A CONSTRUCTIVE PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA

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I

INTRODUCTION

The fundamental theorem of algebra states that every algebraic equation with complex coefficients has at least one complex root. We note that any such equation is equivalent to:

$$1) \qquad \sum_{k=0}^{n} a_k \ z^k = 0,$$

where each a_k is a complex number, $a_n = 1$, and n is a positive integer. The fundamental theorem of algebra is proved by showing that the left side of (1) has a factor $(z-r)$, where r is a complex number.

In this paper it is proved that the left side of (1) is the product of n factors $(z-r_i)$, $i = 1, 2, 3, \ldots, n$. This is done by constructing the sequences:

$$
2)\qquad
\begin{array}{llll}
(r_{k_1})_0, & (r_{k_1})_1, & (r_{k_1})_2, & \ldots \\
(r_{k_2})_0, & (r_{k_2})_1, & (r_{k_2})_2, & \ldots \\
(r_{k_3})_0, & (r_{k_3})_1, & (r_{k_3})_2, & \ldots \\
\vdots & \vdots & \vdots & \\
(r_{k_n})_0, & (r_{k_n})_1, & (r_{k_n})_2, & \ldots
\end{array}
$$

showing that each sequence converges, and that the limits of the sequences are the roots of (1). (For each column of (2), the set, k_1, k_2, ..., k_n, is a reordering of the set 1, 2, ..., n.)

We choose each $(r_k)_o$, k = 1, 2, ..., n), i.e., the first member of each sequence above, to be an arbitrary complex number.

For t = 0, 1, 2, ..., we define $(a_k)_t$ and $(f)_t$ so:

$$3)\qquad \prod_{k=1}^{n}[z-(r_k)_t]=\sum_{k=0}^{n}(a_k)_t\ z^k;$$

$$4)\qquad \begin{aligned}(f)_t &= f\,[(r_1)_t, (r_2)_t, \ldots, (r_n)_t] \\ &= \sum_{k=0}^{n-1}|(a_k)_t-a_k|^2 \geq 0.\end{aligned}$$

The sequences (2) are constructed so that $\lim_{t\to\infty}(f)_t = 0$. This implies that as $t\to\infty$, $(a_k)_t \to a_k$

($k = 0, 1, 2, \ldots, n$), and hence that (making a proper designation of $k_1, k_2, \ldots, k_n$ for each value of t) the sequences in (2) converge, and that the limits thereof are the roots of (1), as is proved in Chapter IV.

The construction of the $(r_k)_t$ ($k = 1, 2, \ldots, n$; $t = 1, 2, 3, \ldots$), is such that

5) $$(f)_{t-1} - (f)_t \geq k_6 (f)_{t-1}^{k_5},$$

where k_5 and k_6 are constant for a given problem, so that $\lim_{t\to\infty} (f)_t = 0$. If (Case 1) the values of $(r_k)_{t-1}$ are all sufficiently distant from each other, it is shown that we can fulfill (5) by letting $(r_k)_t = (r_k)_{t-1}$ for all k, except that the real or imaginary part of some $(r_k)_t$ is chosen so as to minimize $(f)_t$. If this scheme does not work due to a condition (Case 2) that $(r_k)_{t-1}$, $k = 1, 2, \ldots, n$, contains some equal quantities, but yet all unequal quantities are sufficiently distant from each other, then letting $\prod_k [z-(r_k)_{t-1}]$ be the product of some set of equal factors $[z-(r_k)_{t-1}]$, we show that by letting

$$\prod_k [z-(r_k)_t] = \{\prod_k [z-(r_k)_{t-1}]\} + \Delta P,$$

where ΔP is a real or pure imaginary number, but otherwise letting $(r_k)_t = (r_k)_{t-1}$, (5) can be fulfilled. For proof of this see equations (6)-(11) and intervening discussion. (Note that Case 1 is a trivial variation of Case 2.) If (Case 3) some unequal quantities from $(r_k)_{t-1}$, $k = 1, 2, \ldots, n$, are too close to each other, they are adjusted to values equal to or distant from each other and other $(r_k)_{t-1}$. The new values are designated $(r_k)_{\underline{t-1}}$, $k = 1, 2, \ldots, n$. Then the $(r_k)_t$ are determined from the new values $(r_k)_{\underline{t-1}}$. This is done in such a way that in spite of the fact that $(f)_{\underline{t-1}}$ may be greater than $(f)_{t-1}$, $(f)_{\underline{t-1}} - (f)_t$ is large enough so (5) holds.

Using the above method of choosing $[(r_1)_t, (r_2)_t, \ldots, (r_n)_t]$, given the values of $[(r_1)_{t-1}, (r_2)_{t-1}, \ldots, (r_n)_{t-1}]$, then, by induction on t, all values in the sequences (2) may be obtained. As indicated above, by proper ordering of each column of (2), n sequences, each converging to a root of (1), are obtained.

Note: It is assumed throughout that $n \geq 2$.

II

THE ITERATION $(r_k)_{t-1} \rightarrow (r_k)_t$

We shall "round-off" the real and imaginary parts of each $(r_k)_{t-1}$ to the nearest integral multiple of M_4, a quantity defined in Chapter III, and call the result $(r_k)_{\underline{t-1}}$. Thus any two unequal quantities of the set, $[(r_1)_{\underline{t-1}}, (r_2)_{\underline{t-1}}, \ldots, (r_n)_{\underline{t-1}}]$, will have a difference of modulus not less than M_4.

Let the set $[(r_1)_{\underline{t-1}}, (r_2)_{\underline{t-1}}, \ldots, (r_n)_{\underline{t-1}}]$ be divided into equivalence classes so that two elements of the set are in the same class if and only if they are equal. Since the ordering of the set $\{(r_k)_{\underline{t-1}}\}$, $k = 1, 2, \ldots, n$, is arbitrary, let us redistribute the subscripts k so that all the members of any equivalence class are adjacently located in the set $[(r_1)_{\underline{t-1}}, (r_2)_{\underline{t-1}}, \ldots, (r_n)_{\underline{t-1}}]$. We will find the following alternative notation useful. The v-th element of the u-th equivalence class we call $(r_{u,v})_{\underline{t-1}}$. If $(r_k)_{\underline{t-1}} = (r_{u_0,v_0})_{\underline{t-1}}$, we say that $k = m(u_0,v_0)$, that $u_0 = u(k)$, and that $v_0 = v(k)$. Also, in general, we let m_0 denote $m(u_0,v_0)$. Consequently: $m(u_2,v_2) > m(u_1,v_1)$ if and only if $u_2 > u_1$, or $u_2 = u_1$ and $v_2 > v_1$; and, if $m(u,v_1 + 1)$ exists, it equals $m(u,v_1) + 1$.

For any g, $(g)_t$ and $(g)_{\underline{t}}$ $(t = 0, 1, 2, \ldots)$ shall be considered as particular values of a variable, g^*. Conversely, if g or g^* is some function, and given some s, $(g)_s$ means the value of said function when $r_m^* = (r_m)_s$, $m = 1, 2, \ldots, n$. We also define:

6)
$$P_{u,v} = \prod_{j=1}^{v} (z - r_{u,j}^*).$$

An $m_0 (1 \leq m_0 \leq n)$, and consequently, $u_0 = u(m_0)$, and $v_0 = v(m_0)$, are chosen. We define:

$$(r_{u,v})_t = (r_{u,v})_{\underline{t-1}} \text{ if } u \neq u_0 \text{ or } v > v_0.$$

The other $(r_m)_t$ are so defined that the increment of P_{m_0}, as each $r_{u_0,j}^*$ $(j = 1, 2, \ldots, v_0)$ goes from $(r_{u_0,j})_{\underline{t-1}}$ to $(r_{u_0,j})_t$, is ΔP_{m_0}, which is defined as the optimum real or optimum pure imaginary value of the increment of P_{m_0}, for the minimization of $(f)_t$, under the given conditions. A wise choice of m_0, and whether ΔP_{m_0} should be real or imaginary, are discussed in Chapter III.

Let $f = f(r_1^*, r_2^*, \ldots, r_n^*)$. The symbol

$\frac{\partial f}{\partial \operatorname{Re} P_{u_o, v_o}}$ or $\frac{\partial f}{\partial \operatorname{Im} P_{u_o, v_o}}$ shall indicate differentiation with all $r^*_{u,v}$ for which $u \neq u_o$ or $v > v_o$, held constant, but with the real or imaginary part of the term, which does not involve z, of the polynomial expansion of P_{u_o, v_o}, variable.

Using the notation:

7) $$\Delta_{t,1}\, r_m = (r_m)_t - (r_m)_{\underline{t-1}},$$

we apply the increment $\Delta P_{u_o, v_o}$ (here called Δ) to P_{u_o, v_o} by letting (when $v = 1, 2, \ldots, v_o - 1$):

8) $$\Delta_{t,1}\, r_{u_o, v} = \sqrt[v_o]{\Delta} \operatorname{cis} \left(\frac{2v-1}{v_o}\pi\right), \text{ if } \Delta > 0;$$

9) $$\Delta_{t,1}\, r_{u_o, v} = \sqrt[v_o]{-\Delta} \operatorname{cis} \left(\frac{2v}{v_o}\pi\right), \text{ if } \Delta < 0;$$

10) $$\Delta_{t,1}\, r_{u_o, v} = \sqrt[v_o]{\Delta/i} \operatorname{cis} \left(\frac{2v - 1/2}{v_o}\pi\right), \text{ if } \Delta/i < 0;$$

11) $$\Delta_{t,1}\, r_{u_o, v} = \sqrt[v_o]{\Delta/-i} \operatorname{cis} \left(\frac{2v + 1/2}{v_o}\pi\right), \text{ if } \Delta/i < 0.$$

Theorem 1: If ΔP_{m_o} is real,

12) $$\Delta P_{m_o} = \frac{-\left(\frac{\partial f}{\partial \operatorname{Re} P_{m_o}}\right)_{\underline{t-1}}}{\frac{\partial^2 f}{\partial (\operatorname{Re} P_{m_o})^2}}$$

and if ΔP_{m_o} is a pure imaginary,

$$\text{13)} \qquad \Delta P_{m_o} = \frac{-i\left(\dfrac{\partial f}{\partial \operatorname{Im} P_{m_o}}\right)_{t-1}}{\dfrac{\partial^2 f}{\partial(\operatorname{Im} P_{m_o})^2}} .$$

Proof: We define $a^*_{k,m(u,v)}$ $(k = 0, 1, \ldots, n-1)$

in such a manner that:

$$\text{14)} \qquad \frac{\prod_{k=1}^{n}(z-r^*_k)}{P_m} = \sum_{k=0}^{n-1} a^*_{k,m}\, z^k .$$

