AN ABSTRACT OF THE THESIS OF

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(Name) (Degree) (Major)

Date thesis is presented Aug 13, 1964

Title A NUMERICAL SOLUTION TO THE DYNAMIC RESPONSE OF A SIMPLY SUPPORTED BEAM UNDER A MOVING MASS LOAD

Abstract approved Redacted for privacy
(Major Professor)

The following work is concerned with a numerical solution to the dynamic response of a simply supported beam under a moving mass load.

Basis for these investigations is a linear integral equation of the first kind developed by C. E. Smith. Its general form is:

$$\int_0^t R(\tau) K(t, \tau) d\tau = G(t)$$

where $R(\tau)$ is the unknown reaction between the mass and the beam, and $K(t, \tau)$ and $G(t)$ are known functions.

Solution of the above equation was accomplished by approximating the integral by a finite sum, and setting N equations for N different values of t. Trajectories of the mass were then obtained once the reactive force was known.

Results were also obtained for approximate solutions obtained previously by Stokes and Inglis and comparisons drawn

between these and the results from the above integral equation.

The numerical solution of the integral equation presented its difficulties because of the nature and peculiar behavior of the kernel, which includes an infinite series whose terms are products of sine functions. Solutions for a large number of terms in this series are impracticable because of the increasing "waviness" of the function and the excessive amount of computer time involved. However, it is possible for some ranges of the parameters to determine by trial runs a number of terms that will yield sufficiently accurate results without using an excessive amount of computer time.

The numerical procedure presented here requires large computing facilities (digital) and it can become impracticable beyond a certain range of the parameters. However, in spite of these limitations this method presents a definite improvement over the approximate solutions of Stokes and Inglis.

A NUMERICAL SOLUTION TO THE DYNAMIC RESPONSE OF
A SIMPLY SUPPORTED BEAM UNDER A
MOVING MASS LOAD

by

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A THESIS

submitted to

OREGON STATE UNIVERSITY

in partial fulfillment of
the requirements for the
degree of

MASTER OF SCIENCE

August 1964

APPROVED:

Redacted for privacy

Associate Professor of Mechanical Engineering

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Date thesis is presented Aug 13, 1964

Typed by Carol Baker

ACKNOWLEDGMENT

I wish to express my gratitude to professor C. E. Smith who suggested the topic of these studies and whose guidance and advice greatly contributed to its successful completion.

I also extend my sincere appreciation to the Engineering Experiment Station for sponsoring this investigation and to the Western Data Processing Center, University of California, Los Angeles, whose services and computing facilities made this work possible.

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LIST OF SYMBOLS

A_n . . . . Arbitrary constant

B_n . . . . Fourier series coefficient defined in Equation(19a)

C_{in} . . . . Matrix term defined in Equation (28a)

D. . . . . $\sqrt{\frac{1}{4} - \beta}$

F_i . . . . $F(\zeta_i)$

$F(\zeta_i)$. . . . . $\frac{1}{2}\zeta_i^2$

EI. . . . . Flexural rigidity of beam

G(t). . . . . Function defined in Equation (5b)

$G_s(t)$. . . . . Function defined in Equation (11b)

$H(\zeta, z)$. . . . . . . .Kernel in Equation (9)

H_s . . . . Function defined in Equation (9b)

$K(t, \tau)$. . . . . . . Kernel in Equation (5)

$K_1(t, \tau)$. . . . . . Kernel in Equation (27)

$K_s(t, \tau)$. . . . . . .Kernel in Equation (11)

L. . . . . Distance between beam's supports

M. . . . . Total mass of the beam

N. . . . . Number of increments

NT. . . . . .Number of terms in series H_s

P(s). . . . . Laplace transform of p(t)

R(t). . . . . Reaction on the beam from the moving mass

S. . . . . $\sqrt{\beta - \frac{1}{4}}$

$T(z)$ .... Ratio of reactive force to the weight of the moving mass, $\frac{R(\tau)}{mg}$

$T(\zeta)$ .... $\frac{R(t)}{mg}$

$W(x)$ .... Loading function (Figure 2)

$Y(s)$ .... Laplace transform of $y(t)$

$Y_1(s)$ .... Laplace transform of $y_1(t)$

c .... Speed of wave travel for first mode, $\pi\sqrt{\frac{EI}{ML}}$

f_1, f_2 .... Functions defined in Equations (16a) and (16b)

g .... Acceleration due to gravity

h .... Load per unit length

i .... An integer

k .... Spring constant

m .... Mass of moving particle

n .... An integer

$p(t)$ .... $mg - R(t)$

$q(x, t)$ .... Time varying distributed load

s .... Laplace transform parameter

t .... Time

u .... Damping constant

v .... Horizontal speed of moving mass

$w(x, t)$ .... Beam displacements

$w_o(x, t)$ .... Free vibration displacements

$w_p(x, t)$ .... Forced vibration displacements

$w_s(x)$ .... Initial static displacements

x .... Horizontal coordinate (distance from left support)

$y(t)$ .... Vertical displacements of unsprung moving mass (positive downward), $w(vt, t)$

$y_1(t)$ .... Vertical displacements of sprung mass (Figure 1)

$y_o(t)$ .... $w(vt, t)$. See Equation (3)

$y_p(t)$ .... $w(vt, t)$. See Equation (3)

$y_s(t)$ .... $w(vt)$. See Equation (3)

z .... Integration variable

z_i .... $\frac{1}{2}(\zeta_i - \zeta_{i-1})$

β .... $\frac{3EI}{mv^2 L} = \frac{3}{\pi^2 \lambda^2 \rho}$

δ .... Mid span static deflection due to the weight of the moving mass acting at $x = \frac{L}{2}$, $\frac{mgL^3}{48EI}$

ζ .... $\frac{vt}{L}$

ζ_i .... $\frac{i}{N}$

$\eta(\zeta)$ .... $\frac{y(t)}{\delta}$

λ .... $\frac{v}{c}$

ρ .... $\frac{m}{M}$

τ .... Integration variable

ψ_n .... Arbitrary constant

ω_n .... Natural angular frequency of the nth oscillation mode of the beam, $\frac{n^2 \pi c}{L}$

$(\dot{\ })$, $(\ddot{\ })$ ..... $\frac{d}{dt}$, $\frac{d^2}{dt^2}$

A NUMERICAL SOLUTION TO THE DYNAMIC RESPONSE OF A SIMPLY SUPPORTED BEAM UNDER A MOVING MASS LOAD

I. INTRODUCTION

The following studies will be concerned with a numerical solution to the dynamic behavior of a simply supported beam over which a heavy mass particle is constrained to move at constant speed.

