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A technique of differentiation with respect to the distance to the boundary of an outer parallel-body is applied to known measures of sets of p-dimensional linear spaces which intersect a general convex body in n-dimensional euclidean space in order to obtain an appropriate definition of the measures of sets of p-dimensional linear spaces which are tangent to a general convex body in n-space. A few side results are obtained along the way, and there are included two applications of these measures of tangents. The first is a simple application to geometric probabilities in 3-space, and the second yields a new and integro-geometric proof of Kubota's formula.

THE INTEGRO-GEOMETRIC TANGENT MEASURES OF EUCLIDEAN N-SPACE

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THE INTEGRO-GEOMETRIC TANGENT MEASURES OF EUCLIDEAN N-SPACE.

Chapter 1. Introduction.

If the euclidean plane is covered by an infinity of lines drawn at random, it seems reasonable to suppose that the measure of the set of these lines which meet a given line segment is proportional to the length of the line segment, and that it is the same for all positions of the segment. If we take s as the length of the segment and $2s$ as the measure of this set of lines, then for a convex curve C whose boundary has length L, each line intersecting the interior of C meets its boundary in two points and the measure of the set of straight lines which meet the interior of C is L. When the intuition is made rigorous we get a well-known integro-geometric measure.

We will deviate from the established integral geometry at this point by asking for a measure of lines which are tangent to C, or for a measure of lines tangent to C at a subset of the boundary of C. In two dimensions these questions are settled easily, and the first really interesting case is the determination of the measure of the set of lines tangent to a convex body in three-dimensional space.

If K is a convex body in three-space then Santaló (8, p. 32) obtained a measure for the set of lines which meet the interior of K, namely $\frac{\pi}{2}S(K)$, where $S(K)$ is the surface area of K.

To find the desired definition for the measure of lines tangent to K we use the following technique: consider an outer-parallel body $K(t)$ whose boundary is determined by the endpoints of the vectors of length t normal to the surface of K and originating from points of

this surface. Steiner's formula gives us the surface area $S(K(t))$ of $K(t)$:
$$S(K(t)) = S(K) + 2tM(K) + 4\pi t^2,$$
where $M(K)$ is the total mean curvature (sometimes called total integral curvature). Thus, the measure of the lines that pass only through the shell of width t between the boundaries of K and $K(t)$ is simply the difference

$$\frac{\pi}{2}S(K(t)) - \frac{\pi}{2}S(K).$$

Next we divide by the factor t and let $t \to 0^+$.

Thus,

$$(1/t)[\frac{\pi}{2}S(K(t)) - \frac{\pi}{2}S(K)] = \pi M(K) + 2\pi^2 t$$

$$\to \pi M(K) \quad \text{as } t \to 0^+.$$

Intuition hence dictates that the proper definition of the measure of lines tangent to K is $\pi M(K)$.

The purpose of this dissertation is to utilize the above technique applied to the previously mentioned results of Santaló to obtain the definition of the measures of sets of p-dimensional linear spaces tangent to a general convex body in n-dimensional euclidean space, and to obtain some new results with these definitions. A few side results are obtained along the way. Chapter two establishes some of the unorthodox notation and use of terms and delineates briefly the basic mathematical tools necessary for the sequel. Chapter three is a presentation of the theory of mixed surface area functions to the extent, more or less, it will be used later in the paper. The results obtained in sections 3.8 and 3.9 the author has not been able to find in the literature. Chapter four develops a description of the mixed surface area functions of chapter three in terms of higher-dimensional volumes and angles, and culminates in 4.5, which formula is part of mathematical folklore, although there has been no proof given in the literature. Chapter five consists of a compilation of the basics of integral geometry for use in chapter six, in which is developed the measure for sets of linear spaces tangent to a convex body. These results, as mentioned above, are not yet part of previously established integral geometry. Chapter seven obtains a generalization of the results in chapter six with some added detail for polytopes by using the representation 4.5 of chapter four. Chapter eight contains two applications of these results, the first is a simple application of notions of chapter seven to geometric probabilities, and the second is an application of the results of chapter six which yields a new and integro-geometric proof of Kubota's formula.

Chapter 2. Preliminaries.

This chapter contains the notations, terminology and basic known facts necessary for the understanding of this paper.

I. The Underlying Space.

Throughout this paper the space under consideration will be the n-dimensional real euclidean space, denoted by E_n, in which points will be represented by ordered n-tuples of real numbers as follows:

$$x = (x_i) \qquad (i=1,2,\cdots,n).$$

It is assumed that the reader is familiar with the elementary geometric, topological, and measure-theoretic properties of E_n, and that he understands the elementary concepts involved in considering E_n as a vector space.

The origin of E_n is the point $x \in E_n$ for which $x_i = 0$, $(i=1, 2,\cdots,n)$. The usual basis vectors of E_n are denoted by the symbol e_k, $(k=1,2,\cdots,n)$. We will identify vectors with their endpoints, whence,

$$e_k = (\delta_{ik}) \qquad (i=1,2,\cdots,n),$$

where

$$\delta_{ik} = \begin{cases} 1 \text{ if } k = i, \\ 0 \text{ if } k = i. \end{cases}$$

2.1. Definition. A p-flat L is a subset of E_n such that for some fixed $y \in L$ the set $\{x-y: x \in L\}$ is a linear vector subspace of E_n

having the vector dimension p. p-flats are sometimes called p-dimensional (linear) varieties. Note that a p-flat is simply a translate of a linear subspace. It can be shown that a p-flat is the solution space to a set of n-p linearly independent linear equations in n unknowns and, equivalently, a p-flat is the set of all vectors originating from a fixed point $y \in E_n$ which are perpendicular to a fixed set of n-p mutually orthogonal unit vectors also originating from y. For a final equivalent formulation, a p-flat is the set of all linear combinations of p mutually orthogonal unit vectors originating from y, i.e., the space spanned by these vectors. An (n-1)-flat in E_n is called a hyperplane.

The (euclidean) distance between two points $x, y \in E_n$ is

$$|x-y| = [\Sigma_{i=1}^{n} (x_i-y_i)^2]^{1/2},$$

where $x = (x_i)$, $y = (y_i)$, $(i=1,2,\cdots,n)$.

The norm or length of $x \in E_n$ (considered as a vector) is

$$|x| = [\Sigma\, x_i^2]^{1/2}.$$

2.2. Definition. The p-sphere (p-hypersphere) of radius r is $\Omega_p(r) = \{x \in L_{p+1}: r = x\}$, where L_{p+1} is a (p+1)-flat through the origin $(p < n)$. The unit p-sphere $\Omega_p(1)$ will be denoted by Ω_p.

2.3. Convention. The unit (n-1)-sphere of E_n will be denoted by Ω. We will consider more general spheres, viz., Ω_p may first be imbedded in E_n if $p < n$, and then moved about within the space. This suggests the following:

2.4. Definition. The p-sphere of radius r lying in a (p+1)-flat L_{p+1} and centered at $y \in L_{p+1}$ is

$$\Omega_p(y;r) = \{x \in L_{p+1} : |x-y| = r\}.$$

Note that $\Omega_p(y;r) = \{x \in L_{p+1} : (x-y) \in \Omega_p(r)\}$.

II. Convexity.

2.5. Definition. A set K E_n is *convex* if for each pair of points P,Q in K, the line segment joining P and Q lies entirely within K.

2.6. Definition. A *convex body* is a bounded, closed, convex subset of E_n.

2.7. Convention. Unless otherwise specified, convex bodies are assumed to have a non-empty topological interior.

2.8. Definition. For a bounded set $K \subset E_n$, the set-theoretic intersection of all the convex bodies in E_n containing K as a subset is called the *convex closure* of K, denoted by cvx(K). Further notions from the theory of convex bodies such as *support plane*, *support function*, *mixed volume*, etc., will be used here without formal definition. For details see (2), (3), or (5). We include next some basic notions in the setting of the n-dimensional euclidean space E_n.