From equation (4),

$$\text{15)} \qquad f = \sum_{k=0}^{n-1} \left\{[\operatorname{Re}(a^*_k - a_k)]^2 + [\operatorname{Im}(a^*_k - a_k)]^2\right\}.$$

These equations follow:

$$\frac{\partial f}{\partial \operatorname{Re} P_m} = \sum_{k=0}^{n-1} \left[2\operatorname{Re}(a^*_k - a_k)\frac{\partial \operatorname{Re} a^*_k}{\partial \operatorname{Re} P_m} + 2\operatorname{Im}(a^*_k - a_k)\frac{\partial \operatorname{Im} a^*_k}{\partial \operatorname{Re} P_m}\right]$$

$$\text{16)} \qquad = 2\sum_{k=0}^{n-1} \left[\operatorname{Re}(a^*_k - a_k)\operatorname{Re} a^*_{k,m} + \operatorname{Im}(a^*_k - a_k)\operatorname{Im} a^*_{k,m}\right];$$

$$\frac{\partial f}{\partial \operatorname{Im} P_m} = \sum_{k=0}^{n-1} \left[2\operatorname{Re}(a^*_k - a_k)\frac{\partial \operatorname{Re} a^*_k}{\partial \operatorname{Im} P_m} + 2\operatorname{Im}(a^*_k - a_k)\frac{\partial \operatorname{Im} a^*_k}{\partial \operatorname{Im} P_m}\right]$$

$$\text{17)} \qquad = 2\sum_{k=0}^{n-1} \left[-\operatorname{Re}(a^*_k - a_k)\operatorname{Im} a^*_{k,m} + \operatorname{Im}(a^*_k - a_k)\operatorname{Re} a^*_{k,m}\right];$$

18) $$\frac{\partial^2 f}{\partial\,(\text{Re } P_m)^2} = \frac{\partial^2 f}{\partial\,(\text{Im } P_m)^2}$$
$$= 2\sum_{k=0}^{n-1} \left\{[\text{Re } a^*_{k,m}]^2 + [\text{Im } a^*_{k,m}]^2\right\}.$$

We note that $a^*_{k,m}$, and the two second partial derivatives above, are independent of P_m. Since $a^*_{k,m}$ ($k = 0, 1, \ldots, n-v(m)$) are the coefficients of the polynomial representation of the product of all $(z-r^*_k)$ for which $u(k) \neq u(m)$ or $v(k) > v(m)$, $a^*_{n-v(m),m} = 1$, and due to equation (18),

19) $$\frac{\partial^2 f}{\partial\,(\text{Re } P_m)^2} = \frac{\partial^2 f}{\partial\,(\text{Im } P_m)^2} \geq 2 > 0.$$

The function f will be minimized with respect to the variable $\Delta P_{u_o,v_o}$, and hence with respect to Re P_{m_o} or Im P_{m_o}, if and only if $\frac{\partial f}{\partial \text{ Re } P_{m_o}}$ or $\frac{\partial f}{\partial \text{ Im } P_{m_o}}$ vanishes, due to (16) - (19). Then by definition of $r_{m,t}$:

20) $$\left(\frac{\partial f}{\partial \text{ Re } P_{m_o}}\right)_t = 0 \text{ if } \Delta P_{m_o} \text{ is real;}$$

and

21) $$\left(\frac{\partial f}{\partial \text{ Im } P_{m_o}}\right)_t = 0 \text{ if } \Delta P_{m_o} \text{ is imaginary.}$$

Hence if ΔP_{m_o} is real:

22) $\frac{\partial^2 f}{\partial (\mathrm{Re}\, P_{m_o})^2} [(P_{m_o})_{\underline{t-1}} - (P_{m_o})_t] = (\frac{\partial f}{\partial \mathrm{Re}\, P_{m_o}})_{\underline{t-1}}$,

and if ΔP_{m_o} is imaginary:

23) $\frac{\partial^2 f}{\partial (\mathrm{Im}\, P_{m_o})^2} [\mathrm{Im}\, (P_{m_o})_{\underline{t-1}} - \mathrm{Im}\, (P_{m_o})_t] = (\frac{\partial f}{\partial \mathrm{Im}\, P_{m_o}})_{\underline{t-1}}$.

The theorem follows from equations (22) and (23) and the fact that $\Delta P_{m_o} = (P_{m_o})_t - (P_{m_o})_{\underline{t-1}}$.

Theorem 2:

$(f)_t = (f)_{\underline{t-1}} - 1/2\, (\frac{\partial f}{\partial \mathrm{Re}\, P_{m_o}})^2_{\underline{t-1}} \quad \frac{\partial^2 f}{\partial (\mathrm{Re}\, P_{m_o})^2}$, if

ΔP_{m_o} is real;

$(f)_t = (f)_{\underline{t-1}} - 1/2\, (\frac{\partial f}{\partial \mathrm{Im}\, P_{m_o}})^2_{\underline{t-1}} \quad \frac{\partial^2 f}{\partial (\mathrm{Im}\, P_{m_o})^2}$, if

ΔP_{m_o} is imaginary.

Proof: The expansion of f, considered as a function of the real variable, Re P_{m_o} or Im P_{m_o}, by Taylor's series, about $P_{m_o} = (P_{m_o})_t$, yields:

24) $(f)_{\underline{t-1}} = \sum_{k=0}^{\infty} \Big\{ (\frac{\partial^k f}{\partial (\mathrm{Re}\, P_{m_o})^k})_t \cdot \mathrm{Re}\, [(P_{m_o})_{\underline{t-1}} -$

$(P_{m_o})_t]^k \div k! \Big\}$ if ΔP_{m_o} is real;

25) $(f)_{t-1} = \sum_{k=0}^{\infty} \left\{ \left(\frac{\partial^k f}{\partial (\mathrm{Im}\ P_{m_o})^k} \right)_t \cdot [\mathrm{Im}\ (P_{m_o})_{t-1} - \mathrm{Im}\ (P_{m_o})_t]^k \div k! \right\}$ if ΔP_{m_o} is imaginary. By equations (20) and (21), the terms in (24) and (25) for which $k = 1$ vanish. The terms for which $k > 2$ vanish by the remark following (18). Hence:

26) $(f)_{t-1} = (f)_t + 1/2 \dfrac{\partial^2 f}{\partial (\mathrm{Re}\ P_{m_o})^2} (\Delta P_{m_o})^2$ if ΔP_{m_o} is real; and

27) $(f)_{t-1} = (f)_t - 1/2 \dfrac{\partial^2 f}{\partial (\mathrm{Im}\ P_{m_o})^2} (\Delta P_{m_o})^2$ if ΔP_{m_o} is imaginary. Theorem 2 then follows from equations (12), (13), (26), and (27).

III

THE CONVERGENCE OF f TO ZERO,

Let $A^*_{k,m}$ denote the matrix of elements $a^*_{k,m}$, with n rows ($k = 0, 1, \ldots n-1$) and n columns ($m = 1, 2, \ldots n$). We denote by $A^*_{k,m;p}$ and $a^*_{k,m;p}$ the resulting matrix, and an element thereof, after p elementary transformations of the first type on $A^*_{k,m}$, as follows ($p = 0, 1, 2, \ldots$): If $p < m \leq n$ and $1 < v(m) \leq p$:

28) $$a^*_{k,m;p} = (a^*_{k,m;p-1} - a^*_{k,m-1;\,p-1}) \div (r^*_m - r^*_{p+1-v(m)});$$

if $p < m \leq n$ and $v(m) = 1$:

29) $$a^*_{k,m;p} = (a^*_{k,m;p-1} - a^*_{k,p;p-1}) \div (r^*_m - r^*_p);$$

if $1 \leq m \leq p$ or $v(m) > p$:

30) $$a^*_{k,m;p} = a^*_{k,m;p-1}$$

We define:

31) $$S_{m;p} = \sum_{k=0}^{n-1} a^*_{k,m;p} z^k,$$

with the understanding that $A^*_{k,m;o} = A^*_{k,m}$, $a^*_{k,m;o} = a^*_{k,m}$, and $S_{m;o} = S_m$.

Lemma 1:

If $1 \leq m \leq p + 1$,

$$S_{m;p} = \prod_{k=m+1}^{n} (z - r_k^*); \tag{32}$$

if $p < m = n$ and $1 \leq v(m) \leq p + 1$,

$$S_{m;p} = [\prod_{k=p+2-v(m)}^{n} (z - r_k^*)] \div [\prod_{k=m+1-v(m)}^{m} (z - r_k^*)], \tag{33}$$

and if $v(m) > p$,

$$S_{m;p} = [\prod_{k=1}^{n} (z - r_k^*)] \div [\prod_{k=m+1-v(m)}^{m} (z - r_k^*)]. \tag{34}$$

(Note: The product of the elements of a null set is defined to be unity.)

Proof: We note that (6) is equivalent to:

$$P_m = \prod_{k=m+1-v(m)}^{m} (z - r_k^*). \tag{35}$$

We then prove the lemma by induction on p.

Suppose $p = 0$. Then (32), interpreted by (31), says that:

$$\sum_{k=0}^{n-1} a_{k,1}^* \, z^k = \prod_{k=2}^{n} (z - r_k^*). \tag{36}$$

Because $v(1) = 1$, (35) says that $P_1 = (z - r_1^*)$. Using this fact, (14) shows the truth of (36), and hence of (32) when $p = 0$.

The truth of equations (33) and (34) in the case $p = 0$ is established by (35), (14), and (31). Hence Lemma 1 is true in the case $p = 0$.

Assume the lemma is true when $p = p_o - 1$. The following paragraphs prove that this implies the truth of the lemma when $p = p_o$:

First we note that as a consequence of (31), equations (28) - (30) retain their validity when $a^*_{k,m}$ is replaced by S_m, or $a^*_{k,m-1}$ is replaced by S_{m-1}.

