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This thesis presents a general model for the location problem based on integer linear programming with fixed charges. The location problem is concerned with choosing locations for facilities throughout a particular region or area in such a way that total costs and expenses incurred are minimized. Although the location problem presented is basically of second degree, a transformation is used to convert the problem to linear form.

Following the development of the general model, this particular method is applied to a problem of determining the optimum size, location, and combination of incinerators and/or power plants to be installed for the disposal of wood waste. The determination of fixed charges and other relative costs becomes increasingly more complex as deviations from the general model occur.

INTEGER PROGRAMMING FOR OPTIMIZED FACILITY LOCATION

by

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INTRODUCTION

The location problem formulated in this paper is concerned with determining the optimum placement of discrete facilities over a finite number of possible locations. The optimization is accomplished when either (1) the various costs to the system are minimized or, (2) the profit gained from the overall operation of the system is maximized.

A discussion and definition of location problems in general is presented followed by a mathematical development for this class of problems. A fixed charge is introduced for the cost incurred in constructing a facility at some feasible location. These costs are combined with various transportation costs and the location problem formulated as an integer linear program with fixed charges. Through the use of a control variable, the objective function is expressed as a quadratic function of either continuous and integer variables or all integer variables. The objective function as presented involves terms of second degree but a conversion to linear form is effected.

Later an example of this type of optimization problem is found in the placement of waste disposal incinerators and/or power plants throughout the Medford area of the Bear Creek Valley in Oregon. This example involves the determination of optimum sizes, locations and combinations of these facilities to be installed for wood waste disposal. It is noted that the fixed charges and other transportation costs have become more complex due to the many possibilities which must be considered.

Further applications and considerations of the location problem for optimized facility location are presented followed by a brief summary of the mathematical model and some concluding remarks on the Medford study.

THE LOCATION PROBLEM

General Description

The topic for this study developed while considering the location of wood waste disposal plants in the Medford area of the Bear Creek Valley of Oregon during the summer of 1964. While consideration was given to this specific example, it was decided the underlying structure of this fixed charge location problem was basic to a wide class of problems requiring integer solutions for some subset of the variables.

Optimizing a function of several variables subject to linear and/or non-linear constraints is not always accomplished by the determination of a maxima or minima by classical methods such as differential calculus. Furthermore, restriction to integer-valued solutions compounds the difficulty and alternative methods or approximations are necessary. Many remedies to this situation have been developed including linear, non-linear, and dynamic programming. An extensive discussion of such methods may be found in (11, 12). Various alternatives were investigated in an attempt to arrive at a suitable solution to the location problem. The method decided upon may be classified as non-linear programming.

Underlying the general heading of non-linear programming are a wide variety of techniques available for the solution of optimization problems. The particular method to be discussed on the following pages is generally known as integer linear programming with fixed charges. Following the development of the simplex method by Dantzig in 1947, interest and research continued in an attempt to arrive at a method of optimizing linear programs with the added restraint that some or all of the variables be restricted to integer values. Later in 1954, a paper by Dantzig, Fulkerson, and Johnson was published which considered integer solutions to the traveling salesman problem (8). In the four years following work continued to obtain a computational technique which could be guaranteed to converge to a solution in a finite number of steps. Finally in 1958 Gomory developed a method for the all integer case and two years later in 1960 he published the algorithm for the mixed integer-continuous variable problem (9, 10).

In the location problem one is concerned with the placement of items so as to optimize the overall operation of the system. The general problem may be formulated in the following way:

> Distribute a combination of various facilities throughout a local area in such a way as to minimize the expected cost of construction and expenses incurred in transporting resources to these facilities from predetermined source points.

Mathematical Development

The general integer linear programming problem may be formulated as follows:

Given a system of m equalities or inequalities in n variables,

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \quad & \{\leq, =, \geq\}\ b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \quad & \{\leq, =, \geq\}\ b_2 \\
& \cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \quad & \{\leq, =, \geq\}\ b_m \ ,
\end{aligned}
$$

determine the maximum value of the objective function

$$Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \ .$$

The solution to the above may be restricted by various other constraints. With integer linear programming one is concerned with the following:

(1) $x_j \geq 0$ for $j = 1, 2, \cdots, n$;

(2) some or all of the x_j be integers.

The values of a_{ij} and b_i in the system of equations above are known constants. The constants c_j in the objective function are also determined from prior considerations and usually are in the form of cost or profit on a per unit basis.

As in the transportation problem (7, p. 299-313), in the location problem the constants a_{ij} are assigned the values 0 or 1 and the b_i are either in the form of (1) capacities or demand at a particular location for a facility; or (2) supply or availability of resources at the predetermined source points. The cost factors are the costs of shipping one unit of resource from one point to another. In the location problem a fixed charge is made to account for the expense of constructing a facility at any location. The following variables and constants define the parameters of the location problem:

Let

S = the number of predetermined locations from which the resources are to be shipped,

L = the number of possible locations for the facilities,

x_{ij} = the amount to be shipped from source point i to the facility at location j, where $i = 1, 2, \cdots, S$; $j = 1, 2, \cdots, L$,

c_{ij} = the cost incurred during shipment of one unit from source point i to facility j, where $i = 1, 2, \cdots, S$; $j = 1, 2, \cdots, L$ (The units of measurement of x_{ij} and c_{ij} may be given in any convenient basis),

C_{Bj} = the cost of building a facility at location j, where $j = 1, 2, \cdots, L$.

The cost C_{Bj} is known as a fixed charge. For example, regardless of the quantity of resources shipped from source point i to the facility at location j there is a constant cost charged to the system. The cost C_{Bj} is positive and will in general change from location to location.

From the definitions developed so far, one can now consider the total cost incurred in shipping various quantities of resources x_{iv} from the several source points i to a given prospective location v for a facility. This total cost C_{Tv} may be written:

$$C_{Tv} = \sum_{i=1}^{S} c_{iv} x_{iv} + C_{Bv} .$$

This relationship may be interpreted graphically as in Figure I.

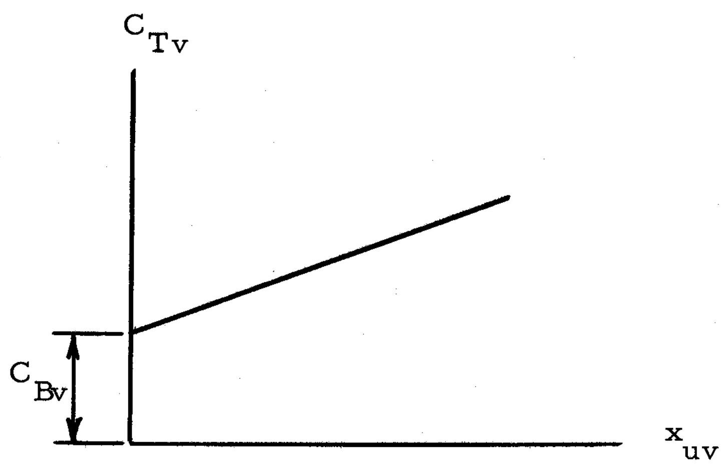

Figure I. Fixed Charge Cost of Construction .

Here it is assumed that x_{iv} is positive for $i = u$ only. That is, this relationship reflects only the shipment from one source point to this particular facility. As shipments from other source points are introduced, additional terms of the form $c_{iv} x_{iv}$ would be added, where $i = 1, 2, \cdots, S \ (i \neq u)$. The graphical representation would become more complex in three dimensions and impossible in four and higher dimensions. However, although the total cost C_{Tv} would increase due to shipment from other source points, the fixed charge C_{Bv} would remain the same.

There are several other variables requiring definitions:

Let

M_i = the supply of resources available for distribution at source point i, $i = 1, 2, \cdots, S$,

N_j = the capacity of the facility at location j, $j = 1, 2, \cdots, L$,

$$\delta_j = \begin{cases} 0, & \text{if } \sum_{i=1}^{S} x_{ij} = 0 \text{ for } j = 1, 2, \cdots, L, \\ 1, & \text{if any } x_{ij} > 0 . \end{cases}$$

An important relationship exists between the cost C_{Bj} and the variable δ_j. In the above definition there would be no shipment of resources to location j if $\sum_{i=1}^{S} x_{ij} = 0$ and would imply the corresponding cost $C_{Bj} = 0$. Now the product $\delta_j C_{Bj}$ may be used as an expression for the fixed charge where $\delta_j = 1$ would correspond to constructing a facility at location j and $\delta_j = 0$ would imply a facility is not to be built at location j. In this model, the sequence $\{\delta_j\}$ is a sequence of zeroes and ones describing a building program where $\delta_j = 1$ implies that a facility is to be built at location j, and $\delta_j = 0$ implies that no such facility is to be built. The sequence $\{x_{ij}\}$ constitutes a shipping schedule where x_{ij} is the amount shipped from source point i to facility location j.

Therefore, the objective function $F(\{x_{ij}\}, \{\delta_j\})$ may be expressed as a function of the building program $\{\delta_j\}$ and shipping schedule $\{x_{ij}\}$ --allowing the location problem to be formulated as follows:

(1) The system of constraints for the supply at each source point i are:

$$\sum_{j=1}^{L} x_{ij} = M_i; \quad i = 1, 2, \cdots, S.$$

This set of equations says that all of the resources at each source point must be distributed among the various facilities.

(2) The constraints for demand are as follows:

$$\sum_{i=1}^{S} x_{ij} \leq N_j; \quad j = 1, 2, \cdots, L.$$

This set of inequalities says that the amount shipped to the facility at location j must not exceed N_j.