The problem of moving masses and loads over beams has been under investigation for over a hundred years and even at the present time is still the subject of much research. Although a solution has been found for the case of a traveling constant force, no exact* analytic solution exists as yet for the more complicated case where the inertia of the moving mass is taken into account. The difficulty arises from the interaction between the mass and the beam, since beam displacements depend on the reactive force which in turn depends on the acceleration imparted to the mass by the beam in motion.

Several approximate methods have been devised since interest in the problem was first aroused in 1847. All of these methods involve some sort of simplifying assumption made either in the

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* An exact solution usually means that assumptions of the Bernoulli-Euler beam theory are made, and that rotatory inertia of the beam and deflections due to shear are neglected.

differential equation, in its solution, or both.

Earliest analytical work was motivated by railway bridge vibrations under passing locomotives and it was started by R. Willis (9) who set up a differential equation for a massless beam system; he assumed that for heavy locomotives and very light bridge structures the mass of the latter could be neglected without introducing an appreciable error in the solution. Stokes (9) found a rigorous solution to this equation and on further work analyzed a system in which the mass of the beam was accounted for; however, he constrained the beam to deflect retaining the shape of a simply supported beam under an uniformly distributed load which is very similar to a half sine curve.

Kriloff (5) also included the mass of the beam in his analysis, but neglected the inertia of the moving mass, thus reducing his solution to that of a moving constant force, which he assumed to be a close approximation for the case of a slowly moving mass.

Schallemkamp (7) approximated the reaction by a finite sine series with n unknown coefficients and obtained independently, expressions for mass displacements and beam deflections; he then converted the latter into an expression for mass deflections, equated it to the first one for n different time intervals and solved for the unknown coefficients.

One of the most extensive studies ever made on the subject is due to C. E. Inglis (3). His analysis was based on the assumption that the support structure deflects retaining the general shape of a half sine curve, that is, he neglected all modes of vibration higher than the first. Most of his work was oriented towards railway bridge vibrations where excitation due to the out-of-balance forces of the locomotive's mechanism plays a dominant role. However, he also studied the response to the moving weight itself and from measurements of natural frequencies of an average railway bridge concluded that higher modes than the first could not be excited by a locomotive even at its highest speed. His solutions for mid span deflections were found in fairly close agreement with experimental values; however, he did not investigate the reactive force nor the trajectory of the mass which in Stokes' massless beam solution approached the end of the span vertically, thus suggesting the existence of an infinite reaction at that point.

With the advent of the electronic digital computer emphasis has shifted to numerical solutions of rigorously developed equations of motion. Basis for this study is a linear integral equation of the first kind rigorously developed by C. E. Smith (8, p. 3, 14), which, although it appears that it cannot be solved analytically, lends itself to a rather straight forward numerical solution that yields the reactive force acting on the beam. Trajectories and bending moments can be readily obtained by a similar numerical procedure once the force is known.

This work investigates the advantages and possible limitations of this method and attempts are also made to evaluate the approximate solutions by Stokes and Inglis.

II. DERIVATION OF INTEGRAL EQUATION

Unsprung Traveling Mass
(8, p. 3-30)

A brief outline of the derivation of C. E. Smith's integral equation shall be presented here as it applies to a simply supported beam.

The differential equation for the general response of a simply supported beam to a general loading function is obtained from the beam theory and Newton's second law:

$$EI \frac{\partial^4 w}{\partial x^4} + \frac{M}{L} \frac{\partial^2 w}{\partial t^2} = q(x, t) \tag{1}$$

where $q(x, t)$ is a general time-varying distributed load which with a suitable limiting process can be made to represent a variable concentrated force traveling at constant speed (8, p. 3).

By means of the theory of forced vibrations of statically coupled, linear systems (4, p. 170-178) an expression is obtained for the beam displacements:

$$w(x, t) = w_o(x, t) + w_p(x, t) + w_s(x). \tag{2}$$

The term

$$w_o(x, t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \cdot \cos \left(\frac{n^2 \pi ct}{L} - \psi_n\right),$$

represents initial free vibrations of the beam;

$$w_p(x,t) = \frac{2L}{M\pi c}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi x}{L}\int_0^t R(T)\sin\frac{n\pi vT}{L}\cdot$$

$$\sin\frac{n^2\pi c(t-T)}{L}\,dT$$

represents forced vibrations; and $w_s(x)$ is the shape of the beam when unloaded and at rest.

Letting $x=vt$ in equation (2) gives an expression for mass displacements as a function of time:

$$y(t) = y_o(t) + y_p(t) + y_s(t) \qquad (3)$$

which is a superposition of mass displacements due to initial vibrations, forced vibrations and static beam deflections respectively:

$$y_p(t) = \frac{2L}{M\pi c}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi vt}{L}\int_0^t R(T)\sin\frac{n\pi vT}{L}\cdot$$

$$\sin\frac{n^2\pi c(t-T)}{L}\,dT \qquad (3a)$$

$$y_o(t) = w_o(vt,t)$$

$$y_s(t) = w_s(vt)$$

A second expression for $y(t)$ is obtained independently by superposition of mass displacements due to its initial vertical velocity plus those caused by each force acting on the mass.

Assuming $y(o) = 0$:

$$y(t) = \dot{y}(o)t + \frac{1}{2}gt^2 - \frac{1}{m}\int_o^t R(\tau)(t-\tau)\,d\tau \tag{4}$$

Combination and re-arrangement of (3) and (4) gives:

$$\int_o^t R(\tau)K(t,\tau)\,d\tau = G(t) \tag{5}$$

where

$$K(t,\tau) = \frac{t-\tau}{m} + \frac{2L}{M\pi c}\sum_{n=1}^{\infty} \sin\frac{n\pi vt}{L} \cdot \sin\frac{n\pi v\tau}{L} \cdot \sin\frac{n^2\pi c(t-\tau)}{L} \tag{5a}$$

and

$$G(t) = \dot{y}(o)t + \frac{1}{2}gt^2 - [y_o(t) + y_s(t)] \tag{5b}$$

For an unsprung mass, $\dot{y}(o)$ cannot be set arbitrarily. If the reaction is to remain finite at $t = o$ the mass must be moving tangent to the beam as it passes over the first support.

Therefore, $$\dot{y}(o) = [\,\dot{y}_s(t) + \dot{y}_o(t)]_{t=o} \tag{5c}$$

In order to draw meaningful comparisons between solutions by this method and those by Stokes where the beam cannot have any initial deflection nor motion, only solutions for these initial conditions shall be obtained here.