Due to the assumption that (32) is true when $p = p_o - 1$, and applying (30):

37) $$\text{if } 1 \leq m \leq p_o,\ S_{m;p_o} = \prod_{k=m+1}^{n} (z - r^*_k).$$

By the assumption that (33) is true when $p = p_o - 1$, and applying (28), and the fact that $v(m) > 1$, $v(m-1) = v(m) - 1$: if $p_o < m \leq$ and $1 < v(m) \leq p_o$,

$$S_{m;p_o} = (S_{m;p_o-1} - S_{m-1;p_o-1}) \div (r^*_m - r^*_{p+1-v(m)})$$

$$= \Big\{ [\prod_{k=p_o+1-v(m)}^{n} (z - r^*_k)] \div [\prod_{k=m+1-v(m)}^{m} (z - r^*_k)]$$

$$- [\prod_{p_o+2-v(m)}^{n} (z - r^*_k)] \div [\prod_{k=m+1-v(m)}^{m-1} (z - r^*_k)] \Big\}$$

$$\div (r^*_m - r^*_{p_o+1-v(m)})$$

$$= \Big\{ [\prod_{p_o+2-v(m)}^{n} (z - r^*_k)] \div [\prod_{k=m+1-v(m)}^{m} (z - r^*_k)] \Big\}$$

$$\cdot[(z - r^*_{p_o+1-v(m)}) - (z - r^*_m)] \div (r^*_m - r^*_{p_o+1-v(m)}.$$

So, if $p_o < m \leq n$ and $1 < v(m) \leq p_o$,

$$38)\quad S_{m;p_o} = [\prod_{k=p_o+2-v(m)}^{n} (z - r^*_k)] \div$$

$$[\prod_{k=m+1-v(m)}^{m} (z - r^*_k)]$$

Consider the case $p_o < m \leq n$ and $v(m) = 1$. By (33) and (29):

$$S_{m;p_o} = (S_{m;p_o-1} - S_{p_o;p_o-1}) \div (r^*_m - r^*_{p_o})$$

$$= \Big\{[\prod_{k=p_o}^{n} (z - r^*_k)] \div (z - r^*_m)$$

$$- [\prod_{k=p_o+1-v(p_o)}^{n} (z - r^*_k)] \div [\prod_{k=p_o+1-v(p_o)}^{p_o} (z - r^*_k)\Big\}$$

$$\div (r^*_m - r^*_{p_o})$$

$$= \Big\{[\prod_{k=p_o+1}^{n} (z - r^*_k)] \div (z - r^*_m)\Big\}.$$

$$[(z - r^*_{p_o}) - (z - r^*_m)] \div (r^*_m - r^*_{p_o}).$$

Since this is equivalent to (38) in the case $v(m) = 1$, the validity of (38) is extended to the case: $p_o < m \leq n$ and $1 \leq v(m) \leq p_o$.

By the assumption that (34) is true when $p = p_o - 1$, and applying (30): if $v(m) > p_o$,

39) $$S_{m;p_o} = S_{m;p_o-1} = [\prod_{k=1}^{n} (z - r_k^*)] \div [\prod_{k=m+1-v(m)}^{m} (z - r_k^*)].$$

Suppose $v(m) = p_o + 1$. Then, by (39):

$$S_{m;p_o} = [\prod_{k=1}^{n} (z - r_k^*)] \div [\prod_{k=m-p_o}^{m} (z - r_k^*)],$$

which, if $v(m) = p_o + 1$, is equivalent to (38). So equation (38) is true if $p_o < m \leq n$ and $1 \leq v(m) \leq p_o + 1$.

Suppose $m = p_o + 1$. Then $1 \leq v(m) \leq p_o + 1$, so (38) holds. In this case (38) implies:

$$S_{m;p_o} = [\prod_{k=p_o+2-v(m)}^{n} (z - r_k^*)] \div [\prod_{k=p_o+2-v(m)}^{m} (z - r_k^*)],$$

which is

$$S_{m;p_o} = \prod_{k=m+1}^{n} (z - r_k^*).$$

Hence equation (37) is true when $1 \leq m \leq p_o + 1$.

Due to equations (37) - (39), and the cases for which their validity is established, the lemma is true when $p = p_o - 1$. Hence the lemma is proved by induction on p.

As a consequence of Lemma 1,

40) $$S_{m;n-1} = \prod_{k=m+1}^{n} (z - r_k^*), \quad m = 1, 2, \ldots, n.$$

We let:

41) $$a_{k,m}^{*(0)} = a_{k,m;n-1}^*; \text{ and } A_{k,m}^{*(0)} = A_{k,m;n-1}^*$$

We then define elementary transformations of the second type as follows:

42) $$\text{If } m \leq n - p, \; a_{k,m}^{*(p)} = a_{k,m}^{*(p-1)} + r_{m+p}^* \, a_{k,m+1}^{*(p-1)};$$

43) $$\text{if } m > n - p, \; a_{k,m}^{*(p)} = a_{k,m}^{*(p-1)}.$$

We make a definition similar to (31):

44) $$S_m^{(p)} = \sum_{k=0}^{n-1} a_{k,m}^{*(p)} \, z^k.$$

Lemma 2:

45) $$\text{If } m \leq n - p, \; S_m^{(p)} = z^p \prod_{k=m+p+1}^{n} (z - r_k^*);$$

46) $$\text{if } m \geq n - p, \; S_m^{(p)} = z^{n-m}.$$

Proof: A comparison of (31), (41), and (44) shows that $S_m^{(0)} = S_{m;n-1}$. Hence when $p = 0$, (40) establishes the truth of (45). If $p = 0$, and $m \geq n - p$, then $m = n$, since m is limited to $(1, 2, \ldots, n)$. Hence in the case $p = 0$, statement (46) reduces to $S_n^{(0)} = 1$. $S_{n;n-1} = 1$, by (40). Since $S_{n;n-1} = S_n^{(0)}$, (46) and hence the lemma, is true when $p = 0$.

Suppose the lemma is true when $p = p_0 - 1$. The lemma will be proved for the case $p = p_0$, completing its proof by induction.

We note that (44) permits the substitution of $S_m^{(p)}$ for $a_{k,m}^{*(p)}$, etc., in (42) and (43). By (42) and (45), letting $p = p_0 - 1$ in (45), if $m \leq n - p_0$,

$$S_m^{(p_0)} = S_m^{(p_0-1)} + r_{m+p_0}^* S_{m+1}^{(p_0-1)}$$

$$= z^{p_0-1} \prod_{k=m+p_0+1}^{n} (z - r_k^*)$$

$$+ r_{m+p_0}^* z^{p_0-1} \prod_{k=m+p_0+1}^{n} (z - r_k^*)$$

$$= [z^{p_0-1} \prod_{k=m+p_0+1}^{n} (z - r_k^*)] [z - r_{m+p_0}^* + r_{m+p_0}^*],$$

or

47) $$S_m^{(p_0)} = z^{p_0} \prod_{k=m+p_0+1}^{n} (z - r_k^*)$$

By (43) and (46), letting $p = p_0 - 1$ in (46), if $m > n - p_0$ (This inequality is equivalent to $m \geq n - (p_0 - 1)$.),

48) $$S_m^{(p_0)} = S_m^{(p_0-1)} = z^{n-m}.$$

If $m = n - p_o$, then by (47): $S_m^{(p_o)} = z^{p_o}$, which is equivalent to (48) in the case $m = n - p_o$. So equation (48) is true when $m \geq n - p_o$. Equations (47) and (48), and the cases for which they are established, prove that the truth of the lemma in the case $p = p_o - 1$ implies its truth in the $p = p_o$. This completes the proof of the lemma by induction.

Lemma 3: The matrix $A^{*(n-1)}$ has one's everywhere in the non-principal diagonal and zeros everywhere else, i.e.,

$$a_{k,m}^{*(n-1)} = \begin{cases} 1 \text{ if } k = n - m. \\ 0 \text{ if } k \neq n - m. \end{cases}$$

Proof: By (46), if $m = 1, 2, \ldots, n$,

$$S_m^{(n-1)} = z^{n-m}.$$

But by (44),

$$S_m^{(n-1)} = \sum_{k=0}^{n-1} a_{k,m}^{*(n-1)} z^k.$$

In order to reconcile the above equations $a_{k,m}^{*(n-1)}$ must be unity when $k = n-m$, but otherwise zero. The lemma is proved.

Lemma 4:

49) $(f)_{\underline{t-1}} - (f)_t \geq$

$$[(f)_{\underline{t-1}}\ M_2^{2n-2}] \div 2^{2n-3}\ n^2\ M_3^2\ (M_3 + 1)^{4n-6},$$

for at least one choice of m_0 and whether ΔP_{m_0} is real or imaginary, where:

50) $$M_1 = \max |a_k|,\ k = (0,\ 1,\ \ldots,\ n-1);$$

51) $$M_2 = \min \{2,\ |(r_j)_{\underline{t-1}} - (r_k)_{\underline{t-1}}|\},$$

$(r_j)_{\underline{t-1}} \neq (r_k)_{\underline{t-1}},\ 1 \leq j \leq n,\ 1 \leq k \leq n;$

52) $$M_3 = \max \{1,\ (n-1)\ (\sqrt{(f)_{t-1}} + M_1)\}.$$

Further calculations depend on the fact that the M's are positive.

Proof: Due to (4):

$$\text{Max}\ |(a_k)_{\underline{t-1}} - a_k|^2 \geq (f)_{\underline{t-1}} / n;$$

53) $$\text{Max}\ |(a_k)_{\underline{t-1}} - a_k| \geq \sqrt{(f)_{\underline{t-1}} / n}.$$

Due to (3):

$$\sum_{k=0}^{n} a_k^*\ (r_m^*)^k = 0,\ \text{if}\ m = 1,\ 2,\ \ldots,\ n.$$

Since $a_n^* = 1$, if $r_m^* \neq 0$:

$$1 = \sum_{k=0}^{n-1} (-a_k^*)\ (r_m^*)^{k-n} \leq \sum_{k=0}^{n-1} |a_k^*| \div |(r_m^*)^{n-k}|.$$

Then, if $|(r_m^*)| \geq 1$:

$$1 \leq (n - 1)\ \max\ |a_k^*| \div |r_m^*|;$$

$$|r_m^*| \leq (n - 1)\ \max\ |a_k^*|.$$

Hence, in particular,

54) $$|(r_m^*)_{t-1}| \leq \max \left\{1, (n-1) \; |(a_k)_{t-1}|\right\}.$$

But due to (4):

$$\max | a_k^* - a_k|^2 \leq f;$$

55) $$\max | a_k^* - a_k| \leq \sqrt{f};$$

$$\max \left\{|a_k^* | - |a_k|\right\} \leq \sqrt{f};$$

$$\max | a_k^* | - \max |a_k| \leq \sqrt{f};$$

$$\max | a_k^* | \leq \sqrt{f} + \max | a_k|;$$

In particular:

$$\max |(a_k)_{t-1}| \leq \sqrt{(f)_{t-1}} + M_1.$$

Hence (54) becomes, with the aid of (52), and the fact that each $(r_m)_{\underline{t-1}}$ will be defined so $\max |(r_m)_{\underline{t-1}}| \leq \max |(r_m)_{t-1}|$, $(m = 1, 2, \ldots, n)$:

56) $\max |(r_m)_{\underline{t-1}}| \leq \max |(r_m)_{t-1}| \leq M_3$, $(m = 1, 2, \ldots, n)$.

