider the carbon sequestration problem, if foresters choose never to cut down their trees ($\tau = \infty$), then they incur no carbon penalty. In this case, foresters never got the timber revenue and the land would not grow more than one generation of trees for carbon sequestration, so the marginal benefits and marginal costs to both foresters and society are zero. Thus, it makes more sense that if over the events $[\tau = \infty]$ the person owning the pay-off to the objective function takes a zero pay-off for the problem in these parameter regimes as well.

Thus, let there be the following cases:

Case Ia: $p_1 > 0,\ p_2 < 0,$ and $0 < p_1 + \frac{1}{2}a_{11} \leq p_2 + \frac{1}{2}a_{22},\ T = 0,$ (within the Θ_{EXTHO}.)

Case Ib: $p_1 > 0,\ p_2 < 0,$ and $p_1 + \frac{1}{2}a_{11} > p_2 + \frac{1}{2}a_{22} > 0,\ T = 0,$ (within the Θ_{EXTC}.)

Case II: $p_1 > 0,\ p_2 < 0,$ and $p_1 + \frac{1}{2}a_{11} > 0 > p_2 + \frac{1}{2}a_{22},\ T = 0,$ (within the Θ_{EXTC}.)

The union of parameter sub-spaces under Case Ib and II is Θ_{EXTC}, as defined on page 11. Let us try to solve each case in turn.

5.2 Discussion of the Cases:

Let $g(x_2, x_1) = x_1 - x_2$.

For Case Ia, even though $p_2 < 0$, $p_2 + \frac{1}{2}a_{22} \geq p_1 + \frac{1}{2}a_{11} > 0$ is still possible as long as a_{22} is positive enough. Since $p_1 + \frac{1}{2}a_{11} \leq p_2 + \frac{1}{2}a_{22}$, we can use the free boundary pde method of Hu, Oksendal [13] to solve the problem in this region of the parameter space.

According to Hu, Oksendal [13], the joint conditions $L\psi_{\mu^*} = 0$, whenever $(x_2, x_1) \in S^c_{\mu^*}$, and $\psi_{\mu^*} = x_1 - x_2$, and $\nabla\psi_{\mu^*} = \nabla(x_1 - x_2)$ whenever $x_1 = \mu^* x_2$, give the following solution for ψ_{μ^*} : $\psi_{\mu^*} = \dfrac{1}{\lambda}\mu^{*1-\lambda}x_1^{\lambda}x_2^{1-\lambda} = \dfrac{\mu^*-1}{\mu^*}\mu^{*1-\lambda}\left(\dfrac{x_1}{x_2}\right)^{\lambda}x_2 = \dfrac{\mu^*-1}{\mu^{*\lambda}}h^{\lambda}x_2$, where $\lambda = \dfrac{\frac{1}{2}\gamma + (p_1 - p_2) + \sqrt{\left(p_1 - p_2 + \frac{1}{2}\gamma\right)^2 + 2\gamma p_2}}{\gamma}$, and $\mu^* = \dfrac{\lambda}{\lambda - 1}$.

Looking at the above solution, because $x_1, x_2 > 0$, if $\frac{1}{\lambda}\mu^{*1-\lambda} = \frac{\mu^* - 1}{\mu^*}\mu^{*1-\lambda} = \frac{1}{\lambda}\Big(\frac{\lambda}{\lambda-1}\Big)^{1-\lambda} < 0$, then the policy-choice function ψ_{μ^*} must be negative. This can happen when $\lambda < 1$. The $\lambda > 1$ condition is then necessary in order for ψ_{μ^*} to have the correct sign. Thus, we look to satisfy this condition in this region of the parameter space as well.

For Cases 1b and 2, the free boundary pde method will not necessarily give us the correct solution; we must rely on our integral method to give a solution that applies to part of the space as given above in the statement for the Cases (we will see that an additional constraint over and above those in Chapters 3 and 4 will be needed). The integral method does give the same functional form as the solution given in Case 1a over S^c, and therefore, the same comment about the need for $\lambda > 1$ also applies. Thus, examining the value for λ is crucial.

First, we have to make sure we are dealing with real roots for the radical in the λ expression:

$\Big(p_1 - p_2 + \frac{1}{2}\gamma\Big)^2 + 2\gamma p_2 = (p_1 - p_2)^2 + (p_1 - p_2)\gamma + \frac{1}{4}\gamma^2 + 2\gamma p_2 = (p_1 - p_2)^2 + (p_1 + p_2)\gamma + \frac{1}{4}\gamma^2 = (p_1 + p_2)^2 + (p_1 + p_2)\gamma + \frac{1}{4}\gamma^2 - 4p_1p_2 = \Big(p_1 + p_2 + \frac{1}{2}\gamma\Big)^2 - 4p_1p_2 > 0$, since $p_1 > 0, p_2 < 0, \gamma > 0$. This condition applies throughout this Chapter.

Let $\lambda_1 = \dfrac{\frac{1}{2}\gamma + p_1 - p_2 + \sqrt{\Big(p_1 - p_2 + \frac{1}{2}\gamma\Big)^2 + 2\gamma p_2}}{\gamma}$,

and $\lambda_2 = \dfrac{\frac{1}{2}\gamma + p_1 - p_2 - \sqrt{\Big(p_1 - p_2 + \frac{1}{2}\gamma\Big)^2 + 2\gamma p_2}}{\gamma}$.

We look at possible range restrictions on both λ_1, λ_2.

Lemma 5.2.1. *The root that takes the minus sign, λ_2, is always less than 1.*

Proof. Without loss of generality, let $p_1 = a\gamma$, $p_2 = -b\gamma$, with $a, b > 0$. The root that takes the negative sign, λ_2, can then be written as

$$\lambda_2 = \frac{1}{2} + \frac{p_1 - p_2 - \sqrt{\left(\frac{1}{2}\gamma + p_1 - p_2\right)^2 + 2\gamma p_2}}{\gamma} = \frac{1}{2} + \frac{a\gamma + b\gamma - \sqrt{\left(\frac{1}{2} + a + b\right)^2 \gamma^2 - 2\gamma b\gamma}}{\gamma}$$
$$= \frac{1}{2} + a + b - \sqrt{\left(\frac{1}{2} + a + b\right)^2 - 2b}.$$

We want to see if $a + b - \sqrt{\left(\frac{1}{2} + a + b\right)^2 - 2b}$ can exceed the value of $\frac{1}{2}$ or not. Note that

$$\sqrt{\left(\frac{1}{2} + a + b\right)^2 - 2b} = \sqrt{a^2 + 2a\left(b + \frac{1}{2}\right) + \left(b + \frac{1}{2}\right)^2 - 2b}$$
$$= \sqrt{a^2 + 2a\left(b + \frac{1}{2}\right) + \left(b - \frac{1}{2}\right)^2} > 0.$$

Suppose

$$\lambda_2 \geq 1 \Leftrightarrow a + b - \sqrt{\left(\frac{1}{2} + a + b\right)^2 - 2b} \geq \frac{1}{2} \Leftrightarrow a + b - \frac{1}{2} \geq \sqrt{\left(\frac{1}{2} + a + b\right)^2 - 2b}$$
$$\Leftrightarrow \left(a+b-\frac{1}{2}\right)^2 \geq \left(\frac{1}{2}+a+b\right)^2 - 2b = (a+b)^2 + (a+b) + \frac{1}{4} - 2b = (a+b)^2 + (a-b) + \frac{1}{4}$$
$$\Leftrightarrow (a + b)^2 - (a + b) + \frac{1}{4} = (a + b)^2 - a - b + \frac{1}{4} \geq (a + b)^2 + (a - b) + \frac{1}{4}$$
$$\Leftrightarrow -a \geq a$$

The last line indicates a contradiction; therefore, $\lambda_2 < 1$ as a is strictly positive because both p_1 and γ are. Over all parameter values $p_1 > 0, p_2 < 0, \gamma > 0$ this result holds true and therefore applies to the entire chapter. □

Note that as $\lambda_2 < 1$, the discussion under Section 5.2 says that λ_2 will not be the solution chosen to calculate the slope of the optimal boundary: μ^*, the slope of the optimal boundary, will not be equal to $\frac{\lambda_2}{\lambda_2 - 1}$.

Lemma 5.2.2. *Over Case 1a, 1b, and 2, when* $p_1 + p_2 + \frac{1}{2}\gamma > 0$*, we have* $\lambda_1 > 1$.

Proof. By its definition,

$$\lambda_1 = \frac{\frac{1}{2}\gamma + p_1 - p_2 + \sqrt{\left(p_1 - p_2 + \frac{1}{2}\gamma\right)^2 + 2\gamma p_2}}{\gamma}$$

$$= \frac{1}{2} + \frac{(p_1 - p_2) + \sqrt{\left(p_1 - p_2 + \frac{1}{2}\gamma\right)^2 + 2\gamma p_2}}{\gamma}$$
$$= \frac{1}{2} + \frac{(p_1 - p_2) + \sqrt{(p_1 - p_2)^2 + (p_1 - p_2)\gamma + \frac{1}{4}\gamma^2 + 2\gamma p_2}}{\gamma}$$
$$= \frac{1}{2} + \frac{(p_1 - p_2) + \sqrt{(p_1 - p_2)^2 + (p_1 + p_2)\gamma + \frac{1}{4}\gamma^2}}{\gamma}$$
$$= \frac{1}{2} + \frac{(p_1 - p_2) + \sqrt{(p_1 + p_2)^2 + (p_1 + p_2)\gamma + \frac{1}{4}\gamma^2 - 4p_1p_2}}{\gamma}$$
$$= \frac{1}{2} + \frac{(p_1 - p_2) + \sqrt{\left[(p_1 + p_2) + \frac{1}{2}\gamma\right]^2 - 4p_1p_2}}{\gamma}$$
$$> \frac{1}{2} + \frac{(p_1 - p_2) + \sqrt{\left[(p_1 + p_2) + \frac{1}{2}\gamma\right]^2}}{\gamma}.$$

If $p_1 + p_2 + \frac{1}{2}\gamma > 0$, then

$$\frac{(p_1 - p_2) + \sqrt{\left[(p_1 + p_2) + \frac{1}{2}\gamma\right]^2}}{\gamma} = \frac{(p_1 - p_2) + p_1 + p_2 + \frac{1}{2}\gamma}{\gamma} = \frac{2p_1}{\gamma} + \frac{1}{2} > \frac{1}{2}.$$

Thus, $\lambda_1 > 1 + \frac{2p_1}{\gamma} > 1$. □

If $p_1 + p_2 + \frac{1}{2}\gamma \leq 0$, we cannot assert that $\lambda_1 > 1$.

