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Title: THE STRUCTURE OF PREORDERED SETS AND THEIR TOPOLOGICAL PROPERTIES

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Let X be a set. Given any preorder $\leq$ on the set X, there corresponds a family of subsets of X, namely, $\{\bar{L}_x | x \in X\}$ where $\bar{L}_x = \{y | y \in X, y \leq x\}$ such that, for all elements x and y of X, $x \leq y$ iff $\bar{L}_x \subseteq \bar{L}_y$. In this thesis, it is shown that, conversely, the preorders on a set can be derived from arbitrary families of subsets of that set. Thus, an upper bound for the cardinality of all preorders on any set is obtained. Moreover, the cardinal number of the set of all partial orders on any finite set is odd.

An equivalence relation is defined on the collection of all families of subsets of a set by calling two families equivalent iff they define the same preorder. All equivalence classes of this equivalence relation are closed under arbitrary nonempty unions. Therefore, every equivalence class has a greatest element with respect to set inclusion. In fact, a family of subsets of X is the greatest element

of an equivalence class if and only if it is a topology for X which is also closed under arbitrary intersections. Consequently, the cardinal number of the set of all preorders on a set X is equal to the cardinal number of the set of all topologies for X which are also closed under arbitrary intersections and, in particular, if X is a finite set, then the cardinal number of the set of all preorders on the set X is equal to the cardinal number of the set of all topologies for X.

The Structure of Preordered Sets and Their Topological Properties

by

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THE STRUCTURE OF PREORDERED SETS AND THEIR TOPOLOGICAL PROPERTIES

1. PREORDERED SETS

In the first section of this introductory chapter, we give the definition of a preordered set, and then define some order properties of a preordered set. As we shall see in the second section, a preordered set can be considered as a topological space. We shall describe the relationships between some order properties and the topological properties such as connectedness and compactness in the second section. Finally, in the last section, we define partially ordered set, linearly ordered set, well-ordered set, lattice and directed set.

1-1. Preorders

Definition 1.1. A preorder on a set X is a relation on X which is reflexive and transitive; that is, a subset, call it $\leq$, of $X \times X$ such that

(1) $(x,x) \in \leq$ for all elements x of X.

(2) If $(x,y) \in \leq$ and $(y,z) \in \leq$, then $(x,z) \in \leq$ for all elements x, y and z of X.

Remark. The symbol "$\leq$" will always denote a preorder on a set. For the usual order on the real numbers, we shall use the symbol "$\leq_{(R)}$."

Example 1.2. (1) In the set R of all real numbers, the subset $\{(x, y) \mid x \leq_{(R)} y\}$ of $R \times R$ is a preorder on R, whereas $\{(x, y) \mid x \leq_{(R)} y, x \neq y\}$ is not since it is not reflexive.

(2) Let X be any set and set $\leq = \{(x, x) \mid x \in X\}$; then $\leq$ is a preorder on X.

(3) If X is a set, then $\leq = X \times X$ is a preorder on X.

Definition 1.3. (1) An ordered pair $(X, \leq)$ is called a preordered set if $\leq$ is a preorder on the set X.

(2) Let $(X, \leq)$ be a preordered set and let a, b be elements of X, we say that a is less than or equal to b (b is greater than or equal to a) iff $(a, b) \in \leq$ and we denote this relation by $a \leq b$; we say that a is less than b (b is greater than a) iff $a \leq b$ and $a \neq b$, and we denote this relation by $a < b$.

(3) Let a and b be elements of a preordered set $(X, \leq)$. The elements a and b are said to be $\leq$-related iff at least one of $a \leq b$ and $b \leq a$ holds. If exactly one of $a \leq b$ and $b \leq a$ holds, we say that a and b are $\leq$-simply related. If both of $a \leq b$ and $b \leq a$ hold, we say that a and b are $\leq$-doubly related and denote this fact by $a \underline{\Delta} b$. In case there is no confusion, we shall just say two elements are related, simply related or doubly related without indicating the preorder $\leq$.

(4) Let $(X, \leq)$ be a preordered set. For every pair of elements a and b of X, we define

$$[a, b] = \{x \mid x \in X,\ a \leq x \leq b\}$$

$$(a, b) = \{x \mid x \in X,\ a < x < b\}$$

$$L_a = \{x \mid x \in X,\ x < a\}$$

$$\overline{L}_a = \{x \mid x \in X,\ x \leq a\}$$

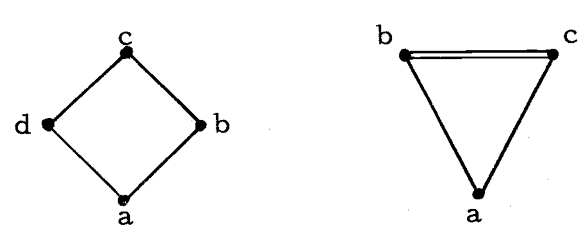
$$R_a = \{x \mid x \in X,\ a < x\}$$

$$\overline{R}_a = \{x \mid x \in X,\ a \leq x\}.$$

Remark. (1) The relation $<$ is irreflexive and not always transitive. Irreflexivity is clear. For intransitivity, let $X = \{a, b\}$ where a, b are distinct and let $\leq\ = X \times X$, then $a < b$ and $b < a$, but $a \not< a$.

(2) The set L_a is also called the initial segment of X determined by the element a of X.

(3) For the sake of convenience in giving examples, we shall use graphical representation for preordered sets. Two examples are shown below:



The left hand figure represents the preordered set $(X, \leq)$, where $X = \{a, b, c, d\}$ and

$\leq = \{(a,a),(b,b),(c,c),(d,d),(a,b),(a,c),(a,d),(b,c),(d,c)\}.$

The right hand figure represents the preordered set $(X, \leq)$, where $X = \{a, b, c\}$ and

$$\leq = \{(a,a),(b,b),(c,c),(a,b),(a,c),(b,c),(c,b)\}.$$

Note that there are two bars connecting b and c in the figure on the right; these mean b and c are doubly related.

(4) If $(X, \leq)$ is a preordered set and $A \subseteq X$, then the relative relation on A:

$$\leq | A = \{(x,y) \mid (x,y) \in \leq \text{ and } x, y \in A\}$$

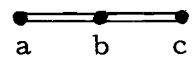
is clearly a preorder on A. If X is a set with a preorder defined on it, then whenever we consider a subset A of X as a preordered set, the preorder on A will always be the relative preorder.

<u>Definition 1.4.</u> (1) A preorder on a set X is called the <u>discrete order</u> on X if no two distinct elements of X are related.

(2) A preorder on a set X is called the <u>indiscrete order</u> on X if all pairs of elements of X are doubly related.

<u>Example 1.5.</u> (1) In Example 1.2(2), $\leq$ is the discrete order on X. If $X = \{a, b, c\}$, then the graphical representation of $(X, \leq)$ is

(2) In Example 1.2(3), $\leq$ is the indiscrete order on X. If $X = \{a, b, c\}$, then the graphical representation of $(X, \leq)$ is



<u>Theorem 1.6</u>. Let $(X, \leq)$ be a preordered set and let x, y be elements of X, then $x \leq y$ iff $\bar{L}_x \subseteq \bar{L}_y$ or $\bar{R}_y \subseteq \bar{R}_x$.

Proof. Suppose $\bar{L}_x \subseteq \bar{L}_y$. Since $x \in \bar{L}_x$, we have $x \in \bar{L}_y$. Therefore $x \leq y$. Conversely, suppose $x \leq y$. Since $z \leq x$ for all $z \in \bar{L}_x$, by transitivity, we have $z \leq y$ for all $z \in \bar{L}_x$. Hence, $z \in \bar{L}_y$ for all $z \in \bar{L}_x$; that is, $\bar{L}_x \subseteq \bar{L}_y$.

Similar arguments can be made to prove

$$x \leq y \text{ iff } \bar{R}_y \subseteq \bar{R}_x.$$

Q.E.D.

<u>Corollary</u>. If x and y are doubly related in a preordered set $(X, \leq)$, then $\bar{L}_x = \bar{L}_y$ and $\bar{R}_x = \bar{R}_y$.

<u>Remark</u>. The statements of the above theorem and corollary would not be true if $\bar{L}_x$, $\bar{L}_y$, $\bar{R}_x$ and $\bar{R}_y$ were replaced by L_x, L_y, R_x and R_y respectively. For example, let X be a set containing more than one element and let x, y be two distinct elements of X. If $\leq$ is the indiscrete order on X, then $x \underline{\Delta} y$, but

$L_x = R_x = X - \{x\}$ is not comparable to $L_y = R_y = X - \{y\}$. On the other hand, if $\leq$ is the discrete order on X, then $L_x = L_y = R_x = R_y = \emptyset$, but x and y are not related.

Theorem 1.7. Let $(X, \leq)$ be a preordered set and let x, y be elements of X which are simply related, then the following statements are equivalent:

(1) $x \leq y$

(2) $\bar{L}_x \subseteq L_y$ $(\bar{R}_y \subseteq R_x)$

(3) $L_x \subset L_y$ $(R_y \subset R_x)$.

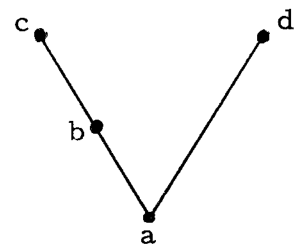
Proof. (1) $\Rightarrow$ (2): Suppose $x \leq y$. Since x and y are simply related, $y \not\leq x$. By Theorem 1.6, $\bar{L}_x \subseteq \bar{L}_y$. If there is an element z of $\bar{L}_x$ such that $z = y$, then we would have $y \leq x$ which is a contradiction. Hence for all elements z of $\bar{L}_x$, $z < y$; that is, $z \in L_y$. Consequently, $\bar{L}_x \subseteq L_y$.

(2) $\Rightarrow$ (3): Since $L_x \subset \bar{L}_x$, it is clear that if $\bar{L}_x \subseteq L_y$ then $L_x \subset L_y$.

(3) $\Rightarrow$ (1): If $x \not\leq y$, then since x and y are simply related, $y \leq x$. From (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3), we have $L_y \subset L_x$. Since L_x and L_y are simply related by the proper set inclusion $\subset$, $L_x \not\subset L_y$.

Similar arguments can be made to prove the remaining parts written in the parentheses. Q.E.D.

Remark. It could happen that $L_x \subset L_y$ and x, y are not related. For example, if the graphical representation of a preordered set $(X, \leq)$ is



then $L_d = \{a\} \subset \{a, b\} = L_c$ but c and d are not related.

Next, we shall give definitions of maximal element, minimal element, greatest element, least element of a preordered set; and upper bound, lower bound, least upper bound, greatest lower bound for a subset of a preordered set.

Definition 1.8. Let $(X, \leq)$ be a preordered set.

(1) An element α of X is called a minimal (maximal) element of X if

$$x \leq \alpha \ (\alpha \leq x) \text{ implies } \alpha \leq x \ (x \leq \alpha) \text{ for all } x \in X;$$

that is, α is doubly related to every element of X which is less (greater) than or equal to α.

(2) An element α of X is called a minimax element of X if α is both a minimal and a maximal element of X; that is, if

α is doubly related to every element of X which is related to α. Symbolically, an element α of X is a minimax element of X, if, for all $x \in X$, $\alpha \leq x$ iff $x \leq \alpha$.

(3) An element α of X is called a greatest (least) element of X if

$$x \leq \alpha \, (\alpha \leq x) \quad \text{for all} \quad x \in X.$$

(4) An element α of X is called an upper (a lower) bound for a subset A of X if

$$x \leq \alpha \, (\alpha \leq x) \quad \text{for all} \quad x \in A.$$

(5) Let A be a subset of X and let $R_A (L_A)$ be the set of all upper (lower) bounds for A. An element α of $R_A (L_A)$ is called a least upper (greatest lower) bound for A if

$$\alpha \leq x \; (x \leq \alpha) \quad \text{for all} \quad x \in R_A (L_A).$$

Example 1.9. (1) In Example 1.2(1), the preordered set $(R, \{(x, y) \mid x \leq_{(R)} y\})$ has no maximal element or minimal element. Every bounded subset of R has many upper (lower) bounds and every subset of R which is bounded above (below) has a least upper (greatest lower) bound.

(2) If the preorder is the discrete order on a set X, then every element of X is a minimax element of X. Moreover, no subset of X containing more than one element has an upper bound or

a lower bound. Thus a maximal (minimal) element of X need not be an upper (a lower) bound for X. Furthermore, every subset of X containing more than one element, with the relative preorder, has neither a greatest element nor a least element.

(3) If the preorder is the indiscrete order on a set X, then every element of X is a minimax element of X; and both a least upper bound and a greatest lower bound for any subset of X. Note that in this example, if A is a subset of X containing more than one element, then least upper bounds and greatest lower bounds for A are not unique. Moreover, every element of a subset of X, with the relative preorder, is both a greatest element and a least element for that subset.

(4) Let C be the set of all complex numbers and define

$$z_1 \leq z_2 \quad \text{iff} \quad |z_1| \leq_{(R)} |z_2|$$

for all elements z_1 and z_2 of C. Then $(C, \leq)$ is a preordered set. Every element of the unit circle $\{z \mid z \in C, |z| = 1\}$ is a least upper bound for the subset $A = \{z \mid z \in C, |z| \leq_{(R)} 1\}$ of C, but no element of the unit circle is a maximal element of C. If we take A as our universal set and use the relative preorder, then every element of the unit circle is both maximal and a least upper bound for A.

Definition 1.10. A preordered set $(X, \leq)$ is called order-complete if every nonempty subset of X which has an upper bound has a least upper bound.

Example 1.11. From Example 1.9, we see that every preordered set with the discrete order or the indiscrete order is order-complete.

Theorem 1.12. A preordered set is order-complete iff every nonempty subset which has a lower bound has a greatest lower bound.

Proof. Suppose that the preordered set $(X, \leq)$ is order-complete and that A is a nonempty subset of X which has a lower bound a. Let B be the set of all lower bounds for A. Then, every element of A is an upper bound for B. Since $a \in B$, $B \neq \emptyset$. By hypothesis, $(X, \leq)$ is order-complete, hence B has a least upper bound, say, a_0. Since every element of A is an upper bound for B, a_0 is a lower bound for A. Consequently, a_0 is a greatest lower bound for A.

The converse statement can be proved by similar arguments or, directly, we can apply the result just proved to the preorder inverse to $\leq$. Q.E.D.

Definition 1.13. A preordered set $(X, \leq)$ is called a prechain if every pair of elements is related.

Example 1.14. Every set with the indiscrete order is a prechain.

Remark. We do not require the preorder $\leq$ to be antisymmetric in a prechain.

Theorem 1.15. If the preordered set $(X, \leq)$ is a prechain, then every nonempty finite subset A of X (with the relative preorder) has both a least element and a greatest element.

Proof. Suppose A has no least element and let $a_1 \in A$. Since a_1 is not a least element of A and $(X, \leq)$ is a prechain, there exists an element a_2 of A such that $a_2 < a_1$ and $a_1 \nless a_2$. The element a_2 is, again, not a least element of A, hence there exists an element a_3 of A such that $a_3 < a_2$ and $a_2 \nless a_3$. Note that, a_1, a_2 and a_3 are all distinct. Continuing this process, we see that A is an infinite subset of X which is a contradiction.

Similarly, we can prove that A also has a greatest element.

Q.E.D.

Remark. The hypothesis that the preordered set $(X, \leq)$ is a prechain is essential. For example, if $\leq$ is the discrete order on X, as indicated in Example 1.9(2), every subset of X containing more than one element has neither a greatest element nor a least element with respect to the relative preorder.

Definition 1.16. A preordered set $(X, \leq)$ is called dense if for every pair of distinct elements x and y of X such that $x < y$, there exists an element z of X such that $x < z < y$.

Example 1.17. Every preordered set with the discrete order is dense. If $\leq$ is the indiscrete order on a set X which does not contain exactly two distinct elements, then $(X, \leq)$ is also dense. But, if X is a finite set with the preorder $\leq$ such that every pair of distinct elements of X is simply related (this kind of preordered set is called a linearly ordered set; we shall define it in 1-3.2), then $(X, \leq)$ is not dense.

1-2. Preorders and the Order Topologies

Definition 2.1. Let $(X, \leq)$ be a preordered set. The order topology for X induced by $\leq$ is the topology having a subbase consisting of all sets of the form L_a or R_a for some element a of X. If T is the order topology for X induced by $\leq$, we say that (X, T) is the ordered topological space induced by $\leq$.

Example 2.2. (1) The order topology T induced by the discrete order on a set X is the topology having $T = \{\phi, X\}$; that is, the order topology T is the indiscrete topology for X.

(2) If $X = \{a, b, c\}$ and $\leq$ is the indiscrete order on X, then the subbase for the order topology T induced by $\leq$ is

$$\{\{b, c\}, \{c, a\}, \{a, b\}\}.$$

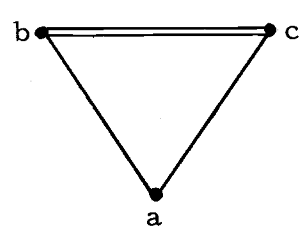
Hence

$$T = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, X\}$$
$$= P(X),$$

the family of all subsets of X. Note that T is the discrete topology for X.

If X is an infinite set, then the order topology T induced by the indiscrete order on X consists of $\emptyset$ and all complements of finite subsets of X; that is, T is the cofinite topology for X.

(3) If the graphical representation of a preordered set $(X, \leq)$ is as follows



then

$$R_a = \{b, c\}, \quad R_b = \{c\}, \quad R_c = \{b\}$$
$$L_a = \emptyset, \quad L_b = \{c, a\}, \quad L_c = \{a, b\}.$$

Hence the order topology T induced by $\leq$ is also the discrete topology for X. Notice that different preorders may induce the same order topology.

(4) In the set R of all real numbers, the preorder $\{(x, y) | x \leq_{(R)} y\}$ is the usual order on R. The subbase for the induced order topology, therefore, consists of all sets of the form $\{x | x <_{(R)} a\}$ or $\{x | a <_{(R)} x\}$ for some real number a. Hence the induced order topology is the usual topology for R.

Now we are going to establish some relationships between the order properties and the topological properties.

<u>Theorem 2.3.</u> Let the preordered set $(X, \leq)$ be a prechain. If the ordered topological space (X, T) induced by $\leq$ is connected, then $(X, \leq)$ is dense.