Define the matrices:

57) $$A_k = (a_0, a_1, \ldots, a_{n-1});$$

58) $$F_m = \left(\frac{\partial \tilde{f}}{\partial P_1}, \frac{\partial \tilde{f}}{\partial P_2}, \ldots \frac{\partial \tilde{f}}{\partial P_n}\right),$$

59) $$\text{where } \frac{\partial \tilde{f}}{\partial P_m} = \frac{\partial f}{\partial \operatorname{Re} P_m} + i \frac{\partial f}{\partial \operatorname{Im} P_m}.$$

Due to (16) and (17):

60) $$2\,(A_k^* - A_k)\,\overline{A}^*_{k,m} = F_m,$$

61) $$\text{where } A_k^* = (a_o^*, a_1^*, \ldots, a_{n-1}^*).$$

Hence:

62) $$2\ [A_k^* - A_k]\,\overline{A}_{k,m}^{*(n-1)} = F_{m,}^{*(n-1)},$$

where $\overline{A}_{k,m}^{*(n-1)}$ and $F_m^{*(n-1)}$ are the results after (n-1) elementary column transformations of the first type followed by (n-1) elementary column transformations of the second type on $\overline{A}^*_{k,m}$ and F^*_m, respectively. $\overline{A}_{k,m}^{*(n-1)}$ is the complex conjugate of $A_{k,m}^{*(n-1)}$ and hence, by Lemma 3, consists of one's in the non-principal diagonal and zeros everywhere else. Hence if the elements of $2\,[A^*_k - A_k]$ are arranged in reverse order, $F_m^{*(n-1)}$ results. Hence by (53), the greatest magnitude of any element of $F_m^{*(n-1)}$ is at least $2\sqrt{f \div n}$. Hence,

63) $$\max\,(F_m^{(n-1)})_{\underline{t-1}} \geq 2\sqrt{(f)_{\underline{t-1}} \div n}.$$

Let

64) $$\varepsilon = \max\,\left(\left|\frac{\partial\,\tilde{f}}{\partial\,P_m}\right|\right)_{\underline{t-1}},\ m = 1, 2, \ldots, n.$$

Then an upper bound for the magnitude of the elements of $(F_{m;o})_{\underline{t-1}} = (F_m)_{\underline{t-1}}$ is ε. A corresponding bound for $(F_{m;p_0-1})_{\underline{t-1}}$ is $\varepsilon\,(\frac{2}{M_2})^p$. To show this, we remark that the statement is true when $p = 0$. Suppose it is true for $p = p_0 - 1$, i.e., that a corresponding bound for $(F_{m;p_0-1})_{\underline{t-1}}$ is $\varepsilon\,(\frac{2}{M_2})^{p_0-1}$. The p_0-th transformation of the first type consists of altering some elements by subtracting other elements from them, then dividing the results by some $(r_j)_{\underline{t-1}} - (r_k)_{\underline{t-1}}$, $r_j \neq r_k$, in magnitude at least M_2, see (51). Since $M_2 \geq 2$, a corresponding bound for the resulting $(F_{m;p_0})_{\underline{t-1}}$ is $\varepsilon\,(\frac{2}{M_2})^{p_0-1}\,(\frac{2}{M_2}) = \varepsilon\,(\frac{2}{M_2})^{p_0}$. This proves the earlier statement, and that a corresponding bound for $(F_{m;n-1})_{\underline{t-1}} = (F_m)^{(0)}_{\underline{t-1}}$ is $\varepsilon\,(\frac{2}{M_2})^{n-1}$. A corresponding bound for $(F_m)^{(0)}_{\underline{t-1}}$ is $\varepsilon(\frac{2}{M_2})^{n-1}\,(M_3 + 1)^p$. To show this statement, we observe that it is true for $p = 0$ and suppose it is true for $p = p_0 - 1$. Then the corresponding bound for $(F_m)^{(p_0-1)}_{\underline{t-1}}$ is $\varepsilon\,(\frac{2}{M_2})^{n-1}\,(M_3 + 1)^{p_0-1}$. The p_0-th transformation of the second type consists of altering some elements by adding to each of them the product of some $(r_m)_{\underline{t-1}}$ and some other element. Since $(r_m)_{\underline{t-1}} \leq M_3$, the corresponding bound for the resulting $(F_m)^{(p_o)}_{\underline{t-1}}$ is $\varepsilon\, (\frac{2}{M_2})^{n-1}\, (M_3 + 1)^{p_o - 1}$ +

$M_3\, \varepsilon\, (\frac{2}{M_2})^{n-1}\, (M_3 + 1)^{p_o - 1}$, which is $\varepsilon\, (\frac{2}{M_2})^{n-1}\, (M_3 + 1)^{p_o}$.

Said statement is true for $p = p_o$, and is therefore true, by induction. Hence a corresponding bound for $(F_m)^{(n-1)}_{\underline{t-1}}$ is $\varepsilon\, [\frac{2}{M_2}(M_3 + 1)]^{n-1}$. Then, by (63):

$$\varepsilon\, [\frac{2}{M_2}\, (M_3 + 1)]^{n-1} \geq 2\sqrt{(f)_{\underline{t-1}} \div n};$$

$$\varepsilon = \max\, (|\frac{\partial \tilde{f}}{\partial\, P_m}|)_{\underline{t-1}} \geq 2\sqrt{\frac{(f)_{\underline{t-1}}}{n}}\, [\, \frac{M_2}{2(M_3+1)}\,]^{n-1};$$

$$\max\, (|\frac{\partial \tilde{f}}{\partial\, P_m}|^2)_{\underline{t-1}} \geq 4\, \frac{(f)_{\underline{t-1}}}{n}\, [\, \frac{M_2}{2(M_3+1)}\,]^{2n-2};$$

and, due to (59):

$$\text{65)} \quad \max\left\{(\frac{\partial\, f}{\partial\, \mathrm{Re}\, P_m})^2_{\underline{t-1}},\ (\frac{\partial\, f}{\partial\, \mathrm{Im}\, P_m})^2_{\underline{t-1}}\right\} \geq 2\, \frac{(f)_{\underline{t-1}}}{n}\, [\, \frac{M_2}{2(M_3+1)}\,]^{2n-2}.$$

Note: The importance of (65) lies in the fact that we may choose m_o such that

$$(\frac{\partial\, f}{\partial\, \mathrm{Re}\, P_{m_o}})^2_{\underline{t-1}} \text{ or } (\frac{\partial\, f}{\partial\, \mathrm{Im}\, P_{m_o}})^2_{\underline{t-1}}$$

equals or exceeds the right side of the inequality.

Let $\tilde{r}_k$ be chosen arbitrarily in the closed interval: $[(r_k)_{t-1}, (r_k)_{\underline{t-1}}]$, $k = 1, 2, \ldots, n$. By (56), $\max |\tilde{r}_k| \leq M_3$. Hence the statement, "The magnitudes of the coefficients of the polynomial expansion of the product of p factors $[x - \tilde{r}_k]$, are bounded by $M_3(M_3 + 1)^{p-1}$," is true when $p = 1$. Suppose it is true when $p = p_0$. If the polynomial

$$\sum_{k=0}^{p_0} b_k z^k = \prod_{k=1}^{p_0} [z - \tilde{r}_{m_k}],$$

then

$$|b_k| \leq M_3(M_3 + 1)^{p_0 - 1}, \; k = 0, 1, \ldots, p_0.$$

If

$$\sum_{k=0}^{p_0+1} c_k z^k = \prod_{k=1}^{p_0+1} [z - r_{m_k}],$$

then

$$c_k = b_{k-1} + b_k (r_{p_0+1})_{\underline{t-1}}, \text{ where } b_{p_0+1} = 0.$$

Since

$$|(r_{p_0+1})_{\underline{t-1}}| \leq M_3,$$

$$|c_k| \leq M_3(M_3 + 1)^{p_0 - 1}\,(M_3 + 1) = M_3(M_3 + 1)^{p_0}$$

where $k = 0, 1, \ldots, p_0 - 1$. Hence said statement is true when $p = p_0 + 1$, and is always true. Since

$\tilde{a}_{k,m}$ is a coefficient of a polynomial equal to the product of not more than $(n - 1)$ factors $(z - \tilde{r}_{m_k})$, where $\tilde{a}_{m,k} = a^*_{m,k}$ when $r^*_k = \tilde{r}_k$, $k = 1, 2, \ldots, n$,

66) $$\max |\tilde{a}_{k,m}| \leq M_3(M_3 + 1)^{n-2}$$

Noting that if we choose $\tilde{r}_k = (r_k)_{\underline{t-1}}$, $k = 1, 2, \ldots, n$, then $\tilde{a}_{k,m} = (a_{k,m})_{\underline{t-1}}$, and due to (16) and (66):

67) $$\max \left[\frac{\partial^2 f}{\partial (\mathrm{Re}\ P_m)^2}\right]_{\underline{t-1}} = \max \left[\frac{\partial^2 f}{\partial (\mathrm{Im}\ P_m)^2}\right]_{\underline{t-1}} \leq 2n\, M_3^2\, (M_3 + 1)^{2n-4}$$

Due to (65), (67), and Theorem 2; the lemma is proved.

The decrease of f from $(f)_{\underline{t-1}}$ to $(f)_t$ can be made sufficiently large by making M_2 sufficiently large. This is why, when each $(r_k)_{t-1}$ is "rounded off" to $(r_k)_{\underline{t-1}}$, the possibility of a too small but non-zero $|(r_{k_1})_{\underline{t-1}} - (r_{k_2})_{\underline{t-1}}|$ must be eliminated.

We will define $M_4 \geq$

68) $$2 \max \left\{|\mathrm{Re}\ (r_k)_{\underline{t-1}} - (r_k)_{t-1}|,\ |\mathrm{Im}\ (r_k)_{\underline{t-1}} - (r_k)_{t-1}|\right\},\ k = 1, 2, \ldots, n.$$

We will define M_4 more exactly later in this chapter.

Lemma 5:

$$(f)_{\underline{t-1}} - (f)_{t-1} \leq n^2 M_3 M_4 (M_3 + 1)^{n-2}\sqrt{2(f)_{\underline{t-1}}}.$$

Proof: If $(f)_{\underline{t-1}} - (f)_{t-1} \leq 0$, lemma is proved, so only the case, $(f)_{\underline{t-1}} - (f)_{t-1} > 0$ is considered. For purpose of this proof, let

$$f(\theta) = f[r_1^*(\theta), r_2^*(\theta), \ldots, r_n^*(\theta)]$$

where

$$r_k^*(\theta) = (r_k)_{t-1} + \theta[(r_k)_{\underline{t-1}} - (r_k)_{t-1}], \; k = 1, 2, \ldots, n.$$

Then by (4),

$$f(0) = (f)_{t-1} \text{ and } f(1) = f_{\underline{t-1}},$$

so, in the case being considered,

$$f(0) < f(1). \tag{69}$$

Since, due to (4), f is continuous in the $(a_1^*, a_2^*, \ldots, a_n^*)$ hyperplane, and hence, by (3), in the $(r_1^*, r_2^*, \ldots, r_n^*)$ hyperplane, and likewise$\left(\frac{\partial f}{\partial r_k^*}\right)$, $k = 1, 2, \ldots, n$, exist:

$$\frac{df(\theta)}{d\theta} = \sum_{k=1}^{n} \left[\frac{\partial f(\theta)}{\partial \operatorname{Re}[r_k^*(\theta)]} \frac{d\operatorname{Re}[r_k^*(\theta)]}{d\theta} + \frac{\partial f(\theta)}{\partial \operatorname{Im}[r_k^*(\theta)]} \frac{d\operatorname{Im}[r_k^*(\theta)]}{d\theta}\right].$$

The existence of $\frac{df(\theta)}{d\theta}$ for $0 \leq \theta \leq 1$ is also assured.