As already mentioned in the 2nd paragraph under Section 5.2, Hu, Oksendal's [13] free boundary pde approach can still be used for the problem in Case 1a. The following section is the solution according to their method.

5.3 Solution to Case Ia.

Proposition 5.3.1. *Let the parameter restrictions be as in Case Ia. When* $p_1 + p_2 + \frac{1}{2}\gamma > 0$*, the solution to this problem in this region of the parameter space is then given by*

$$\sup_\tau E^{(x_2,x_1)}\Big\{(X_{1,\tau} - X_{2,\tau})\boldsymbol{I}_{[\tau<\infty]}\Big\} = \begin{cases} \frac{(\mu_1-1)}{\mu^{\lambda_1}} x_1^{\lambda_1} x_2^{1-\lambda_1} & \textit{whenever} \quad (x_2, x_1) \in S^c \\ x_1 - x_2 & \textit{whenever} \quad (x_2, x_1) \in S \end{cases} \tag{5.1}$$

where $S = \Big\{(x_2, x_1) : x_1 \geq \mu_2 x_2\Big\}$, $S^c = \Big\{(x_2, x_1) : x_1 < \mu_2 x_2\Big\}$, $\mu_1 = \dfrac{\lambda_1}{\lambda_1 - 1}$.

Proof. Considering the parameter restrictions jointly, we note that if $a_{12} \leq 0$, $p_1 + p_2 + \frac{1}{2}\gamma = p_1 + \frac{1}{2}a_{11} + p_2 + \frac{1}{2}a_{22} - a_{12} \geq p_1 + \frac{1}{2}a_{11} + p_2 + \frac{1}{2}a_{22} > 0$, when $p_2 + \frac{1}{2}a_{22} \geq p_1 + \frac{1}{2}a_{11} > 0$. This is possible as random components of $X_{1,t}, X_{2,t}$ can be uncorrelated or negatively correlated. Further, we can impose restrictions $p_1 > 0, p_2 < 0$ without making the restricted parameter space of Lebesgue measure zero, or as we say in this thesis, without making the restricted parameter space trivial. The necessary condition, that $\gamma = a_{11} + a_{22} - 2a_{12} > 0$, does not contradict any of the above restrictions.

Thus, when $p_1 > 0, p_2 < 0, p_2 + \frac{1}{2}a_{22} \geq p_1 + \frac{1}{2}a_{11}$, and $p_1 + p_2 + \frac{1}{2}\gamma > 0$, we have $\lambda_1 > 1$, while $\lambda_2 < 1$. We verify the conditions from Hu, Oksendal's [13] paper to see if (5.1) is indeed the solution to the optimal stopping problem.

So far, we have used (5.2) to (5.4) to derive the ψ_μ function:

$$\psi_\mu \in C^2\big(S^c\big) \quad \text{and} \quad L\psi_\mu = 0 \text{ over } S^c; \tag{5.2}$$

$$\psi_\mu = x_1 - x_2 \quad \text{when} \quad x_1 = \mu x_2; \tag{5.3}$$

$$\nabla\psi_\mu = \nabla(x_1 = x_2) \quad \text{when} \quad x_1 = \mu x_2. \tag{5.4}$$

We can also invoke Proposition 4.1.6, with $\mu^* = \frac{\lambda_1}{\lambda_1 - 1}$, to get the following result:

$$\psi_{\mu^*}(x_2, x_1) > x_1 - x_2 \quad \text{when} \quad 0 < x_1 < \mu^* x_2. \tag{5.5}$$

The remaining conditions that we need to verify for the verification theorem are as follows:

$$\mathcal{L}(x_1 - x_2) \leq 0 \quad \text{when} \quad x_1 > \mu^* x_2. \tag{5.6}$$

$$\Big\{\psi_{\mu^*}\big(X_{2,\tau}, X_{1,\tau}\big)\Big\}_{\tau \in \Upsilon_{S^c}} \text{is uniformly integrable w.r.t. } P^{(x_2,x_1)}, \forall (x_2, x_1) \in S^c.$$

(Υ_{S^c} is the set of all bounded stopping times $\tau \leq \tau_{S^c}$, where

$$\tau_{S^c} = \inf\{t \geq 0 : X_{1,t} = \mu^* X_{2,t}\}.) \tag{5.7}$$

$$\tau_{S^c} \text{ is a.s. } P^{(x_2,x_1)} \text{ finite } \forall (x_2, x_1) \in R_+^2. \tag{5.8}$$

Verification for (5.6):

$$\mathcal{L}(x_1 - x_2) = -p_1 x_1 + p_2 x_2 < 0, \text{ since } p_1, x_1, x_2 > 0, \text{ and } p_2 < 0.$$

Verification for (5.7):
$\psi_{\mu^*}\left(X_{2,\tau}, X_{1,\tau}\right) = (\mu^* - 1)\left(\frac{\mu^*}{h}\right)^{-\lambda} x_2.$

Using Theorem C.3 in the Appendix of Oksendal [18], pick an $r > 1$. Then,

$$\begin{aligned}
E^{(x_2,x_1)}\left\{\left[\psi_{\mu^*}\left(X_{2,\tau}, X_{1,\tau}\right)\right]^r\right\} &= E^{(x_2,x_1)}\left\{\left[\frac{\mu^*-1}{\mu^*}X_{1,\tau}\right]^r\right\} = \left(\frac{\mu^*-1}{\mu^*}\right)^r E^{(x_2,x_1)}\left\{X_{1,\tau}\right\} \\
&= \left(\frac{\mu^*-1}{\mu^*}\right)^r E^{(x_2,x_1)}\left\{x_1^r e^{-r\left(p_1+\frac{1}{2}a_{11}\right)\tau+q_{11}rB_{1,\tau}+q_{12}rB_{2,\tau}}\right\} \\
&= \left(\frac{\mu^*-1}{\mu^*}\right)^r x_1^r E^{(x_2,x_1)}\left\{e^{-r\left(p_1+\frac{1}{2}a_{11}\right)\tau+q_{11}rB_{1,\tau}+q_{12}rB_{2,\tau}}\right\} \\
&= \left(\frac{\mu^*-1}{\mu^*}\right)^r x_1^r \times \\
&E^{(x_2,x_1)}\left\{e^{-r\left(p_1+\frac{1}{2}a_{11}\right)\tau+r^2\left(p_1+\frac{1}{2}a_{11}\right)\tau-r^2\left(p_1+\frac{1}{2}a_{11}\right)\tau+q_{11}rB_{1,\tau}+q_{12}rB_{2,\tau}}\right\} \\
&= \left(\frac{\mu^*-1}{\mu^*}\right)^r x_1^r \times \\
&E^{(x_2,x_1)}\left\{e^{-r(1-r)\left(p_1+\frac{1}{2}a_{11}\right)\tau-r^2\left(p_1+\frac{1}{2}a_{11}\right)\tau+q_{11}rB_{1,\tau}+q_{12}rB_{2,\tau}}\right\} \\
&= \left(\frac{\mu^*-1}{\mu^*}\right)^r x_1^r \times \\
&E^{(x_2,x_1)}\left\{e^{-r\left[(1-r)\left(p_1+\frac{1}{2}a_{11}\right)+rp_1\right]\tau-r^2\left(\frac{1}{2}a_{11}\right)\tau+q_{11}rB_{1,\tau}+q_{12}rB_{2,\tau}}\right\}
\end{aligned}$$

$$\leq \left(\frac{\mu^*-1}{\mu^*}\right)^r x_1^r E^{(x_2,x_1)}\left\{e^{-r^2\left(\frac{1}{2}a_{11}\right)\tau+q_{11}rB_{1,\tau}+q_{12}rB_{2,\tau}}\right\}$$
$$= \left(\frac{\mu^*-1}{\mu^*}\right)^r x_1^r E^{(x_2,x_1)}\left\{\lim_{k\to\infty} e^{-r^2\left(\frac{1}{2}a_{11}\right)(\tau\wedge k)+q_{11}rB_{1,\tau\wedge k}+q_{12}rB_{2,\tau\wedge k}}\right\}$$
$$\leq \left(\frac{\mu^*-1}{\mu^*}\right)^r x_1^r \liminf_k E^{(x_2,x_1)}\left\{e^{-r^2\left(\frac{1}{2}a_{11}\right)(\tau\wedge k)+q_{11}rB_{1,\tau\wedge k}+q_{12}rB_{2,\tau\wedge k}}\right\}$$
$$= \left(\frac{\mu^*-1}{\mu^*}\right)^r x_1^r \liminf_k 1 = \left(\frac{\mu^*-1}{\mu^*}\right)^r x_1^r < \infty.$$

where the first inequality above holds if $r(1-r)\left(p_1+\frac{1}{2}a_{11}\right)+r^2p_1>0$, or if

$$r\left(p_1+\frac{1}{2}a_{11}\right)-r^2p_1-\frac{r^2a_{11}}{2}+r^2p_1 = r\left(p_1+\frac{1}{2}a_{11}\right)-\frac{r^2a_{11}}{2}$$

$$= rp_1+\frac{a_{11}}{2}(r-r^2)>0$$

$\Leftrightarrow p_1+\frac{a_{11}}{2}(1-r)>0 \Leftrightarrow \frac{a_{11}}{2}(1-r)>-p_1 \Leftrightarrow 1-r>-p_1\frac{2}{a_{11}} \Leftrightarrow r<1+\frac{2p_1}{a_{11}}$.

As $\frac{2p_1}{a_{11}}>0$, $1+\frac{2p_1}{a_{11}}>1$, and we can choose a suitable r such that $1<r<1+\frac{2p_1}{a_{11}}$, and $\sup_{\tau\in\Upsilon_{S^c}} E^{(x_2,x_1)}\left\{\left[\psi_{\mu^*}\left(X_{2,\tau},X_{1,\tau}\right)\right]^r\right\} \leq \left(\frac{\mu^*-1}{\mu^*}\right)^r x_1^r < \infty$. Theorem C.3 in Oksendal's [18] appendix then says that $\left\{\psi_{\mu^*}\left(X_{2,\tau},X_{1,\tau}\right)\right\}_{\tau\in\Upsilon_{S^c}}$ is uniformly integrable, as $e^{-r^2\left(\frac{1}{2}a_{11}\right)\tau\wedge k+q_{11}rB_{1,\tau\wedge k}+q_{12}rB_{2,\tau\wedge k}}$ is a $\mathcal{F}_{\tau\wedge k}$ exponential martingale indexed by k with $E^{(x_2,x_1)}\left\{e^{-r^2\left(\frac{1}{2}a_{11}\right)\tau\wedge k+q_{11}rB_{1,\tau\wedge k}+q_{12}rB_{2,\tau\wedge k}}\right\} = 1$, by Bhattacharya and Waymire [2]. That together with Fatou's Lemma gives the result.

