AN ABSTRACT OF THE THESIS OF

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Title: NONLOCAL ELECTROMAGNETIC THEORY OF FLUIDS WITH APPLICATIONS TO SURFACE WAVES IN A DIELECTRIC FLUID

Abstract approved: M. N. L. Narasimhan

The last two decades have seen many new continuum models for electromagnetic materials which were constructed by merely adding terms to the classical balance laws and constitutive equations. Non-local theories, such as the one developed by A. C. Eringen, however, depart from the traditional approaches by accounting for the effects of distant atomic, molecular and granular interactions.

Although some attention has been given to nonlocal electromagnetic solids no work has been done on nonlocal electromagnetic fluids. The natural occurrence of electromagnetic fluids (atmospheric and oceanic circulations, the earth's core as well as its mantle) and their applications to certain mundane problems such as energy conversion, biofluids, medicine, nuclear and electrical engineering make electromagnetic fluids an important field of study. Moreover, in many laboratory devices the electromagnetic fluids are preferred over
electromagnetic solids since they readily deform and flow as compared to the latter.

In the present thesis, based on Eringen's approach, we develop a nonlocal theory of fluids with electromagnetic constitution, capable of exhibiting electromagnetic interactions. Though the approach is nonrelativistic in nature it is in agreement with relativistic Lorentz theories up to terms of \( O(v^2/c^2) \), where \( v \) is the speed of the material; \( c \) is the speed of light in vacuum. A generalized Clausius-Duhem thermodynamic inequality which encompasses nonlocal effects is derived and is applied to obtain specific forms of the constitutive equations, including the total electromagnetic momentum, stress, and energy without any a priori assumption as to their nature or form. Both local and nonlocal electromagnetic variables, including their time rates are incorporated into the constitutive theory. Full thermodynamic restrictions and admissibility of the constitutive equations is investigated. In order to facilitate practical applications a completely linear constitutive theory is also derived, again including thermodynamic restrictions as well as the full field equations with appropriate boundary conditions.

Surface effects, such as surface waves and surface tension, which occur on free surfaces and between stratified layers are of great importance in many applied fields. Consequently, the linear theory developed above is used to analyze the problem of surface
waves in a dielectric fluid medium.

A theory of nonlocal polar electromagnetic fluids is also developed. Such a theory takes into account internal orientational effects with electromagnetic interactions in a material with internal substructures, e.g. liquid crystals and animal blood.
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with Applications to Surface Waves
in a Dielectric Fluid

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I. INTRODUCTION

1. Limitations of Classical Continuum Mechanics

There are three basic assumptions underlying the foundations of classical continuum mechanics. It is assumed that the nature and response of the body is such that: (1) there exists a continuous mass density function at every material point, (2) the balance laws are valid for every part of the body, and (3) the state of the body at any material point is influenced only by an infinitesimal neighborhood surrounding the point. Although classical continuum mechanics has enjoyed great success in the past the three basic assumptions aforementioned have been recently challenged due to the advent of more sophisticated experiments and measuring devices. Several instances arise where the foundations underlying continuum physics are violated.

The assumption of the existence of a continuous mass density function has experimentally been shown to be false in general. Specifically, one defines a mass density function $\rho(\mathbf{x}, t)$ for each fixed material point $\mathbf{x}$ and time $t$ by

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V}$$
where \( \Delta m \) is the mass of the microvolume element \( \Delta V \). Experiments have shown that as \( \Delta V \) becomes small comparable to some characteristic material dimension, say some critical microvolume \( \Delta V_c \), then \( \rho \) begins to fluctuate. That is, by taking a sequence of microvolumes \( \{ \Delta V_i \} \) such that \( \Delta V_i \to 0 \) as \( i \to \infty \), having mass \( \{ \Delta m_i \} \), the limit \( \Delta V_i / \Delta m_i \) does not exist. Since all bodies consist of subbodies or packets consisting of atoms, molecules or grains having some characteristic size (or distance), the nonexistence of a continuous mass density function at the 'sub' macroscopic level casts severe restrictions on what type of physical phenomena can accurately be described by classical continuum mechanics. Indeed, the response of a body to external forces depends to a great extent on the ratio of, the characteristic length \( \lambda \) associated with the body force to the internal characteristic length \( l \) of the body. When \( \lambda/l \gg 1 \) classical continuum physics gives 'good' predictions whereas when \( \lambda/l \sim 1 \) the classical theory fails to successfully predict the response of the body to the external load. A good example is the inability of classical elasticity to predict a dispersive character of waves with short-wavelengths, a phenomena well borne out by experimentation and lattice dynamics.

The remaining two assumptions cast restrictions on what affects the state and motion of the body at any material point. Assuming the balance laws to be valid for every part of the body eliminates any
long-range effects of forces on the body. The assumption that any material point is influenced solely by an immediate infinitesimal neighborhood ignores the effects of any long-range interatomic interactions. Thus the state and motion of the body at any material point depend solely on local effects. What is taking place at the rest of the body is assumed to have no effect at the point in question.

Materials whose behavior is nonlocal in nature, that is, which do not adhere to the above assumptions have been known for some eighty years. Many experiments have pointed out the fact that the state of the body at a particular material point does depend heavily on the influence of all other material points, that is, nonlocal type effects. Duhem [16], in 1893 was probably the first to notice such nonlocal effects in his study of force and stress at a point and their influence on the state of the entire body. Lord Rayleigh [62] in 1918 also dealt with nonlocal effects as did Oseen [58] in 1933 in his fundamental work on liquid crystals.

Continuum physics has taken four separate avenues to account for nonlocal effects. The first involves assigning to each material point some geometric properties which endow particles with more than three degrees of freedom. Such theories include the director theories of Ericksen [20], Toupin [68], and Mindlin [53], the multipolar theory of Green and Rivlin [41, 42], and the micropolar theory of Eringen [21]. The second approach consists of including higher gradients of
displacement and velocity in the basic postulates, while the third method of attack deals with theories referred to as diffusion theories (see Edelen [17] for references). The last approach considered consists of the category of theories we are concerned with, namely those theories which admit the possibility of nonlocal interactions. The last class of theories has important differences relative to the other three in that no "generalized forces" or additional balance laws are introduced.

2. Nonlocal Theory

Although nonlocal has a specific meaning to each person we will attempt to give a definition of what is meant by nonlocal continuum physics (cf. Edelen [17]). In classical (local) continuum theories the governing field equations consist solely of differential equations in the appropriate field variables. On the other hand nonlocal theories are characterized by the fact that one or more of the field equations consist of integrodifferential equations. That is, integrals of the state variables as functions of space and time would appear in the basic equations describing the state and motion of the body. Although differential equations can be transformed into equivalent integrodifferential equations we are not alluding to this type of equivalence between local and nonlocal theories. Indeed, there exist integrodifferential
equations which have no equivalent finite order differential equations [17].

Such types of nonlocal equations occur in many fields of study. For example, Boltzmann's integrodifferential equations of gas dynamics or Nagumo's equation in the modeling of neurophysiological systems, Nagumo et al. [55],

\[
\frac{\partial e}{\partial t} = \frac{\partial^2 e}{\partial t^2} + e(1-e)(e-a) - b \int e dt
\]

which describes electrical wave phenomena in the cortex and the propagation of nerve impulses, are examples of equations accounting for nonlocal effects. Other areas where nonlocal effects occur include: electromagnetic wave propagation through a bounded medium, electromagnetic and acoustic wave propagation through bounded homogeneous random media, interstellar dynamics, particle physics, radiative transfer, colloidal suspensions and polymers.

Early contributors to nonlocal theories include Green and Rivlin [43], Kröner [50], and Kröner and Datta [51]. Many of these early attempts lacked a systematic approach as well as thermodynamic considerations and hence were to a large extent heuristic in nature. Edelen et al. [18] did, however, develop some thermodynamic aspects while Eringen and Edelen [39] presented a more complete nonlinear nonlocal theory of elasticity based on the balance laws, and
the thermodynamical theory through nonlocal variational principles.
The foundations of nonlocal polar theory, a combination of the micro-
morphic and nonlocal theories were established by Eringen [22].
From this work Eringen developed theories on nonlocal fluid mechanics
[23], microfluid mechanics [24], microelastic solids [25], memory-
dependent nonlocal elastic solids [26], nonlocal thermoelasticity [27],
and nonlocal electromagnetic solids [28].

To explicitly demonstrate the crucial differences between
nonlocal and local continuum theories we must investigate carefully
the process of passage from the global balance laws to the localized
laws.

We shall assume that the body  \( B \) in question has a volume  \( V \)
bounded by a surface  \( \partial B \) enclosing any proper, connected subset  \( P \)
of  \( B \) with a nonzero volume measure. If we make statements con-
cerning the entire body  \( B \) we say such statements are \textbf{global}.
Accordingly, any statement concerning an arbitrary part  \( P \) of the
body will be termed \textbf{local}. If  \( B \) is simply replaced by  \( P \) in any
global statement with the consequence of invalidating the latter then
the global statement is said to be \textbf{nonlocal}.

In general the global balance laws for a body  \( B \) are of the
form
This yields the localized balance law

\[ \Phi = \Sigma + \gamma \]  

(2.5)
since (2.4) must be valid for all parts of the body.

The nonlocal theory does not postulate (2.1) to be valid for all arbitrary parts of the body. Using (2.2) we may write (2.1) as

$$\int_{B} (\phi - \Sigma - \gamma) d\tau = 0. \quad (2.6)$$

Hence, the only conclusion we can draw is that the integrand in (2.6) must be a function having zero mean on the entire body $B$. We denote the class of integrable functions with zero mean on $B$ by $\mathcal{M}(B)$, that is, $R \in \mathcal{M}(B)$ iff

$$\int_{B} R d\tau = 0. \quad (2.7)$$

For each $R \in \mathcal{M}(B)$, (2.6) is satisfied if we take

$$\phi = \Sigma + \gamma + R \quad (2.8)$$

at each point of $B$. Thus for each $R \in \mathcal{M}(B)$ the integration of (2.8) over any arbitrary part $P$ of $B$ yields a local statement

$$\int_{P} \phi d\tau = \int_{P} \Sigma d\tau + \int_{P} \gamma d\tau + \int_{P} R d\tau. \quad (2.9)$$

Assuming (2.2) to be valid for $P$ yields the local balance law
Thus for each global balance law (2.1) an entire equivalence class of local balance laws (2.10) is generated. Here we take the equivalence relation: two local statements are equivalent if they imply satisfaction of the same global statement. Equivalently, each global law (2.1) is a nonlocal statement resulting in an equivalence class of local statements (2.10) for all \( R \in \mathcal{M}(B) \). Only when \( R = 0 \) does the global statement (2.1) hold for every part of the body, that is, yielding the classical continuum theory.

The localization of (2.9) yields (2.8) which clearly differs from the classical localization law (2.5) by the additional factor \( R \). For this reason such quantities as \( R \) are called localization residuals. We can interpret these localization residuals as the rate of production of \( \phi \) at a given material point \( X \) due to all the rest of \( B \). This can be seen from the following. From (2.7) we have

\[
\int_P R d\tau = - \int_{B-P} R d\tau \quad (2.11)
\]

and hence (2.10) can be written as

\[
\frac{D}{Dt} \int_P \phi d\tau = \oint_{\partial P} \mathbf{g} \cdot d\mathbf{a} + \int_P \gamma d\tau - \int_{B-P} R d\tau \quad (2.12)
\]
from which the above interpretation readily follows. Hence we see
that nonlocal theories take into account the effects of distant atomic,
molecular, and granular interactions through the introduction of the
localization residuals. Nonlocal theories thus appear to extend the
validity of the classical continuum theory towards the realm of micro-
scopically phenomena.

Since nonlocal continuum physics is still in its early stages of
development it is difficult to assess its true worth at present.
Although the number of physical examples to which the nonlocal theory
has been applied is small, it has enjoyed great success. Results that
lattice dynamics or quantum theories could predict but classical
continuum mechanics fails to predict have been borne out by the non-
local theories. Among these is the dispersion of plane waves in non-
local elastic solids Eringen([29, 30]) as well as in nonlocal polar elastic
solids Eringen([25]). Both the acoustic and optic modes of the disper-
sion curve for Rayleigh type waves in elastic solids have been obtained
by Eringen[31,32] using nonlocal polar theory. The dispersion curves
are in excellent agreement with those obtained by lattice dynamics
throughout the entire Brillouin zone. Applying nonlocal theory to the
stress concentration at the tip of a crack, Eringen and Kim [40] found
that the nonlocal elasticity eliminates the stress discontinuity arising
from classical continuum treatment. Furthermore, when the hoop
stress obtained is set equal to the cohesive stress the well known Griffith
fracture criteria is obtained in a natural fashion. Nonlocal theories have also been applied to quasi-static dielectrics Demiray [13], and the deformation of a nonlocal elastic layer, Cekirage and Teymur [8]. Nonlocal effects in fluids encompass such effects as surface tension, surface stresses, and surface viscosities (see Eringen [23]). These few examples at least attest to the potential of the nonlocal theories.

3. Electromagnetic Interactions and Nonlocal Effects

The foundations of classical electromagnetic theory rest in the experimental work of Faraday, Ampere, Coulomb, Gauss and many others. In fact Maxwell formulated his celebrated equations of electromagnetic theory based primarily on the work of Faraday. Since their formulation over a century ago Maxwell's equations have been used as postulates governing the behavior of electromagnetic phenomena.

Maxwell's equations have had many far reaching consequences in many theoretical settings. Probably the most remarkable discovery was Maxwell's realization that light should be looked upon as essentially an electromagnetic phenomenon. The vast success of Maxwell's theory also led to the concept of a unified field theory which, although has shown some discrepancies in its predictions, is nevertheless an important concept in theoretical physics. The area of application of electromagnetic theory reaches from phenomena involving short-wave
lengths \((10^{-12}\text{ m})\) to long-wave lengths \((10^{2}\text{ m})\). Thus Maxwell's equations describe behavior of light, x-rays, microwaves, etc. and furnish physicists, engineers, biomedical researchers, and other scientists with the necessary theory to develop such fields as geometrical optics, antenna theory, transmission line theory, energy conversion, etc.

Other areas of application, where the interaction of electromagnetic fields with deformable bodies is of major concern, Maxwell's theory coupled with classical continuum mechanics has not been as successful. A simple example is the fact that classical electromagnetic theory predicts a constant refractive index in an isotropic non-dissipative medium. It is well known, however, that the refractive index does depend on the frequency of propagation in almost all wave propagation phenomena and hence dispersion in deformable materials interacting with electromagnetic fields is the rule rather than the exception. The failure of classical electromagnetic theory to predict such behavior is worthy of attention in problems dealing with dielectric fluids which are known to be dispersive. Many other effects not predictable by classical electromagnetic theory, such as photo-viscoelasticity, streaming birefringence, and neurophysiological impulses, have prompted the development of new theories to account for such phenomena.
With the development of the theory of relativity came the first major departure from classical electromagnetic theory. In fact a totally satisfactory and consistent theory for the interaction of electromagnetic fields with a moving continuum must be at least Lorentz invariant \[45\]. Such a formulation is, however, not particularly useful for the solution of practical problems. Furthermore for many applications the speed \( v \) of the medium is small compared to \( c \), the speed of light in a vacuum so that an approximation to Lorentz invariance to within terms of \( \frac{v^2}{c^2} \) is more than sufficient.

Another independent departure from classical electromagnetic theory arose with the development of quantum physics. Based primarily on the state and motion of elementary particles making up a medium, quantum physics has been successful in predicting the response of a medium at the microscopic level. At the macroscopic level, however, such quantum theories have shown some discrepancies in their predictions and severe limitations as to their realm of application.

If one examines the classical theory of electromagnetic fields interacting with deformable materials the theory is seen to be based on (1) the classical continuum mechanics of materials, (2) the classical electromagnetic theory of Maxwell, and (3) the incorporation of the electromagnetic effects on the body and vice versa, that is, via the balance laws and/or constitutive theory. Our previous discussion
of classical and nonlocal continuum theories has brought out the shortcomings of the classical continuum theory. As for (3), the manner in which the interaction of electromagnetic fields with deformable materials is taken into account has been, in the past, a matter of taste. The inclusion or exclusion of higher order deformation gradients and/or thermal gradients, surface and body couples, multipolar electromagnetic quantities, etc. has been strictly subjective in nature, depending primarily on the phenomena to be studied. Even if one decides on what type of phenomena is to be modeled in many cases it is not clear what is the proper way to introduce the mechanism necessary to describe such effects. One of the classic examples in this regard is, as to how the electromagnetic field should be introduced as an external body force (cf. Penfield and Haus [60]). We give an account of the modern approaches taken within the past two decades when the scope of this thesis is discussed. We now proceed to the discussion of (2), tracing the development of the classical electromagnetic theory of Maxwell.

Examining the original integral formulations presented by Maxwell we can see how the nonlocal theory can be applied. For example Maxwell's first hypothesis is a restatement of the law of electromagnetic induction discovered by Faraday and Lenz [49]. Loosely speaking it states that the electromotive force, or the work done by the electric field \( \mathbf{E} \) on a unit electric charge carried
around a closed curve \( \mathcal{C} \), is equal to the time rate of decrease of
the flux of magnetic induction \( \mathcal{B} \) through any surface \( S \) bounded
by \( \mathcal{C} \). More specifically we have

\[
\oint_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{x} = -\frac{1}{c} \frac{D}{Dt} \int_S \mathbf{B} \cdot d\mathbf{a}
\]  

(3.1)

where \( c \) is the speed of light in vacuum, \( \frac{D}{Dt} \) is the total time
derivative and the electromagnetic quantities are expressed in
Heaviside-Lorentz units. Applying Stokes' theorem to the left hand
side of (3.1) and assuming \( S \) to be fixed in time we may write

\[
\int_S (\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial}{\partial t} \mathbf{B}) \cdot d\mathbf{a} = 0.
\]  

(3.2)

Assuming (3.1) to be true for any closed curve \( \mathcal{C} \) and hence for
arbitrarily small surfaces \( S \) implies the integrand in (3.2) must
vanish which yields Faraday's Law in differential form:

\[
\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial}{\partial t} \mathbf{B} = 0.
\]  

(3.3)

As with the global balance laws for a continuum a nonlocal
theory of electromagnetic fields raises the question of why, or on
what basis, should (3.1) be valid for all parts of the body. The moti-
vation for raising such a question lies in the origin of Maxwell's
equations. First it must be remembered that the laws of Faraday, Ampere, and Gauss were derived from an empirical point of view based on experimental results. Furthermore these integral forms arose strictly from electrostatic and magnetostatic experiments. Later they were postulated as being true for time dependent electromagnetic fields interacting with deformable materials. Though experimental data have confirmed the validity of these postulated laws there are inherent drawbacks involving such experimental evidence. Foremost in this regard is the very nature of electromagnetic fields. The speeds and rates of change of the time dependent electromagnetic fields make their measurement at a macroscopic level exceedingly difficult. In fact, within a continuum interacting with electromagnetic fields, measurements of their intensities within the medium are impossible. Secondly, due to a general feeling among experimentalists that the classical electromagnetic theory had been making great strides in the past coupled with the lack of sophistication in measuring electromagnetic field variables, there existed little motivation to design suitable experiments for testing the validity of the classical postulates of Maxwell. Thus the very nature of the electromagnetic fields coupled with the lack of experimental data leaves the validity of the localization assumption for Faraday's and Ampere's laws somewhat unfounded. It seems more reasonable to employ the nonlocal theory to the integral formulations of the electromagnetic field equations than applying
the classical assumption that the localization residuals vanish identically.

4. Scope of the Thesis

A number of continuum, as well as quantum and statistical theories have been developed to account for the shortcomings of the classical theory of electromagnetic fields interacting with deformable bodies. In order to explain many of the phenomena not borne out by the classical theory such problems were first studied separately involving three main groups: dielectrics, magnetic materials, and conductors. About 1956 many such theories describing one or the other of the above materials began to appear. Toupin [69], Eringen [33], Bragg [4], Grindlay [44], and Nelson and Lax [57] developed theories concerning dielectrics where polarization was important while polarization gradients played important roles in the theories of Mindlin [53], Suhubi [64], and Asker et al. [2]. For magnetic materials various theories have been developed based on the magnetization vector or on such phenomena as spin angular momentum and spin-spin interactions: Tiersten [66, 67], Brown [6, 7], and Maugin and Eringen [52]. A generalization of Ohm's Law was used by Walker [72] in his theory of conductors while Moon [54], and Demiray and Eringen [14] considered a conductor as a mixture of negatively charged plasma and a positively charged elastic lattice. Several unified
theories later appeared, among which are Jordan and Eringen [48], Dixon and Eringen [15], Bressan [5], Grot and Eringen [47], Penfield and Haus [60], DeGroot and Suttorp [11, 12], Grot [46], and Curtiss and Lianis [10]. Many of the similarities and differences of these general theories are pointed out in Grot [45].

All of the above theories deal primarily with the interaction of an elastic material with an electromagnetic field with no concern with the analogous situation for fluids. Since electromagnetic fields arise naturally in phenomena associated with ocean waves (Podney [61]), atmospheric circulations, geological disturbances such as seismic waves, and since there exist fluids exhibiting internal electromagnetic properties which find applications in energy conversion devices such as transducers, biofluids, medicine, nuclear reactors, and electrical engineering a continuum theory for adequately describing such media is necessary. Although Pao and Hutter [59] have developed a continuum theory for fluids interacting with electromagnetic fields, the latter is classical in nature and hence is incapable of bringing out phenomena due to nonlocal effects.

In 1973 Eringen [28] developed a general nonlocal continuum theory for electromagnetic elastic solids. However, there exists no corresponding theory for fluids. The main objective of this current research is to develop a nonlocal continuum theory for electromagnetic fluids. Although a rigorous formulation should be relativistic in
nature, in most applications a \( \frac{v^2}{c^2} \) approximation is quite sufficient. For this reason relativistic effects will be neglected in favor of the simpler \( \frac{v^2}{c^2} \) approximation.

