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Title A STUDY OF THE RESECTION PROBLEM IN SURVEYING

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Three special cases of the resection problem of surveying are examined and solved. The coordinates of unknown points are found with respect to given points in a rectangular coordinate system. This is accomplished in the case of (a) one unknown point and three given points (Snell's problem), (b) two unknown points and two given points (Hansen's problem), and (c) three unknown points and two given points. The method introduced by Dr. Harry Goheen and Vincenzo Orsi is successfully applied in all three cases.

A STUDY OF THE RESECTION PROBLEM IN SURVEYING

by

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A STUDY OF THE RESECTION PROBLEM IN SURVEYING

INTRODUCTION

A fundamental issue in the process of surveying is the problem of ascertaining the coordinates of a position with respect to a given coordinate system. Solutions to this problem have been given by Snell, Pothenot, Hansen and Marek (1). Referred to as the "Forward and Backwards Sections", Snell and Pothenot wrote solutions for the case in which three points are given and a single unknown point is determined. Hansen's problem is formulated in terms of two given points and the measure of four independent angles from two unknown points whose coordinates are to be calculated. Marek considered four given points and two unknown points where four angles were measured from the line joining the two unknown points.

We are concerned here with a common method of solution for three problems of this kind including Snell's and Hansen's problem. The method is one introduced by Dr. Harry Goheen of Oregon State University and Vincenzo Orsi. In each case a system of non-linear equations arises but the system can be treated as a linear homogeneous system with a parameter. As Dr. Goheen has noted, the method is very convenient from a practical viewpoint. We find, in each case considered here, that this same technique can be applied. The solution of each system is written, if it exists, and some of the conditions for which a solution does not exist or is ambiguous are discussed.

The method is applied first to Snell's problem, second to Hansen's problem and third to the problem given two fixed points and three unknown points.

The following notation is used throughout to abbreviate the description of the elementary row and column operations applied to determinants or matrices. C_i and R_i refer to the i^{th} column and i^{th} row respectively. $C_{ij}(k)$ means "replace column i by C_i+kC_j"; $R_{ij}(k)$ means "replace row i by $R_i + kR_j$".

CHAPTER I. SNELL'S PROBLEM

It is clear that the problem of determining the location of a single point with reference to two fixed points has no solution if only the measurement from the unknown point is allowed. (See Figure 1).

In the problem of Pothenot (Figure 2), the coordinates of three distinct points are given $A_1:(a_1, b_1)$, $A_2:(a_2, b_2)$ and $A_3:(a_3, b_3)$ in a plane Cartesian Coordinate System. The coordinates of the point P (oriented with respect to A_1, A_2 and A_3) are to be determined. The measurements of the positively oriented angles ϕ_1 and ϕ_2 are taken so that $\angle A_2PA_1 = \phi_1$ and $\angle A_3PA_2 = \phi_2$.

Denote the unknown positively oriented angle determined by the unique line through P and A_1 and the positive x-axis by ψ. Similarly, the angle of inclination of the line through P and A_2 will be denoted by $\psi + c_1$ and the angle of inclination of the line through P and A_3 by $\psi + c_2$ so that

$$c_1 + \phi_1 = 0,$$

and

$$c_2 + \phi_1 + \phi_2 = 0. \tag{1.0}$$

Then

$$\begin{aligned} y - b_1 &= (x - a_1) \tan \psi, \\ y - b_2 &= (x - a_2) \tan (\psi + c_1), \\ y - b_3 &= (x - a_3) \tan (\psi + c_2). \end{aligned} \tag{1.1}$$

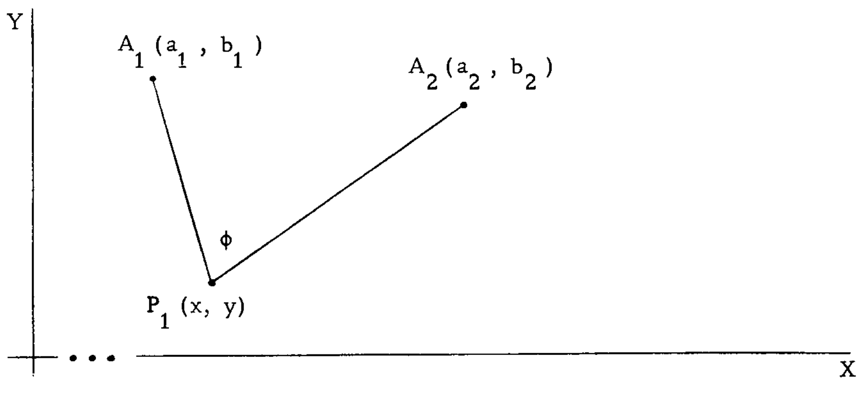

Figure 1

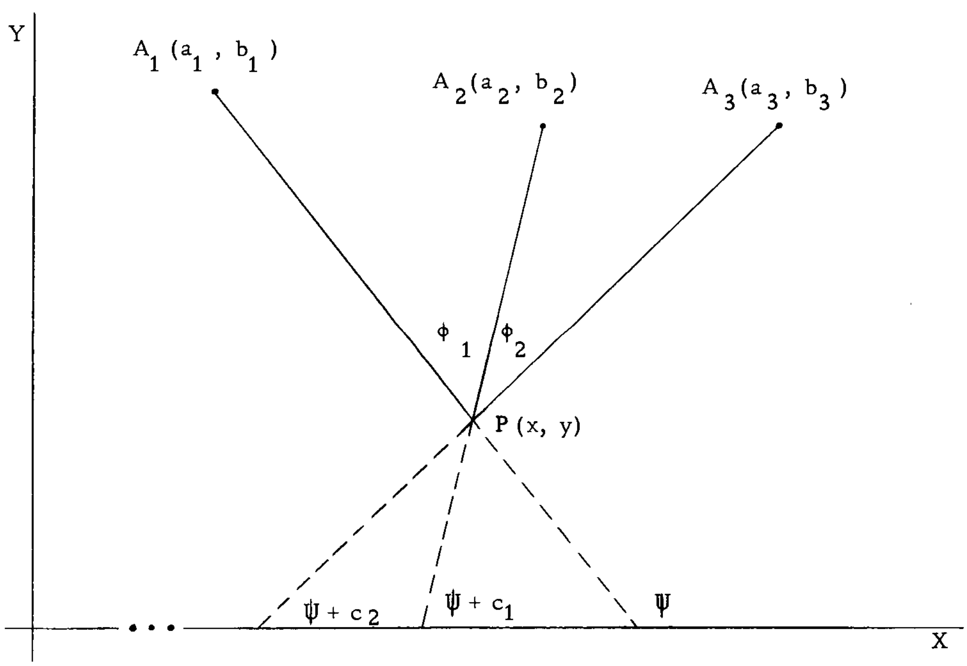

Figure 2

Applying the identity for the tangent of the sum of two angles, letting

$$\tan\psi = z\,,$$

$$\tan c_1 = \gamma_1\,,$$

$$\tan c_2 = \gamma_2\,,$$

and clearing of fractions, the system (1. 1) is equivalent to the system

$$\begin{aligned} zx \qquad\qquad -y - a_1 z + b_1 \qquad\qquad\qquad\qquad &= 0\,, \\ (z+\gamma_1)x - (1 - z\gamma_1)y - a_2 z - b_2\gamma_1 z - a_2\gamma_1 + b_2 &= 0\,, \qquad (1.\,2) \\ (z+\gamma_2)x - (1 - z\gamma_2)y - a_3 z - b_3\gamma_2 z - a_3\gamma_2 + b_3 &= 0\,. \end{aligned}$$

The system of equations (1. 2) may be considered as a system of linear homogeneous equations in the three variables x, y and 1, with z as a parameter. The necessary condition that the system (1. 2) have non-trivial solutions, i. e., that there exist values of x and y for which it is true, is that the determinant of the coefficients be zero. That is

$$\begin{vmatrix} z & -1 & -a_1 z + b_1 \\ z+\gamma_1 & -(1 - z\gamma_1) & -a_2 z - b_2\gamma_1 z - a_2\gamma_1 + b_2 \\ z+\gamma_2 & -(1 - z\gamma_2) & -a_3 z - b_3\gamma_2 z - a_3\gamma_2 + b_3 \end{vmatrix} = 0\,. \qquad (1.\,3)$$

The determinant (1. 3) is a cubic equation in z, but performing the operation $C_{21}(-z)$ and factoring $z^2 + 1$ from the second column results in

$$(z^2+1)\begin{vmatrix} z & -1 & -a_1 z + b_1 \\ z+\gamma_1 & -1 & -a_2 z - b_2\gamma_1 z - a_2\gamma_1 + b_2 \\ z+\gamma_2 & -1 & -a_3 z - b_3\gamma_2 z - a_3\gamma_2 + b_3 \end{vmatrix} = 0 . \tag{1.4}$$

