AN ABSTRACT OF THE THESIS OF

Paul Nelson Cowgill for the M.S. in Electrical Engineering
(Name) (Degree) (Major)

Date thesis is presented July 5, 1967

Title DESIGN-APPROACH EVALUATION OF MULTIPLE-INPUT, MODEL-REFERENCE, ADAPTIVE CONTROL SYSTEMS

Abstract approved [redacted]
(Major professor)

The model-reference, adaptive control concept is based on the precept that desired control system performance is a known design requirement and can be obtained from a representative model. Control system parameters are adjusted by the adaptive controller through a minimizing operation on a function of the error (i.e., performance index) between control system and model response characteristics. By minimizing the performance index, the control system response characteristics will track those of the model.

This thesis presents a unified review of model-reference, adaptive control system design and analysis techniques and a method for analytically determining adaptive control loop sensitivity to multiple forcing functions. The design and analysis techniques presented encompass design approach variations (i.e., configuration deviations) and solutions to commonly encountered application problems. The multiple forcing functions considered consist of reference and disturbance inputs.

The forcing function sensitivity of two adaptive controller configurations is analytically derived and compared for single and simultaneous, sinusoidal inputs. Frequency-domain analysis techniques are thereby extended for application to these two particular configurations of non-linear systems.

DESIGN-APPROACH EVALUATION OF MULTIPLE-INPUT, MODEL-REFERENCE, ADAPTIVE CONTROL SYSTEMS

by

PAUL NELSON COWGILL

A THESIS

submitted to

OREGON STATE UNIVERSITY

in partial fulfillment of
the requirements for the
degree of

MASTER OF SCIENCE

June 1968

APPROVED:

[redacted]

Professor of Electrical Engineering
In Charge of Major

[redacted]

Head of Department of Electrical and
Electronics Engineering

[redacted]

Dean of Graduate School

Date thesis is presented July 5, 1967

Typed by Patricia F. Jones

ACKNOWLEDGEMENT

The author wishes to express appreciation to Professor Solon A. Stone for constructive suggestions offered during the preparation of this thesis.

TABLE OF CONTENTS

TABLE OF CONTENTS (continued)

LIST OF FIGURES

LIST OF FIGURES (continued)

DESIGN-APPROACH EVALUATION OF MULTIPLE-INPUT, MODEL-REFERENCE, ADAPTIVE CONTROL SYSTEMS

I. INTRODUCTION

Automatic control of complex physical processes is becoming increasingly more demanding because of greater restrictions on system-performance characteristics and increasing complexity of the processes being controlled. This trend dictates the use of new and improved control techniques to provide greater performance capabilities than are obtainable with conventional feedback controllers. Illustrations of this trend are the increasing use of computers to control industrial processes and providing control systems with self-improvement and/or optimizing capabilities. Adaptive controllers are being used in many system applications to provide self-improvement where physical process characteristics are either insufficiently known or subject to large unpredictable changes as the process traverses the range of its operating environment.

Adaptive-Control Applications

The adaptive approach is appealing to the designer and applicable to the solution of many current control problems. However, because this approach introduces increased complexities of control system analysis, design, and mechanization, adaptive control should be employed only when suitable control performance is unattainable with conventional controllers. A number of specific applications as suggested by Hagen (12, p. 20-21) are as follows:

1. Aerospace Vehicle Control--to provide improved operational capabilities in the presence of large and rapid changes in physical process dynamics while eliminating elaborate controller-gain programming.

2. Improvement of Reliability--to maintain satisfactory response despite partial failure (malfunction) of the system.

3. Flexible Structure Control--to adjust compensation for time-dependent elastic modes.

4. Dynamic Performance Identification--to adjust a model of the physical process for tracking or model-matching capabilities.

Adaptive-Systems Definition

The following composite definition of adaptive control (controllers) has been gleaned from the reference material: Adaptive control is 1) a system with a means of continuously monitoring its own performance affected by changes in environment, character of forcing functions (e.g., input signals), and/or physical process characteristics, 2) a system that continually evaluates its performance in relation to a given figure of merit or optimal condition, and 3) a system with means of modifying its own parameters and/or state variables via closed-loop action or well-defined changes in dynamic characteristics.

Numerous adaptive control system techniques have been developed and, correspondingly, many classification schemes have been proposed.

However, evaluation of the above definition explicitly yields three basic modes of operation: 1) identification, 2) decision, and 3) modification. Some adaptive systems perform the identification operation implicitly evaluating the modification effectiveness via the performance criterion. This mode of operation is utilized in the model-reference, adaptive control (MRAC) concept as developed by Osburn et al. (10, 12, 14, 15, 16, 19) of the Massachusetts Institute of Technology (MIT) Instrumentation Laboratory. The investigation presented in this document is based on that concept.

MRAC Concept

The basic operational philosophy of MRAC systems is to force the response characteristics of a control system, which contains the physical process, to track the response characteristics of a model. This is accomplished by modifying the controller parameters that provide compensation for variations of the physical process parameters. Conventionally, the model is designed to provide the desired or reference response to reference inputs r(t) thereby representing the desired control system configuration. Parameter modifications result from a minimizing operation on a performance index consisting of a function of the response error e(t) between control system and model responses c(t) and y(t) respectively. The minimizing operation utilizes a weighting function obtained from the variables $\delta_i(t)$ that precede the controller adjustable parameters P_i or the corresponding variables $\delta'_i(t)$ in the model as illustrated in Figure 1.

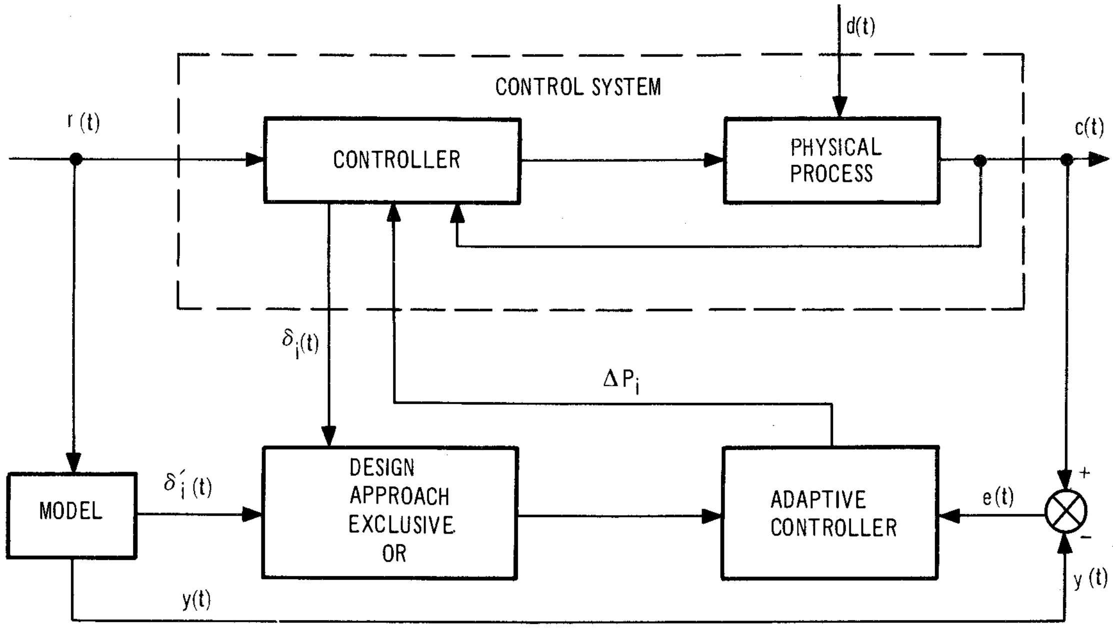


Figure 1. Basic MRAC System Configuration

Research Objectives

This thesis presents the double-valued objective:

1. To provide for adaptive control system designers a unified, useful, scholarly, and documented review of MRAC system design and analysis techniques based on current and pertinent literature. General examples illustrate these techniques and are augmented by specific examples in latter sections.

2. To provide the results of an investigation concerning adaptive-controller, steady-state sensitivity to multiple forcing functions.

Primary emphasis is placed on an analytical method developed to evaluate the sensitivity of the adaptive controller to simultaneous, sinusoidal r(t) and d(t) inputs. The investigation culminated in an evaluation and comparison of adaptive controller, forcing function sensitivities for the two basic MRAC system configurations of using either control system variables $\delta_i(t)$ or equivalent model variables $\delta_i'(t)$ to force the adaptive-loop weighting filters. The comparison is based on the perturbations of P_i required to maintain the adaptive point (i. e., minimum performance index) as a function of r(t) and d(t) amplitudes and frequencies.

II. MRAC CONCEPT REVIEW

The review of design techniques includes a unified basic design procedure and a summary of commonly encountered problems with various proposed solutions.

Basic Design Procedure

The basic MRAC design procedure and philosophy was primarily developed by Osburn (15) and has since been expanded by others (2, 4, 5, 6, 10, 12, 14, 16, 18), although some authors as indicated by Aseltine et al (1) have considered the MRAC concept prior to Osburn's work. To summarize this procedure in a unified manner requires an explicit definition of the functional blocks, performance index, minimization process, and a method of approximations necessary for realizable mechanization.

Functional Blocks

Assuming a quasi-linear and quasi-stationary representation and expanding the basic MRAC configuration shown in Figure 1 results in the system functional-block configuration illustrated in Figure 2. The controller may contain a prefilter, feedforward compensation, and/or feedback compensation where

$$M(s) = \frac{\sum_{j=0}^{\gamma} m_{jn} s^j}{\sum_{j=0}^{\zeta} m_{jd} s^j} \quad ; \quad \zeta \geq \gamma \qquad (2-1)$$

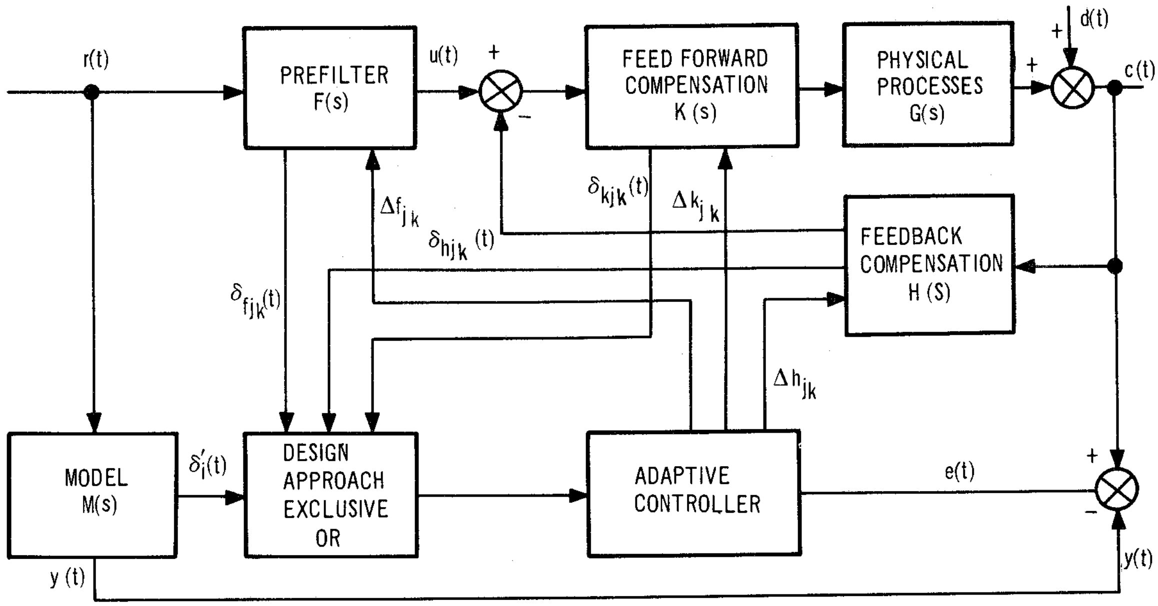


Figure 2. MRAC System Functional Block Diagram

$$F(s) = \frac{\sum_{j=0}^{\theta} f_{jn} s^j}{\sum_{j=0}^{\psi} f_{jd} s^j} \quad ; \quad \psi \geq \theta \tag{2-2}$$

$$K(s) = \frac{\sum_{j=0}^{\rho} k_{jn} s^j}{\sum_{j=0}^{\mu} k_{jd} s^j} \quad ; \quad \mu \geq \rho \tag{2-3}$$

$$G(s) = \frac{\sum_{j=0}^{\eta} g_{jn} s^j}{\sum_{j=0}^{\nu} g_{jd} s^j} \quad ; \quad \nu \geq \eta \tag{2-4}$$

$$H(s) = \frac{\sum_{j=0}^{u} h_{jn} s^j}{\sum_{j=0}^{\xi} h_{jd} s^j} \quad ; \quad \xi \geq u \tag{2-5}$$

and $$k = n \text{ or } d \tag{2-6}$$

In reality, however, the functional blocks contain time-varying parameters and often nonlinearities. A suitable selection of values for controller adjustable parameters P_i(i. e., f_{jk} k_{jk}, and/or h_{jk}) may provide the required compensation for the variation of physical-process parameters from desired values at any time t. The resulting response characteristics of the control system will then match those of the model.

By providing a continuous closed-loop adjustment of the compensating parameters via the adaptive controller, the response of the control system will be slaved to that of the model for all time t. Matching of TF_c coefficients to those of TF_y is then implied for the case of the equal-order control system and model, where

$$TF_c = \text{control system transfer-function}$$

$$= \frac{F(s)\, K(s)\, G(s)}{1 + K(s)\, G(s)\, H(s)}$$

$$= \frac{\sum_{j=0}^{\lambda} \alpha_{jn} s^j}{\sum_{j=0}^{\beta} \alpha_{jd} s^j} \quad ; \quad \beta \geq \lambda \tag{2-7}$$

and $$TF_y = \text{model transfer-function}$$

$$= M(s) \tag{2-8}$$

Specifically, the implication of perfect adaptation for cases of equal-order control system and model refers to equal transfer-function coefficients α_{jk} and m_{jk}.

The following limiting assumptions, as indicated above, are necessary before proceeding with the formal mathematical development:

1. Physical-process parameters g_{jk} vary slowly as compared to the basic time constants of TF_c and TF_y.

2. Physical-process parameters vary slowly as compared to the rates of adjustment of compensating parameters P_i in the controller functional blocks.

3. The adjustment mechanism will provide rapid rates of adjustment $\dot{P}_i$ as compared to the rate of performance index variation resulting from r(t).

Performance Index

As previously noted, desired parameter adjustments are obtained by minimizing a performance index consisting of a function of error between the response of the control system and the model. Forming state vectors from the state variables to express the response error results in

$$\bar{e}(t) = \bar{c}(t) - \bar{y}(t) = \begin{bmatrix} c_1(t) - y_1(t) \\ c_2(t) - y_2(t) \\ \vdots \\ c_n(t) - y_n(t) \end{bmatrix} \tag{2-9}$$

where

$$c_n(t) = \frac{d^{n-1}}{dt^{n-1}} c(t) \tag{2-9a}$$

$$y_n(t) = \frac{d^{n-1}}{dt^{n-1}} y(t) \tag{2-9b}$$

The dependence of $\overline{c}(t)$, $\overline{y}(t)$, and hence $\overline{e}(t)$ upon $r(t)$ should be noted and will be referred to again.

The performance index (PI) is arbitrarily chosen but must be an even function that provides a well-defined minimum condition. The integral of a quadratic function of the response error or just integral-squared-error (ISE) is commonly used for the PI and will be so used here; thus,

$$PI = \int_{t_1}^{t_2} f^2(e)\, dt \tag{2-10}$$

where

$$\begin{aligned} f^2(e) &= \left[f(e)\right]^2 \\ &= \left[\overline{u}^T \underline{Q}\, \overline{e}(t)\right]^2 \\ &= \left[q_1 e_1(t) + q_2 e_2(t) + \dots + q_n e_n(t)\right]^2 \end{aligned} \tag{2-10a}$$

or

$$PI = \int_{t_1}^{t_2} f(e^2)\, dt \tag{2-11}$$

where

$$\begin{aligned} f(e^2) &= \overline{e}^T(t)\, \underline{Q}\, \overline{e}(t) \\ &= \left[q_1 e_1^2(t) + q_2 e_2^2(t) + \dots + q_n e_n^2(t)\right] \end{aligned} \tag{2-11a}$$

and

$\underline{Q}$ = fixed diagonal matrix of response-error weighting coefficients $q_1, q_2, \ldots, q_n$

$\overline{u}$ = unit vectors

n = number of state variables of $\overline{c}(t)$ and $\overline{y}(t)$ used; normally no more than three which could represent the readily measurable states of position, rate, and acceleration.

The work discussed here will be based on the PI as defined by equation (2-10) although the PI of equation (2-11) is just as applicable.

The minimum PI is obtained when minimum or zero response-error occurs, representing a null condition for adaptive controller activity (i. e. , the adaptive point). Thus,

$$\begin{aligned} PI &= 0 \\ &= \text{adaptive point} \end{aligned} \qquad (2\text{-}12)$$

Before proceeding to the next section, it is interesting to note that in most of the work on MRAC originating at MIT, a single-state variable e(t) is used to represent the response-error function. This may be an overly restrictive choice.

Minimization Procedure

The optimum or desired state of the control system may be considered to contain a set of adjustable parameters for which a minimum PI is obtained as expressed in equation (2-12). To accomplish this objective, the slope of the PI (i. e. , ISE) with respect to each adjustable parameter must be used to provide odd-function adaptive control. The derivation may be approached directly in the time-domain by either minimizing errors between corresponding coefficients of equal-order control systems and models, Donalson and Leondes (6), or by a direct application of gradient techniques to the performance index.

The gradient technique is defined and discussed in Leitmann (13, p. 206-211) for both continuous and discrete cases, and is utilized in most all MRAC work originating at MIT. Some desirable aspects associated with this approach are the lack of restrictions on relative control system and model order and the minimization of the PI directly with respect to each adjustable parameter P_i.

Specifically odd-function, closed-loop adaptive control of each P_i (i.e., ΔP_i) is obtained via an error quantity equation defined as the partial derivative of the PI with respect to the particular P_i.

$$\Delta P_i = - a_i EQ_i \tag{2-13}$$

where

$$EQ_i = \frac{\partial}{\partial P_i} \int_{t_1}^{t_2} f^2(e)\, dt \tag{2-13a}$$

$$a_i = \text{adaptive-loop proportionality constant} \tag{2-13b}$$

and at the adaptive point

$$EQ_i = 0 \tag{2-14}$$

The steepest-descent process in the gradient technique is the time derivative of equation (2-13).

$$\dot{P}_i = - a_i \frac{\partial}{\partial P_i} f^2(e)$$

$$= - 2a_i\, f(e) \frac{\partial f(e)}{\partial P_i}$$

$$= - 2a_i \left[\bar{u}^T \underline{Q}\, \bar{e}(t)\right] \left[\bar{u}^T \underline{Q} \frac{\partial \bar{e}(t)}{\partial P_i}\right] \tag{2-15}$$

where

$$\frac{\partial f(e)}{\partial P_i} = \overline{u}^T \underline{Q} \frac{\partial \overline{e}(t)}{\partial P_i} \tag{2-15a}$$

$$= \text{adaptive-loop weighting function}$$

Since the performance of the model is invariant with perturbations of P_i, then

$$\frac{\partial \overline{e}(t)}{\partial P_i} = \frac{\partial \overline{c}(t)}{\partial P_i}$$

$$= \overline{w}_i (t)$$

$$= \text{adaptive-loop weighting state vector} \tag{2-16}$$

and

$$\dot{P}_i = -2a_i \left[\overline{u}^T \underline{Q}\, \overline{e}(t)\right] \left[\overline{u}^T \underline{Q}\, \overline{w}_i(t)\right] \tag{2-17}$$

In the event of $f(e^2)$ being used for the response-error function,

$$\dot{P}_i = -a_i \frac{\partial}{\partial P_i} f(e^2)$$

$$= -2a_i \overline{e}^T(t) \underline{Q} \frac{\partial \overline{e}(t)}{\partial P_i}$$

$$= -2a_i \overline{e}^T(t) \underline{Q}\, \overline{w}_i(t) \tag{2-18}$$

Proceeding with the development, assume the limits of integration in equation 2-13 to be independent of P_i and that $f^2(e)$ and $\partial f^2(e)/\partial P_i$ are continuous functions of both P_i and t, then the order of integration and differentiation is interchangeable. This results in

$$\Delta P_i = -2a_i \int_{t_1}^{t_2} f(e) \frac{\partial f(e)}{\partial P_i}\, dt$$

$$= -2a_i \int_{t_1}^{t_2} \left[\overline{u}^T \underline{Q}\, \overline{e}(t)\right] \left[\overline{u}^T \underline{Q}\, \overline{w}_i(t)\right] dt \tag{2-19}$$

and

$$P_i = -2a_i \int_{t_1}^{t_2} \left[\bar{u}^T \underline{Q} \, \bar{e}(t) \right] \left[\bar{u}^T \underline{Q} \, \bar{w}_i(t) \right] dt + P_i(t_1) \qquad (2\text{-}20)$$

Thus, by employing the control logic of equation (2-17) for each adjustable parameter P_i, the ISE is then minimized via a continuous steepest-descent process with equation (2-20) describing the trajectory of each P_i.