2.9. Definition. A hyperplane ((n-1)-flat) π in E_n is a *support (hyper)plane* to a convex body K in E_n if π contains points of the topological boundary of K and does not meet the interior of K.

<u>2.10. Definition</u>. For a convex body K in E_n with x in its boundary, define

$$\Pi(x) = \{\pi : \pi \text{ is a support plane to } K \text{ and } x \in \pi\}$$

For the remainder of this paper ∂K shall denote the topological boundary of a set K.

<u>2.11. Definition</u>. A p-flat L in E_n is <u>tangent</u> to the convex body K at the point $x \in \partial K$ if the following two conditions are met:

1. $x \in L$.
2. There is a $\pi \in \Pi(x)$ such that if $\xi \in \Omega$ is the normal to π then ξ is normal to L.

Note that the support planes to K are themselves tangent to K in the above sense. In short, 2.11 says a p-flat L is tangent to K at x if L is contained in some support plane to K and contains x.

<u>III. Polytopes</u>.

A convex polytope is the n-dimensional analogue of a convex polygon in E_2 and of a convex polyhedron in E_3. We deal here only with <u>convex</u> polytopes.

<u>2.12. Definition</u>. A <u>polytope</u> is the convex closure of a finite set of points in E_n.

A polytope is <u>proper</u> (or n-dimensional) if it has an interior point or, equivalently, if it is not a subset of a bounding hyperplane.

The following properties of polytopes are well-known (see, for example, (4, p. 127) or (7, pp. 2-6)):

1. A polytope is a bounded convex subset of E_n enclosed by a finite number of hyperplanes. The part of the polytope that lies in one of the bounding hyperplanes is called a face.
2. The faces of an n-dimensional polytope are themselves (n-1)-dimensional polytopes imbedded in the bounding hyperplanes.
3. To each polytope there corresponds a sequence of polytopes of descending dimension where the p-dimensional polytopes are the faces of the (p+1)-dimensional polytopes, $(p=0,1,\cdots,n-1)$.

The polytopes of dimension p described in property three are called the p-dimensional faces of the original polytope.

If $P(j,k)$, $(k=1,2,\cdots,r(j))$, are the j-dimensional faces of P, where $r(j)$ is the number of these faces of dimension j, $(j=0,1,\cdots,n-1)$, and we set $r(n) = 1$, $P(n,1) = P$, we have:

2.13. Definition. The j-skeleton of the polytope P is

$$P(j) = \bigcup \{P(j,k): k = 1,2,\cdots, r(j).\}$$

Thus, we have $P = \bigcup_{j=1}^{n} P(j)$ and $P(k) \subset P(j)$ whenever $j \geq k$.

The next definition allows us to consider the faces of a particular dimension of a polytope without including those of lower dimension.

2.14. Definition. The open j-skeleton $P^{0}(j)$, $(j > 0)$, of the polytope P is $P(j)-P(j-1)$, where the minus denotes set theoretic difference. We adopt the convention that $P^{0}(0) = P(0)$.

IV. Spherical Images.

We next extend the notion of the spherical image of a smooth surface as usually defined in differential geometry to include the polytope case.

2.15. Definition. For the convex body K in E_n with the set $k \subset \partial K$ we define the spherical image of k to be

$$\omega(k) = \{\xi \in \Omega: \text{ for some } x \in k \text{ and some } \pi \in \Pi(x),\ \xi \text{ is normal to } \pi\}.$$

Given ω as a subset of Ω, the inverse spherical image of ω is

$$K(\omega) = \{x \in \partial K: \text{ for some } \xi \in \omega \text{ and some } \pi \in \Pi(x),\ \xi \text{ is normal to } \pi\}.$$

V. Measure.

Throughout this paper the symbol m_p will be used in one of two ways:

1. m_p is the p-dimensional Lebesgue measure on any p-flat which reduces to the usual p-volume if it exists.
2. m_p is the p-dimensional Riemannian measure on the surface of any p-sphere which reduces to the usual p-surface area when it exists (also called "surface content").

2.16. Convention. The zero-dimensional measure of a point is one.

Chapter 3. Mixed Surface Area Functions

This chapter contains the salient aspects of the theory of mixed surface area functions as defined and developed by Fenchel and Jessen in (6). An English language summary of this work may be found in (3, pp. 60-72). Theorems from these sources will be used without proof. In addition, results from (2) will be employed to bridge any gaps separating concepts of chapter two from those to be developed presently.

Let P be a convex polytope, $v_1, v_2, \cdots, v_n$ the outer unit normal vectors of its (n-1)-dimensional faces, and $s_1, s_2, \cdots, s_n$ the corresponding (n-1)-volumes of these faces.

<u>3.1. Definition</u>. The <u>surface area function</u> of P is that set function on Ω whose value, $S(P,\omega)$, on the borel set $\omega \subset \Omega$ is the sum of those s_i for which $v_i \in \omega$, i.e.,

$$S(P,\omega) = \Sigma s_i, \qquad (v_i \in \omega).$$

If K is an arbitrary convex body with support function $H(v)$ then (2, p. 41) the mixed volume is

$$V(K,P,\cdots,P) = \frac{1}{n} \Sigma_{j=1}^{n} H(v_j)s_j.$$

This formula has a representation as the Stieltjes integral

$$V(K,P,\cdots,P) = (1/n)\int_\Omega H(v)\ S(P;\ d\omega)$$

where the integral is taken over Ω with respect to the set function $S(P;\cdot)$.

The formula is true not only for polyhedra, but for an arbitrary convex body. To show this, one would use a limit argument applied to a sequence of polyhedra which approaches the given arbitrary convex body in the Blaschke topology (see (2, pp. 34-35)), keeping in mind the continuity of the mixed volumes with respect to this topology (see (2, p. 40)). This motivates the following:

3.2. Theorem. To each convex body K there corresponds exactly one set function $S(K;\cdot)$ with the property that for each convex body K' with support function $H(v)$

$$V(K',K,\cdots,K) = \frac{1}{n} \int_{\Omega} H(v)\ S(K;d\omega),$$

and this set function depends continuously on K in the sense of set functions. Moreover, the total surface area of K is $S(K;\Omega)$. $S(K;\cdot)$ is called the surface area function of K.

The meaning of "continuity in the sense of set functions" is given in (6, p. 8) and (3, pp. 61-62). It is defined by a type of weak convergence. The proof of 3.2 is given in (6, pp. 11-12) and (3, pp. 62-63).

3.3. Theorem. Two convex bodies (with interior points) have the same surface area function if and only if they are translates of one another. The proof is in (6, pp. 15-16), (3, p. 63).

The idea of a mixed surface area function is similar to that of $S(K;\cdot)$ in the sense that it possesses the same type of non-constructive definition within a theorem as follows:

3.4. Theorem. To a set of (n-1) convex bodies $K_1, K_2, \cdots, K_{n-1}$ there corresponds exactly one set function $S(K_1,\cdots,K_{n-1};\cdot)$ called the mixed surface area function of $K_1, K_2, \cdots, K_{n-1}$, with the property that for each convex body K with support function $H(v)$

$$V(K,K_1,\cdots,K_{n-1}) = \frac{1}{n}\int_{\Omega} H(v)\, S(K_1,\cdots,K_{n-1};\, d\omega),$$

and this set function depends continuously on $K_1, K_2, \cdots, K_{n-1}$ in the sense of convergence of set functions. Also, $S(K,\cdots,K;\omega) = S(K;\omega)$. See (6, pp. 21-22) or (3, p. 69) for the proof.