Proof. If $(X, \leq)$ is not dense, then X contains at least two distinct elements, say, x and y such that $x < y$ and there is no element z of X such that $x < z < y$. Now consider the two open sets L_y and R_x in the topological space (X, T). Since $x \in L_y$ and $y \in R_x$, L_y and R_x are nonempty subsets of X. If there is an element z of X such that $z \in L_y \cap R_x$, then $x < z < y$ which is a contradiction. Hence $L_y \cap R_x = \emptyset$. Moreover, since $(X, \leq)$ is a prechain, it is clear that $X = L_y \cup R_x$. Now we see that X is the union of two nonempty disjoint open sets L_y and R_x. Therefore, (X, T) is not connected. Q.E.D.

<u>Theorem 2.4.</u> Let the preordered set $(X, \leq)$ be a prechain.

If the ordered topological space (X, T) induced by $\leq$ is compact, then $(X, \leq)$ has both a least element and a greatest element, and it can not be written as the union of two nonempty sets A and B (with the relative preorders) such that for all elements a of A and all elements b of B, $a < b$ and $b \nless a$, while A has no greatest element and B has no least element.

Proof. If $(X, \leq)$ has no least element, then for every element x of X, since $(X, \leq)$ is a prechain, there exists an element x_* of X such that $x_* < x$ and $x \nless x_*$. Hence for every element x of X, there exists another element x_* of X such that $x \in R_{x_*}$ and $x_* \notin R_x$. It is clear that, then, the family $F = \{R_x | x \in X\}$ is an open covering of X in the topological space (X, T). Now consider any nonempty finite subfamily, say, $F' = \{R_{x_i} | x_i \in X, i = 1, 2, \ldots, n\}$ of F. The set $S = \{x_1, x_2, \ldots, x_n\}$ is a nonempty finite subset of X. Since $(X, \leq)$ is a prechain, by Theorem 1.15, S has a least element, say, x_ℓ, $\ell \in \{1, 2, \ldots, n\}$. Then for all elements x_i of S, $x_\ell \leq x_i$. By Theorem 1.6, we have $\overline{R}_{x_i} \subseteq \overline{R}_{x_\ell}$ for all elements x_i of S. Hence

$$\cup F' \subseteq \cup \{\overline{R}_{x_i} | x_i \in X, i = 1, 2, \ldots, n\} \subseteq \overline{R}_{x_\ell}$$

But, since there exists an element x_{ℓ_*} of X such that $x_{\ell_*} < x_\ell$ and $x_\ell \nless x_{\ell_*}$, we see that $x_\ell \nleq x_{\ell_*}$, and consequently $x_{\ell_*} \notin \overline{R}_{x_\ell}$.

Then $x_{\ell_*} \notin \cup F'$, and hence F' is not a covering of X. Now $\{R_x | x \in X\}$ is an open covering of X which has no finite subcovering. Therefore, (X, T) is not compact.

Similarly, if X has no greatest element, then (X, T) is not compact.

Now suppose that X can be written as the union of two non-empty sets A and B such that for all elements a of A and all elements b of B, $a < b$ and $b \nless a$, while A has no greatest element and B has no least element. Since $(X, \leq)$ is a prechain and A has no greatest element, for every element a of A, there exists an element a_* of A such that $a < a_*$ and $a_* \nless a$. Therefore, for every element a of A, there exists another element a_* of A such that $a \in L_{a_*}$ and $a_* \notin L_a$ where $L_a, L_{a_*} \subseteq X$. Similarly, for every element b of B, there exists another element b_* of B such that $b \in R_{b_*}$ and $b_* \notin R_b$ where $R_b, R_{b_*} \subseteq X$. Hence

$$A \subseteq \cup\{L_a | a \in A\} \quad \text{and} \quad B \subseteq \cup\{R_b | b \in B\}.$$

Since $X = A \cup B$, it follows that the family

$$\{L_a | a \in A\} \cup \{R_b | b \in B\}$$

of open sets of X is an open covering of X. We now prove that

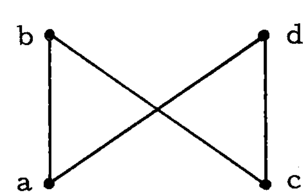
$R_b \subseteq B$ for all elements b of B. If there exists an element x of R_b such that $x \notin B$, then $x \in A$ and it follows that $x \in L_{a_0}$ for some $a_0 \in A$. Hence $b < x$ and $x < a_0$. By transitivity, we have $b \leq a_0$ which contradicts the hypothesis that for all elements a of A and all elements b of B, $a < b$ and $b \nless a$. Consequently, $R_b \subseteq B$ for all elements b of B; that is, if we consider B with the relative preorder, then, for all elements b of B, R_b in X is exactly the same as R_b in B. Hence we can use the same arguments as in the first paragraph of the proof of this theorem to get the conclusion that the covering $\{R_b | b \in B\}$ of B has no finite subcovering. Similarly, the covering $\{L_a | a \in A\}$ of A has no finite subcovering. Therefore, the opening covering

$$\{L_a | a \in A\} \cup \{R_b | b \in B\}$$

of X has no finite subcovering; that is, (X, T) is not compact.

Q.E.D.

<u>Remark.</u> (1) The hypothesis that $(X, \leq)$ is a prechain in the above theorem is essential. For example, if the graphical representation of a preordered set $(X, \leq)$ is as follows

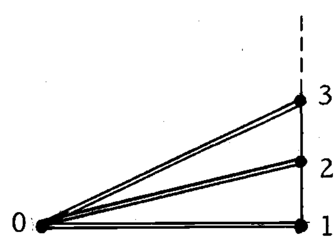


then $(X, \leq)$ has neither a least element nor a greatest element, and it can be written as the union of two nonempty sets $A = \{a, c\}$ and $B = \{b, d\}$ such that every element of A is less than every element of B and no element of B is less than any element of A. However, since X is a finite set, no matter what the topology T is, (X, T) is compact.

(2) The converse of the above theorem is not true. For example, let $\bar{N}$ be the set of all nonnegative integers and

$$\leq \; = \{(m, n) \mid m, n \in \bar{N}, m \leq_{(R)} n \quad \text{or} \quad n = 0\}.$$

The graphical representation of the preordered set $(\bar{N}, \leq)$ is as follows:



Note that 0 is both a least element and a greatest element of $\bar{N}$. Moreover, it is clear that $\bar{N}$ can not be written as the union of two nonempty sets A and B with properties indicated in the above theorem. However, the covering

$$\{L_n \mid n \text{ is a positive integer}\}$$

of $\bar{N}$ has no finite subcovering. Hence the ordered topological space (X, T) induced by $\leq$ is not compact.

1-3. Other Types of Order

Definition 3.1. A preorder $\leq$ on a set X is called a partial order if $\leq$ is antisymmetric; that is, no two distinct elements of X are doubly related. Symbolically, for all elements x and y of X,

$$\text{if } x \,\underline{\Delta}\, y, \text{ then } x = y.$$

An ordered pair $(X, \leq)$ is called a partially ordered set if $\leq$ is a partial order on X.

Definition 3.2. A preorder $\leq$ on a set X is called a linear order if $\leq$ is a partial order and $(X, \leq)$ is a prechain; that is, all pairs of distinct elements of X are simply related. If $\leq$ is a linear order on a set X, then the ordered pair $(X, \leq)$ is called a linearly ordered set.

Example 3.3. (1) Let X be a set containing more than one element. If $\leq$ is the discrete order on X, then $\leq$ is a partial order but not a linear order. If $\leq$ is the indiscrete order on X, then $\leq$ is not a partial order.

(2) The family $P(X)$ of all subsets of a set X is partially ordered by set inclusion. If the set X contains more than one

element, then set inclusion is not a linear order on the family $P(X)$.

(3) The preordered set $(C, \leq)$ in Example 1.9(4) is not a partially ordered set since $i \trianglelefteq 1$ but $i \neq 1$.

(4) Let R be the set of all real numbers and let $P(P(R))$ be the collection of all families of subsets of R. Recall that a family Λ^{α} is a refinement of Λ^{β} iff for every element λ^{α} of Λ^{α} there exists an element λ^{β} of Λ^{β} such that $\lambda^{\alpha} \subseteq \lambda^{\beta}$. Now define $\leq$ on $P(P(R))$ as follows:

$$\Lambda^{\alpha} \leq \Lambda^{\beta} \text{ iff } \Lambda^{\alpha} \text{ is a refinement of } \Lambda^{\beta}.$$

Clearly, the relation $\leq$ is a preorder on the collection $P(P(R))$. It is not a partial order since if we let

$$\Lambda^{\alpha} = \{\bar{L}_{n}^{\alpha} \mid n \text{ is a positive integer}\}$$

$$\Lambda^{\beta} = \{\bar{L}_{n+\frac{1}{2}}^{\beta} \mid n \text{ is a positive integer}\}$$

where

$$\bar{L}_{n}^{\alpha} = \{x \mid x \in R,\ x \leq_{(R)} n\}$$

$$\bar{L}_{n+\frac{1}{2}}^{\beta} = \{x \mid x \in R,\ x \leq_{(R)} n+\tfrac{1}{2}\},$$

then

$$\Lambda^{\alpha} \neq \Lambda^{\beta} \quad \text{and} \quad \Lambda^{\alpha} \trianglelefteq \Lambda^{\beta}.$$

<u>Remark.</u> (1) Let $(X, \leq)$ be a preordered set and let x, y be elements of X. If we define

$$x \sim y \quad \text{iff} \quad x \underline{\Delta} y,$$

then it is clear that $\sim$ is an equivalence relation on X. Let [x] denote the equivalence class of x. Obviously, the preordered set $(X, \leq)$ is a partially ordered set iff $[x] = \{x\}$ for all elements x of X. Now let $X/\sim$ denote the set of all equivalence classes and define

$$\tilde{\leq} = \{([x],[y]) \mid [x],[y] \in X/\sim, x \leq y\}.$$

It is easy to see that $(X/\sim, \tilde{\leq})$ is a partially ordered set.

(2) If $(X, \leq)$ is a partially ordered set, then it is clear that an element α of X is a maximal (minimal) element of X iff $\alpha \leq x (x \leq \alpha)$ implies $x = \alpha$ for all $x \in X$.

(3) In a partially ordered set $(X, \leq)$, there is at most one greatest element or least element, and every nonempty subset of X has at most one least upper bound or greatest lower bound.

In the following two theorems, we prove some equivalent properties. First, we prove a lemma.

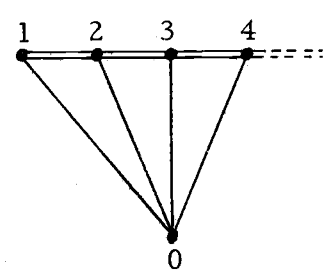
Lemma 3.4. If $(X, \leq)$ is a linearly ordered set, then, for all elements a of X, $\bar{R}_a$ and $\bar{L}_a$ are closed in the ordered topological space (X, T) induced by $\leq$.

Proof. Let $C\bar{R}_a$ be the complement of $\bar{R}_a$. If we can prove that $C\overline{R}_a = L_a$, then $\overline{R}_a$ is closed in the topological space (X, T).

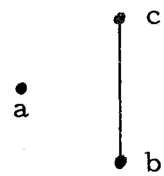
For all elements x of $C\overline{R}_a$, $a \nleq x$. Since $(X, \leq)$ is a prechain, $x < a$. Hence $x \in L_a$. Now suppose $x \in L_a$, then $x \leq a$ and $x \neq a$. If $a \leq x$, by the hypothesis that $\leq$ is antisymmetric, we would have $a = x$ which is a contradiction. Hence $a \nleq x$; that is, $x \notin \overline{R}_a$. Therefore, $x \in C\overline{R}_a$ and it follows that $C\overline{R}_a = L_a$.

Similarly, we can prove that $C\overline{L}_a = R_a$. Hence $\overline{L}_a$ is also a closed set in the topological space (X, T). Q.E.D.

<u>Remark.</u> If $(X, \leq)$ is only a prechain or a partially ordered set, the conclusion of the above theorem is not always true. For example, if the graphical representation of a preordered set $(\overline{N}, \leq)$ where $\overline{N}$ is the set of all nonnegative integers is as follows:



then $(\overline{N}, \leq)$ is a prechain but not a linearly ordered set. Note that, for all positive integer n, $R_n = \overline{N} - \{0, n\}$ and $L_n = \overline{N} - \{n\}$. Moreover, $R_0 = \overline{N} - \{0\}$ and $L_0 = \phi$. Hence the order topology T for X induced by $\leq$ is the cofinite topology for X. Now, for every positive integer n, the set $\overline{R}_n = \overline{N} - \{0\}$ is not closed in the ordered topological space (X, T) since $C\overline{R}_n = \{0\}$ is not open in (X, T). On the other hand, if the preordered set (X, T) has the following graphical representation:



then $(X, \leq)$ is a partially ordered set but not a linearly ordered set. Note that

$$R_a = R_c = L_a = L_b = \emptyset$$

and

$$R_b = \{c\}, \quad L_c = \{b\}.$$

Hence the order topology T induced by $\leq$ is

$$T = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}.$$

The sets $\overline{R}_b = \overline{L}_c = \{b, c\}$, $\overline{R}_c = \{c\}$ and $\overline{L}_b = \{b\}$ are not closed in the ordered topological space (X, T) since their complements are not open in (X, T).

<u>Theorem 3.5.</u> Let $(X, \leq)$ be a linearly ordered set. The ordered topological space (X, T) induced by $\leq$ is connected iff $(X, \leq)$ is dense and order-complete.

Proof. In Theorem 2.3, we proved that if (X, T) is connected then $(X, \leq)$ is dense.

Now we prove that if (X, T) is connected then $(X, \leq)$ is also order-complete. If $(X, \leq)$ is not order-complete, then there exists a nonempty subset A of X such that A has an upper bound, say, α but has no least upper bound. Evidently, A must contain more than one element. Let a_1 and a_2 be two distinct elements of A such that $a_1 \leq a_2$. Since $\leq$ is antisymmetric, $a_2 \not\leq a_1$. So a_1 is not an upper bound for A. If we let B be the set of all upper bounds for A, then $\alpha \in B$ and $a_1 \notin B$. Hence B is a nonempty proper subset of X. Now we are going to prove that the set B is both open and closed in the ordered topological space (X, T) which implies that (X, T) is not connected.

First, we prove that $B = \bigcap_{a \in A} \overline{R}_a$. Then, by Lemma 3.4, we know that B is closed in (X, T).

Since any element b of B is an upper bound for A, we have $a \leq b$ for all $a \in A$. Hence $b \in \overline{R}_a$ for all $a \in A$. Consequently, $b \in \bigcap_{a \in A} \overline{R}_a$ for all $b \in B$; that is, $B \subseteq \bigcap_{a \in A} \overline{R}_a$. For the other inclusion, if x is an element of $\bigcap_{a \in A} \overline{R}_a$, then $x \in \overline{R}_a$ for all $a \in A$. Therefore, $a \leq x$ for all $a \in A$ which means x is an upper bound for A. Since B is the set of all upper bounds for A, we have $x \in B$. Hence $\bigcap_{a \in A} \overline{R}_a \subseteq B$. So, $B = \bigcap_{a \in A} \overline{R}_a$.

Secondly, we prove that $B = \bigcup_{b \in B} R_b$. Then, it follows that B is open in the ordered topological space (X, T).

Since A has no least upper bound and $(X, \leq)$ is a prechain, it is clear that, for every element b of B, there exists an element b_* of B such that $b_* \leq b$ and $b \nleq b_*$. Hence $b_* < b$ which is equivalent to $b \in R_{b_*}$. Consequently, $b \in \bigcup_{b \in B} R_b$ for all elements b of B; that is $B \subseteq \bigcup_{b \in B} R_b$. For the other inclusion, if x is an element of $\bigcup_{b \in B} R_b$, then $x \in R_{b_0}$ for some $b_0 \in B$. Since $b_0 \in B$, b_0 is an upper bound for A. So, $a \leq b_0$ for all elements a of A. Moreover, since $x \in R_{b_0} \subset \overline{R}_{b_0}$, we have $b_0 \leq x$. Then, by transitivity, $a \leq x$ for all $a \in A$; that is, x is an upper bound for A. Consequently, $x \in B$ and $\bigcup_{b \in B} R_b \subseteq B$. So, $B = \bigcup_{b \in B} R_b$.

Now we see that the set B of all upper bounds for A is a nonempty proper subset of X which is both open and closed in the ordered topological space (X, T), hence (X, T) is not connected.

Next, we want to prove that if $(X, \leq)$ is dense and order-complete then the ordered topological space (X, T) is connected.

If (X, T) is not connected, then there exists a nonempty proper subset A of X which is both open and closed in (X, T). Hence the complement CA of A is also a nonempty proper subset of X which is both open and closed in (X, T). Let $a \in A$ and

$a' \in CA$. Since $(X, \leq)$ is a lineraly ordered set and $a \neq a'$, we have either $a < a'$ or $a' < a$.

Case 1. $a < a'$. Let $S = \{x | x \in A,\ a \leq x \leq a'\}$. Since $a \in S$, S is a nonempty subset of X which has an upper bound a'. By order-completeness, there exists a least upper bound, say, α for S.

(1) If $\alpha \in A$, since A is an open set, there exist elements x_1 and x_2 of X such that $\alpha \in (x_1, x_2) \subseteq A$. Hence $x_1 < \alpha < x_2$. Then, by denseness, there is another element α' of X such that

$$x_1 < \alpha < \alpha' < x_2.$$

Since $\leq$ is antisymmetric, $\alpha' \nleq \alpha$. So, if we can prove that $\alpha' \in S$, then α would not be an upper bound for S which is a contradiction. Since $(x_1, x_2) \subseteq A$, it is clear that $\alpha' \in A$. Now we prove that $a < \alpha' < a'$. Since $a \in S$, a' is an upper bound for S and α is the least upper bound for S, we have $a \leq \alpha \leq a'$. If $\alpha' \leq a$, we would have $\alpha' \leq \alpha$ which is a contradiction. If $a' \leq \alpha'$, we would have

$$x_1 < \alpha \leq a' \leq \alpha' < x_2$$

which implies that $a' \in (x_1, x_2) \subseteq A$. This contradicts the fact that $a' \in CA$. Consequently, $\alpha' \nleq a$ and $a' \nleq \alpha'$. Since $(X, \leq)$ is a prechain, we must have $a < \alpha' < a'$. Therefore, $\alpha' \in S$. Thus, we get a contradiction.