Since

$$\frac{d\, \text{Re}\, [r_k^*\,(\theta)]}{d\theta} = \text{Re}\,(r_k)_{\underline{t-1}} - \text{Re}\,(r_k)_{t-1},$$

$$\frac{d\text{Im}\,[r_k^*(\theta)]}{d\theta} = \text{Im}\,(r_k)_{\underline{t-1}} - \text{Im}\,(r_k)_{t-1},$$

and each of these quantities is, by (68), not more than $1/2\ M_4$,

$$70) \quad \frac{d\ f\ (\theta)}{d\ \theta} \leq \frac{M_4}{2} \sum_{k=1}^{n} \left[\frac{\partial\ f\ (\theta)}{\partial\ \text{Re}\ [r_k^*\ (\theta)]} + \frac{\partial\ f\ (\theta)}{\partial\ \text{Im}\ [r_k^*\ (\theta)]}\right].$$

By (6), for each $k (k = 1, 2, \ldots, n)$ there exists an m such that:

$$71) \quad \frac{\partial\ f}{\partial\ (\text{Re}\ r_k^*)} = - \frac{\partial\ f}{\partial\ (\text{Re}\ P_m)} \text{ and } \frac{\partial\ f}{\partial(\text{Im}\ r_k^*)} = - \frac{\partial\ f}{\partial\ (\text{Im}\ P_m)}.$$

Due to (59):

$$72) \quad \left\{\left|\frac{\partial\ f}{\partial\ (\text{Re}\ P_m)}\right| + \left|\frac{\partial\ f}{\partial\ (\text{Im}\ P_m)}\right|\right\} \leq \sqrt{2}\ \left|\frac{\partial^{\sim} f}{\partial\ P_m}\right|.$$

For $k = 1, 2, \ldots, n$, we let $P_k(\theta)$ denote the values of P_k when $r_k^* = r_k^*(\theta)$. Due to (70), (71), and (72),

$$73) \quad \frac{d\ f(\theta)}{d\ \theta} \leq \frac{n\ M_4}{\sqrt{2}} \max \left|\frac{\partial^{\sim} f\ (\theta)}{\partial\ P_m\ (\theta)}\right|.$$

When $0 \leq \theta \leq 1$, $r_k^*(\theta)$ is in the closed interval $[(r_k)_{t-1}, (r_k)_{\underline{t-1}}]$. Under this condition, we have, by applying (55), (66), and (58) to (60):

$$74) \quad \max \left|\frac{\partial\ \tilde{f}(\theta)}{\partial\ P_m(\theta)}\right| \leq 2n\ M_3(M_3 + 1)^{n-2}\sqrt[2]{f}.$$

Due to (73) and (74):

$$\frac{df(\theta)}{d\theta} \leq \sqrt{2}n^2 M_3 M_4 (M_3 + 1)^{n-2}\sqrt{f}.$$

Let

$$L = \sqrt{2}n^2 M_3 M_4 (M_3 + 1)^{n-2} \text{ and } g(\theta) = \sqrt{f(\theta)}.$$

Since $f(\theta) \geq 0$:

$$\frac{d\,[g\,(\theta)]^2}{d\,\theta} = \frac{d\,f\,(\theta)}{d\,(\theta)} \leq L\sqrt{f} = L\,g\,(\theta);$$

$$2\,g\,(\theta)\,\frac{d\,g\,(\theta)}{d\,\theta} \leq L\,g\,(\theta);$$

$$\frac{d\,g\,(\theta)}{d\,\theta} \leq 1/2\ L.$$

By the law of the mean:

$$g(1) - g(0) \leq 1/2\ L.$$

Multiplying by $[g(1) + g(0)]$:

$$[g(1)]^2 - [g(0)]^2 \leq 1/2\ L\ [g(1) + g(0)].$$

Since, due to (69), in the case being considered, $g(1) > g(0)$:

$$f(1) - f(0) < Lg(1) = \sqrt{2}n^2\, M_3 M_4 (M_3 + 1)^{n-2}\sqrt{f(1)}.$$

This implies the lemma.

Lemma 6 : If:

75) $$M_2 \geq 2^{\frac{4n-3}{4n-4}}\, n^{\frac{2}{n-1}}\, M_3^{\frac{3}{2n-2}}\, (f)_{t-1}^{\frac{-1}{4n-4}}\, (M_3 + 1)^{\frac{5n-8}{2n-2}}\, M_4^{\frac{1}{2n-2}},$$

then for at least one choice of m_0 and whether ΔP_{m_0} is real or imaginary,

76) $$(f)_{t-1} - (f)_t \geq [(f)_{t-1} M_2^{2n-2}]$$
$$2^{2n-2} n^2 M_3^2 (M_3 + 1)^{4n-6}$$

Proof: Inequality (76) is true for said choice if

$$(f)_{t-1} - (f)_t \geq 1/2\ [(f)_{\underline{t-1}} - (f)_t],$$

by Lemma 4.

This is true if and only if

$$(f)_{\underline{t-1}} - (f)_{t-1} \leq 1/2\ [(f)_{\underline{t-1}} - (f)_t].$$

By Lemma 4 and 5, this is true for said choice if:

$$n^2 M_3 M_4 (M_3 + 1)^{n-2} \sqrt{2(f)_{\underline{t-1}}} \leq$$
$$1/2\ [(f)_{\underline{t-1}} M_2^{2n-2}] \quad 2^{2n-3} n^2 M_3^2 (M_3 + 1)^{4n-6}.$$

This is true if:

77) $$2^{2n-\frac{3}{2}} n^4 M_3^3 M_4 (M_3 + 1)^{5n-8} \leq \sqrt{(f)_{\underline{t-1}}}\, M_2^{2n-2}.$$

This is true if (75) is true and $(f)_{t-1} \leq (f)_{\underline{t-1}}$, so the lemma is true in this case. We note that:

$$(f)_{\underline{t-1}} \{1 - [M_2^{2n-2} \quad 2^{2n-3} n^2 M_3^2 (M_3 + 1)^{4n-6}]\}$$
$$\geq (f)_t \geq 0,$$

by Lemma 4; and since $(f)_t \geq 0$, the first member in above inequality is positive, and since $(f)_{t-1}$ is positive, the quantity in braces is positive; so, if $(f)_{t-1} > (f)_{\underline{t-1}}$,

$(f)_{t-1} \{1 - [M_2^{2n-2} \quad 2^{2n-3} n^2 M_3^2 (M_3 + 1)^{4n-6}]\} > (f)_t$.

This implies (76). Q. E. D.

Note: We choose m_0 and whether ΔP_{m_0} should be real or imaginary so that $(f)_{t-1} - (f)_t$ is maximized.

We define $M_{4,n-1}$ as the greatest quantity not more than 2 which satisfies (75) if M_2 is replaced by 2 and M_4 is replaced by $M_{4,n-1}$. If j = 1, 2, ... n - 1, we define $M_{4,j-1}$ as the largest number such that $(M_{4,j} - M_{4,j-1})$ is an integral multiple of $M_{4,j-1}$ satisfying (75), when $(M_{4,j} - M_{4,j-1})$ and $M_{4,\ j-1}$ are substituted for M_2 and M_4, respectively.

Lemma 7: If (75) cannot be satisfied, when $M_4 = M_{4,j}$ (for any j = 0, 1, ..., n-2) and $M_2 \geq M_{4,1} - M_{4,0}$, then, given j = 0, 1, ..., n - 1, it is possible to divide the complex plane into squares of side $M_{4,j}$, so that all $(r_k)_{t-1}$, k = 1, 2, ..., n, are contained by at most n - j such squares, (called, in this case, containing squares) where a square consists of its interior, its lower and left sides, and its lower left corner.

Proof by induction: Divide the plane into squares of side $M_{4,0}$, so that the axes from the boundaries of of squares. Since there are n $(r_k)_{t-1}$'s, there are at most n containing squares of side $M_{4,0}$. Hence the lemma is true when $j = 0$.

Suppose the plane is divided into squares of side M_{4,j_0} (where $0 \leq j_0 < n - 1$), not more than $n - j_0$ of which are containing squares. We define the hub of a square as a point z, so that

$$|z| \leq \max |(r_k)_{t-1}|, \; k = 1, 2, \ldots, n,$$

and, for each $(r_k)_{t-1}$ in the square,

$$|\mathrm{Re}\, z - \mathrm{Re}\, (r_k)_{t-1}| \leq 1/2\, M_{4,j_0} \text{ and}$$

$$|\mathrm{Im}\, z - \mathrm{Im}\, (r_k)_{t-1}| \leq 1/2\, M_{4,j_0}.$$

We choose each $(r_k)_{\underline{t-1}}$ at the hub of the square of side $M_{4,0}$, containing $(r_k)_{t-1}$. Then, in accordance with (68), we let $M_4 = M_{4,j_0}$. Hence the hypothesis of the lemma implies that either (75) cannot be satisfied, or that $M_2 < M_{4,1} - M_{4,0}$. In either case,

$$M_2 < M_{4,j_0+1} - M_{4,j_0} < M_{4,n-1} \leq 2,$$

due to the definition of $M_{4,j-1}$. (In applying this definition to the former case, we let $j = j_0 + 1$; in the latter case, we note that $M_{4,j_o} \leq M_{4,j_o+1} - M_{4,j_o}$, where $1 \leq j_o \leq n - 2$). Consequently, we can state that M_{4,j_o+1} is at least the second integral multiple of M_{4,j_o}, greater than M_2. Also, due to (51),

$$M_2 = \min \{|(r_j)_{\underline{t-1}} - (r_k)_{\underline{t-1}}|\},$$
$$(r_j)_{\underline{t-1}} \neq (r_k)_{\underline{t-1}},\ 1 \leq j \leq n,\ 1 \leq k \leq n.$$

Let A and B be distinct squares of side M_{2,j_o} containing $(r_a)_{t-1}$ and $(r_a)_{\underline{t-1}}$, and $(r_b)_{t-1}$ and $(r_b)_{\underline{t-1}}$, respectively, where $|(r_a)_{\underline{t-1}} - (r_b)_{\underline{t-1}}| = M_2$. Then, because M_{4,j_o+1} is at least the second integral multiple of M_{4,j_o}, greater than M_2, the plane can be (and is) divided into squares of side M_{4,j_o+1}, each consisting only of entire squares of side M_{4,j_o}, such that A and B are in the same square of side M_{4,j_o+1}. Hence $(r_a)_{t-1}$ and $(r_b)_{t-1}$, contained by different squares of side M_{4,j_o}, are contained in the same square of side M_{4,j_o+1}. Since there are at most $n - j_o$ containing squares of side M_{4,j_o}, there are at most $n - j_o - 1$ containing squares of side M_{4,j_o+1}. Hence the lemma is true when

$j = j_0 + 1$ if true when $j = j_0$ and $j_0 < n - 1$. Hence the lemma is true.