Adaptive-Loop, Weighting Function Mechanization

To complete the basic MRAC system configuration, a realizable mechanization scheme for obtaining the adaptive weighting state vector $\bar{w}_i(t)$ must be developed. Two design approaches will be discussed: the first method is based on a direct extension of the preceding development, while the second method requires a re-evaluation of the minimization definition.

Control System Variable, Weighting Filter Input. This design approach is utilized in most MRAC work originating at MIT and is based on equation (2-16). Perturbations of each P_i will effect $\bar{c}(t)$ only if nominal values of all parameters are assumed to produce a null response-error. This is based on the nontrivial case of nonzero-valued r(t).

Expressing equation (2-7) in state-variable format and taking the time derivative of equation (2-16) results in

$$\begin{aligned} \dot{\bar{w}}_i(t) &= \frac{\partial \dot{\bar{c}}(t)}{\partial P_i} \\ &= \frac{\partial}{\partial P_i} \left| \underline{\alpha}_d \, \bar{c}(t) + \underline{\alpha}_n \, \bar{r}(t) \right| \end{aligned} \qquad (2\text{-}21)$$

where

$$\dot{\overline{c}}(t) = \underline{\alpha}_d \, \overline{c}(t) + \underline{\alpha}_n \, \overline{r}(t) \qquad (2\text{-}22)$$

Equation (2-21) represents a set of equations with the same coefficients and configuration as the original control system, differing only by the forcing function. Since it is not feasible, except possibly on a discrete, time-share basis, to obtain $\overline{w}_i(t)$ by forcing the control system with self-developed variables resulting from other sources, an alternate method is necessary. The most desirable method is to approximate the system with a filter consisting of model coefficients. This procedure is readily applicable and justifiable when the model is of equal order to the order of predominant control system characteristics, because the control system response is being slaved to the model response. However, justification is not as evident when model and control system are of unequal order, although adaptation will still be achieved but perhaps in a nonoptimum manner.

Osburn (15, p. 25-42) shows, in the complex-variables domain, that the signals required for $\overline{w}_i(t)$ are usually found to be available in the signal paths of the control system without performing the implied partial-differentiation of equation (2-16). The determination of the $\overline{w}_i(t)$ is simplified if all P_i are independent of the transform-variable s, if the function $c(t, P_i)$ is transformable with respect to t with a transform $C(s, P_i)$, and if $\partial c(t, P_i)/\partial P_i$ exists; then

$$\mathcal{L}\left[\frac{\partial}{\partial P_i} \, c(t, P_i)\right] = \frac{\partial}{\partial P_i} \, C(s, P_i) \qquad (2\text{-}23)$$

Osburn then demonstrates that a suitable approximation to $\partial \overline{e}(t)/\partial P_i$ can be generated. This is accomplished by processing the signal found just prior to the adjustable parameter by a filter consisting of model coefficients to approximate the predominant control system closed-loop poles. This is illustrated in the following example.

As example 2.1, consider the general system shown in Figure 3. Assume:

$$q_1 = 1$$

$$q_2, \ldots, q_n = 0 \tag{2-24}$$

Therefore,

$$f(e) = e(t)$$

$$= c(t) - y(t)$$

$$E(s) = (TF_c - TF_y)\, R(s)$$

$$= \left[\frac{P_i\, K(s)\, G(s)}{1 + P_i\, K(s)\, G(s)} - M(s) \right] R(s) \tag{2-25}$$

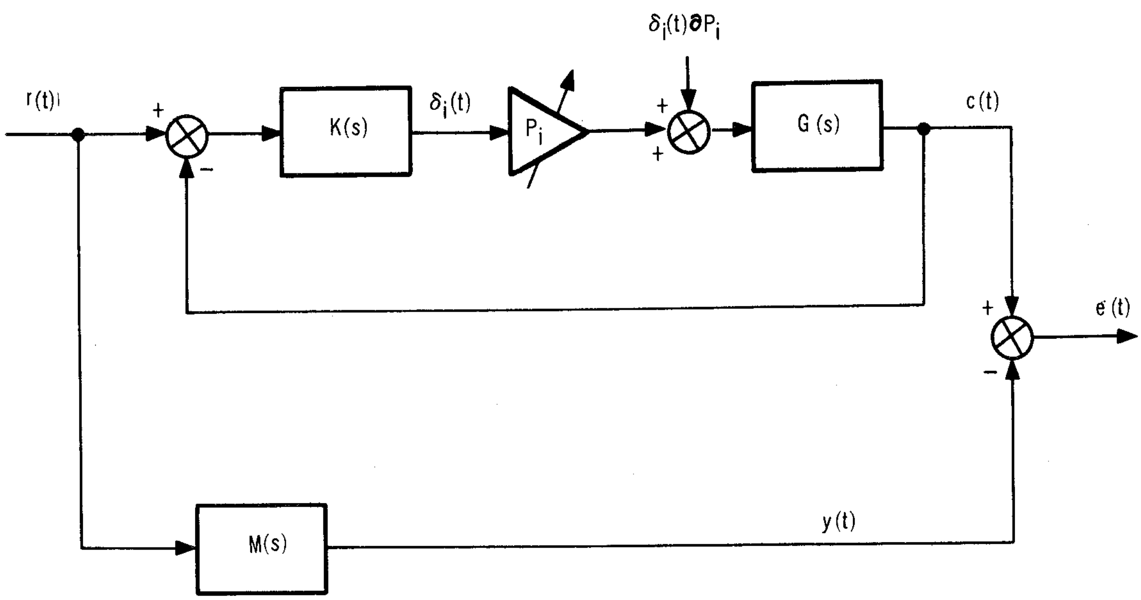


Figure 3. MRAC System with Adjustable Parameter P_i (Example (2.1)

and

$$\frac{\partial f(e)}{\partial P_i} = \frac{\partial e(t)}{\partial P_i}$$

$$\frac{\partial E(s)}{\partial P_i} = \frac{\partial TF_c}{\partial P_i} R(s)$$

$$= \frac{K(s)\,G(s)}{\left[1 + P_i\,K(s)\,G(s)\right]^2} \quad R(s) \tag{2-26}$$

Evaluation of $\Delta_i(s)$ yields

$$\Delta_i(s) = \frac{K(s)}{1 + P_i\,K(s)\,G(s)} \quad R(s) \tag{2-27}$$

Therefore,

$$\frac{\partial E(s)}{\partial P_i} = \Delta_i(s) \quad \frac{G(s)}{1 + P_i\,K(s)\,G(s)} \tag{2-28}$$

Before proceeding, consider the case of parameter perturbations with $\delta_i(t)\,\partial P_i$ as the forcing function.

$$\frac{\partial C(s)}{\Delta_i(s)\,\partial P_i} = \frac{\partial E(s)}{\Delta_i(s)\,\partial P_i}$$

$$= \frac{G(s)}{1 + P_i\,K(s)\,G(s)} \tag{2-29}$$

where equation (2-29) is the same as equation (2-28) and no differentiation operation was explicitly performed.

Now,

$$\frac{G(s)}{1 + P_i\,K(s)\,G(s)} = \text{TF of the control system excluding portions of the controller (i.e., } K(s) \text{ in numerator)} \tag{2-30}$$

Since overall performance characteristics of the control system are being slaved to those of the model, the following approximation is possible:

$$W_i(s) = \frac{\partial E(s)}{\partial P_i}$$

$$\cong M(s)\frac{1}{K(s)}\,\Delta_i(s) = WF_i(s)\Delta_i(s) \qquad (2\text{-}31)$$

where

$$WF_i(s) = \text{adaptive weighting function filter} \qquad (2\text{-}31a)$$

The factor $1/P_i$ is not included in equation (2-31) because the perturbations are secondary effects and it is a nonlinearity which would increase the complexity of mechanization. See Figure 4 for the resulting system.

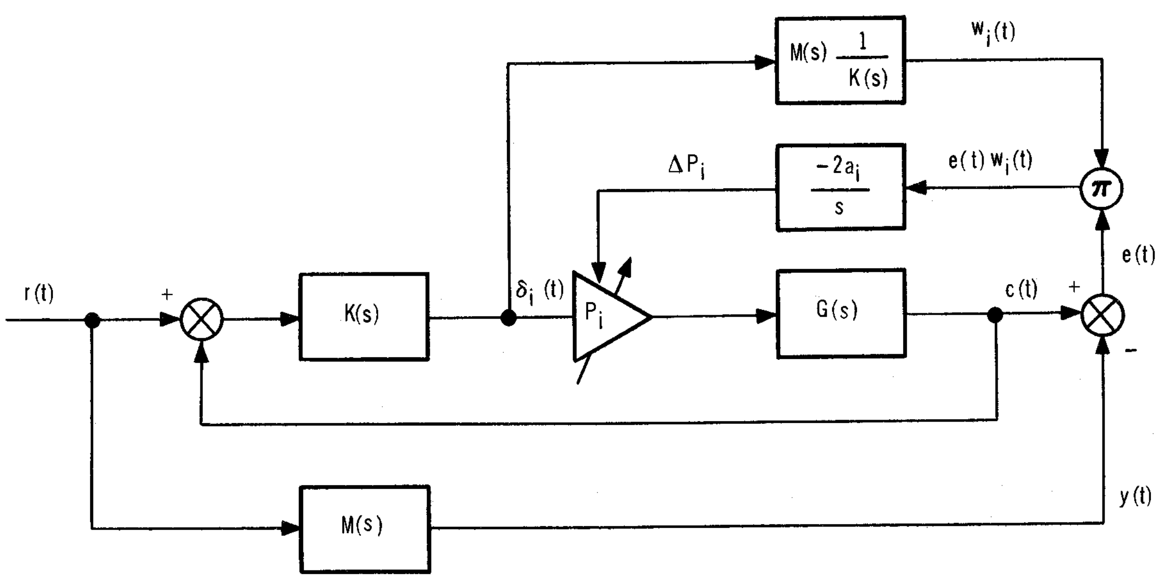


Figure 4. Complete MRAC Configuration for Example 2.1

Model Variable, Weighting Filter Input. Donalson and Leondes (6, 7) utilize this approach and present a thorough development and analysis.

The basic philosophy involved is similar to that of the previous development. The major deviation is the requirement of a steepest-ascent gradient-function to minimize the PI. This deviation results from the minimization of the PI via the partial-derivative of the ISE with respect to a parameter in the model m_i that corresponds to the control system P_i. Thus, mythical perturbations of a fixed parameter are assumed. Since in reality P_i is to be adjusted, not m_i, and the perturbation effects of P_i on the error function $f^2(e)$ are reversed in sign from those of m_i, a steepest-ascent action on $f^2(e)$ is necessary to control P_i such that the ISE is minimized. Thus,

$$\Delta P_i = a_i' \frac{\partial}{\partial m_i} \int_{t_1}^{t_2} f^2(e)\, dt \tag{2-32}$$

and

$$\dot{P}_i = a_i' \frac{\partial}{\partial m_i} f^2(e)$$

$$= 2\, a_i' \left[\bar{u}^T \underline{Q}\, \bar{e}(t)\right]\left[\bar{u}^T \underline{Q}\, \bar{w}_i'(t)\right] \tag{2-33}$$

where

$$\bar{w}_i' = \frac{\partial \bar{e}(t)}{\partial m_i} \tag{2-33a}$$

This approach culminates in an adaptive weighting state vector $\overline{w}_i'(t)$ obtained from a filter $WF_i'(s)$ mechanized with model coefficients and equal to the filter $WF_i(s)$ as derived in the preceding section except for possible steady-state gain variations. However, $WF_i'(s)$ is forced by a state variable from the model $\delta_i'(t)$ that corresponds to the control system variable $\delta_i(t)$ located prior to the adjustable parameter. A sign difference in filter forcing functions or filter transfer-function is required as illustrated in the following example.

As example 2.2, consider the general system shown in Figure 5.

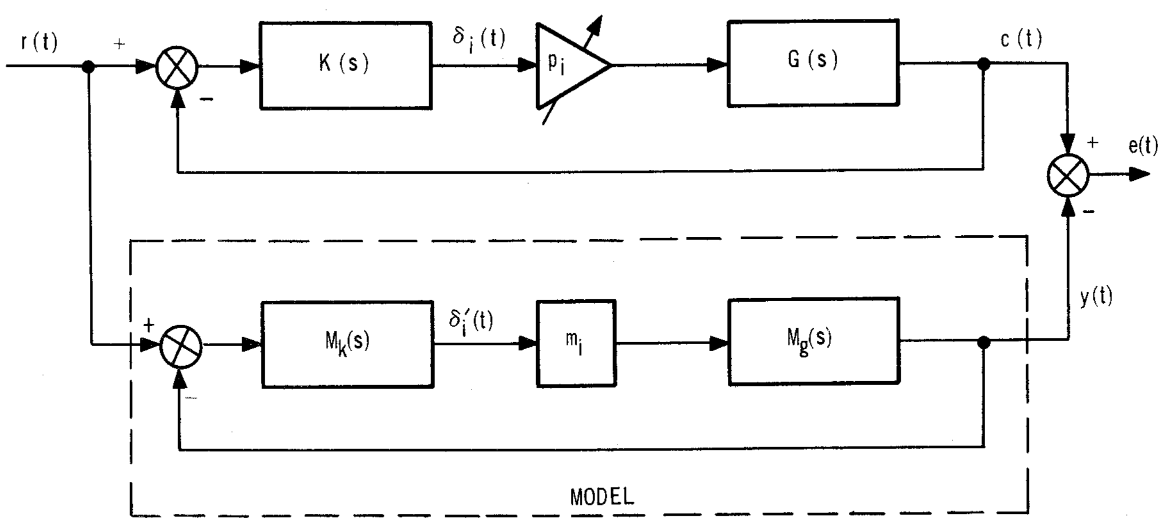


Figure 5. MRAC System with Adjustable Parameter P_i (Example 2.2)

Again assume equation (2-24), then

$$f(e) = e(t)$$

$$E(s) = \left[TF_c - TF_y \right] R(s)$$

$$= \left[\frac{P_i K(s) G(s)}{1 + P_i K(s) G(s)} - \frac{m_i M_k(s) M_g(s)}{1 + m_i M_k(s) M_g(s)} \right] R(s) \qquad (2\text{-}34)$$

Since TF_c is independent of m_i

$$\frac{\partial f(e)}{\partial m_i} = \frac{\partial e(t)}{\partial m_i}$$

$$\frac{\partial E(s)}{\partial m_i} = - \frac{\partial TF_y}{\partial m_i} R(s)$$

$$= - \frac{M_k(s) M_g(s)}{[1 + m_i M_k(s) M_g(s)]^2} R(s) \qquad (2\text{-}35)$$

where

$$\Delta_i'(s) = \frac{M_k(s)}{1 + m_i M_k(s) M_g(s)} R(s) \qquad (2\text{-}36)$$

then

$$W_i'(s) = \frac{\partial E(s)}{\partial m_i}$$

$$= - \frac{M_g(s)}{1 + m_i M_k(s) M_g(s)} \Delta_i'(s)$$

$$= - M(s) \frac{1}{m_i M_k(s)} \Delta_i'(s)$$

$$= - WF_i'(s) \Delta_i'(s) \qquad (2\text{-}37)$$

where

$$M(s) = TF_y \tag{2-37a}$$

and

$$WF_i'(s) = \text{adaptive weighting function filter}$$

$$\cong WF_i(s) \tag{2-37b}$$

The weighting filter WF_i (s) inherently consists of model coefficients and, therefore, is readily mechanized. See Figure 6.

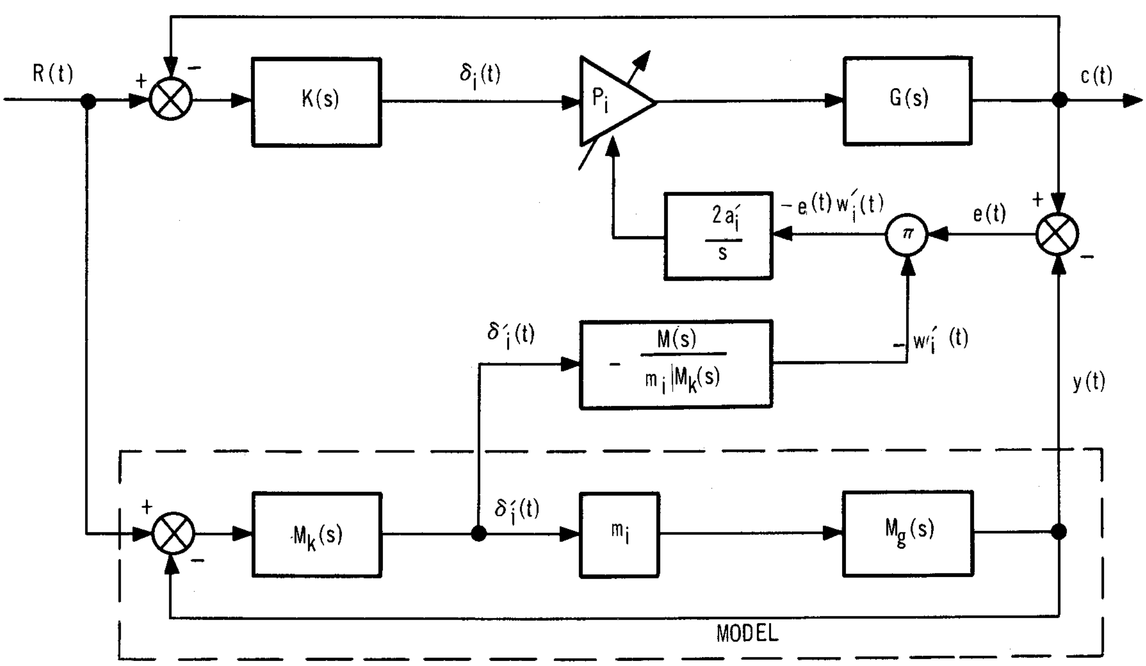


Figure 6. Complete MRAC Configuration for Example 2.2

The approach requires more assumptions and restrictions on application than does the method of forcing the weighting filter with $\delta_i(t)$. An illustration is the assumption of equal-order control systems and models with corresponding parameters and variables. Also, the effects of P_i perturbations do not exist in $\delta_i'(t)$ although Hagen (12, p. 93-148) shows these to be second-order effects.

It is interesting to note that rapid perturbations of any parameter will cause a violation of the limiting Assumptions 1 and 2, described above in the Functional Blocks Section of the Basic Design Procedure. This, in turn, will invalidate some of the derivation procedures. However, intuition, stability analyses, simulations, and actual applications of MRAC systems (2, 4, 5, 6, 7, 9, 10, 12, 14, 15, 18) demonstrate that suitable adaptive action is provided by this concept.

Problem Areas and Available Solutions

Although this system provides satisfactory control in applications where conventional feedback controller capabilities are inadequate, the MRAC system has certain design and operational problem areas. These problem areas may be categorized as follows:

1. Sensitivity to forcing function characteristics.
2. Analytical determination of dynamic response and stability.

A reasonable amount of information has been published concerning these problem areas except for the disturbance and multiple forcing function cases which are the subject of the next chapter.

Reference-Input Magnitude Sensitivity

As noted previously, MRAC systems are inoperative (e. g., trivial case) without forcing functions, whether deterministic, random reference, disturbance, and/or self-contained oscillations. The dependence upon r(t) is noted by

$$\bar{c}(t),\ \bar{y}(t),\ \bar{e}(t) \;=\; f[r(t)] \tag{2-38}$$

and

$$\overline{w}_i(t),\ \overline{w}_i'(t) \;=\; f[r(t)] \tag{2-39}$$

therefore,

$$f(e)\,\frac{\partial f(e)}{\partial P_i},\ f(e)\,\frac{\partial f(e)}{\partial m_i} = f[r^2(t)] \tag{2-40}$$

Equation (2-40) indicates the adaptive loop gain to be directly proportional to the square of r(t). Severe stability and response problems may possibly result, thus dictating the use of a compensation scheme if r(t) contains or is an unknown function.

Although many have contributed to the solution of this problem, perhaps the most significant effort was by Clark (4). The approach was first to modify the adaptive weighted-error function $f(e)\ \partial f(e)/\partial P_i$ by using only the sign of $\overline{w}_i(t)$ (i. e., sgn $\partial f(e)/\partial P_i$), reducing the ISE to be dependent on the first power of r(t). Next, a peak signal detecting/holding controller with a decay time constant much greater than control system time constants was employed to track and hence represent the absolute value of $\partial f(e)/\partial P_i$ which is then a function of the r(t) amplitude.