3.5. Definition. Let $B \subset E_n$ denote the unit ball (Ω and its interior) and p denote an integer with $0 \leq p \leq n-1$. Fenchel-Jessen (6, pp. 25-26) defines the p^{th} surface area function of K:

$$S_p(K;\cdot) = S(\underbrace{K,\cdots,K}_{p},\underbrace{B,\cdots,B}_{n-p-1};\cdot).$$

Note that in particular, $S_{n-1}(K;\omega) = S(K;\omega)$ and $S_0(K;\omega) = m_{n-1}(\omega)$ (cf. chapter 2, V. 2 above), where $\omega \subset \Omega$ is a borel set.

We have a representation for S_p in the smooth case, i.e., where the principal radii of curvature exist continuously on the surface of a convex body. Using the above definition, Fenchel-Jessen gives a proof of:

3.6. Theorem. Let $\{R_1,\cdots,R_p\}$ be the p^{th} elementary symmetric function of the principal radii of curvature on the smooth surface ∂K of the convex body K. Then

$$S_p(K;\omega) = \frac{1}{\binom{n-1}{p}} \int_\omega \{R_1,\cdots,R_p\} dv.$$

The proof follows from the uniqueness of S_p (3.4) and the representations of mixed volumes in (2, p. 59 and p. 63). For two sets $A,B \subset E_n$ and real numbers $a,b \geq 0$ we can define $a\cdot A = \{ax: x \in A\}$, and $a\cdot A + b\cdot B = \{ax+by: x \in A \text{ and } y \in B\}$. This definition can be extended recursively to any finite number of sets and non-negative real numbers, and is simply vector addition and scalar multiplication for sets.

<u>3.7. Theorem.</u> Let $K^1, \cdots, K^q, K_1, \cdots, K_{n-p-1}$ $(0 < p \leq n-1)$ be arbitrary convex bodies and $r_1, r_2, \cdots, r_q$ be non-negative numbers. If $K = r_1K^1 + \cdots + r_qK^q$ then

$$S(K,\cdots,K,K_1,\cdots,K_{n-p-1};\omega) =$$

$$\sum_{a \in I(p,q)} r_{a(1)}\cdots r_{a(p)} S(K^{a(1)},\cdots,K^{a(p)},K_1,\cdots,K_{n-p-1};\omega)$$

where $I(p,q)$ is the set of sequences of length p from the integers $1,2,\cdots,q$, and if $a \in I(p,q)$ then $a(i)$ is the value of the sequence a at the integer i.

The proof is in (6, p. 23).

<u>3.8. Corollary.</u> Let K be an arbitrary convex body, t a non-negative real number and $B \subset E_n$ the unit ball. Then,

$$S_p(K+tB;\omega) = \Sigma_{k=0}^{p} \binom{p}{k} t^k S_{p-k}(K;\omega).$$

Proof: In 3.7 set $K_j = B$, and for $0 \leq q \leq p$,

$$r_j = \begin{cases} 1 & \text{if } j = 1 \\ t & \text{if } j = q \\ 0 & \text{if } 1 < j < q \end{cases} \qquad K^j = \begin{cases} K & \text{if } 1 \leq j \leq q-1 \\ B & \text{if } j = q \end{cases}$$

Thus, $r_1K^1 + \cdots + r_qK^q = K + tB$. The product $r_{a(1)}\, r_{a(2)} \cdots r_{a(p)}$ $= 0$ whenever the sequence a has numbers other than q or 1 in its range, whence $r_{a(1)}\, r_{a(2)} \cdots r_{a(p)} = t^k$, where k is the number of $a(i)$'s equal to q. This occurs in exactly $\binom{p}{k}$ different ways. Now, from 3.4 and the fact that mixed volumes are symmetric we conclude that mixed areas are also symmetric, whence

$$S(K^{a(1)},\ldots,K^{a(p)},B,\cdots,B;\omega) = S_{p-k}(K;\omega).$$

Since $0 \leq k \leq p$ yields all possible terms the result is established.

<u>3.9. Corollary</u>. Let K be an arbitrary convex body, B the unit ball in E_n, and t a non-negative real number, then

$$\lim_{t\to 0^+} (1/t)[S_p(K+tB;\omega)-S_p(K;\omega)] = pS_{p-1}(K;\omega).$$

The proof follows from the fact that

$$(1/t)[S_p(K+tB;\omega)-S_p(K;\omega)] = \Sigma_{k=1}^{p} \binom{p}{k} t^{k-1} S_{p-k}(K;\omega).$$

<u>3.10. Definition</u>. Let K be an arbitrary convex body, ω a borel set on Ω, and t an arbitrary positive number. For $v \in \omega$ there exists a unique support plane $\pi(v)$ to K with v as its outer normal. Let $a \in \pi(v) \cap \partial K$. For each such pair v, a, the line segment

$$L(v,a) = \{x \in E_n\colon x=a+rv,\ 0 < r \leq t\}$$

is called the <u>bristle</u> at a in the direction v of length t. The <u>brush set</u> of K belonging to ω and t is the union of all the corresponding bristles and is denoted by $B_t(K;\omega)$, i.e.,

$$B_t(K;\omega) = \bigcup \{L(v,a)\colon a \in \pi(v) \cap \partial K,\ v \in \omega\}.$$

Finally, we denote the n-dimensional measure of $B_t(K;\omega)$ by $V_t(K;\omega)$.

Note that $B_t(K;\Omega)$ is the set-theoretic difference of the parallel-bodies $K + tB$ and K (where B is the unit ball). Also, $V_t(K;\omega)$ is a set function on Ω satisfying

$$V_t(k;\Omega) = V(K+tB) - V(K),$$

where $V(K+tB)$, $V(K)$ are respectively the n-volumes of $K + tB$ and K.

<u>3.11. Theorem</u>. For each borel set ω on Ω and each arbitrary convex body K,

$$S(K,\omega) = \lim_{t\to 0^+} (1/t)V_t(K;\omega).$$

The proof of this Minkowski relationship between surface area and volume as generalized to set functions on Ω is in (6, pp. 29-30).

From the definition of brush set,

$$V_{t+\Delta t}(K;\omega) = V_t(K;\omega) + V_{\Delta t}(K+tB;\omega),$$

where $\Delta t > 0$ is arbitrary. Thus, from 3.11,

$$\lim_{\Delta t\to 0^+} (1/\Delta t)[V_{t+\Delta t}(K;\omega) - V_t(K;\omega)] = S(K+tB;\omega).$$

Taking $p = n-1$ in 3.8, we obtain

$$S(K+tB;\omega) = \Sigma_{k=0}^{n-1} \binom{n-1}{k} t^k S_{n-1-k}(K;\omega),$$

whence

$$\lim_{\Delta t \to 0^+} (1/\Delta t)[V_{t+\Delta t}(K;\omega) - V_t(K;\omega)] = \Sigma_{k=0}^{n-1} \binom{n-1}{k} t^k S_{n-1-k}(K;\omega).$$

Simultaneously integrating both sides of this last equation with the initial condition $V_0(K;\omega) = 0$ one gets the following:

3.12. Theorem. In the above notation,

$$V_t(K;\omega) = \frac{1}{n} \Sigma_{k=1}^{n} \binom{n}{k} t^k S_{n-k}(K;\omega).$$

This theorem also occurs in (6, p. 31) and is a generalization of the well-known Steiner formula for the volume of a parallel-body.

In the next chapter we shall see the strength of 3.12 as a representation of $V_t(K;\omega)$ as a polynomial in t with unique coefficients: the mixed surface area functions. This allows a new representation of these mixed surface area functions in more geometrical terms.