Lemma 8: The inequality (76) can be satisfied for at least one of the choices of M_2 and M_4:

78) $M_2 \geq M_{4,1} - M_{4,0}$, and

$$M_4 = M_{4,j} \ (j = 0, 1, \ldots, n - 1).$$

Proof: Due to the note at the end of the proof of Lemma 6, (76) is true if (75) is true. Hence: if it is proved that if (75) cannot be satisfied when $M_2 \geq M_{4,1} - M_{4,0}$ and $M_4 = M_{4,j}$ (for some $j = 0, 1, \ldots, n - 2$), (75) can be satisfied by $M_4 = M_{4,n-1}$ and some M_2: $M_2 \geq M_{4,1} - M_{4,0}$; then Lemma 8 follows. Lemma 7 further reduces the proof of Lemma 8 to proving that the conclusion of Lemma 7 implies that (75) can be satisfied by $M_4 = M_{4,n-1}$ and some M_2: $M_2 \geq M_{4,1} - M_{4,0}$.

By the conclusion of Lemma 7, one square of side $M_{4,n-1}$ contains all $(r_k)_{t-1}$, $k = 1, 2, \ldots, n$. We then define each $(r_k)_{\underline{t-1}}$ as the hub of this square. Since each $(r_k)_{t-1}$ is within $1/2\ M_{4,n-1}$ of this hub in each of its real and imaginary parts, the designation $M_4 = M_{4,n-1}$ is permitted. Since all $(r_k)_{\underline{t-1}}$ are equal, and due to (51), $M_2 = 2$: By definition of $M_{4,n-1}$ and

$M_{4,j-1}$, $M_2 = 2 \geq M_{4,n-1} \geq M_{4,1} > M_{4,1} - M_{4,0}$, and (75) is satisfied. Q. E. D.

Note: If other attempts at finding M_2 and M_4 fail, divide the plane into squares of side $M_{4,0}$, $M_{4,1}$, $\cdots$ (the squares of side $M_{4,0}$ are constructed arbitrarily), as in the proof of Lemma 7, until an $M_{4,j}$ is found such that if $M_4 = M_{4,j}$, and all $(r_k)_{\underline{t-1}}$ are computed after the manner of said proof, M_2 and M_4 satisfy (75) and (78). The method of said proof may be used to compute $(r_k)_{\underline{t-1}}$ even if M_4, the sidelength of the squares, is chosen as some number, other than $M_{4,j}$, for some integer j, as long as (75) is satisfied and $M_2 \geq M_{4,1} - M_{4,0}$. So we choose $(r_k)_{\underline{t-1}}$ $(k=1,2,\cdots, n)$ and m_o, and let ΔP_{m_o} be real or imaginary, in such a way as to satisfy (76) and (78). Define

79)
$$k_1 = 2^{\frac{4n-3}{4n-4}} n^{\frac{2}{n-1}} M_3^{\frac{3}{2n-2}} (f)_{t-1}^{-\frac{1}{4n-4}} (M_3+1)^{\frac{5n-8}{2n-2}} .$$

Note that $k_1 > 0$.

Lemma 9:
$$2 k_1^{-1} M_{4,j-1}^{1 - \frac{1}{2n-2}} < \frac{1}{4} , \quad j = 0,1, \cdots, n.$$

Proof: By definition, $M_{4,n-1}$ is the greatest number not more than 2 satisfying:

$$k_1 \, M_{4,n-1}^{\frac{1}{2n-2}} \leq 2,$$

which is equivalent to

$$M_{4,n-1} \leq (2\, k_1^{-1})^{2n-2} \,. \tag{80}$$

$M_{4,j-1}$ $(j = 0,1, \cdots,n)$, may replace $M_{4,n-1}$ in (80), because by definition it is no larger. Then:

$$M_{4,j-1}^{\;1 - \frac{1}{2n-2}} \leq (2\, k_1^{-1})^{2n-3} \;;$$

$$2\, k^{-1}\, M_{4,j-1}^{\;1 - \frac{1}{2n-2}} \leq (2\, k_1^{-1})^{2n-2} \,.$$

Since $n \geq 2$ (Chapter I), $2n-2 \geq 2$, and it need only be proved that $2\, k_1^{-1} < \frac{1}{2}$.

$$2\, k_1^{-1} = 2^{-\frac{1}{4n-4}}\, n^{-\frac{2}{n-1}}\, M_3^{-\frac{3}{2n-2}}\, (f)_{\underline{t-1}}^{\frac{1}{4n-4}} (M_3+1)^{\frac{8-5n}{2n-2}} \tag{81}$$

Note that all factors in (81) are positive if $f \neq 0$.

Because, by (51), $M_3 \geq \max\{1, \sqrt{(f)_{\underline{t-1}}}\,\}$:

$$(f)_{\underline{t-1}}^{\frac{1}{2}}\; M_3^{-1} \leq 1 \quad ; \qquad\qquad M_3^{-2} \leq 1 \quad ;$$

$$(f)_{\underline{t-1}}^{\frac{1}{2}}\; M_3^{-3} \leq 1 \quad ; \qquad\qquad (f)_{\underline{t-1}}^{\frac{1}{4n-4}}\; M_3^{-\frac{3}{2n-2}} \leq 1 \,. \tag{82}$$

Since $\frac{8-5n}{2n-2} \leq -1$, and $M_3 \geq 1$:

83) $$(M_3 + 1)^{\frac{8-5n}{2n-2}} \leq \frac{1}{2} .$$

Since $\frac{1}{4n-4}$ and $\frac{2}{n-1}$ are positive:

84) $$2^{-\frac{1}{4n-4}} < 1 ; \qquad n^{-\frac{2}{n-1}} < 1 .$$

By (81)-(84), $2\, k_1^{-1} < \frac{1}{2}$, and the lemma is proved.

Lemma 10: If $j = 1, 2, \cdots, n-1$,

$$M_{4,j-1} > (\tfrac{4}{5}\, k_1^{-1}\ M_{4,j})^{2n-2} .$$

Proof: By definition, $M_{4,j-1}$ is the largest number such that $M_{4,j}$ is an integral multiple of $M_{4,j-1}$ and

85) $$M_{4,j} - M_{4,j-1} \geq k_1\, M_{4,j-1}^{\frac{1}{2n-2}} ,$$

which is equivalent to

$$M_{4,j} \,/\, M_{4,j-1} \geq k_1\, M_{4,j-1}^{\frac{1}{2n-2} - 1} + 1 .$$

So $M_{4,j} \,/\, M_{4,j-1}$ is the least integer such that

$$\frac{M_{4,j}}{M_{4,j-1}} \geq k_1\, M_{4,j}^{\frac{1}{2n-2} - 1} \left(\frac{M_{4,j}}{M_{4,j-1}} \right)^{1 - \frac{1}{2n-2}} + 1 .$$

Note: Since $M_{4,j} > M_{4,j-1}$, by (85), $M_{4,j} / M_{4,j-1} > 1$.

Therefore

$$\frac{M_{4,j}}{M_{4,j-1}} - 1 < k_1 \, M_{4,j}^{\frac{1}{2n-2} - 1} \left(\frac{M_{4,j}}{M_{4,j-1}} - 1\right)^{1 - \frac{1}{2n-2}} + 1$$

$$< k_1 \, M_{4,j}^{\frac{1}{2n-2} - 1} \left(\frac{M_{4,j}}{M_{4,j-1}} \right)^{1 - \frac{1}{2n-2}} + 1$$

$$= k_1 \, M_{4,j}^{\frac{1}{2n-2} - 1} + 1 \; .$$

Hence

$$\frac{M_{4,j}}{M_{4,j-1}} < k_1 \, M_{4,j-1}^{\frac{1}{2n-2} - 1} + 2 \; ;$$

$$M_{4,j} < k_1 \, M_{4,j-1}^{\frac{1}{2n-2}} + 2 \, M_{4,j-1} \; ;$$

$$M_{4,j} < k_1 \, M_{4,j-1}^{\frac{1}{2n-2}} \left(1 + 2 \, k_1^{-1} \, M_{4,j-1}^{1 - \frac{1}{2n-2}}\right).$$

By Lemma 9:

$$M_{4,j} < \frac{5}{4} \, k_1 \, M_{4,j-1}^{\frac{1}{2n-2}} \; .$$

From this follows Lemma 10.

Lemma 11: If $j = 0, 1, \ldots, n-1,$

$$M_{4,n-j-1} \geq (2k_1^{-1})^{(2n-2)^{j+1}} \left(\frac{4}{5} \, k_1^{-1}\right)^{\sum_{k=1}^{j} (2n-2)^k}$$

Note: The sum of the elements of a null set is defined as zero.

Proof by induction: Since $2k_1^{-1} < \frac{1}{2}$, and $2n-2 \geq 2$, $(2k_1^{-1})^{2n-2} < \frac{1}{4}$. Hence, by (80), $M_{4,n-1} < \frac{1}{4}$, so $M_{4,n-1}$ is the greatest number satisfying (80).

Hence:

$$M_{4,n-1} = (2k_1^{-1})^{2n-2},$$

and the lemma is true when $j = 0$.

Suppose the lemma is true when $j = j_0 < n-1$.

Then, by Lemma 10:

$$M_{4,n-j_0-2} > \left(\frac{4}{5} k_1^{-1} M_{4,n-j_0-1} \right)^{2n-2}$$

$$\geq \left[\frac{4}{5} k_1^{-1} (2k_1^{-1})^{(2n-2)^{j_0+1}} \left(\frac{4}{5} k_1^{-1} \right)^{\sum_{k=1}^{j_c} (2n-2)^k} \right]^{2n-2}$$

$$= (2k_1^{-1})^{(2n-2)^{j_0+2}} \left(\frac{4}{5} k_1^{-1} \right)^{\sum_{k=1}^{j_0+1} (2n-2)^k}.$$

Hence the lemma is true when $j = j_0 + 1$ if true when $j = j_0 < n - 1$. Q.E.D.