The adaptive loop gain for each P_i is then normalized with respect to r(t) in the following manner.

$$\dot{P}_i = - \begin{cases} 2\, a_i \dfrac{f(e)\left[\text{sgn}\,\dfrac{\partial f(e)}{\partial P_i}\right]}{\partial f(e)/\partial P_i}\,; & \left|\dfrac{\partial f(e)}{\partial P_i}\right| > D_o \\[2ex] 2\, a_i \dfrac{f(e)\left[\text{sgn}\,\dfrac{\partial f(e)}{\partial P_i}\right]}{D_o}\,; & \left|\dfrac{\partial f(e)}{\partial P_i}\right| \le D_o \end{cases} \qquad (2\text{-}41)$$

where

$$D_o = \text{arbitrary threshold} \qquad (2\text{-}41\text{a})$$

A problem associated with this method is the hysteresis characteristics dependent upon both the sign and magnitude of r(t); thus, wrong final values of P_i are possible.

It is this author's contention that this method could be adapted to detection and holding of peak absolute values of r(t) and adjustments of $f(e)\ \partial f(e)/\partial P_i$ made by the reciprocal of $r^2_{max}(t)$ for $|r_{max}(t)| > D_o$. An ISE based on a constant amplitude r(t) could then be maintained.

Reference-Input Frequency Sensitivity

Frequency dependence of the adaptive loops in MRAC systems is a predictable phenomenon, but difficult to analyze because of the inherent nonlinearity of time-domain multiplication in each adaptive loop, specifically $f(e)\ \partial f(e)/\partial P_i$.

An analytical approach was developed by Farmelo and Sammon (10, p. 10-14) for determining the steady-state frequency dependence of the value of P_i required to maintain the adaptive point (i. e., minimum PI). This was accomplished for MRAC systems with single forcing functions such as r(t). Using well defined sinusoidal functions, their results were as follows. Let

$$r(t) = R \sin \omega_r t \qquad (2\text{-}42)$$

then

$$c(t) = C_r \sin(\omega_r t + \phi_{cr}) \qquad (2\text{-}43)$$

and

$$y(t) = Y_r \sin(\omega_r t + \phi_{yr}) \qquad (2\text{-}44)$$

therefore,

$$\begin{aligned} e(t) &= e(t) - y(t) \\ &= E_r \sin(\omega_r t + \phi_{er}) \qquad (2\text{-}45) \end{aligned}$$

and

$$w_i(t) = W_{ir} \sin(\omega_r t + \phi_{wr}) \qquad (2\text{-}46)$$

Now forming the product,

$$e(t)\ w_i(t) = \frac{E_r W_{ir}}{2} \left[\cos(\phi_{er} - \phi_{wr}) - \cos(2\omega_r t + \phi_{er} + \phi_{wr})\right] \qquad (2\text{-}47)$$

The product contains a bias term and a sinusoidal term at twice the forcing frequency. Since P_i is adjusted (i. e., controlled) by the integral of the error-quantity, only the bias term contributes to the steady-state value of P_i. The bias component $\cos(\phi_{er} - \phi_{wr})$ will determine the direction of P_i motion and will reverse sign at

$$\phi_{er} - \phi_{wr} = \pm 90° \tag{2-48}$$

Therefore, the value of P_i required to maintain the adaptive point can be determined as a function of ω_r via equation (2-48) in the following manner:

1. Determine

$$E(s) = (TF_c - TF_y)\, R(s) \tag{2-49}$$

2. Derive

$$W_{ir}(s) = \frac{\partial E(s)}{\partial P_i}\, R(s)$$

or

$$W_{ir}'(s) = \frac{\partial E(s)}{\partial m_i}\, R(s) \tag{2-50}$$

3. Form the ratio

$$\frac{E(s)}{W_{ir}(s)} = \frac{E(j\omega_r)}{W_{ir}(j\omega_r)} \tag{2-51}$$

4. Since

$$\tan(\phi_{er} - \phi_{wr}) = \frac{\mathrm{Im}\,\dfrac{E(j\omega_r)}{W_{ir}(j\omega_r)}}{\mathrm{Re}\,\dfrac{E(j\omega_r)}{W_{ir}(j\omega_r)}}$$

$$= \pm\infty \tag{2-52}$$

when

$$\phi_{er} - \phi_{wr} = \pm 90° \tag{2-48}$$

then set

$$\text{Re } \frac{E(j\omega_r)}{W_{ir}(j\omega_r)} = 0 \qquad (2\text{-}53)$$

and solve for P_i as a function of ω_r.

Farmelo and Sammon's results (10, p. 14-58) indicated that often, especially whenever the control system was of higher order than the model, the adaptive weighting filter $WF_i(s)$ mechanized to approximate the control system with model characteristics, did not supply sufficient phase-shift characteristics as a function of ω_r. This results in a shift of P_i required to maintain the adaptive point by decreasing the phase shift of the control system. The converse is also true. This may result in a control system instability or lack of response capability as a function of ω_r. The problem was noted to be the most severe when first-order models and weighting filters $WF_i(s)$ were used with higher-order control systems. A solution for minimizing adaptive-point P_i sensitivity to ω_r was to increase the order of $WF_i(s)$ (e.g., additional filtering) to more closely approximate the control system. Closer matching of model and control system order also aids in minimizing the ω_r sensitivity of P_i.

It is this author's contention that the problem of adaptive-point P_i sensitivity to ω_r will not be encountered, if during the initial design phase, nominal control system parameters and configurations are used in lieu of model characteristics in mechanizing the weighting filter $WF_i(s)$. However, if model characteristics are used for $WF_i(s)$ and additional filtering is required, this author has noted in several examples, that acceptable P_i insensitivity to ω_r is often obtained by employing the rule of thumb that additional filter time constants for $WF_i(s)$ be approximately 6 db below the model time constants.

Dynamic Response and Stability

Various stability analyses of MRAC systems have been performed utilizing nonlinear techniques such as Lyapunov's Direct Method as noted in Bekey and Humphrey (2, p. 22-30) and Donalson and Leondes (7). Restrictions on these applications are equal-order control systems and models along with a compensating P_i for each variable g_{jk} in the physical process.

Hagen (12, p. 35-183) develops a method of representing the adaptive control loops in MRAC systems as a set of parallel, piece-wise linearized, control loops, each for a separate P_i. This makes feasible the application of more conventional analysis techniques. A brief summary of Hagen's thorough development, first includes the assumption that only small parameter perturbations (i.e., approximately 10%) are considered.

Then

$$\bar{c}(t) = \left[\underline{\alpha}_d + \underline{\underline{\alpha}}_d(t)\right] \bar{c}(t) + \left[\underline{\alpha}_n + \underline{\underline{\alpha}}_n(t)\right] \bar{r}(t) \qquad (2\text{-}54)$$

where a convergent series represents the control system output

$$\bar{c}(t) = \sum_{i=1}^{n} \bar{c}_i(t) \qquad (2\text{-}55)$$

Expanding and using only the first-order effects of parameter perturbations or the first two terms of equation(2-55) results in the system shown in Figures 7 and 8. Figure 7 illustrates the separation of the P_i adaptive loop from the nominal-parameter control system dynamics and model which represent null-response error characteristics. Figure 8 illustrates the linearized (i. e., first-order perturbation effects), parameter-adapting or identification-process control loop for a single adjustable parameter.

The operational philosophy of each adaptive control loop may be defined as either a regulator which minimizes e(t) regardless of ΔP_i or an identification process with ΔP_i as the controlled output resulting in eventual minimization of e(t).

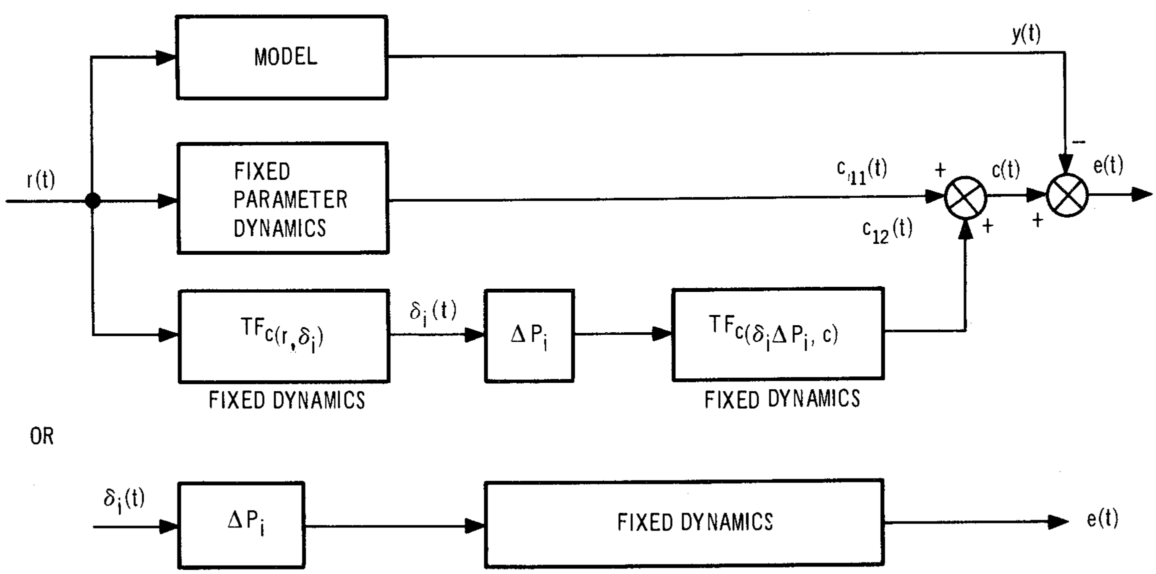


Figure 7. Separation of First-Order Parameter Perturbation Effects

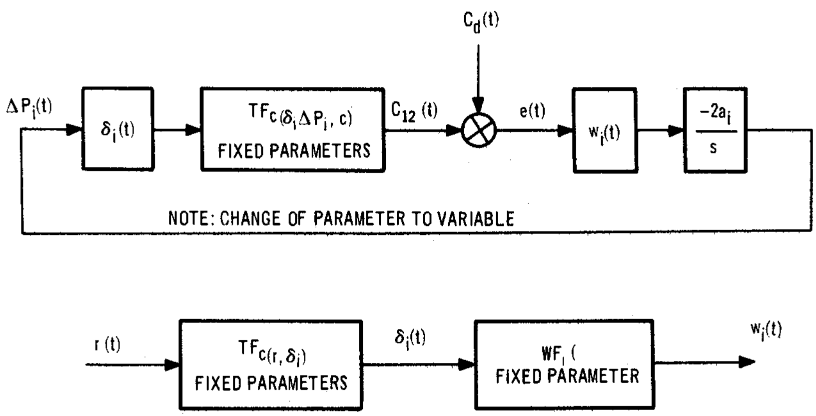


Figure 8. Linearized Adaptive Control Loop

Figure 9 illustrates the representation of a set of linearized, adaptive control loops in a MRAC system with a regulator-type mode of operation. Adaptive control loop activity is induced by a finite response error resulting from a disturbance signal $c_d(t)$ in the control system.

Interaction of adaptive control loops is readily illustrated in Figure 9. Depending on the desired operational philosophy, interaction may or may not be a problem. If decoupling is desired, orthogonalizing the adaptive control loops may be accomplished for a limited range of parameter perturbations by additional crossfeed functions. Gibson (11, p. 30-33) outlines a method of designing noninteracting control loops.

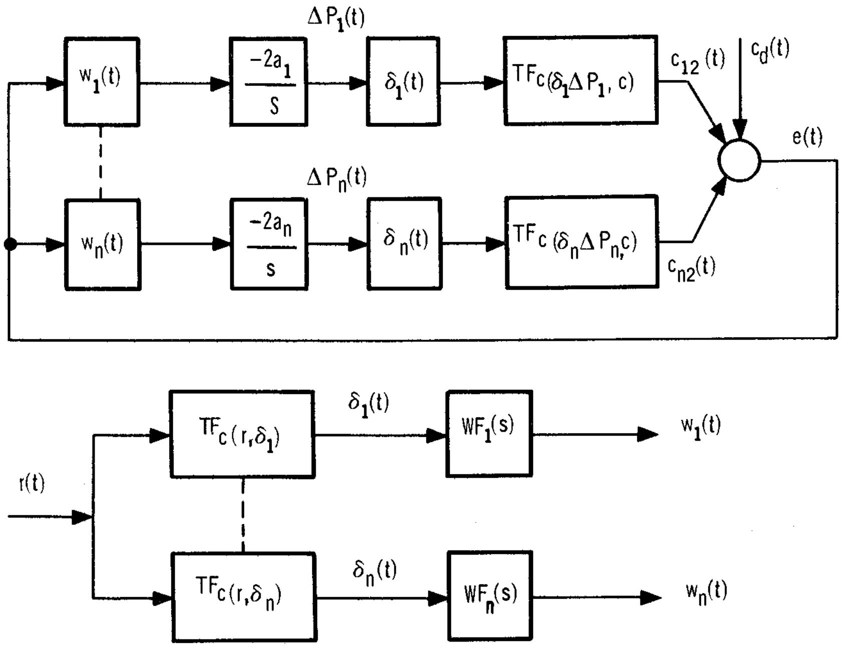


Figure 9. Multiple Linearized Adaptive Control Loops

One of Hagen's general conclusions concerned the possible use of unity transfer-function weighting filters $WF_i(s)$ to improve MRAC system stability and response characteristics. This greatly simplifies the mechanization and analyses; however, there is a corresponding loss of a defined PI.

III. MULTIPLE INPUT SENSITIVITY

As noted in the previous chapter, MRAC systems exhibit sensitivity to forcing function amplitude and frequency characteristics, and although the development was based on a reference input r(t), sensitivity to other forcing functions is implied. Several researchers, Clark (4), Farmelo and Sammon (10), Hagen (12), Osburn (15), Rucker (16), and Whitaker (18), have considered, to a limited extent, the problem of MRAC system response to disturbance forcing functions d(t). A general conclusion was that d(t) only induced motion of the adjustable parameters toward limit values in the MRAC system that exclusively used control system variable, weighting filter inputs. Specifically, Clark evaluates the stochastic case where d(t) is represented by a gaussian distribution; Farmelo and Sammon mention compromise, steady-state, adaptive-point P_i values obtained with an analog computer simulation for simultaneous, sinusoidal inputs r(t) and d(t), and the direction of P_i motion for sinusoidal d(t) excitation only; and Rucker considers the effects of d(t) in the identification problem.

In the case of MRAC system excitation via d(t) only, the response-error will consist of the control system response only and is generally minimum when control system loop gains are either minimum or maximum depending on the particular characteristics of d(t), system configuration, and point of application. By driving the adjustable parameters to their limit values, the MRAC system is then operating as designed by minimizing the PI. However, this author does not share the opinion of some who claim that no real problems result from all P_i being driven to their respective limit values. Since the MRAC system is usually designed to slave the response of the control system to that of the model with respect to r(t), limit-value operation of all P_i resulting from d(t) does not necessarily allow the control system response to any future r(t) to be optimum or even acceptable. Also, unstable control system characteristics may be encountered because of high loop gains resulting from any particular P_i being driven to or toward its limit value. In actual operation, however, simulation observations show that a controlled (i. e., flexible) upper bound is provided by the system for the P_i that induces the destabilizing effects by developing and adaptively maintaining a controlled oscillation (i. e., limit cycle mode) in which the oscillation supplies a reference variable.

The results presented herein consist of both qualitative and quantitative evaluation of the adaptive loop stability and steady-state value of P_i required to maintain the adaptive point as a function of d(t) alone and of simultaneous, sinusoidal r(t) and d(t). A frequency-domain design criterion is to be developed by which the P_i sensitivity to d(t) may be determined and minimized.

The assumption expressed in equation (2-24) will be used throughout this chapter. That is

$$f(e) = e(t) \tag{3-1}$$

$$f^2(e) = e^2(t) \tag{3-2}$$

and

$$\Delta P_i = -2a_i \int_{t_1}^{t_2} e(t) \frac{\partial e(t)}{\partial P_i} dt \tag{3-3}$$

where

$$\dot{P}_i = -2a_i\, e(t)\; w_i(t) \tag{3-3a}$$

or

$$\Delta P_i = 2a_i \int_{t_1}^{t_2} e(t)\, \frac{\partial e(t)}{\partial m_i}\; dt \tag{3-4}$$

where

$$\dot{P}_i = 2a_i\, e(t)\; w_i'(t) \tag{3-4a}$$

and

$e(t)\, w_i'(t)$, $e(t)\, w_i(t)$ = adaptive weighted-error functions

Disturbance Sensitivity of Adaptive Controller

The sensitivity of the adaptive controller in MRAC systems to deterministic, disturbance forcing functions d(t) may be determined analytically in terms of adaptive-point P_i perturbations. Although d(t) may enter the control system at any point with equivalent results, the controlled output c(t) is conventionally considered to be the summation point as illustrated in Figure 2.

Control System Variable, Weighting Filter Input

The response error becomes the control system response for the case of d(t) input only; that is for

$$r(t) = 0 \tag{3-5}$$

and

$$d(t) \neq 0 \tag{3-6}$$

then

$$
\begin{aligned}
e(t) &= c(t) - y(t) \\
&= \left[c_r(t) + c_d(t)\right] - y(t) \\
&= c_d(t) && \text{(3-7)}
\end{aligned}
$$

Transforming to the complex-operator domain and referring to Figure 2,

$$
\begin{aligned}
E(s) &= C_d(s) \\
&= \frac{1}{1 + K(s)\,G(s)\,H(s)}\,D(s) && \text{(3-8)}
\end{aligned}
$$

and each weighting filter, forcing function is

$$
\begin{aligned}
\Delta_i(s) &= -\,f[K(s)]\,H(s)\;C_d(s)\ @\ P_i \text{ in } K(s) && \text{(3-9a)} \\
&= f[H(s)]\;C_d(s) \qquad @\ P_i \text{ in } H(s) && \text{(3-9b)} \\
&= 0 \qquad @\ P_i \text{ in } F(s) && \text{(3-9c)}
\end{aligned}
$$

Recalling that the weighting filter $WF_i(s)$ contains model characteristics plus possible additional fixed filtering to compensate for reference forcing function frequency dependence of the adaptive-point, then

$$
\begin{aligned}
W_i(s) &= WF_i(s)\Delta_i(s) && \text{(3-10)} \\
&= -\,WF_i(s)\,\frac{f[K(s)]\,H(s)}{1+K(s)\;G(s)\;H(s)}\,D(s)\ @\ P_i \text{ in } K(s) && \text{(3-10a)} \\
&= WF_i(s)\,\frac{f[H(s)]}{1+K(s)\;G(s)\;H(s)}\,D(s) \quad @\ P_i \text{ in } H(s) && \text{(3-10b)} \\
&= 0 \qquad @\ P_i \text{ in } F(s) && \text{(3-10c)}
\end{aligned}
$$

Now the method of determining the P_i required to maintain the adaptive point, as discussed above in the Reference-Input Frequency Sensitivity Section, may be applied to the case of d(t) frequency sensitivity. That is

$$e(t)\; w_i(t)_{dc} = \frac{EW_i}{2} \cos\left(\phi_{ed} - \phi_{wd}\right)$$

$$= 0 \qquad (3\text{-}11)$$

for

$$d(t) = D \sin \omega_d t \qquad (3\text{-}11a)$$

which represents the adaptive point and is evaluated by

$$\mathrm{Re}\, \frac{E(j\,\omega_d)}{W_i(j\,\omega_d)} = 0 \qquad (3\text{-}12)$$

where

$$s = 0 + j\,\omega_d \qquad (3\text{-}12a)$$

therefore, since

$$\frac{E(j\,\omega_d)}{W_i(j\,\omega_d)} = \frac{-1}{WF_i(s)\; H(s)\; f[K(s)]} \quad @P_i \text{ in } K(s) \qquad (3\text{-}13a)$$

$$= \frac{1}{WF_i(s)\; f[H(s)]} \quad @P_i \text{ in } H(s) \qquad (3\text{-}13b)$$

and the controller functions f[H(s)] and f[K(s)] consist of known and assumed fixed (i. e., fixed when evaluating each adaptive loop) transfer-functions preceding P_i. The polynomial ratios of equation(3-13a and b) then contain only fixed coefficients without the P_i. Also, equation(3-12) which corresponds to $(\phi_{ed} - \phi_{wd})$ being at ±90 degrees is then dependent only on fixed coefficients and the variable frequency ω_d, and consequently, will provide only the algebraic sign of the bias term $e(t)\; w_i(t)_{dc}$.