Chapter 4. Representations of Mixed Surface Area Functions in Terms of Angles and Volumes.

This chapter contains the development of theorems relating the concepts of brush set and mixed surface area functions to the concepts of n-dimensional angles and volumes. We begin with a brief description of the various angles in E_n that concern us and the way they are measured. The procedure is by analogy, beginning in E_2 and then increasing the dimension. A full treatment of this topic may be found in (10, pp. 152 f.).

In the plane:

There is just one species of angle, the plane angle. The radian is the unit of measure and the complete angle at a point is 2π. The radian measure of an angle is obtained as the arclength cut from the unit circle centered at the vertex of the angle by the legs of the angle.

In dimension three:

In addition to the plane angle as above, we have two species of solid angle:

1. The dihedral angle (angle at an edge).
2. The polyhedral angle (angle at a vertex).

The first is determined by two planes, the second by at least three. The unit of solid angle is chosen so that the measure of the angle is equal to the area cut out on the surface of the unit sphere centered on the edge or vertex of the configuration being considered. Note that we are considering only species of polygonal angles, i.e., derived from intersecting planes. These suffice for our present purposes.

In dimension n.

There are $n-1$ species of angles we consider:

1. Angles bounded by more than $n-1$ hyperplanes. These angles originate at a point.
2. Angles bounded by $n-1$ hyperplanes. These angles originate at a line.

...

n-1. Angles bounded by 2 hyperplanes. These angles originate at an $(n-2)$-flat.

The radian measure of these angles is the $(n-1)$-surface content (Ch. 2, V, 2 above) cut out on the unit hypersphere centered at the origin of the angle. This is equal to the ratio of the surface content cut from an arbitrary hypersphere with the same center to the $(n-1)$ power of its radius, or, equivalently, the n-volume of the hypersector cut from the arbitrary hypersphere divided by the n^{th} power of its radius and multiplied by n (see (10, p. 153) for a discussion of this). This last characterization is useful for the following theorem:

4.1. Theorem. Let F be a p-face in the open p-skeleton of the polytope K in E_n (i.e., a p-face of K with its lower-dimensional faces removed), where $0 \leq p < n$. Let t be an arbitrary positive number. If $\omega(F)$ denotes the spherical image of F (see 2.15 above), then

$$V_t(K;\omega(F)) = \frac{m_p(F) \cdot m_{n-p-1}(\omega(F))}{n-p} t^{n-p}.$$

Proof: We evaluate $V_t(K;\omega(F))$ by integrating the area of the orthogonal sections of F as follows: for each $x \in F$ define $F'(x)$ to be the orthogonal space to F through x, i.e., the set of all combinations of vectors originating at x orthogonal to F. Since the dimension of F is p, the dimension of $F(x)$ is $n-p$. The orthogonal section $G_t(x)$ at x is the intersection of the brush set of K belonging to $\omega(F)$ and t with $F'(x)$, i.e.,

$$G_t(x) = B_t(K;\omega(F)) \cap F'(x).$$

If $\Omega_{n-p}(x;t)$ is the (n-p)-sphere of radius t lying in $F'(x)$ and centered at x, then $G_t(x)$ is the (n-p)-hypersector cut from this sphere by the outer boundary of the brush set. The radius of this (n-p)-sector is t, and its (n-p)-angle is $m_{n-p-1}(\omega(F))$, the (n-p-1)-surface content of $\omega(F)$. This follows because each $v \in \omega(F)$ determines a bristle of $B_t(k;\omega(F))$ lying in the hypersector, and each line segment from x of length t lying in this hypersector is a bristle in $B_t(K;\omega(F))$ and hence has a direction $u \in \omega(F)$. From the previous discussion, the (n-p)-content of a hypersector of radius t and (n-p)-angle θ is $\theta t^{n-p}/(n-p)$, whence

$$m_{n-p}(G_t(x)) = m_{n-p-1}(\omega(F))\, t^{n-p}/(n-p).$$

Thus,

$$V_t(K;\omega(F)) = \int_F m_{n-p}(G_t(x))dm_p(x)$$

$$= m_{n-p-1}(\omega(F))\, m_p(F)\, t^{n-p}/(n-p),$$

where the integration was performed easily since the integrand is independent of x. This completes the proof of 4.1.

If ω is a borel subset of $\omega(F)$, then we shall consider the angle of the subsector of $G_t(x)$ formed by those bristles of $G_t(x)$ having directions in ω to be $m_{n-p-1}(\omega)$. With this extension the proof of 4.1 is valid also for the following:

4.2. Corollary. Under the hypotheses of 4.1, if ω is a borel subset of $\omega(F)$ then

$$V_t(K;\omega) = m_p(F)\, m_{n-p-1}(\omega)\, t^{n-p}/(n-p).$$

4.3. Corollary. If K is a polytope in E_n, ω a borel subset of Ω, and if for $0 \leq p \leq n-1$, $F(p,1)$, $F(p,2)$, $\cdots$, $F(p,q(p))$ are all the p-faces in the open p-skeleton of K, where there are $q(p)$ such p-faces, then

$$V_t(k;\omega) = \Sigma_{p=0}^{n-1} \Sigma_j m_p(F(p,j))\, m_{n-p-1}(\omega(F(p,j)) \cap \omega)\, t^{n-p}/(n-p).$$

Proof: This result is a consequence of three facts:

1. Corollary 4.2.
2. V_t and m_p (as a spherical and as a flat measure) are completely additive.
3. The brush sets of distinct open faces of K are disjoint.

Number 1 has been dealt with already, number 2 is well-known for m_p from measure theory, and hence is true for its particularization, V_t. Number 3 is true by definition, and it is the reason no cross terms (with respect to the index j) occur in the formula.

Corollary 4.3 yields a representation of $V_t(K;\omega)$ as a polynomial in t, thus allowing a comparison of coefficients with 3.12. First we restate 3.12:

<u>4.4. Theorem (restatement of 3.12).</u> If K is a convex body in E_n and ω a borel subset of Ω, then

$$V_t(K;\omega) = \frac{1}{n} \Sigma_{p=0}^{n-1} \binom{n}{p} S_p(K;\omega)\ t^{n-p}.$$

Proof: Replace the k of 3.12 by $n-p$ and use $\binom{n}{p} = \binom{n}{n-p}$.

<u>4.5. Theorem.</u> If K is a polytope in E_n and ω is a borel subset of Ω, and if for $0 \leq p \leq n-p$, $F(p,1)$, $F(p,2)$, $\cdots$, $F(p,q(p))$ are all the p-faces of K in the open p-skeleton, then

$$\binom{n-1}{p} S_p(K;\omega) = \Sigma_j m_p(F(p,j))\ m_{n-p-1}(\omega(F(p,j)) \cap \omega).$$

Proof: Equate the coefficients of t^{n-p} in the two representations of $V_t(K;\omega)$ in 4.3 and 4.4.

Chapter 5. Integro-Geometric Densities and Measures.

The objective of this chapter is to establish the integro-geometric theory pertinent to the rest of this paper. We need examine only a small part of the theory of integral geometry here. We begin with a brief outline of the theory as presented in (9).

Using the orthogonal cartesian coordinates (x,y) of a point $P \in E_2$, the group M of rigid motions of E_2 can be represented by the equations

$$x = a + x_1 \cos \alpha - y_1 \sin \alpha$$

$$y = b + x_1 \sin \alpha + y_1 \cos \alpha.$$

We wish to define a measure for sets X of points which is invariant under the transformations of M. If we limit ourselves to measures which can be expressed by multiple integrals of the form

$$m(X) = \int_X f(x,y)dxdy,$$

then one can show (9, p. 6) that f is a constant function. This leads us to the following:

<u>5.1. Definition</u>. The measure of a set X of points $P \in E_2$ is defined by

$$m(X) = \int_X dxdy,$$

and up to a constant factor this is the only measure of its form invariant under the group M of rigid motions of E_2.