(3) The constraints on δ_j, $j = 1, 2, \cdots, L$, are by definition

$$\delta_j = \begin{cases} 0, & \text{if } \sum_{i=1}^{S} x_{ij} = 0 , \\ 1, & \text{if } \sum_{i=1}^{S} x_{ij} > 0 . \end{cases}$$

(4) The objective function to be optimized takes the form

$$\underset{\{x_{ij}\},\{\delta_j\}}{\text{minimize}}\ F\ (\{x_{ij}\},\ \{\delta_j\})$$

$$= \sum_{j=1}^{L} \sum_{i=1}^{S} \delta_j (c_{ij} x_{ij}) + \sum_{j=1}^{L} C_{Bj} \delta_j$$

with the minimization performed over choices of building programs $\{\delta_j\}$ and shipping schedules $\{x_{ij}\}$.

The objective function presented above is not in linear form since it involves the product of terms δ_j with x_{ij}, however, this may be converted to linear form by the following proposition:

PROPOSITION:

$$F\ (\{x_{ij}\}, \{\delta_j\}) = \sum_{j=1}^{L} \sum_{i=1}^{S} (c_{ij} x_{ij} + \delta_j \frac{C_{Bj}}{S}) .$$

PROOF:

$$F(\{x_{ij}\}, \{\delta_j\}) = \sum_{j=1}^{L} \sum_{i=1}^{S} \delta_j (c_{ij} x_{ij}) + \sum_{j=1}^{L} \delta_j C_{Bj} .$$

Now if $\delta_j = 0$ then by definition $x_{ij} = 0$ for all i. Then, terms of the form

$$\delta_j (c_{1j} x_{1j} + c_{2j} x_{2j} + \cdots + c_{Sj} x_{Sj}) = 0$$

$$= c_{1j} x_{1j} + c_{2j} x_{2j} + \cdots + c_{Sj} x_{Sj} = 0$$

since both $\delta_j = 0$ and $x_{ij} = 0$. Also, if $\delta_j = 1$ then some $x_{ij} > 0$ or

$$\delta_j(c_{1j}x_{1j} + c_{2j}x_{2j} + \cdots + c_{Sj}x_{Sj}) > 0$$

$$= c_{1j}x_{1j} + c_{2j}x_{2j} + \cdots + c_{Sj}x_{Sj} > 0$$

provided $c_{ij} > 0$. Therefore, $F(\{x_{ij}\}, \{\delta_j\})$ may be written

$$\sum_{j=1}^{L} \sum_{i=1}^{S} c_{ij}x_{ij} + \sum_{j=1}^{L} \delta_j C_{Bj}$$

$$= \sum_{j=1}^{L} \sum_{i=1}^{S} c_{ij}x_{ij} + S\sum_{j=1}^{L} \delta_j \frac{C_{Bj}}{S}$$

$$= \sum_{j=1}^{L} \sum_{i=1}^{S} (c_{ij}x_{ij} + \delta_j \frac{C_{Bj}}{S}) .$$

This completes the proof of the proposition.

Therefore, the objective function to be optimized may be written in the form:

$$\underset{\{x_{ij}\},\{\delta_j\}}{\text{minimize}} \; F(\{x_{ij}\}, \{\delta_j\}) = \sum_{j=1}^{L} \sum_{i=1}^{S} (c_{ij}x_{ij} + \delta_j \frac{C_{Bj}}{S}) .$$

The values of δ_j may be restricted as required in the definition by introducing constraints to force δ_j in the interval

$$0 \leq \delta_j \leq 1.$$

To guarantee that δ_j assume only values in the above range consider the following discussion as extended from (12, p. 253).

Suppose there exists an upper bound for the variables $\chi = \sum_{i=1}^{S} x_{ij}$; in this particular application an upper bound does exist, namely N_j, the capacity of the facility at location j. Clearly,

$$\sum_{i=1}^{S} x_{ij} \leq N_j \qquad j = 1, 2, \cdots, L$$

$$\text{or} \quad \sum_{i=1}^{S} x_{ij} \leq \delta_j N_j \qquad j = 1, 2, \cdots, L$$

$$\text{or} \quad \sum_{i=1}^{S} x_{ij} - \delta_j N_j \leq 0 \qquad j = 1, 2, \cdots, L .$$

From this last inequality it follows that $\sum_{i=1}^{S} x_{ij}$ cannot be positive unless $\delta_j = 1$. That is, if $\delta_j = 1$ then

$$\sum_{i=1}^{S} x_{ij} - N_j \leq 0 \qquad \text{or}$$

$$\sum_{i=1}^{S} x_{ij} \leq N_j .$$

Since N_j is an upper bound for $\sum_{i=1}^{S} x_{ij}$, the range of values $\sum_{i=1}^{S} x_{ij}$ may assume is not restricted in any practical sense.

If, on the other hand, $\delta_j = 0$ then the inequality reduces to $\sum_{i=1}^{S} x_{ij} \leq 0$ which means $x_{ij} = 0$ (for $i = 1, 2, \cdots, S$) since it is assumed that $x_{ij} \geq 0$. Also, if a solution is optimum the value of δ_j will not be 1 if $\sum_{i=1}^{S} x_{ij} = 0$ because the objective function $F(\{x_{ij}\}, \{\delta_j\})$ would be reduced further by having $\delta_j = 0$. That is, a fixed charge of the form C_{Bj} could be eliminated.

Now the general mathematical model of location problems may be formulated as follows:

$$\underset{\{x_{ij}\}, \{\delta_j\}}{\text{minimize}} \quad F(\{x_{ij}\}, \{\delta_j\})$$

$$= \sum_{j=1}^{L} \sum_{i=1}^{S} \left(c_{ij} x_{ij} + \delta_j \frac{C_{Bj}}{S}\right),$$

Subject to:

(1) $x_{ij} \geq 0$;

(2) $0 \leq \delta_j \leq 1$, $\quad j = 1, 2, \cdots, L, \quad \delta_j$ integers ;

$$(3) \quad \left[\sum_{i=1}^{S} x_{ij}\right] - \delta_j N_j \leq 0, \qquad j = 1, 2, \cdots, L;$$

$$(4) \quad \sum_{j=1}^{L} x_{ij} = M_i, \qquad i = 1, 2, \cdots, S;$$

Thus, the location problem has been stated in terms of a mixed integer-continuous variable problem where the control variables δ_j are assigned integer values and x_{ij} take on values in a bounded continuum.

The number of variables and constraints may produce some difficulties as far as the computational aspects of such problems is concerned as both of these quantities increase at a rapid rate as the problem size increases. When dealing with inequalities, it is necessary to introduce slack variables to transform these inequalities to equalities.

In this problem it will be necessary to introduce the following slack variables:

(1) L slack variables in $\left[\sum_{i=1}^{S} x_{ij}\right] - \delta_j N_j \leq 0$.

(2) L slack variables in $\delta_j \leq 1$.

Therefore, the number of variables is :

$$N = L(S+2)$$

while the number of constraints is :

$$M = S + 2L .$$

Although algorithms for the solution of integer and mixed integer-continuous variable programs have been devised only recently (9, 10), computer programs do exist for the solution of such problems (4). At the present time, attempts to obtain a program to be run on the computer at Oregon State University have been delayed; however, it is hoped these programs will be available in the near future. An example illustrating the algorithm for the solution of a mixed integer-continuous variable problem is given in Appendix A.

APPLICATIONS

The Medford Study

During recent years increased particulates emitted from wood waste burners in the Medford area have resulted in heavy contamination of the atmosphere throughout the Bear Creek Valley of Oregon. To find a solution to this growing problem a research project was initiated by the Engineering Experiment Station at Oregon State University on June 12, 1964 and a research proposal was submitted to the Forest Industries Air Quality Committee of the Oregon Associated Industries at their request (3). The growing concern by state and local officials prompted group action and investigation into possible methods of improvement or alternative methods of solution to the air pollution problem in this area.

The primary areas of investigation proposed for the original study are (3, p. 2):

> (1) The wood waste burner at each of the mills included in the study will be critically examined. A report of findings will be prepared which will provide mill owners and the industry with information on how to reduce smoke and fly ash emissions as much as possible with existing equipment. The findings of this phase of the study will be incorporated in the final report.

(2) Meteorological and air quality data will be collected. These data will be analyzed to obtain additional information on air pollution factors in the Medford basin. From the collected data, an attempt will be made to predict pollution levels at times of the year other than the period when this study is conducted.

(3) A preliminary study will be conducted to determine the feasibility of alternate methods of wood waste residue disposal. This study will utilize existing information and will require the cooperation of individual mill owners in the furnishing of data.

By more efficient burning of wood waste in the area, possibly under some form of restricted burning, it was hoped that particulate contamination of the atmosphere could be considerably reduced. However, if results from this phase of study indicated that the present level of air pollution could not be reduced to comply with Oregon State regulations, an alternative method of waste disposal might become mandatory. In anticipation of this possibility the following alternatives were suggested to improve existing conditions.

(1) <u>Disposal of Wood Waste by Incineration:</u>

By disposing of mill waste at much higher temperatures and controlled burning, atmospheric contamination would be reduced.

(2) <u>Disposal of Wood Waste Through the Generation of Electric Power:</u>

By burning waste material at high temperatures the possibility exists that electric power would be an efficient by-product of wood waste disposal.

In the event either of these proposals or a combination of the two were adopted, it is assumed that some form of mutual cooperation would exist among the respective mill owners so that incurred costs would be shared on an equal or proportionate basis.

After briefly reviewing the background and nature of the Medford air pollution problem, this location problem may be formulated as follows:

> Distribute a combination of power plants and incinerators throughout the Medford area of the Bear Creek Valley in such a way as to minimize the construction and operating costs of such facilities to respective mill owners.