Since equation (3-12) provides the direction of P_i motion independently of P_i, the P_i will eventually reach its limit value. However, if, as noted in reference material, a control system instability occurs because P_i is driven toward its limit value, a limit-cycle will develop at the point of conditional stability. This will provide a reference variable, and equation (3-12) will no longer describe the state of operation. Consider as example, 3.1, the second-order control system and model conforming with Figure 2, where

$$F(s) = 1 \tag{3-14a}$$

$$K(s) = k_1 \text{ (adjustable parameter)} \tag{3-14b}$$

$$G(s) = \frac{1}{s(\tau_g s + 1)} \tag{3-14c}$$

$$H(s) = 1 \tag{3-14d}$$

$$M(s) = \frac{1}{(\tau_m s + 1)(\frac{\tau_m}{2} s + 1)} \tag{3-14e}$$

Proceeding with the conventional MRAC design

$$\dot{k}_1 = -2a_1\, e(t)\, w_1(t) \tag{3-15}$$

where

$$w_1(t) = \frac{\partial e(t)}{\partial k_1} \tag{3-15a}$$

or

$$W_1(s) = \frac{\partial TF_c}{\partial k_1} R(s) \tag{3-15b}$$

$$= \frac{1}{s^2 \tau_g + s + k_1} \Delta_1(s) \tag{3-15c}$$

with

$$\Delta_1(s) = \frac{s(\tau_g s + 1)}{s^2 \tau_g + s + k_1} R(s) \tag{3-15d}$$

Assuming perfect adaptation, $WF_1(s)$ is mechanized by

$$WF_1(s) = \frac{\tau_m}{\left(\tau_m s + 1\right)\left(\frac{\tau_m}{2} s + 1\right)} \tag{3-16}$$

Also, recall that a_1 is the adaptive loop dimension and dc gain compensation for nominal k_1 and τ_m. Then

$$W_1(s) = \frac{\tau_m}{\left(\tau_m s + 1\right)\left(\frac{\tau_m}{2} s + 1\right)} \Delta_1(s) \tag{3-17}$$

and the system shown in Figure 10 is obtained.

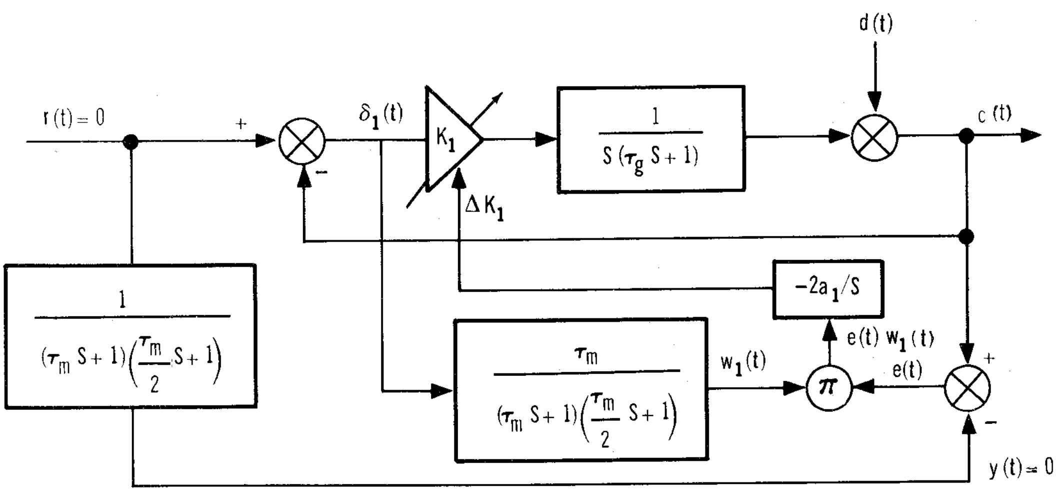


Figure 10. Complete MRAC System Configuration for Example 3.1

Now apply equations (3-8 and 3-9a) for zero-valued r(t).

$$
\begin{aligned}
E(s) &= C_d(s) \\
&= \frac{s(\tau_g s + 1)}{s^2\tau_g + s + k_1} D(s) \\
&= -\Delta_1(s) \qquad (3\text{-}18)
\end{aligned}
$$

and

$$
\begin{aligned}
W_1(s) &= WF_1(s)\Delta_1(s) \\
&= -\left[\frac{\tau_m}{\left(\tau_m s + 1\right)\left(\frac{\tau_m}{2} s + 1\right)}\right]\left[\frac{s(\tau_g s + 1)}{\left(s^2\tau_g + s + k_1\right)}\right] D(s) \qquad (3\text{-}19)
\end{aligned}
$$

Forming the ratio with

$$s = j\omega_d \qquad (3\text{-}12a)$$

$$\frac{E(j\omega_d)}{W_1(j\omega_d)} = -\,(1 + j\,\omega_d\tau_m)\left(1 + j\,\omega_d\frac{\tau_m}{2}\right)\Big/\tau_m \qquad (3\text{-}20)$$

As predicted, equation (3-20) is only a function of fixed coefficients and ω_d. Using equation (3-12) to determine the sign or direction of k_1 motion

$$
\begin{aligned}
\mathrm{Re}\,\frac{E(j\,\omega_d)}{W_1(j\,\omega_d)} &= 0 \\
&= 1 - \frac{\tau_m^2}{2}\,\omega_d^2 \qquad (3\text{-}21)
\end{aligned}
$$

where

$$\tan\left(\phi_{ed}-\phi_{wd}\right) = \frac{I_m}{R_e}\left[\frac{E(j\,\omega_d)}{W_1(j\,\omega_d)}\right] \qquad (3\text{-}21a)$$

The quadrants through which the angle $(\phi_{ed}-\phi_{wd})$ travels as a function of ω_d are defined by the sign of solution perturbations in equation 3-21. Observation of equation (3-20) provides the proper quadrant sequence with respect to increasing ω_d to be from the third to the fourth. It is necessary to refer back to E(s) and $W_1(s)$ since equation (3-21) provides a double quadrant choice. Therefore, the $\cos(\phi_{ed} - \phi_{wd})$ will proceed from a negative value through zero to a positive value with increasing ω_d. Since by equation (3-15), the direction of k_1 is the reverse of $\cos(\phi_{ed} - \phi_{wd})$, then

$$\omega_d \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \frac{T_m}{\sqrt{2}} \begin{cases} k_1 \text{ is driven to maximum limit} & (3\text{-}22a) \\ \text{null condition, unstable equilibrium} & (3\text{-}22b) \\ k_1 \text{ is driven to minimum limit} & (3\text{-}22c) \end{cases}$$

Since the control system is second-order, a highly oscillatory but not unstable mode of operation may possibly result when k_1 is driven to its maximum limit.

These results concur with those obtained by Clark (4) for the stochastic case and by Farmelo and Sammon (10, p. 59-63) from observations of analog-computer simulations.

Model Variable, Weighting Filter Input

Since a model variable, $\delta_i'(t)$ that corresponds to $\delta_i(t)$, is used to force the adaptive weighting filter $WF_i'(s)$, then

$$\begin{aligned} W_i'(s) &= WF_i'(s)\Delta_i'(s) \\ &= 0 \end{aligned} \qquad (3\text{-}23)$$

because

$$\Delta_i'(s) = f[M(s)]\, R(s)$$
$$= 0 \qquad (3\text{-}24)$$

Therefore, the adaptive weighted error $e(t)\,w_i'(t)$ will be zero-valued for any d(t) as long as r(t) is zero-valued, thereby providing an adaptive loop null-condition.

Thus, the method of generating $w_i'(t)$ in lieu of $w_i(t)$ for the adaptive weighting function provides a MRAC system that is insensitive to any disturbance d(t) during periods of reference r(t) inactivity. This is a superior mode of operation for MRAC systems designed to minimize the error resulting from r(t) with duty cycles containing periods of r(t) inactivity and d(t) activity.

Simultaneous, Sinusoidal Input Sensitivity of Adaptive Controller

Again referring to Figure 2 and assuming simultaneous, steady-state, sinusoidal, forcing functions as defined by

$$r(t) = R \sin \omega_r t \qquad (3\text{-}25)$$

$$d(t) = D \sin (\omega_d t + \phi_{dr}) \qquad (3\text{-}26)$$

where ϕ_{dr} is the initial phase angle of d(t) with respect to r(t). The adaptive-controller sensitivity to r(t) and d(t) will be obtained from the steady-state, adaptive, weighted error, bias function.

Steady-State, Adaptive Weighted Error

The control system response will contain two simultaneous functions that result from r(t) and d(t), thus

$$c(t) = c_r(t) + c_d(t) \tag{3-27}$$

where

$$c_r(t) = C_r \sin(\omega_r t + \phi_{cr}) \tag{3-27a}$$

$$c_d(t) = C_d \sin(\omega_d t + \phi_{cd} + \phi_{dr}) \tag{3-27b}$$

Since the model response results only from r(t), then

$$y(t) = Y_r \sin(\omega_r t + \phi_{yr}) \tag{3-28}$$

Therefore, the response error is

$$\begin{aligned} e(t) &= c(t) - y(t) \\ e(t) &= \sin \omega_r t (C_r \cos \phi_{cr} - Y_r \cos \phi_{yr}) \\ &\quad + \cos \omega_r t (C_r \sin \phi_{cr} - Y_r \sin \phi_{yr}) \\ &\quad + C_d \sin(\omega_d t + \phi_{cd} + \phi_{dr}) \\ &= \sin \omega_r t (E_r \cos \phi_{er}) + \cos \omega_r t (E_r \sin \phi_{er}) \\ &\quad + C_d \sin(\omega_d t + \phi_{cd} + \phi_{dr}) \\ &= E_r \sin(\omega_r t + \phi_{er}) + C_d \sin(\omega_d t + \phi_{cd} + \phi_{dr}) \end{aligned} \tag{3-29}$$

The response-error may be represented by the error between the control system and model resulting from r(t) and the control system response to d(t).

Control System Variable, Weighting Filter Input Since, by superposition, the control system contains simultaneous variables from the forcing functions r(t) and d(t), then $\delta_i(t)$ and hence $w_i(t)$ will also contain simultaneous r(t) and d(t) dependent variables. That is

$$\begin{aligned} w_i(t) &= w_{ir}(t) + w_{id}(t) \\ &= W_{ir} \sin(\omega_r t + \phi_{wr}) + W_{id} \sin(\omega_d t + \phi_{wd} + \phi_{dr}) \end{aligned} \quad (3\text{-}30)$$

Forming the adaptive weighted error

$$\begin{aligned} e(t)\, w_i(t) &= \Big[E_r \sin(\omega_r t + \phi_{er}) + C_d \sin(\omega_d t \\ &\quad + \phi_{cd} + \phi_{dr})\Big] \Big[W_{ir} \sin(\omega_r t + \phi_{wr}) \\ &\quad + W_{id} \sin(\omega_d t + \phi_{wd} + \phi_{dr})\Big] \\ &= \frac{E_r W_{ir}}{2} \Big[\cos(\phi_{er} - \phi_{wr}) - \cos(2\omega_r t + \phi_{er} + \phi_{wr})\Big] \\ &\quad + \frac{C_d W_{id}}{2} \Big[\cos(\phi_{cd} - \phi_{wd}) - \cos(2(\omega_d t + \phi_{dr}) + \phi_{cd} \\ &\quad + \phi_{wd})\Big] + \frac{E_r W_{id}}{2} \Big[\cos((\omega_r - \omega_d)t + \phi_{er} - \phi_{wd} - \phi_{dr}) \\ &\quad - \cos((\omega_r + \omega_d)t + \phi_{er} + \phi_{wd} + \phi_{dr})\Big] \\ &\quad + \frac{C_d W_{ir}}{2} \Big[\cos((\omega_d - \omega_r)t + \phi_{cd} + \phi_{dr} - \phi_{wr}) \\ &\quad - \cos((\omega_r + \omega_d)t + \phi_{cd} + \phi_{dr} + \phi_{wr})\Big] \end{aligned} \quad (3\text{-}31)$$

The steady-state, weighted error $e(t)\ w_i(t)$ for the MRAC system with two simultaneous, sinusoidal forcing functions $r(t)$ and $d(t)$ is noted to contain four basic terms. Each term containing a time-independent function such as $\cos(\phi_{er} - \phi_{wr})$ will contribute a bias to $e(t)\ w_i(t)$ while the time-dependent functions are sinusoidal with zero bias. The first two terms of equation 3-31 produce the bias for the case of nonequal forcing frequencies ω_r and ω_d while all four terms contribute to the bias when ω_r and ω_d are equal due to interaction or beat-frequency phenomena. The bias determines the direction and magnitude of $\dot{P}_i$ and over an integral number of cycles, the value of P_i. At the adaptive point, the bias is null or zero valued.

The reason for assuming only single-variable response-error $e(t)$ and adaptive weighting function $w_i(t)$ will be evaluated in the following manner. A complete formal development would use

$$f(e)\,\frac{2\,f(e)}{\partial P_i} = \left[\bar{u}^t\,\underline{Q}\,\bar{e}(t)\right]\left[\bar{u}^t\,\underline{Q}\,\overline{w}_i(t)\right] \tag{3-32}$$

where $\bar{e}(t)$ and $\overline{w}_i(t)$ each represent n state variables. The product would result in an increase in the number of terms as expressed in equation (3-31) by a factor of n^2. Thus, second-order $\bar{e}(t)$ and $\overline{w}_i(t)$ would produce 16 terms when expanded in the same manner as equation (3-31). Since all the information necessary to establish methodology can be obtained from the single-variable representation, simplicity of illustration will be maintained.

Returning to the development and considering only the bias terms,

$$e(t)\, w_i(t)_{dc} = \frac{E_r W_{ir}}{2} \cos\left(\phi_{er} - \phi_{wr}\right) + \frac{C_d W_{id}}{2} \cos\left(\phi_{cd} - \phi_{wd}\right) \Bigg|_{\omega_r \neq \omega_d} \tag{3-33a}$$

$$= \frac{E_r W_{ir}}{2} \cos\left(\phi_{er} - \phi_{wr}\right) + \frac{C_d W_{id}}{2} \cos\left(\phi_{cd} - \phi_{wd}\right) + \frac{E_r W_{id}}{2} \cos\left(\phi_{er} - \phi_{wd} - \phi_{dr}\right) + \frac{C_d W_{ir}}{2} \cos\left(\phi_{cd} + \phi_{dr} - \phi_{wr}\right) \Bigg|_{\omega_r = \omega_d} \tag{3-33b}$$

and

$$e(t)\, w_i(t)_{dc} = 0 \tag{3-33c}$$

when the value of P_i reaches the adaptive point.

An obvious conclusion is that a MRAC system, using $w_i(t)$ and designed to minimize the ISE with respect to r(t), will encounter degradation of performance when d(t) is applied simultaneously. Thus, equation (3-33a, b, and c) is an analytical expression representing the condition observed by Farmelo and Sammon (10, p. 63) on an analog-computer simulation.

The following is a qualitative analysis of equation (3-33a, b, and c). Forcing function amplitude dependence is noted where the two terms of equation (3-33a) and the first two terms of equation (3-33b) are directly proportional to R^2 and D^2 respectively. The remaining two terms of equation (3-33b) result from the beat-frequency phenomena and are directly proportional to RD. Also, all terms are inherently functions of ω_r and/or ω_d. Therefore, adaptive-point P_i would be determined as a function of ω_r, ω_d, R, D and fixed system parameters from $e(t)\, w_i(t)_{dc}$. The bias expression consists of multiple terms each of multiple-order complex functions. For zero-valued d(t), these equations reduce to the single-term functions derived by Farmelo and Sammon and discussed above in the Reference-Input Frequency Sensitivity Section.

Model Variable, Weighting Filter Inputs. Consider each adaptive weighting filter $WF_i(s)$ or $WF_i'(s)$ to be forced by a model variable $\delta_i'(t)$ corresponding to $\delta_i(t)$ in the control system. The following adaptive weighting function results.

$$w_i'(t) = W_{ir}' \sin(\omega_r t + \phi_{wr}') \tag{3-34}$$

where $w_i'(t)$ consists of response characteristics resulting from r(t) only. Now forming the product

$$\begin{aligned} e(t)\, w_i'(t) &= \left[E_r \sin(\omega_r t + \phi_{er}) + C_d \sin(\omega_d t + \phi_{cd} + \phi_{dr})\right]\left[W_{ir}' \sin(\omega_r t + \phi_{wr}')\right] \\ &= \frac{E_r W_{ir}'}{2}\left[\cos(\phi_{er} - \phi_{wr}') - \cos(2\omega_r t + \phi_{er} + \phi_{wr}')\right] \\ &+ \frac{C_d W_{ir}'}{2}\left[\cos\left((\omega_d - \omega_r)t + \phi_{cd} + \phi_{dr} - \phi_{wr}'\right)\right. \\ &\left. - \cos\left((\omega_d + \omega_r)t + \phi_{cd} + \phi_{dr} + \phi_{wr}'\right)\right] \end{aligned} \tag{3-35}$$

Therefore, $e(t)\, w_i'(t)$ contains only two terms as compared to the four in $e(t)\, w_i(t)$. Even more significant is the fact that $e(t)\, w_i'(t)$ contains no bias function resulting from d(t) for unequal ω_r and ω_d and only one additional bias function is encountered during beat-frequency conditions of ω_r equal to ω_d.

Thus,

$$e(t)\, w_i'(t)_{dc} = \frac{E_r W_{ir}'}{2} \cos (\phi_{er} - \phi'_{wr}) \Big|_{\omega_r \neq \omega_d} \tag{3-36a}$$

$$e(t)\, w_i'(t)_{dc} = \frac{E_r W_{ir}'}{2} \cos (\phi_{er} - \phi'_{wr}) + \frac{C_d W_{ir}'}{2} \cos (\phi_{cd} + \phi_{dr} - \phi'_{wr}) \Big|_{\omega_r = \omega_d} \tag{3-36b}$$

and

$$e(t)\, w_i'(t)_{dc} = 0 \tag{3-36c}$$

for adaptive-point operation.

A qualitative analysis illustrates the dependence of $e(t)\, w_i'(t)_{dc}$ on a single-term function consisting of ω_r, R^2, and fixed system parameters for the case of unequal ω_r and ω_d. For this case the method developed by Farmelo and Sammon discussed above in the Reference-Input Frequency Sensitivity Section, is readily applicable in evaluating adaptive-point P_i. However, double terms of multiple-order complex functions are again involved in the beat-frequency case of ω_r equal to ω_d with $e(t)\, w_i'(t)_{dc}$ being dependent on ω_r, R^2, ω_d, RD and fixed system parameters.

Adaptive-Point Evaluation

Solving $e(t)\, w_i(t)_{dc}$ and $e(t)\, w_i'(t)_{dc}$ as functions of ω_r, ω_d, R, D and fixed system parameters for the adaptive-point value of P_i is a difficult problem when the bias functions consist of multiple-terms each of multiple-order complex functions. Also, the possibility exists that the adaptive-point solutions would result in multiple realizable values of P_i. In such a case, the slope of the $e(t)\, w_i(t)_{dc}$ or $e(t)\, w_i'(t)_{dc}$ function with respect to P_i at each solution point would be required in determining whether the solution represents a stable or an unstable equilibrium point. For a system using steepest descent, a positive slope of $e(t)\ w_i(t)_{dc}$ through the adaptive point P_i will provide convergent (i. e., stable equilibrium) action on $\dot{P}_i$. A negative slope would produce an unstable equilibrium and would represent the boundary condition for stable operation of the adaptive controller. The converse is true for the case of steepest ascent as encountered when employing $e(t)\ w_i'(t)_{dc}$.

A method of solution that is straightforward yet yields a maximum of information concerning the adaptive controller would be to compute the adaptive weighted error functions $e(t)\ w_i(t)_{dc}$ and/or $e(t)\ w_i'(t)_{dc}$ as a function of the variables and parameters involved. The effects of any one or combination of parameters and variables could then be graphically displayed. By observing the open-loop adaptive weighted error trends, the system designer could then develop insight of adaptive-loop operational characteristics that would be difficult to ascertain except by analog-computer simulations.

Admittedly, this procedure is time consuming and requires extensive computations; however, use of Fortran or similar programs with a digital computer makes this approach acceptable.

Application of Developments

Two examples representing typical MRAC systems designed by the methods discussed in Chapter II will be used to illustrate the preceding developments. The adaptive weighted errors $e(t)\ w_i(t)_{dc}$ and $e(t)\ w_i'(t)_{dc}$, will be qualitatively and quantitatively evaluated as functions of the variables involved. Since these are specific and not general examples, the procedure and general but not the exact results, are directly applicable to other MRAC system configurations.