The differential form $dxdy$ is called the <u>density</u> for sets of points and is frequently denoted by dP. Densities are always taken in absolute value to insure positive measures.

There is no reason to restrict our endeavor to sets of points; there are other kinds of geometric figures for sets of which we may also define unique measures and densities invariant under M. Indeed, this is the essence of general integral geometry: to define measures and densities for sets of "geometrical elements" in such a way that they are invariant under a certain type of group (see (9), part III, especially section 18). Our present interest does not extend that far, however, and we proceed to discuss only those concepts necessary for this paper.

We seek next to define density and measure invariant under M for sets X of straight lines G. The crucial move here is choice of coordinates for G, the <u>normal</u> coordinates p, φ, i.e., the distance p of G from the origin and the angle φ that G makes with the positive x-axis, so that the equation of G is

$$x \cos \varphi + y \sin \varphi - p = 0.$$

<u>5.2. Definition</u>. The measure $m(X)$ of a set X of straight lines $G(p,\varphi)$ is defined by

$$m(X) = \int_X dp d\varphi,$$

and up to a constant factor, this measure is the only one which is invariant under the group M of motions in the plane. We call the differential form $dpd\varphi$ the <u>density</u> for straight lines and denote it by dG (9, p. 10).

Let K be a convex body in E_2 with rectifiable boundary ∂K, then we have:

5.3. Theorem. If $X = \{G: G \text{ is a line intersecting } K\}$, then $m(X) = m_1(\partial K)$, i.e., the measure of the set of straight lines which intersect a convex body is equal to the length of its perimeter.

The generalization of 5.3 to higher dimensions is important here and we state it without proof.

5.4. Theorem. Let K be an arbitrary convex body in E_n, and $Q_i = m_i(\Omega_i)$, i.e., Q_i is the surface content of the i-sphere Ω_i so that $Q_i = 2\pi^{(i+2)/2}/\Gamma((i+2)/2)$. If X is the set of p-flats which intersect K, then

$$m(X) = C_p^n \, S_{n-p}(K;\Omega),$$

where

$$C_p^n = \frac{Q_{n-2}\cdots Q_{n-p-1}}{2(n-p)Q_{p-1}\cdots Q_1} \quad \text{for } n > p > 0.$$

For the proof, see (8, pp. 31-32).

It is worthwhile to examine 5.4 in the cases of two and three dimensions:

a) $n = 2$, $p = 1$: $m(X) = S_1(X) =$ length of ∂K.

b) $n = 3$, $p = 1$: $m(X) = (\pi/2)\, S_2(K)$ ($S_2(K)$ is the surface area of K).

c) $n = 3$, $p = 2$: $m(X) = S_1(K;\Omega)$ (the total mean curvature for a smooth surface).

d) $n = 3$, $p = 0$: 5.4 does not cover this case, but since a zero-flat is a point, this is simply the measure of points intersecting K, i.e., the volume $V(K)$ of K.

Chapter 6. Total Measures of Tangents to Convex Bodies.

In this chapter we obtain a result similar to 5.4 which will yield a measure of p-flats which, rather than intersect a given convex body, are instead tangent to it.

If X is the set of p-flats intersecting the convex body K in E_n, then the set Y of p-flats tangent to K satisfies $Y \subset X$. One can show directly (see the derivation of 5.4: (8, pp. 31-32)) that Y is a set of measure zero as far as the measure of 5.4 is concerned. However, we can utilize the measure function introduced in 5.4 to lead us to the appropriate definition for the invariant measure of tangent p-flats.

We begin the discussion in the plane. Let K be a convex body in E_2, and $K(t)$ denote the parallel-body $K + tB$, $t > 0$. Taking $p = 1$ in 3.8 one obtains

$$S_1(K+tB;\Omega) = S_1(K;\Omega) + tS_0(K;\Omega).$$

But $S_1(K+tB;\Omega)$ is the length $L(K(T))$ of the boundary of $K(t)$, and $S_1(K;\Omega)$ is the length $L(K)$ of the boundary of K, and $S_0(K;\Omega) = m_1(\Omega_1) = 2\pi$. Thus,

$$L(K(t)) = L(K) + 2\pi t.$$

Define

$$T(t) = \{G: G \text{ is a line intersecting } K(t) \text{ and having no points in common with the interior of } K\},$$

where $t \geq 0$ and $K(0) = K$. Then by 5.4 and the formula for $L(K(t))$ we have

$$m(T(t)) = L(K(t)) - L(K) = 2\pi t.$$

Let $T = \bigcap_{t \geq 0} T(t)$; this is clearly the set of lines tangent to K, and

$$m(T) = \lim_{t \to 0^+} m(T(t)) = 0,$$

an expected result. This is like calculating the area of the perimeter of a square - the dimension of the measure does not match the dimension of the set to be measured. The reduction in dimension is accomplished by dividing $m(T(t))$ by the factor t before proceeding to the limit. Thus,

$$\underline{m}(T) = \lim_{t \to 0^+} m(T(t))/t = 2\pi.$$

This leads us to

6.1. Definition. The measure $\underline{m}(T)$ of the set T of all lines tangent to the convex body K in E_2 is $\underline{m}(T) = 2\pi$.

Note that 2π is also the total turning of the tangent, and also the one-volume (perimeter) of the surface of the one-circle.

The three-dimensional case has an essential difference from the preceding one in that in E_2 tangent lines correspond analogously to tangent planes in E_3, but lines tangent to a convex body in E_3 have no analogue in the plane. The treatment for E_3 is, however, similar to that for E_n, $n > 1$, so we proceed now to the general case and specify later the results for $n = 3$.

Let K be a convex body in E_n and B the unit ball of E_n. If $X(t)$ is the set of p-flats which intersect $K + tB$, then by 5.4,

$$m(X(t)) = C_p^n\, S_{n-p}(K+tB;\Omega) \quad (n > p > 0),$$

and if $X = X(0)$ is the set of p-flats which intersect K, then

$$m(X) = C_p^n\, S_{n-p}(K;\Omega).$$

Define

$$T(t) = \{G\colon G \text{ is a p-flat intersecting } K + tB \text{ and having no point in common with the interior of } K\}.$$

Then by 5.4,

$$m(T(t)) = m(X(t)) - m(X)$$

$$= C_p^n\, [S_{n-p}(K+tB;\Omega) - S_{n-p}(K;\Omega)].$$

Let $T = \bigcap_{t \geq 0} T(t)$; this is the set of p-flats tangent to K, whence using 3.9,

$$\underline{m}(T) = \lim_{t\to 0^+} m(T(t))/t$$

$$= C_p^n \lim_{t\to 0^+} (1/t)[S_{n-p}(K+tB;\Omega) - S_{n-p}(K;\Omega)]$$

$$= (n-p)C_p^n\, S_{n-(p+1)}(K+OB;\Omega).$$

This leads us to the next definition:

6.2. Definition. The measure $\underline{m}_p(T)$ of the set T of all p-flats tangent to the convex body K in E_n is

$$\underline{m}_p(T) = (n-p)C_p^n \, S_{n-(p+1)}(K;\Omega) \qquad (n > p > 0),$$

where C_p^n is the constant given in 5.4.