The problem, therefore, is one of locating a combination of these facilities so as to minimize an objective function similar to the one presented above. However, this type of fixed-charge location problem is much more difficult than the one presented in the previous model. Obvious questions arise when consideration is given to the economic tradeoff between the higher cost incurred in power plant construction and the financial benefits to be gained from the sale or use of electric power produced.

Table I presents the relative costs of various sizes of power plants and incinerators(2, p. 31).

The common unit of wood residue is defined as 200 cubic feet and weighs approximately 2,000 pounds. The residue, consisting of

TABLE I. FIXED CHARGE COSTS OF CONSTRUCTION

Installation	Cost of Construction
2 - 30 unit/hr Incinerators	$ 1,310,000
3 - 20 unit/hr Incinerators	1,305,000
4 - 15 unit/hr Incinerators	1,328,000
2 - 30 Megawatt Generating Stations	12,708,000
3 - 20 Megawatt Generating Stations	13,328,000
4 - 15 Megawatt Generating Stations	13,350,000

sawdust, shavings, chips and bark, is weighed without packing in the containing vehicle of transportation at the mill. The table presents figures for power plants and incinerators designed to handle 15, 20 and 30 units of fuel per hour. It can be seen from studying the table that as far as the initial cost of construction of a facility is concerned an incinerator would be less expensive in all of the 15, 20 and 30 unit sizes.

In Table II are summarized the relative wood waste disposal costs per unit of fuel for the various sizes of power plants and incinerators(2, p. 30).

Here it may be observed that the cost of operating a power plant is considerably less than that for a corresponding incinerator. The reason is that in the case of the power plant, the electric power

TABLE II. OPERATING COST PER INSTALLATION

Installation	Cost, $/Unit
2 - 30 unit/hr Incinerators	$ 2.58
3 - 20 unit/hr Incinerators	2.83
4 - 15 unit/hr Incinerators	3.00
2 - 30 Megawatt Generating Stations	1.67
3 - 20 Megawatt Generating Stations	2.15
4 - 15 Megawatt Generating Stations	2.49

produced by the installation is sold to reduce the operating costs incurred. A more extensive cost breakdown for the various proposals is given in (2, p. 54-57).

From this example it can be seen that as one deviates from the basic model presented above, the fixed charges and other cost considerations become increasingly more complex. The following discussion relates the Medford project to the location problem.

Associated with the construction of a power plant of size h at location j is a fixed charge C_{jPh} where h may assume the values 1, 2, and 3 to correspond respectively with the 15, 20 and 30 unit plants. In a similar manner, a cost C_{jIh} may be defined as the cost of erecting an incinerator of size h at location j. It is assumed that both of these costs are calculated on the unit basis previously discussed. Now suppose the profit to be gained from the sale or use of electric power can be calculated for each of the various sizes of power plants under consideration. That is, let K_{jPh} denote the profit per unit of wood waste derived from the installation of a power plant of size h at location j. Since an incinerator would produce no electric power the "profit" $K_{jIh} \equiv 0$. A new fixed charge may therefore be defined to incorporate this feature. That is, a pair of fixed charges Π_{jPh} and Π_{jIh} are defined as follows:

$$\Pi_{jPh} = C_{jPh} - K_{jPh} ,$$

$$\Pi_{jIh} = C_{jIh} - K_{jIh} = C_{jIh} .$$

The following variables and constants defining the parameters in the Medford location study are as follows:

Let

S = the number of sawmills or sources of wood waste in the Medford area,

L = the number of feasible locations for power plants and/or incinerators in the area.

Consider x_{ij} which would normally denote the amount of waste material to be shipped from sawmill i to the incinerator or power plant at location j. As previously discussed the subscript i has the range $i = 1, 2, \cdots, S$; but special consideration must be given to the range of the subscript j. If no assumptions are made as to the relative merit of the various locations, (i.e. suppose for any m and n in the set of possible locations for a facility, it is equally possible for location m to receive a 15, 20 or 30 unit burner as it is for location n to receive the same installation - in a practical situation an analysis of the area might rule out the possibility of installing a 30 unit installation in an area which is far removed from the majority of sawmills), then for each construction site consideration must be taken for installing 2-30 unit power plants; 2-30 unit incinerators; 3-20 unit power plants; 3-20 unit incinerators; 4-15 unit power plants and 4-15 unit incinerators or a total of 18 distinct facilities. It follows therefore that subscripting of the variables x_{ij} would be convenient. As a consequence, Let

x_{ijkh} = the amount of wood waste per 200 cubic feet which is shipped from sawmill i to an installation of type k and size h at location j, where $i = 1, 2, \cdots, S$ (sawmills), $j = 1, 2, \cdots, 4L$ (sites for facilities), $k = 1, 2$ (facility types), $h = 1, 2, 3$ (facility sizes).

Increasing the range of the subscript j from L to $4L$ will allow for the possibility of having more than one installation at a certain location. The shipping costs and fixed charges associated with these "new locations" will necessarily include some duplication.

For convenience let the number $4L$ be denoted by the symbol L'. Therefore, the range of the subscript j is $j = 1, 2, \cdots, L'$.

Let

c_{ijkh} = the cost involved in shipping one unit of fuel from sawmill i to the installation of type k and size h at location j. The problem as formulated will have $c_{ijkh} \equiv c_{ij}$ since the additional subscripts are only useful in determining what happens to the fuel once it reaches location j,

Π_{jkh} = the fixed charge associated with constructing a facility of type k and size h at location j. This combines into one quantity the two distinct charges Π_{jPh} and Π_{jIh} by the additional subscript k where $k = 1, 2$ for power plants and incinerators respectively. The range of the subscript j is extended to L',

M_i = the supply in 200 cubic feet per day of wood waste available at sawmill i,

N_{jkh} = the capacity of the facility of type k and size h located at point j,

$$\delta_{jkh} = \begin{cases} 0, & \text{if } \sum_{i=1}^{S} x_{ijkh} = 0 , \\ 1, & \text{if } \sum_{i=1}^{S} x_{ijkh} > 0 . \end{cases}$$

In the Medford study the sequence $\{\delta_{jkh}\}$ is a sequence of zeroes and ones used to describe a building program where $\delta_{jkh} = 1$ would imply that a facility of type k and size h is to be constructed at location j, while $\delta_{jkh} = 0$ would imply that no such facility is to be built. In this way the product $\delta_{jkh}\Pi_{jkh}$ may be used as an expression for the fixed charge with $\delta_{jkh} = 1$ or 0 corresponding respectively to building or not building a particular facility. Similarly, the sequence $\{x_{ijkh}\}$ constitutes a shipping schedule where x_{ijkh} is the amount shipped from source point i to facility of type k and size h at location j. Therefore, the objective function $F(\{x_{ijkh}\}, \{\delta_{jkh}\})$ may be expressed as a function of the building program $\{\delta_{jkh}\}$ and shipping schedule $\{x_{ijkh}\}$ as follows:

$$F(\{x_{ijkh}\}, \{\delta_{jkh}\}) =$$

$$\sum_{h=1}^{3}\sum_{k=1}^{2}\sum_{j=1}^{L'}\sum_{i=1}^{S} \delta_{jkh}(c_{ijkh}x_{ijkh}) + \sum_{h=1}^{3}\sum_{k=1}^{2}\sum_{j=1}^{L'} \delta_{jkh}\Pi_{jkh}$$

$$= \sum_{h=1}^{3}\sum_{k=1}^{2}\sum_{j=1}^{L'}\sum_{i=1}^{S} (c_{ijkh}x_{ijkh} + \delta_{jkh}[\frac{\Pi_{jkh}}{S}]) .$$

By extension of the previous theory, assume that there exists an upper bound for the variables $\sum_{i=1}^{S} x_{ijkh}$. A suitable upper bound does exist namely N_{jkh}, that is

$$\sum_{i=1}^{S} x_{ijkh} \leq N_{jkh} \qquad j = 1, 2, \cdots, L'; \; k = 1, 2; \; h = 1, 2, 3$$

or $$\sum_{i=1}^{S} x_{ijkh} \leq \delta_{jkh} N_{jkh}$$

or $$\sum_{i=1}^{S} x_{ijkh} - \delta_{jkh} N_{jkh} \leq 0 \; .$$

In the last inequality $\sum_{i=1}^{S} x_{ijkh}$ will not be positive unless $\delta_{jkh} = 1$. For if $\delta_{jkh} = 0$ then $\sum_{i=1}^{S} x_{ijkh} = 0$ since the constraint $x_{ijkh} \geq 0$ must simultaneously be satisfied. Therefore, this inequality represents a mathematical equation for the desired properties inherent in the definition of δ_{jkh}.

From the above discussion the Medford project defined as a location problem may be expressed in terms of a mixed integer-continuous linear programming problem with fixed charges and defined in the following way:

$$\underset{\{x_{ijkh}\},\{\delta_{jkh}\}}{\text{minimize}} \quad F(\{x_{ijkh}\},\{\delta_{jkh}\})$$

$$= \sum_{h=1}^{3} \sum_{k=1}^{2} \sum_{j=1}^{L'} \sum_{i=1}^{S} (c_{ijkh} x_{ijkh} + \delta_{jkh} [\frac{\Pi_{jkh}}{S}]) .$$

Subject to the following constraints:

(1) $x_{ijkh} \geq 0$;

(2) $0 \leq \delta_{jkh} \leq 1, \quad \delta_{jkh}$ integers;

(3) $\left[\sum_{i=1}^{S} x_{ijkh}\right] - \delta_{jkh} N_{jkh} \leq 0$;

(4) $\sum_{h=1}^{3} \sum_{k=1}^{2} \sum_{j=1}^{L'} x_{ijkh} = M_i$.