In Chapter II we develop the localized balance laws for electromagnetic media as derived from a general set of nonlocal global balance laws. The localized laws thus derived do, of course, incorporate the localization residuals which account for distant atomic and molecular interactions. The second law of thermodynamics which includes nonlocal effects is then formed and postulated as being valid for the entire body.

A general constitutive theory for nonlocal electromagnetic fluids is developed in Chapter III in which the constitutive variables are invariant under Galilean type frame changes and are \( \frac{v^2}{c^2} \) approximations to the Lorentz theory. The thermodynamic admissibility yields the forms of the electromagnetic momenta, electromagnetic stress, and electromagnetic energy without any a priori assumptions being made as to their nature or form. To facilitate practical applications of the theory a completely linear constitutive theory is developed along with the complete thermodynamic restrictions governing the material coefficients. The field equations and boundary conditions are also developed from the resulting linear theory.
Chapter IV is concerned with the development of a nonlocal polar electromagnetic theory for orientable fluids. Although a complete set of balance laws, including the thermodynamic inequality, and general constitutive make-up are presented we do not derive here the linear constitutive theory but merely indicate the general direction and procedures to be taken for further work.

Since all materials, regardless of their constitution, are dispersive in character the dispersion relationship for surface waves propagating through the medium is of great importance. Consequently, in Chapter V the linear constitutive theory developed in Chapter III is applied to electromechanical surface waves in a dielectric fluid. Linearizing the resulting field equations yields a set of decoupled integrodifferential equations governing the mechanical and electromagnetic disturbances. Analyzing the mechanical surface waves we obtain a dispersion relation incorporating nonlocal effects and viscous influences. For purposes of obtaining some insight into the quantitative nature of the dispersion, as influenced by nonlocality, we utilize the fact that some viscoelastic and viscoplastic fluids exhibit solid characteristics. Hence, we have applied our theory to the case of lubricating oil utilizing some results from lattice dynamics for purposes of demonstration of the nonlocal effects. We conclude the chapter by analyzing the electromagnetic surface wave. Due to the lack of experimental data, however, the dispersion curve for the
electromagnetic surface wave is not obtained.

In Chapter VI we briefly discuss our work undertaken here and indicate several areas of further research and application of the theory developed herein.

In the sequel, a symbol such as (II. 2. 5) refers to formula 5 in Section 2 of Chapter II and (2. 5) refers to formula 5 in Section 2 of the current chapter.
II. BALANCE LAWS FOR NONLOCAL ELECTROMAGNETIC MATERIALS

1. Preliminary Remarks

All materials treated as continua are assumed to obey a certain set of fundamental balance laws. After describing the motion of the body we state these well known global balance laws for electromagnetic media. From these nonlocal global balance laws the appropriate localized balance laws containing the localization residuals introduced in Chapter I are then developed. We conclude the chapter with a formulation of the second law of thermodynamics incorporating nonlocal effects.

2. Motion

Let the material points of the body in the undeformed state be determined by a set of rectangular coordinates $X_K$, $K = 1, 2, 3$. At time $t$ the motion carries $X_K$ to the spatial points $x_k$ under the continuous mapping

$$x_k = x_k(X, t)$$  \hspace{2cm} (2.1)

which also possesses continuous first-order partial derivatives with respect to $X$ and $t$. Furthermore, we require that

$$J = \det x_k, K > 0$$  \hspace{2cm} (2.2)
so that (2.1) has a continuous inverse given by

\[ X_K = X_K(x, t) \]  \hspace{1cm} (2.3)

for all points of the body, except possibly a countable set of singular surfaces, lines, and points.

The summation convention over repeated indices is employed. Moreover, a subscript comma shall denote partial differentiation and a superposed dot shall denote material time rate, e.g.

\[ \nabla_k \frac{\partial \psi(x, t)}{\partial x_k} ; \dot{\epsilon} = \frac{D\epsilon}{Dt} = \frac{\partial \epsilon}{\partial t} + \dot{\epsilon}, \nabla_k \epsilon_k. \]  \hspace{1cm} (2.4)

3. Electromechanical Balance Laws

The development of a set of balance laws to describe the interaction of deformable bodies with electromagnetic fields depends primarily on the type of phenomena to be modeled. The physical model to be used will be a nonrelativistic model that is accurate to within a \( \frac{v^2}{c^2} \) (\( v \) speed of the material, \( c \) speed of light in vacuum) approximation of relativistic theory. The balance laws will be formulated in such a manner to exclude electric and magnetic quadrupoles and poles of higher order along with magnetic spin, spin-spin interactions and polarization gradients. The theory is, however, very general in nature and is broad enough to encompass
materials such as dielectrics, magnetic materials, and conductors. The fundamental assumption in this regard is that the electromagnetic fields interact with the material body through contributions to the total momentum, the total stress vector, the total energy density, and the total energy flux, Grot [45]. All of the balance laws will thus be expressed in terms of the total effects due to both electromagnetic and mechanical forces without any arbitrary splitting up of the total stress vector, total energy density, or total energy flux.

The integral form of the mechanical balance laws accompanied by the electromagnetic balance laws may be written in the form [45]:

Conservation of Mass:

\[
\frac{D}{Dt} \int_{V-\sigma} \rho dv = 0 \quad (3.1)
\]

Balance of Linear Momentum:

\[
\frac{D}{Dt} \int_{V-\sigma} \rho (\vec{v}+\vec{g}) dv - \oint_{S-\sigma} t_k da_k - \int_{V-\sigma} \rho \vec{f} dv = 0 \quad (3.2)
\]

Balance of Moment of Momentum:

\[
\frac{D}{Dt} \int_{V-\sigma} \rho \vec{p} \times (\vec{v}+\vec{g}) dv - \oint_{S-\sigma} \vec{p} \times \vec{t} da_k - \int_{V-\sigma} \rho \vec{p} \times \vec{f} dv = 0 \quad (3.3)
\]
Conservation of Energy:

\[
\frac{D}{Dt} \int_{V-\sigma} \rho (\varepsilon + \frac{1}{2} v^2) dv - \int_{S-\sigma} (t_k \cdot \nu + q_k) da_k - \int_{V-\sigma} \rho (\nabla \cdot \nu + h) dv = 0 \tag{3.4}
\]

Faraday's Law:

\[
\int_{C-\gamma} \mathbf{E}_k \cdot dx_k + \frac{1}{c} \frac{D}{Dt} \int_{S-\gamma} B_k da_k = 0 \tag{3.5}
\]

Ampere's Law (modified by Maxwell):

\[
\int_{C-\gamma} \mathbf{H}_k \cdot dx_k - \frac{1}{c} \frac{D}{Dt} \int_{S-\gamma} D_k da_k - \frac{1}{c} \int_{S-\gamma} \mathcal{J}_k da_k = 0 \tag{3.6}
\]

Gauss' Law:

\[
\int_{S-\sigma} D_k da_k - \int_{V-\sigma} q dv = 0 \tag{3.7}
\]

Conservation of Magnetic Flux:

\[
\int_{S-\sigma} B_k da_k = 0 \tag{3.8}
\]

Conservation of Charge:

\[
\frac{D}{Dt} \int_{V-\sigma} q dv + \int_{S-\sigma} \mathcal{J}_k da_k = 0 \tag{3.9}
\]
where,

\[ \rho \equiv \text{mass density} \]
\[ g \equiv \text{electromagnetic momentum density} \]
\[ t_k \equiv \text{total stress vector} \]
\[ q_k \equiv \text{total energy flux} \]
\[ h \equiv \text{energy density supply} \]
\[ q \equiv \text{free charge density} \]
\[ p \equiv \text{position vector} \]
\[ \mathbf{v} \equiv \text{velocity vector} \]
\[ f \equiv \text{total body force density} \]
\[ \varepsilon \equiv \text{total internal energy density} \]

and

\[ \mathbf{\dot{E}}_j \equiv \mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B}, \quad \mathbf{\dot{H}}_j \equiv \mathbf{H} - \frac{1}{c} \mathbf{v} \times \mathbf{D}, \quad \mathbf{\dot{J}}_j \equiv \mathbf{J}^f - q\mathbf{v} \quad (3.10) \]

where,

\[ \mathbf{E} \equiv \text{electric field} \]
\[ \mathbf{H} \equiv \text{magnetic field} \]
\[ \mathbf{J}^f \equiv \text{free current density} \]
\[ \mathbf{D} \equiv \text{electric displacement vector} \]
\[ \mathbf{B} \equiv \text{magnetic induction vector} \]
\[ c \equiv \text{speed of light in vacuum}. \]

We also introduce the following relationships for further reference:
The balance laws (3.1) to (3.4) and (3.7) to (3.9) are expressed over a material volume $V$, enclosed by a surface $S$, which is being swept by a discontinuity surface $\sigma$ having velocity $\mathbf{u}$. Integrals (3.5) and (3.6) are taken over an open material surface $\mathcal{S}$, enclosed within a closed curve $\mathcal{C}$, which is being swept by a discontinuity curve $\gamma$ having velocity $\mathbf{u}'$. The electromagnetic laws are expressed in Heaviside-Lorentz units.

Integrals (3.1) to (3.9) are in either of the following forms:

\begin{align*}
\frac{D}{Dt} \int_{V-\sigma} \phi dv - \int_{S-\sigma} r_k da_k \div - \int_{V-\sigma} g dv &= 0 \\
\frac{D}{Dt} \int_{\mathcal{S}-\gamma} q_k da_k - \int_{\mathcal{C}-\gamma} h_k dx_k \div - \int_{\mathcal{S}-\gamma} r_k da_k &= 0
\end{align*}

which, upon using the generalized Green-Gauss theorem and Stoke's theorem respectively, (Eringen [34] p. 77), may be written as

\begin{align*}
D = E + \mathcal{P} & \quad \mathcal{B} = H + M \\
\mathcal{D} = D + \frac{1}{c} \gamma \times H & \quad \mathcal{E} = D - \frac{1}{c} \gamma \times E \\
\mathcal{P} = P - \frac{1}{c} \gamma \times M & \quad \mathcal{M} = M + \frac{1}{c} \gamma \times P
\end{align*}

where,

$\mathcal{P}$ $\equiv$ polarization vector

$\mathcal{M}$ $\equiv$ magnetization vector.
\[ \int_{V - o} \left[ \frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \nu) - \tau_{k,k} - g \right] dv + \int_{o} \left[ \phi(v - u_{k}) - \tau_{k} \right] n_{k} da = 0, \quad (3.14) \]

\[ \int_{\gamma} (q - \nu \times \rho) \cdot da + \int_{\gamma} (q \times (\nu' - \nu) - \nu') \cdot k ds = 0 \quad (3.15) \]

where \( k \) is the unit tangent vector of \( \gamma \) and \( n \) is the exterior unit normal vector to \( o \). The bold face brackets indicate the jump across \( o \) of the appropriate material properties in (3.14) and in (3.15) and is of the form

\[ [A] = A^{+} - A^{-} \quad (3.16) \]

where \( A^{+} \) and \( A^{-} \) indicate, respectively, the values of \( A \), as the discontinuity surface is approached from positive and negative sides of \( n \) of \( o \). Also, we have defined the convective derivative

\[ \tilde{q} \equiv \frac{\partial q}{\partial t} + (\nu \cdot q)\nu + \nabla x (q \times \nu). \quad (3.17) \]

Using (3.14) and (3.15), (3.1) through (3.9) are transformed in cartesian tensor notation

\[ \int_{V - o} \left[ \frac{\partial p}{\partial t} + (\rho \nu_{k}) \right] dv + \int_{o} (\rho(v - u_{k})) n_{k} da = 0 \quad (3.18) \]
\[ \int_{V-\sigma} \left[ \rho (\dot{v} + \dot{g}) - t_{k}, k - \rho \xi \right] d\nu + \int_{V-\sigma} (v + g) \left[ \frac{\partial \rho}{\partial t} + (\rho v_k), k \right] d\nu \]

(3.19)

\[ + \int_{\sigma} I(v + g)(v_k - u_k) - t_k \ln k da = 0 \]

\[ \int_{V-\sigma} \left( \rho v \times g - p, k \times t_k \right) d\nu + \int_{V-\sigma} p \times (v + g) \left[ \frac{\partial p}{\partial t} + (\rho v_k), k \right] d\nu \]

(3.20)

\[ - \int_{V-\sigma} p \times (t, k, k + \rho \xi - \rho \dot{v} - \rho \dot{g}) d\nu + \int_{\sigma} \rho p \times (v + g)(v_k - u_k) - p p \times t_k \ln k da = 0 \]

\[ \int_{V-\sigma} \left( \rho \dot{v} - t_k \cdot v, k - q_k, k - \rho \xi \cdot v - \rho h \right) d\nu + \int_{V-\sigma} \left( \varepsilon + \frac{1}{2} v^2 \right) \left[ \frac{\partial p}{\partial t} + (\rho v_k), k \right] d\nu \]

(3.21)

\[ - \int_{V-\sigma} \dot{v} \cdot (t, k, k + \rho \xi - \rho \dot{v} - \rho \dot{g}) d\nu + \int_{\sigma} \rho (\varepsilon + \frac{1}{2} v^2)(v_k - u_k) - t_k \cdot v - q_k \ln k da = 0 \]

\[ \int_{\mathcal{G} - \gamma} \left( v \times \xi + \frac{1}{c} \overline{v} \right) \cdot da + \int_{\gamma} \left[ \overline{\mathcal{E}} + \frac{1}{c} \mathcal{U} \times \overline{\mathcal{E}} \right] \cdot kds = 0 \]

(3.22)

\[ \int_{\mathcal{G} - \gamma} \left( v \times \gamma + \frac{1}{c} \overline{\gamma} - \frac{1}{c} \overline{\dot{v}} - \frac{1}{c} \overline{\dot{g}} \right) \cdot da + \int_{\gamma} \left[ \mathcal{H} - \frac{1}{c} \mathcal{U} \times \mathcal{D} \right] \cdot kds = 0 \]

(3.23)

\[ \int_{S-\sigma} (v \cdot D - q) d\nu + \int_{\sigma} |D| \cdot n da = 0 \]

(3.24)

\[ \int_{S-\sigma} v \cdot Bd\nu + \int_{\sigma} |B| \cdot n da = 0 \]

(3.25)
It is readily seen that the classical localized balance laws for electro-mechanical media could be obtained by assuming the above balance laws to be valid for every part of the material volume \( V \). This assumption simply amounts to setting each of the integrands in (3.18) through (3.26) equal to zero.

As was pointed out in Chapter I the nonlocal theory rejects the above assumption and hence takes into account long-range effects due to forces and interatomic interactions. Following the general procedure given in I. 2 we localize the nonlocal global balance laws by introducing the localization residuals. Thus, for nonlocal electro-mechanical media the localized balance laws, as derived from (3.18) to (3.26), are given by:

Conservation of Mass:

\[
\frac{\partial \rho}{\partial t} + (\rho v_k)_k = \hat{\rho}, \quad \text{in } V - \sigma \\
\left[\rho(v_k - u_k) - \hat{\rho}\right]_{n_k} = 0, \quad \text{on } \sigma
\]  

Balance of Linear Momentum:

\[
t_{k\ell},_k + \rho(f_k - \dot{v}_k - \dot{g}_k) = \hat{\rho}(v_k + g_k) - \rho \hat{f}_k, \quad \text{in } V - \sigma \\
\left[t_{k\ell} - \rho(v_k + g_k)(v_k - u_k) + \hat{f}_k\right]_{n_k} = 0, \quad \text{on } \sigma
\]
Balance of Moment of Momentum:

\[ e_{kmn}^p m, \ell \ell n - \rho e_{kmn}^p m g_n = \rho e_{kmn}^p m f_n - \rho f_k, \quad \text{in } V - \sigma \]

\[ [\rho e_{\ell mn}^p m]_n (v_k - u_k) + \hat{f}_k l_n = 0, \quad \text{on } \sigma \]  

(3.29)

Conservation of Energy:

\[ \rho \dot{c} - \rho g_k v_k - t_{\ell \ell} v_k, k - q_k, k - \rho h = \rho \dot{h} - \rho v_k \hat{f}_k \]

\[ - \hat{f}(\epsilon + \frac{1}{2} v^2 - v_g k_k), \quad \text{in } V - \sigma \]  

(3.30)

\[ [t_{\ell \ell} v_k + q_k - \rho (\epsilon + \frac{1}{2} v^2) (v_k - u_k) + \hat{h}_k l_n] = 0, \quad \text{on } \sigma \]

Faraday's Law:

\[ e_{k\ell m} c m, \ell + \frac{1}{c} B_k = \frac{1}{c} b_k, \quad \text{on } \sigma - \gamma \]  

(3.31)

\[ e_{k\ell m} [E_k + \frac{1}{c} e_{krn} u^r B + E_k l_n] = 0, \quad \text{on } \gamma \]

Ampere's Law:

\[ e_{k\ell m} \nabla m, \ell - \frac{1}{c} D_k - \frac{1}{c} \hat{\phi}_k = \frac{1}{c} \hat{\phi}_k, \quad \text{on } \sigma - \gamma \]  

(3.32)

\[ e_{k\ell m} [H_k - \frac{1}{c} e_{krn} u^r D + H_k l_m] = 0, \quad \text{on } \gamma \]

Gauss' Law:

\[ D_k, k - q = \hat{q}, \quad \text{in } V - \sigma \]

\[ [D_k + \hat{D}_k] l_n = 0, \quad \text{on } \sigma \]  

(3.33)
Conservation of Magnetic Flux:

\[ B_{k,k} = \hat{m}, \quad \text{in } V-\sigma \]

(3.34)

\[ [B_{k,k} + \hat{E}_{k}]_{n_k} = 0, \quad \text{on } \sigma \]

Conservation of Charge:

\[ \mathcal{J}_{k,k} + \frac{\partial q}{\partial t} + (q_{\nu,k})_{k} = \hat{\sigma}, \quad \text{in } V-\sigma \]

(3.35)

\[ \int \mathcal{J}_{k,k} + \hat{\Sigma}_{k} \mathcal{N}_{k} = 0, \quad \text{on } \sigma \]

where \( e_{klm} \) is the permutation tensor defined as (a) zero when at least two of the indices are equal; (b) 1 if the sequence of \( k, l, m \) is the sequence 1, 2, 3 or an even permutation of the sequence; and (c) -1, if the sequence \( k, l, m \) is an odd permutation of the sequence 1, 2, 3.

The nonlocal residuals \( \hat{\rho}, \hat{\rho}_{k}, \hat{f}_{k}, \hat{f}_{kl}, \hat{f}_{k}, \hat{h}, \hat{h}_{k}, \hat{b}_{k}, \hat{E}_{k}, \hat{H}_{k}, \hat{q}, \hat{D}_{k}, \hat{B}_{k}, \hat{m}, \hat{\sigma}, \) and \( \hat{\Sigma}_{k} \) being introduced to account for the effects of fields at all other points of the body on the point for which the localized balance laws (3.27) to (3.35) are written. We require that the integrals of the residuals, defined over their manifold of definition to vanish, that is, we must have
Introducing the nonlocal current \( \hat{J} \) by

\[
\hat{\sigma} = \hat{J} - \hat{q}_V,
\]

(3.37)

taking the divergence of (3.32) and comparing the result with (3.35) yields

\[
\hat{\sigma} = -\nabla \cdot \hat{J} - \frac{\partial \hat{q}}{\partial t}.
\]

(3.38)

Thus we may rewrite (3.35) as

\[
\nabla \cdot (\hat{J} + \hat{f}) + \frac{\partial}{\partial t} (q + \hat{q}) = 0,
\]

(3.39)

which is the expression for the law of conservation of nonlocal charge [28]. Similarly, taking the divergence of (3.31) and comparing the result with (3.34) yields

\[
\frac{\partial \hat{m}}{\partial t} + \nabla \cdot (\hat{m} \hat{v}) = \nabla \cdot \hat{b},
\]

(3.40)
which is the expression of the law of the balance of nonlocal pole strength.

The physical significance of the localization residuals can be interpreted from the equations in which they appear. Eringen and Edelen [39] and Eringen [35] discuss the physical significance of the volume and surface residuals arising from purely mechanical sources. For example, \( \hat{\rho} \) is the mass residual which represents the rate at which mass is created or destroyed at the point \( x \) due to the effects of all other material points occupying \( V - \sigma \). Such phenomena could occur either through chemical reactions, through the existence of quasi-particles (e.g. electrons, excitons, phonons) or through dissociation, ionization, and fracture of the subelements of the body [35]. We may interpret \( \hat{f}_k \) as the nonlocal body force at the point \( x \) due to the long-range intermolecular forces produced by all other points of the body (e.g. gravitational, electromagnetic attractions).

The other volume residuals have similar interpretations. The surface residual \( \hat{\rho}_k \) is associated with the production or destruction of mass in crossing the surface \( \sigma \). Similar interpretations are valid for the other surface residuals. The electromagnetic residuals were introduced by Eringen [28]. Here \( \hat{q} \) is the induced nonlocal charge at the point \( x \) due to the charges throughout the body at all other points. Similarly, \( \hat{m} \) (if it exists) is the magnetic pole strength induced at \( x \) by the rest of the body, which is related to \( \hat{b} \), the nonlocal
magnetic induction through (3.40). Again, the remaining electromagnetic residuals have similar physical interpretations.

The distinguishing feature of the nonlocal continuum theory is the appearance of the localization residuals in the localized balance laws. The determination of these residuals is an integral part of the nonlocal theory. Clearly the nature of such residuals depends heavily on the constitutive make-up of the material under consideration. Lastly we note that upon setting all of the nonlocal residuals equal to zero the classical localized balance laws of electromechanical continua are obtained.