Since the effects of g_{jk} parameter perturbations are not of primary interest for this investigation, the following restrictions and assumptions will be used in the two examples.

$$R = 1 \tag{3-37a}$$

$$\tau_m = 1 \quad \text{second} \tag{3-37b}$$

$$\tau_g = 0.5 \quad \text{seconds} \tag{3-37c}$$

$$F(s) = 1 \tag{3-37d}$$

$$H(s) = 1 \tag{3-37e}$$

$$r(t) = R \sin \omega_r t \tag{3-37f}$$

$$d(t) = D \sin \omega_d t \tag{3-37g}$$

$$q_1 = 1, \quad q_2, \ldots, q_n = 0 \tag{3-37h}$$

$$K(s) = k_1 \quad \text{adjustable parameter, positive valued} \tag{3-37i}$$

$$\phi_{dr} = 0 \tag{3-37j}$$

Example 3.2 consists of the following two conditions:

1. Second-Order Control System and First-Order Model
2. Control System Variable, Weighting Filter Input

where

$$G(s) = \frac{1}{s(\tau_g s + 1)} \tag{3-38a}$$

$$M(s) = \frac{1}{\tau_m s + 1} \tag{3-38b}$$

then

$$E_r(s) = \frac{-s(\tau_g s + 1 - k_1 \tau_m)}{\left(\tau_g s^2 + s + k_1\right)\left(\tau_m s + 1\right)} R(s) \tag{3-39a}$$

and

$$E_d(s) = \frac{s(\tau_g s + 1)}{\tau_g s^2 + s + k_1} D(s) \tag{3-39b}$$

To derive $WF_1(s)_1$ temporarily assume

$$d(t) = 0 \tag{3-40}$$

then

$$W_{1r}(s) = \frac{\partial TF_c}{\partial k_1} R(s)$$

$$= \frac{1}{\tau_g s^2 + s + k_1} \Delta_{1r}(s)$$

$$\cong WF_1(s) \Delta_{1r}(s) \tag{3-41}$$

where

$$\Delta_{1r}(s) = \frac{s(\tau_g s + 1)}{\tau_g s^2 + s + k_1} R(s) \tag{3-41a}$$

Assuming perfect adaptation, that is slaving the control system response to that of the model and therefore matching performance functions, and allowing a_1 to be the common proportionality constant in the adaptive loop, then

$$WF_1(s) = \frac{\tau_m}{\tau_m s + 1} \tag{3-42}$$

and

$$W_{1r}(s) = \left[\frac{\tau_m}{\tau_m s + 1}\right]\left[\frac{s(\tau_g s + 1)}{\tau_g s^2 + s + k_1}\right] R(s) \tag{3-43}$$

To determine the possible dependence of adaptive-point k_1 on ω_r, let

$$\mathrm{Re}\,\frac{E_r(j\,\omega_r)}{w_{1r}(j\,\omega_r)} = 0$$

$$= -\left(k_1\,\tau_m - 1 - \tau_g^2\omega_r^2\right) \tag{3-44}$$

where

$$s = j\omega_r \tag{3-44a}$$

Therefore, solving for k_1

$$k_1 \text{ adapt. pt.} = \frac{1 + \tau_g^2\,\omega_r^2}{\tau_m} \tag{3-45a}$$

$$= 1 + \omega_r^2 / 4 \tag{3-45b}$$

which is dependent on and proportional to ω_r^2 for $\omega_r > 1/\tau_g$ which is not a desirable effect. Therefore, additional filtering is required in $WF_1(s)$, and equation (3-42) is redefined as

$$WF_1(s) = \frac{\tau_m}{\left(\tau_m s + 1\right)\left(\frac{\tau_m}{2} s + 1\right)} \tag{3-46}$$

then

$$W_{1r}(s) = \left[\frac{\tau_m}{\left(\tau_m s + 1\right)\left(\frac{\tau_m}{2} s + 1\right)}\right]\left[\frac{s(\tau_g s + 1)}{\tau_g s^2 + s + k_1}\right]R(s) \tag{3-47}$$

and

$$\mathrm{Re}\frac{E_r\,(j\,\omega_r)}{W_{1r}(j\,\omega_r)} = 0$$

$$= -\,k_1\left(\tau_m + \frac{\omega_r^2\,\tau_g\,\tau_m^2}{2}\right) + \left(1 + \omega_r^2\,\tau_g^2\right) \tag{3-48}$$

Again solving for k_1

$$k_{1 \text{ adapt. pt.}} = \frac{1 + \omega_r^2 \tau_g^2}{\tau_m + \omega_r^2 \tau_g \tau_m^2 / 2} \tag{3-48a}$$

$$= 1 \tag{3-48b}$$

which is independent of ω_r as is desired. Therefore, $WF_1(s)$ will be defined by equation (3-46). See Figure 11. Now temporarily assume

$$r(t) = 0 \tag{3-49a}$$

$$d(t) \neq 0 \tag{3-49b}$$

then

$$W_{1d}(s) = -\left[\frac{\tau_m}{(\tau_m s + 1)\left(\frac{\tau_m}{2} s + 1\right)}\right]\left[\frac{s(\tau_g s + 1)}{\tau_g s^2 + s + k_1}\right] D(s) \tag{3-50}$$

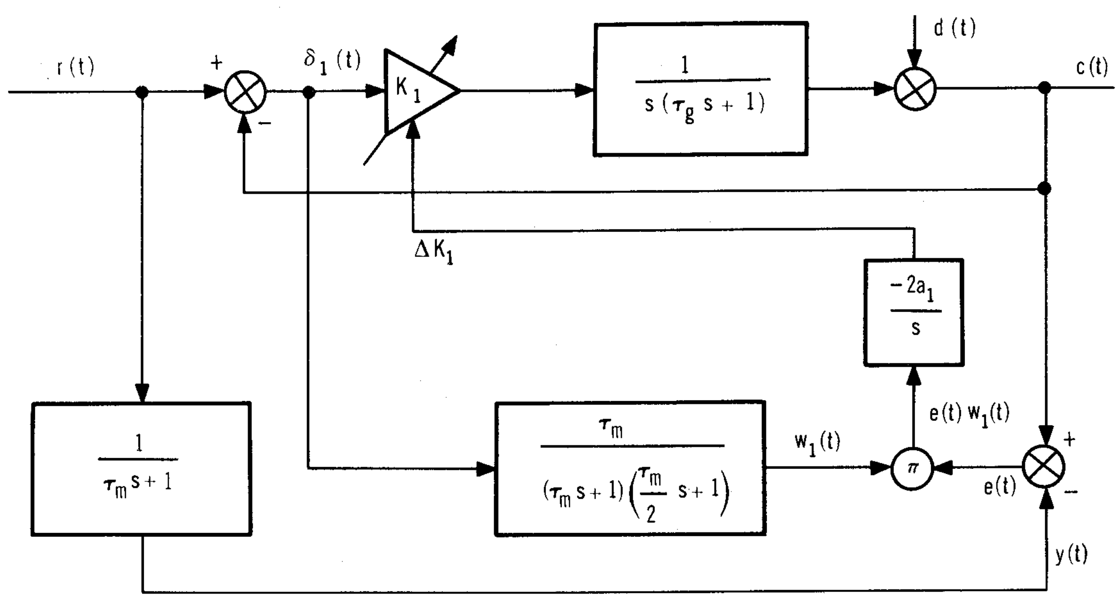


Figure 11. Complete MRAC System Configuration for Example 3.2

and by equation (3-13a)

$$\frac{E_d(j\omega_d)}{W_{1d}(j\omega_d)} = \frac{-(1 + j\omega_d\tau_m)\left(1 + j\omega_d\frac{\tau_m}{2}\right)}{\tau_m} \qquad (3\text{-}51)$$

and

$$\mathrm{Re}\ \frac{E_d(j\omega_d)}{W_{1d}(j\omega_d)} = 0$$

$$= 1 - \omega_d^2\,\frac{\tau_m^2}{2} \qquad (3\text{-}52)$$

From which the following adaptive-loop operations are possible

$$\omega_d \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \frac{\sqrt{2}}{\tau_m} \begin{cases} k_1 \text{ is driven to its maximum limit} & (3\text{-}53a) \\ \text{Null condition, unstable equilibrium} & (3\text{-}53b) \\ k_1 \text{ is driven to its minimum limit} & (3\text{-}53c) \end{cases}$$

Again the maximum possible value of k_1 may cause a highly oscillatory but not unstable control system, since the control system poles tend to move out along the imaginary axis in the s-plane for increasing loop gains.

Returning to the case of simultaneous, steady-state, sinusoidal r(t) and d(t)

$$e(t)\,w_i(t)_{dc} = \frac{E_r W_{1r}}{2}\cos(\phi_{er} - \phi_{wr}) + \frac{C_d W_{1d}}{2}\cos(\phi_{cd} - \phi_{wd})\Big|_{\omega_r \neq \omega_d} \qquad (3\text{-}54a)$$

$$\begin{aligned} = {} & \frac{E_r W_{1r}}{2}\cos(\phi_{er} - \phi_{wr}) + \frac{C_d W_{1d}}{2}\cos(\phi_{cd} - \phi_{wd}) \\ & + \frac{E_r W_{1d}}{2}\cos(\phi_{er} - \phi_{wd} - \phi_{dr}) \\ & + \frac{C_d W_{1r}}{2}\cos(\phi_{cd} + \phi_{dr} - \phi_{wr})\Big|_{\omega_r = \omega_d} \end{aligned} \qquad (3\text{-}54b)$$

where

$$E_r = R \left| \frac{j\omega_r (1 - \tau_m k_1 + j\omega_r \tau_g)}{\left(k_1 - \omega_r^2 \tau_g + j\omega_r\right)\left(1 + j\omega_r \tau_m\right)} \right| \tag{3-54c}$$

$$W_{1r} = R \left| \frac{j\omega_r (1 + j\omega_r \tau_g)}{(1 + j\omega_r \tau_m)\left(1 + j\omega_r \frac{\tau_m}{2}\right)\left(k_1 - \omega_r^2 \tau_g + j\omega_r\right)} \right| \tag{3-54d}$$

$$C_d = D \left| \frac{j\omega_d (1 + j\omega_d \tau_g)}{k_1 - \omega_d^2 \tau_g + j\omega_d} \right| \tag{3-54e}$$

$$W_{1d} = D \left| \frac{j\omega_d (1 + j\omega_d \tau_g)}{(1 + j\omega_d \tau_m)\left(1 + j\omega_d \frac{\tau_m}{2}\right)\left(k_1 - \omega_d^2 \tau_g + j\omega_d\right)} \right| \tag{3-54f}$$

$$(\phi_{er} - \phi_{\omega r}) = (\phi_{er} - \phi_{\omega d} - \phi_{dr})$$

$$= \left[\tan^{-1} \frac{\omega_r \tau_g}{1 - \tau_m k_1} - \tan^{-1} \omega_r \tau_g + \tan^{-1} \frac{\omega_r \tau_m}{2}\right] \tag{3-54g}$$

$$(\phi_{cd} - \phi_{\omega d}) = (\phi_{cd} + \phi_{dr} - \phi_{\omega r})$$

$$= \left[\tan^{-1} \omega_d \tau_m + \tan^{-1} \frac{\omega_d \tau_m}{2}\right] \tag{3-54h}$$

To qualitatively and quantitatively analyze equation (3-54), a Fortran program was written and processed on an IBM 1620 for evaluation of $e(t)w_1(t)_{dc}$ as a function of k_1, ω_r, ω_d, and D. The remaining parameters are defined by equation (3-37). An analog computer simulation on a TR-48 employing the circuit diagram shown in the Appendix was used to verify the theoretical results. Verification was achieved with a maximum variation of 10 percent from theoretical results.

For the case of no disturbance forcing function d(t), Figure 12 illustrates the adaptive-point k_1 sensitivity to ω_r. A comparison of k_1

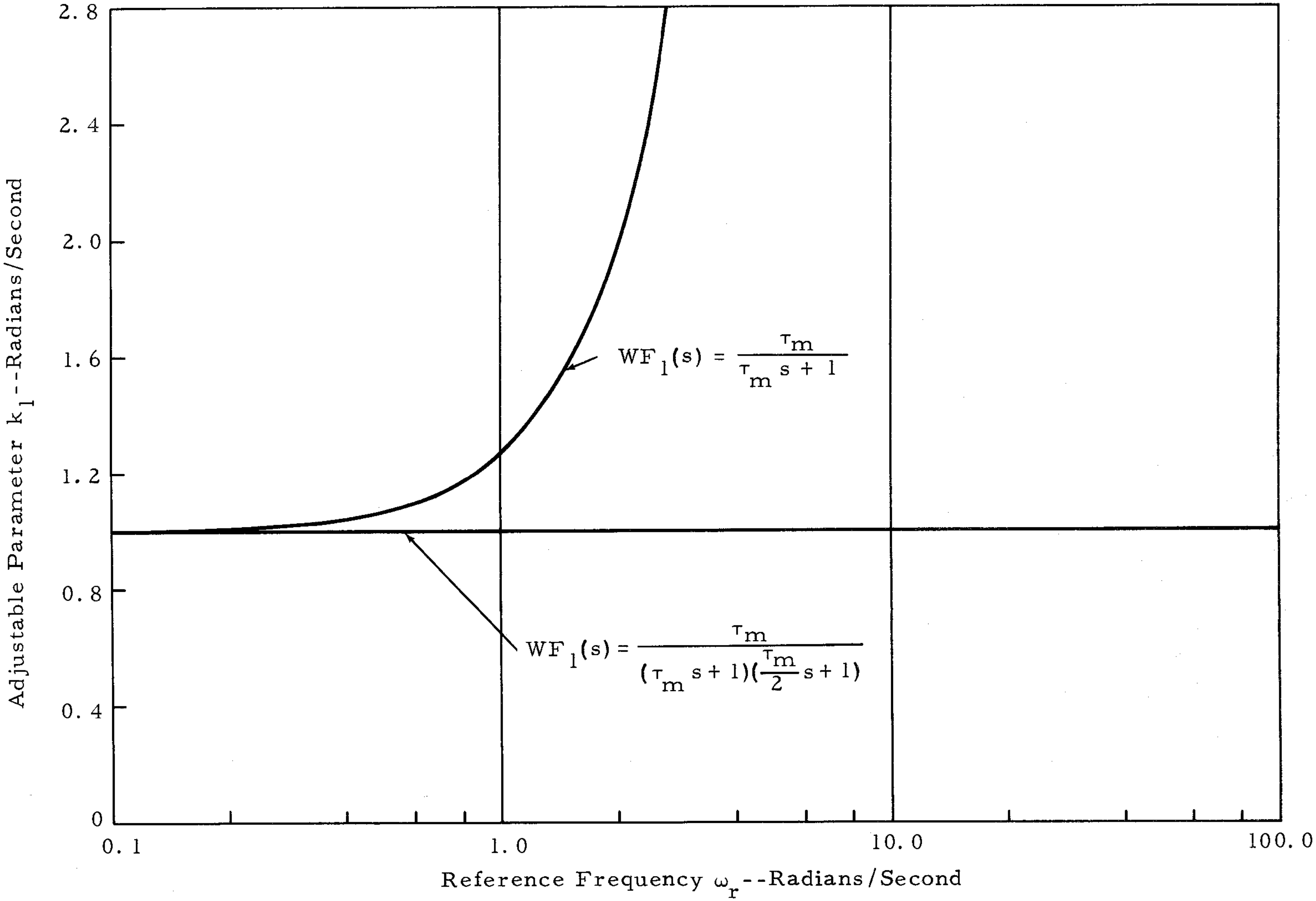


Figure 12. Steady-State Adjustable Parameter Reference Frequency Sensitivity for Example 3. 2, D = 0

sensitivity is displayed for MRAC systems using first- and second-order weighting filters forced by the control system variable $\delta_1(t)$. Equations (3-45) and 3-48) define the corresponding trajectories.

For the case of unequal forcing function frequencies ω_r and ω_d, Figures 13 and 14 illustrate the sustained open adaptive-loop weighted error bias $e(t)w_1(t)_{dc}$ as a function of k_1. The test conditions illustrated are for disturbance levels D of zero, ±50 percent and ±6 db about R with ±6 db variations of ω_d about ω_r. Near adaptive-point values of k_1 (i.e., crossover), the slopes are positive indicating stable equilibrium points necessary for desirable closed adaptive-loop characteristics. Disturbance induced bias effects are noted to cause an increase in adaptive-point values of k_1 proportional to D for conditions of ω_d less than ω_r and decrease with D for ω_d greater than ω_r. Equation (3-53) predicts the reversal to occur at ω_d equal to $\sqrt{2}/\tau_m$ for the case of disturbance only forcing-functions. Therefore, the adaptive loop will minimize the response error by increasing the control loop gain for conditions of ω_d less than $\sqrt{2}/\tau_m$ and decreasing the control loop gain for conditions of ω_d greater than $\sqrt{2}/\tau_m$ for unity ω_r.

For the case of equal forcing function frequencies ω_r and ω_d, Figure 15 illustrates the open, adaptive-loop, weighted error bias $e(t)w_1(t)_{dc}$ as a function of k_1. Stable equilibrium (i.e., positive slope) adaptive-point values of k_1 are noted to exist for conditions of D less than R. However, realizable unstable equilibrium characteristics exist for conditions of D greater than R. Therefore, the condition of D equal to R represents an equilibrium boundary between convergent and divergent closed, adaptive-loop operation.

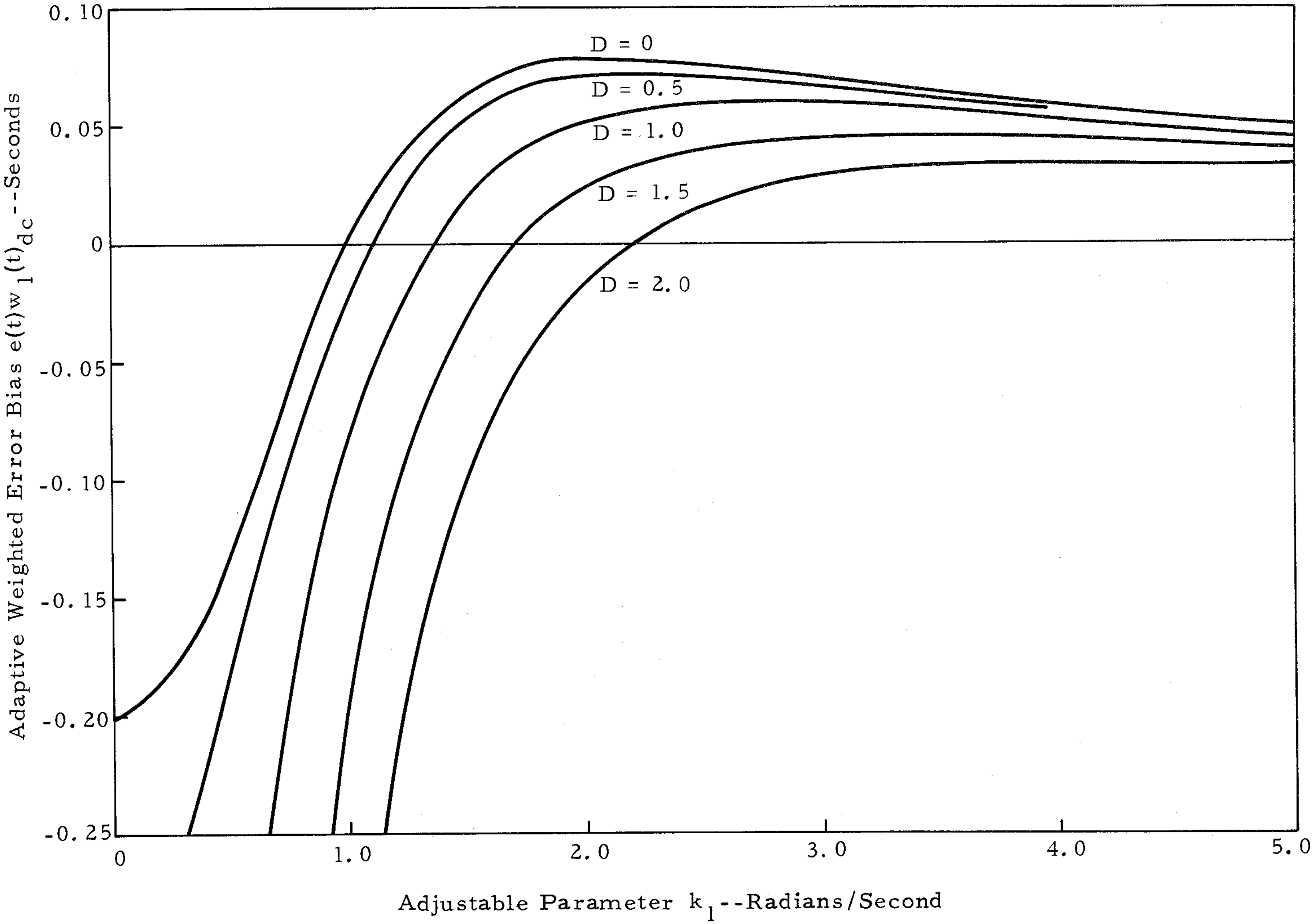


Figure 13. Open Adaptive Loop Bias Characteristics for Example 3.2, $\omega_d = 0.5$, $\omega_r = 1$