Equations (4) and (5) reduce to:

$$y(t) = \frac{1}{2}gt^2 + \frac{1}{m}\int_o^t R(\tau)\,(t-\tau)d\tau \qquad (6)$$

and

$$\int_o^t R(\tau)\,K(t,\tau)\,d\tau = \frac{1}{2}gt^2 \qquad (7)$$

For convenience equations (6) and (7) can be expressed in nondimensional form:

$$\eta(\zeta) = \frac{48}{\pi^2 \rho \lambda^2}\left[\frac{1}{2}\zeta^2 - \int_o^\zeta T(z)\,(\zeta - z)\,dz\right] \qquad (8)$$

where

$$\eta(\zeta) = \frac{y(t)}{\delta}\ , \quad \delta = \frac{mgL^3}{48EI}\ , \quad \rho = \frac{m}{M}$$

$$\lambda = \frac{v}{c}\ , \quad T(z) = \frac{R(\tau)}{mg}\ , \quad \zeta = \frac{vt}{L}$$

and

$$z = \frac{v\tau}{L}$$

Equation (7) transforms into:

$$\int_o^\zeta T(z)\,H(\zeta, z)dz = F(\zeta) \qquad (9)$$

where

$$H(\zeta, z) = \zeta - z + H_s \qquad (9a)$$

$$H_s = \frac{2\rho\lambda}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\ \sin n\pi\zeta \cdot \sin n\pi z \cdot \sin\frac{n^2\pi(\zeta - z)}{\lambda} \qquad (9b)$$

and

$$F(\zeta) = \frac{1}{2}\zeta^2$$

Sprung Traveling Mass

Although no numerical solution shall be obtained for the traveling sprung mass, a brief analysis is included here which leads to the incorporation into the integral equation (5) of terms that will account for a suspension system composed of spring and damper connecting the mass and the beam. This gives a better model of a vehicle traversing a bridge and allows a greater choice of initial conditions* since $\dot{y}(o)$ is no longer constrained to value in equation (5c).

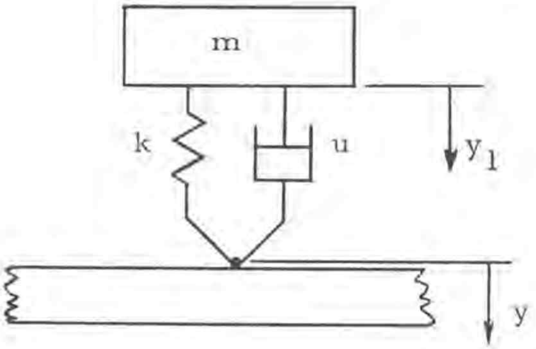

Figure 1

Consider mass m supported by spring and dashpot as shown in Figure 1. The differential equation for the mass is:

$$m\ddot{y}_1 + u\dot{y}_1 + ky_1 - u\dot{y} - ky = 0$$

Also, the reactive force on the beam is:

$$R(t) = mg - u\dot{y} - ky + u\dot{y}_1 + ky_1 \quad ,$$

which can be re-arranged into:

$$u\dot{y} + ky - u\dot{y}_1 - ky_1 = mg - R(t) = p(t)$$

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* Experimental measurements of mid span deflections of highway bridges traversed by heavy trucks have shown that initial motions of both truck and bridge have a great bearing on the structure's response (1, p. 15).

Using the Laplace transform the following set of algebraic equations is obtained:

$$-\left(\frac{u}{m}s + \frac{k}{m}\right) Y(s) + \left(s^2 + \frac{u}{m}s + \frac{k}{m}\right) Y(s) = \dot{y}_1(o)$$

$$(us + k)\, Y(s) = (us + k)\, Y_1(s) = P(s)$$

where $Y(s)$, $Y_1(s)$ and $P(s)$ are the Laplace transforms of $y(t)$, $y_1(t)$ and $p(t)$ respectively.

Solution gives:

$$Y(s) = \frac{\dot{y}_1(o)}{s^2} + \left[\frac{1}{ms^2} + \frac{1}{us+k}\right] P(s)$$

Its inverse transform is:

$$y(t) = \dot{y}_1(o)t + \int_o^t \left[\frac{t-T}{m} + \frac{1}{m} e^{-\frac{k}{u}(t-T)}\right] p(T)dT$$

from which the following expression for $y(t)$ is obtained:

$$y(t) = \dot{y}_1(o)t + \frac{1}{2}gt^2 + \frac{mg}{k}\left(1-e^{-\frac{k}{u}t}\right)$$
$$- \int_o^t R(T)\left[\frac{t-T}{m} + \frac{1}{u} e^{-\frac{k}{u}(t-T)}\right] dT \qquad (10)$$

Combining and rearranging equations (10) and (3) a new integral equation is obtained:

$$\int_o^t R(T)\, K_s(t, T)dT = G_s(t) \qquad (11)$$

where

$$K_s(t, T) = K(t, T) + \frac{1}{u} e^{-\frac{k}{u}(t-T)} \qquad (11a)$$

and

$$G_s(t) = G(t) + \frac{mg}{k}\left(1-e^{-\frac{k}{u}t}\right) \tag{11b}$$

Introduction of a suspension system composed of spring and dashpot leads therefore to an integral equation which is essentially of the same form as that for the unsprung mass; consequently, a numerical solution similar to that described in Chapter IV would also yield the unknown reactive force and the trajectory of the sprung mass for any given set of initial conditions.

III. DISCUSSION OF APPROXIMATE METHODS

Stokes' Massless Beam (9, p. 178-220)

In a hypothetical massless beam system, motion of the beam due to its own inertia does not exist. Therefore, its configuration is at any time that of a simply supported beam statically loaded by a concentrated force.