Only real values of z are of interest here so that $z^2 + 1$ may be suppressed. Performing the row operations $R_{21}(-1)$ and $R_{31}(-1)$ and expanding the determinant by the second column yields

$$\begin{vmatrix} \gamma_1 & (a_1 - a_2 - b_2\gamma_1)z - a_2\gamma_1 - b_1 + b_2 \\ \gamma_2 & (a_1 - a_3 - b_3\gamma_2)z - a_3\gamma_2 - b_1 + b_3 \end{vmatrix} = 0 . \tag{1.5}$$

This linear equation in z has the solution

$$z = \frac{\gamma_1\gamma_2(a_3 - a_2) - \gamma_2(b_1 - b_2) + \gamma_1(b_1 - b_3)}{\gamma_1(a_1 - a_3) - \gamma_2(a_1 - a_2) + \gamma_1\gamma_2(b_2 - b_3)} \tag{1.6}$$

if (1.5) is a conditional equation. Then replacing z in the system (1.2) produces the solution

$$x = \frac{\begin{vmatrix} a_1 z - b_1 & -1 \\ a_2 z + b_2\gamma_1 z + a_2\gamma_1 - b_2 & -1 + z\gamma_1 \end{vmatrix}}{\gamma_1(z^2 + 1)} ,$$

$$y = \frac{\begin{vmatrix} z & a_1 z - b_1 \\ z + \gamma_1 & a_2 z + b_2\gamma_1 z + a_2\gamma_1 - b_2 \end{vmatrix}}{\gamma_1(z^2 + 1)} , \tag{1.7}$$

where $\gamma_1 = \tan c_1 \neq 0$ since A_1, A_2 were taken as distinct points.

If (1.5) is not a conditional equation then (1.5) is a contradiction or an identity. If the former, then $\psi = \frac{\pi}{2}$ so that $x = a_1$. The value of y may be obtained by taking the limit as z tends to infinity in the equation of (1.7). If (1.5) is an identity, i.e.,

$$\begin{aligned} \gamma_1\gamma_2(a_3 - a_2) - \gamma_2(b_1 - b_2) + \gamma_1(b_1 - b_3) &= 0, \\ \gamma_1(a_1 - a_2) - \gamma_2(a_1 - a_2) + \gamma_1\gamma_2(b_2 - b_3) &= 0, \end{aligned} \tag{1.8}$$

then we observe that the equations (1.2) may be considered as linear homogeneous equations in z and 1 where the value of z is not unique. Hence, for all z, the equations

$$\begin{aligned} (x - a_1)z + b_1 - y &= 0, \\ (x + y\gamma_1 - a_2 - b_2\gamma_1)z + \gamma_1 x - y - a_2\gamma_1 + b_2 &= 0, \\ (x + y\gamma_2 - a_3 - b_3\gamma_2)z + \gamma_2 x - y - a_3\gamma_2 + b_3 &= 0, \end{aligned} \tag{1.9}$$

hold and are pairwise linearly dependent. Specifically the first two equations and the first and the last are dependent so that

$$\begin{vmatrix} x - a_1 & b_1 - y \\ x + y\gamma_1 - a_2 - b_2\gamma_1 & \gamma_1 x - y - a_2\gamma_1 + b_2 \end{vmatrix} = 0$$

and

$$\begin{vmatrix} x - a_1 & b_1 - y \\ x + y\gamma_2 - a_3 - b_3\gamma_2 & \gamma_2 x - y - a_3\gamma_2 + b_3 \end{vmatrix} = 0. \tag{1.10}$$

Expanding these determinants and simplifying gives the equations

$$x^2 + y^2 + \frac{(-a_1 - a_2)\gamma_1 + b_2 - b_1}{\gamma_1} x + \frac{a_1 - a_2 + (-b_1 - b_2)\gamma_1}{\gamma_1} y + \frac{(a_1 a_2 + b_1 b_2)\gamma_1 - a_1 b_2 + a_2 b_1}{\gamma_1} = 0 ,$$

$$x^2 + y^2 + \frac{(-a_1 - a_3)\gamma_2 + b_3 - b_1}{\gamma_2} x + \frac{a_1 - a_3 + (-b_1 - b_3)\gamma_2}{\gamma_2} y + \frac{(a_1 a_3 + b_1 b_3)\gamma_2 - a_1 b_3 + a_3 b_1}{\gamma_2} = 0 , \qquad (1.11)$$

which are the equations, respectively, of a circle through the points P, A_1 and A_2 and a circle through the points P, A_2 and A_3. These circles coincide provided the coefficients of x, y and the constant term are equal. The identities (1. 8) imply these equalities, so that if (1.5) is an identity, then P lies on the circle determined by A_1, A_2 and A_3. In this case the point P:(x, y) may lie anywhere on the "danger" circle so that a unique solution is not determined (see Figure 3).

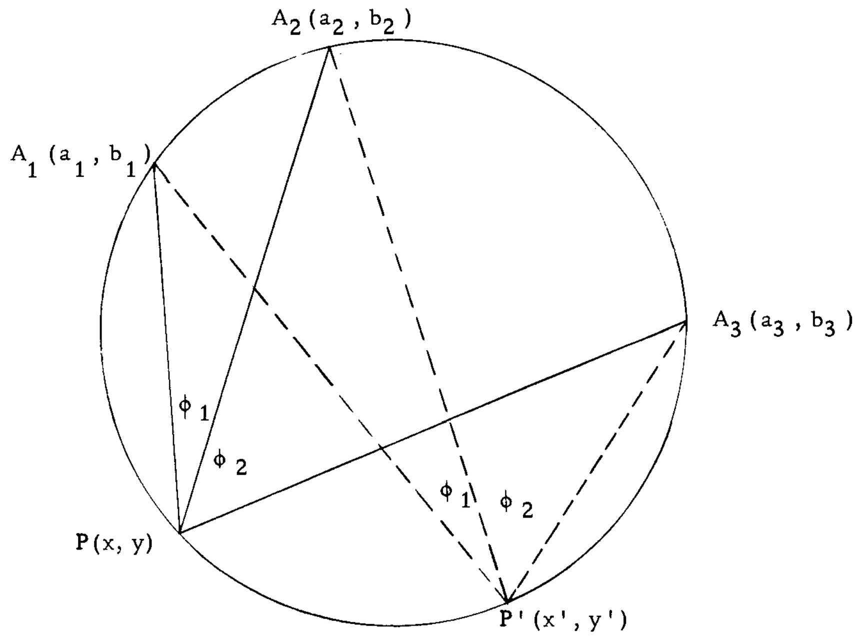

Figure 3

CHAPTER II. HANSEN'S PROBLEM

The problem of Hansen is approached in a similar way (Figure 4). Two points are given $A_1:(a_1, b_1)$ and $A_2:(a_2, b_2)$ and are taken to be distinct. The angles ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are measured positively from the two unknown points $P_1:(x_1, y_1)$ and $P_2:(x_2, y_2)$ so that

$$\angle A_2P_1A_1 = \phi_1, \qquad \angle P_2P_1A_2 = \phi_2,$$

$$\angle A_1P_2P_1 = \phi_3, \qquad \angle A_2P_2A_1 = \phi_4.$$

If the angles of inclination for the lines A_1P_1, A_2P_1 A_2P_2, A_1P_2 and P_1P_2 are designated, respectively, ψ, $\psi + c_1$, $\psi + c_2$, $\psi + c_3$ and $\psi + c_4$ we have the identities

$$\begin{aligned} c_1 + \phi_1 &= 0, \\ c_2 + \phi_1 + \phi_2 + \phi_3 + \phi_4 &= \pi, \\ c_3 + \phi_1 + \phi_2 + \phi_3 &= \pi, \\ c_4 + \phi_1 + \phi_2 &= \pi \quad . \end{aligned} \tag{2.0}$$

The same procedure used in Pothenot's problem is applied here to yield the system of simultaneous equations

$$\begin{aligned} y_1 - b_1 &= (x_1 - a_1) \tan \psi , \\ y_1 - b_2 &= (x_1 - a_2) \tan (\psi + c_1) , \\ y_2 - b_2 &= (x_2 - a_2) \tan (\psi + c_2) , \\ y_2 - b_1 &= (x_2 - a_1) \tan (\psi + c_3) , \\ y_2 - y_1 &= (x_2 - x_1) \tan (\psi + c_4) . \end{aligned} \tag{2.1}$$