Let $M_{\underline{s}}$ be defined as M_s is defined in (52), except that $\sqrt{(f)}_{t-1}$ be replaced by $\sqrt{(f)}_0$. As (76) is established by Lemma 8 and the note following its proof, so $(f)_{t-1} - (f)_t > 0$ if $t \geq 1$, then $(f)_0 \geq (f)_{t-1}$, so $M_{\underline{s}} \geq M_s$.

Define:

86) $k_2 = 2^{\frac{4n-3}{4n-4}} n^{\frac{2}{n-1}} M_{\underline{3}}^{\frac{3}{2n-2}} (M_{\underline{3}} + 1)^{\frac{5n-8}{2n-2}}$;

87) $k_3 = k_2^{1-\sum_{k=0}^{n-1}(2n-2)^k} \, 2^{(2n-2)^{n-1}} \left(\frac{4}{5}\right)^{\sum_{k=0}^{n-2}(2n-2)^k}$;

88) $k_4 = \frac{1}{2}\left[\sum_{k=-1}^{n-2}(2n-2)^k\right] - \frac{1}{4n-4}$;

89) $k_5 = k_4(2n-2) + 1$;

90) $k_6 = k_3^{2n-2} \div 2^{2n-2} n^2 M_{\underline{3}}^2 (M_{\underline{3}} + 1)^{4n-6}$.

Note that all k's are positive.

Lemma 12:

$(f)_{t-1} - (f)_t \geq k_6 (f)_{t-1}^{k_5}$; $t = 1,2,3,\ldots$.

Proof: Due to Lemma 11,

$$M_{4,0} \geq (2k_1^{-1})^{(2n-2)^n} \left(\frac{4}{5} k_1^{-1}\right)^{\sum_{k=1}^{n-1}(2n-2)^k} .$$

Due to (78) and (85),

$$M_2 \geq M_{4,1} - M_{4,0} \geq k_1 (2k_1^{-1})^{(2n-2)^{n-1}} (\tfrac{4}{5} k_1^{-1})^{\sum_{k=0}^{n-2} (2n-2)^k}$$

$$91) \qquad = k_1^{\,1-\sum_{k=0}^{n-1} (2n-2)^k} \, 2^{(2n-2)^{n-1}} (\tfrac{4}{5})^{\sum_{k=0}^{n-2} (2n-2)^k} .$$

Since $\underline{M_3} \geq M_3$, and due to (79), (86)-(88):

$$k_1 \leq k_2 \, (f)_{t-1}^{-\frac{1}{4n-4}} \quad ;$$

by (91),

$$M_2 \geq k_2^{\,1-\sum_{k=0}^{n-1} (2n-2)^k} (f)_{t-1}^{-\frac{1}{4n-4} + \frac{1}{2} \sum_{k=-1}^{n-2} (2n-2)^k}$$

$$2^{(2n-2)^{n-1}} . (\tfrac{4}{5})^{\sum_{k=0}^{n-2} (2n-2)^k} \quad ;$$

$$M_2 \geq k_3 \, (f)_{t-1}^{k_4} \quad ;$$

By the above and (76):

$$(f)_{t-1} - (f)_t \geq (f)_{t-1}^{k_4(2n-2)+1} \, k_3^{\,2n-2} \div 2^{2n-2} \, n^2 \, M_3^2$$

$$(M_3+1)^{4n-6} \; ;$$

then, since $\underline{M_3} \geq M_3$, and due to (89) and (90), the lemma is proved.

The lemma may be written

$$(f)_{t_0-1} - (f)_{t_0} \geq k_6 (f)_{t_0-1}^{\ k_5} \ ; \quad t_0 = 1,2,3, \ldots .$$

Consider $(f)_t$ as a function of t which decreases at a uniform rate as t goes from $t_0 - 1$ to t_0.

Theorem 3: If $t \geq 0$:

92) $(f)_t \leq [(k_5-1)\, k_6 t + (f)_0^{\ 1-k_5}]^{\frac{1}{1-k_5}}$;

if $\delta_1 > 0$ and $t > [\delta_1^{\ 1-k_5} - (f)_0^{\ 1-k_5}] \div k_6(k_5-1)$,

93) then $(f)_t < \delta_1$;

94) $\lim\limits_{t \to \infty} (f)_t = 0$.

Proof: If $t_0 - 1 \leq t_1 < t_2 \leq t$:

$$(f)_{t_1} - (f)_{t_2} \geq (t_2 - t_1)\, k_6\, (f)_{t_0-1}^{k_5} \ ;$$

$$\frac{(f)_{t_1} - (f)_{t_2}}{t_2 - t_1} \geq k_6\, (f)_{t_0-1}^{k_5} \ ;$$

and, since the function on the left is constant, and because $(f)_{t_0-1} \geq (f)_{t_1} > (f)_{t_2} \geq (f)_{t_0}$ due to the way

$(f)_t$ is defined for $t_o - 1 \leq t \leq t_o$,

$$- \lim_{t_2 \to t_1} \frac{(f)_{t_2} - (f)_{t_1}}{t_2 - t_1} \geq k_6 \, (f)_{t_o - 1}^{k_5} \geq \lim_{t_2 \to t_1} k_6 (f)_{t_2}^{k_5} ;$$

$$\frac{-\,d(f)_t}{dt} \geq k_6 (f)_t^{k_5} \quad ;$$

$$\frac{-d(f)_t}{(f)_t^{k_5}} \geq k_6 \, dt \; .$$

Integrating and using the value of $(f)_o$ as a boundary condition,

95) $\frac{1}{k_5 - 1} (f)_t^{1-k_5} \geq k_6 t + \frac{1}{k_5 - 1} (f)_o^{1-k_5}$

Due to (89), and the fact that $k_4 > 0$,

96) $k_5 - 1 > 0.$

Hence

$$(f)_t^{1-k_5} \geq (k_5 - 1) k_6 \, t + (f)_o^{1-k_5}$$

This establishes (92). Because $k_6 > 0$, and due to (95):

97) $t \leq [\, (f)_t^{1-k_5} - (f_o)^{1-k_5} \,] \div k_6 \, (k_5 - 1) \; .$

If the hypothesis of (93) is true, then by (97),

$$[(f)_t^{1-k_5} - (f_o)^{1-k_5}] \div k_6(k_5-1) \geq t > [\delta_1^{1-k_5} - (f)_o^{1-k_5}] \div$$
$$k_6 (k_5 - 1) ;$$

and, because $k_6 > 0$ and due to (96),

$$[(f)_t^{1-k_5} - (f)_o^{1-k_5}] > [\delta_1^{1-k_5} - (f)_o^{1-k_5}] ;$$

and, because $1 - k_5 < 0$,

$$(f)_t < \delta_1 .$$

This proves (93), and from (93) follows (94). Q.E.D.

IV

CONCLUSION

Lemma 1: If, for arbitrary positive ε, and non-negative integers t_1 and t_2,

$\max |(a_k)_{t_1} - (a_k)_{t_2}| < \delta_2$, $(k = 0, 1, \ldots, n)$,

where

$$\delta_2 = \varepsilon^n \div n \, M_{\underline{3}}^{\;n} \; ,$$

then given any m, $(m = 1, 2, \ldots, n)$, there axists at least one j, $(j = 1, 2, \ldots, n)$, such that

$|(r_m)_{t_1} - (r_j)_{t_2}| < \varepsilon$.

Note: Since t_1 and t_2 are interchangeable in the hypothesis, they also are so in the conclusion, of the lemma.

Proof: Under hypothesis of lemma, since $M_{\underline{3}}$ and n are positive,

98) $\quad \varepsilon^n > n \max |(a_j)_{t_2} - (a_j)_{t_1}| \, M_{\underline{3}}^{\;n}$.

Due to (56),

$$M_{\underline{3}}^n \geq M_3^n \geq M_3^j \geq |(r_m)_{t_1}^j|, \text{ for } m = 1, 2, \ldots, n.$$

Hence, by (98),

$$\varepsilon^n > \sum_{j=0}^{n-1} |(a_j)_{t_2} - (a_j)_{t_1}| |(r_m)_{t_1}^j|$$

$$99) \qquad \geq | \sum_{j=0}^{n-1} [(a_j)_{t_2} - (a_j)_{t_1}] (r_m)_{t_1}^j | .$$

The last inequality follows from the triangular inequality. Since $(a_n)_{t_2} = (a_n)_{t_1} = 1$, the upper limit, $n - 1$, of the above summations may be changed to n.

By (3),

$$\sum_{j=0}^{n} (a_j)_{t_1} (r_m)_{t_1}^j = \prod_{j=1}^{n} [(r_m)_{t_1} - (r_j)_{t_1}] = 0 .$$

By adding the first member above (equal to zero) to a quantity in absolute value signs in (99), and applying (3), we obtain,

$$\varepsilon^n > | \sum_{j=0}^{n} (a_j)_{t_2} (r_m)_{t_1}^j | = \prod_{j=1}^{n} |(r_m)_{t_1} - (r_j)_{t_2}| .$$

From this the conclusion of the lemma follows.

Lemma 2: If, for $t = t_1, t_2$,

$$100) \quad t > [(\frac{\delta_2}{2})^{2-2k_5} - (f)_0^{1-k_5}] \div k_6 (k_5 - 1),$$

the hypothesis (and hence the conclusion) of Lemma 1 follow.

Proof: Applying (93) to the hypothesis of this lemma, and letting $\delta_1 = \frac{1}{4}\delta_2^2$,

$$(f)_t < \frac{1}{4}\delta_2^2 .$$

By (4),

$$\max |(a_k)_t - (a_k)|^2 \leq (f)_t < \frac{1}{4}\delta_2^2 , \; k = 0, 1, \ldots, n,$$

$$\max |(a_k)_t - (a_k)| < \frac{1}{2}\delta_2 .$$

Since the above is true for $t = t_1, \; t_2$,

$$\max | (a_k)_{t_1} - (a_k)_{t_2} | \leq |(a_k)_{t_1} - (a_k)| + | (a_k)_{t_2} - (a_k) | < \delta_2 .$$

But this is the hypothesis of Lemma 1. Q.E.D.