Mass deflections can then be expressed in terms of the reactive force and a flexibility coefficient which is a function of time:

$$y(t) = R(t)\frac{L^3}{3EI}\left(\frac{vt}{L}\right)^2\left(1 - \frac{vt}{L}\right)^2 \tag{12}$$

Also from Newton's second law:

$$R(t) = mg - m\frac{d^2y}{dt^2} \tag{13}$$

Substitution of the latter expression for $R(t)$ into (12) leads to the following differential equation:

$$m\frac{d^2y}{dt^2} + \frac{3EI}{L^3\left(\frac{vt}{L}\right)^2\left(1-\frac{vt}{L}\right)^2} \cdot y = mg \tag{14}$$

Expressed in terms of the dimensionless variables defined in the previous chapter this becomes:

$$\frac{d^2\eta}{d\zeta^2} + \frac{\beta}{\zeta^2(1-\zeta)^2} \cdot \eta = 16\beta \tag{15}$$

where

$$\beta = \frac{3EI}{mLv^2}$$

This is, essentially, R. Willis' differential equation for which Stokes found the following particular solution:

$$\eta(\zeta) = 16\beta \left[f_1(\zeta) \int_o^{\zeta} f_2(z)dz - f_2(\zeta) \int_o^{\zeta} f_1(z)\, dz \right] \tag{16}$$

where

$$\left.\begin{aligned} f_1(\zeta) &= \sqrt{\frac{\zeta(1-\zeta)}{2D}} \left(\frac{\zeta}{1-\zeta}\right)^{D} \\ f_2(\zeta) &= \sqrt{\frac{\zeta(1-\zeta)}{2D}} \left(\frac{\zeta}{1-\zeta}\right)^{-D} \end{aligned}\right\} \text{ for } \beta < \frac{1}{4} \tag{16a}$$

$$\left.\begin{aligned} f_1(\zeta) &= \sqrt{\frac{\zeta(1-\zeta)}{S}} \sin\left[S \operatorname{Log} \frac{\zeta}{1-\zeta} \right] \\ f_2(\zeta) &= \sqrt{\frac{\zeta(1-\zeta)}{S}} \cos\left[S \operatorname{Log} \frac{\zeta}{1-\zeta} \right] \end{aligned}\right\} \text{ for } \beta > \frac{1}{4} \tag{16b}$$

$$D = \sqrt{\frac{1}{4} - \beta}\,, \qquad S = \sqrt{\beta - \frac{1}{4}}$$

Similarly, a dimensionless expression can also be obtained from equation (12):

$$T(\zeta) = \frac{\eta(\zeta)}{16\zeta^2(1-\zeta)^2}, \text{ where } T(\zeta) = \frac{R(t)}{mg} \tag{17}$$

Zimmerman (11, p. 249) conducted detailed studies of Stokes' solution in order to determine the general shape of the trajectory of the mass.

For β smaller than 1/4 he found it to be tangent to the horizontal and vertical at $\zeta = 0$ and $\zeta = 1$ respectively. This obviously implies that the reaction approaches infinity as the mass approaches the second support.

For β greater than 1/4 the motion of the mass was found to be oscillatory with its frequency approaching infinity near both ends. The envelope of these oscillations also approached the first and second support tangentially to the horizontal and vertical respectively.

Trajectories and reactions plotted in Graphs I and II in Chapter V are all for values of β greater than 1/4, and were obtained by numerical approximations of equations (16) and (17).

The length of the beam was divided into 200 equal increments and the integrals approximated by finite sums:

$$\eta(\zeta_i) = 16\beta \left[f_1(\zeta_i) \sum_{n=1}^{i} f_2(z_n) \frac{1}{200} - f_2(\zeta_i) \sum_{n=1}^{i} f_1(z_n) \frac{1}{200} \right]$$

where

$$\zeta_i = \frac{i}{200} \qquad 1 \leq i \leq 200$$

and

$$z_n = \frac{1}{2} (\zeta_n + \zeta_{n-1})$$

Consequently:

$$T(\zeta_i) = \frac{\eta(\zeta_i)}{16\zeta_i^2 (1-\zeta_i^2)}$$

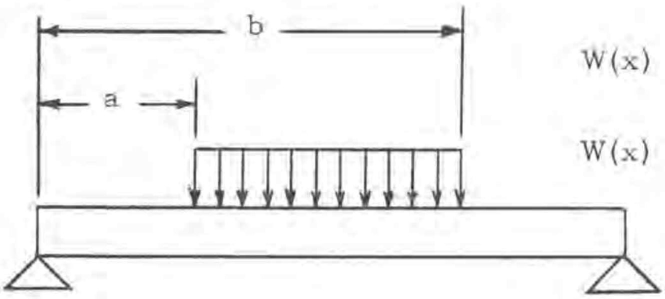
The Fortran program that produced results presented in Chapter V is included in the Appendix.

Inglis' First Mode

As previously stated, the assumptions leading to Inglis' approximate solutions admit beam deflections only in the configuration of its fundamental mode of free vibration. This follows from his assumption that components of the reactive force from the moving mass, in the directions of higher modes, are negligible.

A brief outline of the development of his differential equation for the case of a concentrated mass will be presented here (3, p. 1-9, 45,46), and it will be shown that it is equivalent to C. E. Smith's integral equation when only the first term of the series is retained in its kernel.*

Inglis arrived at a series representation of the time varying traveling load by considering a concentrated force to be a limiting case of a Fourier series representation (2, p. 53-60) of a load uniformly distributed over a segment of the beam (See Figure 2).



$$W(x) = h \qquad a \le x \le b$$

$$W(x) = 0 \qquad \begin{cases} x < a \\ x > b \end{cases}$$

Figure 2

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* In general it can be shown that the number of terms in H_s is equal to the number of vibration modes considered (8, p. 3-30).

The Fourier series representation of load in Figure 2 is:

$$W(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \tag{19}$$

where

$$B_n = \frac{4h}{n\pi} \left[\sin(b+a)\frac{n\pi}{2L} \cdot \sin(b-a)\frac{n\pi}{2L}\right] \tag{19a}$$

By making b approach a and letting

$$h(b-a) = R \quad \text{(constant)}$$

he obtained the following expression for a concentrated force R applied at $x = a$:

$$\frac{2}{L} R \sum_{n=1}^{\infty} \sin \frac{n\pi a}{L} \cdot \sin \frac{n\pi x}{L} \tag{20}$$

For a time varying force traveling with speed v the expression becomes:

$$R(t) \frac{2}{L} \sum_{n=1}^{\infty} \sin \frac{n\pi vt}{L} \cdot \sin \frac{n\pi x}{L} \tag{21}$$

Taking then only the first harmonic of the force he set his differential equation:

$$EI \frac{\partial^4 w}{\partial x^4} + \frac{M}{L} \frac{\partial^2 w}{\partial t^2} = R(t) \frac{2}{L} \sin \frac{\pi vt}{L} \sin \frac{\pi x}{L} \tag{22}$$

A particular solution to this equation is of the form:

$$w(x,t) = f(t) \sin \frac{\pi x}{L}$$

Newton's second law, then, may be written:

$$R(t) = mg - m\frac{d^2}{dt^2}\left[f(t)\sin\frac{\pi vt}{L}\right]$$

Substitution of these two expressions into equation (22) led him to a linear differential equation in $f(t)$ with variable coefficients for which he obtained a series solution by means of a rather lengthy procedure.