6.3. Theorem. If X_{p+1} is the set of (p+1)-flats which intersect the convex body K in E_n and T_p is the set of p-flats tangent to K, then

$$\underline{m}_p(T_p) = (n-p)(C_p^n/C_{p+1}^n) \, m(X_{p+1}).$$

Proof: From 5.4,

$$m(X_{p+1}) = C_{p+1}^n \, S_{n-(p+1)}(K;\Omega)$$

and from 6.2,

$$\underline{m}_p(T_p) = (n-p)C_p^n \, S_{n-(p+1)}(K;\Omega).$$

An application of 6.3 to E_3 yields (for example): the measure of lines tangent to a convex body in E_3 is proportional to the measure of planes intersecting the body, and the constant of proportionality is independent of the choice of the convex body.

6.4. Convention. The set of zero-flats (points) which intersect a convex body K is the volume $V(K)$.

According to 3.11 the procedure we used to obtain 6.2 when used in conjunction with 6.4 will yield

$$\underline{m}_0(T(0)) = \lim_{t\to 0^+} V_t(K;\Omega)/t = S(K;\Omega) = S_{n-1}(K;\Omega).$$

Thus, we can extend formula 6.2 to the case for $p = 0$:

<u>6.5. Definition</u>. The measure $\underline{m}_p(T)$ of the set T of all p-flats tangent to the convex body K in E_n is

$$\underline{m}_p(T) = \begin{cases} (n-p)C_p^n\, S_{n-(p+1)}(K;\Omega) & \text{if } n > p > 0, \\ S_{n-1}(K;\Omega) & \text{if } p = 0, \end{cases}$$

where C_p^n is the constant given in 5.4. (Alternatively one could define $C_0^n = 1/n$).

We examine 6.5 now for the special cases of E_2, E_3.

a) $n = 2$, $p = 0$: $\underline{m}_0(T) = S_1(K) =$ perimeter of K, i.e., the measure of points tangent to a convex curve is the length of curve.

b) $n = 2$, $p = 1$: $\underline{m}_1(T) = 1\cdot C_1^2\, S_0(K) = 2\pi =$ the total turning of the tangent = the perimeter of the one-sphere = Q_1.

c) $n = 3$, $p = 0$: $\underline{m}_0(T) = S_2(K) =$ the surface area of K. Thus, the measure of points tangent to K is its surface area.

d) $n = 3$, $p = 1$: $\underline{m}_1(T) = 2\cdot C_1^3\, S_1(K) = \pi S_1(K)$. The measure of lines tangent to K is π times its total mean curvature.

e) $n = 3$, $p = 2$: $\underline{m}_2(T) = S_0(K;\Omega) = 4\pi$. The measure of planes tangent to K is Q_2, the area of the two-sphere.

Note the similarity between b and e. These two measures do not depend upon the body K, but reflect the fact that K has a unique support hyperplane in each direction (cf. (2, p. 23)).

Chapter 7. Partial Measures of Tangents to Convex Bodies.

We focus our attention now on convex polytopes and begin with a comparison of 6.2 and 4.5.

7.1. Theorem. If P is a convex polytope in E_n, and if for $0 \leq r \leq n-1$, $F(r,1), F(r,2), \cdots, F(r,s)$ denote all the r-faces of P in the open r-skeleton $P^0(r)$ (cf. 4.1), where $s = s(r)$ is the number of such r-faces, and if T is the set of all p-flats tangent to P $(n > p > 0)$, then

$$\underline{m}_p(T) = (n-p)C_p^n\binom{n-1}{p}^{-1} \Sigma_j m_{n-p-1}(F(n-p-1,j))m_p(\omega(F(n-p-1,j))),$$

where C_p^n is the constant of 5.4.

Proof: From 6.2,

$$\underline{m}_p(T) = (n-p)\, C_p^n\, S_{n-p-1}(P;\Omega),$$

and from 4.5,

$$\binom{n-1}{n-p-1}S_{n-p-1}(P;\Omega) = \Sigma_j m_{n-p-1}(F(n-p-1,j))\, m_p(\omega(F(n-p-1,j))).$$

Using $\binom{n-1}{n-p-1} = \binom{n-1}{p}$, the result follows.

7.2. Corollary. Let $\mu = m_{n-p-1} \times m_p$ be the product measure on $P^0(n-p-1) \times \omega(P^0(n-p-1))$, and let

$$R(n-p-1,j) = F(n-p-1,j) \times \omega(F(n-p-1,j)),$$

then

$$\underline{m}_p(T) = (n-p)C_p^n\binom{n-1}{p}^{-1} \mu\left(\bigcup_j R(n-p-1,j)\right).$$

Proof: From the fact that $P^0(n-p-1)$ is a disjoint union of the faces $F(n-p-1,j)$ it follows that $\bigcup_j R(n-p-1,j)$ is a disjoint union, whence,

$$\mu\left(\bigcup_j R(n-p-1,j)\right) = \Sigma_j \mu(R(n-p-1,j))$$

$$= \Sigma_j\, m_{n-p-1}(F(n-p-1,j))\, m_p(\omega(F(n-p-1,j))).$$

The corollary 7.2 indicates that $\underline{m}_p$ can be thought of as a product measure of "dimension" $(n-p-1) + p = n-1$ for the case of a polytope. The following definitions are consistent with our previous results.

7.3. Definition. For a convex polytope P in E_n and the subset k of the boundary ∂P of P, define

$$\underline{k} = \bigcup_j [(k \cap F(n-p-1,j)) \times \omega(k \cap F(n-p-1,j))],$$

where $F(n-p-1,j)$, $(1 \leq j \leq r(n-p-1))$, are all the $(n-p-1)$-faces in $P^0(n-p-1)$, and if $T(k)$ is the set of p-flats tangent to P at points of k, we define

$$\underline{m}_p(T(k)) = (n-p)C_p^n\binom{n-1}{p}^{-1} \mu(\underline{k})$$

$$= (n-p)C_p^n\binom{n-1}{p}^{-1}(m_{n-p-1} \times m_p)\ (\underline{k}).$$

It follows from 7.2 that 7.3 yields the result of 7.1 for the case $k = \partial P$, i.e., $\underline{m}_p(T(\partial P)) = \underline{m}_p(T)$.

7.4. Definition. For the polytope P in E_n with the (n-p-1)-faces $F(n-p-1,j)$, $(1 \leq j \leq r(n-p-1))$, and the borel subset ω of Ω, define

$$\underline{\omega} = \bigcup_j [(P(\omega) \cap F(n-p-1,j)) \times (\omega \cap \omega(F(n-p-1,j)))]$$

and $T(\omega) = \{L_p$: L_p is a p-flat tangent to P and lying in a hyperplane π also tangent to P having the normal $\xi \in \omega\}$.

Then define

$$\underline{m}_p(T(\omega)) = (n-p)C_p^n\binom{n-1}{p}^{-1}(m_{n-p-1} \times m_p)(\underline{\omega}).$$

7.5. Theorem. $\underline{m}_p(T(\Omega)) = \underline{m}_p(T(\partial P)) = \underline{m}_p(T)$.

Proof: From 7.4 and the fact that $P(\Omega) = \partial P$,

$$\underline{m}_p(T(\Omega)) =$$

$$= (n-p)\binom{n-1}{p}^{-1}C_p^n(m_{n-p-1} \times m_p)(\bigcup_j [F(n-p-1,j) \times \omega(F(n-p-1,j))]$$

$$= (n-p)\binom{n-1}{p}^{-1}C_p^n \Sigma_j\, m_{n-p-1}(F(n-p-1,j))\, m_p(\omega(F(n-p-1,j))).$$

Thus, by 7.1 $\underline{m}_p(T(\Omega)) = \underline{m}_p(T)$. The other equality follows similarly and was already noted in the remark following 7.3.

Thus, 7.5 establishes the consistency of 7.3 and 7.4 with 7.1 and hence with the definition 6.2.