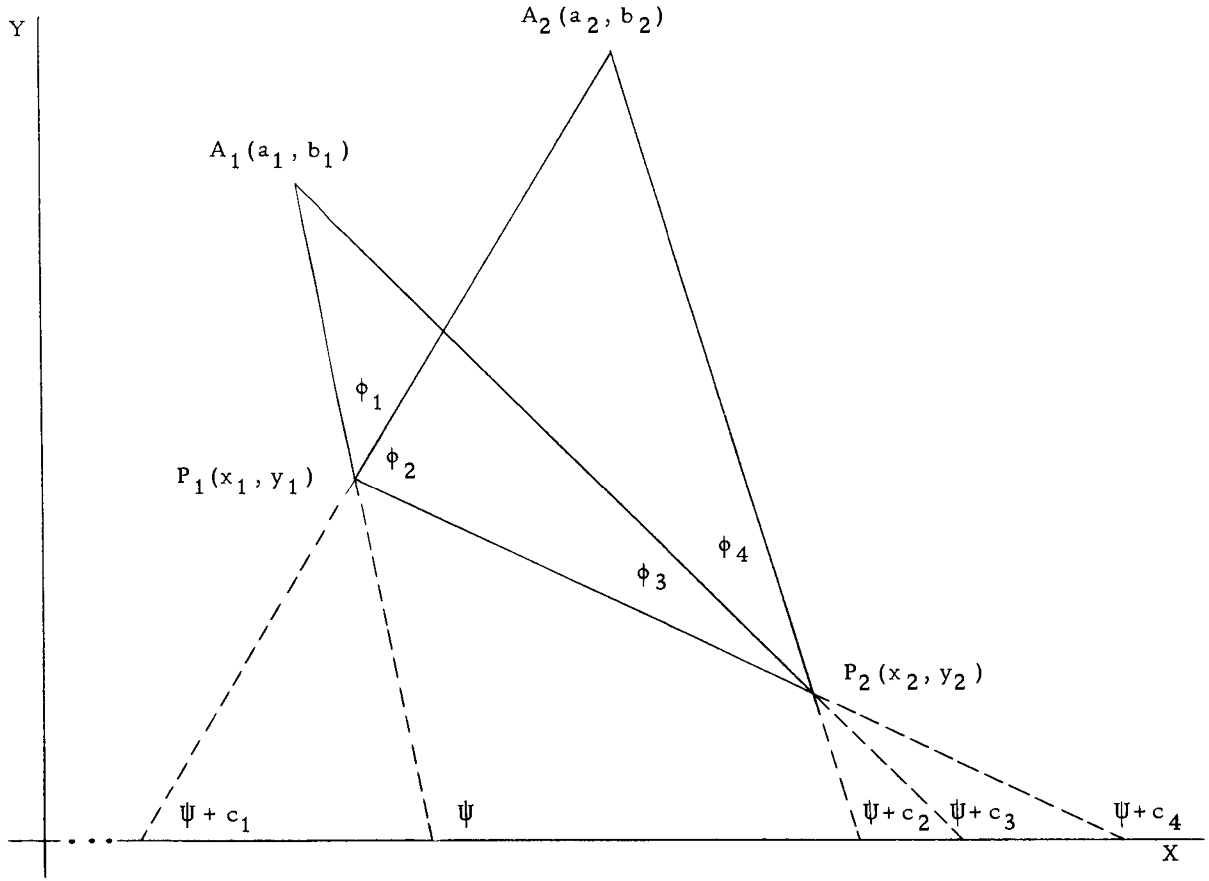
Y
$A_2(a_2, b_2)$
$A_1(a_1, b_1)$
ϕ_1
ϕ_2
$P_1(x_1, y_1)$
ϕ_4
ϕ_3
$P_2(x_2, y_2)$
$\psi + c_1$
ψ
$\psi + c_2$
$\psi + c_3$
$\psi + c_4$
X

Figure 4

Again applying the identity for the tangent of the sum of two angles, replacing tan ψ by z and tan c_i by γ_i ($i = 1, 2, 3, 4$) and simplifying, we have the following system which may be viewed as a system of linear homogeneous equations in the five variables x_1, y_1, x_2, y_2 and 1 with z as a parameter

$$\begin{array}{lr}
z x_1 \quad -y_1 \qquad\qquad - a_1 y + b_1 & = 0\ , \\
(z+\gamma_1)x_1 - (1-z\gamma_1)y_1 \qquad -a_2(z+\gamma_1) + b_2(1-z\gamma_1) & = 0\ , \\
(z+y_4)x_1 - (1-z\gamma_4)y_1 - (z+\gamma_4)x_2 + (1-z\gamma_4)y_2 & = 0\ , \\
(z+\gamma_2)x_2 - (1-z\gamma_2)y_2 - a_2(z+\gamma_2) + b_2(1-z\gamma_2) & = 0\ , \\
(z+\gamma_3)x_2 - (1-z\gamma_3)y_2 - a_1(z+\gamma_3) + b_1(1-z\gamma_3) & = 0\ .
\end{array} \qquad (2.2)$$

In order that there be a non-trivial solution for (2.2), that is, that there exist values of x_1, x_2, y_2 for which the system holds, it is necessary that the determinant of the coefficients vanish:

$$\begin{vmatrix}
z & -1 & 0 & 0 & -a_1 z+b_1 \\
z+\gamma_1 & -(-1-z\gamma_1) & 0 & 0 & -a_2(z+\gamma_1)+b_2(1-z\gamma_1) \\
z+\gamma_4 & -(1-z\gamma_4) & -(z+\gamma_4) & +(1-z\gamma_4) & 0 \\
0 & 0 & z+\gamma_2 & -(1-z\gamma_2) & -a_2(z+\gamma_2)+b_2(1-z\gamma_2) \\
0 & 0 & z+\gamma_3 & -(1-z\gamma_3) & -a_1(z+\gamma_3)+b_1(1-z\gamma_3)
\end{vmatrix} = 0\ . \qquad (2.3)$$

This determinant is reduced in a manner analogous to that used in the problem of Pothenot. The column operations $C_{21}(-z)$ and $C_{43}(-z)$ are performed and $z^2 + 1$ is factored from columns 2 and 4. The factor $(z^2 + 1)^2$ is suppressed so that, again, a linear equation in z remains:

$$\begin{vmatrix} z & -1 & 0 & 0 & -a_1 z+b_1 \\ z+\gamma_1 & -1 & 0 & 0 & -a_2(z+\gamma_1)+b_2(1-z\gamma_1) \\ z+\gamma_4 & -1 & -(z+\gamma_4) & 1 & 0 \\ 0 & 0 & z+\gamma_2 & -1 & -a_2(z+\gamma_2)+b_2(1-z\gamma_2) \\ 0 & 0 & z+\gamma_3 & -1 & -a_1(z+\gamma_3)+b_1(1-z\gamma_3) \end{vmatrix} = 0. \tag{2. 4}$$

The equation (2. 4) is replaced by the equivalent equation

$$\begin{vmatrix} \gamma_1 & 0 & 0 & (a_1-a_2-b_2\gamma_1)z - a_2\gamma_1-b_1+b_2 \\ \gamma_4 & -(z+\gamma_4) & 1 & a_1 z - b_1 \\ 0 & z+\gamma_2 & -1 & (-a_2-b_2\gamma_2)z - a_2\gamma_2+b_2 \\ 0 & z+\gamma_3 & -1 & (-a_1-b_1\gamma_3)z - a_1\gamma_3+b_1 \end{vmatrix} = 0, \tag{2. 5}$$

by performing the operations $R_{21}(-1)$, $R_{31}(-1)$ and expanding the determinant with respect to the second column. Performing the operations $R_{32}(1)$, $R_{42}(1)$ and simplifying the result, we have

$$\begin{vmatrix} \gamma_1 & 0 & a_1-a_2-b_2\gamma_1 \\ \gamma_4 & \gamma_2-\gamma_4 & a_1-a_2-b_2\gamma_2 \\ \gamma_4 & \gamma_3-\gamma_4 & -b_1\gamma_3 \end{vmatrix} z + \begin{vmatrix} \gamma_1 & 0 & -a_2\gamma_1-b_1+b_2 \\ \gamma_4 & \gamma_2-\gamma_4 & -a_2\gamma_2-b_1+b_2 \\ \gamma_4 & \gamma_3-\gamma_4 & -a_1\gamma_3 \end{vmatrix} = 0, \tag{2. 6}$$

which, if it is a conditional equation, may be solved for z and written

$$z = \frac{P(a_1-a_2) + Q(b_1-b_2)}{Q(a_1-a_2) - P(b_1-b_2)} \qquad \text{where}$$

$$P = \begin{vmatrix} \gamma_1 & 0 & 0 \\ \gamma_4 & \gamma_2-\gamma_4 & 0 \\ \gamma_4 & \gamma_3-\gamma_4 & \gamma_3 \end{vmatrix} = \gamma_1\gamma_3(\gamma_2-\gamma_4)$$

and (2. 7)

$$Q = \begin{vmatrix} \gamma_1 & 0 & 1 \\ \gamma_4 & \gamma_2-\gamma_4 & 1 \\ \gamma_4 & \gamma_3-\gamma_4 & 0 \end{vmatrix} = \gamma_4(\gamma_3-\gamma_2) - \gamma_1(\gamma_3-\gamma_4).$$

Provided equation (2. 6) is conditional then the value of z given in (2. 7) furnishes the solution for the system. The remaining unknowns are

$$x_1 = \frac{\begin{vmatrix} a_1z-b_1 & -1 \\ a_2z+b_2\gamma_1z+a_2\gamma_1-b_2 & -1+z\gamma_1 \end{vmatrix}}{\gamma_1(z^2+1)},$$

$$y_1 = \frac{\begin{vmatrix} z & a_1z-b_1 \\ z+\gamma_1 & a_2z+b_2\gamma_1z+a_2\gamma_1-b_2 \end{vmatrix}}{\gamma_1(z^2+1)},$$

$$x_2 = \frac{\begin{vmatrix} a_2z+b_2\gamma_2z+a_2\gamma_2-b_2 & -1+z\gamma_2 \\ a_1z+b_1\gamma_3z+a_1\gamma_3-b_1 & -1+z\gamma_3 \end{vmatrix}}{(\gamma_3-\gamma_2)(z^2+1)}$$

and

$$y_2 = \frac{\begin{vmatrix} z+\gamma_2 & a_2z+b_2\gamma_2z+a_2\gamma_2-b_2 \\ z+\gamma_3 & a_1z+b_1\gamma_3z+a_1\gamma_3-b_1 \end{vmatrix}}{(\gamma_3-\gamma_2)\ (z^2+1)} .$$

This solution exists because $\gamma_1 \neq 0 \neq \gamma_3-\gamma_2$; A_1, A_2 being taken as distinct.