Equation (22) can be readily transformed into the one-term-series integral equation. Substituting $f(t)\sin\frac{\pi x}{L}$ for $w(x,t)$ into (22), differentiating and rearranging gives:

$$\ddot{f} + \omega^2 f = \frac{2}{M}R(t)\sin\frac{\pi vt}{L} \tag{23}$$

where

$$\omega^2 = \frac{EI\pi^4}{ML^3} = \left(\frac{\pi c}{L}\right)^2$$

The solution of equation (23) is:

$$f(t) = \frac{1}{\omega}\int_0^t \frac{2}{M}R(\tau)\sin\frac{\pi v\tau}{L}\sin\omega(t-\tau)\,d\tau \tag{24}$$

for a beam initially straight and at rest (10, p. 49-50).

It then follows that:

$$w(x,t) = \frac{2}{M\omega}\int_0^t R(\tau)\sin\frac{\pi x}{L}\cdot\sin\frac{\pi v\tau}{L}\cdot\sin\omega(t-\tau)\,d\tau \tag{25}$$

and consequently:

$$y(t) = \frac{2}{M\omega}\int_0^t R(\tau)\sin\frac{\pi vt}{L}\cdot\sin\frac{\pi v\tau}{L}\cdot\sin\omega(t-\tau)\,d\tau \tag{26}$$

Combining (26) with (6) yields:

$$\int_0^t R(\tau)\, K_1(t, \tau)\, d\tau = \frac{1}{2} g t^2 \tag{27}$$

where

$$K_1(t, \tau) = \frac{t-\tau}{m} + \frac{2L}{M\pi c} \sin \frac{\pi v t}{L} \cdot \sin \frac{\pi v \tau}{L} \cdot \sin \frac{\pi c(t-\tau)}{L} \tag{27a}$$

Comparison of equations (27) and (27a) with equations (7) and (5a) shows that Inglis' differential equation is equivalent to C. E. Smith's integral equation with only the first term present in the kernel. The results obtained by numerical approximations of equations (26) and (27) are for this reason labeled as Inglis' in graphs in Chapter V.

IV. NUMERICAL SOLUTION OF THE INTEGRAL EQUATION

Procedure

A numerical solution of the integral equation can be obtained by approximating the integral by a finite sum.

Equation (9) can then be expressed as:

$$\sum_{n=1}^{i} T(z_n)\, H(\zeta_i, z_n)(\zeta_n - \zeta_{n-1}) = F(\zeta_i) \tag{28}$$

where, with the length of the beam divided into N equal increments:

$$\zeta_i = \frac{i}{N} \quad , \qquad 0 \leq i \leq N$$

$$z_n = \frac{1}{2}(\zeta_n + \zeta_{n-1})$$

$$\zeta_n - \zeta_{n-1} = \frac{1}{N}$$

By letting $i = 1, 2, 3, \ldots, N$, successively, a system of N equations in N unknowns is then obtained which will yield the reactions at the middle of each increment.

Let $$C_{in} = H(\zeta_i, z_n)\frac{1}{N} \tag{28a}$$

$$T_n = T(z_n)$$

and $$F_i = F(\zeta_i)$$

The system of equations is then:

$$
\begin{aligned}
&C_{11} T_1 && = F_1 \\
&C_{21} T_1 + C_{22} T_2 && = F_2 \\
&C_{31} T_1 + C_{32} T_2 + C_{33} T_3 && = F_3 \\
&\cdots\cdots\cdots\cdots\cdots && \cdots \\
&C_{N1} T_1 + C_{N2} T_2 + C_{N3} T_3 + \ldots + C_{NN} T_N && = F_N
\end{aligned}
$$

Values of the unknown were obtained by solving directly for T_1 in the first equation, substituting into the second and so on. A Fortran program (See Appendix) was specially written for this solution since no subroutines were available for the particular case where $[C_{in}]$ is a triangular matrix. This eliminated the pointless manipulations and storage of the zero terms in the upper right half of the matrix which could be costly in computer time and would also limit the size of the matrix the computer could handle.

Displacements were also obtained by approximating an integral expression by a finite sum.

Equation (8) becomes:

$$\eta_i = \frac{48}{\pi^2 \rho \lambda^2} \left[\frac{1}{2} \zeta_i^2 - \sum_{n=1}^{i} T(z_n)(\zeta_i - z_n) \frac{1}{N} \right] \tag{29}$$

where $\eta_i = \eta(\zeta_i)$, and gives mass displacements at the end of each increment.

Limitations

The procedure outlined above suggests that in theory, values of T and η that are arbitrarily close to the exact solutions can be obtained if a sufficiently large number of increments is used and if coefficients C_{in} are calculated for a large number of terms in the series H_s (9b).

In practice, however, the method proved to have its limitations because of the nature and peculiar behaviour of the kernel in equation (9). Term-by-term differentiation of H_s (9b) with respect to z gives the following:

$$\frac{\partial H_s}{\partial z} = 2\rho \sum_{n=1}^{NT} \sin n\pi\zeta \left[\frac{\lambda}{n} \cos n\pi z \cdot \sin \frac{\pi}{\lambda} n^2 (\zeta - z) - \sin n\pi z \cdot \cos \frac{\pi}{\lambda} n^2 (\zeta - z) \right] \tag{30}$$

The lack of convergence of expression above indicates that for large values of NT $H(\zeta, z)$ is not smooth. Plots of H_s versus z for $\zeta = .5$ and several values of NT showed that the function, although perfectly smooth for NT = 1, became increasingly "wavy" as more terms were added to the series; for NT = 50 H_s exhibited extremely rapid oscillations of significant magnitude.

In order to obtain reasonably accurate results with NT greater than one it became necessary to increase N. This, however, also has a practical limit imposed not only by the speed and storage capacity of the computer, but also by the amount of error the type of solution used introduces when solving for the unknown reaction; this solution is essentially a form of Gauss' "successive elimination of the unknowns" (6, p. 528), which requires for greater accuracy that coefficients C_{ii} be the largest of each row, since each one becomes the denominator of each pivotal equation:

$$T_i = \frac{1}{C_{ii}} \left[F_i - \sum_{n=1}^{i-1} C_{in} T_n \right] \tag{31}$$

It can be noticed by inspection of equations (9a) and (9b) that the diagonal terms of the matrix $[C_{in}]$ are not only the smallest of each corresponding row but they actually approach zero as N becomes very large. Truncation errors are then greatly magnified during computation, and the result is the appearance of an error which is alternately positive and negative; this can be noticed in plots for Inglis' solutions in Graph I. Although also present in computations for $\rho = 1$, the above mentioned error is not visible because of the chosen scale; for reactions much larger than one (as is the case for most of the plots in Graph II) the error is comparatively negligible.