7.6. Lemma. Let $F(n-p-1,j)$ be a face in $P^0(n-p-1)$, ω a borel subset of Ω, then $P(\omega) \cap F(n-p-1,j) = \emptyset$ or $F(n-p-1,j)$.

Proof: The essential part of the proof is that if one point of an open face is in $P(\omega)$ then so is every point of the open face: let

$x, y \in F(n-p-1,j)$ and $x \in P(\omega)$. By 2.15 there is a $\xi \in \omega$ and $\pi \in \Pi(x)$ (see 2.10) such that ξ is normal to π, whence ξ is in the space orthogonal to F at x. But F is open, so the orthogonal space at $y \in F$ is a translate of the orthogonal space at x. Therefore ξ is in the space orthogonal to F at y and $y \in P(\omega)$. Since y was arbitrary, $F(n-p-1,j)$ is a subset of $P(\omega)$, and hence $F(n-p-1,j) = P(\omega) \cap F(n-p-1,j)$.

7.7. Lemma. $P(\omega) \cap F(n-p-1,j) = \emptyset$ if and only if

$$\omega \cap \omega(F(n-p-1,j)) = \emptyset.$$

Proof: $\omega \cap \omega(F(n-p-1,j)) \neq \emptyset$ if and only if there is a $\xi \in \omega$ such that $\xi \in \omega(F(n-p-1,j))$, i.e., if and only if there is a $\xi \in \omega$ and an $x \in F(n-p-1,j)$ such that there is a $\pi \in \Pi(x)$ and ξ is normal to π. Stated slightly differently, there is an $x \in F(n-p-1,j)$ and a $\xi \in \omega$ such that for some $\pi \in \Pi(x)$, ξ is normal to π. The last statement is true if and only if we have $P(\omega) \cap F(n-p-1,j) \neq \emptyset$.

7.8. Theorem. $\underline{m}_p(T(\omega)) = (n-p)C_p^n \, S_{n-p-1}(P;\omega)$.

Proof: From 7.4,

$\underline{m}_p(T(\omega)) =$

$$= (n-p)C_p^n \binom{n-1}{p}^{-1} \Sigma_j \, m_{n-p-1}(P(\omega) \cap F(n-p-1,j)) \, m_p(\omega \cap \omega(F(n-p-1,j))),$$

which by 7.6 and 7.7 is equal to

$$(n-p)C_p^n \binom{n-1}{p}^{-1} \Sigma_j \, m_{n-p-1}(F(n-p-1,j)) \, m_p(\omega \cap \omega(F(n-p-1,j))),$$

which by 4.5 is equal to

$$(n-p)C_p^n\, S_{n-p-1}(P;\omega),$$

where we have also used the fact that $\binom{n-1}{p} = \binom{n-1}{n-p-1}$.

This last theorem generalizes 6.5 for the case of a polytope. We may extend it even further.

7.9. Definition. If K is an arbitrary convex body in E_n and ω a borel subset of Ω, and if

$T(\omega) = \{L_p : L_p$ is a p-flat tangent to K and lying in a hyperplane π also tangent to K having the normal $\xi \in \omega\}$,

then define $\underline{m}_p(T(\omega)) = (n-p)C_p^n\, S_{n-p-1}(K;\omega)$.

The definition 7.9 is clearly consistent with the earlier definition 6.2, for they coincide when $\omega = \Omega$. The following theorem, on the other hand, shows that 7.9 is the logical extension of 7.8 in the sense of limits.

7.10. Theorem. If K is an arbitrary convex body in E_n and $K_1, K_2, \cdots$ is a sequence of polyhedra which converges to K (see the discussion before and after 3.2 concerning these convergences), and if $T(\omega)$ is the set of p-flats tangent to K determined by the borel subset ω of Ω (as in 7.9), and $T_j(\omega)$ the set of p-flats tangent to K_j determined by ω, then as $K_j \to K$,

$$\underline{m}_p(T_j(\omega)) \to \underline{m}_p(T(\omega))$$

for every continuity set ω of $S_{n-p-1}(K;\cdot)$ (cf. (6, p. 8) or (3, pp. 61-68)).

Proof: This is an immediate consequence of the continuity of the mixed area functions as asserted in 3.2 and the representations of $\underline{m}_p$ as mixed area functions in 7.8 and 7.9.

Chapter 8. Applications.

There are two parts to this chapter; the first is an application of the partial measures of chapter seven to the field of geometric probabilities, the second is an integro-geometric proof of Kubota's formula using the total measures of chapter six.

I. Geometric Probabilities.

Let K be a convex body in E_3 and k a subset of the boundary ∂K of K. We also suppose that E_3 is three-dimensional space, and that gravity is operational. Then if one thinks of dropping K onto a horizontal plane π in a random fashion, the probability that the portion of ∂K which will make contact with π lies in k is

$$pr(k) = \underline{m}_2(T(k))/\underline{m}_2(T(\partial K)) = (1/4\pi)\underline{m}_2(T(k)).$$

This is simply the ratio of the measure of planes tangent to K at points of k to the measure of all planes tangent to K.

The case where K is a polyhedron is most interesting, and a cube sufficiently illustrates it: from 7.3 we obtain

$$pr(k) = \frac{1}{4\pi}(3-2)C_2^3 \binom{2}{2}^{-1} \Sigma_j \, m_0(k \cap F(0,j))m_2(\omega(k) \cap \omega(F(0,j)))$$

$$= \frac{1}{4\pi} \Sigma_j \, m_0(k \cap F(0,j)) \, m_2(\omega(k) \cap \omega(F(0,j))).$$

This last formula tells us that $pr(k) \neq 0$ if and only if for some j, $m_0(k \cap F(0,j)) \neq 0$. Hence, we have the following:

8.1. Theorem. The probability that a cube will land on a portion k of its boundary when dropped randomly onto a plane is non-zero if and only if k contains at least one vertex.

There follows an easy corollary:

8.2. Corollary. The probability is one that the cube will land on a vertex and zero that it will land on an entire face or edge.

A similar investigation can be carried out to determine the probabilities of a cube, dropped on a knife edge (or on a pinpoint), landing on a vertex, edge, or face of the cube.

II. Kubota's Formula: An Integro-Geometric Proof.

Let K be a convex body in E_n, $\mu \in \Omega$, and $\pi(\mu)$ a hyperplane through the origin with normal μ. We denote the projection of K onto $\pi(\mu)$ by K^*. K^* is a convex body in the (n-1)-dimensional space $\pi(\mu)$ and we let $S_p^*(K^*;\Omega^*)$ denote the p-th surface area function of K^* as a subset of $\pi(\mu)$, (Ω^* is the unit sphere in $\pi(\mu)$, and hence is the projection of Ω).

8.3. Lemma. The measure $\underline{m}_p^*(T_p^*(K^*))$ of the set $T_p^*(K^*)$ of all (p-1)-flats lying in $\pi(\mu)$ tangent to K^* is

$$\underline{m}_p^*(T_p^*(K^*)) = (n-p)C_{p-1}^{n-1}S^*_{n-p-1}(K^*;\Omega^*).$$

Proof: Apply 6.2.