If the equation (2.6) is a contradiction, then $\psi = \frac{\pi}{2}$ and the solutions may be found by taking limits as in the former problem. If no unique value of z is determined by the system (2.2), that is, if (2.6) is an identity, then from (2.7) we have

$$\begin{aligned} P(a_1-a_2) + Q(b_1-b_2) &= 0, \\ Q(a_1-a_2) - P(b_1-b_2) &= 0, \end{aligned} \qquad (2.9)$$

where $a_1 \neq a_2$ or $b_1 \neq b_2$ if we take A_1, A_2 to be distinct. These equations in (a_1-a_2) and (b_1-b_2) are a linear homogeneous system with no non-trivial solutions P and Q since

$$\begin{vmatrix} P & Q \\ Q & -P \end{vmatrix} = 0 \text{ implies } -P^2 - Q^2 = 0 \text{ or } -P^2 = Q^2 \text{ where}$$

P and Q are real numbers. Therefore

$$P = Q = 0.$$

If $P = 0$ then (from (2.7)) we can distinguish three cases: $\gamma_1 = 0$ or $\gamma_3 = 0$ or $\gamma_2 = \gamma_4$. All three cases imply a contradiction, or that at least three of A_1, A_2, P_1, P_2 are collinear, the "danger" situation. We shall examine only one of these cases in detail.

If $P = 0$ because $\gamma_2 = \gamma_4$ then we conclude that $c_2 = c_4$ or $c_2 = c_4 \pm \pi$ (we omit the details on the latter possibility). If $c_2 = c_4$ then, by the identities (2.0), $\phi_3 + \phi_4 = 0$ so that P_1, P_2, A_2 are collinear. From (2.7) $Q = \gamma_4(\gamma_3 - \gamma_2) - \gamma_1(\gamma_3 - \gamma_4) = (\gamma_2 - \gamma_1)(\gamma_3 - \gamma_2) = 0$ if we replace γ_4 by γ_2, hence either (a) $\gamma_2 = \gamma_1$ or (b) $\gamma_3 = \gamma_2$. If (b) $\gamma_3 = \gamma_2$ then $c_2 = c_3$ or $c_2 = c_3 \pm \pi$ from which $\phi_4 = 0$ or $\phi_4 = \pm \pi$. In either case P_2, A_1, A_2 are collinear so all four points are collinear, clearly a "danger" situation. If (a) $\gamma_2 = \gamma_1$ then $c_2 = c_1$ $(=c_4)$ or $c_2 = c_1 \pm \pi$ $(=c_4)$ from which $\phi_2 = \pi$ or $\phi_2 = 0$ or 2π respectively, confirming that P_1, P_2, A_2 are collinear but adding no new information.

Summarizing, the collinearity of P_1, P_2, A_2 is implied by $\phi_2 = 0$ or π and since $\phi_4 = -\phi_3$, essentially only two angles ϕ_1 and ϕ_3 are known by measurement and these are not sufficient to determine a unique ψ or a unique solution for P_1 and P_2. This is seen in Figure 5 where P_1 may lie anywhere on the circumference of the circle determined by A_1, A_2 and $-\phi_1$, and P_2 lies on the intersection of the line A_2P_1 and the circle determined by A_2, A_1 and ϕ_3.

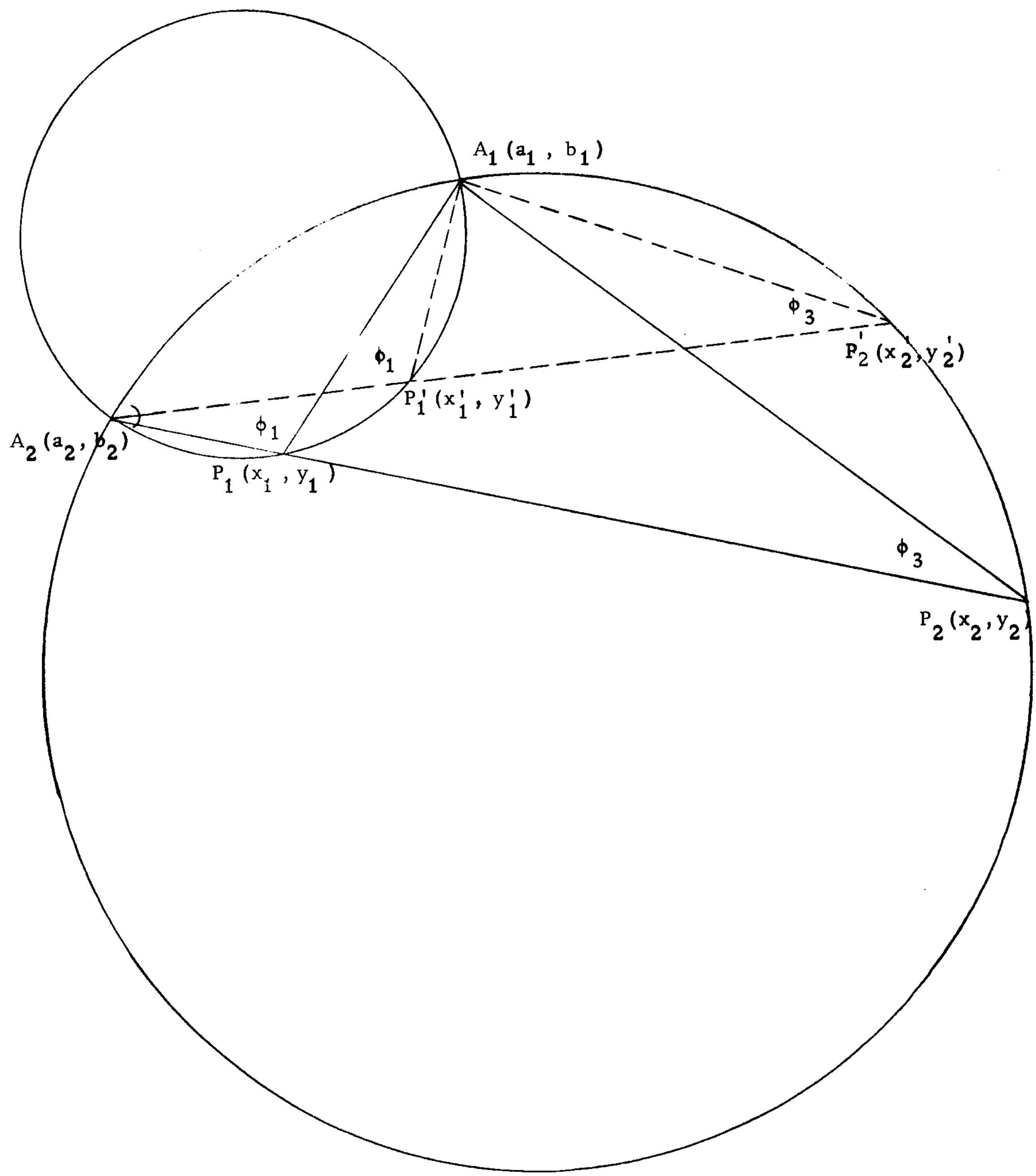

Figure 5

CHAPTER III. DISCUSSION OF THE PROBLEM WITH TWO FIXED POINTS AND THREE UNKNOWN POINTS

We now apply the same procedures to the case of two given points and three unknown points from which a total of nine angles may be measured (Figure 6). These angles are measured positively in the same sense as before. The angles of inclinations of the lines A_1P_1, P_2P_1, A_2P_1, A_2P_2, A_1P_2, A_2P_3, A_1P_3, P_2P_3 and P_1P_3 are respectively designated by ψ, $\psi + c_1$, $\psi + c_2$, $\psi + c_3$, $\psi + c_4$, $\psi + c_5$, $\psi + c_6$, $\psi + c_7$ and $\psi + c_8$. The following identities hold for the c_i and ϕ_i, $i = 1, 2, \ldots, 9$,

$$\begin{aligned}
c_1 + \phi_1 + \phi_2 &= 0 , \\
c_2 + \phi_1 &= 0 , \\
c_3 + \phi_1 + \phi_2 + \phi_4 + \phi_5 &= \pi , \\
c_4 + \phi_1 + \phi_2 + \phi_4 &= \pi , \\
c_5 + \phi_1 + \phi_2 + \phi_3 + \phi_7 + \phi_8 + \phi_9 &= \pi , \\
c_6 + \phi_1 + \phi_2 + \phi_3 + \phi_7 + \phi_8 &= \pi , \\
c_7 + \phi_1 + \phi_2 + \phi_3 + \phi_7 &= \pi , \\
c_8 + \phi_1 + \phi_2 + \phi_3 &= \pi ,
\end{aligned} \tag{3.0}$$

and we note that $\pi + \phi_3 + \phi_7 = \phi_4 + \phi_5 + \phi_6$. Note, also, that ϕ_6 does not appear in any of the identities (3.0) so that no occasion arises in which a measure of ϕ_6 is necessary.