It is believed that the use of double-precision arithmetic*

---

* Double-precision arithmetic is a technique for carrying out calculations with twice the normal number of significant figures.

could eliminate or at least greatly reduce the above difficulty. Since double-precision requires twice as many storage locations as single-precision, a reduction in N would be necessary in order to prevent "core overlap".* Computer runs (single-precision) that yielded results presented in Chapter V used nearly all of the core of a 7094 IBM digital computer.

It is not possible, therefore, with the computing facilities available to obtain exact results (very large NT) by means of the numerical procedure used here.

A series of trial runs was made in order to determine values of N and NT that would give reasonably accurate results without using an excessive amount of computer time; plots of the kernel function were also obtained in order to assure an adequate number of increments.

Results in Chapter V were obtained with $N = 200$ and $NT = 9$ (C. E. Smith solution), and used a total of 6.23 minutes of a 7094 IBM digital computer; .55 minutes of this time was taken by computation for the massless beam.

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* Storage of more than one quantity in one memory location.

V. RESULTS AND DISCUSSION

Results obtained by the numerical approximations described in Chapters III and IV are plotted in graphs in pages 27 and 28.

A discussion of results by Stokes' and Inglis' methods is presented below as they compare with those obtained by C. E. Smith.

Stokes' Method

The results from C. E. Smith's equation indicate first of all that although reactions near the second support became large in some cases, unlike the results implied by Stokes' analysis, they remained finite. For $\rho = .1$ (Graph I) the peak value was 1.84 and, as might be expected, occurred at the highest speed ratio($\lambda = 1.1$). Much larger values were obtained for $\rho = 1$(Graph II) where the reactions reached a maximum of 72 and a minimum of -91, also at the highest speed ratio. In every case reactions for the massless beam reached much greater values than by the other two methods; the highest computed reaction was 1478 at $\zeta = .9975$. (No attempt was made to compute reactions at $\zeta = 1$ in order to avoid certain "overflow"* of the machine.) It can also be noticed that reactions by C. E. Smith's method show, especially for the higher speed ratios, some high frequency, small amplitude oscillations which are absent in Stokes' solution even for the few cases where there is some general agreement between both solutions.

* Computation of a quantity larger than the greatest the computer can handle.

Results by Stokes' method depend exclusively on the parameter $\beta (= \frac{3EI}{mv^2L})$. For the "heavy" beam β can also be expressed as a function of $\rho\ (= \frac{m}{M})$ and $\lambda\ (= \frac{v}{c})$ which are independent from each other. Any given β can, therefore, be obtained by an infinite number of combinations of ρ and λ, each one of them leading to a different solution; there is obviously one combination that will give better correlation with Stokes' results obtained with the same β.

It is possible, by simple visualization of a simply supported beam traversed by a heavy particle, to gain some general idea of what combination of ρ and λ would lead to better agreement between results by both methods. These parameters should:

1. Describe a system very close, physically to a massless beam.
2. Minimize the influence that the beam's mass has on the results.

The first condition requires small M and large m while the second one calls for large $\frac{EI}{L}$ and small v. Large $\frac{EI}{L}$ results in smaller deflections which tend to make inertia effects of the beam's mass negligible. The same effect results from a very low speed (small v); it can be visualized that in the limit as v approaches zero, the beam will have at every time its static equilibrium configuration regardless of whether it is massless or not.

Since all above conditions will result in small λ and large ρ it can be concluded that only in this case will Stokes' solution approximate that of C. E. Smith. Although only two values of ρ

were used in this work, the above conclusion is generally corroborated by the plots: there is fairly good agreement for $\rho = 1$ and $\lambda = .3$ while results for $\rho = .1$ and $\lambda = 1.1$ differ considerably.

Inglis' Method

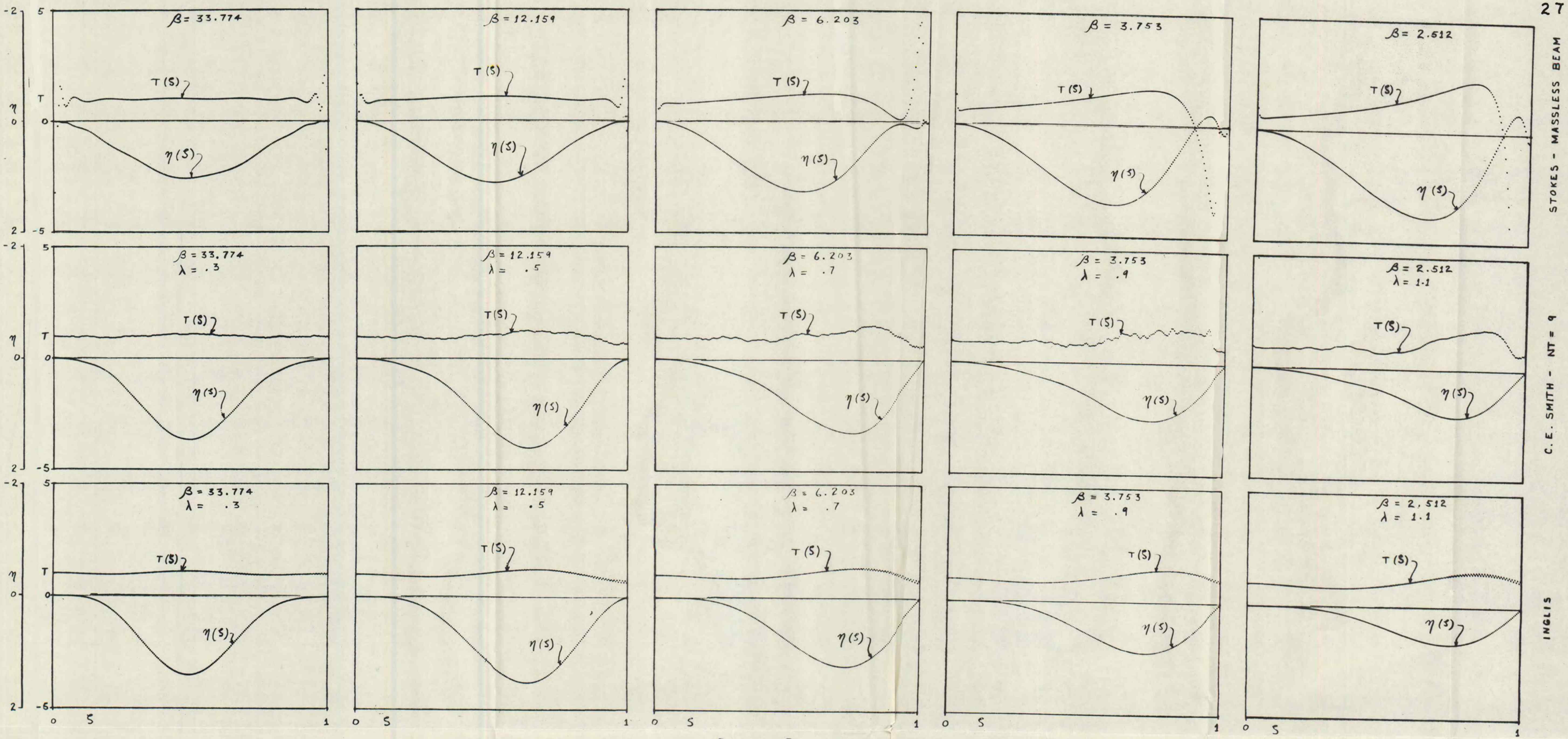
Results by Inglis' method show remarkably good correlation with those by C. E. Smith's method for low values of ρ and λ. For $\rho = .1$ and $\lambda = .3$ the reactive forces and trajectories are almost identical; this is to be expected from simple inspection of the integral equation (9); small values of ρ and λ tend to make H_s negligible thus reducing equation (9) to:

$$\int_o^{\zeta} T(z)\,(\zeta - z)\,dz = \frac{1}{2}\,\zeta^2 \tag{32}$$

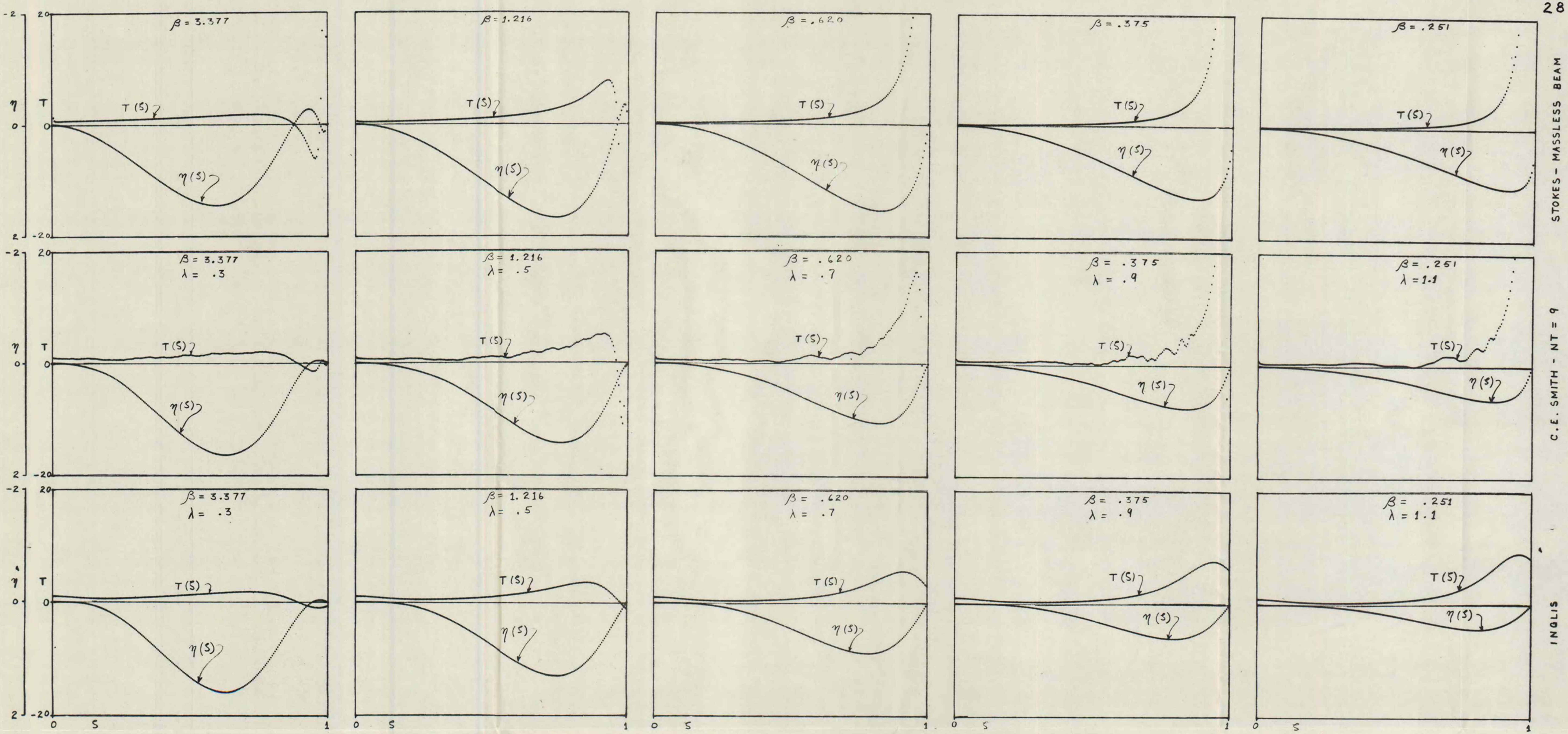
where value of NT becomes immaterial.

The solution of (32) is $T(z) \equiv 1$ which is very nearly the answer arrived at by both methods for $\rho = .1$ and $\lambda = .3$.

It can also be noticed that the high frequency, small amplitude oscillations in the force present in C. E. Smith's solutions are altogether missing in Inglis' results. However, while the discrepancy is relatively small for $\rho = .1$, considerable error can be observed in Inglis' solutions for $\rho = 1$ and higher values of λ, especially at points near the second support.



Graph I - Reactionsand trajectories of a mass crossing a simply suprted beam. $\rho = 0.1$



Graph II - Reactions and trajectories of a mass crossing a simply supported beam. $\rho = 1.0$

VI. SUMMARY AND CONCLUSIONS

The preceeding work shows that C. E. Smith's integral equation, although not suitable for an unlimited range of the parameters (see paragraphs 4 and 5 below), does provide a good basis for obtaining numerical results. Furthermore, as shown in Chapter II (sprung traveling mass), refinements can be easily added to the original equation, permitting the solution to be obtained for more complex systems and with a greater variety of initial conditions.

Conclusions may be summarized as follows:

1. For smaller values of ρ and λ, Kriloff's constant force method (5) provides a much simpler solution that is nevertheless sufficiently accurate.

2. For a combination of a large ρ and a small λ, Stokes' method will give accurate results at a much lesser amount of computer time than it would take by C. E. Smith's method.