(2) If $\alpha \in CA$, since CA is also an open set, there exist elements x_1 and x_2 of X such that $\alpha \in (x_1, x_2) \subseteq CA$. Hence $x_1 < \alpha < x_2$. Then, by denseness, there is another element α' of X such that

$$x_1 < \alpha' < \alpha < x_2.$$

Since $\leq$ is antisymmetric, $\alpha \nleq \alpha'$. So, if we can prove that α' is also an upper bound for S, then α would not be the least upper bound for S which is again a contradiction. Let s be an arbitrary element of S. Then, $s \in A$ and $s \leq \alpha$. If $\alpha' \leq s$, then

$$s \in [\alpha', \alpha] \subseteq (x_1, x_2) \subseteq CA$$

which contradicts the fact that $s \in A$. Since $(X, \leq)$ is a prechain, we must have $s < \alpha'$ for all elements s of S; that is, α' is an upper bound for S. Since $\alpha' < \alpha$, α is not the least upper bound for S, a contradiction.

Case 2. $a' < a$. Define $S = \{x | x \in A,\ a' \leq x \leq a\}$. Since $a \in S$, S is a nonempty subset of X which has a lower bound a'. By Theorem 1.12, there exists a greatest lower bound for S. Using arguments similar to those in Case 1, we can get, again, a contradiction. Q.E.D.

<u>Theorem 3.6</u>. Let $(X, \leq)$ be a linearly ordered set. The

ordered topological space (X, T) induced by $\leq$ is compact iff $(X, \leq)$ has both a least element and a greatest element and it can not be written as the union of two nonempty sets A and B (with the relative preorders) such that each element of A is less than each element of B, while A has no greatest element and B has no least element.

Proof. The necessity was proved in Theorem 2.4. Now we prove the sufficiency.

Let α be the least element and β be the greatest element of X. It is clear that $X = [\alpha, \beta]$. Suppose $G = \{G_\lambda | \lambda \in \Lambda\}$ is an arbitrary open covering of X in the ordered topological space (X, T). Define

$$A = \{x | x \in X, [\alpha, x] \text{ can be covered by a finite subfamily of } G\}.$$

If we can prove that $\beta \in A$, then it is clear that (X, T) is compact. Suppose $\beta \notin A$. If we let B be the complement of A, then $B \neq \emptyset$. Now let a and b be arbitrary elements of A and B respectively. If $b \leq a$, then $[\alpha, b] \subseteq [\alpha, a]$ and hence, $[\alpha, b]$ can be covered by a finite subfamily of G. That is, $b \in A$ which is a contradiction. Therefore, no element of B is less than or equal to an element of A. Since $(X, \leq)$ is a prechain, we must have $a < b$.

Case 1. A has no greatest element and B has no least element. Since $X = A \cup B$, this contradicts the hypothesis.

Case 2. A has a greatest element g and B has no least element. Suppose $g \in G_{\lambda_0}$ for some $\lambda_0 \in \Lambda$. Since G_{λ_0} is an open set, there exist elements x_1 and x_2 of X such that $g \in (x_1, x_2) \subseteq G_{\lambda_0}$. Hence $x_1 < g < x_2$. Since $\leq$ is antisymmetric, we have $x_2 \nleq g$. It follows that $x_2 \notin A$ and hence $x_2 \in B$ since g is the great element of A. Because B has no least element and $(X, \leq)$ is a prechain, there is an element x_3 of B such that $x_3 < x_2$. Hence, we have

$$x_1 < g < x_3 < x_2.$$

Then, $x_3 \in (x_1, x_2) \subseteq G_{\lambda_0}$ and hence, $[a, x_3]$ can be covered by a finite subfamily of G by adding G_{λ_0} to the original finite subfamily of G which covers $[a, g]$. By the definition of A, $x_3 \in A$ which is a contradiction.

Case 3. A has no greatest element and B has a least element ℓ. Suppose $\ell \in G_{\lambda_0}$ for some $\lambda_0 \in \Lambda$. By arguments similar to those in Case 2, we see that $[a, \ell]$ can also be covered by a finite subfamily of G. By the definition of A, $\ell \in A$ which contradicts the fact that ℓ is an element of B.

Case 4. A has a greatest element g and B has a least element ℓ. Suppose $g \in G_{\lambda_0}$ for some $\lambda_0 \in \Lambda$. By the arguments in Case 2, we see that G_{λ_0} contains elements of B. Since $g < \ell$ and ℓ is the least element of B, we see that $\ell \in G_{\lambda_0}$. Consequently, $[\alpha, \ell]$ can be covered by a finite subfamily of G. By the definition of A, $\ell \in A$ which is, again, a contradiction.

From the above arguments, we know that β must be an element of A. Since $X = [\alpha, \beta]$, it follows that (X, T) is compact.

Q.E.D.

In the following, we give definitions for other types of order.

Definition 3.7. A partially ordered set $(X, \leq)$ is called well-ordered if every nonempty subset of X has a least element (with the relative order).

Example 3.8. $\emptyset$ is a well-ordered set. In any set $\{x\}$ consisting of exactly one element, the preorder $\leq$ on $\{x\}$ defined by $x \leq x$, is a well-ordering. The partial order of inclusion in the family $P(X)$ of all subsets of X which contains more than one element is not a well-ordering.

Remark. It is clear that a well-ordered set is a linearly ordered set with a least element. The converse is not always true, the set of all nonnegative real numbers with the usual (relative) order $\leq_{(R)}$ is a linearly ordered set with a least element but not a well-ordered set.

Definition 3.9. A preordered set is called a lattice if every pair of elements (and hence any finite set of elements) has both a least upper bound and a greatest lower bound.

Example 3.10. The family of all subsets of a set ordered by inclusion is a lattice.

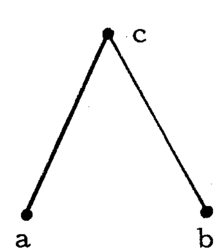
Remark. It is easy to see that any linearly ordered set is a lattice.

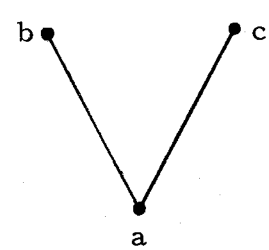
Definition 3.11. A preordered set $(D, \leq)$ is called a directed set if $\leq$ is directive; that is, for all elements a and b of D, there exists an element c of D such that

$$a \leq c \quad \text{and} \quad b \leq c.$$

Remark. Any preordered set with a greatest element is clearly a directed set.

Example 3.12. (1) Consider the following two graphical representations of preordered sets:





The graph on the left represents a directed set since it has a greatest element c, but the graph on the right does not represent a directed set since there exists no element which is greater than or equal to both b and c.

(2) If the preorder $\leq$ is the indiscrete order on a set X, then, clearly, $(X, \leq)$ is a directed set. If $\leq$ is the discrete order on a set X containing more than one element, then $(X, \leq)$ is not a directed set.

(3) The family of all finite subsets of a set is directed by the set inclusion $\subseteq$.

(4) In Example 3.3(4), $(P(P(R)), \leq)$ is a directed set.

(5) Let N_* be the family of all neighborhoods of a point in a topological space and define $\leq$ on N_* as follows.

$$\leq \; = \{(N, M) \mid N, M \in N_*, \; M \subseteq N\}.$$

Since the intersection of two neighborhoods of a point is again a neighborhood of that point. We see that $(N_*, \leq)$ is a directed set.

2. THE STRUCTURE OF PREORDERED SETS

From Theorem 1.6 of Chapter 1, we see that given any preordered set $(X, \leq)$, there is a family $\{\bar{L}_x | x \in X\}$ of subsets of X such that, for all elements x and y of X, $x \leq y$ iff $\bar{L}_x \subseteq \bar{L}_y$. In this chapter, we start from an arbitrary family of subsets of a set and use this family to define a preorder on the set.

2-1. Λ and Λ_x

The first section discusses the general properties in any family of subsets of a set which will be used in defining the preorder.

From now on, Λ (or Λ with any superscript) denotes a subfamily of the family $P(X)$ of all subsets of a set X. An element of Λ which contains the element x of X is denoted by λ_x or μ_x and the set of all elements of Λ containing the element x of X is denoted by Λ_x. Moreover, we set

$$\Lambda_X = \{\Lambda_x | x \in X\}$$

$$\cap \Lambda = \cap \{\lambda | \lambda \in \Lambda\}$$

$$\cup \Lambda = \cup \{\lambda | \lambda \in \Lambda\}$$

$$\cap \Lambda_x = \cap \{\lambda_x | \lambda_x \in \Lambda_x\}$$

$$\cup \Lambda_x = \cup \{\lambda_x | \lambda_x \in \Lambda_x\}$$

$$C\Lambda = \{X - \lambda | \lambda \in \Lambda\}$$

$$C\Lambda_x = \{X - \lambda_x \mid \lambda_x \in \Lambda_x\}$$

$$(C\Lambda)_x = \{\lambda \mid \lambda \in C\Lambda \text{ and } x \in \lambda\}.$$

It is worth noticing that Λ, Λ_x, $C\Lambda$, $C\Lambda_x$ and $(C\Lambda)_x$ are elements of $P(P(X))$, the collection of all families of subsets of X, while $\cap\Lambda$, $\cup\Lambda$, $\cap\Lambda_x$ and $\cup\Lambda_x$ are elements of $P(X)$, the family of all subsets of X; that is, $\cap\Lambda$, $\cup\Lambda$, $\cap\Lambda_x$ and $\cup\Lambda_x$ are subsets of X, but not necessarily elements of Λ. Moreover, all the sets Λ, Λ_x, $C\Lambda$, $C\Lambda_x$, $(C\Lambda)_x$, $\cap\Lambda$, $\cup\Lambda$, $\cap\Lambda_x$ and $\cup\Lambda_x$ are partially ordered by set inclusion.

Theorem 1.1. (1) For every element x of X, $x \in \cap\Lambda_x$.

(2) The set Λ_x is nonempty for all elements x of X iff $\cup\Lambda = X$.

(3) For every element x of X,

$$x \in \cap\Lambda \quad \text{iff} \quad \Lambda = \Lambda_x.$$

(4) $\Lambda - \{\emptyset\} = \bigcup_{x \in X} \Lambda_x$ and $\cap(\Lambda - \{\emptyset\}) = \bigcap_{x \in X} (\cap\Lambda_x)$.

(5) For all elements x of X,

$$(C\Lambda)_x = C\Lambda - C\Lambda_x.$$

Proof. (1) Let x be an element of X. If $\Lambda_x = \emptyset$, then $\cap\Lambda_x = X$. If $\Lambda_x \neq \emptyset$, then $x \in \lambda_x$ for all $\lambda_x \in \Lambda_x$. Consequently, $x \in \cap\Lambda_x$ for every element x of X.

(2) Suppose $\cup \Lambda = X$. Then, for all $x \in X$, there exists an element of Λ containing x; that is, $\Lambda_x \neq \emptyset$. Conversely, suppose $\Lambda_x \neq \emptyset$ for all $x \in X$. Then, for all $x \in X$, there exists an element of Λ_x containing x. Since $\Lambda_x \subseteq \Lambda$, this element is also an element of Λ. Hence $x \in \cup \Lambda$ for all $x \in X$; that is, $X \subseteq \cup \Lambda$. Since $\cup \Lambda \subseteq X$ is obvious, we have $X = \cup \Lambda$.

(3) For every element x of X, if $x \in \cap \Lambda$, then every element of Λ contains the element x. Hence every element of Λ is an element of Λ_x; that is, $\Lambda \subseteq \Lambda_x$. Since $\Lambda_x \subseteq \Lambda$ is obvious, we have $\Lambda = \Lambda_x$. Conversely, if $\Lambda = \Lambda_x$, then $\cap \Lambda = \cap \Lambda_x$. From (1), $x \in \cap \Lambda_x$, hence $x \in \cap \Lambda$.

(4) Case 1. $\Lambda = \emptyset$ or $\Lambda = \{\emptyset\}$. Since $\Lambda_x \subseteq \Lambda$, we have $\Lambda_x = \emptyset$ for all $x \in X$. Hence, $\bigcup_{x \in X} \Lambda_x = \emptyset$ and $\cap \Lambda_x = X$ for all $x \in X$. Therefore, $\Lambda - \{\emptyset\} = \bigcup_{x \in X} \Lambda_x = \emptyset$ and

$$\cap(\Lambda - \{\emptyset\}) = \bigcup_{x \in X} (\cap \Lambda_x) = X.$$

Case 2. $\Lambda \neq \emptyset$ and $\Lambda \neq \{\emptyset\}$. In this case, we have $\Lambda - \{\emptyset\} \neq \emptyset$. Then, any element of $\Lambda - \{\emptyset\}$ contains at least one element, say x, of X and it follows that this element of $\Lambda - \{\emptyset\}$ is an element of Λ_x. Consequently, $\Lambda - \{\emptyset\} \subseteq \bigcup_{x \in X} \Lambda_x$. Since $\bigcup_{x \in X} \Lambda_x \subseteq \Lambda - \{\emptyset\}$ is obvious, we have $\Lambda - \{\emptyset\} = \bigcup_{x \in X} \Lambda_x$.

For the second part, suppose $y \in \bigcap_{x \in X} (\cap \Lambda_x)$, then $y \in \cap \Lambda_x$ for all $x \in X$. Hence $y \in \lambda_x$ for all $\lambda_x \in \Lambda_x$ and all $x \in X$.

Since $\bigcup_{x \in X} \Lambda_x = \{\lambda_x | \lambda_x \in \Lambda_x, x \in X\}$, we see that y is an element of every element of $\bigcup_{x \in X} \Lambda_x$; that is, $y \in \bigcap(\bigcup_{x \in X} \Lambda_x)$. Since $\Lambda - \{\phi\} = \bigcup_{x \in X} \Lambda_x$, as just proved, we have $y \in \bigcap(\Lambda - \{\phi\})$. Therefore, $\bigcap_{x \in X} (\bigcap \Lambda_x) \subseteq \bigcap(\Lambda - \{\phi\})$. Moreover, since $\Lambda_x \subseteq \Lambda - \{\phi\}$ for all $x \in X$, it follows that $\bigcap(\Lambda - \{\phi\}) \subseteq \bigcap \Lambda_x$ for all $x \in X$. Hence $\bigcap(\Lambda - \{\phi\}) \subseteq \bigcap_{x \in X} (\bigcap \Lambda_x)$. Consequently, $\bigcap(\Lambda - \{\phi\}) = \bigcap_{x \in X} (\bigcap \Lambda_x)$.

(5) Notice that, for all elements x of X,

$$
\begin{aligned}
(C\Lambda)_x &= \{\lambda | \lambda \in C\Lambda \text{ and } x \in \lambda\} \\
&= \{X - \lambda | \lambda \in \Lambda \text{ and } x \notin \lambda\} \\
&= \{X - \lambda | \lambda \in \Lambda - \Lambda_x\},
\end{aligned}
$$

while

$$C\Lambda_x = \{X - \lambda | \lambda \in \Lambda_x\}.$$

Hence

$$(C\Lambda)_x = C\Lambda - C\Lambda_x.$$

Q.E.D.

<u>Theorem 1.2.</u> Let x and y be elements of a set X. The following four statements are equivalent:

(1) $\Lambda_y \subseteq \Lambda_x$.

(2) $\bigcap \Lambda_x \subseteq \bigcap \Lambda_y$.

(3) $x \in \bigcap \Lambda_y$.

(4) $C\Lambda_y \subseteq C\Lambda_x$.

Proof. It is evident that if any one of Λ_x and Λ_y is empty then these statements are equivalent. Now we assume that none of Λ_x and Λ_y is empty.

(1) $\Rightarrow$ (2): For every $z \in \cap \Lambda_x$, $z \in \lambda_x$ for all $\lambda_x \in \Lambda_x$. Since $\Lambda_y \subseteq \Lambda_x$, $z \in \lambda_y$ for all $\lambda_y \in \Lambda_y$; that is, $z \in \cap \Lambda_y$. Hence $\cap \Lambda_x \subseteq \cap \Lambda_y$.

(2) $\Rightarrow$ (3): Since, by Theorem 1.1(1), $x \in \cap \Lambda_x$, we have $x \in \cap \Lambda_y$.

(3) $\Rightarrow$ (4): Let $X - \lambda_y$ be an element of $C\Lambda_y$. Since $x \in \cap \Lambda_y$, we have $x \in \lambda_y$. Hence $\lambda_y \in \Lambda_x$ which is equivalent to $X - \lambda_y \in C\Lambda_x$. Consequently, $C\Lambda_y \subseteq C\Lambda_x$.

(4) $\Rightarrow$ (1): Notice that, for every $\lambda_y \in \Lambda_y$, $X - \lambda_y \in C\Lambda_y$. Since $C\Lambda_y \subseteq C\Lambda_x$, we have $X - \lambda_y \in C\Lambda_x$ which is equivalent to $\lambda_y \in \Lambda_x$. Therefore, $\Lambda_y \subseteq \Lambda_x$ as desired. Q.E.D.

Let A be an index set and, for every element a of A, let Λ^a be an element of $P(P(X))$, the collection of all families of subsets of a set X, then $\bigcup_{a \in A} \Lambda^a$ and $\bigcap_{a \in A} \Lambda^a$ are also elements of $P(P(X))$. If $(\bigcup_{a \in A} \Lambda^a)_x$ and $(\bigcap_{a \in A} \Lambda^a)_x$ denote the sets of all elements of $\bigcup_{a \in A} \Lambda^a$ and $\bigcap_{a \in A} \Lambda^a$ containing the element x of X respectively, then we have the following theorem.

<u>Theorem 1.3.</u> If x is an element of X, then

(1) $(\bigcup_{a \in A} \Lambda^a)_x = \bigcup_{a \in A} \Lambda^a_x$

(2) $(\bigcap_{a \in A} \Lambda^a)_x = \bigcap_{a \in A} \Lambda^a_x$

(3) $\cup(\bigcap_{a \in A} \Lambda_x^a) \subseteq \bigcap_{a \in A} (\cup \Lambda_x^a) \subseteq \cup \Lambda_x^{a^0} \subseteq \bigcup_{a \in A} (\cup \Lambda_x^a) = \cup(\bigcup_{a \in A} \Lambda_x^a)$

(4) $\cap(\bigcup_{a \in A} \Lambda_x^a) = \bigcap_{a \in A} (\cap \Lambda_x^a) \subseteq \cap \Lambda_x^{a^0} \subseteq \bigcup_{a \in A} (\cap \Lambda_x^a) \subseteq \cap(\bigcap_{a \in A} \Lambda_x^a)$.