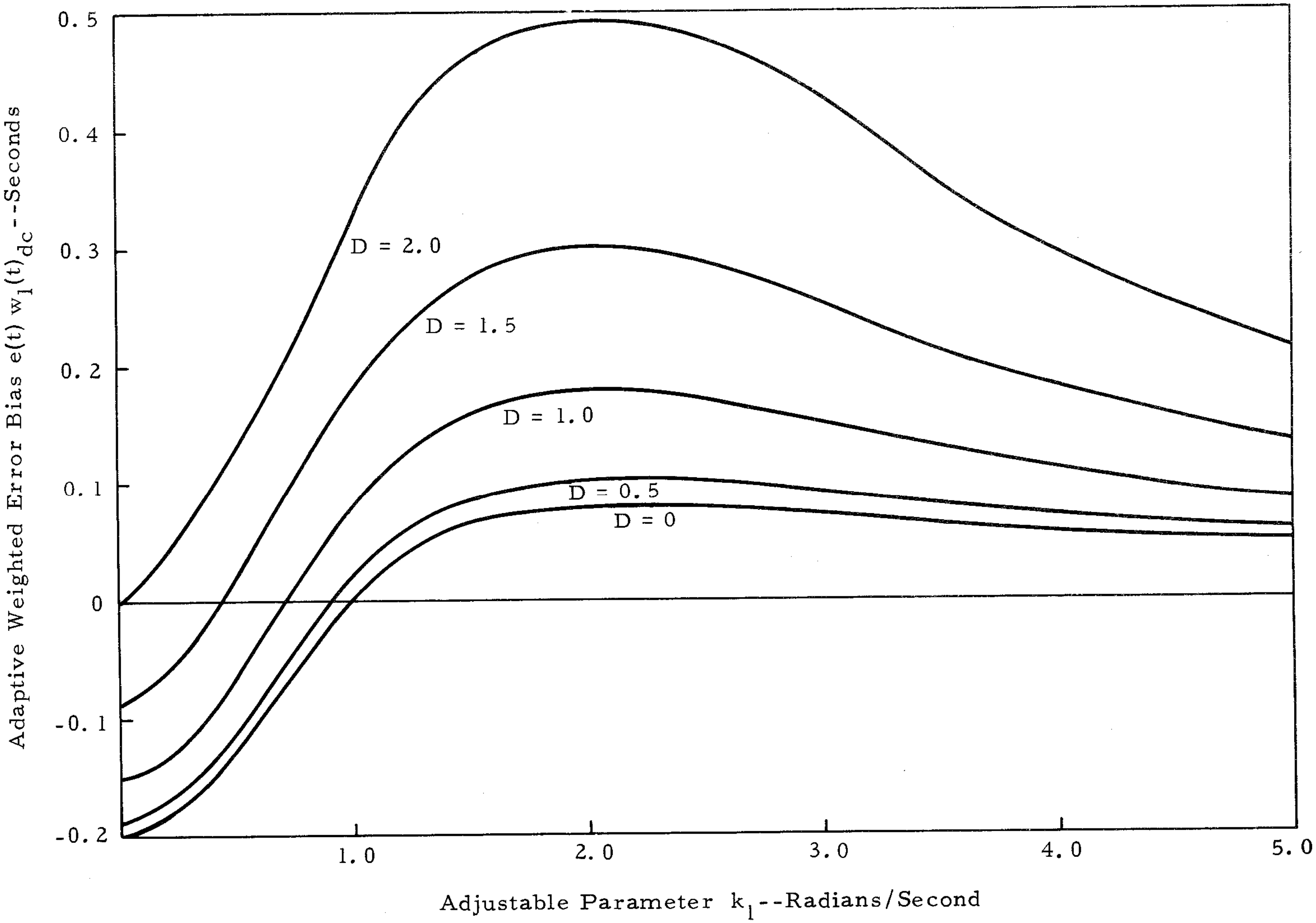


Figure 14. Open Adaptive Loop Bias Characteristics for Example 3.2, $\omega_d = 2$, $\omega_r = 1$

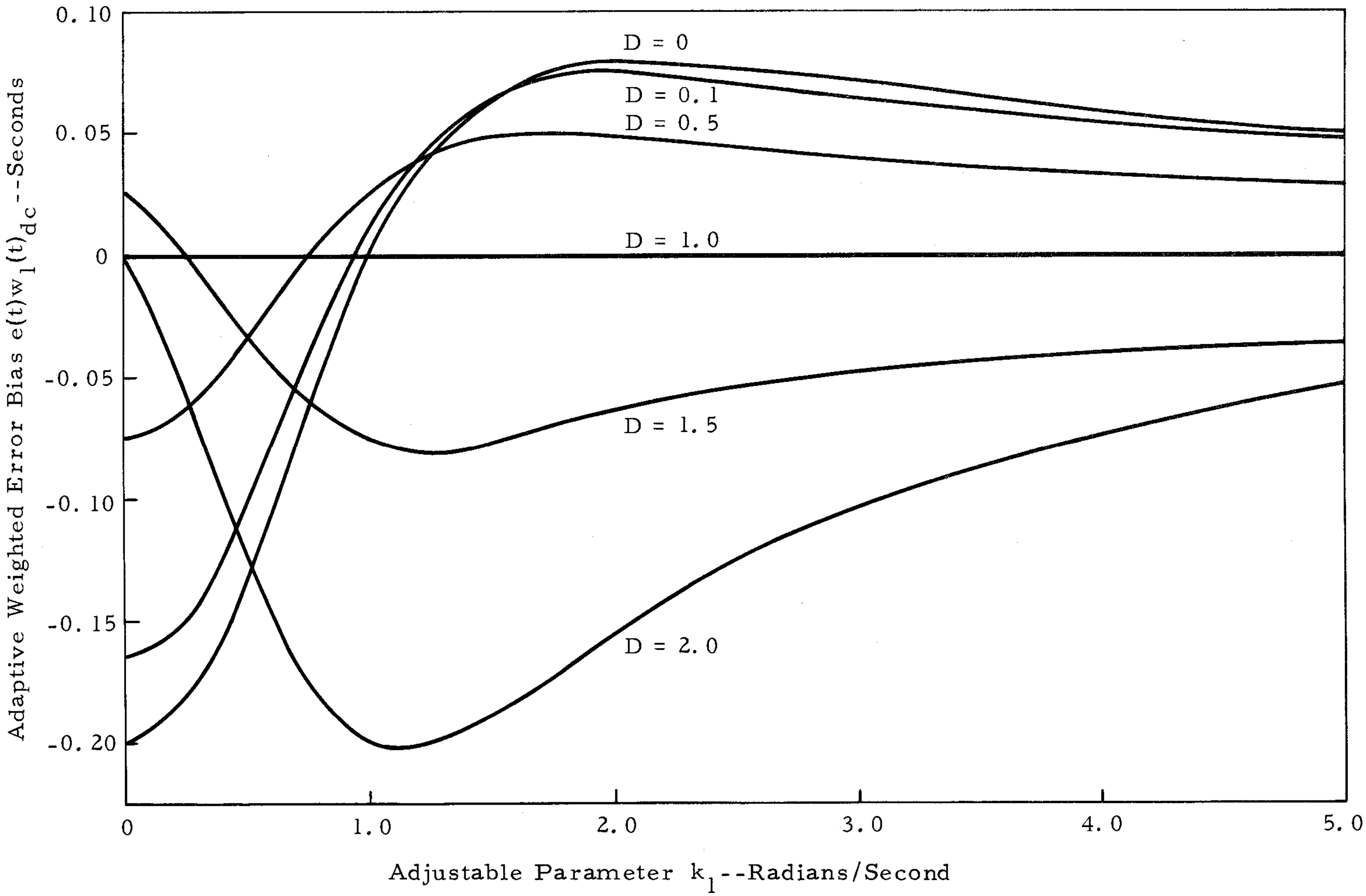


Figure 15. Open Adaptive Loop Bias Characteristics for Example 3. 2, $\omega_r = \omega_d = 1$

For the case of null adaptive weighted error, Figures 16 and 17 illustrate the steady-state, closed-loop, adaptive-point trajectory of k_1 as a function of D, ω_r, and ω_d. The reversal of bias effects is readily noted to occur at an ω_d of $\sqrt{2}/\tau_m$ as previously predicted for the disturbance only case. Peak sensitivity of k_1 to ω_d and D occurs near the control system bandwidth frequency due to the lead-lag control system response characteristics to d(t). However, disturbance bias effects are relatively minimized when ω_r is near the model bandwidth frequency due to the reference bias effects peaking near $1/\tau_m$ as noted in equation (3-54). Figure 16 illustrates the discontinuities encountered while Figure 17 illustrates the multiple values of k_1 that may be maintained for the equilibrium boundary condition of D equal to R when a beat-frequency phenomenon occurs.

Example 3.3 consists of the following two considerations:

1. Second-Order Control System and First-Order Model
2. Model Variable, Weighting Filter Input.

Equations (3-38a and b) and (3-39a and b) are directly applicable. To derive $W_1'(s)$, temporarily assume

$$d(t) = 0 \tag{3-55}$$

then

$$\begin{aligned} W_1'(s) &= -\frac{\partial TF_y}{\partial \frac{1}{\tau_m}} R(s) \\ &= -\frac{\tau_m}{\tau_m s + 1} \Delta_{1r}'(s) \\ &= -WF_1'(s)\, \Delta_{1r}'(s) \end{aligned} \tag{3-56}$$

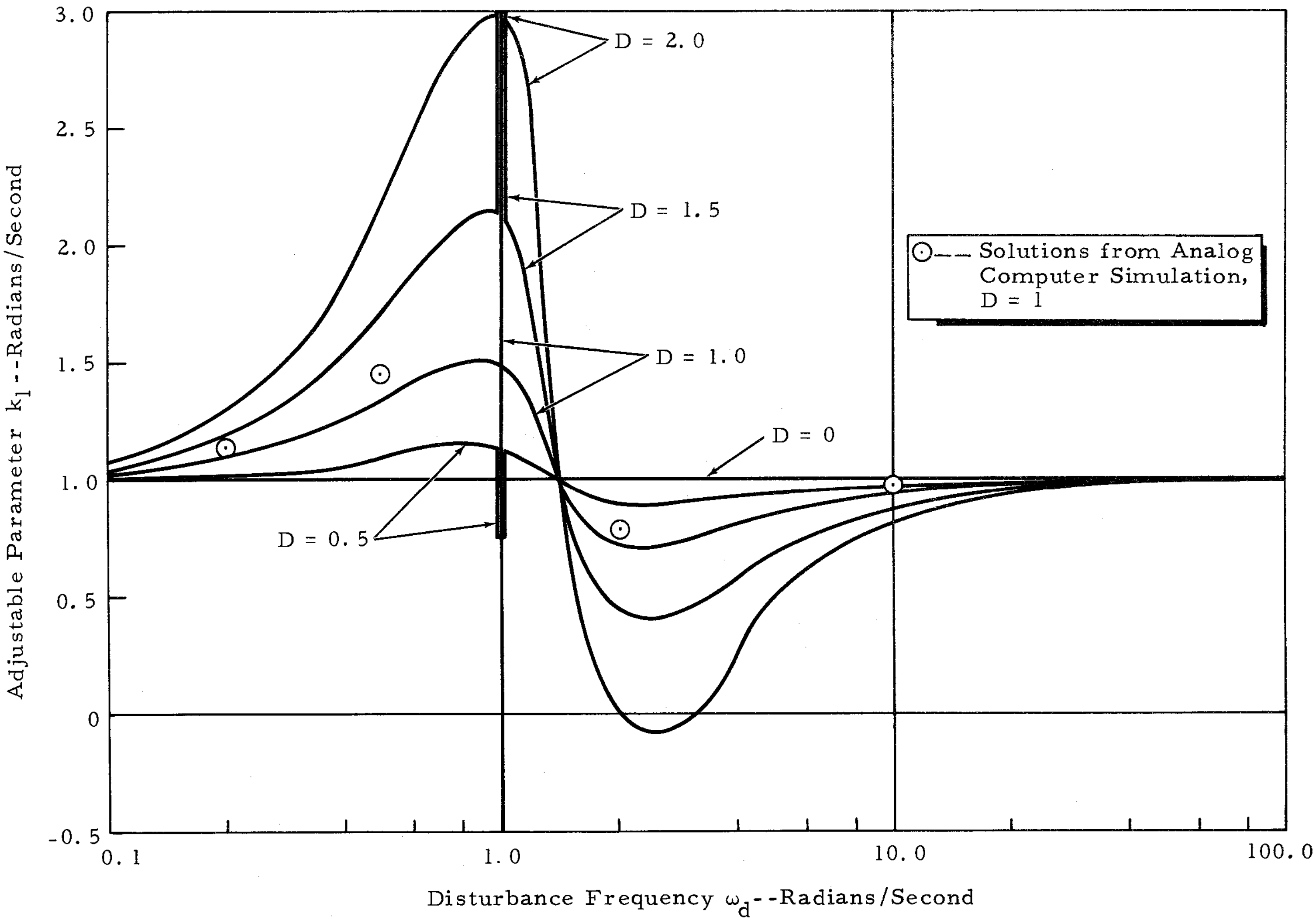


Figure 16. Steady-State Adjustable Parameter Frequency Sensitivity for Example 3.2, $\omega_r = 1$

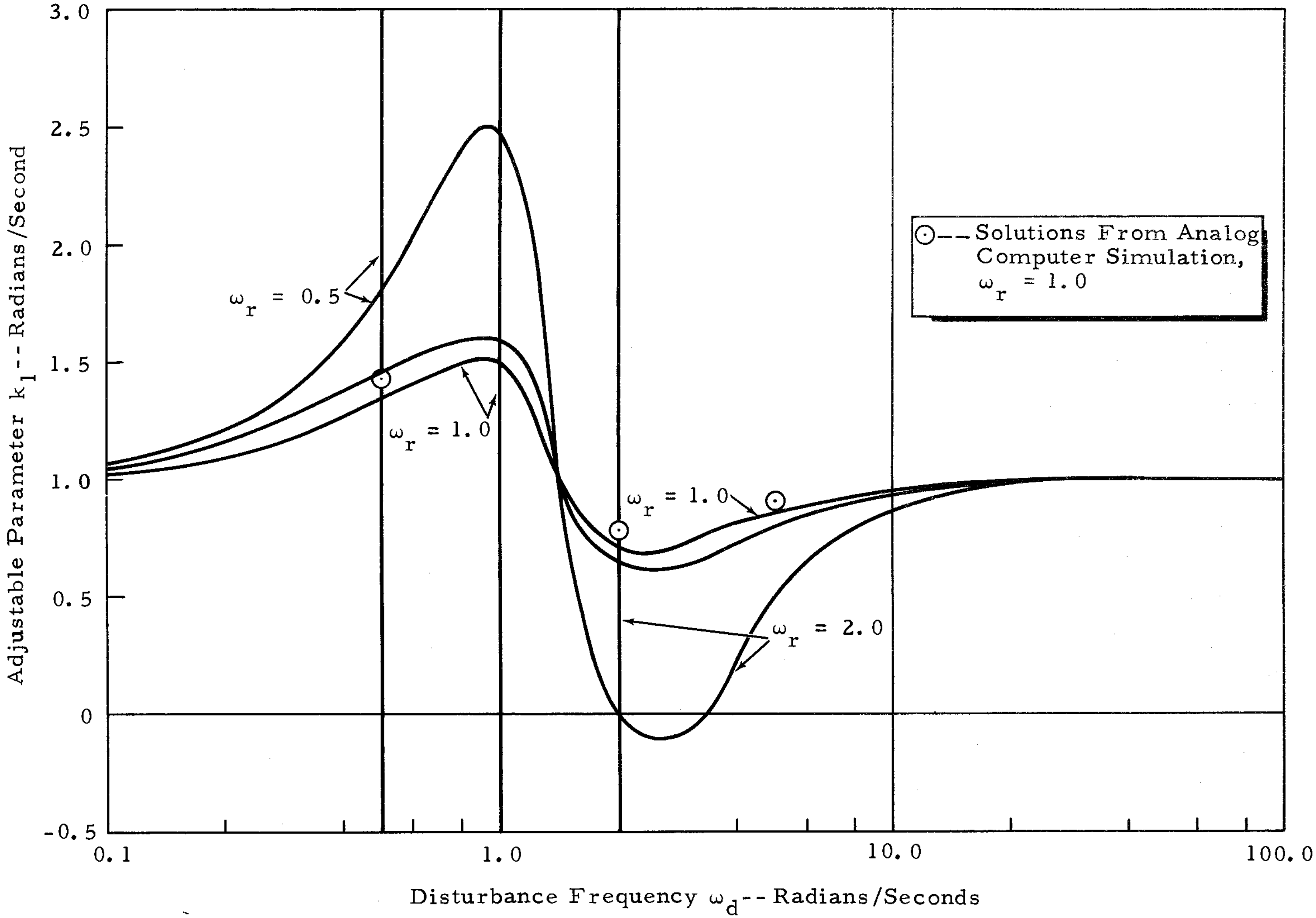


Figure 17. Steady-State Adjustable Parameter Frequency Sensitivity for Example 3.2, D = 1

where

$$\Delta'_{1r}(s) = \frac{\tau_m s}{\tau_m s + 1} R(s) \tag{3-56a}$$

Again the weighting filter $WF'_1(s)$ derived is a first-order function although previous experience in Example 3.2 indicates the use of a second-order filter to provide more desirable characteristics. Therefore, the effectiveness of the two filters must be compared by evaluating the the adaptive-point k_1 sensitivity to ω_r. Let

$$WF'_1(s) = -\frac{\tau_m}{\tau_m s + 1} \tag{3-57}$$

then

$$W'_1(s) = -\left[\frac{\tau_m}{\tau_m s + 1}\right]\left[\frac{s\,\tau_m}{\tau_m s + 1}\right] R(s) \tag{3-58}$$

and

$$\operatorname{Re} \frac{E_r(j\omega_r)}{W_{1r}(j\omega_r)} = 0$$

$$= -\omega_r^2 \left\{\tau_m(1 - k_1\tau_m) + \tau_g - \tau_m\tau_g\left(k_1 - \tau_g\omega_r^2\right)\right\}$$

$$- (1 - k_1\tau_m)\left(k_1 - \tau_g\omega_r^2\right) \tag{3-59}$$

solving for k_1

$$k_{1\ \text{adapt. pt.}} = \frac{1}{2}\left\{-\left(\omega_r^2\tau_m - \frac{1}{\tau_m}\right) \pm \sqrt{\left(\omega_r^2\tau_m - \frac{1}{\tau_m}\right)^2 + 4\omega_r^2\left(1 + \omega_r\tau_g^2\right)}\right\} \tag{3-59a}$$

$$= -\frac{1}{2}\left\{\left(\omega_r^2 - 1\right) \mp \sqrt{\left(\omega_r^2 + 1\right)^2 + \omega_r^4}\right\} \tag{3-60}$$

which is dependent on and proportional to ω_r^2, an undesirable condition.

Next consider equation (3-46) from example 3.4 and let

$$WF_1'(s) = - \frac{\tau_m}{(\tau_m s + 1)\left(\frac{\tau_m}{2} s + 1\right)} \tag{3-61}$$

where

$$W_1'(s) = - \frac{s\,\tau_m^2}{(\tau_m s + 1)^2\left(\frac{\tau_m}{2} s + 1\right)} R(s) \tag{3-62}$$

then

$$\begin{aligned} \mathrm{Re}\,\frac{E_r(j\omega_r)}{W_{1r}'(j\omega_r)} &= 0 \\ &= \omega_r^4 \frac{\tau_g \tau_m^2}{2} - \omega_r^2 \left\{ \frac{3\tau_m}{2}(1 - k_1\tau_m) + \tau_g \right. \\ &\quad \left. - \frac{\tau_m}{2}\left[\tau_m(1 - k_1\tau_m) + 3\tau_g\right]\left[k_1 - \tau_g \omega_r^2\right] \right\} \\ &\quad + (k_1\tau_m - 1)\left(k_1 - \omega_r^2 \tau_g\right) \end{aligned} \tag{3-63}$$

and again solving for k_1

$$\begin{aligned} k_{1\,\mathrm{adapt.pt.}} = - \frac{1}{2\tau_m\left(1 - \frac{\tau_m^2 \omega_r^2}{2}\right)} &\left\{ \frac{\tau_m \omega_r^2}{2}\left[\tau_m\left(4 + \tau_m \tau_g \omega_r^2\right) + \tau_g\right] \right. \\ &\left. \mp \sqrt{\left[\frac{\tau_m \omega_r^2}{2}\left(\tau_m\left(4 + \tau_m\tau_g\omega_r^2\right) + \tau_g\right) - 1\right]^2 + 6\tau_m^2 \omega_r \left(1 - \frac{\tau_m^2 \omega_r^2}{2}\right)\left(1 + \tau_g^2 \omega_r^2\right)} \right\} \end{aligned} \tag{3-64}$$

$$k_{1\,\text{adapt. pt.}} = -\frac{1}{2\left(1-\frac{\omega_r^2}{2}\right)}\left\{\left[\frac{\omega_r^2}{4}\left(9+\omega_r^2\right)-1\right]\right.$$

$$\left.\mp\sqrt{\left[\frac{\omega_r^2}{2}\left(9+\omega_r^2\right)-1\right]^2+4\left(1-\frac{\omega_r^2}{2}\right)\left(\frac{3\,\omega_r^2}{2}\left(1+\frac{\omega_r^2}{4}\right)\right)}\right\} \qquad (3\text{-}65)$$

which is also dependent on ω_r, but not to the same extent as noted in equation(3-60)for the first-order filter case. Therefore, the second-order filter noted in equation(3-62)will be used for $WF_1'(s)$. However, the solutions of both equations(3-60)and(3-64)result in double-valued functions with the possibility of both solutions being realizable.