4. Second Law of Thermodynamics

We shall state the fundamental thermodynamic law in the form of a generalized Clausius-Duhem inequality. It is assumed that this inequality has the form [45]:

\[
\frac{D}{Dt} \int_{V} \rho \eta dv + \oint_{S} \frac{1}{\theta} (\mathbf{q} \cdot \mathbf{E} \times \mathbf{H}) \cdot d\mathbf{a} - \int_{V} \frac{1}{\theta} (\rho \mathbf{h} + \mathbf{J}_e \cdot \mathbf{E}) dv \geq 0. 
\]

(4.1)

Here \( \eta \) is the entropy density, \( \theta > 0 \) is the absolute temperature, \( c \mathbf{E} \times \mathbf{H} \) is the Poynting vector when the body is viewed in the rest frame, and \( \mathbf{J}_e \) is the external current source. Physically, we are assuming that the entropy flux is equal to the total energy flux minus the Poynting vector divided by the temperature \( \theta. \)
Localizing (4.1) as we did in the previous section yields

$$\rho \dot{\eta} + \nabla \cdot \left[ \frac{1}{\theta} (q - c \bar{\rho} \times \bar{\eta}) \right] - \frac{1}{\theta} (\rho \dot{\eta} + \bar{\rho} \cdot \bar{\eta}) + \dot{\rho} \eta - \frac{1}{\theta} \rho \dot{\hat{s}} \geq 0, \text{ in } V - \sigma \quad (4.2)$$

$$[\rho \eta (v_k - u_k) - \frac{1}{\theta} (q - c \bar{\rho} \times \bar{\eta})] \cdot \hat{s}_k \ln k \geq 0, \text{ on } \sigma$$

where $\hat{s}_k$ are entropy localization residuals subject to

$$\int_{V - \sigma} \frac{1}{\theta} \rho \hat{s} \, dv = 0, \quad \int_{\sigma} \hat{s}_k \, d a_k = 0. \quad (4.3)$$

The second term in (4.2) can be simplified by using (3.31)\textsubscript{1} and (3.32)\textsubscript{1} yielding

$$cv \cdot (\bar{\rho} \times \bar{\eta}) = - \bar{\eta} \cdot \hat{s} - \bar{\rho} \cdot \hat{\sigma} - \bar{\rho} \cdot \hat{\sigma} - \hat{\sigma} \cdot \hat{s}.$$ \quad (4.4)

Substituting (4.4) into (4.2)\textsubscript{1} and eliminating $h$ between (4.2)\textsubscript{1} and (3.30)\textsubscript{1} gives

$$\rho (\dot{\eta} - \frac{\nu \hat{g}}{\theta} + \frac{1}{\theta} t_k \cdot v_k) + (q - c \bar{\rho} \times \bar{\eta}) \cdot \nabla (\frac{1}{\theta}) + \frac{1}{\theta} (\bar{\rho} \cdot \hat{\sigma} + \bar{\rho} \cdot \hat{s}) \hat{s} + \bar{\rho} \hat{\sigma} \cdot \hat{s}.$$ \quad (4.5)

which becomes the generalized Clausius-Duhem inequality. Introducing
\[ \psi = \varepsilon - \theta \eta, \quad \bar{\psi} = \psi - \nabla \cdot \mathbf{g} \]
\[ \bar{\mathbf{q}} = \mathbf{q} - \mathbf{q}_0, \quad \mathbf{Q} = q - \mathbf{c} \times \mathbf{H} \]

where \( \psi \) is the Helmholtz free-energy, (4.5) may be written as

\[ - \frac{\rho}{\theta} (\dot{\psi} + \dot{\mathbf{v}} \cdot \mathbf{g} + \dot{\theta} \eta) + \frac{1}{\theta} \varepsilon_k \cdot \mathbf{v}_k + \mathbf{Q} \cdot \nabla (\frac{1}{\theta}) + \frac{1}{\theta} (\varepsilon \cdot \dot{\mathbf{D}} + \mathbf{H} \cdot \dot{\mathbf{B}}) - \frac{\rho}{\theta} \hat{\mathbf{H}} \cdot \mathbf{v} \]
\[ - \frac{\rho}{\theta} (\dot{\mathbf{v}} + \varepsilon_k \cdot \mathbf{v}_k) + \frac{1}{\theta} (\varepsilon \cdot \dot{\mathbf{D}} + \mathbf{H} \cdot \dot{\mathbf{B}}) + \frac{\rho}{\theta} (\hat{\mathbf{H}} - \dot{\mathbf{v}}) \geq 0, \quad (4.7) \]

in \( V - \sigma \).

The balance laws, including the generalized Clausius-Duhem inequality must be satisfied by all materials regardless of their nature, that is, fluids, solids, etc. Thus one needs a different mechanism to incorporate material characterization and thereby bring out the various types of responses experienced by different types of materials. Such a mechanism consists of constructing appropriate theories in order to describe the constitutive make-up of a material, which we shall develop in the subsequent chapter. It is the constitutive make-up of the media along with the balance laws that dictate the nature and response of the material.
In this chapter we develop the general constitutive theory for nonlocal electromagnetic fluids based on the previous works of Eringen [23, 28]. To this end, we designate a set of variables as forming a cause set to represent the constitutive functionals which remain invariant under Galilean type transformations. Upon developing a Hilbert space setting for the nonlocal variables the thermodynamic admissibility of the constitutive equations is analyzed. In order to facilitate practical applications of our theory we next develop the linear constitutive theory, a major feature of which is the derivation of the full thermodynamic restrictions on the material moduli. We conclude the chapter with the development of the appropriate field equations and boundary conditions for nonlocal electromagnetic fluids (for the linear constitutive theory).

2. Development of Constitutive Functions and the Pertinent Thermodynamics

In order to formulate a constitutive theory for a general class of nonlocal electromagnetic fluids an appropriate pair of independent electromagnetic quantities must be chosen as constitutive variables. Although convenient choices exist for particular types of
electromagnetic materials (e.g. dielectrics, conductors, magnetic materials), for a general development no one pair is totally advantageous over another. For a \( \frac{v^2}{c^2} \) approximation to the Lorentz theory it is convenient to choose \((\mathcal{E}, \mathcal{D})\), defined earlier by (II. 3.11), as our independent electromagnetic quantities to appear in the constitutive theory.

In nonlocal theory it is convenient to designate \( x \) and \( \tilde{x} \) (Lagrangian and Eulerian descriptions of points) as local points of interest and \( x' \) and \( \tilde{x}' \) to denote the surrounding points in the material, which form the source of nonlocal influences. For all such points \( x' \) an appropriate set of electromechanical measures for a general class of nonlocal electromagnetic materials is:

\[
\begin{align*}
\mathcal{E}'(x') &= x'(x', t) - x(x, t) = x' - x \\
\mathcal{C}_{KL}(x') &= \tilde{x}' - \tilde{x} = \frac{\partial x'}{\partial x_{\tilde{K}}} - \frac{\partial x}{\partial x_{\tilde{K}}} \\
\mathcal{C}_{KL}(x) &= x_{\tilde{K}} \tilde{x}_{\tilde{L}} \\
\Gamma'(x') &= \mathcal{E}'(x', t) - \mathcal{E}(x, t) \\
\Pi'(x') &= \mathcal{D}'(x', t) - \mathcal{D}(x, t)
\end{align*}
\]

where in (2.1) a slight modification to the nonlocal measures introduced by Eringen [23] and [28], has been made.
Since Stokesian fluids are materials which accept every frame of reference leaving density unchanged as a reference frame we can choose $X \rightarrow x; \ X' \rightarrow x'$ with $\rho$ fixed. Consequently, for fluids, which have such instantaneous configurations we replace the above measures by their material time rates. Thus we have

$$
\dot{x}' = \dot{x} - \dot{x} \\
\dot{x}'_k = \dot{0} \\
\dot{\eta}_k = \dot{x}'_k - \dot{x}_k \\
\dot{C}_{KL} = \dot{0} \\
\dot{\xi}' = \dot{\xi} - \dot{\xi} \\
\dot{\eta}' = \dot{\eta} - \dot{\eta} \\
\dot{\Pi}' = \dot{\Pi} - \dot{\Pi}
$$

where $\delta_{k\ell}$ is the Kronecker delta and for brevity we use

$\dot{\imath}' \equiv \dot{\imath}(x', t), \ \dot{\mathcal{C}}' \equiv \dot{\mathcal{C}}(x', t)$ etc.

We now define our class of nonlocal electromagnetic fluids by the constitutive equations of the form

$$
\overline{\psi}(x, t) = \psi[x^{-1}(x'), \beta_k(x'), \gamma_{k\ell}(x'), \Gamma_k(x', t), \hat{\Gamma}_k(x', t), \Pi_k(x', t), d_{kl}, \mathcal{B}_k, \mathcal{B}_k, \mathcal{G}_k, \mathcal{G}_k, \rho^{-1}, \theta]
$$

(2.3)

where $\psi$ is a scalar valued function of $d_{kl}, \mathcal{B}_k, \mathcal{B}_k, \mathcal{G}_k, \mathcal{G}_k, \rho^{-1}, \theta,$ and a functional of the nonlocal variables $\beta_k, \gamma_{k\ell}, \Gamma_k, \hat{\Gamma}_k.$
All of the nonlocal variables are defined over all points \( x' \) of the body.

For convenience we define the following ordered sets:

\[
\mathcal{F}' = \{ r^{-1}(x'), \rho(x'), \chi(x'), \Gamma(x', t), \Gamma'(x', t), \Pi(x', t), \Pi'(x', t) \} \tag{2.5}
\]

\[
\mathcal{F} = \{ \mathcal{d}, \mathcal{g}, \mathcal{g}', \mathcal{g}, \mathcal{f}, \rho^{-1}, \theta \}
\]

which divides up the cause set into, respectively, nonlocal and local sets. Using this notation the free-energy constitution may be expressed as

\[
\psi = \psi(\mathcal{F}', \mathcal{F}). \tag{2.6}
\]

The other response functionals, such as the stress \( \tau \), the total energy flux \( q_k \), the total internal energy density \( \varepsilon \), the entropy \( \eta \), the electric and magnetic fields \( E \) and \( H \), and the conduction current \( J^c \equiv J^f - \mathcal{Q}^0 \) are also assumed to be of the form given by (2.6).

We next subject the constitutive functionals to certain invariance requirements, namely the axiom of objectivity, also known as material frame indifference. One usually demands that the constitutive functionals be form invariant under the changes of reference frames defined by the transformation

\[
\Pi_k' \quad \text{and} \quad \Pi_k \quad \text{where}
\]

\[
r^{-1}(x') \equiv \rho^{-1}(x') - \rho^{-1}(x). \tag{2.4}
\]
where \( t^+ = t - a \) for some constant \( a \), and where \( Q \) is an orthogonal transformation, that is,

\[
Q Q^T = Q^T Q = I
\]

(2.8)

where \( \det Q = \pm 1 \) and \( Q^T \) denotes the transpose of \( Q \). Electromagnetic quantities, however, do not obey the principle of objectivity as dictated by (2.7). In fact there is no firm agreement as to how electromagnetic quantities should transform under Euclidean transformations to accelerating or rotating frames [45]. The correct theory to apply is the Lorentz invariant theory which, of course, incorporates relativistic effects. Since we are ignoring relativistic effects, a \( v^2/c^2 \) approximation to Lorentz invariant theories shall be utilized. To this end we consider the group of Galilean transformations

\[
x^+ = Qx + v_o t + b_o
\]

(2.9)

where \( Q, v_o, \) and \( b_o \) do not depend on time. We now require that under the Galilean type of frame change given by (2.9) that \( B \) and \( D \) transform in such a way that

\[
B^+ = QB \quad \text{and} \quad D^+ = QD
\]

(2.10)
where \( \mathcal{O} \) and \( \mathcal{Q} \) are given by (II. 3. 11).

With \( \mathcal{O} \) and \( \mathcal{Q} \) transforming as in (2.10) under (2.9) all of the variables in \( \mathcal{F}' \) and \( \mathcal{F} \) given in (2.5) are invariant under the rigid motions defined by (2.9). A typical example to illustrate the objectivity is as follows.

In component notation (2.9) is written as

\[
x^+_k = Q_{k\ell} x^\ell + v^o_k t + b^o_k .
\]

To check the objectivity of the vector \( \beta \) we must show

\[
\beta^+_k = Q_{kn} \beta_n .
\]

From (2.2)_3 we have

\[
\beta^+_k = v^+_k (x'^+_r - x^+_r) + v^+_k - v_k ,
\]

which on applying (2.9) becomes

\[
\beta^+_k = Q_{\ell m} Q_{kn} v^+_m + Q_{r m} (x'^+_r - x^+_r) + Q_{km} v^+_m + v^+_o - Q_{kn} v_m - v^+_o .
\]

Using (2.8) yields

\[
\beta^+_k = Q_{kn} v^+_m (x'^+_m - x^+_m) + Q_{km} (v^+_m - v^+_m)
= Q_{kn} [v^+_m (x'^+_m - x^+_m) + v'_+ - v'_-]
= Q_{kn} \beta^+_n .
\]
Hence \( \vec{p} \) is an objective vector. Similarly, the objectivity of the second rank tensor \( \gamma_{kl} \), namely \( \gamma^+ = Q\gamma Q^T \) can be shown by an analogous procedure.

The objectivity of the electromagnetic quantities follow directly from the invariance of \( \mathcal{B} \) and \( \mathcal{D} \) under (2.9). The nonlocal electromagnetic terms are invariant due to their linear nature. The objectivity of the electromagnetic variables in (2.3) is straightforward and hence we omit the computations.

Since the localization residuals represent the mutual interactions of distant points with the local point \( x \), these interactions remain unaltered when the body is given arbitrary rigid motions. Thus we assume the localization residuals to be objective under the class of Galilean transformations defined by (2.9).

Finally we note that the constitutive functional \( \psi \) is form-invariant under (2.9), that is,

\[
\psi[r^{-1}, \vec{p}, \gamma, \Gamma, \Pi, \vec{q}; \mathcal{D}, \mathcal{E}, \mathcal{G}, \mathcal{H}, \rho^{-1}, \theta] = \psi[r^{-1}, \mathcal{Q}\gamma \mathcal{Q}^T, \mathcal{Q}\Gamma, \mathcal{Q}\Pi, \mathcal{Q}\vec{q}; \mathcal{Q}\mathcal{D}\mathcal{Q}^T, \mathcal{Q}\mathcal{E}, \mathcal{Q}\mathcal{G}, \mathcal{Q}\mathcal{H}, \rho^{-1}, \theta].
\]

At \( x' = x \) we have \( \mathcal{F}' = \{O\} \), that is all of the nonlocal variables in the cause set vanish. Without loss of generality we set

\[
\psi(\{O\}, \mathcal{F}) = \psi_0
\]
as the free-energy of local electromagnetic fluids.

The nonlocal mechanical variables bring out some important physical phenomena. For example, \( r^{-1}(\mathbf{x}') \) has been shown to include surface effects such as surface tension, surface viscosities, and surface energy [23]. The nonlocal deformation measures \( \beta(x') \) and \( \gamma(x') \) contain relative rotational motions, that is,

\[
\beta_{k}(x') = (x'_k - x_k)(d_{k}^{\ell} + w_{k}^{\ell}) + v_k' - v_k
\]

\[
\gamma_{k\ell}(x') = d_{k\ell}^{\ell} - d_{k\ell}^{\ell} + w_{k\ell}^{\ell} - w_{k\ell}
\]

where

\[
w_{k\ell} \equiv \frac{1}{2} (v_k, \ell - v_\ell, k)
\]

is the classical spin tensor which does not appear in the classical constitutive equation.

We now investigate the consequences of the thermodynamic admissibility. To achieve this the material time rate of the functional \( \psi \) in (2.3), with respect to all of its variables, must be investigated. We assume first that all arguments of \( \psi \) are continuously differentiable with respect to \( x', x, \) and \( t \). We also assume that \( \psi \) has continuous Fréchet partial derivatives of order zero with respect to all nonlocal functions appearing in \( \mathcal{F}' \) as well as continuous first-order partial derivatives with respect to all the arguments in \( \mathcal{F} \).
In order to bring out the smoothness requirements explicitly we must introduce a function space whose properties are shared by the elements of $\mathcal{F}'$. Though a Banach space structure could be used it is more natural from a physical point of view to introduce a Hilbert space structure on $\mathcal{F}'$. To this end, for any two such sets of functions $\mathcal{F}'_1$ and $\mathcal{F}'_2$ we define the inner product as

$$
(\mathcal{F}'_1, \mathcal{F}'_2)_{\overline{H}} = \int_{V-\sigma} \overline{H} (\|x' - x\|) \mathcal{F}'_1 \cdot \mathcal{F}'_2 \, dv(x')
$$

(2.17)

and say $\mathcal{F}'_1, \mathcal{F}'_2 \in \mathcal{H}$, the Hilbert space defined by (2.17). The inner product $\mathcal{F}'_1 \cdot \mathcal{F}'_2$ is defined by

$$
\mathcal{F}'_1 \cdot \mathcal{F}'_2 = r_1^{-1} r_2^{-1} + \beta_1 \beta_2 + \cdots + \Pi_1 \Pi_2
$$

(2.18)

while the norm of $\mathcal{F}' \in \mathcal{H}$ is given by

$$
\| \mathcal{F}' \|_{\overline{H}} = (\mathcal{F}' , \mathcal{F}')_{\overline{H}}^{1/2}
$$

(2.19)

The influence function $\overline{\Pi} (\|x' - x\|)$ is a positive decreasing function of $\|x' - x\|$ such that $\overline{\Pi}(0) = 1$. With the influence function introduced as above the axiom of attenuating neighborhoods given in [36] is introduced in a natural fashion. This axiom is a strong continuity requirement arising from the fact that nonlocal effects diminish rapidly as $\|x' - x\| \to \infty$. Thus $\overline{H}$ is defined so as to favor the influence of points $x'$ near $x$ and deemphasize the
effects of points distant from $x$. For example, we may take $\overline{H}$ to be of the form:

$$\overline{H}(\|x' - x\|) = \exp \left[ - \alpha(x) \|x' - x\|^n \right], \quad n > 0, \alpha > 0. \tag{2.20}$$

The influence function $\overline{H}$ may be thought of as belonging to the class of rapidly decreasing functions defined over $\mathbb{R}^3$. Other forms of the influence function may be motivated by physical considerations, such as lattice structures.

We denote the Fréchet derivative of $\psi$ with respect to $x'$ as $\delta \psi(x'|\Omega')$ and define it as

$$\lim_{\|\Omega'\|_H \to 0} \left| \psi(x' + \Omega') - \psi(x') - \delta \psi(x'|\Omega') \right| = 0 \tag{2.21}$$

for $x', \Omega' \in \overline{H}$. Assuming $\psi$ to be Fréchet derivative to be bounded we may apply the following representation due to F. Riesz, (cf. Tapia [65], p. 79)

$$\delta \psi(x'|\Omega) = \int_{V - \partial x'} \frac{\delta \psi}{\delta x'} (x', \lambda) \cdot \Omega'(\lambda) \, dv(\lambda) \tag{2.22}$$

where $\frac{\delta \psi}{\delta x'}$ is called the gradient of $\psi$ at $x'$. We assume the above relations to be true for all the nonlocal variables in $x'$. 
We shall consider nonlocal electromagnetic fluids for which the mass production and heat conduction are negligible, that is, we set

\[ \dot{\rho} = 0, \quad \mathcal{Q} = 0 \quad (2.23) \]

so that the entropy inequality (II. 4.7) may be written as

\[
- \frac{\rho}{\theta} (\dot{\psi} + \dot{\psi} \cdot \partial \eta) + \frac{1}{\theta} t_k \cdot v_{,k} + \frac{1}{\theta} (E \cdot \partial \Phi + H \cdot \partial \Phi) + \frac{1}{\theta} E \cdot \partial v_{,k}
\]

\[
- \frac{\rho}{\theta} \dot{\psi} \cdot v + \frac{1}{\theta} (E \cdot \dot{\Phi} - H \cdot \dot{\Phi}) + \frac{\rho}{\theta} (\dot{h} - \dot{s}) \geq 0, \quad \text{in } V - \sigma. \quad (2.24)
\]

Using (2.3) and (2.22) we now calculate \( \dot{\psi} \). This yields

\[
\dot{\psi} = \frac{\partial \psi}{\partial d_{kl}} d_{kl} + \frac{\partial \psi}{\partial G_k} \dot{G}_k + \frac{\partial \psi}{\partial \xi_k} \dot{\xi}_k + \frac{\partial \psi}{\partial \xi} \dot{\xi}
\]

\[
+ \frac{\partial \psi}{\partial \rho^{-1}} \rho^{-1} + \frac{\partial \psi}{\partial \theta} \dot{\theta} + \int_{V - \sigma} \frac{\delta \psi}{\delta r^{-1}} (\mathcal{F}', \mathcal{F} ; \lambda) r^{-1}(\lambda) \text{dv}(\lambda)
\]

\[
+ \int_{V - \sigma} \left[ \frac{\delta \psi}{\delta \beta_k} (\mathcal{F}', \mathcal{F} ; \lambda) \dot{\beta}_k(\lambda) + \frac{\delta \psi}{\delta \gamma_{kl}} (\mathcal{F}', \mathcal{F} ; \lambda) \dot{\gamma}_{kl}(\lambda) \right] \text{dv}(\lambda)
\]

\[
+ \int_{V - \sigma} \left[ \frac{\delta \psi}{\delta \Gamma_k} (\mathcal{F}', \mathcal{F} ; \lambda) \dot{\Gamma}_k(\lambda) + \ldots + \frac{\delta \psi}{\delta \Pi_k} (\mathcal{F}', \mathcal{F} ; \lambda) \dot{\Pi}_k(\lambda) \right] \text{dv}(\lambda).
\]

Using the equation of continuity (II. 3.27) with \( \dot{\rho} = 0 \) we find that

\[
\rho^{-1} = -\rho \beta \rho^{-1} v_{,k}, \quad k. \quad (2.26)
\]
Also, we may write

\[
\frac{\partial \psi}{\partial d_{k\ell}} \dot{d}_{k\ell} = \frac{1}{2} \left( \frac{\partial \psi}{\partial d_{k\ell}} + \frac{\partial \psi}{\partial d_{\ell k}} \right) \dot{v}_{k, \ell}. \tag{2.27}
\]