Verification for (5.8):

The condition $p_1+\frac{1}{2}a_{11}\leq p_2+\frac{1}{2}a_{22}$ still guarantees the a.s. finiteness of τ_{S^c}, by the law of the iterated logarithm. See Hu, Oksendal [13].

With all the requirements for the verification theorem satisfied, (5.1) above is the solution to this problem, as given in the Hu, Oksendal [13] paper, by use of the verification theorem.

□

5.4 Results for Case 1b:

The parameter restrictions are $p_1 > 0, p_2 < 0$, and $p_1 + \frac{1}{2}a_{11} > p_2 + \frac{1}{2}a_{22} > 0$, $T = 0$. For this class of problem, the verification theorem does not guarantee the free boundary pde approach would work. We resort to our integration method. In order to use that method, we need to impose the additional condition that $0 \leq \left|\mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\right| < \sqrt{\mu_1^2+\mu_2^2}$. To guarantee that $\lambda_1 > 1$, we need to impose an additional condition $p_1 + p_2 + \frac{1}{2}\gamma > 0$. Thus, we impose the following four restrictions for the problem in this section:

i) $p_1 > 0, p_2 < 0$,

ii) $p_1 + \frac{1}{2}a_{11} > p_2 + \frac{1}{2}a_{22} > 0$,

iii) $0 \leq \left|\mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\right| < \sqrt{\mu_1^2+\mu_2^2}$,

iv) $p_1 + p_2 + \frac{1}{2}\gamma > 0$.

As in Section 5.3, we can impose conditions i), ii), and iv) jointly without trivializing the restricted parameter space. This means that there is a possibility that even after adding restriction iii), our solution would still apply to a non-trivial restricted parameter space.

Now we add the final restriction iii); as shown in Section 3.1.4, condition (iii) is true iff

$$\begin{aligned}&2a_{12}\Big(p_2 + \frac{1}{2}a_{22} + p_1 + \frac{1}{2}a_{11}\Big) - a_{12}^2 \\ &\quad < (p_1 - p_2)^2 + (p_1 + p_2)(a_{11} + a_{22}) + \Big(\frac{1}{2}a_{11} + \frac{1}{2}a_{22}\Big)^2,\end{aligned}$$

$\Leftrightarrow$

$$
\begin{aligned}
&2a_{12}\Big(p_2 + \frac{1}{2}a_{22} + p_1 + \frac{1}{2}a_{11} - \frac{a_{12}}{2}\Big)\\
&\qquad < (p_1 - p_2)^2 + (p_1 + p_2)(a_{11} + a_{22}) + \Big(\frac{1}{2}a_{11} + \frac{1}{2}a_{22}\Big)^2\\
&\qquad = (p_1 + p_2)^2 + (p_1 + p_2)(a_{11} + a_{22}) + \Big(\frac{1}{2}a_{11} + \frac{1}{2}a_{22}\Big)^2 - 4p_1p_2\\
&\qquad = \Big(p_1 + p_2 + \frac{1}{2}a_{11} + \frac{1}{2}a_{22}\Big)^2 - 4p_1p_2,
\end{aligned}
$$

$\Leftrightarrow$

$$
\begin{aligned}
&2a_{12}\Big(p_2 + \frac{1}{2}a_{22} + p_1 + \frac{1}{2}a_{11} - \frac{a_{12}}{2}\Big)\\
&= 2a_{12}\Big(p_2 + \frac{1}{2}a_{22} + p_1 + \frac{1}{2}a_{11} - a_{12} + \frac{a_{12}}{2}\Big)\\
&= 2a_{12}\Big(p_2 + p_1 + \frac{\gamma}{2} + \frac{a_{12}}{2}\Big) < \Big(p_1 + p_2 + \frac{1}{2}a_{11} + \frac{1}{2}a_{22}\Big)^2 - 4p_1p_2.
\end{aligned}
$$

As we impose the condition $p_2 < 0$, both terms on the right are positive, and the term on the left is negative whenever $a_{12} < 0$, and $p_2 + p_1 + \frac{\gamma}{2} + \frac{a_{12}}{2} > 0$. Imposing the condition $p_2 + p_1 + \frac{\gamma}{2} > 0$, we note that as long as $-p_1 - p_2 - \frac{1}{2}\gamma < \frac{a_{12}}{2} < 0$, $\Big(p_2 + p_1 + \frac{\gamma}{2} + \frac{a_{12}}{2}\Big)$ is still positive. This guarantees $2a_{12}\Big(p_2 + p_1 + \frac{\gamma}{2} + \frac{a_{12}}{2}\Big) < \Big(p_1 + p_2 + \frac{1}{2}a_{11} + \frac{1}{2}a_{22}\Big)^2 - 4p_1p_2$, or condition iii), with iv) imposed. At this juncture, imposing $p_1 + \frac{1}{2}a_{11} > p_2 + \frac{1}{2}a_{22} > 0$ and $p_1 > 0$ pose no necessary contradiction.

Thus, this restricted parameter space is once again non-trivial, and therefore, our theorem has a meaningful extension to this non-trivial part of the parameter space. Condition (iv) guarantees that within this restricted parameter space, $\lambda_1 > 1$.

As stated above, Lemmas 5.2.1, and 5.2.2 also apply to this region of the parameter space as well.

Proposition 5.4.1. *When the following conditions are satisfied:*

i) $p_1 > 0, p_2 < 0$,

ii) $p_1 + \frac{1}{2}a_{11} > p_2 + \frac{1}{2}a_{22} > 0$, *iii)* $0 \leq \left| \mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}} \right| < \sqrt{\mu_1^2+\mu_2^2}$,

iv) $p_1 + p_2 + \frac{1}{2}\gamma > 0$;

the optimal stopping boundary for this problem is a single straight line emanating from the origin.

Proof. We need to verify the single straight line boundary is still the solution to this problem.

The discussion right before this Proposition states that when conditions i) to iv) are satisfied, $\lambda_1 > 0$ necessarily. We checked our case against lemmas, propositions, and theorems developed in Chapters 3 and 4. We found that:

Proposition 4.1.2 and Corollary 4.1.3 still hold.

Lemma 4.1.5 holds because of condition iv) above guaranteeing that $\lambda > 1$.

In Proposition 4.1.6, we deal only with functions ψ_μ and $g(x_2, x_1) = x_1 - x_2$, and with λ, thus $\mu^* > 1$, the proposition continues to hold. Corollary 4.1.7 then follows immediately.

From the discussion above, there exists a non-trivial region in the parameter space such that restrictions i) to iv) apply. Thus, $\lambda_1 > 1$. All theorems under Section 3.2 of Chapter 3 hold, except for Proposition 3.2.4. But Proposition 3.2.4 still holds true for $i = 1$, even in this region of the parameter space. Lemmas 3.2.5 , 3.2.6, and Proposition 3.2.7 continue to hold. Also, the remark that follows Lemma 3.2.6 continues to hold. Propositions 3.5.1, 3.5.2, 3.5.3, 3.5.4, and Theorem 3.1.1 continue to hold, as long as we are working in the region where $\lambda_1 > 1$. From Theorem 3.1.1 then, we know that

$$\sup_\mu E^{(x_2,x_1)}\Big\{(X_{1,\tau_\mu} - X_{2,\tau_\mu})\mathbf{I}_{[\tau_\mu<\infty]}\Big\} = \begin{cases} \frac{(\mu_1-1)}{\mu^{\lambda_1}} x_1^{\lambda_1} x_2^{1-\lambda_1} & \text{whenever} \quad (x_2,x_1) \in S^c \\ x_1 - x_2 & \textit{whenever} \quad (x_2,x_1) \in S. \end{cases}$$

Proposition 4.1.8 also holds, because we still have an objective function equal to zero as $\tau = \infty$, and a τ_{μ^*} (with Theorem 3.1.1 still valid) policy still giving positive pay-offs everywhere.

Lemma 4.1.9 holds, and Lemma 4.1.10 immediately follows. Proposition 4.1.11 then holds. Proposition 4.1.12 follows.

Since Lemma 4.1.9 still holds, there exists at least one line of the form $\{(x_2, x_1) : x_1 = hx_2\} \subseteq S$. Lemma 4.1.9 holding means that $\inf_{\mu} SL$, as defined in Proposition 4.1.12, exists, and is within $[1, h]$. Proposition 4.1.13 still holds because our parameterization guarantees that $\lambda > 1$, and the stopping boundary is still the unique line of the form $\{(x_2, x_1) : x_1 = \mu^* x_2\}$.

□

Proposition 5.4.2. *When our parameter space satisfies*

i) $p_1 > 0, p_2 < 0,$

ii) $p_1 + \frac{1}{2}a_{11} > p_2 + \frac{1}{2}a_{22} > 0,$

iii) $0 < \left|\mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\right| < \sqrt{\mu_1^2+\mu_2^2},$

iv) $p_1 + p_2 + \frac{1}{2}\gamma > 0,$

the solution to the optimization problem in this thesis is given by

$$\sup_{\tau} E^{(x_2,x_1)}\Big\{(X_{1,\tau} - X_{2,\tau})\boldsymbol{I}_{[\tau<\infty]}\Big\} = \begin{cases} \frac{(\mu_1-1)}{\mu^{\lambda_1}} x_1^{\lambda_1} x_2^{1-\lambda_1} & \textit{whenever} \quad (x_2, x_1) \in S^c \\ x_1 - x_2 & \textit{whenever} \quad (x_2, x_1) \in S, \end{cases}$$

where $S = \{(x_2, x_1) : x_1 \geq \mu_1 x_2\}$, $S^c = \{(x_2, x_1) : x_1 < \mu_1 x_2\}$, $\mu_1 = \dfrac{\lambda_1}{\lambda_1 - 1}$.

Proof. Since $\lambda_1 > 1$, Proposition 4.1.13 holds. Lemma 4.1.14 also holds because the proof of Proposition 4.1.6 still applies, with the optimum $\mu = \mu^*$ because the conclusions from Proposition 4.1.11, Proposition 4.1.12, and 4.1.13 still hold. The statement of this proposition then follows. □

5.5 Results for Case 2:

As in Section 5.3, because we are outside the Hu, Oksendal [13] parameter regime, we need to impose the additional restriction, $0 \leq \left|\mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\right| < \sqrt{\mu_1^2+\mu_2^2}$, to utilize the integral approach. Lemmas 5.2.1 to 5.2.3 continue to hold for this section as well. We also need to impose the additional restriction that $p_1 + p_2 + \frac{1}{2}\gamma > 0$ to guarantee that the condition $\lambda_1 > 1$ holds.