The a^0 in (3) and (4) is an arbitrary but fixed element of the index set A.

Proof. (1) Notice the following equivalences: $\lambda \in (\bigcup_{a \in A} \Lambda^a)_x$ iff $x \in \lambda \in \bigcup_{a \in A} \Lambda^a$, iff there exists $a^0 \in A$ such that $x \in \lambda \in \Lambda^{a^0}$, iff there exists $a^0 \in A$ such that $\lambda \in \Lambda_x^{a^0}$, iff $\lambda \in \bigcup_{a \in A} \Lambda_x^a$. It is clear that

$$(\bigcup_{a \in A} \Lambda^a)_x = \bigcup_{a \in A} \Lambda_x^a.$$

(2) The following statements are equivalent: $\lambda \in (\bigcap_{a \in A} \Lambda^a)_x$, $x \in \lambda \in \bigcap_{a \in A} \Lambda^a$, $x \in \lambda \in \Lambda^a$ for all $a \in A$, $\lambda \in \Lambda_x^a$ for all $a \in A$, $\lambda \in \bigcap_{a \in A} \Lambda_x^a$. Therefore, we have

$$(\bigcap_{a \in A} \Lambda^a)_x = \bigcap_{a \in A} \Lambda_x^a.$$

(3) First, we prove that $\cup(\bigcap_{a \in A} \Lambda_x^a) \subseteq \bigcap_{a \in A} (\cup \Lambda_x^a)$. Notice the following implications: $y \in \cup(\bigcap_{a \in A} \Lambda_x^a)$ implies there exists λ such that $y \in \lambda \in \bigcap_{a \in A} \Lambda_x^a$, implies there exists λ such that

$y \in \lambda \in \Lambda_x^\alpha$ for all $\alpha \in A$, implies $y \in \cup \Lambda_x^\alpha$ for all $\alpha \in A$, implies $y \in \bigcap_{\alpha \in A} (\cup \Lambda_x^\alpha)$. Thus, we have

$$\cup(\bigcap_{\alpha \in A} \Lambda_x^\alpha) \subseteq \bigcap_{\alpha \in A} (\cup \Lambda_x^\alpha).$$

Secondly, it is clear that, for an arbitrary but fixed element α^0 of the index set A,

$$\bigcap_{\alpha \in A} (\cup \Lambda_x^\alpha) \subseteq \cup \Lambda_x^{\alpha^0} \subseteq \bigcup_{\alpha \in A} (\cup \Lambda_x^\alpha).$$

Finally, we prove that $\bigcup_{\alpha \in A} (\cup \Lambda_x^\alpha) = \cup(\bigcup_{\alpha \in A} \Lambda_x^\alpha)$. The following statements are equivalent: $y \in \bigcup_{\alpha \in A} (\cup \Lambda_x^\alpha)$, there exists $\alpha^0 \in A$ such that $y \in \cup \Lambda_x^{\alpha^0}$, there exist an element α^0 of A and a subset λ of X such that $y \in \lambda \in \Lambda_x^{\alpha^0}$, there exists a subset λ of X such that $y \in \lambda \in \bigcup_{\alpha \in A} \Lambda_x^\alpha$, $y \in \cup(\bigcup_{\alpha \in A} \Lambda_x^\alpha)$. Therefore, we have

$$\bigcup_{\alpha \in A} (\cup \Lambda_x^\alpha) = \cup(\bigcup_{\alpha \in A} \Lambda_x^\alpha).$$

(4) First, we prove that $\cap(\bigcup_{\alpha \in A} \Lambda_x^\alpha) = \bigcap_{\alpha \in A} (\cap \Lambda_x^\alpha)$. Since $\bigcup_{\alpha \in A} \Lambda_x^\alpha = \{\lambda^\alpha | \lambda^\alpha \in \Lambda_x^\alpha, \alpha \in A\}$, the following statements are equivalent: $y \in \cap(\bigcup_{\alpha \in A} \Lambda_x^\alpha)$, $y \in \lambda^\alpha$ for all $\lambda^\alpha \in \Lambda_x^\alpha$ and all $\alpha \in A$, $y \in \cap \Lambda_x^\alpha$ for all $\alpha \in A$, $y \in \bigcap_{\alpha \in A} (\cap \Lambda_x^\alpha)$. Therefore, we have

$$\cap(\bigcup_{a\in A}\Lambda_x^a) = \bigcap_{a\in A}(\cap\Lambda_x^a).$$

Secondly, it is clear that, for an arbitrary but fixed element a^0 of the index set A,

$$\bigcap_{a\in A}(\cap\Lambda_x^a) \subseteq \cap\Lambda_x^a \subseteq \bigcup_{a\in A}(\cap\Lambda_x^a).$$

Finally, we prove that $\bigcup_{a\in A}(\cap\Lambda_x^a) \subseteq \cap(\bigcap_{a\in A}\Lambda_x^a)$. If $y \in \bigcup_{a\in A}(\cap\Lambda_x^a)$, then there exist $a^0 \in A$ such that $y \in \cap\Lambda_x^{a^0}$. Since $\bigcap_{a\in A}\Lambda_x^a \subseteq \Lambda_x^{a^0}$, we have $\cap\Lambda_x^{a^0} \subseteq \cap(\bigcap_{a\in A}\Lambda_x^a)$. Consequently, if $y \in \bigcup_{a\in A}(\cap\Lambda_x^a)$, then $y \in \cap(\bigcap_{a\in A}\Lambda_x^a)$. So,

$$\bigcup_{a\in A}(\cap\Lambda_x^a) \subseteq \cap(\bigcap_{a\in A}\Lambda_x^a).$$ Q.E.D.

Remark. The set inclusion "$\subseteq$" in the theorem can not be replaced by "=". For example, let $X = \{x, y, z\}$, $\Lambda^1 = \{\{x\}\}$ and $\Lambda^2 = \{\{x, y\}\}$. Since $\Lambda_x^1 = \Lambda^1$ and $\Lambda_x^2 = \Lambda^2$, we have $\cup\Lambda_x^1 = \cap\Lambda_x^1 = \{x\}$, $\cup\Lambda_x^2 = \cap\Lambda_x^2 = \{x, y\}$, $\bigcap_{a\in\{1,2\}}\Lambda_x^a = \emptyset$ and $\bigcup_{a\in\{1,2\}}\Lambda_x^a = \{\{x\}, \{x, y\}\}$. Note that

$$\cup(\bigcap_{a\in\{1,2\}}\Lambda_x^a) = \cup\emptyset = \emptyset$$

$$\bigcap_{a\in\{1,2\}}(\cup\Lambda_x^a) = \{x\} \cap \{x, y\} = \{x\}$$

$$\bigcup_{a \in \{1, 2\}} (\cup \Lambda_x^a) = \{x\} \cup \{x, y\} = \{x, y\}$$

$$\bigcap_{a \in \{1, 2\}} (\cap \Lambda_x^a) = \{x\} \cap \{x, y\} = \{x\}$$

$$\bigcup_{a \in \{1, 2\}} (\cap \Lambda_x^a) = \{x\} \cup \}x, y\} = \{x, y\}$$

$$\cap (\bigcap_{a \in \{1, 2\}} \Lambda_x^a) = \cap \phi = X = \{x, y, z\}.$$

<u>Theorem 1.4.</u> Λ is linearly ordered by set inclusion iff Λ_X is linearly ordered by set inclusion.

Proof. If Λ is not linearly ordered by set inclusion, then there exists a pair of elements, say, λ and μ of Λ such that

$$\lambda - \mu \neq \phi \quad \text{and} \quad \mu - \lambda \neq \phi.$$

Let $x \in \lambda - \mu$ and $y \in \mu - \lambda$. Then, $\lambda \in \Lambda_x - \Lambda_y$ and $\mu \in \Lambda_y - \Lambda_x$. Hence,

$$\Lambda_x - \Lambda_y \neq \phi \quad \text{and} \quad \Lambda_y - \Lambda_x \neq \phi.$$

Consequently, Λ_x and Λ_y are not related in the set Λ_X with set inclusion; that is, Λ_X is not linearly ordered by set inclusion.

If Λ_X is not linearly ordered by set inclusion, then there exists a pair of elements, say, Λ_x and Λ_y of Λ_X such that

$$\Lambda_x - \Lambda_y \neq \phi \quad \text{and} \quad \Lambda_y - \Lambda_x \neq \phi.$$

Let $\lambda \in \Lambda_x - \Lambda_y$ and $\mu \in \Lambda_y - \Lambda_x$. Then, $x \in \lambda - \mu$ and $y \in \mu - \lambda$. Hence,

$$\lambda - \mu \neq \emptyset \quad \text{and} \quad \mu - \lambda \neq \emptyset.$$

Consequently, λ and μ are not related in the set Λ with set inclusion; that is, Λ is not linearly ordered by set inclusion.

Q.E.D.

2-2. The Preorder Derived from Λ

In this section, we define a preorder $\leq_\Lambda$ on a set from an arbitrary family Λ of subsets of that set. Moreover, we shall show that if $\leq$ is a preorder on a set X and the family $\Lambda = \{\overline{L}_x | x \in X\}$, then the preorder $\leq_\Lambda$ derived from Λ is exactly the original preorder $\leq$ on X. From this fact, we get an upper bound for the cardinality of preorders on any set. In the last part of this section, we prove that if $\leq_\Lambda$ is the preorder derived from the family Λ of subsets of a set X, then the relative preorder $\leq_\Lambda | A$ of any subset A of X is the same as the preorder derived from the family $\Lambda | A$ of subsets of A where $\Lambda | A = \{\lambda \cap A | \lambda \in \Lambda\}$.

Definition 2.1. Let X be a set and let Λ be an arbitrary family of subsets of X. With respect to Λ, we define a relation $\leq_\Lambda$ on X as follows:

$$x \leq_\Lambda y \quad \text{iff} \quad \Lambda_y \subseteq \Lambda_x \quad \text{for all} \quad x, y \in X.$$

Remark. It is clear that $x \Delta_\Lambda y$ iff $\Lambda_x = \Lambda_y$.

Theorem 2.2. $(X, \leq_\Lambda)$ is a preordered set.

Proof. (1) Since $\Lambda_x \subseteq \Lambda_x$ for all $x \in X$, $x \leq_\Lambda x$ for all $x \in X$. So, $\leq_\Lambda$ is reflexive.

(2) If $x \leq_\Lambda y$ and $y \leq_\Lambda z$, then $\Lambda_y \subseteq \Lambda_x$ and $\Lambda_z \subseteq \Lambda_y$. Therefore $\Lambda_z \subseteq \Lambda_x$ which is equivalent to $x \leq_\Lambda z$. Thus, $\leq_\Lambda$ is transitive.

By virtue of (1) and (2), we conclude that $\leq_\Lambda$ is a preorder on X. So, $(X, \leq_\Lambda)$ is a preordered set. Q.E.D.

From now on, when we speak of the preorder derived from a family Λ of subsets of a set, we always mean the preorder defined in Definition 2.1. We use the symbol "$\leq_\Lambda$" to denote the preorder derived from Λ. If the family of subsets of the set is Λ^a, then the preorder derived from Λ^a will be denoted by $\leq_\Lambda^a$.

Theorem 2.3. For the preordered set $(X, \leq_\Lambda)$, the following four statements are equivalent:

(1) $x \leq_\Lambda y$.

(2) $\cap \Lambda_x \subseteq \cap \Lambda_y$.

(3) $x \in \cap \Lambda_y$.

(4) $C\Lambda_y \subseteq C\Lambda_x$.

Proof. By Theorem 1.2. Q.E.D.

Corollary. The preorder $\leq_\Lambda$ is the indiscrete order (discrete order) on a set X iff, for all elements x of X,

$$\cap \Lambda_x = X \quad (\cap \Lambda_x = \{x\}).$$

Proof. By the theorem,

$$x \leq_\Lambda y \quad \text{is equivalent to} \quad x \in \cap \Lambda_y;$$

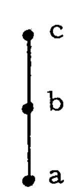
clearly, we have the result. Q.E.D.

Example 2.4. Let $X = \{a, b, c\}$.

(1) $\Lambda = \{\{a\}, \{a, b\}\}$. Then $\Lambda_a = \{\{a\}, \{a, b\}\}$, $\Lambda_b = \{\{a, b\}\}$ and $\Lambda_c = \phi$. Since $\Lambda_c \subset \Lambda_b \subset \Lambda_a$, we have

$$a \leq_\Lambda b \leq_\Lambda c.$$

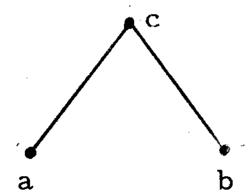
The graphical representation of $(X, \leq_\Lambda)$ is



Notice that this preordered set is a linearly ordered set.

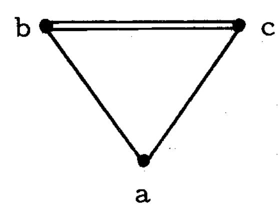
(2) $\Lambda = \{\{a\}, \{b\}, \{a, b\}\}$. Then $\Lambda_a = \{\{a\}, \{a, b\}\}$, $\Lambda_b = \{\{b\}, \{a, b\}\}$ and $\Lambda_c = \phi$. Note that $\Lambda_c \subset \Lambda_a$ and $\Lambda_c \subset \Lambda_b$,

but Λ_a and Λ_b are not related by set inclusion. Hence the graphical representation of $(X, \leq_\Lambda)$ is



This preordered set is a partially ordered set but not a linearly ordered set.

(3) $\Lambda = \{\{a\}\}$. Then $\Lambda_a = \{\{a\}\}$ and $\Lambda_b = \Lambda_c = \emptyset$. Since $\Lambda_b = \Lambda_c \subset \Lambda_a$, the graphical representation of $(X, \leq_\Lambda)$ is



In this example, the preordered set $(X, \leq_\Lambda)$ is not a partially ordered set.

Example 2.5. Let X be any set.

(1) $\Lambda = \{\emptyset, X\}$. Then $\Lambda_x = \{X\}$ for all elements x of X. Hence $x \trianglelefteq_\Lambda y$ for all elements x and y of X; that is, $\leq_\Lambda$ is the indiscrete order on X. Similarly, if we take $\Lambda = \{X\}, \{\emptyset\}$ or $\emptyset$, we shall obtain the same (indiscrete) order on X.

(2) $\Lambda = P(X)$, the family of all subsets of X. Then $\{x\} \in \Lambda_x$ for all elements x of X. Consequently, for any distinct elements x and y of X, Λ_x and Λ_y are not related by set

inclusion. So, $\leq_\Lambda$ is the discrete order on X.

Example 2.6. Let R be the set of all real numbers and let

$$\bar{L}_x = \{y \mid y \in R, y \leq_{(R)} x\}.$$

(1) $\Lambda = \{\bar{L}_x \mid x \in R\}$. Notice that $\bar{L}_x \in \Lambda_y$ iff $y \in \bar{L}_x$, iff $y \leq_{(R)} x$. Since, by Theorem 1.6 of Chapter 1, $y \leq_{(R)} x$ is equivalent to $\bar{L}_y \subseteq \bar{L}_x$, we see that the element $\bar{L}_y$ of Λ_y is a subset of any element of Λ_y; that is, $\cap \Lambda_y = \bar{L}_y$. Thus, by Theorem 2.3, we have

$$x \leq_\Lambda y \quad \text{iff} \quad x \in \cap \Lambda_y = \bar{L}_y.$$

Since $x \in \bar{L}_y$ iff $x \leq_{(R)} y$, it follows that

$$x \leq_\Lambda y \quad \text{iff} \quad x \leq_{(R)} y.$$

That is, the preorder $\leq_\Lambda$ is the same as the usual order $\leq_{(R)}$ on the set R of all real numbers.

(2) $\Lambda = \{\bar{L}_n \mid n \text{ is an integer}\}$. Note that $\cap \Lambda_x = \bar{L}_N$ where N is the least integer which is greater than or equal to x. If we let $(N, N+1] = \{x \mid x \in R, N <_{(R)} x \leq_{(R)} N+1\}$ where N is an integer, then, for all elements x and y of $(N, N+1]$, $x \underline{\Delta}_\Lambda y$. For example, $\frac{2}{3} \underline{\Delta}_\Lambda \frac{1}{2}$. If x and y are not elements of $(N, N+1]$ for any integer N, then

$$x \leq_{\Lambda} y \quad \text{iff} \quad x \leq_{(R)} y.$$

(3) For any real number x, let

$$\bar{I}_{|x|} = \{y \mid y \in R, -|x| \leq_{(R)} y \leq_{(R)} |x|\}.$$

Take $\Lambda = \{\bar{I}_{|x|} \mid x \in R\}$. Then $\bar{I}_{|x|} \in \Lambda_y$ iff $y \in \bar{I}_{|x|}$, iff $-|x| \leq_{(R)} y \leq_{(R)} |x|$, iff $|y| \leq_{(R)} |x|$. Since $|y| \leq_{(R)} |x|$ iff $\bar{I}_{|y|} \subseteq \bar{I}_{|x|}$, we see that the element $\bar{I}_{|y|}$ of Λ_y is a subset of any element of Λ_y; that is, $\cap \Lambda_y = \bar{I}_{|y|}$. Thus, by virtue of Theorem 2.3, we have

$$x \leq_{\Lambda} y \quad \text{iff} \quad x \in \cap \Lambda_y = \bar{I}_{|y|}.$$

Since $x \in \bar{I}_{|y|}$ iff $|x| \leq_{(R)} |y|$, it follows that

$$x \leq_{\Lambda} y \quad \text{iff} \quad |x| \leq_{(R)} |y|.$$

Notice that, for all elements x of X, $-x \trianglelefteq_{\Lambda} x$.

Theorem 2.7. In the preordered set $(X, \leq_{\Lambda})$, if $\cap \Lambda_x$ is an element of Λ, then $x \leq_{\Lambda} y$ iff there exists an element λ_x of Λ_x such that $\lambda_x \subseteq \cap \Lambda_y$.