Thus, the Medford study formulated above illustrates the general model for location problems presented in the previous section in determining the optimum location of power plants and/or incinerators to be installed for the purpose of wood waste disposal. Although numerical results were not obtained for this particular problem, Appendix B presents an example similar to the Medford study where results were obtained by direct enumeration of the possible cases. It is noted that direct enumeration would be computationally unfeasible for the entire Medford study illustrating the application for the above model.

Other Examples

The model presented could be applied to a variety of other situations. The relative location of industrial plants would be one such example. Here the resources would correspond to the particular raw natural resources used by the plant and the source points to the sources of supply for such resources. A fixed charge similar to the one in the Medford study may be introduced for the initial cost of construction. In this example care must be used when defining the relative cost factor c_{ij}; for here consideration must be made not only for transportation costs between the source points and the facility but also for the transportation costs between the facility and final destination of the product.

The relative location of warehouses could be described in a similar way. Again, additional care must be taken when considering the costs involved in shipping the inventory from the source points to the warehouses and shipping the inventory from the warehouses to the retail outlets.

An application similar to the Medford study is the location of a refuse disposal plant. Here resources might include personal and industrial refuse but need not be considered on an individual "house" basis. Rather, an orderly division of the area might be accomplished either by placing the region on a rectangular grid to give equal division of the area or by splitting the region into groups on the basis of the amount of refuse to be disposed of.

These examples illustrate various applications of the location problem for the model developed. In considering any particular example, care must be used in defining the various parameters as illustrated in the Medford study.

SUMMARY

The placement or distribution of discrete facilities over a discrete region is defined as a location problem. This problem does not lend itself to solution by linear methods and hence some nonlinear method or form of approximation must be used. The method of solution discussed in this paper may be classified as integer linear programming with fixed charges.

The main points in the integer model developed may be listed as follows:

1. There exists a fixed charge C_{Bj} which is incurred if a facility is to be constructed at location j. This charge is independent of the amount of resources to be shipped to location j from the various source points, that is, once it is decided to build an installation of size k at location j a constant cost is assessed to the system regardless if the installation is operating at full capacity in relation to the rest of the system.

2. Through the definition of the control variable δ_j, the objective function $F(\{x_{ij}\}, \{\delta_j\})$ may be expressed as a quadratic function of the shipping schedule $\{x_{ij}\}$ and the building program $\{\delta_j\}$ as follows:

$$F(\{x_{ij}\},\{\delta_j\}) = \sum_{j=1}^{L} \sum_{i=1}^{S} \delta_j(c_{ij}x_{ij}) + \sum_{j=1}^{L} \delta_j C_{Bj} .$$

Although the above expression is not in linear form it may be reduced to

$$F(\{x_{ij}\},\{\delta_j\}) = \sum_{j=1}^{L} \sum_{i=1}^{S} (c_{ij}x_{ij} + \delta_j \frac{C_{Bj}}{S}) .$$

3. Following the presentation of the general integer programming model, a location problem dealing with the distribution of power plants and/or incinerators in the Medford area of the Bear Creek Valley of Oregon was formulated as follows:

$$\underset{\{x_{ijkh}\},\{\delta_{jkh}\}}{\text{minimize}} \quad F(\{x_{ijkh}\},\{\delta_{jkh}\})$$

$$= \sum_{h=1}^{3} \sum_{k=1}^{2} \sum_{j=1}^{L'} \sum_{i=1}^{S} (c_{ijkh}x_{ijkh} + \delta_{jkh}[\frac{\Pi_{jkh}}{S}]) .$$

Subject to the following constraints:

(1) $x_{ijkh} \geq 0$;

(2) $0 \leq \delta_{jkh} \leq 1, \quad \delta_{jkh}$ integers ;

(3) $\left[\sum_{i=1}^{S} x_{ijkh} \right] - \delta_{jkh} N_{jkh} \leq 0$;

$$(4) \quad \sum_{h=1}^{3} \sum_{k=1}^{2} \sum_{j=1}^{L'} x_{ijkh} = M_i \, ;$$

4. Various other examples were presented illustrating the application of the general integer programming model for the solution of location problems.

It is intended the integer linear programming model developed on the preceding pages will be a useful approach to the solution of the problem of locating discrete facilities over a finite number of possible locations. With advances in high speed computers and sophisticated algorithms just recently developed, the solution to the location problem as formulated above will be possible.

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APPENDIX

APPENDIX A

The following example illustrates Gomory's algorithm for the solution of a mixed integer-continuous variable problem (12, p. 290). Although this example is not a location problem the same procedure repeated in higher dimensions with additional constraints would be used to obtain a solution in the Medford study. Computation was performed by hand using the dual-simplex method after addition of the cutting plane constraints and the results checked on the IBM 1410 Computer at Oregon State University.

<u>Problem:</u>

maximize $z = 8x_1 + 6x_2$

Subject to:

(1) $x_1, x_2 \geq 0$,

(2) x_1 an integer,

(3) $3x_1 + 5x_2 \leq 11$,

(4) $4x_1 + x_2 \leq 8$.

The feasible set (without consideration of the integer constraint) and objective function are presented in Figure II. It is clear geometrically that the maximum solution ignoring the integer constraint occurs at point C, whereas it will be shown the

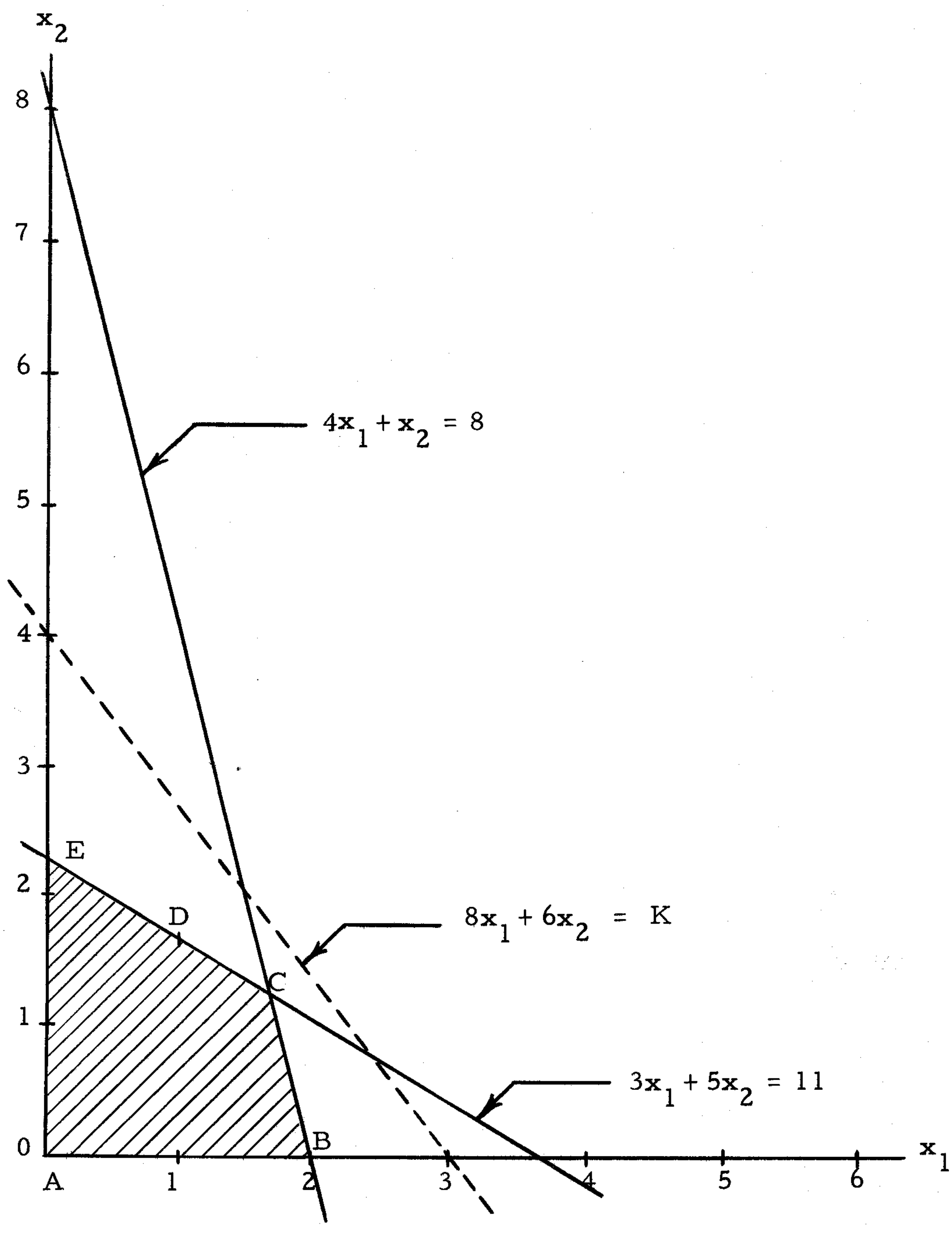


Figure II. Feasible Set for the Mixed Integer-Continuous Variable Problem.

optimum solution for the system with x_1 restricted to integer values will occur at point D.

In attempting to maximize the function $z = 8x_1 + 6x_2$ subject to the given constraints, the following procedure will be carried out. First, a solution will be obtained using the simplex method and the basic set of variables will be examined. If the basic set has x_1 an integer, the algorithm is terminated. However, if an integer solution has not been obtained a "cutting plane" constraint is then introduced to further restrict the feasible set and the problem is again solved using the simplex method. This procedure is repeated until all integer restrictions have been met. The theory for determining the cutting plane is given in (12, p. 282-285), along with a proof for the convergence of the algorithm.