The restrictions in this section are then

i) $p_1 > 0, p_2 < 0$,

ii) $p_1 + \frac{1}{2}a_{11} > 0 > p_2 + \frac{1}{2}a_{22}$,

iii) $0 \leq \left|\mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\right| < \sqrt{\mu_1^2+\mu_2^2}$,

iv) $p_1 + p_2 + \frac{1}{2}\gamma > 0$.

Considering the parameter restrictions jointly, we start with the most complicated restriction equivalent to iii): $2a_{12}\Big(p_2 + p_1 + \frac{\gamma}{2} + \frac{a_{12}}{2}\Big) < \Big(p_1 + p_2 + \frac{1}{2}a_{11} + \frac{1}{2}a_{22}\Big)^2 - 4p_1p_2$. Once again, the r.h.s. is positive, as we impose $p_2 < 0$. It is certainly possible for $p_1 + p_2 + \frac{\gamma}{2} = p_1 + \frac{1}{2}a_{11} + p_2 + \frac{1}{2}a_{22} - a_{12} > 0$ by having $p_1 + \frac{1}{2}a_{11} + \frac{1}{2}a_{22} - a_{12}$ be positive and large enough. Once again, $a_{12} < 0$ can make the term even more positive. Then, even as we impose condition ii), $p_2 + \frac{1}{2}a_{22} < 0 < p_1 + \frac{1}{2}a_{11}$, as long as $\left|p_2 + \frac{1}{2}a_{22}\right| < \left|p_1 + \frac{1}{2}a_{11}\right|$, we get $p_1 + p_2 + \frac{\gamma}{2} > 0$, and also $p_2 + p_1 + \frac{\gamma}{2} + \frac{a_{12}}{2} > 0$. The term $2a_{12}\Big(p_2 + p_1 + \frac{\gamma}{2} + \frac{a_{12}}{2}\Big)$ is then negative, while the r.h.s. on the first inequality of this paragraph remains positive. Imposing $p_1 > 0$ does not necessarily produce any contradiction. Once again, the restricted parameter space is non-trivial.

Proposition 5.5.1. *When our parameter space satisfies:*

i) $p_1 > 0, p_2 < 0$,

ii) $p_1 + \frac{1}{2}a_{11} > 0 > p_2 + \frac{1}{2}a_{22}$,

iii) $0 \leq \left| \mu_2 + \frac{\sigma_2 \sigma_1 \sqrt{1-\rho^2}}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}} \right| < \sqrt{\mu_1^2 + \mu_2^2}$,

iv) $p_1 + p_2 + \frac{1}{2}\gamma > 0$,

the solution for the optimization problem in this thesis in this parameter regime is as follows:

$$\sup_{\tau} E^{(x_2,x_1)}\Big\{(X_{1,\tau} - X_{2,\tau})\boldsymbol{I}_{[\tau<\infty]}\Big\} = \begin{cases} \frac{(\mu_1-1)}{\mu^{\lambda_1}} x_1^{\lambda_1} x_2^{1-\lambda_1} & \textit{whenever} \quad (x_2,x_1) \in S^c \\ x_1 - x_2 & \textit{whenever} \quad (x_2,x_1) \in S \end{cases}$$

where $S = \big\{(x_2,x_1) : x_1 \geq \mu_1 x_2\big\}$, $S^c = \big\{(x_2,x_1) : x_1 < \mu_1 x_2\big\}$, $\lambda = \lambda_1$, $\mu_1 = \dfrac{\lambda_1}{\lambda_1 - 1}$,

(i.e. the same as the solution in Section 5.3).

Proof. By previous lemmas, $\lambda = \lambda_1$ is the only viable solution for λ. All of the Lemmas and Propositions under Section 3.2 hold. Theorem 3.1.1 also holds.

Proposition 4.1.2 to Corollary 4.1.7 hold without modification.

Proposition 4.1.8 holds, as the terminal pay-off is still 0.

Lemma 4.1.9 and Lemma 4.1.10 still hold. The conclusions for Proposition 4.1.11 to Theorem 4.1.2 also hold. The optimal policy-choice function ψ_μ, chosen by the region $\big\{(x_2,x_1) : x_1 < x_2\big\}$, and the overall solution, are once again given by the statement of this Theorem, as $T = 0$ is still the pay-off associated with the event $[\tau = \infty]$. □

5.6 Discussion of Chladna's [9] case:

Combining results from Proposition 5.4.2 and Proposition 5.5.1, we have the following Theorem:

Theorem 5.6.1. *When the following conditions are satisfied:*

i) $p_1 > 0, p_2 < 0,$

ii) $0 \leq \left| \mu_2 + \frac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}} \right| < \sqrt{\mu_1^2+\mu_2^2},$

iii) $p_1 + p_2 + \frac{1}{2}\gamma > 0.$

the solution for the optimization problem in this thesis in this parameter regime is as follows:

$$\sup_{\tau} E^{(x_2,x_1)}\Big\{(X_{1,\tau} - X_{2,\tau})\boldsymbol{I}_{[\tau<\infty]}\Big\} = \begin{cases} \frac{(\mu_1-1)}{\mu^{\lambda_1}} x_1^{\lambda_1} x_2^{1-\lambda_1} & \textit{whenever} \quad (x_2,x_1) \in S^c \\ x_1 - x_2 & \textit{whenever} \quad (x_2,x_1) \in S \end{cases}$$

where $S = \big\{(x_2,x_1) : x_1 \geq \mu_1 x_2\big\},\ S^c = \big\{(x_2,x_1) : x_1 < \mu_1 x_2\big\},\ \lambda = \lambda_1,\ \mu_1 = \dfrac{\lambda_1}{\lambda_1 - 1}.$

As far as the approximation to Chladna's [9] problem, our results are not definitive because we utilized an approximation to her mean-reverting timber price process, but there is evidence pointing in the direction that, under Case 1b, and Case 2, the optimal stopping times can be non-a.s. finite. If this continues to hold true in her actual problem, then posing a boundary condition that automatically harvests might not be quite appropriate. In any case, our approximation gives the optimal stopping time distribution as given by Proposition 3.2.7, and the valuation function as given by Theorem 3.1.1, as long as we admit an additional restriction (restriction iii) under Theorem 5.6.1.

6 CONCLUSION

This thesis has introduced another solution method to the swaption problem described in Section 1.3. We verified, albeit only on a restricted set of Hu, Oksendal's [13] parameter space, that their solution is correct. As we went through our new solution method, we gained several valuable insights.

First, de-correlating jointly normal random variables and letting one variable in two dimensions govern the stopping time while the other is independent of the first can lead to the correct solution in these optimal stopping problems. Second, we successfully evaluated an integral involving a disintegrated measure involving the product of a Cauchy distribution and a generalized hyperbolic distribution over one dimension to produce a power function reminiscent of solutions to these types of perpetual swaption problems, under a mild restriction of the parameter space. We gave that result in Corollary 3.5.5. For this specific problem, we could also, without knowing the exact form of the optimal function in the continuation region, derive the geometry of the stopping boundary.

We then use this geometry to convert the entire problem into a maximization problem that gives the exact boundary and exact optimal continuation value function. This is the variational approach referred to by Arkin and Slastnikov [1]. Third, we extended the results of Hu, Oksendal [13] to a region outside their restriction. Notably, we solve the problem without the property that stopping times are a.s. finite, using only mathematical arguments.

We drew comparisons between the two parameter regimes (Hu, Oksendal [13] and ours), and discovered that as the mean trajectory for revenue damps down relatively more quickly than the mean trajectory for the cost term, the swaption holder becomes less finicky in holding onto swaptions in order to realize larger pay-offs later. He/she would tend to exercise earlier over a lower set of pay-offs. But if volatility for the revenue term is increased, relative to the volatility for the cost term, the opposite result could apply in some cases. The swaption holder can take advantage of this increase in volatility of the benefit term by being more patient in exercising (i.e. waiting for a larger net profit to come along before exercising). This situation also occurs when volatility for the cost term gets higher; the swaption holder can be more patient in waiting for a low cost to occur, together with an acceptable level of revenue, before he/she chooses to exercise. However, these conclusions only apply when the rate of exponential damping for asset 1 is greater than the same rate for asset 2.

Finally, we examine a model in carbon sequestration. The model posits zero or moderately positive revenue growth for forest revenues, but high exponentiating costs for carbon emissions for either a central planner or for a forester who has to pay these damage costs. We found that over this region, after adding a relatively mild restriction to what we have imposed in Chapter 3, the usual bi-variate power function for the continuation region is also the solution. Since the cost model, in the context of a stock swaption, would imply that the company's stock representing asset 2 would pay a continuous stream of negative dividends, we believe that other authors have not explored the continuous-time solution in this region of the parameter space. However, the actual utility of our approximate model to the forestry optimal rotation problem is that it has a closed form solution and we can calculate values of the real swaption quickly. Whether the approximation is a good one is an empirical issue. It is definitely not the exact solution to the mean-reversion timber price model: we need to compare our values and stopping time distributions with those from the actual mean-reverting model in order to assess the size of the approximation error. This is outside the scope of this thesis.

Further research in extending findings of this thesis can proceed along the direction of relaxing the $0 \leq \left|\mu_2 + \dfrac{\sigma_2\sigma_1\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}}\right| < \sqrt{\mu_1^2+\mu_2^2}$ constraint. We can try to extend our new-found insights on the non-a.s. finite stopping version of this problem to the single-revenue and multiple-cost cases, similar to what Hu, Oksendal [13] had done. Extending this model with a front end continuous pay-off would proceed along Chladna's [9] direction. We can also consider other forms of pay-off functions, particularly those that would make our objective functions remain as homothetic functions. Also, we can utilize our method to study problems in sequential compound swaptions, where exercising a swap at one time begets the swappers new options in the future. An example in Research and Development might be that a company that successfully swaps a sum of money for know-how finds itself in position to exploit the know-how in new markets. The company now owns an option to expand into the new market by swapping set-up costs in order to sell in the new market in exchange for revenue gained from selling in that new market. Another example from natural resource economics might be that after the forester swaps the forest for carbon penalties plus other costs, he/she has acquired a new real option on how to use the bare land subsequent to the harvest: urban development, crop planting, or other uses. Both take our basic problem and transform it into sequential swaption pricing problems, as the asset includes compound option values instead of just the simple option value considered in this thesis.