Proof. Suppose $x \leq_{\Lambda} y$. By theorem 2.3, we have $\cap \Lambda_x \subseteq \cap \Lambda_y$. Since $x \in \cap \Lambda_x \in \Lambda$, $\cap \Lambda_x \in \Lambda_x$. Therefore, if we let $\lambda_x = \cap \Lambda_x$, then $\lambda_x \subseteq \cap \Lambda_y$.

Conversely, suppose that there exists an element λ_x of Λ_x such that $\lambda_x \subseteq \cap \Lambda_y$. Since $\cap \Lambda_x \subseteq \lambda_x$, it follows that $\cap \Lambda_x \subseteq \cap \Lambda_y$. By Theorem 2.3, we see that $x \leq_\Lambda y$. Q.E.D.

Corollary. If Λ is a family of subsets of X such that $\cap \Lambda_x \in \Lambda$ for all elements x of X; that is, for all elements x of X, there exists a minimal element of Λ (with respect to set inclusion) containing x, then, for all elements x and y of X, $x \leq_\Lambda y$ iff there exists an element λ_x of Λ_x such that $\lambda_x \subseteq \cap \Lambda_y$.

Theorem 2.8. In the preordered set $(X, \leq_\Lambda)$, if $\Lambda_y \neq \phi$ and $\cup \Lambda_y$ is an element of Λ for some $y \in X$, then

$$x \leq_\Lambda y \quad \text{implies} \quad x \in \cup \Lambda_y .$$

Proof. If $x \leq_\Lambda y$, then $\Lambda_y \subseteq \Lambda_x$. Since $\Lambda_y \neq \phi$ and $\cup \Lambda_y \in \Lambda$, we have $y \in \cup \Lambda_y \in \Lambda_y$. Therefore, $\cup \Lambda_y \in \Lambda_x$; that is, $x \in \cup \Lambda_y$. Q.E.D.

Corollary. In the preordered set $(X, \leq_\Lambda)$, suppose $\Lambda_y \neq \phi$, $\cup \Lambda_y$ is an element of Λ and x is an element of X such that $x \notin \cup \Lambda_y$.

If $\Lambda_x = \phi$ then $y \leq_\Lambda x$.

If $\Lambda_x \neq \phi$ then x and y are not related.

Proof. Clearly, if $\Lambda_x = \phi$, then $\Lambda_x \subseteq \Lambda_y$. Hence $y \leq_\Lambda x$.

Now suppose $\Lambda_x \neq \phi$. Since $x \notin \cup\Lambda_y$, by the theorem, it is impossible that $x \leq_\Lambda y$. Moreover, if $y \leq_\Lambda x$, then $\Lambda_x \subseteq \Lambda_y$ which implies $\cup\Lambda_x \subseteq \cup\Lambda_y$. Since $\Lambda_x \neq \phi$, $x \in \cup\Lambda_x$. Thus, we would have $x \in \cup\Lambda_y$ which is a contradiction. Therefore, it is also impossible that $y \leq_\Lambda x$; in other words, x and y are not related. Q.E.D.

Definition 2.9. Two preorders $\leq^1$ and $\leq^2$ on the same set X are said to be equal or the same, denoted by

$$\leq^1 = \leq^2,$$

provided that

$$x \leq^1 y \quad \text{iff} \quad x \leq^2 y$$

for all elements x and y of X.

Given any preordered set $(X, \leq)$, we have the family Λ of subsets of X defined by

$$\Lambda = \{\bar{L}_x \mid x \in X\}$$

where $\bar{L}_x = \{y \mid y \in X \text{ and } y \leq x\}$. Corresponding to Λ, there is a preorder $\leq_\Lambda$ on X derived from Λ. Example 2.6(1) tells us that if $X = R$, then $\leq_\Lambda = \leq_{(R)}$. Is $\leq_\Lambda = \leq$ true for any preordered set $(X, \leq)$? The answer is yes, as shown in the next theorem.

Before giving the theorem, we prove a lemma.

Lemma 2.10. Let $(X, \leq)$ be a preordered set and let Λ be the family of subsets of X defined by

$$\Lambda = \{\bar{L}_x \mid x \in X\}$$

where $\bar{L}_x = \{y \mid y \in X \text{ and } y \leq x\}$. Then we have

(1) Λ is a base for a topology for the set X; that is, for all elements $\bar{L}_x$ and $\bar{L}_y$ of Λ and all elements z of X such that $z \in \bar{L}_x \cap \bar{L}_y$, there exists an element λ of Λ such that

$$z \in \lambda \subseteq \bar{L}_x \cap \bar{L}_y.$$

(2) For all elements x of X,

$$\cap \Lambda_x = \bar{L}_x.$$

That is, for all elements x of X, $\bar{L}_x$ is the least element (with respect to set inclusion) of Λ containing x.

Proof. (1) For all elements $\bar{L}_x$ and $\bar{L}_y$ of Λ, if z is an element of X such that $z \in \bar{L}_x \cap \bar{L}_y$, then

$$z \leq x \quad \text{and} \quad z \leq y.$$

Hence, for all elements w of X such that $w \leq z$, by transitivity, we have

$$w \leq x \quad \text{and} \quad w \leq y.$$

Therefore, $w \in \bar{L}_x \cap \bar{L}_y$. Consequently,

$$z \in \bar{L}_z \subseteq \bar{L}_x \cap \bar{L}_y.$$

Let $\lambda = \bar{L}_z$ and we have proved (1).

(2) For every $\bar{L}_z \in \Lambda_x$, $x \in \bar{L}_z$, so $x \leq z$. By Theorem 1.6 of Chapter 1, we have $\bar{L}_x \subseteq \bar{L}_z$. Since $\bar{L}_x \in \Lambda_x$, it is clear that

$$\cap \Lambda_x = \bar{L}_x. \qquad \text{Q.E.D.}$$

Theorem 2.11. Let $(X, \leq)$ be a preordered set and let Λ be defined as in Lemma 2.10. Then we have

$$\leq_\Lambda = \leq .$$

Proof. By Theorem 2.3, for all elements x and y of X, $x \leq_\Lambda y$ iff $\cap \Lambda_x \subseteq \cap \Lambda_y$. By Lemma 2.10(2), this is equivalent to $\bar{L}_x \subseteq \bar{L}_y$. Finally, by Theorem 1.6 of Chapter 1, this is equivalent to $x \leq y$. Q.E.D.

Note that Theorem 2.11 tells us that any preorder on a set X can be derived from a family of subsets of X by using Definition 2.1. Hence we have the following theorem.

Theorem 2.12. Let $\bar{\bar{X}}$ be the cardinal number of any set X

and let θ be the cardinal number of the set of all preorders on X. If $\leq_{(C)}$ denotes the usual order on cardinal numbers, then

$$\theta \leq_{(C)} 2^{2^{\overline{\overline{X}}}} .$$

In particular, if X is a finite set with n distinct elements, then

$$\theta <_{(C)} 2^{2^{n}} .$$

Proof. Since the cardinal number of the collection $P(P(X))$ of all families of subsets of X is $2^{2^{\overline{\overline{X}}}}$, it is clear that

$$\theta \leq_{(C)} 2^{2^{\overline{\overline{X}}}} .$$

If X is a finite set with n distinct elements, then $\overline{\overline{X}} = n$. Since the two families $\Lambda^1 = \emptyset$ and $\Lambda^2 = \{\emptyset\}$ of subsets of X always define the indiscrete preorder, we have

$$\theta <_{(C)} 2^{2^{n}} .$$ Q.E.D.

Now if A is a subset of a set X and Λ is a family of subsets of X, then we can get the family $\Lambda|A$ of subsets of A from Λ as follows:

$$\Lambda|A = \{\lambda \cap A | \lambda \in \Lambda\}.$$

A question arises: Is the relative preorder $\leq_\Lambda | A$ on A equal to the preorder $\leq_{\Lambda | A}$ on A derived from $\Lambda | A$? The affirmative answer is given in the following theorem.

Theorem 2.13. Let A be a subset of X and let Λ be a family of subsets of X. For any elements a and b of A,

$$a \leq_\Lambda | A \; b \text{ iff } a \leq_{\Lambda | A} b$$

where $\leq_\Lambda | A$ is the relative preorder on A and $\leq_{\Lambda | A}$ is the preorder on A derived from the family $\Lambda | A$ of subsets of A.

Proof. Notice that, for all elements a of A,

$$(\Lambda | A)_a = \{\lambda_a \cap A | \lambda_a \in \Lambda_a\}.$$

Moreover, if a and b are elements of A, then the following statements are equivalent: $a \leq_\Lambda | A \; b$, $a \in \cap \Lambda_b$, $a \in \lambda_b$ for all $\lambda_b \in \Lambda_b$, $a \in \lambda_b \cap A$ for all $\lambda_b \in \Lambda_b$, $a \in \cap (\Lambda | A)_b$, $a \leq_{\Lambda | A} b$; hence we have

$$a \leq_\Lambda | A \; b \text{ iff } a \leq_{\Lambda | A} b. \qquad \text{Q.E.D.}$$

2-3. Characterization of Orders and Some Order Properties

The third section is to discuss conditions on a family Λ of subsets of a set X which are necessary and sufficient for the preordered set $(X, \leq_\Lambda)$ to be a partially ordered set, a prechain, a linearly ordered set or a directed set. We also discuss conditions for an element of the preordered set $(X, \leq_\Lambda)$ to be a maximal element, a minimal element, a greatest element or a least element etc.

Definition 3.1. A family Λ of subsets of a set X is said to distinguish elements of X provided that, for all elements x and y of X,

$$\Lambda_x = \Lambda_y \quad \text{iff} \quad x = y.$$

Remark. Since $x = y$ always implies $\Lambda_x = \Lambda_y$, we see that Λ distinguishes elements of X iff, for every pair of distinct elements x and y of X, $\Lambda_x \neq \Lambda_y$.

Example 3.2. In Example 2.4(3), the family $\Lambda = \{\{a\}\}$ does not distinguish elements of the set $\{a, b, c\}$ since $\Lambda_b = \Lambda_c = \emptyset$ but $b \neq c$.

Theorem 3.3. The family Λ of subsets of X distinguishes elements of X iff, for every pair of distinct elements x and y of X.

$$\{x, y\} \not\subseteq (\cap \Lambda_x) \cap (\cap \Lambda_y).$$

Proof. If Λ does not distinguish elements of X, then there exists a pair of distinct elements x and y of X such that

$\Lambda_x = \Lambda_y$. Then $\cap \Lambda_x = \cap \Lambda_y$ and hence,

$$(\cap \Lambda_x) \cap (\cap \Lambda_y) = \cap \Lambda_x = \cap \Lambda_y .$$

Since $x \in \cap \Lambda_x$ and $y \in \cap \Lambda_y$, it follows that

$$\{x, y\} \subseteq (\cap \Lambda_x) \cap (\cap \Lambda_y).$$

Conversely, suppose that there exists a pair of distinct elements x and y of X such that

$$\{x, y\} \subseteq (\cap \Lambda_x) \cap (\cap \Lambda_y)$$

Then $x \in \cap \Lambda_y$ and $y \in \cap \Lambda_x$. By Theorem 2.3, $x \leq_\Lambda y$ and $y \leq_\Lambda x$. Hence $\Lambda_y \subseteq \Lambda_x$ and $\Lambda_x \subseteq \Lambda_y$. Since Λ_X is partially ordered by set inclusion, we have $\Lambda_x = \Lambda_y$. Therefore, Λ does not distinguish elements of X. Q.E.D.

Theorem 3.4. The preorder $\leq_\Lambda$ is a partial order on a set X iff Λ distinguishes elements of X.

Proof. The preorder $\leq_\Lambda$ is a partial order on a set X iff, for all elements x and y of X,

$$x \underline{\Delta} y \quad \text{implies} \quad x = y;$$

that is, for all elements x and y of X,

$$\Lambda_x = \Lambda_y \quad \text{implies} \quad x = y.$$

Hence the preorder $\leq_\Lambda$ is a partial order on X iff Λ distinguishes elements of X. Q.E.D.

Theorem 3.5. The following three statements are equivalent:

(1) The preordered set $(X, \leq_\Lambda)$ is a prechain.

(2) $(\Lambda_X, \subseteq)$ is a linearly ordered set.

(3) $(\Lambda, \subseteq)$ is a linearly ordered set; that is, Λ is nested.

Proof. The definition of $\leq_\Lambda$ shows (1) is equivalent to (2). By Theorem 1.4, (2) is equivalent to (3). Hence these three statements are equivalent. Q.E.D.

Corollary. The preordered set $(X, \leq_\Lambda)$ is a linearly ordered set iff Λ is nested and distinguishes elements of X.

Proof. By the theorem and Theorem 3.4. Q.E.D.

Theorem 3.6. The preordered set $(X, \leq_\Lambda)$ is a directed set iff, for all elements x and y of X, there exists an element z of X such that

$$\{x, y\} \subseteq \cap \Lambda_z .$$

Proof. The preordered set $(X, \leq_\Lambda)$ is a directed set iff, for all elements x and y of X, there exists an element z of X

such that both $x \leq_\Lambda z$ and $y \leq_\Lambda z$; that is, for all elements x and y of X, there exists an element z of X such that both $x \in \cap \Lambda_z$ and $y \in \cap \Lambda_z$. Consequently, the preordered set $(X, \leq_\Lambda)$ is a directed set iff, for all elements x and y of X, there exists an element z of X such that $\{x, y\} \subseteq \cap \Lambda_z$.

Q.E.D.

<u>Theorem 3.7.</u> In the preordered set $(X, \leq_\Lambda)$, an element a of X is a maximal (minimal) element of X iff $x \in \cap \Lambda_a$ $(a \in \cap \Lambda_x)$ for all elements x of X such that $a \in \cap \Lambda_x$ $(x \in \cap \Lambda_a)$. Moreover, if Λ distinguishes elements of X, then a is a maximal (minimal) element of X iff $x = a$ for all elements x of X such that $a \in \cap \Lambda_x$ $(x \in \cap \Lambda_a)$.

Proof. Clearly, by Theorem 2.3 and the definition of maximal (minimal) element, we have the first part of the theorem. If Λ distinguishes elements of X, by Theorem 3.4, $(X, \leq_\Lambda)$ is a partially ordered set and hence, we have the second part of the theorem.

Q.E.D.

<u>Theorem 3.8.</u> In the preordered set $(X, \leq_\Lambda)$, if

$$a \notin \bigcup_{x \in X - \{a\}} (\cap \Lambda_x),$$

then a is a maximal element of X.

Proof. If

$$a \notin \bigcup_{x \in X - \{a\}} (\cap \Lambda_x),$$

then, for all elements x of X such that $x \neq a$, we have $a \notin \cap \Lambda_x$. Since

$$a \leq_\Lambda x \quad \text{iff} \quad a \in \cap \Lambda_x,$$

it follows that, for all elements x of X such that $a \leq_\Lambda x$, we must have $x = a$, otherwise, we would get a contradiction. Hence a is a maximal element of X. Q.E.D.

<u>Corollary</u>. If Λ distinguishes elements of X, then a is a maximal element of X iff

$$a \notin \bigcup_{x \in X - \{a\}} (\cap \Lambda_x).$$

Proof. The sufficiency is proved by the theorem.

For the necessity, if a is a maximal element of X, then, by Theorem 3.7, for all elements x of X such that $a \in \cap \Lambda_x$, we have $x = a$; that is, for all elements x of X such that $x \neq a$, we have $a \notin \cap \Lambda_x$. Hence

$$a \notin \bigcup_{x \in X - \{a\}} (\cap \Lambda_x).$$ Q.E.D.

<u>Theorem 3.9</u>. In the preordered set $(X, \leq_\Lambda)$, if

$$\cap \Lambda_a = \{a\},$$

then a is a minimal element of X.

Proof. If

$$\cap \Lambda_a = \{a\},$$

then, for all elements x of X such that $x \neq a$, we have $x \notin \cap \Lambda_a$. Consequently, for all elements x of X such that $x \leq_\Lambda a$, we must have $x = a$, otherwise, we would get a contradiction. Hence a is a minimal element of X. Q.E.D.

<u>Corollary 1.</u> In the preordered set $(X, \leq_\Lambda)$, if $\{a\} \in \Lambda$, then a is a minimal element of X.

Proof. If $\{a\} \in \Lambda$, then $\cap \Lambda_a = \{a\}$. By the theorem, a is a minimal element of X. Q.E.D.

<u>Corollary 2.</u> If Λ distinguishes elements of X, then a is a minimal element of X iff

$$\cap \Lambda_a = \{a\}.$$

Proof. The sufficiency is proved by the theorem.

For the necessity, if a is a minimal element of X, then, by Theorem 3.7, for all elements x of X such that $x \in \cap \Lambda_a$, we have $x = a$; that is, $\cap \Lambda_a \subseteq \{a\}$. Since $\{a\} \subseteq \cap \Lambda_a$ was proved in Theorem 1.1(1), it follows that

$$\cap \Lambda_a = \{a\}. \qquad \text{Q.E.D.}$$

Remark. The converses of Theorems 3.8 and 3.9 are not true. For example, if $\leq_\Lambda$ is the indiscrete order on a set X containing more than one element, then, as mentioned in Example 1.9(3) of Chapter 1, each element of X is both maximal and minimal, but from the corollary of Theorem 2.3, we see that $\cap \Lambda_x = X$ for all elements x of X.

Theorem 3.10. In the preordered set $(X, \leq_\Lambda)$, an element a of X is a greatest element of X iff

$$\cap \Lambda_a = X.$$

Proof. An element a of X is a greatest element of X iff $x \leq_\Lambda a$ for all elements x of X; that is, iff $x \in \cap \Lambda_a$ for all elements x of X. Hence a is a greatest element of X iff $\cap \Lambda_a = X$. Q.E.D.

Corollary 1. In the preordered set $(X, \leq_\Lambda)$, an element a of X is a greatest element of X iff

$$\text{either } \Lambda_a = \emptyset \text{ or } \Lambda_a = \{X\}.$$

Proof. Notice that $\cap \Lambda_a = X$ iff either $\Lambda_a = \emptyset$ or $\Lambda_a = \{X\}$.

Hence, from the theorem, we have the result. Q.E.D.