Upon completion of any of the linear programs not satisfying the integer restrictions, the inequality $\sum_{j \in R} (-d_{uj})x_j \leq -f_{ub}$ is introduced as a cutting plane constraint and added to the set of "original" constraints. R is the set of non-basic variables, f_{ub} is the fractional part of the right hand side of the u^{th} equation in a particular basic set (u represents a variable in the basic set which must be an integer but is not an integer at the current stage) and

d_{uj}

$= \bar{a}_{uj}$, if x_j is nonbasic, is not required to be an integer and $\bar{a}_{uj} \geq 0$,

$= \frac{f_{ub}}{1-f_{ub}} \left| \bar{a}_{uj} \right|$, if x_j is nonbasic, is not required to be an integer and $\bar{a}_{uj} < 0$,

$= f_{uj}$, if x_j is nonbasic, required to be an integer and $f_{uj} \leq f_{ub}$,

$= \frac{f_{ub}}{1-f_{ub}} (1-f_{uj})$, if x_j is nonbasic, required to be an integer and $f_{uj} > f_{ub}$,

where $f_{uj} = \bar{a}_{uj} - \delta_{uj}$, and δ_{uj} is the largest integer less than or equal to $\bar{a}_{uj}$,

$f_{ub} = b_u - \delta_{ub}$, and δ_{ub} is the largest integer less than or equal to b_u ,

$\bar{a}_{uj}$, is the entry in the u^{th} equation for the j^{th} nonbasic variable in the simplex tableau,

b_u, is the right hand side of the u^{th} equation in the simplex tableau.

Some motivation may be gained for the above procedure by considering the line of reasoning presented in (12, p. 273). Any basic variable u not yet an integer and required to be an integer may be rewritten as

$$x_u = b_u - \sum_{j \in R} \bar{a}_{uj} x_j .$$

If $\bar{a}_{uj} = \delta_{uj} + f_{uj}$ and $b_u = \delta_{ub} + f_{ub}$ then since b_u is not an integer, $f_{ub} > 0$. Now the above expression may be written

$$x_u = \delta_{ub} - \sum_{j \in R} \delta_{uj} x_j + f_{ub} - \sum_{j \in R} f_{uj} x_j$$

or

$$x_u - \delta_{ub} + \sum_{j \in R} \delta_{uj} x_j = f_{ub} - \sum_{j \in R} f_{uj} x_j .$$

If a solution is to be an integer then $x_u - \delta_{ub} + \sum_{j \in R} \delta_{uj} x_j$ must be an integer and hence $f_{ub} - \sum_{j \in R} f_{uj} x_j$ must be an integer. Since $\sum_{j \in R} f_{uj} x_j$ must be greater than or equal to zero and $0 < f_{ub} < 1$, $f_{ub} - \sum_{j \in R} f_{uj} x_j \leq 0$ or $\sum_{j \in R} (-f_{uj}) x_j \leq -f_{ub}$. Therefore a solution to the original integer program must now satisfy the constraint

$$\sum_{j \in R} (-f_{uj}) x_j \leq -f_{ub} .$$

In the mixed-integer case, the various possibilities must be examined separately producing the various alternative definitions for d_{uj}.

The function to be maximized was solved using the simplex method and the solution is illustrated in the following tableau:

	x_1	x_2	x_3	x_4	$-z$	b
x_2	0	1	$\frac{4}{17}$	$-\frac{3}{17}$	0	$\frac{20}{17}$
x_1	1	0	$-\frac{1}{17}$	$\frac{20}{68}$	0	$\frac{29}{17}$
$-z$	0	0	$\frac{16}{17}$	$\frac{22}{17}$	1	$\frac{352}{17}$

Although x_1 and x_2 are in the basic solution, x_1 is not an integer and therefore a cutting plane constraint must be added. The non-basic variables are x_3 and x_4, neither of which is required to be an integer and hence by the above discussion

$$d_{23} = \frac{f_{2b}}{1-f_{2b}} \left| \bar{a}_{23} \right| = \frac{12}{85}$$

and

$$d_{24} = \bar{a}_{24} = \frac{20}{68} .$$

Thus, the constraint for the cut is

$$-\frac{12}{85} x_3 - \frac{20}{68} x_4 \leq -\frac{12}{17} \quad .$$

This inequality may be combined with the previous tableau to produce

	x_1	x_2	x_3	x_4	e_1	$-z$	b
x_2	0	1	$\frac{4}{17}$	$-\frac{3}{17}$	0	0	$\frac{20}{17}$
x_1	1	0	$-\frac{1}{17}$	$\frac{20}{68}$	0	0	$\frac{29}{17}$
e_1	0	0	$-\frac{12}{85}$	$-\frac{20}{68}$	1	0	$-\frac{12}{17}$
$-z$	0	0	$\frac{16}{17}$	$\frac{22}{17}$	0	1	$\frac{352}{17}$

The dual simplex method is now applied to obtain

	x_1	x_2	x_3	x_4	e_1	$-z$	b
x_2	0	1	$\frac{136}{425}$	0	$-\frac{12}{20}$	0	$\frac{136}{85}$
x_1	1	0	$-\frac{19}{85}$	0	1	0	1
x_4	0	0	$\frac{204}{425}$	1	$-\frac{68}{20}$	0	$\frac{12}{5}$
$-z$	0	0	$\frac{136}{425}$	0	$\frac{22}{5}$	1	$\frac{1496}{85}$

whereupon the final solution is:

$$x_1 = 1.0000$$

$$x_2 = 1.5999$$

$$x_3 = 0.0000$$

$$x_4 = 2.3998$$

$$e_1 = 0.0000$$

$$\max z = 17.6002$$

The effect of the cutting plane on the feasible set is illustrated in Figure III. It is noticed the feasible set is reduced and a corner is introduced at point D, allowing the final result to be obtained by the simplex method. The solution to the location problem presented in this thesis would proceed in a similar manner and be repeated until all control variables δ_j assumed the integer values zero or one for choices of the various building programs.

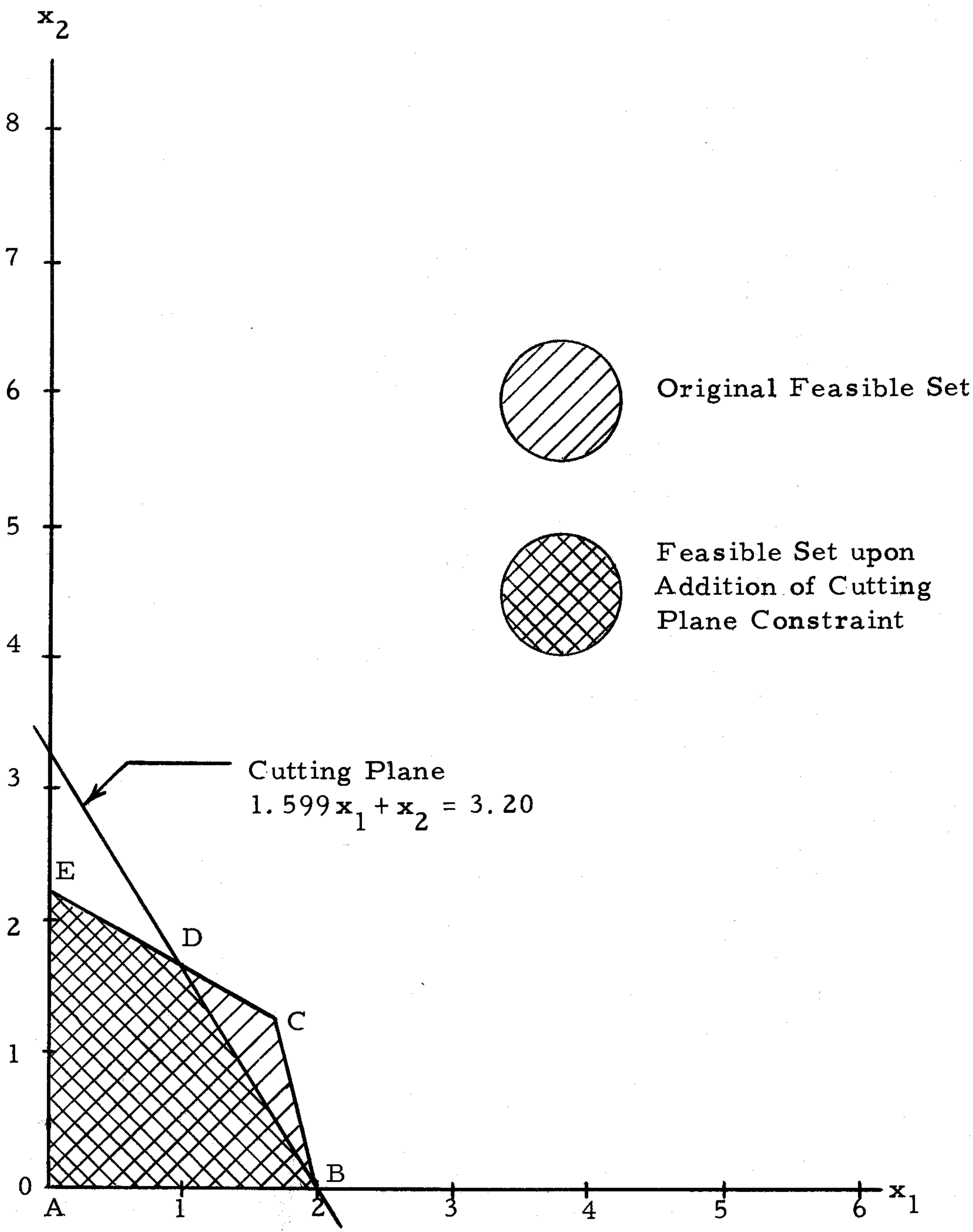


Figure III. Effect of Cutting Plane Constraint on the Feasible Set.