BIBLIOGRAPHY

1. V. I. Arkin and A.D. Slastnikov. A variational approach to optimal stopping problems for diffusion processes. *Theory of Probability and its Applications*, 53:467–480, 2009.
2. R. Bhattacharya and Waymire E.C. *A Basic Course in Probability*. Springer, 2007.
3. R. Bhattacharya and E.C. Waymire. *Stochastic Processes with Applications*. John Wiley & Sons Inc., 1990.
4. F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.
5. R. Brazee and R. Mendelsohn. Timber harvesting with fluctuating prices. *Forest Science*, 34(2):359–372, 1988.
6. M. Broadie and J. DeTemple. The valuation of american options on multiple assets. *Journal of Mathematical Finance*, 7(3):241–286, 1997.
7. P. Carr. The valuation of sequential exchange opportunities. *Journal of Finance*, 58:1235–1256, 1988.
8. J.T. Chang and D. Pollard. Conditioning as disintegration. *Statistica Neerlandica*, 51:287–317, 1997.
9. A.K. Dixit and R.S. Pindyck. *Investment under Uncertainty*. Princeton University Press, 1994.
10. E. Eberlein and E.A. von Hammerstein. Generalized hyperbolic and inverse gaussian distributions: Limiting cases and approximation of processes. *In Seminar on Stochastic Analysis, Random Fields and Applications IV*, 58(4):221–264, 2004.
11. M. Faustmann. Calculation of the value of which forest land and immature stands possess for forestry. *Journal of Forest Economics*, 1(1):7–44, 1849.
12. H. U. Gerber and E.S.W. Shiu. martingale approach to pricing perpetual American options on two stocks. *Journal of Mathematical Finance*, 6:303–322, 1996.
13. Y. Hu and B. Oksendal. Optimal time to invest when the price processes are geometric Brownian motion. *Finance and Stochastics*, 2:295–310, 1998.
14. J.E. Janisch and M.E. Harmon. Successional changes in live and dead wood carbon stores: implications for net ecosystem productivity. *Tree Physiology*, 22:77–89, 2002.
15. W. Margrabe. The value of an option to exchange one asset for another. *Journal of Finance*, 33:177–186, 1978.

16. R. McDonald and D. Siegel. The value of waiting to invest. *Quarterly Journal of Economics*, 101:707–727, 1986.

17. K. Nishide and L.C.G. Rogers. Optimal time to exchange two baskets. *Journal of Applied Probability*, 48:21–30, 2011.

18. B. Oksendal. *Stochastic Differential Equations: An Introduction with Applications, 6th Edition*. Heideberg Press, 2003.

19. T.E. Olsen and G. Stensland. On optimal timing of investment when cost components are additive and follow geometric diffusions. *Journal of Economic Dynamics and Control*, 16:39–51, 1992.

20. A.J. Plantinga. The optimal timber rotation: An option value approach. *Forest Science*, 44:192–202, 1998.

21. G.E. Roberts and H. Kaufman. *Table of Laplace Transforms*. W.B. Saunders Company, 1966.

22. J. Sadowsky. *The Value of Learning in Product Development Stage: A Real Options Approach*. London Business School, 2005.

23. J.T. Williams. Real estate development as an option. *Journal of Real Estate Finance and Economics*, 4:191–208, 1991.

24. C. Won. Valuation of investments in natural resources using contingent-claim framework with application to bituminous coal developments in korea. *Energy*, 34:1215–1224, 2009.

25. K. Yosida. *Functional Analysis*. Heideberg, 1978.

1 APPENDIX

A Appendix to Chapter 1

A.1 Simple Problem as an illustration

Suppose we were to solve the following optimization problem, with $Y_{1,t}$ signifying the revenue that the swap-option-holding company will receive when it chooses to exercise the option at time t, and $Y_{2,t}$ signifying the cost that company has to pay at the same time of exercise. The company is to choose an optimum time to exercise so as to maximize its objective function, revenue over costs, on a one-time basis. Given time-differentiable (smooth) processes $(Y_{2,t}, Y_{1,t})$, let the following function of the joint process give the objective function, as we try to maximize the function with respect to the choice variable t; that is, $f_1(Y_{2,t}, Y_{1,t}) = e^{-rt}(Y_{1,t} - Y_{2,t})$ defines the objective function the company is trying to maximize. Typically, there is a discount factor associated with these pay-off problems. It is of the form e^{-rt}, where r is the required rate of return for the company. Hu, Oksendal [13] assumed that the discount rate r, as we call the required rate of return for the shareholders, is greater than either of the growth rates for the revenue or cost terms, as the prices that the stocks sell for represent the revenue and cost terms. As Hu, Oksendal [13] were following McDonald and Siegel [16], authors of the latter claimed that discounted price processes were super-martingales because they resembled the returns for high-technology companies in industries where barriers to competitive entries were relatively minor. Positive revenues damped downwards because of competitive entries into the industries once their profitability became known. Thus, for projects, the discount rate should be greater than the price appreciation rate; we note that this also models for technological obsolescence. Of course, the same r would be used to discount the net profit, revenue minus cost. Over on the cost side, it is conceivable that the growth rate of costs is either the same, lower or higher than the discount rate. If it is higher, we are in agreement with Chladna's [9] model; if it is lower, we are in agreement with Hu, Oksendal's [13] model. We solve the latter model first. We will provide solutions to the former model in Chapter 5.

In this simple example, we can think of both terms as deterministic processes with growth in the revenue and cost terms. The rate of price appreciation on revenue is $a_{1,t}$, and the rate of return on costs is $a_{2,t}$. Both are less than the market equilibrium required rate of return in the Hu, Oksendal [13] problem. Specifically, let r_t be the required rate of return on company's profit and losses, let $a_{i,t}$ be the price appreciation rate for stock i; then $r_t > a_{i,t}$ for both the cost and revenue terms. Both $r_t, a_{i,t}$ are time-invariant in their model, though each asset can have its own price appreciation rate: $r_t = r$, and $a_{i,t} = a_i$, when $i = 1, 2$. This is the model on which that authors such as McDonald and Siegel [16], Olsen and Stensland [19], and Hu and Oksendal [13] wrote their papers, except that we will not deal with their stochasticities until later. Thus, if we assume that the governing dynamical equations for $(Y_{2,t}, Y_{1,t})$ follow the following differential equations: $dY_{1,t} = -a_1 Y_{1,t} dt$ and $dY_{2,t} = -a_2 Y_{2,t} dt$ [1], then $e^{-rt} Y_{i,t} = e^{-rt} y_i e^{a_i t} = y_i e^{(a_i - r)t}$, with the discount rate absorbed by the growth rate of either of the processes, and if we define

$$(X_{2,t}, X_{1,t}) = e^{-rt}(Y_{2,t}, Y_{1,t}) = (e^{-rt} Y_{2,t}, e^{-rt} Y_{1,t}),$$

we see that $y_i e^{(a_i - r)t} = x_i e^{-p_i t}$ with the following identification: $x_i = y_i$, and $p_i = r - a_i > 0$. Note that we have gone against the usual mathematical convention that the variable with the first index should be listed first, and the second, second. This is in keeping with Hu, Oksendal's [13] treatment of their variables: x_2 is their x-axis variable, and x_1 is their y-axis variable. Because of the form of the solution, we can easily see that $(X_{2,t}, X_{1,t})$ obey the following differential equations: $dX_{1,t} = -p_1 X_{1,t} dt$, and $dX_{2,t} = -p_2 X_{2,t} dt$, with $p_1, p_2 > 0$, and with initial conditions $X_{1,0} = x_1 = y_1$, $X_{2,0} = x_2 = y_2$. Since we do not deal with assets with prices equal to 0, nor do we ever observe them to be infinite in real life, (x_2, x_1) is in the interior of the first quadrant in $\mathbb{R}^2$. We name this space for initial

[1] We will explain the notation in full generality in Section A.4

conditions $\mathbb{R}_+^2$, the initial state-space.

Taken altogether, if $f_1(Y_{2,t}, Y_{1,t}) = e^{-rt}(Y_{1,t} - Y_{2,t}) = X_{1,t} - X_{2,t} = f_2(X_{2,t}, X_{1,t})$, then the company will try to solve the following problem:

Find $t = t^*$ such that $f_2(X_{2,t^*}, X_{1,t^*}) = \sup_t f_2(X_{2,t}, X_{1,t})$, and give the value for $f_2(X_{2,t^*}, X_{1,t^*})$. In other words, solve the following mathematical program:

$$\sup_t f_2(X_{2,t}, X_{1,t})$$

s.t

$$dX_{1,t} = -p_1 X_{1,t} dt, \quad dX_{2,t} = -p_2 X_{2,t} dt,$$

$$X_{1,0} = x_1, \quad X_{2,0} = x_2, \text{ and } 0 < x_1, x_2, p_1, p_2 < \infty.$$

The company can observe present benefits and costs, as given by the initial conditions: $X_{1,0} = x_1$, $X_{2,0} = x_2$. Further, the company knows the revenue and cost time-trajectories, following the deterministic evolution model: $dX_{1,t} = -p_1 X_{1,t} dt$, $dX_{2,t} = -p_2 X_{2,t} dt$. The conditions $p_1, p_2 > 0$ serve as the constraints specific to the Hu, Oksendal [13] problem.

Thus, we can think of the above problem as trying to find the optimal t so as to realize the maximal difference in values between two processes, i.e. determining the t when $X_{1,t}$ exceeds $X_{2,t}$'s value the most. This is the essence of trying to find an optimal time to exercise a swap contract.

From the two differential equations and initial conditions, one can quickly calculate solutions for the terms inside the objective function: $X_{1,t} = x_1 e^{-p_1 t}$, and $X_{2,t} = x_2 e^{-p_2 t}$. Substituting back into the original objective function gives $\sup_t (x_1 e^{-p_1 t} - x_2 e^{-p_2 t})$.