Corollary 2. If the set G is the greatest element of the partially ordered set $(\Lambda\text{-}\{X\}, \subseteq)$; that is,

$$G = \cup(\Lambda\text{-}\{X\}) \in \Lambda,$$

then, in the preordered set $(X, \leq_\Lambda)$,

(1) α is a greatest element of X iff

$$\alpha \in X - G.$$

(2) For all elements α and β of $X - G$,

$$\alpha \,\underline{\Delta}_\Lambda\, \beta.$$

Proof. (1) Notice the following equivalences: α is a greatest element of X, by Corollary 1, iff either $\Lambda_\alpha = \emptyset$ or $\Lambda_\alpha = \{X\}$, iff $\alpha \notin \lambda$ for all $\lambda \in \Lambda - \{X\}$, iff

$$\alpha \notin \cup(\Lambda\text{-}\{X\}) = G.$$

Therefore, α is a greatest element of X iff

$$\alpha \in X - G.$$

(2) By (1), α and β are elements of $X - G$ iff both α and β are greatest elements of X. Hence, for all elements α and β of $X - G$, $\alpha \,\underline{\Delta}_\Lambda\, \beta$. Q.E.D.

<u>Corollary 3.</u> In the preordered set $(X, \leq_\Lambda)$ each element a of $X - \cup\Lambda$ is a greatest element of X.

Proof. If a is an element of $X - \cup\Lambda$, then $\Lambda_a = \emptyset$. Hence, by Corollary 1, a is a greatest element of X.

Q.E.D.

<u>Corollary 4.</u> If $(X, \leq_\Lambda)$ is a preordered set such that $\cup\Lambda$ is a proper subset of X, then $(X, \leq_\Lambda)$ is a directed set.

Proof. If $\cup\Lambda$ is a proper subset of X, then there exists at least one element, say, a of X such that

$$a \in X - \cup\Lambda.$$

By Corollary 3, a is a greatest element of X. Since $(X, \leq_\Lambda)$ has a greatest element, it is a directed set. Q.E.D.

<u>Theorem 3.11.</u> In the preordered set $(X, \leq_\Lambda)$, an element a of X is a least element of X iff

$$a \in \cap(\Lambda - \{\emptyset\}).$$

Proof. Notice the following equivalences: An element a of X is a least element of X iff, for all elements x of X,

$\alpha \leq_\Lambda x$; iff, for all elements x of X, $\alpha \in \cap \Lambda_x$; iff $\alpha \in \bigcap_{x \in X} (\cap \Lambda_x)$; iff, by Theorem 1.1(4), $\alpha \in \cap(\Lambda - \{\emptyset\})$. Therefore, α is a least element of X iff $\alpha \in \cap(\Lambda - \{\emptyset\})$. Q.E.D.

<u>Corollary 1.</u> In the preordered set $(X, \leq_\Lambda)$ an element α of X is a least element of X iff

$$\Lambda - \{\emptyset\} = \Lambda_\alpha.$$

Proof. By the theorem and Theorem 1.1(3). Q.E.D.

<u>Corollary 2.</u> If the set L is the least element of the partially ordered set $(\Lambda - \{\emptyset\}, \subseteq)$; that is,

$$L = \cap(\Lambda - \{\emptyset\}) \in \Lambda,$$

then, in the preordered set $(X, \leq_\Lambda)$,

(1) α is a least element of X iff

$$\alpha \in L.$$

(2) For all elements α and β of L,

$$\alpha \trianglelefteq_\Lambda \beta.$$

Proof. (1) By Corollary 1, α is a least element of X iff $\Lambda - \{\emptyset\} = \Lambda_\alpha$ which is equivalent to $\alpha \in \lambda$ for all elements λ of $\Lambda - \{\emptyset\}$. Hence α is a least element of X iff

$$\alpha \in \cap(\Lambda - \{\emptyset\}) = L.$$

(2) By (1), α and β are elements of L iff both α and β are least elements of X. Hence, for all elements α and β of L

$$\alpha \trianglelefteq_{\Lambda} \beta . \qquad \text{Q.E.D.}$$

Let A be a subset of X and define

$$\Lambda_A = \{\Lambda_a \mid a \in A\}.$$

It is clear that, from the definitions of upper bounds, lower bounds and the definition of $\leq_{\Lambda}$, we have the following theorem.

Theorem 3.12. If A is a subset of X, then, in the preordered set $(X, \leq_{\Lambda})$, the following statements L1 to L4 and U1 to U3 are equivalent respectively:

L1. α is a lower bound for A.

L2. $\alpha \in \bigcap_{a \in A} (\cap \Lambda_a)$.

L3. $\alpha \in \cap\{\lambda \mid \lambda \in \Lambda \text{ and } \lambda \cap A \neq \emptyset\}$.

L4. $\cup \Lambda_A \subseteq \Lambda_\alpha$.

U1. α is an upper bound for A.

U2. $A \subseteq \cap \Lambda_\alpha$.

U3. $\Lambda_\alpha \subseteq \cap \Lambda_A$.

2-4. Order Preserving Mappings and Similarities

In this section, we discuss some order preserving mappings and similarities between preordered sets.

Definition 4.1. Let $(X, \leq)$ and $(X', \leq')$ be two preordered sets. A mapping f from X into X' is called order preserving if, for all elements x and y of X,

$$x \leq y \quad \text{implies} \quad f(x) \leq' f(y).$$

If f is a one-to-one mapping from X onto X' such that for all elements x and y of X,

$$x \leq y \quad \text{iff} \quad f(x) \leq' f(y),$$

then f is called a similarity between X and X'.

Theorem 4.2. Let $(X, \leq_\Lambda)$ be a preordered set and let Λ^* be the subfamily of the family of all subsets of X defined by

$$\Lambda^* = \{\cap \Lambda_x \mid x \in X\}.$$

The ordered pair $(\Lambda^*, \subseteq)$ is a partially ordered set. If φ is the mapping from X onto Λ^* defined by

$$\varphi(x) = \cap \Lambda_x \quad \text{for all} \quad x \in X,$$

then φ is an order preserving mapping; that is, for all elements

x and y of X,

$$x \leq_{\Lambda} y \quad \text{iff} \quad \varphi(x) \subseteq \varphi(y).$$

Proof. From Theorem 2.3,

$$x \leq_{\Lambda} y \quad \text{iff} \quad \cap \Lambda_x \subseteq \cap \Lambda_y,$$

it is clear that φ is an order preserving mapping from X onto Λ^*.

Q.E.D.

Remark. The mapping φ is not necessarily a similarity between X and Λ^*. For example, if $X = \{a, b, c\}$ and $\Lambda = \{\{a, b\}, \{c\}\}$, then $\Lambda^* = \Lambda$. Since $\varphi(a) = \varphi(b) = \{a, b\}$ and $\varphi(c) = \{c\}$, φ is not a one-to-one mapping from X onto Λ^*, so it is not a similarity between X and Λ^*. However, if Λ distinguishes elements of X, then φ is necessarily a one-to-one mapping from X onto Λ^* and hence, a similarity between X and Λ^*. In fact, Λ distinguishes elements of X iff φ is a similarity between X and Λ^*.

From the above theorem, clearly, we have the following corollary.

Corollary. Consider the preordered set $(X, \leq_{\Lambda})$ and the partially ordered set $(\Lambda^*, \subseteq)$ where $\Lambda^* = \{\cap \Lambda_x \mid x \in X\}$.

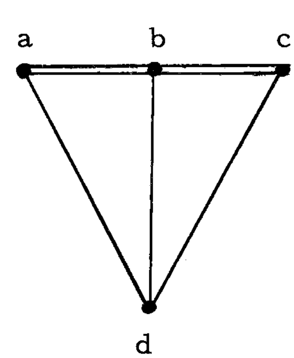
(1) The preordered set $(X, \leq_{\Lambda})$ is a prechain, an order-

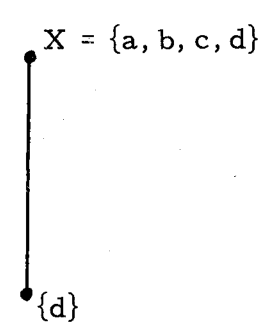
complete set, a lattice or a directed set iff the partially ordered set $(\Lambda^*, \subseteq)$ is a linearly ordered set, an order-complete set, a lattice or a directed set respectively.

(2) An element a of X is a maximal element, a minimal element, a least element or a greatest element of X iff the set $\cap \Lambda_a$ is a maximal element, a minimal element, a least element or a greatest element of Λ^* respectively.

(3) An element a of X is an upper bound, a lower bound, a least upper bound or a greatest lower bound for a subset A of X iff the set $\cap \Lambda_a$ is an upper bound, a lower bound, a least upper bound or a greatest lower bound for the subset A^* of Λ^* respectively where $A^* = \{\cap \Lambda_a | a \in A\}$.

<u>Remark.</u> If $(\Lambda^*, \subseteq)$ is dense then, obviously, $(X, \leq_\Lambda)$ is dense. But the converse is not true. For example, if $X = \{a, b, c, d\}$ and $\Lambda = \{\{d\}\}$, then $\Lambda^* = \{\{d\}, X\}$. The graphical representations of $(X, \leq_\Lambda)$ and $\Lambda^*, \subseteq)$ are in the following:





Notice that $(X, \leq_\Lambda)$ is dense, while $(\Lambda^*, \subseteq)$ is not.

Recall that, for the preordered set $(X, \leq_\Lambda)$, the relation $\sim$ on X defined by

$$x \sim y \quad \text{iff} \quad x \,\underline{\Delta}_\Lambda\, y$$

is an equivalence relation on X and the set $X/\sim$ of all equivalence classes is partially ordered by the order relation $\lesssim_\Lambda$ defined by

$$[x] \lesssim_\Lambda [y] \quad \text{iff} \quad x \leq_\Lambda y \quad \text{for all} \quad [x], [y] \in X/\sim .$$

Now let ψ be the mapping from $X/\sim$ onto Λ^* defined by

$$\psi([x]) = \cap \Lambda_x \quad \text{for all} \quad [x] \in X/\sim .$$

Then we have the following theorem.

<u>Theorem 4.3.</u> For the two partially ordered sets $(X/\sim, \lesssim_\Lambda)$ and $(\Lambda^*, \subseteq)$, the mapping ψ is a similarity between $X/\sim$ and Λ^* such that

$$\psi([x]) = \cap \Lambda_x = \cup \bar{L}_{[x]}$$

where

$$\bar{L}_{[x]} = \{[y] \mid [y] \in X/\sim, [y] \lesssim_\Lambda [x]\}.$$

Proof. Clearly ψ is onto. For one-to-oneness, suppose $\cap \Lambda_x = \cap \Lambda_y$, then by Theorem 2.3, we have $x \,\underline{\Delta}_\Lambda\, y$. Therefore

$[x] = [y]$ and hence, ψ is one-to-one. Moreover, since $\cap\Lambda_x \subseteq \cap\Lambda_y$ iff $x \leq_\Lambda y$, iff $[x] \precsim_\Lambda [y]$, we see that ψ is a similarity between $X/\sim$ and Λ^*.

Now notice the following equivalences: $y \in \cap\Lambda_x$ iff $y \leq_\Lambda x$, iff $[y] \precsim_\Lambda [x]$, iff $[y] \in \overline{L}_{[x]}$. Since $y \in [y]$, it is clear that, if y is an element of $\cap\Lambda_x$, it is an element of $\cup\overline{L}_{[x]}$. Therefore $\cap\Lambda_x \subseteq \cup\overline{L}_{[x]}$. For the other inclusion, if z is an element of $\cup\overline{L}_{[x]}$, then there exists an element, say, $[y]$ of $\overline{L}_{[x]}$ such that $z \in [y]$. Since $z \leq_\Lambda y$ and $y \leq_\Lambda x$, we have $z \leq_\Lambda x$. Consequently, z is an element of $\cap\Lambda_x$; that is, $\cup\overline{L}_{[x]} \subseteq \cap\Lambda_x$. Thus,

$$\psi([x]) = \cap\Lambda_x = \cup\overline{L}_{[x]}$$

and the theorem is proved. Q.E.D.

2-5. An Equivalence Relation on P(P(X))

In this last section of the second chapter, we define an equivalence relation on the collection $P(P(X))$ of all families of subsets of a set X and discuss properties of the elements of equivalence classes. Moreover, we prove that the cardinal number of the set of all partial orders on any finite set is odd.

Definition 5.1. Let $P(P(X))$ be the collection of all families of subsets of a set X. Define a relation $\sim$ on $P(P(X))$ as follows: for all elements Λ^1 and Λ^2 of $P(P(X))$,

$$\Lambda^1 \sim \Lambda^2 \quad \text{iff} \quad \leq^1_\Lambda = \leq^2_\Lambda .$$

Obviously, we have the following theorem.

Theorem 5.2. The relation $\sim$ defined in Definition 5.1 is an equivalence relation on $P(P(X))$.

From now on, the symbol "$[\Lambda]$" will always denote the equivalence class of Λ with respect to the equivalence relation $\sim$ on $P(P(X))$ defined by Definition 5.1, and whenever we refer to an order on $[\Lambda]$, we always mean the set inclusion $\subseteq$.

From the following theorem, we see that the elements Λ and $\Lambda^* = \{\cap \Lambda_x | x \in X\}$ of $P(P(X))$ are in the same equivalence class.

Theorem 5.3. Let Λ be an element of $P(P(X))$ and let $\Lambda^* = \{\cap \Lambda_x | x \in X\}$, Then

$$\leq_\Lambda = \leq^*_\Lambda .$$

Proof. Notice the following equivalences: For any elements x and y of X, $x \leq^*_\Lambda y$ iff $\Lambda^*_y \subseteq \Lambda^*_x$, iff $y \in \cap \Lambda_z$ implies $x \in \cap \Lambda_z$ for all elements z of X.

Now suppose $x \leq^*_\Lambda y$. Since $y \in \cap \Lambda_y$, by the equivalences above, we have $x \in \cap \Lambda_y$; that is, $x \leq_\Lambda y$. Conversely, suppose $x \leq_\Lambda y$. For all elements z of X, if $y \in \cap \Lambda_z$, then $y \leq_\Lambda z$.

By transitivity, we have $x \leq_{\Lambda} y$ and hence $x \in \cap \Lambda_z$. From the equivalences above, $x \overset{*}{\leq}_{\Lambda} y$. Consequently, for all elements x and y of X, $x \overset{*}{\leq}_{\Lambda} y$ iff $x \leq_{\Lambda} y$; that is, $\leq_{\Lambda} = \overset{*}{\leq}_{\Lambda}$. Q.E.D.

Corollary 1. $\Lambda \sim \Lambda^*$; that is, Λ and Λ^* are in the same equivalence class.

Corollary 2. $(\Lambda^*)^* = \Lambda^*$.

Proof. Notice that, for all elements x of X,

$$\cap \Lambda^*_x = \cap \{\cap \Lambda_z \mid z \in X, x \in \cap \Lambda_z\}.$$

Since $x \in \cap \Lambda_x$ and, $x \in \cap \Lambda_z$ iff $\cap \Lambda_x \subseteq \cap \Lambda_z$, we see that $\cap \Lambda^*_x = \cap \Lambda_x$ for all elements x of X. Hence

$$(\Lambda^*)^* = \{\cap \Lambda^*_x \mid x \in X\} = \{\cap \Lambda_x \mid x \in X\} = \Lambda^*. \qquad \text{Q.E.D.}$$

Corollary 3. If $\Lambda^* \subseteq \Lambda$, then, for all elements x and y of X, $x \leq_{\Lambda} y$ iff there exists an element λ_x of Λ_x such that $\lambda_x \subseteq \cap \Lambda_y$.

Proof. If $\Lambda^* \subseteq \Lambda$, then $\cap \Lambda_x \in \Lambda$ for all elements x of X. By the corollary to Theorem 2.7 we have the result. Q.E.D.

In the following, we shall find another way of describing the equivalence relation $\sim$ on $P(P(X))$.

<u>Theorem 5.4.</u> Let Λ^1 and Λ^2 be elements of $P(P(X))$. Then $\cap \Lambda^1_x \subseteq \cap \Lambda^2_x$ for all elements x of X iff $y \leq^1_\Lambda x$ implies $y \leq^2_\Lambda x$ for all elements x and y of X.

Proof. Notice that $\cap \Lambda^1_x \subseteq \cap \Lambda^2_x$ for all elements x of X iff $y \in \cap \Lambda^1_x$ implies $y \in \cap \Lambda^2_x$ for all elements x and y of X which is equivalent to $y \leq^1_\Lambda x$ implies $y \leq^2_\Lambda x$ for all elements x and y of X. Hence we have the result. Q.E.D.

<u>Corollary.</u> Let Λ^1 and Λ^2 be elements of $P(P(X))$ such that $\phi \notin \Lambda^2 - \Lambda^1$. Then $\Lambda^2 \subseteq \Lambda^1$ iff $y \leq^1_\Lambda x$ implies $y \leq^2_\Lambda x$ for all elements x and y of X.

Proof. Since $\phi \notin \Lambda^2 - \Lambda^1$, it follows that $\Lambda^2 \subseteq \Lambda^1$ iff $\Lambda^2_x \subseteq \Lambda^1_x$ for all elements x of X, iff $\cap \Lambda^1_x \subseteq \cap \Lambda^2_x$ for all elements x of X. Hence, by the theorem, $\Lambda^2 \subseteq \Lambda^1$ iff $y \leq^1_\Lambda x$ implies $y \leq^2_\Lambda x$ for all elements x and y of X.

Q.E.D.

<u>Theorem 5.5.</u> Let Λ^1 and Λ^2 be elements of $P(P(X))$. Then $\Lambda^1 \sim \Lambda^2$ iff $\cap \Lambda^1_x = \cap \Lambda^2_x$ for all elements x of X.

Proof. $\Lambda^1 \sim \Lambda^2$ iff $\leq^1_\Lambda = \leq^2_\Lambda$, iff, by Theorem 5.4, $\cap \Lambda^1_x = \cap \Lambda^2_x$. Q.E.D.

<u>Corollary.</u> For all elements Λ of $P(P(X))$,

$$\Lambda \sim \Lambda \cup \{\phi\} \sim \Lambda \cup \{X\} \sim \Lambda \cup \{\phi, X\}.$$

Proof. Notice that $\phi \notin \Lambda_x$ and $\cap \Lambda_x \subseteq X$ for all elements x of X. By the theorem, clearly, we have the result. Q.E.D.

<u>Example 5.6.</u> (1) From Example 2.5(1), we see that, for any set X,

$$\{X, \phi\} \sim \{X\} \sim \{\phi\} \sim \phi.$$

They all define the indiscrete order on X.