Using (2.27) along with (2.26), (II. 3.10)\textsuperscript{3}, and (II. 3.37) in (2.25) and then substituting into the entropy inequality (2.24) we have

\[
- \frac{\rho}{\theta} \left( \eta + \frac{\partial \psi}{\partial \theta} \right) \dot{\theta} + \frac{\rho}{\theta} \left[ \dot{v} - \frac{1}{\rho} \mathcal{E}_k (q + \dot{q}) + \int \frac{\delta \psi}{\delta \beta} v'_{k, \ell} \right]
\]

\[
- \frac{1}{\rho c} e_{klm} \left( \dot{E}_l H_m + \dot{E}_m H_l \right) v_{k} - \frac{\rho}{\theta} (g_k - \int \frac{\delta \psi}{\delta \beta} ) \dot{v}'_{k} + \frac{1}{\theta} \left[ t_{kl} - D_k e_{l} - B_k \dot{\mathcal{H}}_k - \delta_{k \ell} \left( \frac{\partial \psi}{\partial \rho} - 1 \right) - \mathcal{E}_m D_m - \mathcal{H}_m B_m - \int \frac{\delta \psi}{\delta r} \right] v_{\ell, k}
\]

\[
- \frac{\rho}{\theta} \left[ \frac{1}{2} \left( \frac{\partial \psi}{\partial d_{k\ell}} + \frac{\partial \psi}{\partial d_{\ell k}} \right) - \int \frac{\delta \psi}{\delta \Pi_{k\ell}} \right] v_{k, \ell} - \frac{\rho}{\theta} \left( \frac{\partial \psi}{\partial \mathcal{E}_k} - \int \frac{\delta \psi}{\delta \mathcal{H}_k} - \frac{1}{\rho} \mathcal{H}_k \right) \dot{\mathcal{E}}_k
\]

\[
- \frac{\rho}{\theta} \left( \frac{\partial \psi}{\partial \mathcal{E}_k} - \int \frac{\delta \psi}{\delta \h_k} \right) \dot{\mathcal{H}}_k - \frac{\rho}{\theta} \dot{\mathcal{H}} + \frac{1}{\theta} \mathcal{E}_k (J^f_k - \mathcal{H}_k)\]

\[
+ \frac{1}{\theta} \left( \mathcal{E}_k \dot{\mathcal{H}}_k - \mathcal{H}_k \dot{\mathcal{E}}_k \right) + \frac{\rho}{\theta} (h - \dot{\mathcal{H}}) + O(v^2 / c^2) \geq 0, \text{ in } V \cdot \mathcal{E}
\]

where
\[
\hat{D}^h = \int_{V-\sigma} \left[ \frac{\delta \psi}{\delta \Gamma} v_i - \frac{1}{\rho} \delta v_i - k + \frac{\delta v_i}{\delta \Gamma} \left( \frac{\delta v_i}{\delta \Gamma} + \frac{\delta v_i}{\delta \xi} \right) + v_i \delta v_i + \frac{\delta v_i}{\delta \gamma_{kl}} \right] dv^i
\]  
\tag{2.29}

and for brevity we have written

\[\int \frac{\delta \psi}{\delta \beta_k} = \int_{V-\sigma} \frac{\delta \psi}{\delta \beta_k} (\xi', \xi; \lambda) dv(\lambda), \text{ etc.}\]

Ignoring the terms of \(O(v^2/c^2)\) the inequality (2.28) is linear in \(\theta, v_k, v'_k, \dot{v}_k, \ddot{v}_k, \dot{\theta}_k, \ddot{\theta}_k, \text{ and } \dddot{\theta}_k\). If \(\Delta_k^f, \Delta_k^b, \Delta_k^h, \text{ and } \Delta_k^s\) are independent of these linearly independent quantities then (2.28) cannot be maintained for all possible variations of these quantities unless

\[
\eta = -\frac{\partial \psi}{\partial \theta}, \quad \hat{\Delta}_k^f = \int_{V-\sigma} \left[ \frac{\delta \psi}{\delta \beta_k} v_i - \frac{1}{\rho} \delta v_i - k + 1 \right] E_k (q + \dot{q}) - \frac{1}{\rho} E_{kl} (E_k \hat{H} + E_{kl} \hat{H}) dv^i
\]

\[
g_k = \int_{V-\sigma} \frac{\delta \psi}{\delta \beta_k} dv^i, \quad \frac{1}{2} \left( \frac{\partial \psi}{\partial \beta_k} + \frac{\partial \psi}{\partial \beta_k} \right) = \int_{V-\sigma} \frac{\delta \psi}{\delta \gamma_{kl}} dv^i
\]  
\tag{2.30}

\[
\gamma_k = \rho \left( \frac{\partial \psi}{\partial \beta_k} - \int_{V-\sigma} \frac{\delta \psi}{\delta \beta_k} dv^i \right), \quad \varepsilon_k = \rho \left( \frac{\partial \psi}{\partial \beta_k} - \int_{V-\sigma} \frac{\delta \psi}{\delta \beta_k} dv^i \right)
\]

\[
\frac{\partial \psi}{\partial \beta_k} = \int_{V-\sigma} \frac{\delta \psi}{\delta \beta_k} dv^i, \quad \frac{\partial \psi}{\partial \beta_k} = \int_{V-\sigma} \frac{\delta \psi}{\delta \beta_k} dv^i
\]

and
where

\[
\begin{align*}
\frac{1}{\rho} D_{kl} \psi + \frac{1}{\rho} \mathcal{E}_k (J - \mathcal{J}_k - \mathcal{J}_k^f) + \frac{1}{\rho} (\mathcal{E}_k \mathcal{H}_k^f - \mathcal{H}_k^f \mathcal{H}_k) \\
- \frac{1}{\rho} \mathcal{D} \mathcal{H} + \frac{1}{\rho} (\mathcal{H} - \mathcal{H}) \geq 0, \quad \text{in } V - \sigma
\end{align*}
\]

is satisfied.

We may rewrite (2.33) and (2.34) in terms of the constitutive variables,

\[
\begin{align*}
M_{kl} &= \mathcal{E}_k \mathcal{D}_l + \mathcal{H}_k \mathcal{B}_l - \frac{1}{2} (\mathcal{E} \cdot \mathcal{D} + \mathcal{H} \cdot \mathcal{B}) \delta_{kl} \\
M^h &= \frac{1}{2} (\mathcal{E} \cdot \mathcal{D} + \mathcal{H} \cdot \mathcal{B}).
\end{align*}
\]

We have thus proved:

**Theorem:** The constitutive equations of nonlocal electromagnetic fluids are thermodynamically admissible if and only if (2.30) and (2.31) are satisfied.
Using (2.30) and (2.32)-(2.34) the energy equation (II.3.30) may be written as

\[-(\mathcal{D}_t (k/ ) + \mathcal{M}_t (k/ ) - \mathcal{M}_{h5k1})d_{kl} + \rho(\hat{\eta} - g \cdot \hat{\psi})
+ \mathcal{H} \cdot \hat{\mathcal{B}} + \mathcal{E} \cdot \hat{\mathcal{D}} + \rho\mathcal{V} \cdot \hat{f} + \rho(\hat{h} - \hat{\mathcal{h}}) = 0\]

(2.37)

where parentheses enclosing indices indicates symmetrization, that is,

\[\mathcal{D}_t^{(kl)} \equiv \frac{1}{2} (\mathcal{D}_t^{kl} + \mathcal{D}_t^{lk}).\]

We now posit the energy equation (2.37) to be invariant under all rigid motions (2.9). Since \(\psi\), and hence \(\eta\) and \(\dot{\eta}\) are invariant by (2.30) and \(D^{t,kl}, M^{t,kl}\) and \(d_{kl}\) are invariant so are \(D^{t,(kl)}d_{kl}\) and \(M^{t,(kl)}d_{kl}\) as well as the vector products in (2.37). Moreover, since the energy source \(h\) is considered invariant we must have

\[\hat{h} = \hat{h}_0 + D^h + \mathcal{V} \cdot \hat{f}\]

(2.38)

where \(\hat{h}_0\) is a scalar valued constitutive functional of the same type as \(\psi\), satisfying the same invariance under (2.9). Thus we may write the energy equation as

\[\rho\dot{\hat{\eta}} - (D^{t,(kl)} + M^{t,(kl)} - h_{kl})d_{kl} + \mathcal{H} \cdot \hat{\mathcal{B}} + \mathcal{E} \cdot \hat{\mathcal{D}} - \rho(\hat{\psi} + h - \hat{\mathcal{h}}) = 0.\]

(2.39)
It is important to note that no assumptions concerning the nature of the electromagnetic momentum, energy or stress have been made in our analysis. The crucial assumption lies in the form of the second law of thermodynamics given in (II. 4. 1). With the choice of \((\mathbf{E}, \mathbf{F})\) as our pair of independent electromagnetic variables we see that the electromagnetic stress and electromagnetic energy given in (2. 33) and (2. 34), respectively, are analogous (when viewed from the rest frame) to the classical Minkowski expressions. If other pairs of electromagnetic quantities had been chosen, say \((\mathbf{N}, \mathbf{Q})\), a seemingly different structure describing the electromagnetic stress and energy would have appeared. Physically these expressions are, of course, equivalent which can be shown by applying a Legendre-type transformation on the free-energy \(\psi\) \([45]\).

3. Linear Constitutive Theory

The application of the general theory developed in the previous sections to phenomenological problems is mathematically untractable. Hence we now develop a linear constitutive theory, which from a mathematical point of view makes the physical problems tractable.

In accordance with the axiom of equipresence all of the response functions must be assumed to depend on the cause set \((\mathcal{F}', \mathcal{F})\) given in (2. 5). Thus we assume the dissipative stress tensor to be a tensor-valued functional \(f_{\kappa \ell}\) given by
For a linear constitutive theory we may write

\[
D^{kl} = f_{kl}(r^{-1}, \beta_k, \gamma_{kl}, \Gamma_k, \Gamma_{kl}, \Gamma_{ll}; d_{kl}, \theta_k, \theta_{kl}, \theta_{ll}; \rho^{-1}, \theta).
\]  

(3.1)

where the term involving \( r^{-1}(x') \) has been dropped since it may be absorbed in (3.8) which follows. The tensors \( a_i, i = 1, 2, 3, 4, 5 \) are functions of \( \rho^{-1} \) and \( \theta \) while all \( b_j, j = 1, 2, \ldots, 6 \) depend on \( \rho^{-1}, \theta, \) and \( \|x'-x\| \).

The invariance of (3.2) under the full group of orthogonal transformations \( Q \) and translations (2.9) dictates that \( a_i \) and \( b_j \) are isotropic tensors for all \( i \) and \( j \). It is well known, (cf. Spencer [63]), that in rectangular cartesian coordinates all even order isotropic tensors may be written as a linear combination of Kronecker delta terms, whereas all odd order isotropic tensors must reduce identically to the zero tensor. Thus the local moduli \( a_i \) may be written as
\[ \frac{k\ell m}{a_i} = 0 \quad i = 2, 3, 4, 5 \]  
(3.3)  
\[ a_1 \frac{k\ell mn}{d\delta} = \lambda_1 \delta \frac{k\ell mn}{d\delta} + \mu_1 (\delta \frac{k\ell}{d\delta} \frac{mn}{d\delta} + \delta \frac{kn}{d\delta} \frac{lm}{d\delta}) \]  
where \( \lambda_1 \) and \( \mu_1 \) both depend on \( \rho^{-1} \) and \( \theta \), and \( \delta_{ij} \) is the Kronecker delta. The nonlocal moduli \( b_j \) are functions of \( \|x'-x\| \) and hence, according to the theory of invariants are given by

\[ b_1 \frac{k\ell mn}{d\delta} = \lambda_1 \delta \frac{k\ell mn}{d\delta} + \mu_1 (\delta \frac{k\ell}{d\delta} \frac{mn}{d\delta} + \delta \frac{kn}{d\delta} \frac{lm}{d\delta}) + \mu_2 (x^i - x^k)(x^l - x^\ell) \delta \frac{mn}{d\delta} \]  
\[ + \mu_3 (x^i - x^k)(x^m - x^m) \delta \frac{kn}{d\delta} \frac{lm}{d\delta} \]  
\[ + \mu_4 (x^i - x^k)(x^m - x^m) \delta \frac{kn}{d\delta} \frac{lm}{d\delta} \]  
\[ + \mu_5 (x^i - x^l)(x^m - x^m) \delta \frac{kn}{d\delta} \frac{lm}{d\delta} \]  
\[ + \mu_6 (x^i - x^l)(x^m - x^m) \delta \frac{kn}{d\delta} \frac{lm}{d\delta} \]  
\[ + \mu_7 (x^i - x^m)(x^m - x^m) \delta \frac{kl}{d\delta} \]  
\[ + \mu_8 (x^i - x^m)(x^m - x^m) \delta \frac{kl}{d\delta} \]  
(3.4)  
\[ b_1 \frac{k\ell m}{d\delta} = \alpha_1 (x^i - x^m) \delta \frac{k\ell m}{d\delta} + \alpha_2 (x^i - x^m) \delta \frac{kl}{d\delta} + \alpha_3 (x^i - x^m) \delta \frac{kl}{d\delta} \]  
(3.5)  
where \( \lambda_1', \mu_1' \) and \( \alpha_1' \) are all functions of \( \rho^{-1}, \theta' \), and \( \|x'-x\| \) for all \( k = 1, 2, \ldots, 8 \), \( i = 1, 2, 3, 4 \), \( j = 2, 3, 4, 5, 6 \).

We invoke at this stage the axiom of attenuating neighborhoods surrounding a given local point and as a consequence all electromagnetic interactions are confined to a small neighborhood of that point. This amounts to ignoring all of the terms involving \( x'-x \) in (3.4) and (3.5) since the coefficients \( \lambda_1', \mu_1', \) and \( \alpha_1' \) are also
functions of $x' \pm \xi$. Hence, with the use of this axiom the nonlocal moduli reduce to

$$b^{kmn}_j = 0 \quad j = 2, 3, 4, 5, 6$$

and

$$b^{klmn}_1 = \lambda \delta^{kl} \delta^{mn} + \mu \delta^{km} \delta^{ln} + \delta^{kn} \delta^{lm}. \tag{3.6}$$

Now, letting

$$\lambda_v = \lambda_1 - \int_{V_{-\sigma}} \lambda'_1 dv(x'), \quad \mu_v = \mu_1 - \int_{V_{-\sigma}} \mu'_1 dv(x') \tag{3.7}$$

and using (3.3), (3.6), and (3.7) in (3.2) we may write (2.32) as

$$t_{kl} = (-\pi - h + \lambda \frac{d}{\nabla \nabla}) \delta_{kl} + 2\mu \frac{d}{\nabla} \nabla_{kl} + M^t_{kl} \tag{3.8}$$

where

$$\pi \equiv - \frac{\delta \psi}{\delta \rho^{-1}}, \quad \sigma' \equiv \frac{\delta \psi}{\delta x^{-1}} \tag{3.9}$$

and

$$\lambda'_v(x, x') = \lambda'_v(\rho^{-1}, \theta', ||x'-\xi||) \tag{3.10}$$

$$\mu'_v(x, x') = \mu'_v(\rho^{-1}, \theta', ||x'-\xi||)$$

all of which agrees with Eringen [23].

To obtain linear constitutive equations for $\mathcal{F}$ and $\mathcal{E}$ we assume that the free-energy density $\psi$ to have a constitutive
make-up involving linear functionals. Using this expression for $\psi$ to evaluate $(2.30)_5, 6$ will yield the desired linear constitutive equations for $\mathcal{N}$ and $\mathcal{E}$ respectively. To this end, we take

$$\psi = \psi_o + \int_{V_{-\sigma}} \left( \psi_1 \Gamma_{-k} + \psi_2 \Pi_{-k} + \psi_3 \Pi_{-k} + \psi_4 \Pi_{-k} + \psi_5 \beta_k + \psi_6 \gamma_k \right) dv(x')$$  \hspace{1cm} (3.11)

where

$$\psi_0 = \psi_o (\mathcal{B}, \dot{\mathcal{B}}, \mathcal{C}, \dot{\mathcal{C}}, d_k, \rho^{-1}, \theta)$$

$$\psi_i = \psi_i (\mathcal{B}, \dot{\mathcal{B}}, \mathcal{C}, \dot{\mathcal{C}}, d_k, \rho^{-1}, \theta', \|x' - x\|), \quad i = 1, 2, \ldots, 6. \hspace{1cm} (3.12)$$

Using (3.11) and (3.12) in $(2.30)_5$ yields

$$\frac{1}{\rho} \mathcal{H}_k = \frac{\partial \psi_0}{\partial \mathcal{B}_k} + \int_{V_{-\sigma}} \left( \frac{\partial \psi_1}{\partial \mathcal{B}_k} \Gamma_{-l} + \frac{\partial \psi_2}{\partial \mathcal{B}_k} \Pi_{-l} + \frac{\partial \psi_3}{\partial \mathcal{B}_k} \Pi_{-l} + \frac{\partial \psi_4}{\partial \mathcal{B}_k} \Pi_{-l} 
+ \frac{\partial \psi_5}{\partial \mathcal{B}_k} \beta_5 + \frac{\partial \psi_6}{\partial \mathcal{B}_k} \gamma_{lm} \right) dv(x') - \int_{V_{-\sigma}} \psi_1^k dv(x'). \hspace{1cm} (3.13)$$

For a completely linear theory the coefficients in (3.12) are taken in the form

$$\psi_0 = \alpha_1 \mathcal{B}_k + \alpha_2 \dot{\mathcal{B}}_k + \alpha_3 \mathcal{C}_k + \alpha_4 \dot{\mathcal{C}}_k + \alpha_5 d_k \hspace{1cm} (3.14)$$

$$\psi_i^k = \gamma_{i1} \mathcal{B}_l + \gamma_{i2} \dot{\mathcal{B}}_l + \gamma_{i3} \mathcal{C}_l + \gamma_{i4} \dot{\mathcal{C}}_l + \gamma_{i5} d_{lm}, \quad i = 1, 2, 3, 4, 5$$

$$\psi_{6k} = \gamma_{61} \mathcal{B}_m + \gamma_{62} \dot{\mathcal{B}}_m + \gamma_{63} \mathcal{C}_m + \gamma_{64} \dot{\mathcal{C}}_m + \gamma_{65} d_{mn}$$
where in \( \psi_0 \) only the pertinent quadratic terms needed to evaluate (3.13) have been written and

\[
\begin{align*}
\alpha_i &= \alpha_i (\rho^{-1}, \theta_i), \quad \beta_i = \beta_i (\rho^{-1}, \theta_i), \quad i = 1, 2, 3, 4, 5 \\
\gamma_{ij} &= \gamma_{ij} (\rho^{-1}, \theta, ||\mathbf{x}' - \mathbf{x}||) \quad i = 1, 2, \ldots, 6 \\
&\quad j = 1, 2, 3, 4, 5.
\end{align*}
\]

(3.15)

Using (3.14) in (3.13) yields the linear constitutive equation for \( \mathcal{H}_k \)

\[
\frac{1}{\rho} \mathcal{H}_k = A_1 + \beta_1 k \dot{e}_l + \beta_2 k \dot{e}_l + \beta_3 k \dot{e}_l + \beta_4 k \dot{e}_l + \beta_5 k \dot{e}_l + \beta_{klm} k \dot{e}_l
\]

\[
+ \int_{V_\sigma} \left( \gamma_{11} \dot{e}_l + \gamma_{12} \dot{e}_l + \gamma_{13} \dot{e}_l + \gamma_{14} \dot{e}_l + \gamma_{15} \dot{e}_l \right) \partial \mathcal{H}_k \, dv(x')
\]

\[
- \int_{V_\sigma} \left( \gamma_{11} \dot{e}_l + \gamma_{12} \dot{e}_l + \gamma_{13} \dot{e}_l + \gamma_{14} \dot{e}_l + \gamma_{15} \dot{e}_l \right) \partial \mathcal{H}_k \, dv(x').
\]

Defining

\[
\mathcal{H}_k' \equiv \mathcal{H}_k - \rho \alpha_1 k
\]

and dropping the prime for convenience, and by letting

\[
\begin{align*}
\sigma_{kl}^i &= \beta_i^{kl} - \int_{V_\sigma} (\gamma_{11}^{kl} + \gamma_{11}^{kl}) \, dv(x'), \quad i = 1, 2, 3, 4 \\
\sigma_{klm}^{5} &= \beta_5^{klm} - \int_{V_\sigma} \gamma_{15}^{klm} \, dv(x')
\end{align*}
\]

(3.17)

we may write (3.16) as
Since $\mathbf{H}$ is assumed to be an isotropic vector all of the coefficients appearing in (3.18) must be isotropic in nature. Hence we have

$$
\begin{align*}
\sigma_{i}^{k \ell} &= b_{i} \delta_{i}^{k \ell} \quad i = 1, 2, 3, 4 \\
\gamma_{i1}^{k \ell} &= b'_{i} \delta_{i}^{k \ell} + c'_{i}(x^{k} - x_{i})(x^{\ell} - x_{i}) \quad i = 1, 2, 3, 4, 5 \\
\sigma_{5}^{k \ell m} &= 0 \\
\gamma_{61}^{k \ell m} &= g'_{1}(x^{k} - x_{i}) \delta_{m}^{\ell} m + g'_{2}(x^{\ell} - x_{i}) \delta_{k}^{m} m + g'_{3}(x^{m} - x_{i}) \delta_{k}^{\ell} m + g'_{4}(x^{k} - x_{m})(x^{\ell} - x_{i})(x^{m} - x_{i}) \\
\end{align*}
$$

where the local moduli $b_{i}$ depend on $\rho^{-1}$ and $\theta$ while the non-local moduli $b'_{i}$, $c'_{i}$, and $g'_{i}$ depend on $\rho'^{-1}$, $\theta'$, and $\|x' - x\|$. As in the case of the stress constitutive expression we apply the axiom of attenuating neighborhoods, thus ignoring any terms involving $x' - x$ in (3.19) since the coefficients $b_{i}'$, etc. also are functions of $x' - x$. In this case the moduli in (3.19) reduce to

\[ \mathbf{H} = \rho \left( \sigma_{1}^{k \ell} \mathbf{G}_{k \ell} + \sigma_{2}^{k \ell} \mathbf{G}_{k \ell} + \sigma_{3}^{k \ell} \mathbf{G}_{k \ell} + \sigma_{4}^{k \ell} \mathbf{G}_{k \ell} + \sigma_{5}^{k \ell m} \mathbf{G}_{k \ell m} \right) + \sum_{\mathbf{V} = \mathbf{0}} \left( \gamma_{11}^{k \ell} \mathbf{B}^{k \ell} + \gamma_{21}^{k \ell} \mathbf{B}^{k \ell} + \gamma_{31}^{k \ell} \mathbf{B}^{k \ell} + \gamma_{41}^{k \ell} \mathbf{B}^{k \ell} + \gamma_{51}^{k \ell \mu} \mathbf{B}^{k \ell \mu} \right) + \sum_{\mathbf{V} = \mathbf{0}} \left( \gamma_{61}^{k \ell m} \mathbf{B}^{k \ell m} \right) dV(x') \]
\[ \sigma_{kl}^m = b_i \delta_{kl} \quad i = 1, 2, 3, 4 \]
\[ \gamma_{kl} = b_i \delta_{kl} \quad i = 1, 2, 3, 4, 5 \]
\[ \sigma_{5}^{klm} = 0 = \gamma_{61} \quad \quad (3.20) \]

Using (3.20) in (3.18) yields the linear constitutive expression for \( \mathcal{H}_k \),

\[ \mathcal{H}_k = \rho \left[ b_1 \dot{\mathcal{B}}_k + b_2 \dot{\mathcal{B}}_k + b_3 \mathcal{D}_k + b_4 \dot{\mathcal{D}}_k \right. \\
\left. + \int_{V_{-\sigma}} \left( b_1 \dot{\mathcal{B}}_k \dot{\mathcal{B}}_k + b_2 \mathcal{D}_k \mathcal{D}_k + b_3 \dot{\mathcal{D}}_k \dot{\mathcal{D}}_k + b_4 \dot{\mathcal{D}}_k \dot{\mathcal{D}}_k \right) dv(x') \right] \quad (3.21) \]

Performing a similar analysis using (2.30) yields the linear constitutive equation for \( \mathcal{E}_k \),

\[ \mathcal{E}_k = \rho \left[ s_1 \dot{\mathcal{B}}_k + s_2 \dot{\mathcal{B}}_k + s_3 \mathcal{D}_k + s_4 \mathcal{D}_k \right. \\
\left. + \int_{V_{-\sigma}} \left( s_1 \dot{\mathcal{B}}_k \dot{\mathcal{B}}_k + s_2 \mathcal{D}_k \mathcal{D}_k + s_3 \dot{\mathcal{D}}_k \dot{\mathcal{D}}_k + s_4 \mathcal{D}_k \mathcal{D}_k \right) dv(x') \right] \quad (3.22) \]

where again the local moduli \( s_i \) depend on \( \rho^{-1} \) and \( \theta \), and the nonlocal moduli are functions of \( \rho^{-1}, \theta', \) and \( \|x' - x\| \).