This is a one-variable unconstrained optimization problem. The solution can be found by techniques in elementary calculus:

Let $f_2(t) = (x_1 e^{-p_1 t} - x_2 e^{-p_2 t})$. Then $f_2'(t) = -p_1 x_1 e^{-p_1 t} + p_2 x_2 e^{-p_2 t} = 0$ at the stationary point. Solving, the stationary point $t^* = \frac{1}{p_1-p_2}\ln(\frac{p_1x_1}{p_2x_2})$. Taking the second derivative, and evaluating it at the stationary point t^*,

$$\begin{aligned}
f_2''(t^*) &= p_1^2x_1e^{-p_1t^*} - p_2^2x_2e^{-p_2t^*} \\
&= p_1^2x_1e^{-p_1(\frac{1}{p_1-p_2})\ln(\frac{p_1x_1}{p_2x_2})} - p_2^2x_2e^{-p_2(\frac{1}{p_1-p_2})\ln(\frac{p_1x_1}{p_2x_2})} \\
&= p_1^2x_1\Big(\frac{p_1x_1}{p_2x_2}\Big)^{\frac{-p_1}{p_1-p_2}} - p_2^2x_2\Big(\frac{p_1x_1}{p_2x_2}\Big)^{\frac{-p_2}{p_1-p_2}} \\
&= p_1p_1x_1\Big(\frac{p_1x_1}{p_2x_2}\Big)^{\frac{-p_1}{p_1-p_2}} - p_2p_2x_2\Big(\frac{p_1x_1}{p_2x_2}\Big)^{\frac{-p_2}{p_1-p_2}} \\
&= p_1\frac{(p_1x_1)^{1-\frac{p_1}{p_1-p_2}}}{(p_2x_2)^{\frac{-p_1}{p_1-p_2}}} - p_2(p_1x_1)^{\frac{-p_2}{p_1-p_2}}(p_2x_2)^{1+\frac{p_2}{p_1-p_2}} \\
&= p_1\frac{(p_1x_1)^{\frac{-p_2}{p_1-p_2}}}{(p_2x_2)^{\frac{-p_1}{p_1-p_2}}} - p_2(p_1x_1)^{\frac{-p_2}{p_1-p_2}}(p_2x_2)^{\frac{p_1}{p_1-p_2}} \\
&= \frac{(p_1x_1)^{\frac{-p_2}{p_1-p_2}}}{(p_2x_2)^{\frac{-p_1}{p_1-p_2}}}(p_1-p_2).
\end{aligned}$$

The second derivative, evaluated at the stationary point, is negative if $p_1 < p_2$. This would give the optimum stopping time at $t^* = \frac{1}{p_1-p_2}\ln\Big(\frac{p_1x_1}{p_2x_2}\Big)$, and the worth of the swap, as a holder gives up asset two worth at X_{2,t^*} for asset one worth X_{1,t^*}, at time t^*, is then

$$\begin{aligned}
x_1e^{-p_1\frac{1}{p_1-p_2}\ln(\frac{p_1x_1}{p_2x_2})} - x_2e^{-p_2\frac{1}{p_1-p_2}\ln(\frac{p_1x_1}{p_2x_2})} &= x_1\Big(\frac{p_1x_1}{p_2x_2}\Big)^{\frac{-p_1}{p_1-p_2}} - x_2\Big(\frac{p_1x_1}{p_2x_2}\Big)^{\frac{-p_2}{p_1-p_2}} \\
&= \Big(\frac{p_1x_1}{p_2x_2}\Big)^{\frac{-p_1}{p_1-p_2}}\left\{x_1 - x_2\Big(\frac{p_1x_1}{p_2x_2}\Big)^{-\frac{p_2-p_1}{p_1-p_2}}\right\} \\
&= \Big(\frac{p_1x_1}{p_2x_2}\Big)^{\frac{-p_1}{p_1-p_2}}\left\{x_1 - x_2\Big(\frac{p_1x_1}{p_2x_2}\Big)\right\}.
\end{aligned}$$

We also check the boundary solutions to verify that we have found the global maximum above. At $t = \infty$, $f_2(\infty) = (x_1e^{-p_1\infty} - x_2e^{-p_2\infty}) = 0$. At $t = 0$, $f_2(0) = x_1 - x_2$. The solution to the objective function is $\Big(\frac{p_1x_1}{p_2x_2}\Big)^{\frac{-p_1}{p_1-p_2}}\left\{x_1 - x_2\Big(\frac{p_1x_1}{p_2x_2}\Big)\right\}$.

Notice that the situation differs when $p_1 > p_2$. t^* now becomes the worst time to exercise in the sense that exercising at that time would give the holder of the option the least worth. One then will have to check the boundary solutions to see which one is really the maximum. Let us assume that $p_1 > p_2$ for the next few paragraphs. We have

$$\lim_{t\to\infty}(x_1e^{-p_1t} - x_2e^{-p_2t}) = 0, \text{ while } (x_1e^{-p_1t} - x_2e^{-p_2t})\Big|_{t=0} = x_1 - x_2.$$

The solution for the objective function is now $\max(x_1 - x_2, 0)$. So, then $x_1 \geq x_2$, $t^* = 0$, and the swap is worth $x_1 - x_2$. When $x_1 = x_2$, all choice of exercise times, or stopping times, yield the same swap value of 0, and thus there is no unique optimum exercise time.

When $x_1 < x_2$, the optimum choice is to let $t^* = \infty$. If we define the swap value to be continuous there, then choosing to swap at $t^* = \infty$ yields the person owning the swap a value of 0, consistent with the limiting condition $\lim_{t\to\infty}(x_1e^{-p_1t} - x_2e^{-p_2t}) = 0$.

Notice that this terminal condition, if defined in some other arbitrary way, might cause problems in this region of the parameter space (i.e. $p_1 > p_2$ and $x_1 < x_2$). Suppose one were to define the pay-off to the swap exercising at $t^* = \infty$ to be, say, -1. Then one would intuitively try to exercise the swap before $t^* = \infty$ so as to avoid taking a pay-off of -1 (objective function of -1) at $t = \infty$, but still taking a negative pay-off equal to $x_1e^{-p_1t} - x_2e^{-p_2t} < 0$, which gets arbitrarily close to 0 from below (from negative values). There is no finite time $\bar{t}$ such that exercising at a later time $t > \bar{t}$ won't lessen the magnitude of the negative pay-out to the swap holder. So the swap holder postpones exercising indefinitely, but still needs to exercise before $t = \infty$. An optimal time to exercise, then, does not exist!

With this very simple non-stochastic example, we can already see that different regions of the parameter space yield different solutions for this simple problem, and we need to take care in defining a terminal condition for the swap payoff at $t = \infty$ in order for the problem to be well-posed (for an optimal time to exercise to exist).

Now, consider the case $p_1 = p_2$, where the objective function value equals $(x_1 - x_2)e^{-p_1 t}$. Exercising immediately with $t^* = 0$ yields $(x_1 - x_2)$ when $x_1 > x_2$. The holder of the swap option is indifferent to exercising any time or never exercising when $x_1 = x_2$, which yields a swap value of 0. Finally, a person never exercises the option when $x_1 < x_2$ so as to take a pay-off of 0 at $t^* = \infty$.

A.2 From simple example to real problem

In order to address the stochastic optimization problem in a proper framework, we need to develop some preliminaries regarding Brownian motion and stochastic integration. A brief treatment follows.

A.3 Brownian Processes

In what follows, we will model pricing uncertainties as functions of Brownian motion (also known as the Wiener process) rather than the deterministic $(X_{2,t}, X_{1,t})$ considered previously. We will start with a definition of a 1-dimensional Brownian Process.

$\{B_{1,t} : t \geq 0\}$ is a standard Brownian Process if it satisfies the following properties:

(i) $B_{1,0} = 0$ a.s.

(ii) $\{B_{1,t} : t \geq 0\}$ has P-a.s. continuous sample paths (the function $t \mapsto B_{1,t}$ is P-a.s. continuous).

(iii) $\{B_{1,t} : t \geq 0\}$ has independent increments; for any $t_0 < t_1 < t_2 < ... < t_n$, and for any n, $(B_{1,t_1} - B_{1,t_0}, ..., B_{1,t_n} - B_{1,t_{n-1}})$ are jointly independent.

(iv) $\{B_{1,t} : t \geq 0\}$ has stationary increments; for any $0 \leq s \leq t' \leq t$,

$B_{1,t} - B_{1,t'} \overset{d}{\sim} B_{1,t-s} - B_{1,t'-s}$, where for random variables $A, C, A \overset{d}{\sim} C$, means that A is equal in distribution to C.

(v) $B_{1,t} - B_{1,s} \overset{d}{\sim} N(0, t-s)$ for any $s \in \mathbb{R}^+ \cup \{0\}$, and $t \in \mathbb{R}^+$ such that $0 \leq s < t$.

The common probability space that $B_{1,t}$ shares is the probability space $(\Omega, \mathcal{F}, P)$ for all t. $\{B_{1,t} : t \geq 0\}$ generates a natural increasing filtration $\mathcal{F}_{1,t}$ to which the process is adapted.

A 2-dimensional Brownian motion is a 2-dimensional process whose coordinate components are i.i.d. one-dimensional standard Brownian Motions. Let the first coordinate process $\{B_{1,t} : t \geq 0\}$ generate a natural increasing filtration $\mathcal{F}_{1,t}$. Let the second coordinate process $\{B_{2,t} : t \geq 0\}$ generate a natural increasing filtration $\mathcal{F}_{2,t}$.

The joint process is then adapted to the smallest sigma algebra generated by the Cartesian product of sets in $\mathcal{F}_{1,t}$ and in $\mathcal{F}_{2,t}$: $\sigma(\mathcal{F}_{2,t} \times \mathcal{F}_{1,t})$. We call this sigma field $\mathcal{F}_t$. Generalizing our previous problem, we postulate that $(X_{2,t}, X_{1,t})$ satisfy the following stochastic differential equations:

$$dX_{1,t} = -p_1 X_{1,t} dt + q_{11} X_{1,t} dB_{1,t} + q_{12} X_{1,t} dB_{2,t},$$

$$dX_{2,t} = -p_2 X_{2,t} dt + q_{21} X_{1,t} dB_{1,t} + q_{22} X_{2,t} dB_{2,t},$$

where $B_{1,t}$ and $B_{2,t}$ are independent Brownian Processes. These pairs of equations are not ordinary or partial differential equations. They are stochastic differential equations that need further explanation.

A.4 Stochastic differential as a short-hand mathematical notation

It can be shown, via Feller's approximation for the functional form of the tail of the normal density function, that Brownian Motion is non-differentiable (Bhattacharya and Waymire [2].) In a strict sense, $dB_{i,t}$ make no sense as ordinary differentials. An equation such as

$$dX_{i,t} = -p_i X_{i,t} dt + q_{i1} X_{i,t} dB_{1,t} + q_{i2} X_{i,t} dB_{2,t},$$

after being transformed into

$$\frac{dX_{i,t}}{X_{i,t}} = -p_i dt + q_{i1} dB_{1,t} + q_{i2} dB_{2,t},$$

is really a short hand for the following equation:

$$\int_0^t \frac{dX_{i,t}}{X_{i,t}} = -p_i \int_0^t dt + q_{i1} \int_0^t dB_{1,t} + q_{i2} \int_0^t dB_{2,t}. \tag{A.1}$$

The first integral on the right hand side of (A.1) uses the Lebesgue measure, while the last two are stochastic integrals as defined by Itô that are not the same as either the Riemann or Lebesgue Integral. See Oksendal [18] for a reference. The first equation in this section is a stochastic differential equation, and its equivalent representation is (A.1). We give a detailed definition below.