(2) Let $X = \{a, b, c\}$ and let

$$\Lambda^1 = \{\{a, b\}, \{a, c\}\},$$
$$\Lambda^2 = \{\{a\}, \{a, b\}, \{a, c\}\},$$
$$\Lambda^3 = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}.$$

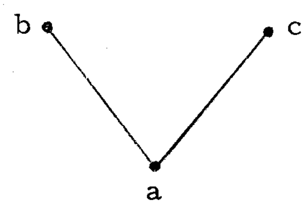
Since

$$\cap \Lambda^1_a = \cap \Lambda^2_a = \cap \Lambda^3_a = \{a\},$$
$$\cap \Lambda^1_b = \cap \Lambda^2_b = \cap \Lambda^3_b = \{a, b\},$$
$$\cap \Lambda^1_c = \cap \Lambda^2_c = \cap \Lambda^3_c = \{a, c\},$$

by virtue of Theorem 5.5, we see that

$$\Lambda^1 \sim \Lambda^2 \sim \Lambda^3.$$

The graphical representation of $(X, \leq^1_\Lambda)$ is as follows:



(3) Let R be the set of all real numbers. Define

$$\Lambda^{\bar{r}} = \{\overline{L}_x \mid x \text{ is a real number}\},$$

$$\Lambda^{\bar{q}} = \{\overline{L}_x \mid x \text{ is a rational number}\},$$

$$\Lambda^{\bar{e}} = \{\overline{L}_x \mid x \text{ is an irrational number}\},$$

$$\Lambda^{r} = \{L_x \mid x \text{ is a real number}\},$$

$$\Lambda^{q} = \{L_x \mid x \text{ is a rational number}\},$$

$$\Lambda^{e} = \{L_x \mid x \text{ is an irrational number}\}$$

where

$$\overline{L}_x = \{y \mid y \in R, y \leq_{(R)} x\}$$

$$L_x = \{y \mid y \in R, y <_{(R)} x\}.$$

It is easy to see that, for all elements x of X,

$$\cap \Lambda^i_x = \overline{L}_x \qquad i = \bar{r}, \bar{q}, \bar{e}, r, q, e.$$

Hence we have

$$\Lambda^{\bar{r}} \sim \Lambda^{\bar{q}} \sim \Lambda^{\bar{e}} \sim \Lambda^{r} \sim \Lambda^{q} \sim \Lambda^{e}.$$

From Example 2.6(1), we know that they all define the usual order on R.

Now suppose Λ is an element of $P(P(X))$, then $C\Lambda$ is also an element of $P(PX))$. We would like to ask: Is there any connection between the two preorders $\leq_{\Lambda}$ and $\leq_{C\Lambda}$ on X? Moreover, if for every element α of an index set A, Λ^{α} is an element of $P(P(X))$, then $\bigcup_{\alpha \in A} \Lambda^{\alpha}$ and $\bigcap_{\alpha \in A} \Lambda^{\alpha}$ are also elements of $P(P(X))$. Another question arises: What are the properties of the preorders derived from $\bigcup_{\alpha \in A} \Lambda^{\alpha}$ and $\bigcap_{\alpha \in A} \Lambda^{\alpha}$? How are they related to the preorders derived from the various Λ^{α}? The rest of this section is devoted to answer these questions.

<u>Theorem 5.7</u>. For all elements x and y of X,

$$x \leq_{\Lambda} y \quad \text{iff} \quad y \leq_{C\Lambda} x.$$

That is, the preorder $\leq_{C\Lambda}$ is the inverse order of $\leq_{\Lambda}$.

Proof. Let x and y be arbitrary elements of X. From Theorem 2.3, we have

$$x \leq_{\Lambda} y \quad \text{iff} \quad C\Lambda_y \subseteq C\Lambda_x$$

and

$$y \leq_{C\Lambda} x \quad \text{iff} \quad (C\Lambda)_x \subsetneq (C\Lambda)_y .$$

Moreover, Theorem 1.1(5) tells us that

$$(C\Lambda)_x = C\Lambda - C\Lambda_x .$$

Therefore

$$y \leq_{C\Lambda} x \quad \text{iff} \quad C\Lambda - C\Lambda_x \subsetneq C\Lambda - C\Lambda_y .$$

Since $C\Lambda - C\Lambda_x \subsetneq C\Lambda - C\Lambda_y$ iff $C\Lambda_y \subsetneq C\Lambda_x$, we see that

$$x \leq_\Lambda y \quad \text{iff} \quad y \leq_{C\Lambda} x .$$

Q.E.D.

<u>Corollary</u>. $\leq_\Lambda = \leq_{C\Lambda}$ iff each element of X is a minimax element of X (with respect to either $\leq_\Lambda$ or $\leq_{C\Lambda}$).

Proof. Let a be an arbitrary but fixed element of X. If $\leq_\Lambda = \leq_{C\Lambda}$, then, for all elements x of X, $a \leq_\Lambda x$ iff $a \leq_{C\Lambda} x$. Since $a \leq_{C\Lambda} x$ is equivalent to $x \leq_\Lambda a$, we see that if $\leq_\Lambda = \leq_{C\Lambda}$, then, for all elements x of X, $a \leq_\Lambda x$ iff $x \leq_\Lambda a$. This means a is a minimax element of X.

Conversely, suppose each element of X is a minimax element of X with respect to $\leq_\Lambda$; that is, for all elements x and y of X, $x \leq_\Lambda y$ iff $y \leq_\Lambda x$. Since $y \leq_\Lambda x$ is equivalent to $x \leq_{C\Lambda} y$, it is clear that if each element of X is a minimax element of X, then, for all elements x and y of X, $x \leq_\Lambda y$ iff $x \leq_{C\Lambda} y$; that is $\leq_\Lambda = \leq_{C\Lambda}$.

Q.E.D.

Theorem 5.8. The cardinal number of the set of all partial orders on any finite set X is odd.

Proof. Notice that, in a partially ordered set, an element is a minimax element iff it is not related to any other elements. From the corollary to Theorem 5.7, it is clear that the elements Λ and $C\Lambda$ of $P(P(X))$ define the same partial order on X iff they both define the discrete order on X. Therefore, if the element Λ of $P(P(X))$ defines a partial order $\leq_\Lambda$ on X which is not the discrete order, then the element $C\Lambda$ of $P(P(X))$ will define the partial order $\leq_{C\Lambda}$ on X and $\leq_\Lambda \neq \leq_{C\Lambda}$. Since each partial order on X can be derived from an element of $P(P(X))$, it follows that the nondiscrete partial orders occur in pairs: any partial order and its inverse partial order. Hence the cardinal number of the set of all partial orders on a finite set is odd. Q.E.D.

Theorem 5.9. Let A be any index set and let $\leq_{\bigcup_{\alpha \in A} \Lambda^\alpha}$ be the preorder derived from $\bigcup_{\alpha \in A} \Lambda^\alpha$. Then, for all elements x and y of X,

$$y \leq_{\bigcup_{\alpha \in A} \Lambda^\alpha} x \quad \text{iff} \quad y \leq^\alpha_\Lambda x \quad \text{for all} \quad \alpha \in A.$$

Proof. From Theorem 1.3(1) and (4), we have

$$\cap\left(\bigcup_{\alpha\in A}\Lambda^{\alpha}\right)_x = \cap\left(\bigcup_{\alpha\in A}\Lambda^{\alpha}_x\right) = \bigcap_{\alpha\in A}(\cap\Lambda^{\alpha}_x).$$

Since, for all elements x and y of X, $y \leq^{\alpha}_{\Lambda} x$ for all $\alpha \in A$ iff $y \in \cap\Lambda^{\alpha}_x$ for all $\alpha \in A$, iff $y \in \bigcap_{\alpha\in A}(\cap\Lambda^{\alpha}_x)$, and $y \leq_{\bigcup_{\alpha\in A}\Lambda^{\alpha}} x$ iff $y \in \cap\left(\bigcup_{\alpha\in A}\Lambda^{\alpha}\right)_x$; we see that, for all elements x and y of X,

$$y \leq_{\bigcup_{\alpha\in A}\Lambda^{\alpha}} x \text{ iff } y \leq^{\alpha}_{\Lambda} x \text{ for all } \alpha \in A. \qquad \text{Q.E.D.}$$

<u>Corollary.</u> All equivalence classes $[\Lambda]$ are closed under arbitrary nonempty unions.

Proof. Let A be an arbitrary nonempty index set and let $\Lambda^{\alpha} \in [\Lambda]$ for all $\alpha \in A$. It is clear that, for all elements x and y of X, $y \leq_{\Lambda} x$ iff $y \leq^{\alpha}_{\Lambda} x$ for all $\alpha \in A$. Therefore, by the theorem, for all elements x and y of X, $y \leq_{\Lambda} x$ iff $y \leq_{\bigcup_{\alpha\in A}\Lambda^{\alpha}} x$. Hence $\bigcup_{\alpha\in A}\Lambda^{\alpha} \in [\Lambda]$. Q.E.D.

<u>Theorem 5.10.</u> Each equivalence class $[\Lambda]$ has a greatest element (with respect to set inclusion).

Proof. From the above corollary, it is clear that the element $\bigcup_{\Lambda^{\alpha}\in[\Lambda]}\Lambda^{\alpha}$ of $P(P(X))$ is the greatest element of the equivalence class $[\Lambda]$. Q.E.D.

Remark. In general, the equivalence class $[\Lambda]$ has no minimal element and hence no least element. In Example 5.6(3), we have $\Lambda^i \in [\Lambda^r]$, $i = \bar{r}, \bar{q}, \bar{e}, r, q, e$. Since $\Lambda^q \cap \Lambda^e = \emptyset$, it follows that if $[\Lambda^r]$ has a minimal element, it must be $\emptyset$, but $\emptyset$ is clearly not an element of $[\Lambda^r]$.

Theorem 5.11. Let A be any index set and let $\leq_{\bigcap_{\alpha \in A} \Lambda^\alpha}$ be the preorder derived from $\bigcap_{\alpha \in A} \Lambda^\alpha$. For any elements x and y of X, if there exists an element α^0 of A such that $y \leq_\Lambda^{\alpha^0} x$, then $y \leq_{\bigcap_{\alpha \in A} \Lambda^\alpha} x$.

Proof. If there exists an element α^0 of A such that $y \leq_\Lambda^{\alpha^0} x$, then $y \in \cap \Lambda_x^{\alpha^0}$. Since, by Theorem 1.3(2),

$$\left(\bigcap_{\alpha \in A} \Lambda^\alpha\right)_x = \bigcap_{\alpha \in A} \Lambda_x^\alpha \subseteq \Lambda_x^{\alpha^0},$$

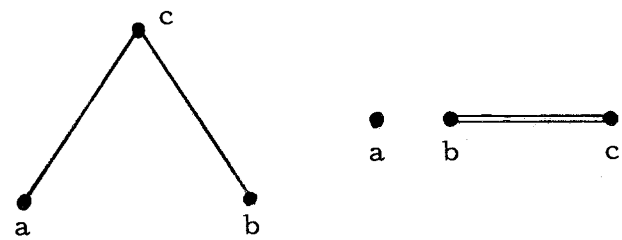
we have

$$\cap \Lambda_x^{\alpha^0} \subseteq \cap \left(\bigcap_{\alpha \in A} \Lambda^\alpha\right)_x.$$

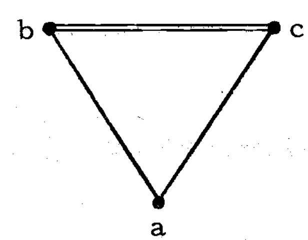
Therefore, if there exists an element α^0 of A such that $y \leq_\Lambda^{\alpha^0} x$, then $y \in \cap\left(\bigcap_{\alpha \in A} \Lambda^\alpha\right)_x$ which is equivalent to $y \leq_{\bigcap_{\alpha \in A} \Lambda^\alpha} x$.

Q.E.D.

Remark. The converse of the theorem is not true. For example, let $X = \{a, b, c\}$, $\Lambda^1 = \{\{a\}, \{b\}, \{a, b\}\}$ and $\Lambda^2 = \{\{a\}, \{b, c\}\}$. The graphical representations of $(X, \leq_\Lambda^1)$ and $(X, \leq_\Lambda^2)$ are in the following:



Notice that $a \not\leq_\Lambda^1 b$ and $a \not\leq_\Lambda^2 b$. However, since $\Lambda^1 \cap \Lambda^2 = \{\{a\}\}$, $a \leq_{\Lambda^1 \cap \Lambda^2} b$. The graphical representation of $(X, \leq_{\Lambda^1 \cap \Lambda^2})$ is



3. TOPOLOGIES AND PREORDERS

From Chapter 2, we know that any family Λ of subsets of a set X defines a preorder $\leq_\Lambda$ on X and the relation $\sim$ defined by

$$\Lambda^1 \sim \Lambda^2 \quad \text{iff} \quad \leq_\Lambda^1 = \leq_\Lambda^2$$

is an equivalence relation on the collection $P(P(X))$ of all families of subsets of X. In this chapter, we discuss the case in which the family Λ of subsets of X is a topology for X.

3-1. The Greatest Element of $[\Lambda]$ and the Cardinality of Preorders

From the proof of Theorem 5.10 of Chapter 2, we see that the element $\bigcup_{\Lambda^a \in [\Lambda]} \Lambda^a$ of $P(P(X))$ is the greatest element of the equivalence class $[\Lambda]$. We shall prove that an element of $[\Lambda]$ is the greatest element of $[\Lambda]$ iff it is a topology for X which is also closed under arbitrary intersections. From this fact, we can find the greatest family of subsets of the set R of all real numbers which defines the usual order on R.

First, we prove that any topology and its bases all define the same preorder.

Theorem 1.1. If Λ^b is a base for the topology Λ^t for a set X, then $\Lambda^b \sim \Lambda^t$.

Proof. Since $\Lambda^b \subseteq \Lambda^t$, it is clear that, for all elements x of X,

$$\cap \Lambda_x^t \subseteq \cap \Lambda_x^b.$$

For the other inclusion, since Λ^b is a base for the topology Λ^b, it follows that, for every element λ_x^t of Λ_x^t, there exists an element λ_x^b of Λ_x^b such that $\lambda_x^b \subseteq \lambda_x^t$. Now it is clear that if y is an element of $\cap \Lambda_x^b$, then

$$y \in \lambda_x^t \text{ for all elements } \lambda_x^t \text{ of } \Lambda_x^t;$$

that is,

$$y \in \cap \Lambda_x^t.$$

Consequently, for all elements x of X,

$$\cap \Lambda_x^b \subseteq \cap \Lambda_x^t.$$

By Theorem 5.5 of Chapter 2, we have $\Lambda^b \sim \Lambda^t$. Q.E.D.

Now we start proving our main theorem of this section.

Lemma 1.2. If a family of subsets of X is the greatest element of an equivalence class, then it is a topology for X which is also closed under arbitrary intersections.

Proof. Let Λ be the greatest element of the equivalence class $[\Lambda]$.

(1) $\emptyset, X \in \Lambda$: By the corollary to Theorem 5.5 of Chapter 2, we have

$$\Lambda \sim \Lambda \cup \{\emptyset, X\}.$$

Since Λ is the greatest element of $[\Lambda]$, it is clear that

$$\emptyset \in \Lambda \quad \text{and} \quad X \in \Lambda.$$

(2) Λ is closed under arbitrary intersections: Let $\{\lambda^a | a \in A\}$ be an arbitrary subfamily of Λ.

Case 1. $\bigcap_{a \in A} \lambda^a = \emptyset$. By (1), $\bigcap_{a \in A} \lambda^a \in \Lambda$.

Case 2. $\bigcap_{a \in A} \lambda^a \neq \emptyset$. Notice that if $x \in \bigcap_{a \in A} \lambda^a$, then $\lambda^a \in \Lambda_x$ for all $a \in A$. Consider the family $\Lambda' = \Lambda \cup \{\bigcap_{a \in A} \lambda^a\}$. Since $\{\lambda^a | a \in A\} \subseteq \Lambda$, it is clear that if $x \in \bigcap_{a \in A} \lambda^a$, then

$$\{\lambda^a | a \in A\} \subseteq \Lambda_x.$$

Consequently, for all elements x of $\bigcap_{a \in A} \lambda^a$,

$$\bigcap \Lambda_x \subseteq \bigcap_{a \in A} \lambda^a.$$

Hence, for all elements x of $\bigcap_{a \in A} \lambda^a$,

$$\cap \Lambda'_x = (\cap \Lambda_x) \cap (\bigcap_{a \in A} \lambda^a) = \cap \Lambda_x .$$

As for elements $x \in X - \bigcap_{a \in A} \lambda^a$, since $\bigcap_{a \in A} \lambda^a \notin \Lambda'_x$, clearly,

$$\cap \Lambda'_x = \cap \Lambda_x .$$

Therefore, for all elements x of X, we have

$$\cap \Lambda'_x = \cap \Lambda_x .$$

By Theorem 5.5 of Chapter 2,

$$\Lambda \sim \Lambda' = \Lambda \cup \{ \bigcap_{a \in A} \lambda^a \}.$$

But, Λ is the greatest element of $[\Lambda]$, accordingly,

$$\bigcap_{a \in A} \lambda^a \in \Lambda .$$

(3) Λ is closed under arbitrary unions: Let $\{\lambda^a | a \in A\}$ be an arbitrary subfamily of Λ. Consider the familv $\Lambda' = \Lambda \cup \{ \bigcup_{a \in A} \lambda^a\}$. If $x \in \bigcup_{a \in A} \lambda^a$, then there exists an element a^0 of A such that

$$x \in \lambda^{a^0} \subseteq \bigcup_{a \in A} \lambda^a$$

and hence, $\lambda^{a_0} \in \Lambda_x$. Consequently, for each element x of $\bigcup_{a \in A} \lambda^a$,

$$\cap \Lambda_x \subseteq \lambda^{a_0} \subseteq \bigcup_{a \in A} \lambda^a .$$

Hence, for all $x \in \bigcup_{a \in A} \lambda^a$,

$$\cap \Lambda'_x = (\cap \Lambda_x) \cap (\bigcup_{a \in A} \lambda^a) = \cap \Lambda_x .$$

As for elements $x \in X - \bigcup_{a \in A} \lambda^a$, since $\bigcup_{a \in A} \lambda^a \notin \Lambda_x$, clearly,

$$\cap \Lambda'_x = \cap \Lambda_x .$$

Therefore, for all elements x of X, we have

$$\cap \Lambda'_x = \cap \Lambda_x .$$

By Theorem 5.5 of Chapter 2,

$$\Lambda \sim \Lambda' = \Lambda \cup \{ \bigcup_{a \in A} \lambda^a \}.$$

But, Λ is the greatest element of $[\Lambda]$, accordingly,

$$\bigcup_{a \in A} \lambda^a \in \Lambda .$$

From (1), (2) and (3), we conclude that the greatest element of $[\Lambda]$ is a topology for X which is also closed under arbitrary intersections. Q.E.D.