The linear constitutive expression for the conduction current \( J^c \) cannot be derived from (2.30). As with all response functionals we assume that \( J^c \) is a vector-valued functional \( f_k \) of the cause set \( (\mathcal{F}', \mathcal{F}) \) given by
\[
\mathcal{J}_k^c = f_k (r^{-1}, \beta_k, \gamma_{kl}, \Gamma_k, \Pi_k, \Pi_k, \hat{d}_k, \beta_k, \delta_k, \rho^{-1}, 0) \quad (3.23)
\]

subject to the restrictions arising from (2.31). The linear expansion of (3.23) may be written as

\[
\mathcal{J}_k^c = e_1 \mathcal{B}_k + e_2 \mathcal{B}_k + e_3 \mathcal{D}_k + e_4 \mathcal{D}_k + e_5 \mathcal{D}_k \quad (3.24)
\]

\[
+ \int_{V-\sigma} \left( n_0 + n_1 \mathcal{B}_k + n_2 \mathcal{B}_k + n_3 \mathcal{B}_k + n_4 \mathcal{B}_k + n_5 \mathcal{B}_k + n_6 \mathcal{B}_k \right) \, dv(x')
\]

where \( e_i, i = 1, 2, 3, 4, 5 \) depend on \( \rho^{-1} \) and \( \theta \), and \( n_j, j = 0, 1, \ldots, 6 \) depend on \( \rho^{-1}, 0', \) and \( \|x' - x\| \). Assuming \( \mathcal{J}_c \) to be an isotropic vector and again applying the axiom of attenuating neighborhoods as we did in deriving the linear expressions for \( \mathcal{H} \) and \( \mathcal{E} \), the moduli in (3.24) may be written as

\[
\mathcal{E}_5 = \mathcal{E}_6 = n_0
\]

\[
e_i^{kl} = \alpha_i^{\delta_k} \quad i = 1, 2, 3, 4 \quad (3.25)
\]

\[
n_j^{kl} = \alpha_j^{\delta_k} \quad j = 1, 2, 3, 4, 5
\]

Setting

\[
\mathcal{J}_k^c \equiv \mathcal{J}_k^c - \int_{V-\sigma} n_0 \, dv(x') \quad (3.26)
\]

and

\[
\mathcal{E}_i \equiv \alpha_i \, dv(x') \quad i = 1, 2, 3, 4 \quad (3.27)
\]
and then dropping the over-bars for convenience we have

\[
J^c_k = \alpha_1 \beta_k + \alpha_2 \dot{\beta}_k + \alpha_3 \dot{\theta}_k + \alpha_4 \ddot{\theta}_k
\]

(3.28)

\[
+ \int_{V-\sigma} \left( \alpha_1' \beta_k' + \alpha_2' \dot{\beta}_k' + \alpha_3' \dot{\theta}_k' + \alpha_4' \ddot{\theta}_k' \right) dv(x')
\]

as the linear constitutive expression for the conduction current.

This completes the linear constitutive theory for nonlocal electromagnetic fluids.

4. Thermodynamic Restrictions

All of the material moduli appearing in (3.8), (3.21), (3.22), and (3.28) are subject to restrictions arising from the entropy inequality (2.31). Integrating (2.31) over \(V-\sigma\) and recalling that the conduction current, \(\dot{J^c} = \dot{J^f} - \dot{\theta}_o\), we obtain the classical dissipation inequality for the entire body:

\[
\int_{V-\sigma} \frac{1}{\theta} D_{k\ell} d_k d_{\ell} dv + \int_{V-\sigma} \frac{1}{\theta} \varepsilon_k J^c_k dv
\]

\[
+ \int_{V-\sigma} \frac{1}{\theta} \left[ \varepsilon_k \dot{J}^c_k - \gamma_k \dot{b}_k + \rho(\dot{h} - \dot{\theta}) \right] dv \geq 0.
\]

The above inequality is postulated to be valid for all independent mechanical as well as electromagnetic processes, and as a
consequence the last integral in (4.1) may be set equal to zero without loss of generality. Thus we have

\[ \int_{V-\sigma} \frac{1}{\theta} D^{t\ell}_{k\ell} d_{k\ell} dv + \int_{V-\sigma} \frac{1}{\theta} \epsilon_i^c_{k} j^c_{k} dv \geq 0. \tag{4.2} \]

Using (3.8) we obtain

\[ D^{t\ell}_{k\ell} d_{k\ell} = \lambda_v d_{rr} d_{k\ell} \delta_{k\ell} + 2\mu_v d_{k\ell} d_{k\ell} \tag{4.3} \]

\[ + \int_{V-\sigma} (\lambda'_v d_{rr} d_{k\ell} \delta_{k\ell} + 2\mu'_v d_{k\ell} d_{k\ell}) dv(x'). \]

Hence through (3.21), (3.22), and (4.3) we see that the two integrals in (4.2) vary independently of each other.

We first consider the case in which the second integral in (4.2) vanishes. Thus we require that

\[ \int_{V-\sigma} \frac{1}{\theta} D^{t\ell}_{k\ell} d_{k\ell} dv \geq 0 \tag{4.4} \]

where it is assumed that the absolute temperature \( \theta > 0 \). We establish only sufficient pointwise satisfaction of (4.4) by requiring the integrand to be everywhere nonnegative. Considering the local and nonlocal terms separately we demand that

\[ \lambda_v d_{rr} d_{k\ell} \delta_{k\ell} + 2\mu_v d_{k\ell} d_{k\ell} \geq 0 \tag{4.5} \]
\[ \int_{V - \sigma} (\lambda_v d_{rr} d_{kl} \delta_{kl} + 2\mu_v d_{rr} d_{kl}) dv(x') \geq 0. \quad (4.6) \]

To satisfy (4.5) we set

\[
\begin{align*}
    y_1 &= d_{11}, \quad y_2 = d_{22}, \quad y_3 = d_{33}, \quad y_4 = d_{12}, \quad y_5 = d_{23}, \quad y_6 = d_{31} \\
    a_{11} &= a_{22} = a_{33} = \lambda_v + 2\mu_v \\
    a_{12} &= a_{21} = a_{13} = a_{31} = a_{23} = a_{32} = \lambda_v \\
    a_{44} &= a_{55} = a_{66} = 4\mu_v \\
    a_{ij} &= 0 \text{ for all other } i, j, \quad 1 \leq i, j \leq 6
\end{align*}
\]

and write (4.5) in an equivalent quadratic form, that is,

\[ \lambda_v d_{rr} d_{kl} \delta_{kl} + 2\mu_v d_{rr} d_{kl} \geq 0 \iff a_{ij} y_i y_j \geq 0. \quad (4.8) \]

Using the step exchange method, (cf. Nef [56]), the quadratic form in (4.8) will be positive definite if and only if

\[ 3\lambda_v + 2\mu_v \geq 0. \quad (4.9) \]

Using the continuity of \( d \) we set

\[
\begin{align*}
    y_1 &= (d_{11}^t d_{11})^{1/2}, \quad y_2 = (d_{22}^t d_{22})^{1/2}, \quad y_3 = (d_{33}^t d_{33})^{1/2} \\
    y_4 &= (d_{12}^t d_{12})^{1/2}, \quad y_5 = (d_{23}^t d_{23})^{1/2}, \quad y_6 = (d_{31}^t d_{31})^{1/2}
\end{align*}
\]

\[
\begin{align*}
    y_1 &= (d_{11}^t d_{11})^{1/2}, \quad y_2 = (d_{22}^t d_{22})^{1/2}, \quad y_3 = (d_{33}^t d_{33})^{1/2} \\
    y_4 &= (d_{12}^t d_{12})^{1/2}, \quad y_5 = (d_{23}^t d_{23})^{1/2}, \quad y_6 = (d_{31}^t d_{31})^{1/2}
\end{align*}
\]
and defining $a_{ij}$ as in (4.7) we find that (4.6) will be satisfied pointwise (sufficiency condition only) if and only if

$$3\lambda'_{v} + 2\mu'_{v} \geq 0. \quad (4.11)$$

Thus our sufficiency thermodynamic restrictions for both the local and nonlocal material coefficients $\lambda'_{v}$, $\mu'_{v}$, $\lambda'_{v}$, and $\mu'_{v}$ are analogous to the classical results and Eringen’s results [23].

We now take the first integral in (4.2) to be zero and require

$$\int_{V_{-\sigma}} \frac{1}{\theta} \varepsilon_{k} \dot{\mathcal{J}}_{k}^{c} dv \geq 0. \quad (4.12)$$

Using (3.22) and (3.28) we simply require a pointwise sufficiency condition to insure (4.12), namely

$$\varepsilon_{k} \dot{\mathcal{J}}_{k}^{c} > 0. \quad (4.13)$$

Using (3.22) and (3.28), (4.13) is satisfied if and only if

$$s_{1}a_{1}^{2}\beta_{k}^{2} + s_{2}a_{2}^{2}\beta_{k}^{2} + s_{3}a_{3}^{2}\beta_{k}^{2} + s_{4}a_{4}^{2}\beta_{k}^{2} + (s_{1}a_{2}^{2} + a_{1}s_{2})(\frac{1}{2} \beta_{k}^{2})$$

$$+ (s_{3}a_{4}^{2} + a_{3}s_{4})(\frac{1}{2} \beta_{k}^{2}) + (s_{2}a_{1}^{2} + a_{4}s_{1})\dot{\beta}_{k} \beta_{k} + (s_{2}a_{3}^{2} + a_{2}s_{3})\dot{\beta}_{k} \beta_{k}$$

$$+ (s_{1}a_{4}^{2} + a_{1}s_{4})\dot{\beta}_{k} \beta_{k} + (s_{2}a_{3}^{2} + a_{2}s_{3})\dot{\beta}_{k} \beta_{k}$$

$$+ \int_{V_{-\sigma}} [(s_{1}a_{1}^{2} + a_{1}s_{1}) \beta_{k} \beta_{k}^{c} + (s_{2}a_{2}^{2} + a_{2}s_{2}) \beta_{k} \beta_{k}^{c}] dv(x') + \quad (4.14)$$
+ \int_{V^-} \left[ \left( s_1 \alpha_1^t + \alpha_1 s_1^t \right) B_k \dot{\theta}_k + (s_3 \alpha_3^t + \alpha_3 s_3^t) B_k \dot{\beta}_k \right] d\nu(x') \\

+ \int_{V^-} \left[ \left( s_2 \alpha_2^t + \alpha_2 s_2^t \right) \dot{B}_k \dot{\theta}_k + (s_4 \alpha_4^t + \alpha_4 s_4^t) \dot{B}_k \dot{\beta}_k \right] d\nu(x') \\

+ \int_{V^-} \left[ \left( s_2 \alpha_2^t + \alpha_2 s_2^t \right) \dot{B}_k \dot{\beta}_k + (s_4 \alpha_4^t + \alpha_4 s_4^t) \dot{B}_k \dot{\beta}_k \right] d\nu(x') \\

+ \int_{V^-} \left[ \left( s_1 \alpha_1^t + \alpha_1 s_1^t \right) B_k \dot{\theta}_k + (s_2 \alpha_2^t + \alpha_2 s_2^t) B_k \dot{\theta}_k \right] d\nu(x') \\

+ \int_{V^-} \left[ \left( s_1 \alpha_1^t + \alpha_1 s_1^t \right) B_k \dot{\beta}_k + (s_2 \alpha_2^t + \alpha_2 s_2^t) B_k \dot{\beta}_k \right] d\nu(x') \\

+ \int_{V^-} \left[ \left( s_1 \alpha_1^t + \alpha_1 s_1^t \right) B_k \dot{\beta}_k + (s_2 \alpha_2^t + \alpha_2 s_2^t) B_k \dot{\beta}_k \right] d\nu(x') \\

+ \int_{V^-} \left[ \left( s_1 \alpha_1^t + \alpha_1 s_1^t \right) B_k \dot{\beta}_k + (s_2 \alpha_2^t + \alpha_2 s_2^t) B_k \dot{\beta}_k \right] d\nu(x') \\

+ \int_{V^-} \left[ \left( s_1 \alpha_1^t + \alpha_1 s_1^t \right) B_k \dot{\beta}_k + (s_2 \alpha_2^t + \alpha_2 s_2^t) B_k \dot{\beta}_k \right] d\nu(x') \\

+ \int_{V^-} \left[ \left( s_1 \alpha_1^t + \alpha_1 s_1^t \right) B_k \dot{\beta}_k + (s_2 \alpha_2^t + \alpha_2 s_2^t) B_k \dot{\beta}_k \right] d\nu(x') \\

+ \int_{V^-} \left[ \left( s_1 \alpha_1^t + \alpha_1 s_1^t \right) B_k \dot{\beta}_k + (s_2 \alpha_2^t + \alpha_2 s_2^t) B_k \dot{\beta}_k \right] d\nu(x') \\

\times \int_{V^-} \left[ \left( s_1 \alpha_1^t + \alpha_1 s_1^t \right) B_k \dot{\beta}_k + (s_2 \alpha_2^t + \alpha_2 s_2^t) B_k \dot{\beta}_k \right] d\nu(x') \geq 0 \\

for all independent processes. Thus, demanding each of the first four terms of (4.14) to be nonnegative yields

\[ s_{i} \alpha_{i} \geq 0 \quad i = 1, 2, 3, 4 \]  

(4.15)
where the underscored subscript suspends the summation convention.

Through (2.34) the next two terms can be related to the magnetic and electric energies stored in the medium. Assuming no external electromagnetic sources to be present the electromagnetic energy stored within the medium must be dissipating so that

\[
\left( \frac{1}{2} \mathbf{B}_k^2 \right) \leq 0, \quad \left( \frac{1}{2} \mathbf{J}_k^2 \right) \leq 0
\]  

(4.16)

which implies that

\[
\begin{align*}
s_1 \alpha_2 + \alpha_1 s_2 & \leq 0, \quad s_3 \alpha_4 + \alpha_3 s_3 \leq 0.
\end{align*}
\]  

(4.17)

The next four terms in (4.14) are nonnegative for all independent processes if and only if they vanish, that is, we must have

\[
\begin{align*}
s_1 \alpha_3 + \alpha_1 s_3 &= 0, \quad s_2 \alpha_3 + \alpha_2 s_3 = 0, \\
s_1 \alpha_4 + \alpha_1 s_4 &= 0, \quad s_2 \alpha_4 + \alpha_2 s_4 = 0.
\end{align*}
\]  

(4.18)

In view of (4.15) we find that (4.18) can be satisfied only if one of the following pairs of relations holds true:

\[
\begin{align*}
\alpha_1 &= 0 = s_1 \quad \text{or} \quad s_3 = 0 = \alpha_3 \\
\alpha_2 &= 0 = s_2 \quad \text{or} \quad s_4 = 0 = \alpha_4.
\end{align*}
\]  

(4.19)

In classical electromagnetic theory \( J^c \) is proportional to \( \mathbf{E} \) while
\( \mathcal{C} \) and \( \mathcal{D} \) are related to each other in a linear fashion also. Hence we require the first pair of relations in (4.19) to hold true, thus dropping the local dependency of \( \mathcal{J}^C \) and \( \mathcal{C} \) on \( \mathcal{D} \) and \( \mathcal{D}' \).

To analyze the integral terms we need only appeal to the continuity of the electromagnetic quantities and the axiom of attenuating neighborhoods. For example, examining

\[
\int_{V-\sigma} (s_3 \alpha'_3 + \alpha_3 s'_3) \mathcal{D}_k \mathcal{D}'_k \, dv(x')
\]  

(4.20)

we see that the main contribution to the integral will arise in a small neighborhood about \( x \) since \( \alpha'_3 \) and \( s'_3 \) both satisfy the axiom of attenuating neighborhoods. Also, within a small neighborhood the continuity of \( \mathcal{D} \) requires that \( \mathcal{D} \cdot \mathcal{D}' \geq 0 \) so that (4.20) will be nonnegative if

\[
s_3 \alpha'_3 + \alpha_3 s'_3 \geq 0.
\]  

(4.21)

For such a term as

\[
\int_{V-\sigma} (s_3 \alpha'_4 + \alpha_3 s'_4) \mathcal{D}_k \mathcal{D}'_k \, dv(x')
\]  

(4.22)

we expand \( \mathcal{D}' \) into a Taylor series

\[
\mathcal{D}' = \mathcal{D} + \|x' - x\| \cdot \nabla \mathcal{D} + O(\|x' - x\|^2)
\]  

(4.23)

and ignore terms of \( O(\|x' - x\|^2) \) or higher. Furthermore, it is
physically reasonable to assume for a wide class of materials that the
gradients of material rates of fields such as $\mathcal{L}$ are small enough to
permit the neglect of their products with $\|\mathbf{x}' - \mathbf{x}\|$, in comparison
with $\mathcal{G}$. Thus, using this assumption we have

\[
(\|\mathbf{x}' - \mathbf{x}\| \cdot \nabla \mathcal{G}) \cdot \mathcal{G} \ll \mathcal{G} \cdot \mathcal{G} = \left( \frac{1}{2} \mathcal{G}^2 \right) \leq 0 \quad (4.24)
\]

which implies, using the above argument, that (4.22) will be nonnegative if

\[
s_3 \alpha_4' + \alpha_3 s_4' \leq 0. \quad (4.25)
\]

Applying similar analyses to all of the integral terms, except for the
last term, and collecting all of the results yields that

\[
s_3 \alpha_3' + \alpha_3 s_3' \geq 0, \quad s_4 \alpha_4' + \alpha_4 s_4' \geq 0, \quad (4.26)
\]

\[
s_3 \alpha_4' + \alpha_3 s_4' \leq 0, \quad s_4 \alpha_3' + \alpha_4 s_4' \leq 0,
\]

and

\[
s_3 \alpha_1' + \alpha_3 s_1' = 0, \quad s_4 \alpha_1' + \alpha_4 s_1' = 0, \quad (4.27)
\]

\[
s_3 \alpha_2' + \alpha_3 s_2' = 0, \quad s_4 \alpha_2' + \alpha_4 s_2' = 0,
\]

\[
s_3 \alpha_5' + \alpha_3 s_5' = 0, \quad s_4 \alpha_5' + \alpha_4 s_5' = 0.
\]

Since $s_3, \alpha_3', s_4'$, and $\alpha_4'$ cannot be zero we have that

\[
s_1' = 0 = \alpha_1', \quad s_2' = 0 = \alpha_2', \quad s_5' = 0 = \alpha_5' \quad (4.28)
\]
and hence the dependency on the magnetic field is dropped altogether, as well as the dependency on the nonlocal mechanical term $\beta$, in the linear constitutive theory. Finally, to evaluate the last term involving the iterated integral in (4.14) each term in the expression is written in an equivalent double integral, e.g.,

$$
\int_{V'} s'_3(\rho^{-1}, \theta', ||x'-x||) \mathcal{O}_k(x') \, dv(x') \int_{V'} a'_3(\rho^{-1}, \theta', ||y'-x||) \mathcal{O}_k(y') \, dv(y')
$$

(4.29)

$$
= \int_{V'} \int_{V'} s'_3(\rho^{-1}, \theta', ||x'-x||) a'_3(\rho^{-1}, \theta', ||y'-x||) \mathcal{O}_k(x') \mathcal{O}_k(y') \, dv(x') \, dv(y').
$$

Again, using the continuity of the electromagnetic variables and the fact that $s'_i$ and $a'_i$ obey the law of attenuating neighborhoods, we have

$$
\begin{align*}
  s'_3a'_3 &\geq 0, \quad s'_4a'_4 \geq 0, \\
  s'_3a'_4 + a'_3s'_4 &\leq 0.
\end{align*}
$$

(4.30)

Thus, summarizing our results the linear constitutive equations for $\mathcal{E}$ and $\mathcal{J}^c$ are given by

$$
\mathcal{E}_k = \rho \left[ s'_3 \mathcal{Q}_k + s'_4 \mathcal{O}_k + \int_{V'} (s'_3 \mathcal{Q}_k + s'_4 \mathcal{O}_k) \, dv(x') \right]
$$

(4.31)
\[ J_k^C = a_3 \mathbf{J} \mathbf{J}_k + a_4 \mathbf{J}_k + \int_{V_{-\sigma}} (a_3 \mathbf{J}_k + a_4 \mathbf{J}_k) \, dv(x') \]  \hfill (4.32)

subject to

\[ s_3 a_3 > 0, \quad s'_3 a'_3 > 0 \]
\[ s_4 a_4 > 0, \quad s'_4 a'_4 > 0. \]  \hfill (4.33)

5. Field Equations and Boundary Conditions

The field equations for nonlocal electromagnetic fluids consist of the equation of continuity (II. 3.27), the equations describing the motion of the media as derived using the relations for \( t_{k\ell}, k \) found in (3.8) in (II. 3.28), and the electromagnetic equations derived from (II. 3.31-3.34).