The system of equations in this case involves nine equations in seven variables.

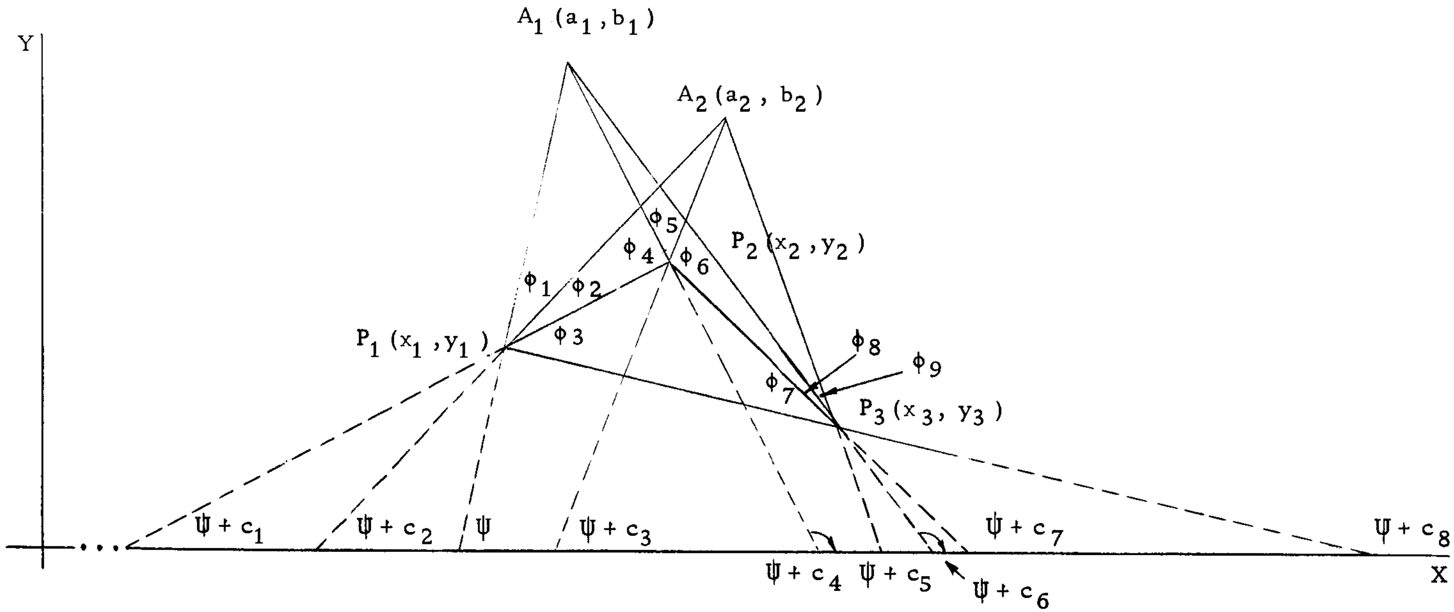
A1 (a1 , b1)
A2 (a2 , b2)
Y
P2 (x2 , y2)
P1 (x1 , y1)
P3 (x3 , y3)
φ1
φ2
φ3
φ4
φ5
φ6
φ7
φ8
φ9
ψ + c1
ψ + c2
ψ
ψ + c3
ψ + c4
ψ + c5
ψ + c6
ψ + c7
ψ + c8
X

Figure 6

$$y_1 - b_1 = (x_1 - a_1) \tan \psi ,$$

$$y_2 - y_1 = (x_2 - x_1) \tan (\psi + c_1) ,$$

$$y_1 - b_2 = (x_1 - a_2) \tan (\psi + c_2) ,$$

$$y_3 - y_1 = (x_3 - x_1) \tan (\psi + c_8) ,$$

$$y_2 - b_2 = (x_2 - a_2) \tan (\psi + c_3) ,$$

$$y_2 - b_1 = (x_2 - a_1) \tan (\psi + c_4) ,$$

$$y_3 - y_2 = (x_3 - x_2) \tan (\psi + c_7) ,$$

$$y_3 - b_2 = (x_3 - a_2) \tan (\psi + c_5) ,$$

$$y_3 - b_1 = (x_3 - a_1) \tan (\psi + c_6) .$$

Substituting the identity for the tangent of the sum of two angles, letting,

$$\tan \psi = z ,$$

$$\tan c_i = \gamma_i ,$$

and simplifying as in the preceding cases, the system can be written

$$\begin{aligned}
& z x_1 - y_1 + b_1 - a_1 z = 0, \\
& (z+\gamma_1)x_1 - (1-z\gamma_1)y_1 - (z+\gamma_1)x_2 + (1-z\gamma_1)y_2 = 0, \\
& (z+\gamma_2)x_1 - (1-z\gamma_2)y_1 + b_2(1-z\gamma_2) - a_2(z+\gamma_2) = 0, \\
& (z+\gamma_8)x_1 - (1-z\gamma_8)y_1 - (z+\gamma_8)x_3 + (1-z\gamma_8)y_3 = 0, \\
& -(z+\gamma_3)x_2 + (1-z\gamma_3)y_2 - b_2(1-z\gamma_3) + a_2(z+\gamma_3) = 0, \\
& -(z+\gamma_4)x_2 + (1-z\gamma_4)y_2 - b_1(1-z\gamma_4) + a_1(z+\gamma_4) = 0, \\
& -(z+\gamma_7)x_2 + (1-z\gamma_7)y_2 + (z+\gamma_7)x_3 - (1-z\gamma_7)y_3 = 0, \\
& (z+\gamma_5)x_3 - (1-z\gamma_5)y_3 + b_2(1-z\gamma_5) - a_2(z+\gamma_5) = 0, \\
& (z+\gamma_6)x_3 - (1-z\gamma_6)y_3 + b_1(1-z\gamma_6) - a_1(z+\gamma_6) = 0.
\end{aligned} \tag{3.1}$$

The matrix of the coefficients of the system is (3.2)

$$
\begin{bmatrix}
z & -1 & 0 & 0 & 0 & 0 & b_1 - a_1 z \\
z+\gamma_1 & -(1-z\gamma_1) & -(z+\gamma_1) & 1-z\gamma_1 & 0 & 0 & 0 \\
z+\gamma_2 & -(1-z\gamma_2) & 0 & 0 & 0 & 0 & b_2(1-z\gamma_2)-a_2(z+\gamma_2) \\
z+\gamma_8 & -(1-z\gamma_8) & 0 & 0 & -(z+\gamma_8) & 1-z\gamma_8 & 0 \\
0 & 0 & -(z+\gamma_3) & 1-z\gamma_3 & 0 & 0 & -b_2(1-z\gamma_3)+a_2(z+\gamma_3) \\
0 & 0 & -(z+\gamma_4) & 1-z\gamma_4 & 0 & 0 & -b_1(1-z\gamma_4)+a_1(z+\gamma_4) \\
0 & 0 & -(z+\gamma_7) & 1-z\gamma_7 & z+\gamma_7 & -(1-z\gamma_7) & 0 \\
0 & 0 & 0 & 0 & z+\gamma_5 & -(1-z\gamma_5) & b_2(1-z\gamma_5)-a_2(z+\gamma_5) \\
0 & 0 & 0 & 0 & z+\gamma_6 & -(1-z\gamma_6) & b_1(1-z\gamma_6)-a_1(z+\gamma_6)
\end{bmatrix}
$$

The matrix may be simplified by the column operations $C_{21}(-z)$, $C_{43}(-z)$ and $C_{65}(-z)$ to yield the equivalent matrix

$$
\begin{bmatrix}
z & -(1+z^2) & 0 & 0 & 0 & 0 & b_1-a_1z \\
z+\gamma_1 & -(1+z^2) & -(z+\gamma_1) & 1+z^2 & 0 & 0 & 0 \\
z+\gamma_2 & -(1+z^2) & 0 & 0 & 0 & 0 & b_2(1-z\gamma_2)-a_2(z+\gamma_2) \\
z+\gamma_8 & -(1+z^2) & 0 & 0 & -(z+\gamma_8) & 1+z^2 & 0 \\
0 & 0 & -(z+\gamma_3) & 1+z^2 & 0 & 0 & -b_2(1-z\gamma_3)+a_2(z+\gamma_3) \\
0 & 0 & -(z+\gamma_4) & 1+z^2 & 0 & 0 & -b_1(1-z\gamma_4)+a_1(z+\gamma_4) \\
0 & 0 & -(z+\gamma_7) & 1+z^2 & z+\gamma_7 & -(1+z^2) & 0 \\
0 & 0 & 0 & 0 & z+\gamma_5 & -(1+z^2) & b_2(1-z\gamma_5)-a_2(z+\gamma_5) \\
0 & 0 & 0 & 0 & z+\gamma_6 & -(1+z^2) & b_1(1-z\gamma_6)-a_1(z+\gamma_6)
\end{bmatrix}. \tag{3.3}
$$

Certainly it is superfluous to measure all nine angles in order to solve for P_1, P_2 and P_3. Any seven of the equations in the system (3. 1) is a system of seven equations in seven unknowns. There are $\binom{9}{7} = 36$ possible subsystems. Which of these require the measure of eight angles? (ϕ_6 can be determined from the others.) Which, if any, of these systems yields a solution not reducible to Hansen's problem?