Consider a p-flat L_p in E_n tangent to K. It is spanned by p ortho-normal vectors $v_1, \cdots, v_p$, originating from some point $x \in L_p$. Also, L_p contains a unit (p-1)-hypersphere $\Omega_{p-1}(x;1)$ centered at x. If $v \in \Omega_{p-1}(x;1)$, then $v-x = \mu \in \Omega$, and the projection

$(L_p)^*$ of L_p onto $\pi(\mu)$ is a (p-1)-flat in the (n-1)-dimensional space $\pi(\mu)$, i.e., if $(L_p)^*$ is the projection of L_p onto $\pi(\mu)$ then

$$(L_p)^* = L^*_{p-1}.$$

Moreover, L^*_{p-1} is tangent to K^*. Since this is true for any $v \in \Omega_{p-1}(x;1)$, it is true for

$$m_{p-1}(\Omega_{p-1}(x;1)) = Q_{p-1}$$

different directions. Hence we have

8.4. Lemma. There are Q_{p-1} directions for which the projection of L_p is a (p-1)-flat tangent to the projection of K. Equivalently, to each L_p tangent to K corresponds a total of Q_{p-1} (p-1)-flats L^*_{p-1} in the projection spaces.

This leads to the next theorem.

8.5. Theorem. $\underline{m}_p(T_p(K)) = (1/Q_{p-1}) \int_\Omega \underline{m}^*_{p-1}(T^*_{p-1}(K^*))d\omega$, where $d\omega$ is the element of area on Ω.

Proof: The left hand side is the measure of all p-flats tangent to K. The integral on the right is the total measure of (p-1)-flats in the projection spaces which are tangent to the corresponding projection of K. Thus, by 8.4 the integral counts each p-flat tangent to K exactly Q_{p-1} times.

8.6. Corollary. $C^n_p S_{n-p-1}(K;\Omega) = (C^{n-1}_{p-1}/Q_{p-1}) \int_\Omega S^*_{n-p-1}(K^*;\Omega^*)d\omega$.

Proof: This is simply a translation of 8.5 in terms of area functions using the equations of 6.2 and 8.3.

8.7. Definition. Let $\mu \in \Omega$, and $\pi(\mu)$ be a hyperplane through the origin normal to μ. Then $\pi(\mu)$ is an (n-1)-dimensional euclidean space and we define

(A) $V^*(K_1^*,\cdots,K_{n-1}^*)$ = the (n-1)-dimensional mixed volume of the bodies $K_1^*, \cdots, K_{n-1}^*$ lying in $\pi(\mu)$.

(B) $W_{p-1}^*(K^*) = V^*(\underbrace{K^*,\cdots,K^*}_{n-p},\underbrace{B^*,\cdots,B^*}_{p-1})$,

where B^* is the unit ball of $\pi(\mu)$.

(C) Let $\mu \in \Omega$, then $\bar{\mu}$ is to be the segment joining the origin with μ.

(D) Define $W_{p-1}(K,\bar{\mu}) = V(\underbrace{K,\cdots,K}_{n-p},\underbrace{B,\cdots,B}_{p-1},\bar{\mu})$, where B is the unit ball of E_n.

8.8. Lemma. If K^* is the projection of K in the direction $\mu \in \Omega$, then

$$W_{p-1}^*(K^*) = nW_{p-1}(K,\bar{\mu}).$$

Proof: From (2, p. 45) we have

$$V^*(K^*,\cdots,K^*) = \sigma_\mu = nV(K,\cdots,K,\bar{\mu}).$$

Let $K_1, \cdots, K_{n-1}$ be convex bodies in E_n, $K_1^*, \cdots, K_{n-1}^*$ their respective projections and $r_1, \cdots, r_{n-1}$ non-negative real numbers. If we replace K by $\Sigma_{i=1}^{n-1} r_i K_i$ in the above formula, we get

$$\Sigma r_{i(1)}\cdots r_{i(n-1)}V^*(K^*_{i(1)},\cdots,K^*_{i(n-1)}) =$$

$$= V^*((\Sigma r_iK_i)^*,\cdots,(\Sigma r_iK_i)^*)$$

$$= nV((\Sigma r_iK_i),\cdots,(\Sigma r_iK_i),\bar{\mu})$$

$$= n\Sigma r_{i(1)}\cdots r_{i(n-1)}V(K_{i(1)},\cdots,K_{i(n-1)},\bar{\mu}).$$

By the uniqueness of the mixed volumes (2, p. 38f.) we have

$$V^*(K^*_{i(1)},\cdots,K^*_{i(n-1)}) = nV(K_{i(1)},\cdots,K_{i(n-1)},\bar{\mu}).$$ Thus,

$$V^*(\underbrace{K^*,\cdots,K^*}_{n-p},\underbrace{B^*,\cdots,B^*}_{p-1}) = nV(\underbrace{K,\cdots,K}_{n-p},\underbrace{B,\cdots,B}_{p-1},\bar{\mu}).$$

8.9. Definition. (A) $W_p(K) = W_p = V(\underbrace{K,\cdots,K}_{n-p},\underbrace{B,\cdots,B}_{p}),$

(B) $S'_{n-p}(K,\bar{\mu};\Omega) = S(\underbrace{K,\cdots,K}_{n-p},\underbrace{B,\cdots,B}_{p-2},\bar{\mu};\Omega).$

8.10. Lemma. (A) $W_p(K) = (1/n)S_{n-p}(K;\Omega).$

(B) $W'_{p-1}(K,\bar{\mu}) = (1/n)S'_{n-p}(K,\bar{\mu};\Omega).$

Proof: By definition,

$$W_p = V(\underbrace{K,\cdots,K}_{n-p},\underbrace{B,\cdots,B}_{p})$$

$$= V(B,\underbrace{K,\cdots,K}_{n-p},\underbrace{B,\cdots,B}_{n-1})$$ (mixed volumes are symmetric),

$$= (1/n) \int_\Omega H(\mu) S(\underbrace{K, \cdots, K}_{n-p}, \underbrace{B, \cdots, B}_{p-1}; d\omega) \quad \text{(see (6, p. 21.))},$$

$$= (1/n) \int_\Omega S_{n-p}(K; d\omega) = (1/n) S_{n-p}(K;\Omega),$$

where $H(\mu) \equiv 1$ is the support function of the unit ball B. This proves (A); the proof of (B) is identical.

8.11. Theorem. $(n-1)S'_{n-p}(K,\bar{\mu};\Omega) = S^*_{n-p}(K^*;\Omega^*)$.

Proof: By 8.10,

$$S'_{n-p}(K,\bar{\mu};\Omega) = nW'_{p-1}(K,\bar{\mu}), \quad \text{and}$$

by 8.8,

$$nW'_{p-1}(K,\bar{\mu}) = W^*_{p-1}(K^*),$$

and by 8.10 again,

$$W^*_{p-1}(K^*) = \frac{1}{n-1} S^*_{(n-1)-(p-1)}(K^*;\Omega^*), \quad \text{Q.E.D.}$$

If we combine 8.6 with 8.11 we get

$$S_{n-p-1}(K;\Omega) = \frac{C^{n-1}_{p-1} \cdot (n-1)}{C^n_p \cdot Q_{p-1}} \int_\Omega S'_{n-p-1}(K,\bar{\mu};\Omega) d\omega.$$

But

$$\frac{C^{n-1}_{p-1} \cdot (n-1)}{C^n_p \cdot Q_{p-1}} = \frac{\dfrac{Q_{n-3} \cdots Q_{n-p-1}}{Q_{p-2} \cdots Q_1} \cdot (n-1)}{\dfrac{Q_{n-2} \cdots Q_{n-p-1}}{Q_{p-1} \cdots Q_1} \cdot Q_{p-1}} = \frac{n-1}{Q_{n-2}} = \frac{1}{V_{n-1}},$$

where V_{n-1} is the volume of the unit (n-1)-ball. Thus, using 8.10 once again, we get

8.12. Theorem (Kubota).

$$W_{p+1}(K) = \frac{1}{V_{n-1}} \int_{\Omega} W_p'(K,\bar{\mu})d\omega.$$

For a standard proof of 8.12 see (2, p. 49).

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