A.5 Definition of a stochastic integral

With minor adjustment to make the exposition flow more smoothly, we take the following definitions and results for the stochastic integral from Chapter 3, pg. 25, of Oksendal [18]. Given a one-dimensional Brownian motion $\{B_{1,t} : t \geq 0\}$, denote $\mathcal{F}_{1,t}$ by the natural filtration of $\mathcal{F}$ generated by $\{B_{1,t} : t \geq 0\}$. Then a function $f : (0, \infty) \times \Omega \to \mathbb{R}$, $(t, \omega) \mapsto z$ belongs to function space $\mathcal{V}$ provided that (i) to (iii) below hold.

(i) $(t, \omega) \mapsto f(t, \omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel σ-algebra on $[0, \infty)$.

(ii) $f(t, \omega)$ is $\mathcal{F}_{1,t}$-adapted, i.e. $f(t, \omega)$ is $\mathcal{F}_{1,t}$-measurable for each t.

(iii) $E\Big[\int_0^t [f(s, \omega)]^2 ds\Big] < \infty$.

It is then over this class of functions that the (Itô) Stochastic Integral $\int f dB_{1,t}$ can be defined.

Let $f \in \mathcal{V}$, $0 = t_0 < t_1 < t_2 < ... < t_n = t$, and denote $(t_{k+1} - t_k)$ by Δ_{t_k}. Let $\Delta B_j = B_{1,t_{j+1}} - B_{1,t_j}$. Let e_j be a $\mathcal{F}_{1,t_j}$-measurable function such that $e_j(\omega)$ is constant on $[t_j, t_{j+1})$. Let $\phi_n(t,\omega) = \sum_{j=0}^{n-1} e_j(\omega) I_{[t_i, t-j+1)}(t)$ and define

$$\int_0^t \phi_n(t,\omega) dB_{1,t}(\omega) = \sum_{j=0}^{n-1} e_j(\omega) \Delta B_j(\omega).$$

Repeating Definition 3.1.6 in Oksendal [18], we say that for $f \in \mathcal{V}$, if

$$\lim_{\Delta t_j \to \infty} E\Big\{ \int [f(t,\omega) - \phi_n(t,\omega)]^2 dt \Big\} = 0$$

as $n \to \infty$, then

$$\int f(t,\omega) dB_{1,t}(\omega) = \lim_{n\to\infty} \int \phi_n(t,\omega) dB_{1,t}(\omega).$$

In particular, let $\phi_n(t,\omega) = \sum_{j=0}^{n-1} f(t_j,\omega) I_{[t_j,t_{j+1})}(t)$.

If

$$\lim_{\Delta t_j \to \infty} E\Big\{ \int [f(t,\omega) - \phi_n(t,\omega)]^2 dt \Big\} = 0$$

as $n \to \infty$, then

$$\int f(t,\omega) dB_{1,t}(\omega) = \lim_{n\to\infty} \int \phi_n(t,\omega) dB_{1,t}(\omega) = \lim_{n\to\infty} \sum_{j=0}^{n-1} f(t_j,\omega)[B_{1,t_j+1}(\omega) - B_{1,t}(\omega)].$$

Then convergence is in $L^2(P)$. The stochastic integral $\int f(t,\omega) dB_{1,t}(\omega)$ exists for any $f \in \mathcal{V}$. Oksendal [18] lists the important properties of $\int f dB_{1,t}$ in op. cit., p. 30.

With the stochastic integral defined, we can make further modifications to the basic program under A.1 in order for the stochastic problem to make sense.

A.6 Changes from deterministic to stochastic optimization

We need to generalize our problem in another way from that of Section A.1: in addition to considering exercise policies that prescribe exercising at fixed times t, we also consider policies that exercise at random times.

Because the price processes $X_{i,t}$ now follow random trajectories, the optimal time to stop is no longer deterministic. The swaption holder observes $(X_{2,t}, X_{1,t})$ at time t, and decides whether he/she should exercise the option at that time, on each and every t, before making the exercise-and-stop or continue decision. The objective function can no longer be $X_{1,t}(\omega) - X_{2,t}(\omega)$. In addition to the last term being a random quantity, "t" is no longer a deterministic quantity, but a random variable $\tau(x_2, x_1, \omega)$. Since there are no obvious ways to rank-order random quantities, we might then consider an objective function of the form $\sup_{\tau} E^{(x_2,x_1)}(X_{1,\tau} - X_{2,\tau})$, where τ is any stopping time within some class under consideration. Uncertainties are added to our model via the state variables:

$$\begin{aligned}
dX_{1,t} &= -p_1 X_{1,t}dt + q_{11}X_{1,t}dB_{1,t} + q_{12}X_{1,t}dB_{2,t}, &\text{(A.2)}\\
dX_{2,t} &= -p_2 X_{2,t}dt + q_{21}X_{1,t}dB_{1,t} + q_{22}X_{2,t}dB_{2,t}, &\text{(A.3)}\\
X_{1,0} &= x_1\\
X_{2,0} &= x_2.
\end{aligned}$$

Once again, these are just the shorthand for the stochastic integration problem:

$$\begin{aligned}
X_{1,t} - x_1 &= -p_1 \int_0^t X_{1,t}dt + q_{11}\int_0^t X_{1,t}dB_{1,t} + q_{12}\int_0^t X_{1,t}dB_{2,t}\\
X_{2,t} - x_2 &= -p_2 \int_0^t X_{2,t}dt + q_{21}\int_0^t X_{2,t}dB_{1,t} + q_{22}\int_0^t X_{2,t}dB_{2,t}.
\end{aligned}$$

We utilize the technique in the next section to come up with their solutions.

A.7 Existence and Uniqueness Result

In order to ensure that system (A.1.2), (A.1.3) has a solution, we can check that Theorem 5.2.1 in Oksendal's [18] textbook guarantees a unique solution to our problem.

To arrive at the actual solution, we use the Lemma in the following section.

A.8 It$\hat{o}$'s Lemma

One theorem that helps in evaluating solutions to stochastic differential equations is It$\hat{o}$'s Lemma. We take this from Oksendal's textbook [18], Definition 4.1.1 and Theorem 4.2.1, and various auxiliary definitions.

Now, we can work with function space $\mathcal{V}$, but we can also work with a larger function space $\mathcal{W}_H$ as Oksendal[18] in his text suggested. A function belongs to $\mathcal{W}_H$ if

(i) $(t,\omega) \mapsto f(t,\omega)$ is $\mathcal{B}\times\mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel σ-algebra on $[0,\infty)$.

(ii) There exists an increasing family of σ-algebras $\mathcal{H}_t$; $t \geq 0$ such that

   (a) $\begin{bmatrix} B_{1,t} \\ B_{2,t} \end{bmatrix}$ is $\mathcal{H}_t$-adapted, and both components are martingales with respect to the same filtration,

   (b) f_t is also $\mathcal{H}_t$-adapted.

   We can take $\mathcal{H}_t$ as the σ-algebra generated by $\sigma(\mathcal{F}_{1,t} \times \mathcal{F}_{2,t})$, or even some other larger super-σ-algebra, as long as the adaptation rules are satisfied.

(iii) $P\Big[\int_0^T f(s,\omega)^2 ds < \infty\Big] = 1.$

Since $\mathcal{V}$ is a proper subset of $\mathcal{W}_H$, whatever applies to the latter also applies to the former. Let a function v belong to $\mathcal{W}_H$.

Definition A.1. *(1-dimensional Itô Process) Let* $\{B_t : t \geq 0\}$ *be 1-dimensional Brownian motion on* $(\Omega, \mathcal{F}, P)$*. A 1-dimensional Itô process (or stochastic integral) is a stochastic process* X_t *on* $(\Omega, \mathcal{F}, P)$ *of the form*

$$X_t = X_0 + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dB_s,$$

where $v \in \mathcal{W}_H$*, such that*

$$\begin{aligned} P\Big[\int_0^t [v(s,\omega)]^2 ds < \infty \quad \forall t \geq 0\Big] &= 1, \text{ and } u \text{ is } \mathcal{F}_t\text{-adapted, and} \\ P\Big[\int_0^t |u(s,\omega)| ds < \infty \quad \forall t \geq 0\Big] &= 1. \end{aligned}$$

Definition A.2. *(2-dimensional Itô Process) Let* $(B_{2,t}, B_{1,t})$ *denote 2-dimensional independent Brownian motion on* $(\Omega, \mathcal{F}, P)$*. A 2-dimensional Itô process (or stochastic integral) is a stochastic process* $(X_{2,t}, X_{1,t})$ *on* $(\Omega, \mathcal{F}, P)$ *of the form*

$$X_{2,t} = X_{2,0} + \int_0^t u_2(s,\omega)ds + \int_0^t v_{21}(s,\omega)dB_{1,s} + \int_0^t v_{22}(s,\omega)dB_{2,s},$$

$$X_{1,t} = X_{1,0} + \int_0^t u_1(s,\omega)ds + \int_0^t v_{11}(s,\omega)dB_{1,s} + \int_0^t v_{12}(s,\omega)dB_{2,s},$$

where $v_{11}, v_{12}, v_{21}, v_{22} \in \mathcal{W}_H$*, and* $P\Big[\int_0^t v_{ij}^2 ds < \infty,\ \forall t \geq 0\Big] = 1$ *for* $i, j = 1, 2$ *and* u_1, u_2 *are* $\mathcal{H}_t$*-adapted and* $P\Big[\int_0^t |u_i| ds < \infty,\ \forall t \geq 0\Big] = 1$ *for* $i = 1, 2$.

Following is a 2-dimensional version of Theorem 4.2.1 in Oksendal's [18] book.

Itô's Lemma: Let the 2-dimensional Itô process $(Y_{2,t}, Y_{1,t})$ be defined by the following stochastic differential equations:

$$\begin{aligned} dY_{1,t} &= -p_1 dt + q_{11} dB_{1,t} + q_{12} dB_{2,t}, \text{ and} \\ dY_{2,t} &= -p_2 dt + q_{21} dB_{1,t} + q_{22} dB_{2,t}. \end{aligned}$$

Let $g(t, Y_{1,t}, Y_{2,t})$ be a C^2 map into $\mathbb{R}$ (i.e. g is twice continuously differentiable on $[0, \infty) \times \mathbb{R} \times \mathbb{R}$). Then the process $Z_t = g(t, Y_{1,t}, Y_{2,t})$ is again an Itô process, and

$$\begin{aligned} dZ_t &= \frac{\partial g}{\partial t}(t, Y_{1,t}, Y_{2,t})dt + \frac{\partial g}{\partial y_1}(t, Y_{1,t}, Y_{2,t})dY_{1,t} + \frac{\partial g}{\partial y_2}(t, Y_{1,t}, Y_{2,t})dY_{2,t} \\ &+ \frac{1}{2}\frac{\partial^2 g}{\partial y_1^2}(t, Y_{1,t}, Y_{2,t})dY_{1,t}^2 + \frac{\partial^2 g}{\partial y_1 \partial y_2}(t, Y_{1,t}, Y_{2,t})dY_{1,t}dY_{2,t} + \frac{1}{2}\frac{\partial^2 g}{\partial y_1^2}(t, Y_{1,t}, Y_{2,t})dY_{2,t}^2 \end{aligned}$$

where $dB_{i,t}dB_{j,t} = \delta_{ij}dt$, $dB_{i,t}dt = dtdB_{i,t} = 0$, $dtdt = 0$, for either i= 1 or 2.