Evaluation of the slope of $e(t)w_1'(t)_{dc}$ through the particular adaptive-point k_1 is necessary to determine the stable and unstable (i. e., boundary values) equilibrium conditions. Also, at $\omega_r = \sqrt{2}$, equation (3-65) becomes an indeterminant; however, this results from the method employed for evaluating the adaptive-point value of k_1. See Figure 18 for the resulting system.

As noted previously, there is no adaptive loop activity for the case of zero-valued r(t).

Returning to the case of simultaneous, steady-state, sinusoidal r(t) and d(t)

$$e(t)w_1'(t)_{dc} = \frac{E_r W_{1r}'}{2}\cos\left(\phi_{er}-\phi_{wr}'\right)\Big|_{\omega_r \neq \omega_d} \qquad (3\text{-}66a)$$

$$= \frac{E_r W_{1r}'}{2}\cos\left(\phi_{er}-\phi_{wr}'\right)+\frac{C_d W_{1r}'}{2}\cos\left(\phi_{cd}+\phi_{dr}-\phi_{wr}'\right)\Big|_{\omega_r = \omega_d} \qquad (3\text{-}66b)$$

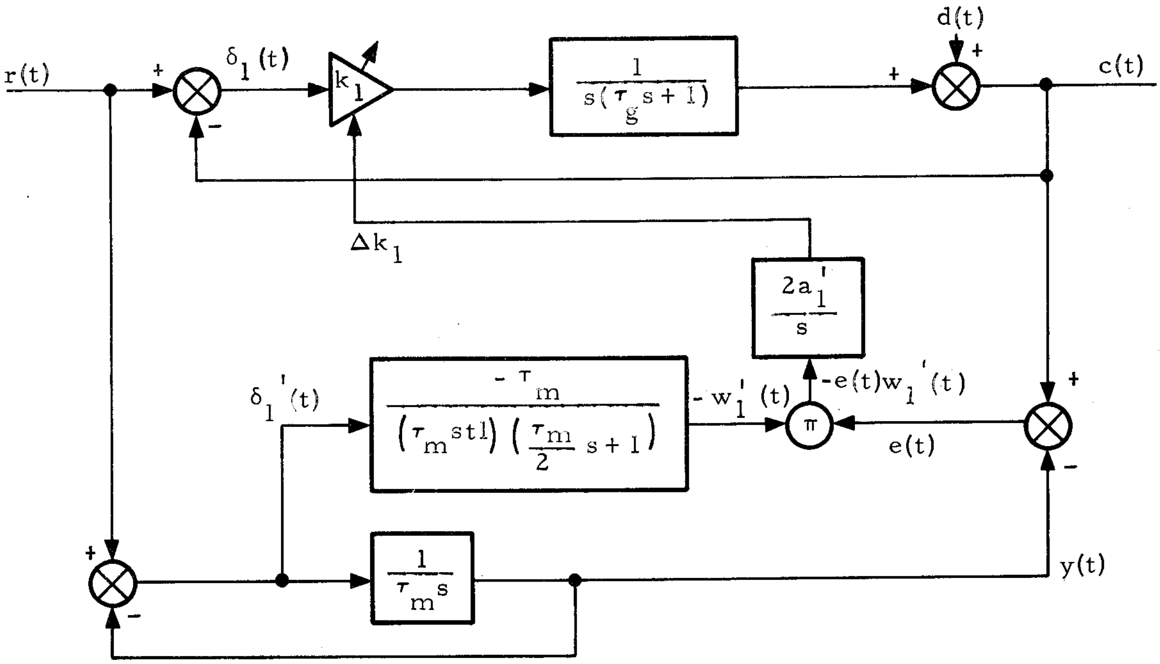


Fig. 18. Complete MRAC System Configuration for Example 3.3.

where

$$E_r = R\left|\frac{j\omega_r(1-k_1\tau_m+j\omega_r\tau_g)}{\left(k_1-\omega_r^2\tau_g+j\omega_r\right)\left(1+j\omega_r\tau_m\right)}\right| \quad (3\text{-}66c,$$

$$W_{1r}' = R\left|\frac{j\omega_r\tau_m^2}{\left(1+j\omega_r\tau_m\right)^2\left(1+j\omega_r\frac{\tau_m}{2}\right)}\right| \quad (3\text{-}66d)$$

$$C_d = D\left|\frac{j\omega_d(1+j\omega_d\tau_g)}{k_1-\omega_d^2\tau_g+j\omega_d}\right| \quad (3\text{-}66e)$$

$$\left(\phi_{er}-\phi_{wr}'\right) = \left[\tan^{-1}\frac{\omega_r\tau_g}{1-k_1\tau_m} - \tan^{-1}\frac{\omega_r}{k_1-\omega_r^2\tau_g} + \tan^{-1}\omega_r\tau_m + \tan^{-1}\omega_r\frac{\tau_m}{2}\right] \quad (3\text{-}66f)$$

$$(\phi_{cd}+\phi_{dr}-\phi_{wr}) = \left[\tan^{-1}\omega_d\tau_g - \tan^{-1}\frac{\omega_d}{k_1-\omega_d^2\tau_g} + 2\tan^{-1}\omega_r\tau_m + \tan^{-1}\frac{\omega_r\tau_m}{2}\right] \qquad (3\text{-}66g)$$

A FORTRAN program was used again to qualitatively and quantitatively analyze equation(3-82). The results are shown and discussed in the following.

For the case of unequal forcing function frequencies ω_r and ω_d, Figures 19 and 20 respectively illustrate the adaptive point k_1 sensitivity to ω_r and the sustained open, adaptive-loop weighted error bias as a function of k_1. Figure 19 exhibits the desirability for using the second-order filter in lieu of the first-order filter for $WF_1'(s)$ to produce relative insensitivity of k_1 to ω_r. An additional constraint on the maximum tolerable value of k_1 is necessary when using the second-order filter for $WF_1'(s)$. This results from the realizable double-valued solution when ω_r is greater than $\sqrt{2}/\tau_m$ with the upper value representing the stability boundary condition.

For the case of equal forcing function frequencies ω_r and ω_d, Figures 21 and 22 illustrate the open, adaptive-loop weighed error bias $e(t)w_i'(t)_{dc}$ as a function of k_1. Negative slopes of $e(t)w_i'(t)_{dc}$ through adaptive point values of k_1 provide stable equilibrium solutions in this system. Since multiple-valued solutions are feasible, solutions with positive slope functions will represent unstable equilibrium conditions that are stability boundaries. When ω_r is greater than $\sqrt{2}/\tau_m$, the stable solution values of the adaptive-point k_1 increase with D while the unstable solution values decrease resulting in a common equilibrium point for a particular value of ω_r and D. This shift of the unstable

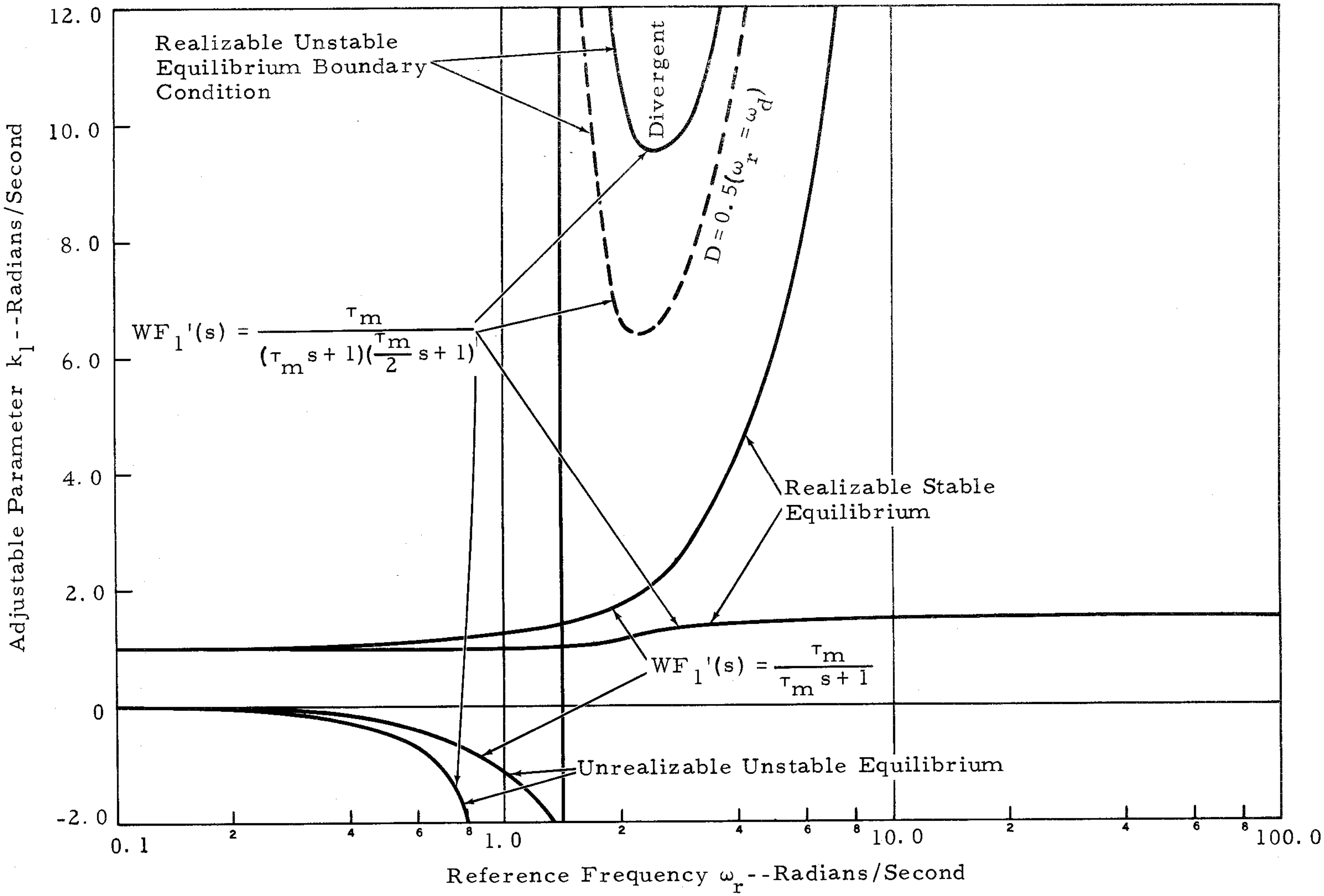


Figure 19. Steady-State Adjustable Parameter Reference Frequency Sensitivity for Example 3.3, D = 0

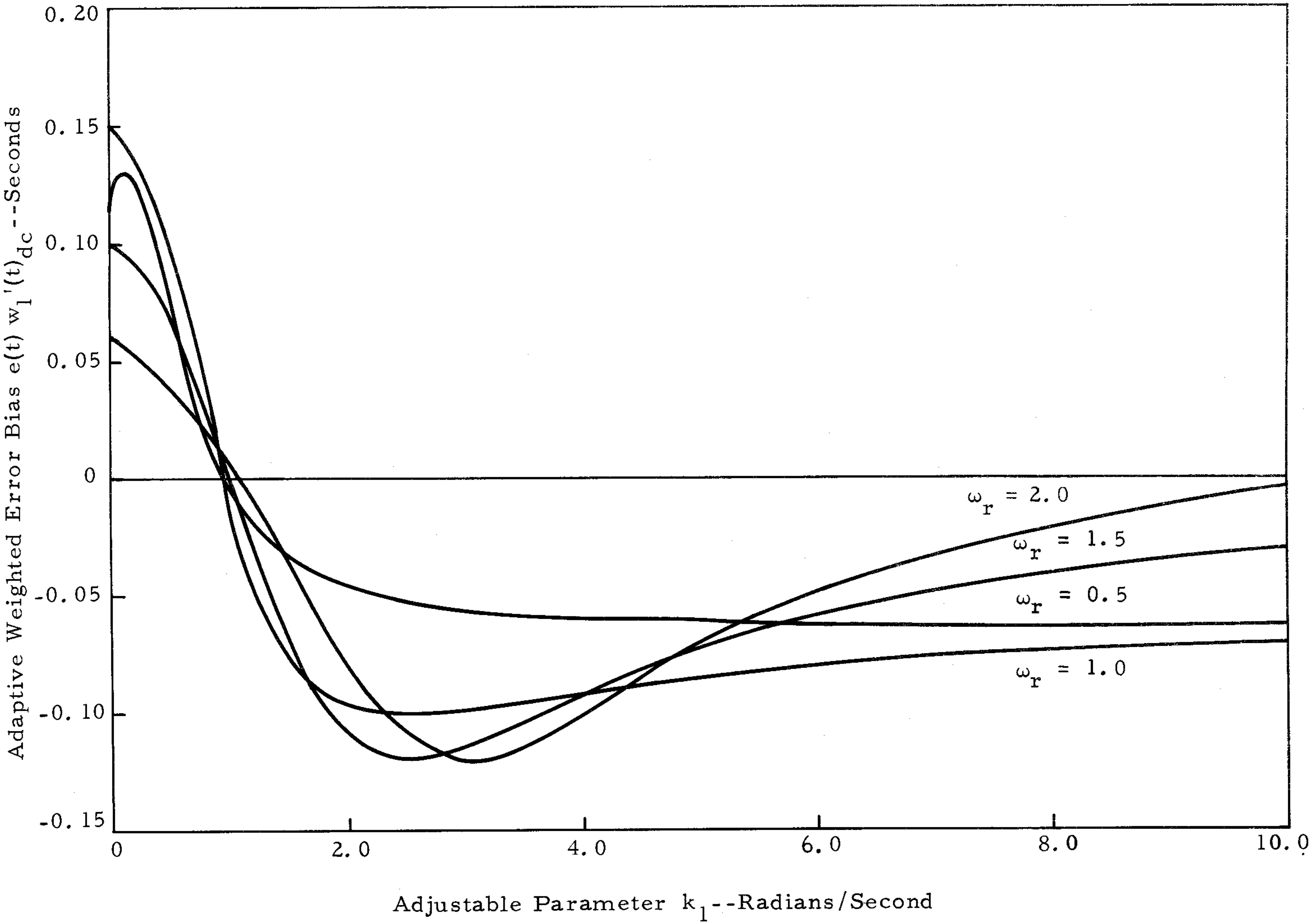


Figure 20. Open Adaptive Loop Bias Characteristics for Example 3.3, D = 0

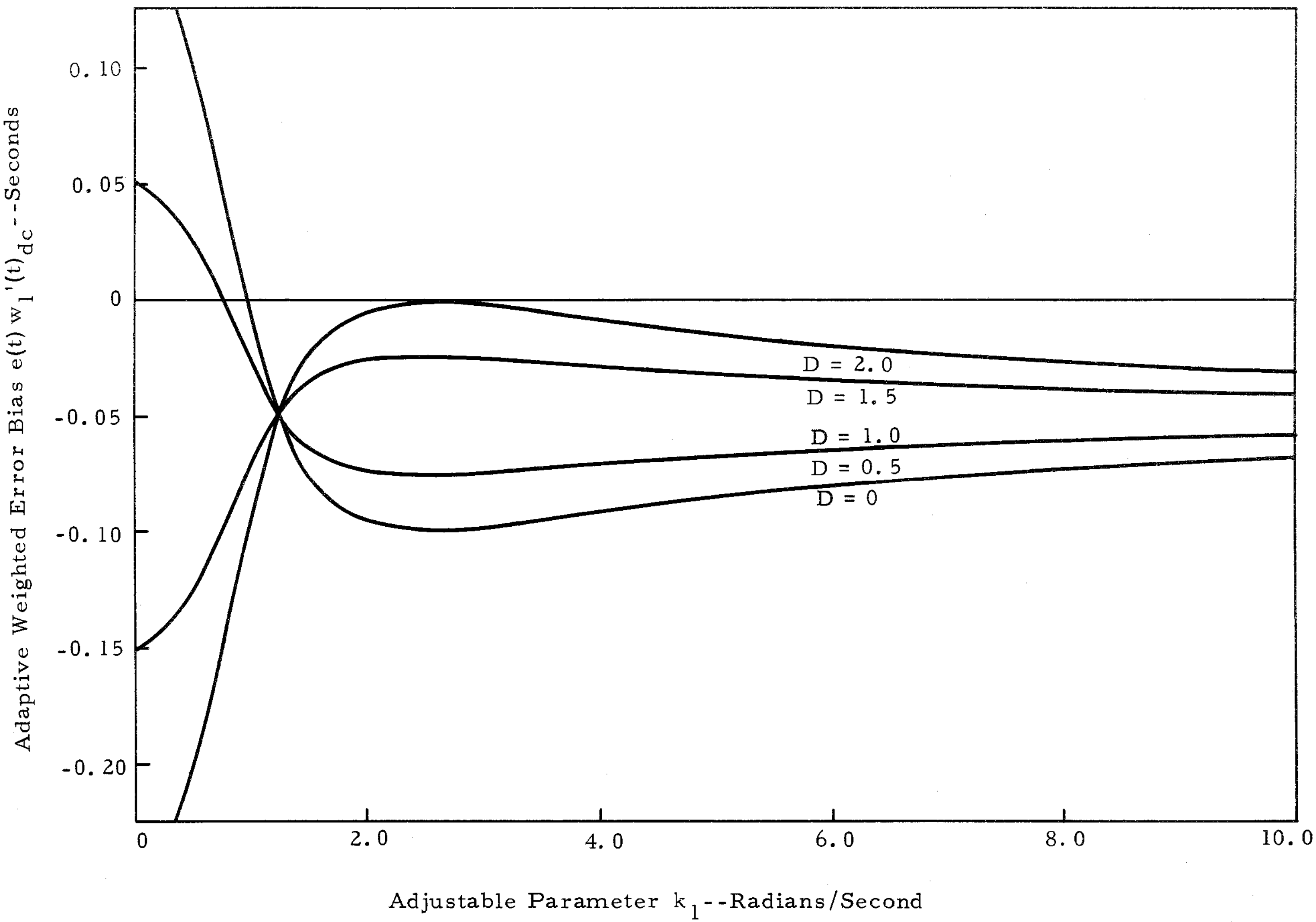


Figure 21. Open Loop Bias Characteristics for Example 3.3, $\omega_r = \omega_d = 1$

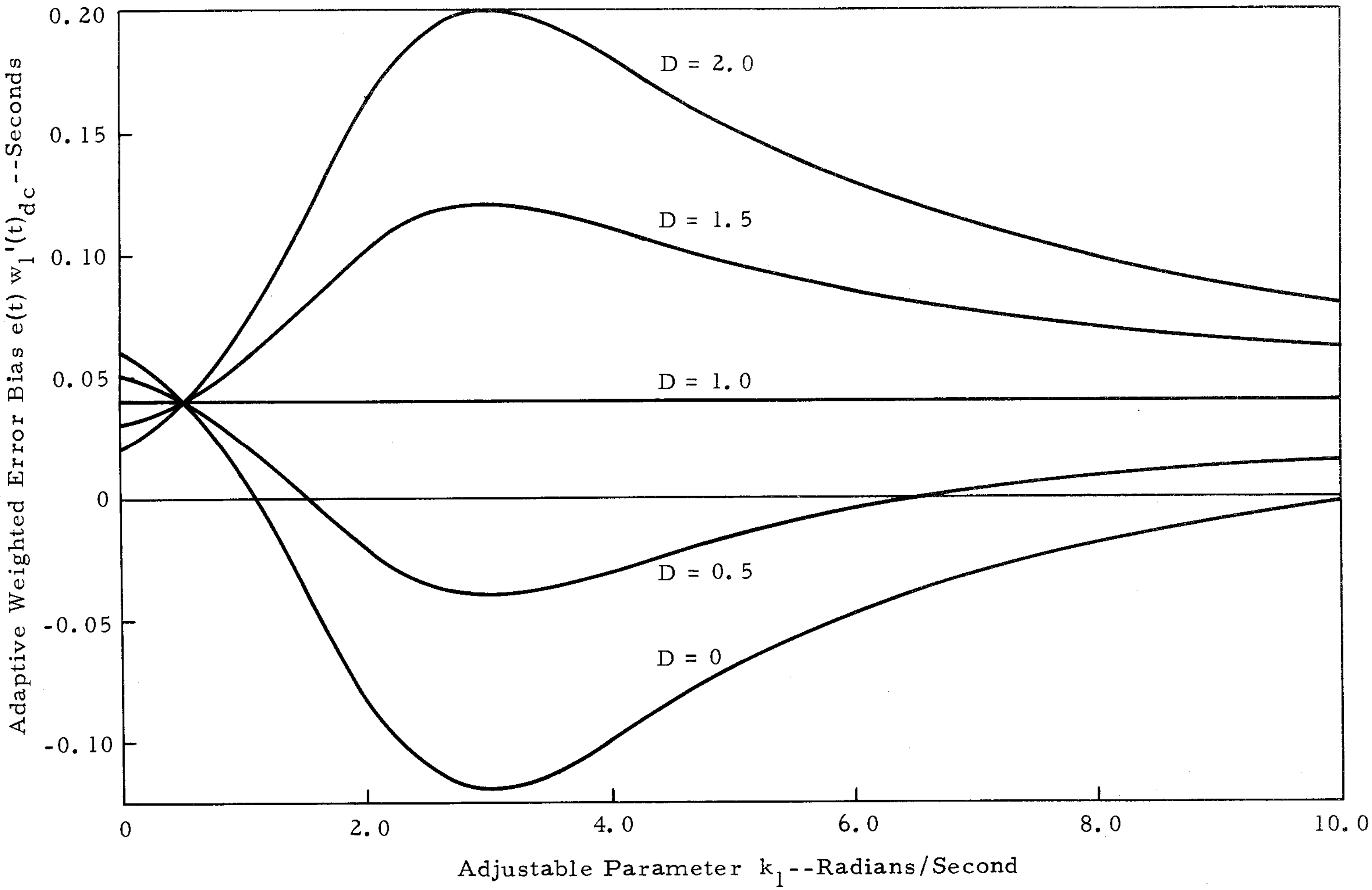


Figure 22. Open Adaptive Loop Bias Characteristics for Example 3.3, $\omega_r = \omega_d = 2$

equilibrium point with D is also noted by the dotted curve on Figure 19. Since $e(t)w_i'(t)_{dc}$ is normalized with respect to R, the single power of D provides a greater disturbance-bias component than does the D^2 function in the preceding example for D less than R. Therefore, the disturbance-bias effects induce a stability boundary occurring at a value of D less than R and as a function of ω_r

For the case of null adaptive weighted error, Figure 23 illustrates the steady-state, closed-loop adaptive-point trajectory of k_1 as a function of D and ω_r. In the event of this system operating in a beat-frequency, forcing function environment, restricting ω_r to be within ±6 db of the model-bandwidth frequency and D less than 0.5R is observed to result in reasonable values of k_1. Also, the operation at ω_r equal to $\sqrt{2}/\tau_m$ provides a null of disturbance-bias effects.