Those subsystems which require the measure of all eight angles are of little interest since any of the 36 systems are available in such a case and, in fact, the problem can be reduced to that of Hansen, with respect to any two of P_1, P_2 and P_3.

Consider the measure of six of the angles ϕ_i, i = 1, 2, 3, 4, 5, 7, 8, 9. By examining the relations (3. 0) we find that if any fewer than six are measured, less than six of the values of c_i (i = 1, 2, . . ., 9) are known, hence the problem has no solution. There are $\binom{8}{6} = 28$ sets of six angles which may be measured.

If ϕ_1, ϕ_2, ϕ_4, ϕ_5 are measured, the solution can be found by reduction to Hansen's problem with respect to P_1 and P_2. There are $\binom{4}{2} = 6$ sets of the 28 in which these four angles occur. If ϕ_1, ϕ_2, ϕ_3, ϕ_7, ϕ_8, ϕ_9 are measured, the problem reduces to Hansen's with respect to P_1 and P_3. In none of the 28 subsets does the necessary ϕ_i occur for reduction of the problem with respect to P_2 and P_3.

By examination of the 28 subsets containing six angles it is found that only three of these yield values of c_i which lead to a solution of the problem:

1. If ϕ_1, ϕ_2, ϕ_3, ϕ_7, ϕ_8, ϕ_9 are measured, the constants in all equations but those in rows 5 and 6 of the system (3. 1) are known.

2. If ϕ_1, ϕ_2, ϕ_3, ϕ_4, ϕ_5, ϕ_7 are measured, the constants in all equations except those in rows 8 and 9 are known.

3. If ϕ_1, ϕ_2, ϕ_3, ϕ_4, ϕ_7, ϕ_8 are measured, the constants in all equations but those in rows 5 and 8 are known.

The solutions to the problem in these cases are given in the next chapter. Note that in the first case above the problem reduces to Hansen's with respect to P_1 and P_3. If the solution to Hansen's problem is used to find P_1 and P_3, measurements from P_2 must also be used while here P_2 is part of the solution. Similarly, in the second case P_1 and P_2 may be found by reducing to Hansen's problem though here P_3 may also be found. Only in one instance, the third case above, does the problem have a solution, no part of which can be found by the considerations in Chapters I and II.

Now consider the measure of any seven of the angles (again omitting ϕ_6). There are $\binom{8}{7} = 8$ subsets, each containing seven angles. Examination of these shows that there are four cases in which solutions may be found, but each of the four has a six element subset which has been discussed above so that the solutions are the same as those above.

CHAPTER IV. SOLUTIONS FOR THE PROBLEM WITH TWO FIXED POINTS AND THREE UNKNOWN POINTS

We exhibit here the solution to the problem discussed in Chapter III in the three cases listed on page 24.

1. The necessary condition that the system of equations obtained by omitting equations 5 and 6 in the system (3. 1) have a solution, is (4. 1)

$$\begin{vmatrix} z & -(1+z^2) & 0 & 0 & 0 & 0 & b_1-a_1z \\ z+\gamma_1 & -(1+z^2) & -(z+\gamma_1) & 1+z^2 & 0 & 0 & 0 \\ z+\gamma_2 & -(1+z^2) & 0 & 0 & 0 & 0 & b_2(1-z\gamma_2)-a_2(z+\gamma_2) \\ z+\gamma_8 & -(1+z^2) & 0 & 0 & -(z+\gamma_8) & 1+z^2 & 0 \\ 0 & 0 & -(z+\gamma_7) & 1+z^2 & z+\gamma_7 & -(1+z^2) & 0 \\ 0 & 0 & 0 & 0 & z+\gamma_5 & -(1+z^2) & b_2(1-z\gamma_5)-a_2(z+\gamma_5) \\ 0 & 0 & 0 & 0 & z+\gamma_6 & -(1+z^2) & b_1(1-z\gamma_6)-a_1(z+\gamma_6) \end{vmatrix} = 0.$$

Factoring $1+z^2$ from columns 2, 4 and 6, we have the factor $(1+z^2)^3$ which is suppressed as in Chapters I and II. Performing the row operations $R_{21}(-1)$, $R_{31}(-1)$ and $R_{41}(-1)$ and expanding the determinant with respect to the second column, simplifies it to

$$\begin{vmatrix} \gamma_1 & -(z+\gamma_1) & 1 & 0 & 0 & -b_1+a_1z \\ \gamma_2 & 0 & 0 & 0 & 0 & b_2(1-z\gamma_2)-a_2(z+\gamma_2)-b_1+a_1z) \\ \gamma_8 & 0 & 0 & -(z+\gamma_8) & 1 & -b_1+a_1z \\ 0 & -(z+\gamma_7) & 1 & z+\gamma_7 & -1 & 0 \\ 0 & 0 & 0 & z+\gamma_5 & -1 & b_2(1-z\gamma_5)-a_2(z+\gamma_5) \\ 0 & 0 & 0 & z+\gamma_6 & -1 & b_1(1-z\gamma_6)-a_1(z+\gamma_6) \end{vmatrix} = 0. \qquad (4.2)$$

The result of $R_{41}(-1)$ followed by expansion with respect to the third column is:

$$\begin{vmatrix} \gamma_2 & 0 & 0 & 0 & b_2(1-z\gamma_2)-a_2(z+\gamma_2)-b_1+a_1z \\ \gamma_8 & 0 & -(z+\gamma_8) & +1 & -b_1+a_1z \\ -\gamma_1 & \gamma_1-\gamma_7 & z+\gamma_7 & -1 & b_1-a_1z \\ 0 & 0 & z+\gamma_5 & -1 & b_2(1-z\gamma_5)-a_2(z+\gamma_5) \\ 0 & 0 & z+\gamma_6 & -1 & b_1(1-z\gamma_6)-a_1(z+\gamma_6) \end{vmatrix} = 0. \qquad (4.3)$$

Rather than expand by column 2 here we follow the general procedure which applies in every instance. Apply $R_{32}(1)$, $R_{42}(1)$, $R_{52}(1)$ and expand first by column 4 and then by column 2. We may then write

$$\begin{vmatrix} \gamma_2 & 0 & a_1-a_2-b_2\gamma_2 \\ \gamma_8 & \gamma_5-\gamma_8 & a_1-a_2-b_2\gamma_5 \\ \gamma_8 & \gamma_6-\gamma_8 & -b_1\gamma_6 \end{vmatrix} z + \begin{vmatrix} \gamma_2 & 0 & a_2\gamma_2+b_1-b_2 \\ \gamma_8 & \gamma_5-\gamma_8 & a_2\gamma_5+b_1-b_2 \\ \gamma_8 & \gamma_6-\gamma_8 & a_1\gamma_6 \end{vmatrix} = 0. \qquad (4.4)$$

If (4.4) is a conditional equation solving for z yields

$$z = \frac{P(a_1-a_2) + Q(b_1-b_2)}{Q(a_1-a_2) - P(b_1-b_2)} \qquad \text{where}$$

$$P = \begin{vmatrix} \gamma_1 & 0 & 0 \\ \gamma_8 & \gamma_5-\gamma_8 & 0 \\ \gamma_8 & \gamma_6-\gamma_8 & \gamma_6 \end{vmatrix} = \gamma_6\gamma_8(\gamma_6-\gamma_5)$$

and $\qquad (4.5)$

$$Q = \begin{vmatrix} \gamma_2 & 0 & 1 \\ \gamma_8 & \gamma_5-\gamma_8 & 1 \\ \gamma_8 & \gamma_6-\gamma_8 & 0 \end{vmatrix} = \gamma_8(\gamma_6-\gamma_5)-\gamma_2(\gamma_6-\gamma_8) .$$