It is easy to check that the process $(Y_{1,t}, Y_{2,t})$ of Theorem 4.2.1 satisfies the condition of Definition 4.2 to be a 2-dimensional Itô process. Thus, all requirements in order to apply the Lemma are satisfied, and we can use Itô's Lemma to evaluate the stochastic differential equations:

$$\begin{aligned} dX_{i,t} &= -p_i X_{i,t} dt + q_{i1} X_{i,t} dB_{1,t} + q_{i2} X_{i,t} dB_{2,t}, \text{ or} \\ \frac{dX_{i,t}}{X_{i,t}} &= -p_i dt + q_{i1} dB_{1,t} + q_{i2} dB_{2,t}, \text{ for i=1,2.} \end{aligned} \tag{A.4}$$

Thus, $\frac{dX_{i,t}}{X_{i,t}}$ is the differential of an Itô process, for both $i = 1, 2$. We apply Itô's Lemma to the function $\ln(x)$, which is in $C^2(0, \infty)$.

As we are only interested in assets with non-zero initial prices, $x \neq 0$. We apply Itô's Lemma:

$$d\ln(X_{i,t}) = \frac{\partial \ln(X_{i,t})}{\partial t}dt + \frac{\partial \ln(X_{i,t})}{\partial X_{i,t}}dX_{i,t} + \frac{\partial \ln(X_{i,t})}{\partial X_{j,t}}dX_{j,t}$$

$$\begin{aligned}
&+ \quad \frac{1}{2}\frac{\partial^2 \ln(X_{i,t})}{\partial X_{i,t}^2} dX_{i,t}dX_{i,t} + \frac{\partial^2 \ln(X_{i,t})}{\partial X_{i,t}X_{j,t}} dX_{i,t}dX_{j,t} + \frac{1}{2}\frac{\partial^2 \ln(X_{i,t})}{\partial X_{j,t}^2} dX_{j,t}dX_{j,t} \\
&= \quad 0dt + \frac{dX_{i,t}}{X_{i,t}} + 0dX_{j,t} - \frac{1}{2}\frac{q_{i1}^2 + q_{i2}^2}{X_{i,t}^2} X_{i,t}^2 dt + 0dX_{i,t}dX_{j,t} + 0dX_{j,t}dX_{j,t} \\
&= \quad \frac{dX_{i,t}}{X_{i,t}} - \frac{(q_{i1}^2 + q_{i2}^2)}{2} dt \\
&= \quad \frac{dX_{i,t}}{X_{i,t}} - \frac{a_{ii}}{2} dt.
\end{aligned}$$

In keeping with Hu, Oksendal's [13] notation, $a_{ii} = (q_{i1}^2 + q_{i2}^2)$, Thus, after substituting (A.1.2) and (A.1.3) into the last expression,

$$\begin{aligned}
d\ln(X_{i,t}) = \frac{dX_{i,t}}{X_{i,t}} - \frac{a_{ii}}{2} dt \quad &= \quad -p_i dt + q_{i1}B_{1,t} + q_{i2}dB_{2,t} - \frac{a_{ii}}{2} dt \\
&= \quad -\big(p_i + \frac{a_{ii}}{2}\big)dt + q_{i1}dB_{1,t} + q_{i2}dB_{2,t}.
\end{aligned}$$

Integrating from 0 to t on both sides of the stochastic differential equation gives

$$\ln(X_{i,t}) - \ln(X_{i,0}) = -\big(p_i + \frac{a_{ii}}{2}\big)t + q_{i1}B_{1,t} + q_{i2}B_{2,t} = \ln(X_{i,t}) - \ln(x_i) = \ln\left(\frac{X_{i,t}}{x_i}\right).$$

Exponentiating on both sides gives

$$\frac{X_{i,t}}{x_i} = e^{-\left(p_i + \frac{a_{ii}}{2}\right)t + q_{i1}B_{1,t} + q_{i2}B_{2,t}}, \text{ or } X_{i,t} = x_i e^{-\left(p_i + \frac{a_{ii}}{2}\right)t + q_{i1}B_{1,t} + q_{i2}B_{2,t}}, \text{ for } i = 1, 2.$$

The stochastic evolution equations, as given by the stochastic differential equations, give their dynamic behavior, with uncertainty attached. For example, the average trajectory of both processes are equal to $E(X_{1,t}) = x_1 e^{-p_1 t}$, and $E(X_{2,t}) = x_2 e^{-p_2 t}$, while an additional exponential and a stochastic component are superimposed on top of this mean time-trajectory as follows:

$$X_{1,t} = x_1 e^{-\left(p_1 + \frac{a_{11}}{2}\right)t + q_{11}B_{1,t} + q_{12}B_{2,t}} = E(X_{1,t}) e^{-\frac{1}{2}a_{11}t + q_{11}B_{1,t} + q_{12}B_{2,t}}$$

$$= [E(X_{1,t})e^{-\frac{1}{2}a_{11}t}]e^{q_{11}B_{1,t}+q_{12}B_{2,t}}.$$

$$X_{2,t} = x_2 e^{-\left(p_2+\frac{a_{22}}{2}\right)t+q_{21}B_{1,t}+q_{22}B_{2,t}} = E(X_{2,t})e^{-\frac{1}{2}a_{22}t+q_{21}B_{1,t}+q_{22}B_{2,t}}$$

$$= [E(X_{2,t})e^{-\frac{1}{2}a_{22}t}]e^{q_{21}B_{1,t}+q_{22}B_{2,t}}.$$

The identities $a_{11} \equiv q_{11}^2 + q_{12}^2$ and $a_{22} \equiv q_{21}^2 + q_{22}^2$ are as defined in Hu, Oksendal's [13] paper. The right-most terms above highlight the interpretation that the stochastic solutions are a product of a deterministic trajectory with a random trajectory, driven by Brownian motion processes. Therefore, $X_{1,\tau} - X_{2,\tau}$ is a random variable. One way to evaluate its numerical size is by looking at its expectation, $E(X_{1,\tau} - X_{2,\tau})$. Including infinite stopping times forces us to re-define the objective function to $E\{(X_{1,\tau} - X_{2,\tau})I_{[\tau<\infty]}\}$, as given in 1.4 in the main body of Chapter 1.

B Appendix to Chapter 3

Lemma (3.2.4)**.**

Let $a_{ii} = q_{i1}^2 + q_{i2}^2$, $i = 1, 2$. $Z_{i,t} = e^{-\left(p_i+\frac{1}{2}a_{ii}\right)t+q_{i1}B_{1,t}+q_{i2}B_{2,t}}$ *are supermartingales for* $i = 1, 2$.

Proof of Lemma 3.2.4.

$Z_{i,t}$ is clearly adapted to the filtration $\Im_t$.

$E[|Z_{i,t}|] = E[Z_{i,t}] = e^{-p_i t} < \infty$ for all t.

For any $s < t < \infty$,

$$\begin{aligned} E[Z_{i,t}|\Im_s] &= E\left[e^{-\left(p_i+\frac{1}{2}a_{ii}\right)t+q_{i1}B_{1,t}+q_{i2}B_{2,t}}|\Im_s\right] \\ &= E\left[e^{-\left(p_i+\frac{1}{2}a_{ii}\right)s+q_{i1}B_{1,s}+q_{i2}B_{2,s}}e^{-\left(p_i+\frac{1}{2}a_{ii}\right)(t-s)+q_{i1}(B_{1,t}-B_{1,s})+q_{i2}(B_{2,t}-B_{2,s})}|\Im_s\right] \\ &= e^{-\left(p_i+\frac{1}{2}a_{ii}\right)s+q_{i1}B_{1,s}+q_{i2}B_{2,s}}e^{-\left(p_i+\frac{1}{2}a_{ii}\right)(t-s)}E\left[e^{q_{i1}(B_{1,t}-B_{1,s})+q_{i2}(B_{2,t}-B_{2,s})}|\Im_s\right] \end{aligned}$$

$$= Z_{i,s} e^{-\left(p_i + \frac{1}{2} a_{ii}\right)(t-s)} e^{0 + \frac{1}{2}(q_{i1}^2 + q_{i2}^2)(t-s)}$$

$$= Z_{i,s} e^{-\left(p_i + \frac{1}{2} a_{ii}\right)(t-s)} e^{\frac{1}{2} a_{ii}(t-s)}$$

$$= Z_{i,s} e^{-p_i(t-s)}$$

$$\leq Z_{i,s},$$

as $e^{q_{i1}(B_{1,t} - B_{1,s}) + q_{i2}(B_{2,t} - B_{2,s})}$ is log-normally distributed with mean $e^{0 + \frac{1}{2}(q_{i1}^2 + q_{i2}^2)(t-s)}$. The inequality above shows that $Z_{i,t} = e^{-\left(p_i + \frac{1}{2} a_{ii}\right)t + q_{i1} B_{1,t} + q_{i2} B_{2,t}}$ are supermartingales for $i = 1, 2$. □

Lemma (3.2.5)**.**

The boundary $\partial S \subseteq S$. S is therefore closed, and S^{c} is open.

Proof of Lemma 3.2.5.

Over ∂S, $\sup_{\tau} E^{(x_2, x_1)} \left[(X_{1,\tau} - X_{2,\tau}) \, I_{[\tau < \infty]} \right] = x_1 - x_2$ satisfies the value-matching condition. This means that every point in ∂S is in S: $\partial S \subseteq S$. Since S contains all its boundary points, it is a closed set. S^{c}, the complementary set of S, is then open. □

Proof of Lemma 3.2.6.

The region $\{(x_2, x_1) : x_1 < x_2\} \subseteq S^{\mathrm{c}}$, by Lemma 3.2.1. Since S^{c} and S are mutually exclusive, the region $\{(x_2, x_1) : x_1 < x_2\}$ cannot contain any point of S or ∂S. □