APPENDIX B

The following example is formulated as a location problem and an optimum solution involving integers is obtained by direct enumeration of the various possibilities. Although the data are hypothetical, the problem is intended to represent a subset of the Medford study previously discussed and involves the placement of incinerators over a finite number of possible locations. Consideration is given for two installations to be constructed at two of the three possible locations illustrating the possibility for the various multiple facility assignments in the Medford study. In considering the integer solutions for this particular example, distinct combinations of locations for facilities are examined taking three, four, and five locations at a time.

A discussion of the number of cases involved in the entire Medford study follows and it is apparent that direct enumeration of the possibilities involved in this case is impractical; clearly illustrating the application for the model developed in this thesis which systematically searches and determines an optimum solution if one exists and in a reasonable amount of time.

Problem:

Assume the region under investigation consists of five sawmills or sources of wood waste and that a preliminary study of the surrounding area revealed that three locations denoted by A, B and C are adequate for incinerator construction. Also, because of the proximity of two of the locations to the source of the fuel supply, it seems feasible that an optimal solution might consist of two incinerators constructed at these two locations. Assuming the existence of these possibilities, the following values determine the supply, capacity, transportation and fixed charge costs for the various situations:

SUPPLY (Units of fuel/day)

$$M_1 = 15.376$$

$$M_2 = 9.155$$

$$M_3 = 10.275$$

$$M_4 = 7.215$$

$$M_5 = 16.429$$

CAPACITY (Units of fuel/day)

$$N_{A1} = 20.$$

$$N_{A2} = 20.$$

$$N_{B1} = 20.$$

$$N_{B2} = 20.$$

$$N_{C1} = 20.$$

TRANSPORTATION COSTS (\$. 50/unit of fuel/day)

$$
\begin{array}{ll|ccccc|}
 & & \multicolumn{5}{c}{\text{Incinerator No.}} \\
 & & A1 & A2 & B1 & B2 & C1 \\
 & 1 & 8 & 8 & 16 & 16 & 48 \\
 & 2 & 19 & 19 & 5 & 5 & 62 \\
\text{Source No.} & 3 & 31 & 31 & 7 & 7 & 41 \\
 & 4 & 29 & 29 & 25 & 25 & 32 \\
 & 5 & 69 & 69 & 59 & 59 & 8
\end{array}
$$

FIXED CHARGE (\$. 50/unit of fuel/day)

$$\pi_{A1} = 5281$$

$$\pi_{A2} = 5281$$

$$\pi_{B1} = 5524$$

$$\pi_{B2} = 5524$$

$$\pi_{C1} = 5775$$

A diagram of the area showing the relative location of the sawmills to the sites for incinerator construction along with the various supplies, capacities, and relative costs are given in Figure IV.

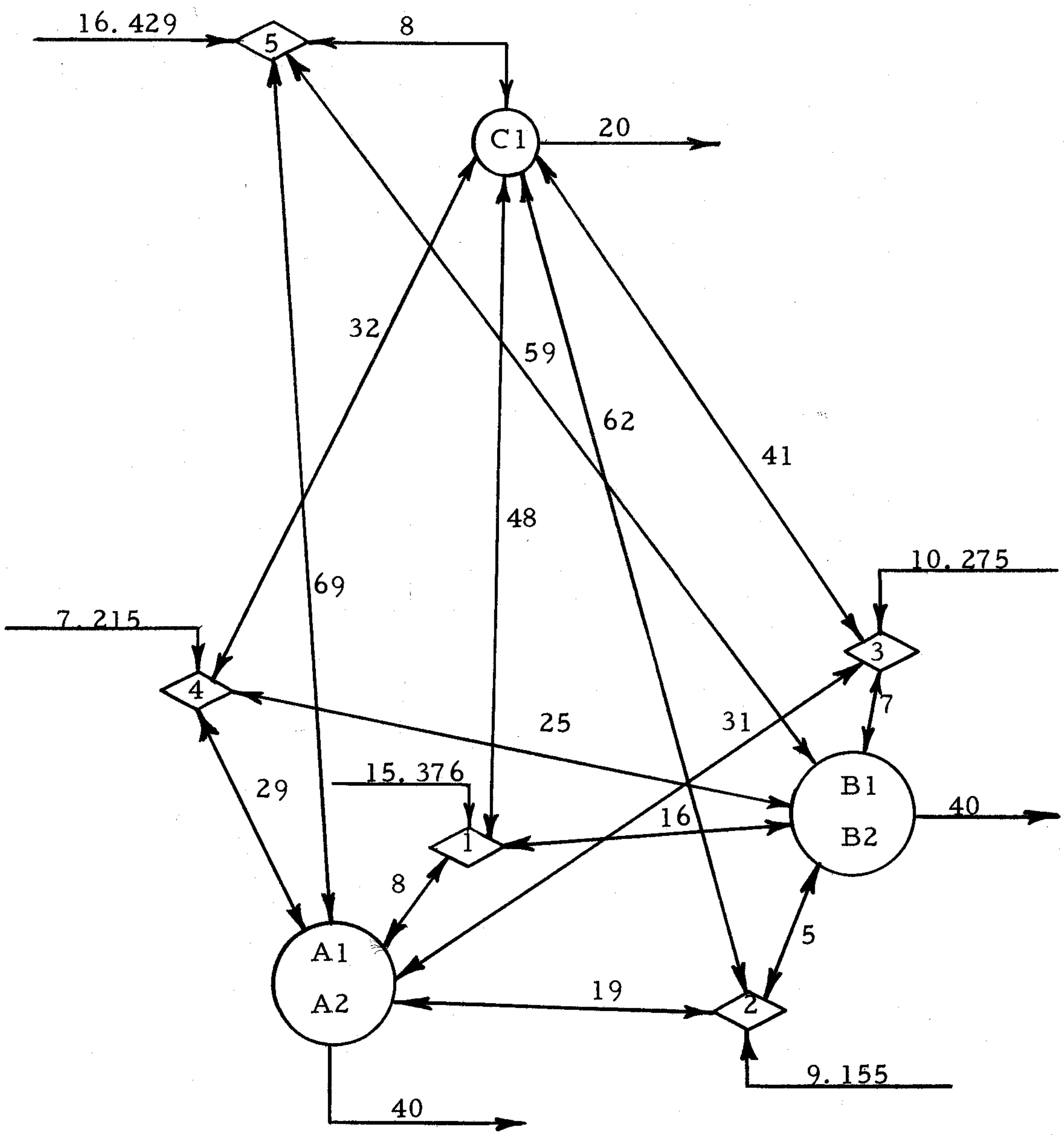


Figure IV. Relative Location of Facilities.

Solution:

One approach to the solution of this problem is to consider a linear programming formulation using the model developed and compare the values obtained with the optimal integer solution. The following discussion characterizes the linear programming approach:

$$\underset{\{x_{ij}\},\{\delta_j\}}{\text{minimize}} \quad F(\{x_{ij}\},\{\delta_j\}) = \sum_{j=1}^{5} \sum_{i=1}^{5} \left(c_{ij}x_{ij} + \delta_j \frac{\pi_j}{S}\right).$$

Subject to :

(1) $x_j \geq 0$,

(2) $0 \leq \delta_j \leq 1$,

(3) $\sum_{j=1}^{5} x_{ij} = M_i$, for $i = 1, 2, \cdots, 5$,

(4) $\left[\sum_{i=1}^{5} x_{ij}\right] - \delta_j N_j \leq 0$, for $j = 1, 2, \cdots, 5$.

It is to be noted that the above formulation is similar to the model presented in the thesis except that in this case no integer restrictions have been placed on the variables δ_j. The purpose of considering this possibility is to compare the values of the control variables δ_j obtained above to those obtained by restricting δ_j

to integer values. This problem was run on the IBM 1410 Computer at Oregon State University and the following values were obtained for the control variables δ_j:

$$\delta_{A1} = 0.7688$$

$$\delta_{A2} = 0.3608$$

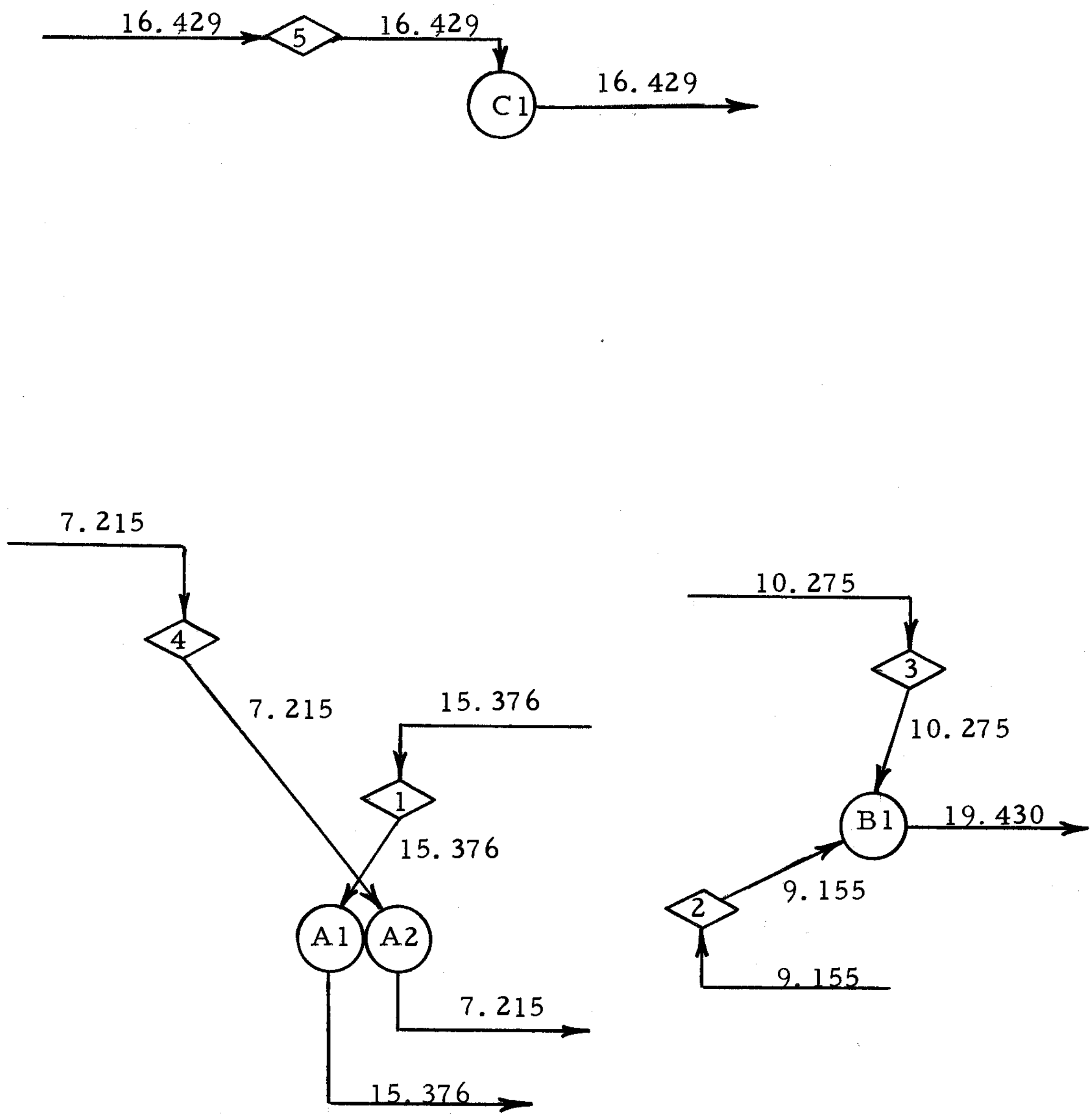
$$\delta_{B1} = 0.9715$$

$$\delta_{B2} = 0.0000$$

$$\delta_{C1} = 0.8125$$

The flow of wood waste through the system for this possibility is presented in Figure V along with the optimal value of the objective function.

The combinatorial nature of the example given is smaller in order of magnitude in comparison with the entire Medford study. If L denotes the number of locations for the incinerators and r denotes a subset of these sites to be considered for the construction of a facility then $\binom{L}{r}$ will be the number of possibilities for L locations taken r at a time. Since it is only necessary to install three incinerators to dispose of the wood waste from all sawmills, only $\binom{5}{3}$ or 10 possibilities for locating facilities need be considered. However, it is of interest to consider the distribution of wood waste throughout the system with four and five installations and therefore



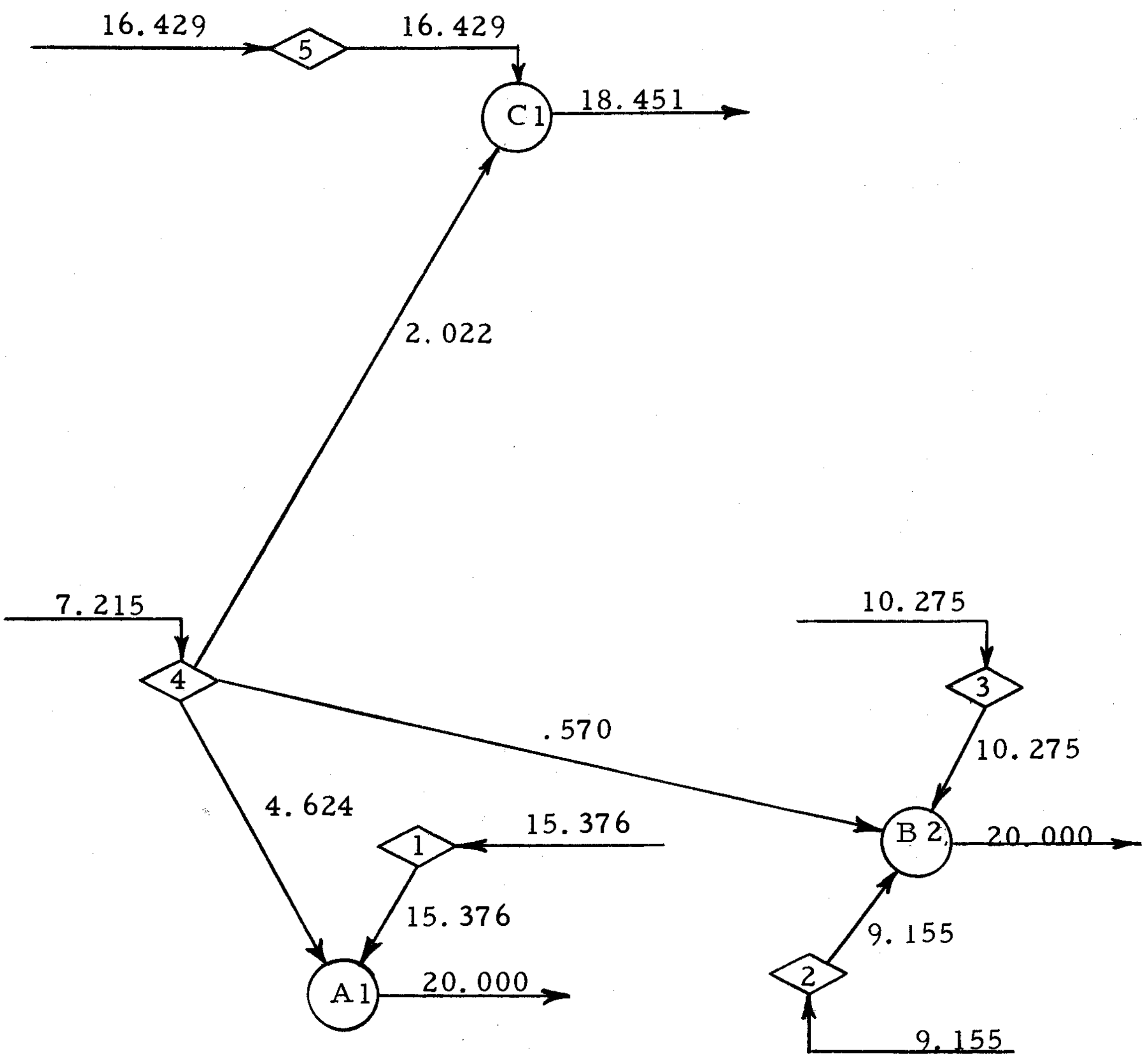
$F(\{x_{ij}\}, \{\delta_j\}) = 16657.277$ $.50/unit of fuel/day

Figure V. Linear Programming Solution.

the total number of possibilities is increased to $\binom{5}{3}+\binom{5}{4}+\binom{5}{5}$ or 16. Since the fixed charges and transportation costs are identical for the multiple facility installations the total number of cases may be reduced to the following nine possibilities (the value of the objective function $F(\{x_{ij}\}, \{\delta_j\})$ is also given in \$. 50/unit of fuel/day):

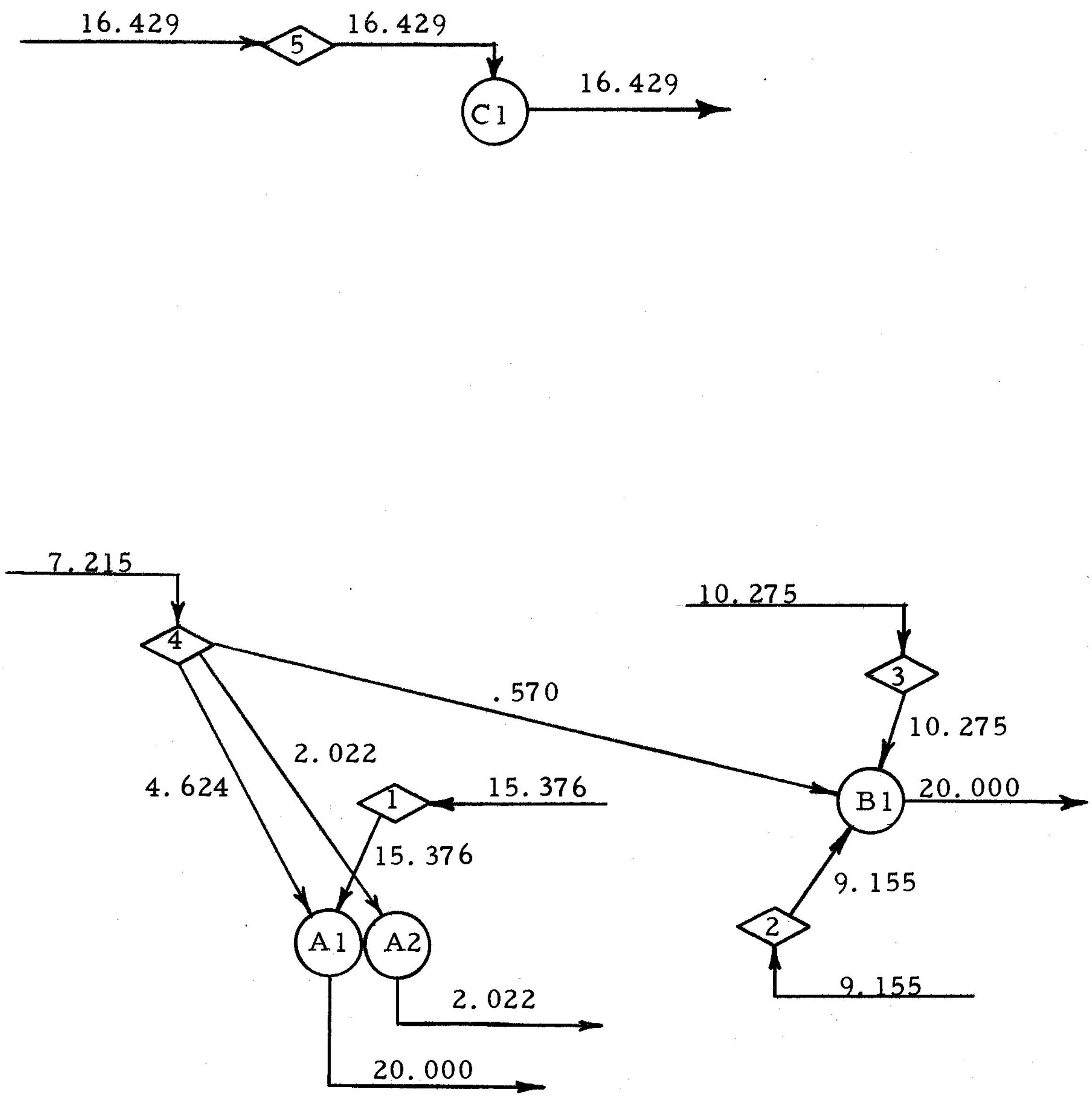
{A1, A2, B1}	-	17,663.868
{A1, B1, B2}	-	17,731.710
{A1, B2, C1}	-	17,165.166
{A1, A2, C1}	-	17,299.225
{B1, B2, C1}	-	17,512.684
{A1, A2, B1, B2}	-	23,012.708
{A1, A2, B1, C1}	-	22,440.101
{A1, B1, B2, C1}	-	22,656.519
{A1, A2, B1, B2, C1}	-	27,937.519