Lemma 2, together with the note after Lemma 1, implies that if t_1 and t_2 satisfy (100), there is a one-to-one correspondence between the sets $\{(r_k)_{t_1}\}$ and $\{(r_k)_{t_2}\}$, $(k = 1, 2, \ldots, n)$, such that given any corresponding elements, such as $(r_m)_{t_1}$ and $(r_j)_{t_2}$, $|(r_m)_{t_1} - (r_j)_{t_2}| < \varepsilon$. Reorder each set, $(r_1)_t, (r_2)_t, \ldots, (r_n)_t$ $(t = 0, 1, 2, \ldots)$, as $(r_{k_1})_t$,

$(r_{k_2})_t, \ldots, (r_{k_n})_t$, so that for each $m (m = 1,2,\ldots,n)$, $(r_{k_m})_{t_1}$ and $(r_{k_m})_{t_2}$ correspond with each other in the above one-to-one correspondence. Then, if t_1 and t_2 satisfy (100) , for any k_m,

101) $$|(r_{k_m})_{t_1} - (r_{k_m})_{t_2}| < \varepsilon \ .$$

Since, given $(r_k)_0$, $M_{\underline{3}}$, k_2, k_3, k_4, k_5, k_6, and δ_2 may be determined, using only their respective definitions and (3) and (4). Hence the right member of (100) is computable, and if t_1 and t_2 exceed this member, (100) is satisfied. Hence for each m ($m = 1, 2, \ldots, n$), the sequence $\{(r_{k_m})_t\}$, $t=0,1,2,\ldots$, converges. Define

102) $$\lim_{t \to \infty} (r_{k_m})_t = (r_{k_m})_\infty \ , \quad m = 1, 2, \ldots, n \ .$$

Then, if t_1 satisfies (100), and since any $t_2 : t_2 > t_1$, satisfies (100) and (101),

$$|(r_{k_m})_{t_1} - (r_{k_m})_\infty| \leq \varepsilon \ .$$

Theorem 4: If $a_n = 1$,

$$\sum_{k=0}^{n} a_k Z^k = \prod_{m=1}^{n} [Z - (r_{k_m})_\infty] \ ,$$

so that $(r_{k_1})_\infty, (r_{k_2})_\infty, \ldots, (r_{k_n})_\infty$ are the zeros of this polynomial.

Proof: Due to (4) and (94):

103) $\lim_{t \to \infty} f[(r_{k_1})_t, (r_{k_2}), \ldots, (r_{k_n})_t] = 0$

Because $(r_1)_t, (r_2)_t, \ldots, (r_n)t$, may be reordered as $(r_{k_1})_t, (r_{k_2})_t, \ldots, (r_{k_n})_t$, (3) may be written:

104) $\prod_{m=1}^{n} [z - (r_{k_m})_t] = \sum_{k=0}^{n} (a_k)_t z^k$.

We may similarly rewrite the second member of (4).

Due to (4) and (104), f is a continuous function of $(r_{k_m})_t$, $(m=1, 2, \ldots, n)$. Due to this fact, (102), and (103),

105) $f[(r_{k_1})_\infty, (r_{k_2})_\infty, \ldots, (r_{k_n})_\infty] = \lim_{t \to \infty} f[(r_{k_1})_t,$

$$(r_{k_2})_t, \ldots, (r_{k_n})_t] = 0.$$

We define $(a_k)_\infty$ by (104). Due to (4) and (105),

$$\sum_{k=0}^{n-1} |(a_k)_\infty - a_k|^2 = 0 .$$

Hence $(a_k)_\infty = a_k$ when $k=1,2,\ldots,n-1$, and $(a_n)_\infty = a_n = 1$, by (1). Substituting a_k for $(a_k)_\infty$

in the definition of the latter, the theorem is proved.

We define $r_m = (r_{k_m})_\infty$, $(m = 1, 2, \ldots, n)$.

If we wish to insure that

$$|r_m - (r_{k_m})_t| \leq \varepsilon ,$$

we iterate the method outlined in section 2, t times, where t satisfies (100).

Theorem 4 not only implies the Fundamental Theorem of Algebra, (that a polynomial of n-th degree has at least one zero in the complex field), but also that said polynomial has n linear factors, a method for the approximation of which is given in this paper.

V

EXAMPLE

Suppose we wish to solve:

106) $z^3 + 3.14\, i\, z^2 + (-1-2.5\, i)\, z + 1+2i = 0$.

We begin with the trial roots:

$(r_1)_0 = 1, \quad (r_2)_0 = 1+3i, \quad (r_3)_0 = 1-i$.

The method applicable to Case 1 in the Introduction is used. In each cycle of six iterations, first the real and then the imaginary parts of first r_1^*, and then r_2^* and r_3^*, are "improved".

Since Case 1 is used, $P_m = r_m^*$ for $m=1,2,3$. (See (6) and preceeding discussion.) In each iteration, we use (12) or (13), (16) or (17), and (18).

Calculations were done on Alwac III-E, a digital computer, to four decimal places past the decimal point.

The result after one cycle (six iterations) was

$(r_1)_6 = -1.0000-0.3496i, (r_2)_6 = 0.8489 + 0.4010i,$

$(r_3)_6 = 1.0954-1.0842i.$

The algebraic equation having the above roots was found to be

$$z^3 + (-0.9444 + 1.0329i)\ z^2 + (-0.8186 - 0.4779i)\ z + (1.5329 - 0.0041i) = 0\ .$$

After some more cycles, the result was

$$r_1^* = -0.9396-2.8359i,\quad r_2^* = 0.4473 + 0.5857i,$$

$$r_3^* = 0.4954 - 0.8874i\ .$$

These are the roots of

$$z^3 + (-0.0030 + 3.1376\ i)\ z^2 + (-1.0000 - 2.4965\ i)\ z + (0.9991 + 2.0020i) = 0\ .$$

After more cycles, the result was

$$(r_1)^* = -0.9422-2.8399i,\quad (r_2)^* = 0.4476 + 0.5853i,$$

$$(r_3)^* = 0.4946-0.8854i\ .$$

These are the roots of

$$z^3 + 3.1400iz^2 + (-1.0000-2.4999i)z + 1.0000+2.0001i=0.$$

After yet more cycles, the trial roots and the corresponding equation remain the same, so the last given values for $(r_k)^*$, $k = 1,2,3$, are accepted as the final approximations to the roots of (106). In fact, when one computes the equation having said approximations for roots, one does, as is seen above, obtain coefficients differing by not more than 0.0001 from those in (106).

VI

A NON-CONSTRUCTIVE PROOF OF THE THEOREM

Another proof of the Fundamental Theorem of Algebra, is as follows [1,p.201-207] .

Consider the polynomial:

107) $$p(z) = \sum_{k=0}^{n} a_k z^k \; ; \; n = 1,2,3,\ldots ; \; a_n \neq 0 \; .$$

We show that $|p(z)|$ attains a minimum value for some complex value of z. Suppose $p(z_N) = N$. Consider all z such that $p(z) \leq N$. Then, if $|z| \geq 1$

$$\left| \sum_{k=0}^{n} a_k \, z^k \right| \leq N \; ;$$

$$| a_n \, z^n | - \sum_{k=0}^{n-1} | a_k \, z^k | \leq N \; ;$$

$$| a_n \, z^n | \leq N + \sum_{k=0}^{n-1} | a_k \, z^k | \; ;$$

$$| z | \leq [N + \sum_{k=0}^{n-1} | a_k |] \div | a_n | \; .$$

Call the right member of the last above inequality, N_1. So, if $|p(z)| \leq N$, then $|z| < 1$ or $|z| \leq N_1$. Then the closed region $|z| \leq \max \{1, N_1\}$, called region A, contains all z such that $|p(z)| \leq N$. Since A is a closed region and $|p(z)|$ a continuous function of z, the Bolzano-Weierstrass Theorem establishes that there exists $z_0 \varepsilon A$ such that for all other $z \varepsilon A$, $|p(z)| \geq |p(z_0)|$. Since $z_n \varepsilon A$, $N = |p(z_n)| \geq |p(z_0)|$. Since, for z not in A, $|p(z)| > N$,

108) $$|p(z)| \geq |p(z_0)| ,$$

whether or not z is in A. Let $p_1(z) = p(z + z_0)$. Then, by (108), for any z,

109) $$|p(z+z_0)| = |p_1(z)| \geq |p_1(0)| = |p(z_0)| .$$

The function $p_1(z)$ is, due to its definition, a polynomial of n-th degree, say

110) $$p_1(z) = \sum_{k=0}^{n} b_k z^k , \quad b_n \neq 0 .$$

Then:

111) $$p(z_0) = p_1(0) = b_0 ;$$

112) $$|p_1(z)| \geq |b_0| .$$

If $b_0 = 0$, then (111) establishes the theorem. So we consider the case, $b_0 \neq 0$. Due to (110), there is a least k such that $b_k \neq 0$ and $n \geq k > 0$. Call this k, k_0. Then, by (112), for all z:

$$|b_0| \leq |p_1(z)| = |b_0 + \sum_{k=k_0}^{n} b_k z^k| ;$$

113) $|b_o| \leq |b_o + b_{k_o} z^{k_o}| + \sum_{k=k_o+1}^{n} |b_k z^k|$.

Since $b_{k_o} \neq 0$, and $b_o \neq 0$, there exists z such that:

114) $$\arg z = (\pi + \arg b_o - \arg b_{k_o}) \div k_o ;$$

115) $$|z| < \min \left\{ \frac{|b_{k_o}|}{\sum_{k=k_o+1}^{n} |b_k|} , \left|\frac{b_o}{b_{k_o}}\right|^{\frac{1}{k_o}} , 1 \right\}$$

If, because $k = n$, $\sum_{k=k_o+1}^{n} |b_k| = 0$, the first quantity in braces is ∞ and the minimum of the other quantities is taken. If (114) and (115) are satisfied:

$$|z| \sum_{k=k_o+1}^{n} |b_k| < |b_{k_o}| ;$$

and, because $|z| < 1$,

$$\sum_{k=k_o+1}^{n} |b_k z^{k-k_o}| - |b_{k_o}| < 0 ;$$

Multiplying by $|z^{k_o}|$,

$$[\sum_{k=k_o+1}^{n} |b_k z^k|] - |b_{k_o} z^{k_o}| < 0 ;$$

116) $[\sum_{k=k_o+1}^{n} |b_k z^k|] - |b_{k_o} z^k| + |b_o| < b_o|$.

Since

$$|z| < \left| \frac{b_o}{b_{k_o}} \right|^{\frac{1}{k_o}} ,$$

117) $|b_{k_o} z^{k_o}| < |b_o|$.

By (114),

$$\arg b_o = -\pi + \arg b_{k_o} + k_o \arg z$$

$$= -\pi + \arg [b_{k_o} z^{k_o}] .$$

Due to (117) and (118) ,

119) $|b_o + b_{k_o} z^{k_o}| = |b_o| - |b_{k_o} z^{k_o}|$.

Due to (116) and (119),

$$|b_o + b_{k_o} z^{k_o}| + \sum_{k=k_o+1}^{n} |b_k z^k| < |b_o| .$$

This contradicts (113), so the case, $b_o \neq 0$, is impossible. Hence $b_o = 0$. The proof is completed.

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