3. Inglis' method can give fairly accurate results at low values of ρ and λ. Use of large values of these parameters results in a considerable amount of error.

4. For very large values of ρ and λ, resulting "waviness" of the kernel function could possibly require a higher N thus making this type of solution impracticable. However, values of ρ and λ larger than those used in this work are not very commonly encountered in engineering applications of these investigations.

5. Because of the time involved in the computation of the series terms this numerical approximation is not suited for small, nor even medium sized digital computers. Large computers like the IBM 7094 or the UNIVAC 1107 are necessary.

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APPENDIX

FORTRAN PROGRAM FOR C. E. SMITH'S AND INGLIS' METHODS

```
      DIMENSION X(200),Z(200),SNX(9,200),SNZ(9,200),
     1SNXZ(9,200), C(20100), P(200), T(200), H(200)
   40 FORMAT (10F11.6)
   50 FORMAT (10H REACTIONS F8.2,I6,F8.2)
   52 FORMAT (14H DISPLACEMENTS F8.2,I6,F8.2)
      PI=3.1415927
C           CALCULATION OF SINES
      DO 1 I=1,200
      R=I
      X(I)=R/200.
      Z(I)=X(I)-.0025
      P(I)=X(I)**2/2.
      DO 1 N=1,9
      U=N
      SNX(N,I)=SINF(U*PI*X(I))
    1 SNZ(N,I)=SINF(U*PI*Z(I))
      CALL PAGE
      DO 32 IVR=3,11,2
      UVR=IVR
      VR=UVR/10.
      Q=PI/VR
      QJ=(PI*VR)**2/24.
      DO 2 I=1,200
      DO 2 N=1,9
      U=N
    2 SNXZ(N,I)=SINF(U**2*Q*Z(I))
C           CALCULATION OF SERIES TERMS
      DO 30 NT=1, 9, 8
      K=0
      DO 6 I=1,200
      DO 6 J=1,I
      S=0.
      M=I-J+1
      DO 4 N=1,NT
      U=N
    4 S=S+SNX(N,I)*SNZ(N,J)*SNXZ(N,M)/U**2
      K=K+1
    6 C(K)=S
C           CALCULATION OF REACTIONS AND DISPLACEMENTS
      DO 32 IW=1,10,9
      UW=IW
      QI=2.*WR*VR/PI
      K=0
      DO 16 I=1,200
      DO 16 J=1,I
      K=K+1
      IF(IW-5)14,14,15
```

(continuation)

```
15    C(K)=(C(K)-(X(I)-Z(J))*.005)*10.+(X(I)-Z(J))*.005
      GO TO 16
   14 C(K)=(C(K)*QI+X(I)-Z(J))*.005
16    CONTINUE
      T(1)=P(1)/C(1)
      N=2
      DO 20 I=1,199
      S=0.
      DO 18 J=1,I
      S=S+C(N)*T(J)
   18 N=N+1
      T(I+1)=(P(I+1)-S)/C(N)
      N=N+1
   20 CONTINUE
      DO 24 I=1,200
      S=0.
      DO 22 J=1,I
   22 S=S+T(J)*(X(I)-Z(J))*.005
      H(I)=(X(I)**2/2.-S)/(8.*QJ*WR)
      H(I)=16.*H(I)
   24 CONTINUE
C            PRINT RESULTS
      WRITE OUTPUT TAPE 6,50,VR,NT,WR
      DO 26 I=1,200,10
      WRITE OUTPUT TAPE 6,40,T(I),T(I+1),T(I+2),T(I+3),
     1 T(I+4), T(I+5), T(I+6), T(I+7), T(I+8), T(I+9)
   26 CONTINUE
      WRITE OUTPUT TAPE 6,52,VR,NT,WR
      DO 28 I=1,200,10
      WRITE OUTPUT TAPE 6,40,H(I),H(I+1),H(I+2),H(I+3),
     1 H(I+4), H(I+5), H(I+6), H(I+7), H(I+8), H(I+9)
   28 CONTINUE
   32 CALL PAGE
      CALL EXIT
      END
```

FORTRAN PROGRAM FOR STOKES' METHOD

```
      DIMENSION H(200),T(200),X(200),Z(200),SC(200),SS(200),
     1RAX(200),RAZ(200),SLX(200),SLZ(200),CLX(200),CLZ(200)
 40    FORMAT (2F15.6,5XI2)
 50   FORMAT(30H       REACTIONS  DISPLACEMENTS 2F9.1,F11.3)
      DO 2 I=1,200
       R = I
      X(I) = R/200.
      Z(I) = X(I)- .0025
  2    CONTINUE
       DO 20 IVR=3,11,2
       UVR = IVR
       VR=UVR/10.
       DO 20 IW=1,10,9
       UW = IW
       WR=UW/10.
       B =.3./((3.141593*VR)**2*WR)
       R=SQRTF(B-1./4.)
      DO 8 I=1,199
       RAX(I)=SQRTF(X(I)*(1.-X(I))/R)
       RAZ(I)=SQRTF(Z(I)*(1.-Z(I))/R)
       SLX(I)=SINF(R*LOGF(X(I)/(1.-X(I))))
       SLZ(I)=SINF(R*LOGF(Z(I)/(1.-Z(I))))
       CLX(I)=COSF(R*LOGF(X(I)/(1.-X(I))))
       CLZ(I)=COSF(R*LOGF(Z(I)/(1.-Z(I))))
  8    CONTINUE
       SC(1)=RAZ(1)*CLZ(1)*.01
       SS(1)=RAZ(1)*SLZ(1)*.01
       H(1) = B*RAX(1)*(SLX(1)*SC(1)-CLX(1)*SS(1))
      T(1)=H(1)/((X(1)*(1.-X(1)))**2)
      H(1)= 16.*H(1)
      DO 24 I=2,199
       SC(I)=RAZ(I)*CLZ(I)*.01+SC(I-1)
       SS(I)=RAZ(I)*SLZ(I)*.01+SS(I-1)
       H(I)=B*RAX(I)*(SLX(I)*SC(I)-CLX(I)*SS(I))
       T(I)=H(I)/((X(I)*(1.-X(I)))**2)
      H(I)= 16.*H(I)
 24    CONTINUE
      CALL PAGE
       WRITE OUTPUT TAPE 6,50,VR,WR,B
      K=60
      DO 20 I=1,199
       WRITE OUTPUT TAPE 6,40,T(I),H(I),I
      IF(K-I)20,32,20
 32    CALL PAGE
      K=K+60
 20    CONTINUE
      CALL EXIT
       END
```