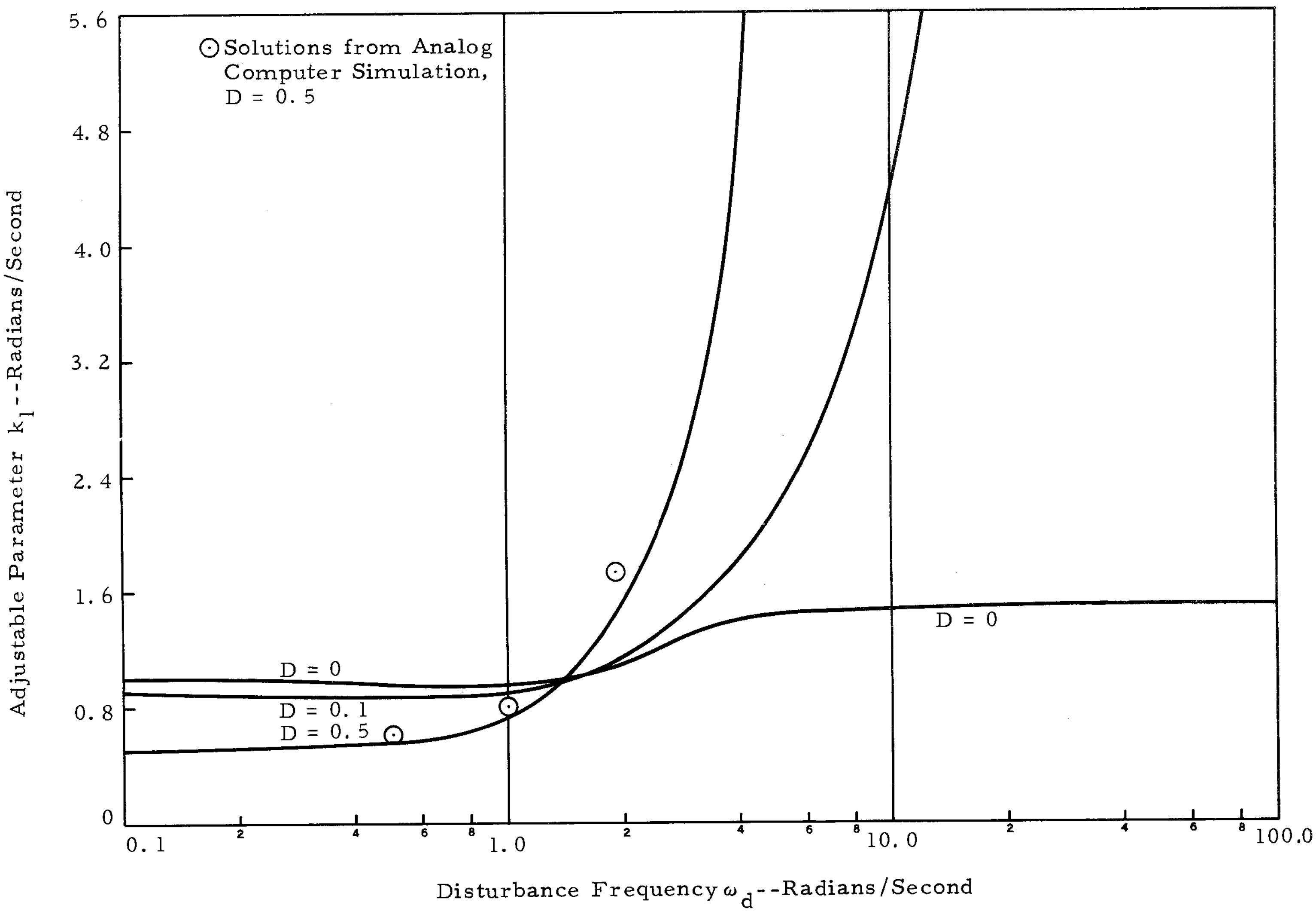


Figure 23. Steady-State Adjustable Parameter Frequency Sensitivity for Example 3. 3, $\omega_r = \omega_d$

IV. CONCLUSIONS

A unified review of MRAC system design and analysis techniques derived from pertinent literature has been presented. The review contains a design format with two basic variations and includes methods of solving commonly encountered problems. The two basic MRAC systems considered employed control system variables $\delta_i(t)$ and corresponding model variables $\delta_i'(t)$ to force the weighting filters in each adaptive loop. The sensitivity of the adjustable parameters P_i to reference r(t) and disturbance d(t) forcing functions were evaluated. A method was developed for analytically determining the P_i sensitivity to simultaneous steady-state, sinusoidal r(t) and d(t), thereby extending the results of Farmelo and Sammon (10) obtained for the single forcing-function case. This also represents an extension of frequency-domain analysis techniques to two particular inherently nonlinear system configurations. Three specific examples illustrated the application of the procedures developed.

In general, MRAC systems that force the weighting filter $WF_i(s)$ with $\delta_i(t)$ to obtain the weighting function $\overline{w}_i(t)$ displayed various sensitivity characteristics to all and any d(t). Adjustable parameters P_i are driven to limit values for cases of excitation by d(t) only and during simultaneous r(t) and d(t) excitation when D is greater than R with ω_r equal to ω_d (i. e., beat-frequency case). Otherwise, a compromise adaptive-point value of P_i with respect to desired values will be obtained as a function of ω_r, ω_d, R and D.

MRAC systems that force the weighting filter $WF_i'(s)$ with $\delta_i'(t)$ to obtain $\overline{w}_i'(t)$ displayed no sensitivity to d(t) unless a beat-frequency condition of ω_r equal to ω_d was encountered during simultaneous r(t) and d(t) excitation. The control of the adjustable parameter P_i would then encounter a beat-frequency sensitive stability boundary for a value of D greater than zero but less than R. Operation above this boundary was noted to cause P_i to be driven to a limit value. This type of system displayed greater adaptive-point sensitivity to excitation frequencies for differences between control system and model order than did a MRAC system employing $\delta_i(t)$ to obtain $\bar{w}_i(t)$. However, the use of $\delta_i'(t)$ to obtain $\bar{w}_i'(t)$ will offer the advantage of P_i insensitivity to disturbances except during beat-frequency cases.

It is recommended by this author that this investigation be extended in future work to include stochastic r(t) and d(t) forcing functions.

Symbol	Units	Definition
a_i, a_i'		General adaptive-loop proportionality constants
a_1, a_1'	$1/\text{seconds}^2$	Example adaptive-loop proportionality constants
C, C_r, C_d		Amplitude of sinusoidal $c(t)$
$c(t)$, $C(s)$, $c_r(t)$, $c_d(t)$, $\overline{c}(t)$		Control system response (output) state variables and the state vector
D		Amplitude of sinusoidal $d(t)$
$d(t)$, $D(s)$, $\overline{d}(t)$		Disturbance input state variables and the state vector
E, E_r		Amplitude of sinusoidal $e(t)$
$e(t)$, $E(s)$, $\overline{e}(t)$		Control system and model response error state variables and the state vector
EQ_i		General adaptive-loop error quantity
$F(s)$		Prefilter transfer-function
f_{jk}		$F(s)$ coefficients
$f^2(e)$, $f(e^2)$		Quadratic functions of response error
$G(s)$		Physical process transfer-function
g_{jk}		$G(s)$ coefficients
$H(s)$		Feedback compensation transfer-function
h_{jk}		$H(s)$ coefficients
$K(s)$		Feedforward compensation transfer-function
k_{jk}		$K(s)$ coefficients
k_1	$\frac{\text{Radians}}{\text{second}}$,	Example adjustable parameter
$\dot{k}_1$	$\frac{\text{Radians}}{\text{second}^2}$	Rate of change of k_1

Symbol	Units	Definition
$M(s)$		Model transfer-function
m_{jk}		$M(s)$ coefficients
m_i		Specific m_{jk} corresponding to P_i
MRAC		Model-reference, adaptive control
n		Number of state variables
P_i, ΔP_i, $\dot{P}_i$		General adaptive-loop adjustable parameter state, incremental change, and rate of change
PI		Performance index
$\underline{Q}$		Fixed diagonal matrix of response-error weighting coefficients
$q_1, q_2, \cdots q_n$		Diagonal elements of Q
R		Amplitude of sinusoidal $r(t)$
$r(t)$, $R(s)$, $\bar{r}(t)$		Reference input state variables and the state vector
s	$\frac{\text{radians}}{\text{seconds}}$	Laplacian operator (i.e., complex variable)
t	seconds	Time variable
TF_c, TF_m		Transfer-functions
$\bar{u}$		Unit vectors
W_i, W_i', W_{ir}, W_{id}, W'_{ir}		Amplitude of sinusoidal $w_i(t)$, $w_i'(t)$
$w_i(t)$, $W_i(s)$, $w_i'(t)$, $W_i'(s)$, $\bar{w}_i(t)$, $\bar{w}_i'(t)$		General adaptive-loop weighting state-variables and the state vectors
W_1, W_1', W_{1r}, W_{1d}, W'_{1r}	seconds	Amplitude of sinusoidal $w_1(t)$, $w_1'(t)$
$w_1(t)$, $W_1(s)$, $w_1'(t)$, $W_1'(s)$	seconds	Example adaptive-loop weighting state-variables

Symbol	Units	Definition
$WF_i(s)$, $WF_i'(s)$		General adaptive-loop weighting filter transfer-functions
$WF_1(s)$, $WF_1'(s)$	seconds	Example adaptive-loop weighting filter transfer-functions
Y, Y_r		Amplitude of sinusoidal $y(t)$
$y(t)$, $Y(s)$, $\bar{y}(t)$		Model response state variables and the state vector
α_{jk}, $\underline{\alpha}_d$, $\underline{\alpha}_n$		Control system transfer-function coefficients for polynomial representation and square matrices for first-order, time-domain representation
$\delta_i(t)$, $\Delta_i(s)$		General control system variable preceding P_i
$\delta_i'(t)$, $\Delta_i'(s)$		General model variable corresponding to $\delta_i(t)$
$\delta_1(t)$, $\Delta_1(s)$, $\delta_1'(t)$, $\Delta_i(s)$		Example control system variable preceding k, and corresponding model variable
τ_g, τ_m	seconds	Example time-constants
ϕ_{cr}, ϕ_{cd}, ϕ_{er}, ϕ_{wd}, ϕ_{wr}, ϕ_{wr}', ϕ_{yr}	radians	Functional-block response phase shift angles for sinusoidal forcing functions
ϕ_{dr}	radians	Initial phase shift of sinusoidal d(t) with respect to sinusoidal r(t)
ω_r, ω_d	$\frac{\text{radians}}{\text{second}}$	Forcing function frequencies

Subscripts

c		Control system response
d		Disturbance input (i.e., with respect to d(t))
e		Response error
g		Physical process coefficients

Symbol	Units	Definition
i		Subscript for general adaptive loop connotation
j		Subscript for polynomial coefficients corresponding to operator order
k		Subscript for numerator or denominator coefficients of polynomial ratio
m		Model coefficients
r		Reference input (i. e., with respect to r(t))
w		Weighting filter response
y		Model response

The prime symbol indicates adaptive-loops with model variable, weighting filter inputs.

V. BIBLIOGRAPHY

1. Aseltine, J. A., A. R. Mancini, and C. W. Sarture. A Survey of Adaptive Control Systems. Institute of Radio Engineers, Transactions on Automatic Control PGAC-6: 102-109. Dec. 1958.

2. Bekey, G. A. and R. E. Humphrey. Review of Model Matching Techniques for the Determination of Parameters in Human Pilot Models. November 8, 1962. 32 numb. leaves. (Space Technology Laboratories. Report No. 9865-6002-MU000.)

3. Birta, L. and M. D. Mesarovic. Synthesis of Interaction in Multivariable Control Systems. Automatica 2 (Great Britain): 15-39. June 1964.

4. Clark, Daniel C. The Model-Reference, Self-Adaptive Control System as Applied to the Flight Control of a Supersonic Transport. In: Preprints of Papers: 1964 Joint Automatic Controls Conference, Stanford, California. June 1964. p. 23-28.

5. Cowgill, P. N., M. R. Prichard, and W. K. Waymeyer. Minimizing Instrumentation by Using Model Feedback and Adaptive Control-System Techniques. Transactions of the Instrument Society of America 5: 156-164. April 1966.

6. Donalson, Dale D. and C. T. Leondes. A Model Referenced Parameter Tracking Technique for Adaptive Control Systems. Part 1: Principle of Adaptation. Institute of Electrical and Electronics Engineers, Transactions on Industry and General Applications, No. 68: 241-252. September 1963.

7. Donalson, Dale D. and C. T. Leondes. A Model Referenced Parameter Tracking Technique for Adaptive Control Systems. Part II: Stability Analysis by the Second Method of Lyapunov. Institute of Electrical and Electronics Engineers, Transactions on Industry and General Applications No. 68: 252-262. September 1963.

8. Dorf, Richard C. Time-Domain Analysis and Design of Control Systems. Reading, Massachusetts, Addision-Wesley Publishing Co., 1965. Various pagings.

9. Eveleigh, Virgil W. Adaptive Control Systems. Electro-Technology, Science and Engineering, Series 52: 79-98. April 1963.

10. Farmelo, Lee A. and John W. Sammon, Jr. Criteria for the Design of the Model of a Model Reference Adaptive Control System. Master's thesis. Cambridge, Massachusetts Institute of Technology, June 1962. 70 numb. leaves.

11. Gibson, John E. Nonlinear Automatic Control. New York, McGraw-Hill, 1963. Various pagings.

12. Hagen, Kenneth Edward. The Dynamic Behavior of Parameter-Adaptive Control Systems. Ph.D. Thesis. Cambridge, Massachusetts Institute of Technology, June 1964. 225 numb. leaves.

13. Leitmann, George (Ed.). Optimization Techniques with Applications to Aerospace Systems. New York, Academic Press, 1962. p. 206-214.

14. Li, Y. T. and H. P. Whitaker. Performance Characterization for Adaptive Control Systems. In: Proceedings of the First International Symposium on Optimizing and Adaptive Control, Rome, Italy, April 1962. Pittsburg, Instrument Society of America, 1962. p. 109-123.

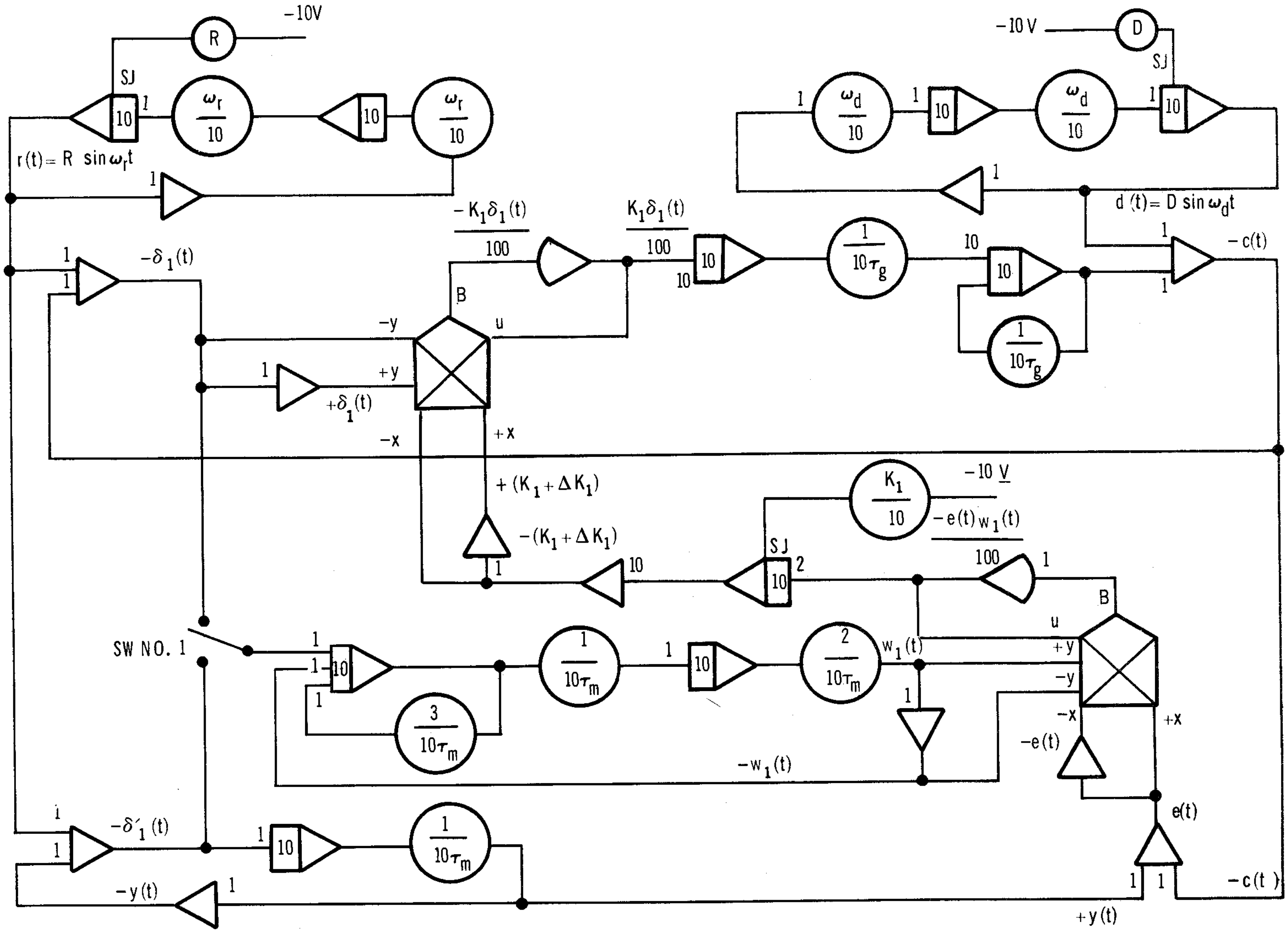
15. Osburn, Paul V. Investigation of a Method of Adaptive Control. Ph.D. Thesis. Cambridge, Massachusetts Institute of Technology, September 1961. 142 numb. leaves.

16. Rucker, Richard A. Real Time System Identification in the Presence of Noise. In: Proceedings of the 1963 Western Electronic Show and Convention, Section 2.3. San Francisco, California, August 1963. p. 1-10.

17. Seifert, William W. and Carl W. Steeg, Jr. Control Systems Engineering. New York, McGraw-Hill, 1960. Various pagings.

18. Whitaker, H. P. Design Capabilities of Model Reference Adaptive Systems. In: Proceedings of the National Electronics Conference XVIII, Chicago, Ill., October 1962. p. 240-249.

VI. APPENDIX



Analog Computer Simulation Circuit Diagram