<u>Lemma 1.3.</u> Let Λ^1 and Λ^2 be two topologies for X which are also closed under arbitrary intersections. If $\Lambda^1 \sim \Lambda^2$, then $\Lambda^1 = \Lambda^2$; that is, there is only one element in $[\Lambda]$ which is a topology for X and is also closed under arbitrary intersections.

Proof. Suppose $\Lambda^1 - \Lambda^2 \neq \emptyset$. Let $\lambda^1 \in \Lambda^1 - \Lambda^2$ and set

$$\Lambda^{2'} = \{\lambda^2 | \lambda^2 \in \Lambda^2, \lambda^2 \subseteq \lambda^1\}.$$

Notice that $\cup \Lambda^{2'} \subseteq \lambda^1$ and, since Λ^2 is a topology for X, $\cup \Lambda^{2'} \in \Lambda^2$. Because $\lambda^1 \notin \Lambda^2$, $\cup \Lambda^{2'}$ is a proper subset of λ^1, so we can choose an element, say x_0, of $\lambda^1 - \cup \Lambda^{2'}$. It is clear that $\cap \Lambda^1_{x_0} \subseteq \lambda^1$ and every element of $\Lambda^2_{x_0}$ is not contained in λ^1. Since Λ^2 is closed under arbitrary intersections, we have $\cap \Lambda^2_{x_0} \in \Lambda^2_{x_0}$. Hence $\cap \Lambda^2_{x_0} \not\subseteq \lambda^1$. From $\cap \Lambda^1_{x_0} \subseteq \lambda^1$ and $\cap \Lambda^2_{x_0} \not\subseteq \lambda^1$, we see that $\cap \Lambda^2_{x_0} - \cap \Lambda^1_{x_0} \neq \emptyset$.

Similarly, if $\Lambda^2 - \Lambda^1 \neq \emptyset$, then there exists an element x'_0 of X such that $\cap \Lambda^1_{x'_0} - \cap \Lambda^2_{x'_0} \neq \emptyset$.

Consequently, if $\cap \Lambda^1_x = \cap \Lambda^2_x$ for all elements x of X, then $\Lambda^1 = \Lambda^2$. By Theorem 5.5 of Chapter 2, it follows that if

$\Lambda^1 \sim \Lambda^2$, then $\Lambda^1 = \Lambda^2$. Q.E.D.

From Lemmas 1.2 and 1.3, clearly, we have the following theorem and corollaries.

Theorem 1.4. A family of subsets of X is the greatest element of an equivalence class $[\Lambda]$ iff it is a topology for X which is also closed under arbitrary intersections.

Corollary 1. If X is a finite set, then a family of subsets of X is the greatest element of an equivalence class $[\Lambda]$ iff it is a topology for X.

Proof. Notice that a topology for X is a subfamily of the family P(X) of all subsets of X. If X is finite, P(X) is finite and hence, any topology for X is finite. Therefore, if X is a finite set, then any topology for X is automatically closed under arbitrary intersections. From the theorem, clearly, we have the result. Q.E.D.

Corollary 2. The family $\{\emptyset, X\}$ is the greatest family of subsets of a set X which defines the indiscrete order on X.

Proof. From Example 2.5(1) of Chapter 2, we know that $\{\emptyset, X\}$ defines the indiscrete order on X. Since the family $\{\emptyset, X\}$ is a topology for X which is also closed under arbitrary intersections, from the theorem, we have the result. Q.E.D.

Corollary 3. The family P(X), the set of all subsets of a set X, is the greatest family of subsets of X which defines the discrete order on X.

Proof. The family P(X) is a topology for X which is also closed under arbitrary intersections. From Example 2.5(2) of Chapter 2 and the theorem, we have the result. Q.E.D.

Suppose A is an index set such that for every element α of A, Λ^{α} is a topology for a set X which is also closed under arbitrary intersections. It is clear that $\bigcap_{\alpha \in A} \Lambda^{\alpha}$ is also a topology for X and is also closed under arbitrary intersections. Hence we have the following corollary.

Corollary 4. The intersection of an arbitrary number of greatest elements of equivalence classes is again the greatest element of some equivalence class; that is, the set of all greatest elements of equivalence classes is closed under arbitrary intersections.

In Example 5.6(3) of Chapter 2, we let R be the set of all real numbers and defined

$$\Lambda^{\bar{r}} = \{\bar{L}_x \mid x \text{ is a real number}\},$$

$$\Lambda^{\bar{q}} = \{\bar{L}_x \mid x \text{ is a rational number}\},$$

$$\Lambda^{\bar{e}} = \{\bar{L}_x \mid x \text{ is an irrational number}\},$$

$$\Lambda^{r} = \{L_x \mid x \text{ is a real number}\},$$

$$\Lambda^{q} = \{L_x \mid x \text{ is a rational number}\},$$

$$\Lambda^{e} = \{L_x \mid x \text{ is an irrational number}\}$$

where

$$\bar{L}_x = \{y \mid y \in R, y \leq_{(R)} x\}$$

$$L_x = \{y \mid y \in R, y <_{(R)} x\}.$$

We know that these families Λ^i $(i = \bar{r}, \bar{q}, \bar{e}, r, q, e)$ are all equivalent and define the usual order on R. Notice that

$$\bigcup_{n=1}^{\infty} \bar{L}_{1-\frac{1}{n}} = \bigcup_{n=1}^{\infty} \bar{L}_{1-\frac{\sqrt{2}}{n}} = \bigcup_{n=1}^{\infty} L_{1-\frac{\sqrt{2}}{n}} = L_1$$

and

$$\bigcap_{n=1}^{\infty} L_{\frac{1}{n}} = \bar{L}_0.$$

Hence $\Lambda^{\bar{r}}$, $\Lambda^{\bar{q}}$, $\Lambda^{\bar{e}}$ and Λ^{e} are not closed under arbitrary unions and Λ^{r} is not closed under arbitrary intersections. For Λ^{q}, let $\{x_n\}$ be an increasing sequence of rational numbers such that $\lim_{n\to\infty} x_n = \sqrt{2}$; e.g., $\{x_n\} = \{1, 1.4, 1.41, 1.414, \ldots\}$, then $\bigcup_{n=1}^{\infty} L_{x_n} = L_{\sqrt{2}}$ which tells us that Λ^{q} is not closed under arbitrary unions. Consequently, by Theorem 1.4, none of these families is the greatest family defining the usual order on R even if we add the two elements $\emptyset$ and R to each family. The greatest family defining the usual order on R is given by the following theorem.

<u>Theorem 1.5.</u> Let R be the set of all real numbers and define

$$\bar{L}_x = \{y \mid y \in R, y \leq_{(R)} x\}$$

$$L_x = \{y \mid y \in R, y <_{(R)} x\}.$$

Let

$$\Lambda = \Lambda^{\bar{r}} \cup \Lambda^{r} \cup \{\emptyset, R\}$$

where

$$\Lambda^{\bar{r}} = \{\bar{L}_x \mid x \text{ is a real number}\}$$

$$\Lambda^{r} = \{L_x \mid x \text{ is a real number}\}.$$

Then Λ is the greatest family of subsets of R which defines the usual order on R.

Proof. First of all, from Chapter 2, Example 5.6(3) and corollaries to Theorems 5.5 and 5.9, we know that Λ defines the usual order on R. Moreover, it is clear that Λ is a topology for R which is also closed under arbitrary intersections. Hence, by Theorem 1.4, Λ is the greatest family of subsets of R which defines the usual order on R. Q.E.D.

From Theorem 1.4, it is clear that if we choose the greatest elements as the representatives of equivalence classes, then the number of different preorders on a set is exactly the number of different topologies for X which are also closed under arbitrary intersections. We state this fact as a theorem.

Theorem 1.6. The cardinal number of the set of all preorders on a set X is equal to the cardinal number of the set of all topologies for X which are also closed under arbitrary intersections.

Corollary 1. If X is a finite set, then the cardinal number of the set of all preorders on X is equal to the cardinal number of the set of all topologies for X.

Corollary 1 says different topologies for a finite set X define different preorders on X. Hence we have the following corollary.

Corollary 2. If X is a finite set, then each equivalence class $[\Lambda]$ contains a unique topology for X.

Notice that if X is a finite set and Λ is a topology for X, then $C\Lambda$ is also a topology for X. Thus, from Theorem 5.7 of Chapter 2, we have the following corollary.

Corollary 3. If X is a finite set and Λ is the greatest element of some equivalence class, then $C\Lambda$ is the greatest element of the equivalence class in which each element defines the inverse order of $\leq_\Lambda$.

3-2. The Preorder Derived from a Topology

Let the family Λ of subsets of a set X be a topology for X. In this section, we discuss the relationship between the topological space (X, Λ) and the preordered set $(X, \leq_\Lambda)$.

First of all, from Example 2.5(1) and (2) of Chapter 2, we know that the indiscrete topology defines the indiscrete order and the discrete topology defines the discrete order. Next, we have the following theorems.

Theorem 2.1. Let Λ be a topology for a set X. The topological space (X, Λ) is a T_0-space iff the preordered set $(X, \leq_\Lambda)$ is a partially ordered set.

Proof. Notice that the family Λ distinguishes elements of X iff, for every pair of distinct elements of X, there exists an element of Λ which contains one of them but not the other. By Theorem 3.4 of Chapter 2 and the definition of T_0-space, we have the result. Q.E.D.

From the definitions of T_i-spaces, $i = 1, 2, 3, 4, 5$, clearly, we have the following theorem.

Theorem 2.2. Let Λ be a topology for a set X. If the topological space (X, Λ) is a T_i-space, $i = 1, 2, 3, 4, 5$, then Λ defines the discrete order on X.

Corollary 1 to Theorem 3.10 of Chapter 2 tells us that, in any preordered set $(X, \leq_\Lambda)$, an element a of X is a greatest element of X iff either $\Lambda_a = \emptyset$ or $\Lambda_a = \{X\}$. Therefore, if a preordered set $(X, \leq_\Lambda)$ has a greatest element a and $\Lambda_a \neq \emptyset$, then $\Lambda_a = \{X\}$; that is, X is the only set in Λ which contains the greatest element a. Accordingly, any subfamily Λ' of Λ which covers X must include the element X of Λ. From this fact, we have the following theorem.

Theorem 2.3. If Λ is a topology for a set X such that the preordered set $(X, \leq_\Lambda)$ has a greatest element, then the topological space (X, Λ) is compact.

For connectedness, we have the following theorem.

Theorem 2.4. Let Λ be a topology for a set X which is also closed under arbitrary intersections. The topological space (X, Λ) is disconnected iff there exists a nonempty proper subset S of X such that no element of S is related to any element of $X - S$

with respect to the preorder derived from Λ.

Proof. For the sufficiency, let x be an arbitrary element of S. For any element y of $X - S$, since x and y are not related, there exists an element $\lambda_{x,y}$ of Λ_x such that $y \notin \lambda_{x,y}$. Let

$$\lambda_x = \cap\{\lambda_{x,y} \mid y \in X - S\}.$$

Since Λ is closed under arbitrary intersections, $\lambda_x \in \Lambda$. Moreover, it is clear that no element of $X - S$ is an element of λ_x. Hence we have $\lambda_x \subseteq S$. Now define

$$S' = \cup\{\lambda_x \mid x \in S\}.$$

It is obvious that $S = S'$. Since Λ is a topology for X and $\lambda_x \in \Lambda$ for all elements x of S, it follows that $S' \in \Lambda$ and, consequently, $S \in \Lambda$. The set $X - S$ is also a nonempty proper subset of X having the property indicated. So we also have $X - S \in \Lambda$. Now the set S is a nonempty proper subset of X which is both open and closed, therefore the topological space (X, Λ) is disconnected.

For the necessity, if the topological space (X, Λ) is disconnected, then there exists a nonempty proper subset S of X which is both open and closed. Thus, $X - S \in \Lambda$. Now, let x and y be arbitrary elements of S and $X - S$ respectively. Since $S \in \Lambda_x$

and $X - S \in \Lambda_y$, we see that x and y are not related. Hence the subset S of X is the required set. Q.E.D.

Remark. In Theorem 2.4, to prove the sufficiency it is essential that Λ is closed under arbitrary intersections. For example, let X be an infinite set and let Λ be the cofinite topology for X; that is, Λ consists of $\emptyset$ and all complements of finite subsets of X. It is clear that the topological space (X, Λ) is a T_1-space. Hence, by Theorem 2.2, Λ defines the discrete order on X. Therefore, no element of any nonempty proper subset of X is related to any element of its complement. But, since X is an infinite set, no nonempty proper subset of X is both open and closed; that is, the topological space (X, Λ) is connected. The point is, the cofinite topology for an infinite set is not closed under arbitrary intersections as seen in the following. Let $\langle x_n \rangle$ be a sequence of distinct elements of X. Since X is an infinite set, there does exist such a sequence. Define $S_n = X - \{x_n\}$ for every positive integer n. Consider the set

$$S = \bigcap_{n=1}^{\infty} S_n = X - \{x_n \mid n \text{ is a positive integer}\}.$$

By the definition of Λ, it is clear that $S_n \in \Lambda$ for all positive integers n and $S \notin \Lambda$. Therefore, Λ is not closed under arbitrary intersections.

finite set and induces the cofinite topology for X if X is an infinite set, as shown in Example 2.2(2) of Chapter 1. Since a topological space (X, T) with the discrete topology or the cofinite topology is a T_1-space, by Theorem 2.2, T defines the discrete order on X. Therefore, if the original preorder $\leq$ is the indiscrete order on X, then the preorder $\leq_T$ is the discrete order on X. In general, we have the following theorems and corollaries.

Theorem 3.1. Let $(X, \leq)$ be a preordered set. For every pair of distinct elements x and y of X, if they are $\leq$-related, then they are not $\leq_T$-related.

Proof. Suppose that the distinct elements x and y of X are $\leq$-related; without losing any generality, we may assume $x < y$. Then we see that $x \in L_y$ and $y \in R_x$. Since $x \notin R_x$, $y \notin L_y$ and L_y, R_x are elements of T, it follows that

$$L_y \in T_x - T_y \quad \text{and} \quad R_x \in T_y - T_x.$$

Consequently, x and y are not $\leq_T$-related. Q.E.D.

Corollary 1. If X is a set containing more than one element, then any preorder $\leq$ on X is not equal to the corresponding preorder $\leq_T$ on X.

Proof. If $\leq$ is the discrete order on X then $\leq_T$ is the indiscrete order on X. If there exist at least two distinct elements of X which are $\leq$-related, then, by the theorem, they are not $\leq_T$-related. Hence, $\leq \neq \leq_T$. Q.E.D.

Corollary 2. Let X be a set containing more than one element. If Λ is a topology for X, then the order topology T for X induced by the preorder $\leq_\Lambda$ on X is never equal to Λ.

Proof. If the order topology T for X induced by the preorder $\leq_\Lambda$ on X were equal to Λ, then we would have $(\leq_\Lambda)_T = \leq_\Lambda$ where $(\leq_\Lambda)_T$ is the preorder on X derived from T.

By Corollary 1, this is impossible. Q.E.D.

From Theorem 3.1, the following corollaries are obvious.

Corollary 3. The preordered set $(X, \leq_T)$ is a prechain iff $\leq$ is the discrete order on X.

Corollary 4. If the preorder set $(X, \leq)$ is a prechain, then the preorder $\leq_T$ is the discrete order on X.

Remark. The converse of Corollary 4 is not true. For example, consider the preordered set $(X, \leq)$ with the following graphical representation:

Since

$$R_a = \{c\},\ R_b = \{d\},\ L_c = \{a\} \text{ and } L_d = \{b\};$$

it follows that the order topology T for X induced by $\leq$ is the discrete topology for X. Consequently, the preorder $\leq_T$ on X is the discrete order on X. However, the elements a and b are not $\leq$-related.

<u>Theorem 3.2.</u> Let $(X, \leq)$ be a preordered set. For every pair of distinct elements x and y of X, $y \leq_T x$ iff

$$L_x \subseteq L_y \text{ and } R_x \subseteq R_y.$$

Proof. Suppose the distinct elements x and y of X satisfy $y \leq_T x$. If $L_x \not\subseteq L_y$, then there exists an element z of X such that $z \in L_x - L_y$. Hence $z < x$ and $z \not< y$. So, $x \in R_z$ and $y \notin R_z$. Since R_z is an element of T it follows that $R_z \in T_x - T_y$. Therefore, $T_x \not\subseteq T_y$ which means $y \not\leq_T x$. Thus, we get a contradiction. Similarly, if $R_x \not\subseteq R_y$ then there exists an element z of X such that $L_z \in T_x - T_y$ which means $y \not\leq_T x$, again, a contradiction.

For the sufficiency, suppose $L_x \subseteq L_y$ and $R_x \subseteq R_y$. Notice that if $x \in L_z$ for some element z of X, then $x < z$ and hence, $z \in R_x \subseteq R_y$. So, $y < z$ which implies $y \in L_z$. Similarly, if $x \in R_z$ for some element z of X, then y is also an element of R_z. Therefore, in the order topology T for X, any open set containing x must contain y; that is $T_x \subseteq T_y$. Hence $y \leq_T x$. Q.E.D.

<u>Corollary.</u> If $R_x \neq R_y$ or $L_x \neq L_y$ for every pair of elements x and y of a set X which are not $\leq$-related, then the preorder $\leq_T$ is a partial order on X.

Proof. Let x and y be two distinct elements of X.

Case 1. x and y are $\leq$-related. By Theorem 3.1, they are not $\leq_T$-related.

Case 2. x and y are not $\leq$-related. Since $R_x \neq R_y$ or $L_x \neq L_y$, it is impossible that $R_x = R_y$ and $L_x = L_y$. Hence, by the theorem, it is impossible that $x \,\underline{\Delta}_T\, y$.

From Case 1 and Case 2, the corollary is proved. Q.E.D.

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