To generate the equation of motion we must compute the stress divergence, \( t_{k\ell}, k \) using (3.8) and substitute the resulting expression into the balance of linear momentum equation (II. 3.28). To this end, from (3.8) we have

\[ t_{k\ell}, k = -\mathbf{M}, \ell - M_{\mathbf{M}}, \ell + \lambda \mathbf{v}_k \mathbf{d}_{k\ell}, \ell + 2\mu \mathbf{v}_k \mathbf{d}_{k\ell}, k + M_t \mathbf{t}_{k\ell}, k \]
\[ + \int_{V_{-\sigma}} (\sigma' \mathbf{J}_k + \lambda' \mathbf{J}_k) \mathbf{d}_{k\ell} \mathbf{d}(x'). \]  \hfill (5.1)

Using (3.21) and (4.31) in (2.35) we find that
\[ M^{kl} = \rho \left\{ s_3 (D_k D_l)^{.}, k + s_4 (\dot{D}_k D_l)^{.}, k + (r_k \beta^l)^{.}, k + \tau_{kl}, k \right\} \]

\[ + D_l, k \int_{V_0} (s_3' D_k^{.} + s_4' \dot{D}_k^{.}) dv(x') \]

\[ + D_l \int_{V_0} (s_3' D_k^{.} + s_4' \dot{D}_k^{.}) dv(x') \]

\[ + B_l, k \int_{V_0} r_k^{.} dv(x') + B_l \int_{V_0} r_k^{.} dv(x') \]

where

\[ \tau_{kl} = \frac{1}{c} \left[ \mathcal{H}_k (v \times E)_l - \mathcal{E}_k (v \times H)_l \right] \]

\[ r_k = b_1 \mathcal{B}_k + b_2 \dot{\mathcal{B}}_k + b_3 \dot{D}_k + b_4 \dot{\dot{D}}_k \]

\[ r_k' = b_1' \mathcal{B}_k' + b_2' \dot{\mathcal{B}}_k' + b_3' \mathcal{D}_k' + b_4' \dot{\mathcal{D}}_k' + b_5' \dot{\dot{D}}_k ' \]

Since \( \sigma^l, \lambda^l, \mu^l, s_3^l, s_4^l, b_1^l, b_2^l, b_3^l, b_4^l, \) and \( b_5^l \) are functions of \( ||x' - x|| \), we may write

\[ \sigma^l, l = \frac{\partial \sigma^l}{\partial x'_l} = - \frac{\partial \sigma^l}{\partial x'_l} = - \sigma, l', \quad \lambda^l, l = - \lambda, l', \quad s_3, l = - s_3, l' \]

etc. Using (5.4) in (5.1) and (5.2) and substituting (5.2) into (5.1) yields
Applying the generalized Green-Gauss theorem to the last integral in (5.5) we may write this term as

\[ -\oint_{S_{\sigma}} s^i_{kl} \, d\alpha_k (\xi') + \int_{S_{\sigma}} I s^i_{kl} \, d\alpha_k (\xi') \]  \hspace{1cm} (5.6)
\[ \frac{\partial \rho}{\partial t} + (\rho v_k), k = 0 \]  

\[ - (\pi + M) + \lambda v_k, k + \mu v_l, k + \mu v_l, k \]

\[ + \rho[s_3(\mathcal{D}_k \mathcal{B}_l), k + s_4(\mathcal{D}_k \mathcal{B}_l), k + (r_k \mathcal{B}_l), k + \tau kl, k] \]

\[ + \int_{V - \sigma} \left[ (\lambda' + \mu') v_{k'}, k' + \lambda' v_{l'}, k' + \rho[\mathcal{D}_l (s_3 \mathcal{B}_k, k) + s'_4 \mathcal{B}_k, k] \right] \, dv(x') \]

\[ - \int_{S - \sigma} \mathcal{S}_l \, da_{l}(x') + \int_{l} \mathcal{S}_l \, da_{k}(x') + \rho(f_k - \dot{v}_k - \dot{g}_k) - \rho \hat{f}_k = 0. \]

To derive the electromagnetic field equations we must utilize the constitutive relations between \((\mathcal{H}, \mathcal{B})\) and \((\mathcal{E}, \mathcal{D})\). From (3.21) and (4.31) we have

\[ \mathcal{H} = \rho \left[ b_1 \mathcal{B} + \vec{r} + \int_{V - \sigma} \mathcal{I}' \, dv(x') \right] \]

\[ \mathcal{E} = \rho \left[ s_3 \mathcal{D} + s_4 \mathcal{D}' + \int_{V - \sigma} \mathcal{P}' \, dv(x') \right] \]

where

\[ \vec{r} = b_2 \mathcal{D} + b_3 \mathcal{D} + b_4 \mathcal{D} \]

\[ \mathcal{P}' = s'_3 \mathcal{D}' + s'_4 \mathcal{D}' \]

and \(\mathcal{I}'\) is given by (5.3)\textsubscript{3}. 
Substituting \((5.10)_2\) and \((\Pi.3.33)_1\) into the curl of \((\Pi.3.31)_1\) and using the fact that

\[
\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}
\]  

(5.12)

for any vector \(\mathbf{A}\), the resulting expression may be written as

\[
\nabla^2 \mathbf{D} - \alpha^2 \frac{\partial^2 \mathbf{D}}{\partial t^2} = \nabla(q + \dot{q}) + \frac{s_4}{s_3} \nabla \times \nabla \times \mathbf{\mathcal{G}} + \frac{1}{\rho \sigma s_3} \nabla \times \mathbf{\mathcal{G}} - \frac{1}{\rho \sigma s_3} \nabla \times \mathbf{\mathcal{B}}
\]

\[
+ \frac{1}{\rho s_3} \nabla \times \nabla \int_{V - \sigma} p' dv(x')
\]  

(5.13)

where we have written \(q^*\) defined in \((\Pi.3.17)\) as

\[
q^* = \frac{\partial q}{\partial t} + \ddot{q}
\]

and \(\alpha^2 = 1/(\rho c)^2 b_1 s_3\). Similarly, substituting \((5.10)_1\) and \((\Pi.3.34)_1\) into the curl of \((\Pi.3.32)_1\) and using \((5.12)\) yields

\[
\nabla^2 \mathbf{D} - \alpha^2 \frac{\partial^2 \mathbf{D}}{\partial t^2} = \nabla \hat{m} + \frac{1}{\rho b_1} \nabla \times \nabla \times \hat{\mathbf{F}} - \frac{1}{\rho \sigma b_1} \nabla \times \hat{\mathbf{G}} - \frac{1}{\rho \sigma b_1} \nabla \times \hat{\mathbf{B}}
\]

\[
- \frac{1}{\rho \sigma b_1} \nabla \times \hat{\mathbf{B}} + \frac{1}{\rho b_1} \nabla \times \nabla \int_{V - \sigma} \hat{\mathbf{p}}' dv(x')
\]  

(5.14)

\[
+ \alpha^2 \frac{\partial}{\partial t} \left[ \rho s_4 \nabla \times \hat{\mathbf{F}} + \nabla \int_{V - \sigma} p' dv(x') + \frac{1}{c} \hat{\mathcal{F}} - \frac{1}{c} \hat{\mathcal{B}} \right].
\]
Along with equations (5.8) and (5.9), equations (5.13) and (5.14) yield the field equations for nonlocal electromagnetic fluids.

The boundary conditions may be obtained by setting \( \hat{\rho} = 0 \) in (II.3.27) and using (II.3.31) through (II.3.35). Thus, on the moving surface of discontinuity, we require the following jump conditions:

\[
\begin{align*}
\{ \rho (v-u) \} \cdot \mathbf{n} &= 0 \\
\{ E + \frac{1}{c} (u \times B + \hat{E}) \} \times \mathbf{n} &= 0 \\
\{ H - \frac{1}{c} (u \times D + \hat{H}) \} \times \mathbf{n} &= 0 \\
\{ D + \hat{D} \} \cdot \mathbf{n} &= 0 \\
\{ B + \hat{B} \} \cdot \mathbf{n} &= 0 \\
\{ \mathcal{E} + \hat{\mathcal{E}} \} \cdot \mathbf{n} &= 0.
\end{align*}
\]

(5.15)

We note that if all of the electromagnetic terms and nonlocal terms are set to zero (5.9) is nothing more than the classical Navier-Stokes equations. Furthermore, we obtain the classical Maxwell's equations for free space, in the rest frame, from (5.13) and (5.14) by setting all of the nonlocal terms to zero and using the classical constitutive relations for \( (H, B) \) and \( (E, D) \).
IV. THEORY OF NONLOCAL POLAR ELECTROMAGNETIC FLUIDS

1. Preliminary Remarks

We now develop a general theory of nonlocal polar electromagnetic fluids. The micropolar structure we introduce here is a special case of the micromorphic theory introduced by Eringen [37]. In the micromorphic theory, the body is envisioned as composed of a large collection of small elements known as microvolume elements. Each microvolume element is thought of as an aggregate of a large number of molecules and atoms, which is considered as a continuum, yet small enough to incorporate the effects of local substructures within the material. These microvolume elements can undergo deformation and rotational motions independent of the motions of the body as a whole. Thus they can support such effects as couple stress, body couples, and spin-inertia. This theory is, however, mathematically complicated for purposes of applications to most physical problems. Consequently, Eringen [38] introduced the micropolar theory as a special case of the micromorphic theory. The micropolar theory thus constructed is found to govern the behavior of a large class of real materials such as liquid crystals, animal blood, colloidal suspensions, polymeric fluid additives, fibrous and granular media, solid rocket propellents, fats, emulsions, and gels, which are all constituted by
bar-like elements or dumbbell shaped molecules. In such media the aggregates can only support microrotations along with couple stress, body couples, and spin-inertia.

We first describe the motion of a micropolar (orientable) medium and then state the global balance laws governing a nonlocal orientable electromagnetic fluid. From the nonlocal global balance laws the localized balance laws and a generalized Clausius-Duhem inequality are then developed.

Based on our previous work in Chapter III and that of Eringen [37] a suitable formulation of the constitution of nonlocal polar electromagnetic fluids is derived. Next, the thermodynamic admissibility of these constitutive equations are investigated.

2. Motion and Balance Laws

In a micropolar continuum we assume that each point of the material body can rotate independently of the motion of the body, as given by (II. 2. 1). The rotation of the particle, which may be called the micromotion, is described by

$$\xi = \chi \Xi \Leftrightarrow \xi_k = \chi_{kK}(X, t) \Xi_k$$

(2. 1)

where $\Xi$ and $\xi$ are, respectively, the position vectors of the centers of mass of a microvolume element before and after deformation. The micromotion defined by (2. 1) possesses a unique inverse
denoted by
\[ \chi_{Kk}^{-1}(x_k, t) \] (2.2)
as well as continuous first-order partial derivatives at all points of the body.

Using (II. 2. 1-2. 3) the motion and micromotion of the body may be expressed, respectively, as

\[ x_k = x_k(x, t) \] (2.3)
\[ \xi_k = \chi_{kK}(x, t) \frac{x_k}{x} \]

where
\[ \det x_{k, K} > 0, \quad \chi_{Kk}^{-1} = \chi_{kK} \] (2.4)

for all material points \( x \) of the body and for all time \( t \). Consequently,

\[ x_{k, K} x_{k, L} = \delta_{kL}, \quad x_{K, k} x_{k, L} = \delta_{KL} \] (2.5)
\[ x_{kK} x_{kL} = \delta_{kL}, \quad x_{L, L} x_{kK} = \delta_{KL} \] (2.6)

Using (2.6) we can solve (2.3) for \( \Xi_K \) yielding
\[ \Xi_K = \chi_{kK} \xi_k \] (2.7)
To find the local angular velocity vector $\nu_k$ of the rotating triad formed by the three vectors $X_K = \chi_{kK}g_k$, where $g_k$, $k = 1, 2, 3$ define the basis vectors of the space, we differentiate (2.3) with respect to time (with $X$ and $\Xi$ fixed), that is,

$$\dot{\chi}_k = \dot{\chi}_{kK}(X, t)\Xi_K.$$ (2.8)

Substituting (2.7) into (2.8) yields

$$\dot{\xi}_k = \nu_{kl}\dot{\xi}_l,$$ (2.9)

where $\nu_{kl}$ is the skew-symmetric gyration tensor defined by

$$\nu_{kl}(x, t) = \dot{\chi}_{kK}(X, t)\chi_{lK}, \quad \nu_{kl} = -\nu_{lk}. \quad (2.10)$$

The local angular velocity vector $\nu_k$ is given by

$$\nu_k = -\frac{1}{2}e_{k lm}\nu_l m, \quad \nu_{kl} = -e_{kl m} \nu_m$$ (2.11)

where $e_{kl m}$ is the permutation tensor defined in II.3.

Since each point of a micropolar body is envisioned as a rigid particle we ascribe to $X$ a material inertia tensor $J_{KL}(X)$, which is symmetric and positive definite. We denote the dual of $J_{KL}$ at $x$ as $j_{kl}(x, t)$ and introduce the following relationship for further use:
The balance laws for micropolar materials are given by Eringen [35]:

Conservation of Mass:

\[
\frac{D}{Dt} \int_{V-\sigma} \rho \, dv = 0 \tag{2.13}
\]

Conservation of Microinertia:

\[
\frac{D}{Dt} \int_{V-\sigma} \rho i_{kl} x_k x_l \, dv = 0 \tag{2.14}
\]

Balance of Linear Momentum:

\[
\frac{D}{Dt} \int_{V-\sigma} \rho (v+g) \, dv - \int_{S-\sigma} t_k a_k \, da_k - \int_{V-\sigma} \rho f \, dv = 0 \tag{2.15}
\]

Balance of Moment of Momentum:

\[
\frac{D}{Dt} \int_{V-\sigma} [\rho p x (v+g) + \rho \omega] \, dv - \int_{S-\sigma} (p x t_k + m_k) a_k \, da_k - \int_{V-\sigma} (\rho p x f + \rho \ell) \, dv = 0 \tag{2.16}
\]
Conservation of Energy:

\[ \frac{D}{Dt} \int_{V} \rho (c + \frac{1}{2} v^2 + \frac{1}{2} \bar{\varepsilon} \cdot v) \, dv - \int_{S} (t_k \cdot v + q_k + m_k \cdot v) \, da_k \]

\[ - \int_{V} \rho (f \cdot v + h + f \cdot v) \, dv = 0 \]  \hspace{1cm} (2.17)

where

\[ \sigma \equiv \text{spin-inertia tensor} \]

\[ I \equiv \text{total body couple density} \]

\[ m \equiv \text{total body couple stress tensor}, \]

and where the remaining symbols have their usual meanings introduced in II. 3, and we have used the following relations

\[ X_k \equiv X_{kK} G_K \]

\[ \sigma \equiv i_k \nu_k \]  \hspace{1cm} (2.18)

\[ G_K \] being the basis vectors in the undeformed space. Since the electromagnetic laws are the same as in (II. 3. 1-3. 5) we do not repeat them here.

The most striking feature of the balance laws is the appearance of a new conservation law for the micromoment of inertia (2.14) which deals solely with the micromotion of the material points. Due to the micropolar nature of the body, couple stresses and body couples arising from both mechanical and electromagnetic sources are included in the balance laws. Such terms allow surface couples and spin
interactions to be taken into account.

To localize these laws one first applies the localization to an infinitesimal tetrahedron three of whose surfaces are taken as coordinate surfaces and whose fourth face coincides with \( S \), (cf. Eringen [35]). For (2.15) and (2.16) this yields

\[
\begin{align*}
\frac{\partial}{\partial t} t_{k} = t_{k} n_{k} = t_{k} n_{k} g_{k}, & \quad m_{k} = m_{k} n_{k} = m_{k} n_{k} g_{k}, \\
\end{align*}
\]

(2.19)

Carrying out the localization as described in Chapter II we find the localized balance laws for nonlocal orientable electromagnetic materials to be given by:

Conservation of Mass:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + (\rho v_{k})_{k} = \frac{\hat{\rho}}{k}, & \quad \text{in } V-\sigma \\
\left[\rho(v_{k} - u_{k}) - \frac{\hat{\rho}}{k}\right] n_{k} = 0, & \quad \text{on } \sigma \\
\end{align*}
\]

(2.20)

Conservation of Microinertias:

\[
\begin{align*}
\rho(j_{k}^{R} + e_{k} m^{r})_{k} = \frac{j_{k}^{R} + e_{k} m^{r}}{k}, & \quad \text{in } V-\sigma \\
\left[\rho j_{k}^{R}(v_{r} - u_{r}) + \frac{\hat{j}_{k}^{R}}{rkl}\right] n_{r} = 0, & \quad \text{on } \sigma \\
\end{align*}
\]

(2.21)

Balance of Linear Momentum:

\[
\begin{align*}
t_{k} = t_{k} n_{k} = t_{k} n_{k} g_{k}, & \quad m_{k} = m_{k} n_{k} = m_{k} n_{k} g_{k}, \\
\left[t_{k} - \rho(v_{r} + g_{r}) (v_{k} - u_{k}) + \hat{t}_{k} \right] n_{k} = 0, & \quad \text{on } \sigma \\
\end{align*}
\]

(2.22)
Balance of Moment of Momentum:

\[
e_{k m n} p_m, l^n - \rho e_{k m n} m g_n + m_{l k}, l + \rho (l_k - \dot{\sigma}_k)
\]

\[
= \rho e_{k m n} \hat{m} \hat{f} + \rho \sigma - \rho \hat{\rho} k, \text{ in } V - \sigma
\]

\[
[\rho e_{l m n} \hat{m} \hat{t} \hat{k} - (v_n + g_n)(v_k - u_k)] + \rho (v_k - u_k) - m_{l k} + \hat{\rho} k, l \hat{m} \hat{n} k
\]

\[= 0, \text{ on } \sigma
\]

Conservation of Energy:

\[
\rho \dot{\epsilon} - \rho \dot{g}_k v_k - t_{k l} (v_k + u_k) - q_k, k - \rho h - m_{k l} v_l, k
\]

\[
= \rho \hat{h} - \rho \nu_{k l} v_k - \rho (\epsilon + \frac{1}{2} v^2 - \nu g_k) + \frac{1}{2} \rho \hat{\nu}_{k l} v_k v_l - \rho (\hat{\nu}_{k l} - \nu_{k l} m_{k l} p_{l m} \nu_{k m})
\]

\[\text{in } V - \sigma
\]

\[
\int_{k l} v_l + q_k - \rho (\epsilon + \frac{1}{2} v^2 + \frac{1}{2} j_{l m} v_k \nu_{k m})(v_k - u_k) + m_{k l} v_l + \hat{\rho} k, l \hat{m} \hat{n} k
\]

\[= 0, \text{ on } \sigma
\]

where we have introduced a new localization residual \( \hat{j}_{k l} \) for the microinertia and corresponding surface residual \( \hat{\rho}_{r k l} \). As with all localization residuals we require that

\[
\int_{V - \sigma} \rho \hat{j}_{k l} dv = 0, \int_{S - \sigma} \hat{\rho}_{r k l} da_r = 0.
\]

The localized electromagnetic laws (II. 3. 31-3. 35) as well as the entropy inequality (II. 4. 1) remain the same. Upon eliminating \( h \) between (II. 4. 2) and (2. 24), and upon using (II. 4. 6), we obtain the
following generalized Clausius-Duhem inequality:

\[
-\frac{\theta}{\theta} (\dot{\psi} + \dot{v}_k g_{kk} + \dot{\eta}) + \frac{1}{\theta} t_{k\ell} (v_{\ell k} + v_{k\ell}) + \frac{1}{\theta} m_{k\ell} v_{\ell k} - \frac{1}{\theta} Q_{k\ell} \theta_k
\]

\[
+ \frac{1}{\theta} (E_{k}\frac{\epsilon_{k\ell}}{k} + \mu_{k}\frac{\epsilon_{k\ell}}{k}) - \frac{\theta}{\theta} \hat{f}_k v_k - \frac{\theta}{\theta} (\psi + \frac{1}{2} v^2) + \frac{1}{\theta} E_k \theta_k
\]

\[
+ \frac{1}{\theta} (E_{k}\frac{\epsilon_{k\ell}}{k} - \mu_{k}\frac{\epsilon_{k\ell}}{k}) + \frac{\theta}{\theta} (\hat{h} - \delta) + \frac{1}{\theta} \hat{f}_k v_k + \frac{\theta}{\theta} (\hat{f}_k - e_{k\ell m} p_{\ell m} \hat{f}_{km}) v
\]

\[\geq 0, \text{ in } V-\sigma\]

which we posit to be valid for all possible independent motions of the body.

Having generated the localized balance laws for nonlocal polar electromagnetic materials we now develop the general constitutive theory for nonlocal polar electromagnetic fluids.