The optimal solution to the various integer problems taking three, four, and five locations at a time is given in Figures VI, VII and VIII respectively. A network describing the flow of wood waste through the system is presented along with the values of the objective function $F(\{x_{ij}\}, \{\delta_j\})$. Figure VI represents the optimum integer solution and involves constructing incinerators at locations A1, B2,



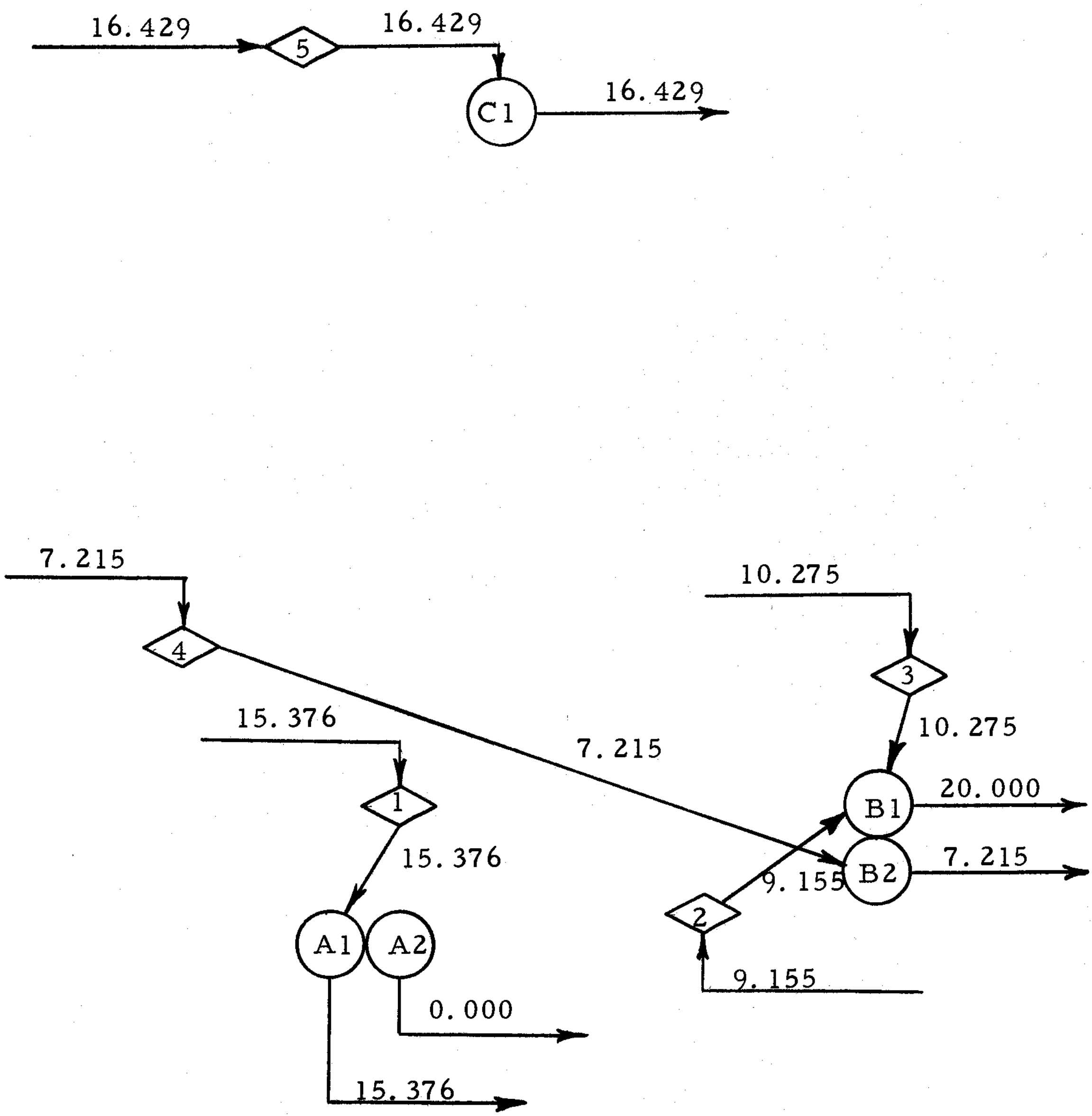
$F(\{x_{ij}\},\{\delta_j\}) = 17165.166$ \$.50/unit of fuel/day

Figure VI. Optimum Solution Involving Three Installations.



$F(\{x_{ij}\}, \{\delta_j\}) = 22440.101$ $.50/unit of fuel/day

Figure VII. Optimum Solution Involving Four Installations.



$F(\{x_{ij}\}, \{\delta_j\}) = 27937.519$ \$.50/unit of fuel/day

Figure VIII. Optimum Solution Involving Five Installations.

and C1. The minimum value of the objective function is

$$F(\{x_{ij}\}, \{\delta_j\}) = 17,165.166 \quad \$.50/\text{unit of fuel/day.}$$

It is of interest to notice that rounding the values of δ_j from the linear programming solution in this example results in an equivalent optimum integer solution.

Determination of an optimum solution for the location of power plants and/or incinerators for the entire Medford study by this method would not be computationally feasible due to the increased number of cases which must be considered. Assuming 20 sources of wood waste in the Medford area of the Bear Creek Valley and seven possible locations for power plants and/or incinerators, consider the following discussion relevant to the total number of cases which must be considered.

The following possibilities for power plant and/or incinerator construction satisfy the constraint that

$$\sum_{j=1}^{L} N_j = \sum_{i=1}^{S} M_i .$$

<u>Case 1:</u> Construction of 2-30 unit installations.

<u>Case 2:</u> Construction of 3-20 unit installations.

<u>Case 3:</u> Construction of 4-15 unit installations.

<u>Case 4:</u> Construction of 1-30 and 2-15 unit installations.

If L denotes the total number of distinct locations for the facilities, r the number of locations included in a particular subset under consideration, N_i the number of ways of distributing the L facilities among the i locations where $i = 1, 2, \cdots, r$, then

<u>Case 1:</u> $(L = 7,\ r = 2)$

$$T_1 = \sum_{i=1}^{2} N_i = \binom{7}{1} + \binom{7}{2} = 28 .$$

<u>Case 2:</u> $(L = 7,\ r = 3)$

$$T_2 = \sum_{i=1}^{3} N_i = \binom{7}{1} + \binom{7}{1}\binom{6}{1} + \binom{7}{3} = 84.$$

<u>Case 3:</u> $(L = 7,\ r = 4)$

$$T_3 = \sum_{i=1}^{4} N_i = \binom{7}{1} + \left[\binom{7}{2} + \binom{7}{1}\binom{6}{1}\right] + \binom{7}{1}\binom{6}{1}\binom{5}{1} + \binom{7}{4} = 315 .$$

<u>Case 4:</u> $(L = 7,\ r = 3)$

$$T_4 = \sum_{i=1}^{3} N_i = \binom{7}{1} + \left[\binom{7}{2} + \binom{7}{2}\right] + \binom{7}{3} = 84.$$

If T denotes the total number of cases to be considered, then

$$T = 2 \sum_{i=1}^{4} T_i$$
$$= 1022.$$

Because of the increased size of the Medford study and the various possibilities which must be considered, it is clear that direct enumeration of the possible cases would be impractical. The model developed in this thesis is intended to solve this problem in a systematic manner and in a reasonable amount of time. Estimates place the running time on the IBM 1410 Computer in excess of 50 hours after preparation of the input for direct enumeration of the cases (this figure is based on the fact that the nine integer solutions for the example computed in the appendix ran approximately 25 minutes). It is estimated a computer solution for the entire Medford study using integer programming with fixed charges could be found in less than four hours -- illustrating the application of the integer model in reducing a computationally unfeasible problem to a solution.