If z has the value given in (4.4) then the solution to the system obtained by omitting equations 5 and 6 from (3.1) is given by

$$x_1 = \frac{\begin{vmatrix} -b_1 + a_1 z & -1 \\ a_2 z + b_2\gamma_2 z + a_2\gamma_2 - b_2 & -1 + \gamma_1 z \end{vmatrix}}{\gamma_2(1 + z^2)} , \qquad (4.6)$$

$$y_1 = \frac{\begin{vmatrix} z & -b_1 + a_1 z \\ z + \gamma_2 & a_2 z + b_2\gamma_2 z + a_2\gamma_2 - b_2 \end{vmatrix}}{\gamma_2(1 + z^2)} ,$$

$$x_3 = \frac{\begin{vmatrix} a_2z + b_2\gamma_5z + a_2\gamma_5 - b_2 & -1 + z\gamma_5 \\ a_1z + b_1\gamma_6z + a_1\gamma_6 - b_1 & -1 + z\gamma_5 \end{vmatrix}}{(\gamma_6 - \gamma_5)(1 + z^2)},$$

$$x_3 = \frac{\begin{vmatrix} z + \gamma_5 & a_2z + b_2\gamma_5z + a_2\gamma_5 - b_2 \\ z + \gamma_6 & a_1z + b_1\gamma_6z + a_1\gamma_6 - b_1 \end{vmatrix}}{(\gamma_6 - \gamma_5)(1 + z^2)},$$

(4.6 cont'd)

$$x_2 = \frac{\begin{vmatrix} -(z + \gamma_1)x_1 + (1-z\gamma_1)y_1 & 1 - z\gamma_1 \\ -(z + \gamma_7)x_3 + (1-z\gamma_7)y_3 & 1 - z\gamma_7 \end{vmatrix}}{(\gamma_7 - \gamma_1)(z^2 + 1)},$$

$$y_2 = \frac{\begin{vmatrix} -z - \gamma_1 & -(z + \gamma_1)x_1 + (1 - z\gamma_1)y_1 \\ -z - \gamma_7 & -(z + \gamma_7)x_3 + (1 - z\gamma_7)y_3 \end{vmatrix}}{(\gamma_7 - \gamma_1)(z^2 + 1)}.$$

This solution exists and is the unique solution provided (4.4) is a conditional equation and $\gamma_2 \neq 0$, $\gamma_6 \neq \gamma_5$ and $\gamma_7 \neq \gamma_1$. If $\gamma_2 = 0$ then $\phi_1 = 0$ and P_1, A_1, A_2 are collinear. If $\gamma_6 = \gamma_5$ then $\phi_9 = 0$ and P_3, A_1, A_2 are collinear, and if $\gamma_7 = \gamma_1$ then $\phi_3 + \phi_7 = \pi$ so either P_2P_3 is parallel to P_2P_1 or P_1, P_2, P_3 are collinear. Each of these possibilities leads to a "danger" situation similar to the one illustrated below in the third case. In case (4.4) is an identity (which will occur for example if $\gamma_6 = \gamma_5 = \gamma_8$) similar "danger" configurations arise.

2. Omitting rows 8 and 9 from the matrix (3. 3), setting the corresponding determinant equal to zero and proceeding in a manner analogous to that used in case 1, we have

$$\begin{vmatrix} \gamma_2 & 0 & a_1-a_2-b_2\gamma_2 \\ -\gamma_1 & \gamma_1-\gamma_3 & -a_1+a_2+b_2\gamma_3 \\ -\gamma_1 & \gamma_1-\gamma_4 & b_1\gamma_4 \end{vmatrix} z + \begin{vmatrix} \gamma_2 & 0 & -a_2\gamma_2-b_1+b_2 \\ -\gamma_1 & \gamma_1-\gamma_3 & a_2\gamma_3+b_1-b_2 \\ -\gamma_1 & \gamma_1-\gamma_4 & a_1\gamma_4 \end{vmatrix} = 0. \qquad (4.7)$$

If (4. 7) is not a contradiction or an identity then

$$z = \frac{P(a_1 - a_2) + Q(b_1 - b_2)}{Q(a_1 - a_2) - P(b_1 - b_2)}, \qquad \text{where}$$

$$P = \begin{vmatrix} \gamma_2 & 0 & 0 \\ -\gamma_1 & \gamma_1 - \gamma_3 & 0 \\ -\gamma_1 & \gamma_1 - \gamma_4 & -\gamma_4 \end{vmatrix} = -\gamma_2\gamma_4(\gamma_1 - \gamma_3)$$

and (4. 8)

$$Q = \begin{vmatrix} \gamma_2 & 0 & 1 \\ -\gamma_1 & \gamma_1 - \gamma_3 & -1 \\ -\gamma_1 & \gamma_1 - \gamma_4 & 0 \end{vmatrix} = \gamma_1(\gamma_2 - \gamma_3) - \gamma_4(\gamma_2 - \gamma_1).$$

Replacing z by the value in (4. 8), the solution to the system of equations obtained by omitting equations 8 and 9 from (3. 1) is given by:

(x_1, y_1) same as in (4.6),

$$x_2 = \frac{\begin{vmatrix} -a_2z-b_2\gamma_3z-a_2\gamma_3+b_2 & 1-z\gamma_3 \\ -a_1z-b_1\gamma_4z-a_1\gamma_4+b_1 & 1-z\gamma_4 \end{vmatrix}}{(\gamma_4-\gamma_3)(1+z^2)};$$

$$y_2 = \frac{\begin{vmatrix} -z-\gamma_3 & -a_2z-b_2\gamma_3z-a_2\gamma_3+b_2 \\ -z-\gamma_4 & -a_1z-b_1\gamma_4z-a_1\gamma_4+b_1 \end{vmatrix}}{(\gamma_4-\gamma_3)(1+z^2)} \tag{4.9}$$

and (x_3, y_3) same as in (4.6).

These are solutions provided $\gamma_4 \neq \gamma_3$, $\gamma_2 \neq 0$, $\gamma_7 \neq \gamma_1$ and z is determined by (4.7). The two latter situations are described above. If $\gamma_4 = \gamma_3$ then $\phi_5 = 0$ and A_1, A_2, P_2 are collinear. Here the solution is ambiguous in a manner similar to that illustrated below in case 3 where A_1, P_2 and P_3 are collinear. There are several positions for points P_1, P_2, P_3 where $\phi_5 = 0$ and ϕ_1, ϕ_2, ϕ_3 and ϕ_7 are the same, for example when P_1 and P_3 are reflected in the line determined by A_1, A_2, P_2.

3. Here we omit rows 5 and 8 from the matrix (3.3) and proceed as in the previous cases to get

$$\begin{vmatrix} \gamma_2 & 0 & 0 & a_1-a_2-b_2\gamma_2 \\ -\gamma_1 & \gamma_1-\gamma_4 & 0 & b_1\gamma_4 \\ \gamma_8-\gamma_1 & \gamma_1-\gamma_7 & \gamma_7-\gamma_8 & 0 \\ \gamma_8 & 0 & \gamma_6-\gamma_8 & -b_1\gamma_6 \end{vmatrix} z + \begin{vmatrix} \gamma_2 & 0 & 0 & -a_2\gamma_2-b_1+b_2 \\ -\gamma_1 & \gamma_1-\gamma_4 & 0 & a_1\gamma_4 \\ \gamma_8-\gamma_1 & \gamma_1-\gamma_7 & \gamma_7-\gamma_8 & 0 \\ \gamma_8 & 0 & \gamma_6-\gamma_8 & -a_1\gamma_6 \end{vmatrix} = 0. \tag{4.10}$$

and solve for $\mathbf{z}$ if (4. 10) is conditional.

$$\mathbf{z} = \frac{[P(a_1 - a_2) + Q(b_1 - b_2)]R}{[Q(a_1 - a_2) - P(b_1 - b_2)]R} \quad \text{where } P = -\gamma_2,\ Q = 1 \quad \text{and}$$

$$R = \begin{vmatrix} -\gamma_1 & \gamma_1-\gamma_4 & 0 \\ \gamma_8-\gamma_1 & \gamma_1-\gamma_7 & \gamma_7-\gamma_8 \\ \gamma_8 & 0 & \gamma_6-\gamma_8 \end{vmatrix} = \gamma_1\gamma_4\gamma_8 + \gamma_1\gamma_6\gamma_7 + \gamma_4\gamma_6\gamma_8 - \gamma_1\gamma_4\gamma_6 - \gamma_1\gamma_6\gamma_8 - \gamma_4\gamma_7\gamma_8.$$

If $R \neq 0$, $\mathbf{z}$ is uniquely determined by (4. 10). Substituting $\mathbf{z}$ in the system of equations obtained by omitting equations 5 and 8 from (3. 1) yields a system of equations whose solution may be written:

(x_1, y_1) same as in (4. 6)

$$x_2 = \frac{\begin{vmatrix} -x_1\mathbf{z}-\gamma_1 y_1\mathbf{z}-x_1\gamma_1+y_1 & 1-\mathbf{z}\gamma_1 \\ -a_1\mathbf{z}-b_1\gamma_4\mathbf{z}-a_1\gamma_4\mathbf{z}+b_1 & 1-\mathbf{z}\gamma_4 \end{vmatrix}}{(\gamma_4-\gamma_1)(1+\mathbf{z}^2)},$$

$$y_2 = \frac{\begin{vmatrix} -\mathbf{z}-\gamma_1 & -x_1\mathbf{z}-\gamma_1 y_1\mathbf{z}-x_1\gamma_1+y_1 \\ -\mathbf{z}-\gamma_4 & -a_1\mathbf{z}-b_1\gamma_4\mathbf{z}-a_1\gamma_4+b_1 \end{vmatrix}}{(\gamma_4-\gamma_1)(1+\mathbf{z}^2)},$$

$$x_3 = \frac{\begin{vmatrix} -x_1\mathbf{z}-\gamma_8 y_1\mathbf{z}-x_1\gamma_8+y_1 & 1-\mathbf{z}\gamma_8 \\ -a_1\mathbf{z}-b_1\gamma_6\mathbf{z}-a_1\gamma_6+b_1 & 1-\mathbf{z}\gamma_6 \end{vmatrix}}{(\gamma_6-\gamma_8)(1+\mathbf{z}^2)} \qquad (4.\ 12)$$

and

$$y_3 = \frac{\begin{vmatrix} -\mathbf{z}-\gamma_8 & -x_1\mathbf{z}-\gamma_8 y_1\mathbf{z}-x_1\gamma_8+y_1 \\ -\mathbf{z}-\gamma_6 & -a_1\mathbf{z}-b_1\gamma_6\mathbf{z}-a_1\gamma_6+b_1 \end{vmatrix}}{(\gamma_6-\gamma_8)(1+\mathbf{z}^2)}$$

The "danger" situation arises if $R = 0$. In this case (4.10) is an identity. No attempt is made here to exhaust all of the "danger" configurations. Clearly, if all five points are collinear no solution to the system exists. We describe two other possibilities.

If $\gamma_1 = \gamma_4 = \gamma_7$ then $R = 0$. This occurs, for instance, if (as seen from (3.0)) $c_1 = c_4 = c_7$ and hence $\phi_4 = \pi$ and $\phi_3 + \phi_7 = \pi$ so that P_1, P_2, A_2 are collinear and the line P_1P_2 is parallel to the line P_3P_2 (Figure 7). This is a contradiction since P_2 is common to both lines, unless P_1, P_2, P_3 and A_1 are collinear so that $\phi_3 = \pi$ and $\phi_7 = 0$, an obvious "danger" position.

A more interesting case is the one in which $\gamma_4 = \gamma_6 = \gamma_7$. This is the case if $c_4 = c_6 = c_7$ which implies that $\phi_8 = 0$ and $\phi_3 + \phi_7 = \phi_4$. Both of the latter statements indicate that A_1, P_2 and P_3 are collinear. That this condition is ambiguous is illustrated in Figure 8 where P_1 may lie anywhere on the circumference of the circle determined by A_1, A_2 and ϕ_1. The points P_2 and P_3 are then positioned so as to be collinear with A_1 where the angles ϕ_4 and ϕ_7 with vertices P_2 and P_3 respectively are the same in each case.

The preceding discourse, admittedly, leaves much work to be done on the problem. The author has completed the discussion of the "backwards section" for the case of three unknown points and two known points.

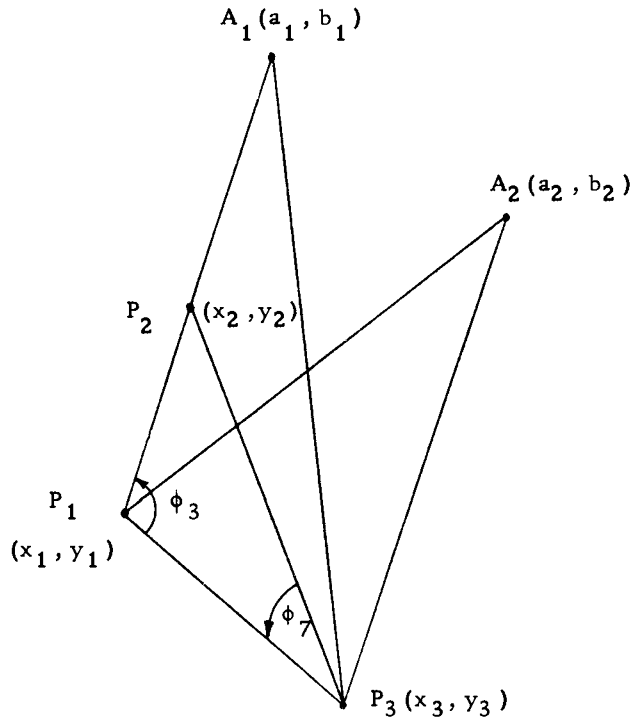

Figure 7

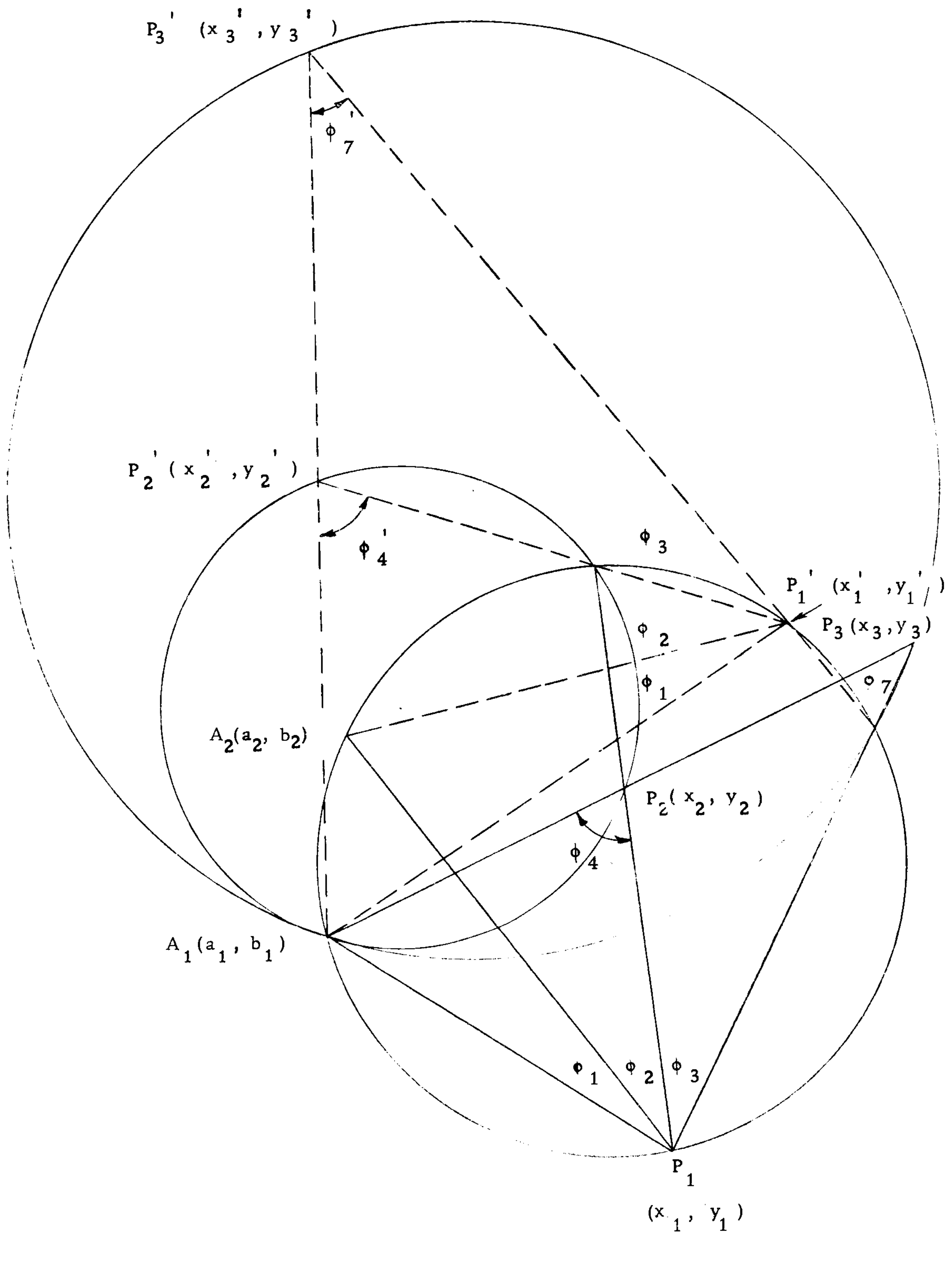
P3' (x3', y3')
φ7'
P2' (x2', y2')
φ4'
φ3
P1' (x1', y1')
P3 (x3, y3)
φ2
φ1
φ7
A2(a2, b2)
P2(x2, y2)
φ4
A1(a1, b1)
φ1
φ2
φ3
P1
(x1, y1)

Figure 8

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