3. Constitutive Theory

An appropriate cause set for the constitution of nonlocal polar electromagnetic fluids can be formulated with the use of the constitutive set of variables developed by Eringen [35], for nonlocal polar fluids. Motivated by these results we first define two ordered sets analogous to \( \mathcal{F}^1 \) and \( \mathcal{F} \) introduced in Chapter III:
\[ F' = \{r^{-1}(x'), h_{k\ell}(x'), \beta_{k\ell}(x'), \nu_{k\ell}(x'), \alpha_{k\ell m}(x') \} \]

\[ \Gamma_{k}(x', t), \hat{\Gamma}_{k}(x', t), \Pi_{k}(x', t), \hat{\Pi}_{k}(x', t) \} \]

\[ F = \{d_{k\ell}^{ij}, j_{k\ell}, b_{k\ell}, a_{k\ell m}, \beta_{k}, \dot{\beta}_{k}, \dot{\beta}_{k}, \rho^{-1}, \theta \} \]

where

\[ h_{k\ell} = j_{k\ell}^{i} - j_{k\ell} \]

\[ \beta_{k\ell} = \nu_{k\ell}^{i} - \nu_{k\ell} \]

\[ \alpha_{k\ell m} = \nu_{k\ell m} - \nu_{k\ell m} \]

\[ b_{k\ell} = \nu_{k\ell} + \nu_{k\ell} \]

(3.1)

(3.2)

and for brevity we have used \( j_{k\ell}^{i} = j_{k\ell}(x') \), \( \nu_{k\ell}^{i} = \nu_{k\ell}(x', t) \) etc. We define our class of nonlocal polar electromagnetic fluids by the constitutive equations of the form:

\[ \bar{\psi} = \psi(F', F) \]  

(3.3)

with the other response functions \( t, m, q, \eta, \varepsilon, \bar{\varepsilon}, \bar{\varepsilon}, \) and \( j^{c} \) having similar forms as (3.3).

According to the axiom of objectivity all of the arguments appearing in (3.1) must be objective under the rigid motions defined by the Galilean type transformations in (III. 2. 9). The fact that all the constitutive variables appearing in (3.1) are objective can be established by the procedure shown below for the case of one of the variables, \( \nu_{k\ell} \).
The second rank tensor $\nu_{k\ell}$ is objective if

$$\nu_{k\ell}^+ = Q_{km} Q_{\ell n} \nu_{mn}. \quad (3.4)$$

Since it is assumed that

$$\chi_{kK}^+ = Q_{k\ell} \chi_{\ell K} \quad (3.5)$$

under (II.2.9), upon using (2.10) and the above we have

$$\nu_{k\ell}^+ = Q_{km} \chi_{mK} Q_{\ell n} \chi_{nK}$$

$$= Q_{km} Q_{\ell n} \nu_{mn}$$

and hence $\nu_{k\ell}$ is objective. The objectivity of the remaining quantities in (3.1) follow similarly.

To investigate the consequences of the thermodynamic admissibility we assume that all of the constitutive variables are continuously differentiable with respect to all their arguments and all of the nonlocal variables in $F'$ belong to a Hilbert space defined analogous to (III.2.17). Calculating $\dot{\psi}$ from (3.3) yields
\[
\dot{\psi} = \frac{\partial \psi}{\partial d_{k\ell}} \dot{d}_{k\ell} + \frac{\partial \psi}{\partial j_{k\ell}} \dot{j}_{k\ell} + \frac{\partial \psi}{\partial b_{k\ell}} \dot{b}_{k\ell} + \frac{\partial \psi}{\partial a_{k\ell m}} \dot{a}_{k\ell m} + \frac{\partial \psi}{\partial B_k} \dot{B}_k
\]
\[
+ \frac{\partial \psi}{\partial \theta_k} \dot{\theta}_k + \frac{\partial \psi}{\partial \rho_{k\ell}} \dot{\rho}_{k\ell} + \frac{\partial \psi}{\partial \phi_k} \dot{\phi}_k + \frac{\partial \psi}{\partial \rho_{-1}} \dot{\rho}_{-1} + \frac{\partial \psi}{\partial \theta} \dot{\theta}
\]
\[
+ \int_{V-\sigma} \left[ \frac{\delta \psi}{\delta \tau^{-1}_{k\ell}} (F', F; \lambda) \tau^{-1}_{k\ell} (\lambda) \right] \, d\mu(\lambda)
\]
\[
+ \int_{V-\sigma} \left[ \frac{\delta \psi}{\delta \Gamma_k} (F', F; \lambda) \Gamma_k (\lambda) \right] \, d\mu(\lambda).
\]

Using the facts that [35]
\[
\frac{D}{D_t} (j_{k\ell}) = v_{kr} j_{rl} + v_{lr} j_{rk}
\]
and
\[
\frac{\partial \psi}{\partial j_{k\ell}} \dot{j}_{k\ell} = \left( \frac{\partial \psi}{\partial j_{k\ell}} + \frac{\partial \psi}{\partial j_{k\ell}} \right) j_{rl} v_{kr}
\]
and substituting these, along with (III.2.26), (III.2.27), (II.3.10), and (II.3.37), into (3.6) and then substituting the result into the entropy inequality (2.26) yields
\[ - \frac{\rho}{\theta} \left( \eta + \frac{\partial \psi}{\partial \theta} \right) \hat{\theta} + \frac{\rho}{\theta} \left[ - \frac{f_k}{\rho} - \frac{1}{\rho} \varepsilon_k(q + \hat{q}) + \int \frac{\delta \psi_{k'}}{\delta \beta_{k'}} v_{k', l} \right. \\
\left. - \frac{1}{\rho c} e_{k' \ell m} \left( \varepsilon_{k' \ell} + \varepsilon_{\ell m} \right) \right] v_k \\
- \frac{\rho}{\theta} \left( g_k - \int \frac{\delta \psi}{\delta \beta_k} \right) \hat{v}_k + \frac{1}{\theta} \left[ t_{k' \ell} - D_k \varepsilon_{k' \ell} - B_k \eta_{k' \ell} \\
- \delta_{k' \ell} \left( \frac{\partial \psi}{\partial \rho} - \varepsilon_m D_m - \varepsilon_m B_m - \int \frac{\delta \psi}{\delta r} \right) \right] v_{k', k} \\
- \frac{\rho}{\theta} \left[ \frac{1}{2} \left( \frac{\partial \psi}{\partial d_{k' \ell}} + \frac{\partial \psi}{\partial d_{k \ell k'}} \right) - \int \frac{\delta \psi}{\delta \gamma_{k' \ell}} - \int \frac{\delta \psi}{\delta \beta_{k' \ell}} \right] \hat{v}_{k, \ell} \\
+ \frac{1}{\theta} \left[ t_{k' \ell} - \rho j_{k' \ell} \left( \frac{\partial \psi}{\partial j_{k' \ell}} + \frac{\partial \psi}{\partial j_{k \ell k'}} - \int \frac{\delta \psi}{\delta h_{k' \ell}} + \int \frac{\delta \psi}{\delta h_{k \ell k'}} \right) \right] v_{k, k} \right] \right] v_{k, \ell} \right] \\
- \frac{\rho}{\theta} \left( \frac{\partial \psi}{\partial b_{k' \ell}} - \int \frac{\delta \psi}{\delta \beta_{k' \ell}} \right) \hat{v}_{k, \ell} - \frac{\rho}{\theta} \left( \frac{\partial \psi}{\partial \alpha_{k' \ell m}} - \int \frac{\delta \psi}{\delta \alpha_{k' \ell m}} \right) \hat{a}_{k' \ell m} \\
- \frac{\rho}{\theta} \left( \frac{\partial \psi}{\partial \alpha_k} - \int \frac{\delta \psi}{\delta \Gamma_{k}} - \frac{1}{\rho} \eta_k \right) \hat{G}_k - \frac{\rho}{\theta} \left( \frac{\partial \psi}{\partial \beta_k} - \int \frac{\delta \psi}{\delta \Gamma_{k}} - \frac{1}{\rho} \varepsilon_k \right) \hat{G}_k, \\
- \frac{\rho}{\theta} \left( \frac{\partial \psi}{\partial \beta_k} - \int \frac{\delta \psi}{\delta \Gamma_{k}} \right) \hat{G}_k - \frac{\rho}{\theta} \left( \frac{\partial \psi}{\partial \beta_k} - \int \frac{\delta \psi}{\delta \Gamma_{k}} \right) \hat{G}_k + \frac{1}{\theta} m_{k' \ell} v_{k', k} \\
+ \frac{1}{\theta} \varepsilon_k \left( J_{k}^{f} - \theta_{f_{o_{k}}} \right) + \frac{1}{2} \frac{\rho}{\theta} \varepsilon_{k' \ell m} \left( f_{n} + e_{n k' \ell m} m_{k' \ell} \right) v_{k' \ell m} + \frac{1}{\theta} \left( \varepsilon_k \hat{G}_k - \eta_k \right) \hat{G}_k \\
- \frac{\rho}{\theta} \left( \hat{h} - \hat{\Phi} \right) - \frac{\rho}{\theta} \hat{D}_{\ell} + \frac{1}{2} \frac{\rho}{\theta} \hat{j}_{k' \ell m} v_{k' \ell m} + \mathcal{O}(v^2/c^2) \geq 0, \quad \text{in } V-\sigma \]

where
\[ D^\h = \int_{V-\sigma} \left[ \frac{\delta \psi}{\delta \beta} v_k, k + \frac{\delta \psi}{\delta \beta} k_{i, k} \right] dv \]
\[ + \left( \frac{\delta \psi}{\delta \alpha \| \right) v_k', k, k + \frac{\delta \psi}{\delta \alpha \| k} \right] dk, k \]
\[ + \frac{\delta \psi}{\delta \Gamma} k_{i, k}' + \frac{\delta \psi}{\delta \Gamma} k_{i, k}' + \frac{\delta \psi}{\delta \Gamma} k_{i, k}' + \frac{\delta \psi}{\delta \Gamma} k_{i, k}' \] (3.10)

and where for brevity we have written

\[ \int \frac{\delta \psi}{\delta \beta} = \int_{V-\sigma} \frac{\delta \psi}{\delta \beta} (F', F; \lambda) \beta_k (\lambda) dv(\lambda), \quad \text{etc.} \] (3.11)

Ignoring the terms of \( O(v^2/c^2) \) the inequality (3.9) is linear

in \( \dot{v}_k, \dot{v}_k', \dot{v}_k, \dot{v}_k', \dot{a}_{klm}, \dot{a}_{k}, \dot{a}_k', \dot{a}_k', \dot{a}_k', \) and \( \dot{a}_k \). If \( \dot{\ell}, \dot{\ell}, \dot{\ell}, \dot{\ell}, \dot{\ell}, \dot{\ell}, \dot{\ell}, \dot{\ell}, \) are independent of these linearly independent quantities then (3.9) cannot be maintained for all possible variations of these quantities unless

\[ \eta = -\frac{\partial \psi}{\partial \theta}, \quad \epsilon_k = \int_{V-\sigma} \left[ \frac{\delta \psi}{\delta \beta} v_k, k \right] + \frac{1}{\rho} \mathcal{E}_k (\dot{\theta} + \dot{\theta}) - \frac{1}{\rho c} e_{klm} (E_{1\| m} + E_{1\| m}) \right] dv' \]
\[ g_k = \int_{V-\sigma} \frac{\delta \psi}{\delta \beta} k_{i, k} dv', \quad \frac{1}{2} \left( \frac{\partial \psi}{\partial a_{klm}} + \frac{\partial \psi}{\partial a_{klm}} \right) = \int_{V-\sigma} \frac{\delta \psi}{\delta \alpha_{klm}} dv' + \int_{V-\sigma} \frac{\delta \psi}{\delta \alpha_{klm}} dv' \]
\[ \frac{\partial \psi}{\partial b_{klm}} = \int_{V-\sigma} \frac{\delta \psi}{\delta \beta_{klm}} dv', \quad \frac{\partial \psi}{\partial a_{klm}} = \int_{V-\sigma} \frac{\delta \psi}{\delta \alpha_{klm}} dv' \] (3.12)
\[
\mathcal{H}_k = \rho \left( \frac{\partial \psi}{\partial \mathcal{B}_k} - \int_{V-\sigma} \frac{\delta \psi}{\delta \mathcal{B}_k} \, dv' \right), \quad \mathcal{E}_k = \rho \left( \frac{\partial \psi}{\partial \mathcal{E}_k} - \int_{V-\sigma} \frac{\delta \psi}{\delta \mathcal{E}_k} \, dv' \right)
\]

(3.12, cont)

\[
\frac{\partial \psi}{\partial \mathcal{B}_k} = \int_{V-\sigma} \frac{\delta \psi}{\delta \mathcal{B}_k} \, dv', \quad \frac{\partial \psi}{\partial \mathcal{E}_k} = \int_{V-\sigma} \frac{\delta \psi}{\delta \mathcal{E}_k} \, dv'
\]

and

\[
\frac{1}{\theta} D^{t}_{kl} \nu_{l,k} + \frac{1}{\theta} m_{kl} \nu_{l,k} + \frac{1}{\theta} (t_{kl} + \pi_{kl} - \sigma_{kl}) \nu_{l,k} + \frac{1}{\theta} \mathcal{E}_k (J_f \mathcal{J}_k - \mathcal{J}_k \mathcal{J}_k) + \frac{1}{\theta} D_{kl} \nu_{l,k} + \frac{1}{\theta} \mathcal{E}_k (\mathcal{J}_f \mathcal{J}_k - \mathcal{J}_k \mathcal{J}_k) - \frac{\rho}{\theta} \mathcal{H}_k \nu_{l,k} + \frac{1}{\theta} \mathcal{E}_k (\mathcal{J}_f \mathcal{J}_k - \mathcal{J}_k \mathcal{J}_k) - \frac{\rho}{\theta} \mathcal{H}_k \nu_{l,k}
\]

(3.13)

\[
+ \frac{\rho}{\theta} (\pi - \mathcal{H}_k \nu_{l,k}) + \frac{1}{2} \frac{p}{\theta} \mathcal{H}_k \nu_{l,k}, \quad \text{in } V-\sigma
\]

where

\[
D^{t}_{kl} = t_{kl} - M^{t}_{kl} + \delta_{kl} (\pi + \sigma + M^h)
\]

(3.14)

\[
\pi_{kl} = -\rho j_{km} \left( \frac{\partial \psi}{\partial j_{ml}} + \frac{\partial \psi}{\partial j_{lm}} \right)
\]

(3.15)

\[
\sigma_{kl} = -\rho j_{km} \int_{V-\sigma} \left( \frac{\delta \psi}{\delta h_{ml}} + \frac{\delta \psi}{\delta h_{lm}} \right) \, dv'
\]

(3.16)

and \(M^{t}_{kl}, M^h, \pi, \) and \(\sigma \) are given by (III.2.33), (III.2.34), and (III.3.9) respectively. Thus we have proved:

**Theorem.** The constitutive equations of nonlocal polar electromagnetic fluids are thermodynamically admissible if and only if (3.12) and (3.13) are satisfied.
At this stage, since it is not our main goal to pursue this theory any further, we will simply indicate the lines along which further progress could be made. In order to utilize the theory developed here for practical applications it is necessary to obtain a constitutively linear theory and an appropriate thermodynamical theory. This can be accomplished by following procedures identical to those already developed by us in Chapter III, Sections 3 and 4.
V. DISPERSION OF SURFACE WAVES IN A DIELECTRIC FLUID

1. Preliminary Remarks

As we pointed out in Chapter I, since the nonlocal continuum theories are of relatively recent origin, it is to be expected that very little work exists in this area. In this chapter we employ the theory developed in Chapter III to study the dispersive character of electromagnetic surface waves propagating in a dielectric fluid. Due to the lack of experimental work at present, especially for fluids, our results must await experimental verification. Aside from the theoretical importance of the dispersion relation developed here, we hope our investigation will stimulate intense experimental work in this area.

Due to the electromagnetic constitution of the material, a disturbance propagating through the medium can invoke responses which are electromechanical in nature. Thus, in general, the surface wave propagating in such a dielectric medium is composed of disturbances arising from the interactions between electromagnetic and mechanical constituents of the medium. However, the linear theory developed here leads to the decoupling of the electromechanical surface waves into two independent surface phenomena: one being purely mechanical in nature (the Rayleigh type wave), the other being electromagnetic in
nature (the Zenneck type wave). Hence, these two disturbances may be studied separately.

The analysis of the Rayleigh type wave yields a dispersion relation which takes into account the viscous effects of the fluid as well as the nonlocal influences. Since the classical work in this field is restricted to simply an "inviscid" fluid, (cf. Whitham [73]), our theoretical results obtained here cannot be compared with the classical results. However, in order to gain some insight into the quantitative nature of the nonlocal influences and also demonstrate the power of the nonlocal theory for fluids developed here, we fix our attention on some common fluids such as lubricating oils. Since these media are viscoelastic in nature, at least to some extent their solid characteristics will be comparable to those for which lattice dynamical results are available.

The corresponding problem for the electromagnetic disturbance is, unlike its mechanical counterpart, unmanageable from a mathematical point of view and hence the dispersion relation incorporating the nonlocal effects is not specifically obtained. It is found that in the classical context, that is, when we ignore the nonlocal effects no dispersion relation can be derived, as is to be expected.

2. Assumptions

Before formulating the problem we wish to specialize the field equations for deformable bodies with electromagnetic constitution
given in Chapter III. We shall accomplish this through certain assumptions imposed on an incompressible, homogeneous, isotropic, dielectric material and by linearizing the resultant field equations.

We shall restrict our considerations to the case of dielectric, nonconducting materials and the situation where \( \nu^2/c^2 \ll 1 \). We shall therefore assume [45]:

i) Nonconducting Assumption

\[ \Theta = 0. \]  

(2.1)

ii) Dielectric Assumption

\[ M = \frac{1}{c} \mathbf{P} \times \nu. \]  

(2.2)

iii) Nonrelativistic Assumption

\[ g = 0. \]  

(2.3)

The nonconducting assumption states that there can be no current flow in a dielectric medium. For this to be true we should have no free charges, consequently we take \( q = 0 \) and thus, through (II. 3.35) we also have \( \hat{\sigma} = 0 \). For nonconduction at all parts of the body the localization residuals, \( \hat{\Theta} = 0 \) and \( \hat{q} = 0 \).

The dielectric assumption states that the dielectric material has no magnetic moment when viewed in the rest frame. Thus, the magnetic pole strength induced on a material point by all of the other material points is zero. Hence, we take \( \hat{m} = 0 \), which through (II. 3.40) and (II. 3.36) implies that \( \hat{b} = 0 \).
The nonrelativistic assumption follows from the fact that the total energy-momentum tensor is symmetric, from which one can deduce the equivalence of momentum flow and energy flux given by Truesdell and Toupin [71]:

\[
g = \frac{Q}{c^2} - \frac{t \cdot \nu}{c^2} + \rho \left( \frac{e-c^2}{c^2} \right) \nu, \quad (2.4)
\]

where \(e\) represents the internal energy density. Since we have assumed the heat conduction \(Q = 0\) and in view of the Lorentz approximation \(v^2/c^2 \ll 1\), which leads to (cf. Grot [45]),

\[
(\text{tr} \; \tau^2)^{1/2} \leq \rho c^2, \quad |\rho(e-c^2)| \leq \rho c^2 \quad (2.5)
\]

assumption (iii) follows.

In classical electromagnetic theory the constitutive relations for a linear, homogeneous, isotropic medium are given by

\[
\begin{align*}
D &= \varepsilon E \\
B &= \mu H
\end{align*} \quad (2.6)
\]

where \(\varepsilon\) and \(\mu\) are, respectively, the electric permittivity and magnetic permeability of the medium. Following along these lines we shall simplify our constitutive relations given in (III. 3. 21) and (III. 4. 31) to
where

\[ \mathcal{C} = \epsilon^{-1} \mathcal{E} + \sum_{\mathcal{V} - \sigma} \epsilon^* \mathcal{E}' d\mathcal{V}' \]

\[ \mathcal{Y} = \mu^{-1} \mathcal{B} + \sum_{\mathcal{V} - \sigma} \mu^* \mathcal{B}' d\mathcal{V}' \]

so that the classical theory is included as a special case of our non-local theory.

The linearization of the field equations, as we have already pointed out, results in the decoupling of the electromechanical fluid equations into two systems, one governing a purely mechanical disturbance and the other a purely electromagnetic wave propagation. Furthermore, since we are studying wave phenomena it is customary to consider the displacement vector \( \mathbf{u} (\mathbf{\ddot{u}} = \mathbf{v}) \), as opposed to the velocity vector \( \mathbf{\dot{v}} \). Using the conservation of mass (II.3.27), the balance of linear momentum, and the expression for the stress (III.3.8), the linearized field equations in the absence of all body forces may be written as

\[ \ddot{u}_k, k = 0 \quad (2.8) \]

\[ t_{k\ell}, k = \rho \ddot{u}_\ell \quad (2.9) \]

where
\[ t_{kl} = (-p + \lambda \hat{u}_r) \delta_{kl} + \mu \left( \hat{u}_k, l + \hat{u}_l, k \right) + \int_{V_{-\sigma}} \left[ \sigma' + \lambda' \hat{u}'_r \right] \delta_{kl} + \mu' \left( \hat{u}'_k, l + \hat{u}'_l, k \right) \text{dv}(\mathbf{x}') \] (2.10)

and \( \delta_{kl} \) is Kronecker delta, \( p \) is total pressure, \( \lambda \) is dilatational viscosity, \( \mu \) is shear viscosity, and \( \sigma', \lambda', \) and \( \mu' \) are nonlocal material moduli which depend on \( ||\mathbf{x} - \mathbf{x}'|| \) for a homogeneous fluid and obey the axiom of attenuating neighborhoods. Furthermore, since we have linearized our field equations

\[ \ddot{u}_k = \frac{\partial u_k(x, t)}{\partial t}, \quad \ddot{u}'_k = \frac{\partial u'_k(x', t)}{\partial t}, \quad \text{etc.} \]

Linearizing (III. 5.13) and (III. 5.14) and using the aforementioned assumptions the electromagnetic field equations become

\[ D_{k, ll} - \alpha^2 \ddot{D}_{k} = \int_{V_{-\sigma}} \varepsilon' \varepsilon'_{k, l} \text{dv}(\mathbf{x}') - \int_{V_{-\sigma}} \varepsilon' \varepsilon'_{k, l} \text{dv}(\mathbf{x}') \]

(2.11)

\[ - c \alpha^2 e_{k l m} \int_{V_{-\sigma}} \mu' \dot{B}'_{m} \text{dv}(\mathbf{x}') \]

\[ B_{k, ll} - \alpha^2 \ddot{B}_{k} = \int_{V_{-\sigma}} \mu' \mu'_{k, l} \text{dv}(\mathbf{x}') - \int_{V_{-\sigma}} \mu' \mu'_{k, l} \text{dv}(\mathbf{x}') \]

(2.12)

\[ + c \alpha^2 e_{k l m} \int_{V_{-\sigma}} \varepsilon' \varepsilon'_{k, l} \text{dv}(\mathbf{x}') \]
where
\[ a^2 = \varepsilon^r / c^2, \quad \varepsilon^r = \varepsilon \varepsilon^*, \quad \mu^r = \mu \mu^* \]  \hspace{1cm} (2.13)

and \( e_{k\ell m} \) is the permutation tensor.

3. Formulation of the Problem

We shall consider a dielectric fluid in a half-space given in rectangular coordinates \( x_1, x_2, x_3 \) (or simply \( x_k, k = 1, 2, 3 \)). The half-space occupies the region \( x_1 > 0, x_2 > 0 \) with the free surface \( x_2 = 0 \) being the plane of propagation of the surface waves. We assume that the surface disturbance is set into motion along the line \( x_1 = 0 = x_2 \) at time \( t = 0 \). The resulting mechanical and electromagnetic waves have the property of losing their energy in an exponential fashion as they penetrate the medium \( (x_2 > 0) \). As in the classical treatment the problem (for the Rayleigh wave) will be considered as a two dimensional problem in the domain \( x_1 \in (0, \infty), \)
\( x_2 \in (0, \infty), \) everything being uniform in the \( x_3 \)-direction. As for the Zenneck wave, only three components are necessary to describe its propagation, (cf. Barlow and Brown [3]), namely \( D_1, D_2, \) and \( B_3. \)

The field equations for the Rayleigh type waves thus take the form:
\[ \dot{u}_{1,1} + \dot{u}_{2,2} = 0 \]
\[ t_{11,1} + t_{21,2} - \rho \ddot{u}_{1} = 0 \]  \hspace{1cm} (3.1)
\[ t_{12,1} + t_{22,2} - \rho \ddot{u}_{2} = 0 \]

where

\[ t_{11} = -p + (\lambda v + 2\mu v)\dot{u}_{1,1} + \lambda v \dot{u}_{2,2} \]
\[ + \int_{0}^{\infty} \int_{0}^{\infty} [\sigma'(v) + (\lambda v + 2\mu v)\dot{u}_{1,1}' + \lambda v \dot{u}_{2,2}'] dx_{1}' dx_{2}' \]

\[ t_{12} = t_{21} = \mu v \dot{u}_{1,2} + \mu v \dot{u}_{2,1} \]  \hspace{1cm} (3.2)
\[ + \int_{0}^{\infty} \int_{0}^{\infty} (\mu v \dot{u}_{1,2}' + \mu v \dot{u}_{2,1}') dx_{1}' dx_{2}' \]

\[ t_{22} = -p + (\lambda v + 2\mu v)\dot{u}_{2,2} + \lambda v \dot{u}_{1,1} \]
\[ + \int_{0}^{\infty} \int_{0}^{\infty} [\sigma'(v) + (\lambda v + 2\mu v)\dot{u}_{2,2}' + \lambda v \dot{u}_{1,1}'] dx_{1}' dx_{2}' \]

Since all tractions at the free surface, \( x_2 = 0 \), must vanish, the boundary conditions for the Rayleigh type waves take the form:

\[ t_{22} = 0 = t_{21} \quad \text{at} \quad x_2 = 0 \]  \hspace{1cm} (3.3)
\[ u_k = 0 \quad \text{as} \quad x_2 \to \infty. \]

The field equations for the Zenneck type waves may be written as
\[ D_{1,11} + D_{1,22} - \alpha^2 \cdot D_1' = \int_0^\infty \int_0^\infty (\epsilon'_{12} D_{2,11}' - \epsilon',_{22} D_{1,11}') dx_1 dx_2 \]
\[-c \alpha^2 \int_0^\infty \int_0^\infty \mu',_{22} \hat{B}_3' dx_1 dx_2 \]

\[ D_{2,11} + D_{2,22} - \alpha^2 \cdot D_2' = \int_0^\infty \int_0^\infty (\epsilon'_{12} D_{1,11}' - \epsilon',_{22} D_{1,22}') dx_1 dx_2 \]
\[+c \alpha^2 \int_0^\infty \int_0^\infty \mu',_{22} \hat{B}_3' dx_1 dx_2 \]

\[ B_{3,11} + B_{3,22} - \alpha^2 \cdot B_3' = -\int_0^\infty \int_0^\infty (\mu',_{11} \mu',_{22}) B_3' dx_1 dx_2 \]
\[-c \alpha^2 \int_0^\infty \int_0^\infty (\epsilon',_{11} \dot{D}_{2,11}' + \epsilon',_{22} \dot{D}_1') dx_1 dx_2 \]

Again, since all surface tractions must vanish at the free surface and since the dielectric medium is nonconducting, the boundary conditions for the electromagnetic waves are

\[ D_1 = D_2 = 0 = \frac{\partial B_3}{\partial x_2} \text{ at } x_2 = 0 \]
\[ D_1 = D_2 = 0 = B_3 \text{ as } x_2 \rightarrow \infty. \]
4. Dispersion of Rayleigh Type Waves

We consider a solution field satisfying (3.1) through (3.3) in the form of a double Fourier integral:

\[ u_k(x_1, x_2, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{u}_k(\xi, x_2, \omega)e^{-i(\xi x_1 + \omega t)} d\xi d\omega \]

(4.1)

\[ p(x_1, x_2, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{p}(\xi, x_2, \omega)e^{-i(\xi x_1 + \omega t)} d\xi d\omega \]

Substituting (4.1) into (3.1) yields

\[ -i\xi \bar{u}_1 + \bar{u}_2, 2 = 0 \]

(4.2)

\[ -i\xi \bar{t}_{11} + \bar{t}_{21}, 2 + \omega^2 \rho \bar{u}_1 = 0 \]

\[ -i\xi \bar{t}_{12} + \bar{t}_{22}, 2 + \omega^2 \rho \bar{u}_2 = 0, \]

while using (4.1) in (3.2) gives

\[ \bar{t}_{11} = -\bar{p} + \omega \xi (\lambda v + 2\mu v) \bar{u}_1 - i\omega \lambda v \bar{u}_2, 2 \]

\[ + \int_0^\infty [\bar{\sigma} \delta(\xi)\delta(\omega) - \omega \xi (\bar{\lambda} v + 2\bar{\mu} v) \bar{u}_1 - i\omega \bar{\lambda} v \bar{u}_2, 2] dx_2^i \]

\[ \bar{t}_{12} = -i\mu v \bar{u}_1, 2 - \xi \mu v \bar{u}_2 + \int_0^\infty -i(\bar{\sigma} v \bar{u}_1^i v _2 - i\xi \bar{\lambda} v \bar{u}_2^i ) dx_2^i \]

(4.3)
\[ \bar{\tau}_{22} = -\bar{\rho} - i\omega (\lambda + 2\mu) \bar{\mu}_{2,2} - \omega \xi \lambda \bar{\mu}_{1} \]

\[ + \int_{0}^{\infty} \left[ \sigma'(\xi)\delta(\omega) + (\lambda' + 2\mu')(-i\omega \bar{\mu}_{2,2} - \omega \xi \bar{\mu}_{1}) \right] dx', \]

where a superposed bar on letters indicates the Fourier transformation.

The nonlocal material moduli \( \sigma', \lambda', \) and \( \mu' \) are expected to change very sharply as we move from the surface, \( x_2 = 0, \) to within the medium. Since they obey the axiom of attenuating neighborhoods they must die out quickly as \( ||x' - x|| \to \infty. \) We idealize this situation mathematically by considering the behavior of these nonlocal moduli to be \( \delta \)-functions in the \( x_2 \)-variable, that is,

\[ \bar{\sigma}'(\xi, |x_2' - x_2|) = \bar{\sigma}(\xi) \delta(\omega) \delta(\omega) \]

\[ \bar{\lambda}'(\xi, |x_2' - x_2|) = \bar{\lambda}(\xi) \delta(\omega) \delta(\omega) \]

\[ \bar{\mu}'(\xi, |x_2' - x_2|) = \bar{\mu}(\xi) \delta(\omega) \delta(\omega). \]

Using (4.4) in (4.3) and substituting the result into (4.2), the field equations take the following form:

\[ -i\xi \bar{\mu} + \bar{\mu}_{2,2} = 0 \]

\[ -i\xi \bar{\mu} + i\omega \xi^{2}(\lambda + 2\mu) \bar{\mu} + 2\bar{\mu}) \bar{\mu}_{1} - i\omega (\mu + \bar{\mu}) \bar{\mu}_{1,2} \]

\[ - \xi \omega (\lambda + \bar{\lambda}) \bar{\mu}_{2,2} - \omega \xi (\mu + \bar{\mu}) \bar{\mu}_{2,2} + \omega^{2} \rho \bar{\mu}_{1} = 0 \]
Letting

\[ \vec{P} = -\vec{p} + \sigma(\xi) \delta(\xi) \delta(\omega) \] (4.6)

(4.5) may be written as

\[ -i\xi \vec{u}_1 + \vec{u}_2, 2 = 0 \] (4.8)

Since Rayleigh waves decay exponentially as they penetrate the medium we have

\[ \vec{u}_k(\xi, x_2) = U_k(\xi)e^{-\alpha x_2} \] (4.9)

\[ \vec{P}(\xi, x_2) = P(\xi)e^{-\alpha x_2} \]

where \( \text{Re}(\alpha) > 0 \). Substituting (4.9) into (4.8) yields the following homogeneous system of linear equations:
where

\[ \gamma_{2j} = \frac{-\xi}{\alpha_j}, \quad \gamma_{31} = \frac{-i\omega \xi}{\xi}. \] (4.13)

Using (4.12) in (4.10), the boundary conditions \( t_{22} = 0 = t_{21} \) can be satisfied if and only if

\[ \frac{2\xi^2}{\alpha_1} U_{11} + \frac{2\xi^2}{\alpha_2} U_{12} = 0 \]
\[ (2\xi^2 - ik\omega) U_{11} + 2\xi^2 U_{12} = 0 \] (4.14)
which has nontrivial solutions if and only if the following frequency equation is satisfied:

\[ \gamma^3 - \frac{3}{2} \gamma^2 + \frac{1}{2} \gamma - \frac{1}{16} = 0 \]  \hspace{1cm} (4.15)

where

\[ \gamma = \frac{\xi^2}{i \omega \rho k} \] \hspace{1cm} (4.16)

The solutions of (4.15) yield two roots that are physically tenable.

The resulting possible dispersion curves may therefore be written as

\[ \omega = K \left( \frac{\mu_v}{\rho} \right) k^2 \left( \frac{\bar{\mu}(\xi)}{\mu_v} \right) \] \hspace{1cm} (4.17)

where \( k = \text{Re}(\xi) \) and \( K \) takes on one of the two values

\[ K = 1.017 \quad \text{or} \quad K = 4.433. \] \hspace{1cm} (4.18)

We may write (4.17) as

\[ c = \frac{\omega}{k} = K \left( \frac{\mu_v}{\rho} \right) k \left( 1 + \frac{\bar{\mu}(\xi)}{\mu_v} \right) \] \hspace{1cm} (4.19)

where \( c \) is the surface wave velocity for the nonlocal viscous fluid.

Letting

\[ c_R = K \left( \frac{\mu_v}{\rho} \right) k \]
where \( c_R \) is the classical Rayleigh type wave velocity, we may write the dispersive relation in the form

\[
\frac{c}{c_R} = 1 + \frac{\bar{\mu}(\xi)}{\bar{\mu}_v} . \tag{4.21}
\]

Using the dispersion relation from one-dimensional lattice dynamics [29]:

\[
\frac{c^2}{c_R^2} = \pi d^2 (\lambda + 2\mu) \sum_{n=1}^{N} a_n (1 - \cos n\pi k') \tag{4.22}
\]

where \( d \) is the distance between atoms, \( \lambda \) and \( \mu \) are material coefficients, \( a_n \) is the force on a given atom (per unit volume of the atom) due to unit displacement of all atoms located at a distance \( nd \), and \( k' \) is the reduced wave number defined by

\[
k' \equiv \frac{k}{kd}, \quad -1 \leq k' \leq 1. \tag{4.23}
\]

We may write (4.22) as

\[
\frac{c}{c_R} = \left[ \sum_{n=1}^{N} A_n \left( \frac{2}{\pi k'} \right)^2 (1 - \cos n\pi k') \right]^{1/2} . \tag{4.24}
\]

Considering only the two nearest neighbor interactions \( (N = 2) \), and matching \( c/c_R \) at the long wave limit \( (k' = 0) \) and at the end of the Brillouin zone \( (k' = 1) \) we have:
\[
\frac{c}{c_R} = \frac{c_1}{c_R k'} \left[ \sin^2 \left( \frac{\pi k'}{2} \right) + \left( \frac{c_R}{c_1} \pi \right)^2 \frac{1}{4} \sin^2 (\pi k') \right]^{1/2}
\] (4.25)

where \( c_1 \equiv 1.88 \) for the case of two nearest neighbor interactions.

Using (4.19) and (4.20) yields

\[
\omega d = c_1 \left[ \sin^2 \left( \frac{\pi k'}{2} \right) + \left( \frac{K \mu}{\rho c_1 \pi} \right)^2 \frac{1}{4} \sin^2 (\pi k') \right]^{1/2}
\] (4.26)

which explicitly brings out the frequency dependence on the wave number \( k' \).

Figure 1 illustrates the dispersion relationship developed here applied to lubricating oil.

We wish to stress the point that the dispersion relation obtained here can only be an approximation to the true dispersive character of a fluid since we have used results from lattice dynamics to predict the behavior of an essentially randomly oriented medium. However, since the fluids we are considering here are viscoelastic in nature they possess some solid-like characteristics, which justifies to a significant extent the comparisons we have made here. Furthermore, it is of interest to note that the phonon dispersion curves obtained by Yarnell et al. [74] for various modes in the case of aluminum at 300°K bear a close resemblance to the dispersion curve obtained by us here. It is hoped that such comparisons will stimulate further experimental as well as theoretical work in the field.
Figure 1. Dispersion curve for lubricating oil.
5. Dispersion of Zenneck Type Waves

We again assume a solution field to (3.4) and (3.5) in the form of a double Fourier integral:

\[
D_k(x_1, x_2, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D_k(\xi, x_2, \omega) e^{-i(\xi x_1 + \omega t)} \, d\xi \, d\omega 
\]

\[
B_3(x_1, x_2, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} B_3(\xi, x_2, \omega) e^{-i(\xi x_1 + \omega t)} \, d\xi \, d\omega .
\]

Substituting (5.1) into the field equations (3.4) yields

\[
(\alpha^2 \omega^2 - \xi^2)\overline{D}_1 + \overline{D}_1, 22 = -\int_0^\infty (i\xi \overline{\overline{D}}_1 + \xi \overline{\overline{D}}_1) \, dx_2' 
\]

\[
+ ic\omega^2 \int_0^\infty \overline{\overline{D}}_1 \, dx_2' 
\]

\[
(\alpha^2 \omega^2 - \xi^2)\overline{D}_2 + \overline{D}_2, 22 = -\int_0^\infty (i\xi \overline{\overline{D}}_2 + \xi \overline{\overline{D}}_2) \, dx_2' 
\]

\[
- ic\omega^2 \int_0^\infty \overline{\overline{D}}_3 \, dx_2' 
\]

\[
(\alpha^2 \omega^2 - \xi^2)\overline{B}_3 + \overline{B}_3, 22 = \int_0^\infty (\xi \overline{\overline{B}}_3 + \overline{\overline{B}}_3) \, dx_2' 
\]

\[
+ ic\omega^2 \int_0^\infty (-i\xi \overline{\overline{B}}_2, 22) \, dx_2' .
\]
Assuming the nonlocal electromagnetic material moduli to vary rapidly as we move into the media, \( x_2 > 0 \), we idealize this change mathematically by considering their behavior to be \( \delta \)-functions in the \( x_2 \)-variable. Thus we have

\[
\tilde{\varepsilon}'(\xi, |x_2 - x'_2|) = \tilde{\varepsilon}(\xi) \delta(|x_2 - x'_2|) \quad \text{(5.3)}
\]
\[
\tilde{\mu}'(\xi, |x_2 - x'_2|) = \tilde{\mu}(\xi) \delta(|x_2 - x'_2|) .
\]

Since the \( \delta \)-function has compact support, upon using (5.3) in (5.2) yields

\[
(k+1)\overline{D}_{1,2} + m^2 k \overline{D}_1 - i \xi \overline{D}_{2,2} + ic\omega k \frac{2}{\ell} \overline{B}_{3,2} = 0
\]
\[
-ic\omega k \frac{2}{\ell} \overline{D}_{1,2} - \omega k \frac{2}{\ell} \overline{D}_2 + (\ell - 1) \overline{B}_{3,2} + (\ell m^2 - \xi^2) \overline{B}_3 = 0
\]

where

\[
k = \varepsilon^{-1}, \quad \ell = \mu^{-1}, \quad m^2 = \omega^2 - \xi^2 .
\]

Zenneck type waves die out exponentially as they penetrate the medium so that

\[
\overline{D}_k(\xi, x_2) = d_k(\xi)e^{-\gamma x_2}
\]
\[
\overline{B}_3(\xi, x_2) = b(\xi)e^{-\gamma x_2}
\]

\[
(5.6)
\]
where \( \text{Re}(\gamma) > 0 \). Using (5.6) in (5.4) yields the following linear, homogeneous system:

\[
\begin{aligned}
[y^2(k+1)+m^2k]d_1 + i\xi \gamma d_2 - ic\omega \gamma a^2 \frac{k}{\ell} b &= 0 \\
i\xi \gamma d_1 + [k(\gamma^2+m^2) - \xi^2]d_2 + ic\omega \gamma a^2 \frac{k}{\ell} b &= 0 \\
ic\omega \gamma a^2 d_1 - c\omega \xi a^2 d_2 + \frac{k}{\ell} [\ell(\gamma^2+m^2) - \gamma^2 + \xi^2] b &= 0.
\end{aligned}
\]

(5.7)

A nontrivial solution to (5.7) occurs for values of \( \gamma \) satisfying

\[
\begin{vmatrix}
y^2(k+1)+m^2k & i\xi \gamma & -ic\omega \gamma a^2 \\
i\xi \gamma & k(\gamma^2+m^2) - \xi^2 & ic\omega \gamma a^2 \\
ic\omega \gamma a^2 & -c\omega \xi a^2 & \ell(\gamma^2+m^2) - \gamma^2 + \xi^2
\end{vmatrix} = 0
\]

(5.8)

Since (5.8) results in a sixth degree algebraic equation one must resort to numerical means for the determination of \( \gamma \). However, to follow such a procedure the dependence of \( k \) and \( \ell \) on \( \xi \) must be known. At present, due to the lack of experimental data the nature of this dependence remains undetermined. Thus, we shall not pursue the determination of \( \gamma \) and hence the dispersion relation for the Zenneck type waves will not be derived here.

It is important to note that if we ignore all of the nonlocal effects, that is, examine the problem in the classical setting then no dispersion relation can be obtained. In other words, the classical
electromagnetic theory fails to predict any dispersive character in the Zenneck type waves. The dispersive nature of these waves is brought out only by the inclusion of the nonlocal effects.
VI. SUMMARY, DISCUSSION AND SCOPE OF FUTURE WORK

The work herein is the only theoretical development involving electromagnetic interactions with viscous fluids which accounts for nonlocal effects. With the growing importance and interest in electromagnetic phenomena and the encouraging success of the nonlocal theory in predicting material responses not borne out by the classical continuum theories the present theoretical results should prove invaluable for further research in this area.

Subsequent work that is closely related to the theory we have developed here, as well as the surface phenomena we have studied, can take many forms. For example, the mathematical nature and form of the localization residuals, both mechanical and electromagnetic, is yet to be determined. Such a determination must necessarily come about from experimental work as well as theoretical investigations. Also, the study of the propagation of the electromechanical surface waves undertaken in Chapter V should be analyzed in a more general sense. By this we mean the retention, of at least to some measure, the nonlinear terms so that the decoupling effect of the surface waves does not occur, and hence allowing one to study the electromechanical interactions and their subsequent dispersion. Because of the incorporation of the electromagnetic constitution and the nonlocal effects in our theory, other surface phenomena, such as the
effects of electromagnetic fields on surface tension may be investigated. Another problem of interest is the study of the effects of external electromagnetic fields on such a medium as the one we have characterized, e.g. nonlocal magnetohydrodynamics.

Other applications encompass such areas as ferromagnetism, neurophysiological systems, high-current flow through conductors, and energy conversion. Of particular interest would be the development of the linear constitutive theory of the nonlocal polar electromagnetic materials developed in Chapter IV and its subsequent application to derive both the acoustic and optic dispersion modes of electromechanical surface waves.

As we have previously mentioned, the experimental work designed to detect and measure nonlocal effects and hence determine empirically the nonlocal material moduli is very scarce at this point in time. The thermodynamic restrictions on these moduli, which we developed in Chapter III, should prove invaluable to experimentalists when interpreting the outcomes of such experiments. Furthermore, such applications as those in Chapter V should encourage experimentalists to pursue the determination of the nonlocal material moduli since only through nonlocal effects can many such phenomena be successfully predicted.
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