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TITLE: EFFECTS OF COUPLED ATOMIC STATES ON THE RESONANCE SCATTERING
OF RADIATION

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The excitation and decay probabilities for resonance scattering
of radiation from an atom with two coupled excited states and one
ground state in an external static field are calculated as a function
of time and frequency. Various oscillatory behaviors are found in all
the probabilities. These oscillations depend on the frequency of
radiation, the external coupling and the energy difference between
the excited states. For the case where one of the excited states is
non-decaying, a "hole" is observed in the emission line at the
frequency equal to the frequency difference between the ground state
and the unperturbed non-decaying excited state. For non-degenerate
excited states, the two lines in the emitted beam have unequal
linewidths which depend on the energy difference between the excited
states and the external coupling. For degenerate excited states, the
two lines have equal linewidth which is half of the linewidth of the
decaying state. For two decaying excited states, one of the two
lines in the scattered beam is suppressed and the other is enhanced by the external coupling. For this case crossing and anticrossing signals are calculated in the total intensity of the scattered beam. These signals are computed for the $2^2P$ state in Li$^7$ as a function of an external magnetic field.
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Resonance Scattering of Radiation

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The study of scattering processes has always been of significant importance in Physics. The photon scattering, in particular, has provided information about energies, life-times and fine and hyperfine interactions of atoms, molecules and solids. The scattering of radiation has also been studied in the presence of an external perturbation.

Colegrove and others (1) demonstrated in 1959 that, under certain conditions, the crossing of two Zeeman levels of two excited states of an atom in a magnetic field produces an interference term in the scattered beam. This interference term has spatial dependence and also a dependence on the energy difference between the two levels. The study of such signals provides experimental values for the fine and the hyperfine structure constants of the levels involved. In 1963, Eck and others (2) found another kind of signal called anticrossing signal. This signal arises due to the presence of a perturbation which couples the two Zeeman levels which cross in the absence of the perturbation. Himmell and Fontana (3) have investigated such signals in atomic Hydrogen.

Hearn and Fontana (4) used the quantum theory of radiation developed by Heitler and Ma (5) to investigate the resonance scattering of radiation from a two-atom system. They have studied the frequency distribution of the scattered radiation as a function of the inter-
atomic distance. The present work investigates the resonance scattering of radiation from a single atom with two excited states and one ground state. The two excited states are coupled by an external static perturbation. The incident radiation is assumed to be of constant intensity $I_0$ per unit area per unit solid angle per unit frequency (white light). The method of solving the wave equation is the same as used by Hearn and Fontana (4). This approach gives all the probabilities including the final state probability as a function of time whereas the Wigner Weisskopf (6) approach does not give the probability for the final state directly.

The absorption and the emission of radiation is studied here as a function of time and frequency for different special cases. These special cases are: (i) one of the excited states is non-decaying. (ii) both the excited states decay with equal decay constants. The effects of the external coupling on the lineshape and linewidth of the scattered radiation is investigated.

The intensity of the scattered radiation is calculated for a given direction and polarization of the incident and the scattered beams. As an application, this intensity is computed for the $2^2P$ state in Li$^+$ and is compared with the experimental signals observed by Wieder and Eck (7).
II. GENERAL THEORY

The system, under study, consists of an isolated atom which interacts with a radiation field and a time independent external perturbation. The atom is assumed to have two excited states and one ground state. The external perturbation couples only the two excited states. Initially, the atom is in the ground state and a beam of white light (a continuous spectrum of radiation with constant energy $I_0$ per unit area per unit solid angle and per unit frequency) is incident on it.

A. Method of Solution of the Wave Equation

The method used for solving the wave equation is a transformation of the wave equation into energy space (8). The interaction representation is used. The units are such that $\hbar = c = m = 1$ where $\hbar$ is the Plank constant divided by $2\pi$, $c$ the speed of light in vacuum and $m$ the mass of the electron at rest.

If $\Psi(t)$ is the wave vector in the Schrödinger representation, then

$$|\Psi(t)\rangle = e^{i\mathcal{H}_0 t} |\Psi(0)\rangle$$  \hspace{1cm} (2-1)

is the wave vector in the interaction representation. $\mathcal{H}_0$ is the Hamiltonian of the atom plus that of the radiation field, when the two are non-interacting and there is no external perturbation field. The Hamiltonian of the whole system in the Schrödinger representation then is
\[ \mathcal{H} = H_0 + H' \]  

(2-2)

where \( H' \) consists of \( V \), the external perturbation, plus \( H \), the interaction of the atom with radiation field.

In the interaction representation, the wave equation is given by the expression

\[ i \frac{\partial}{\partial t} |\Psi(t)\rangle = e^{iH_0 t} H' e^{-iH_0 t} |\Psi(t)\rangle . \]  

(2-3)

Considering only processes involving one photon, the state vector of the system in the interaction representation is given by

\[ |\Psi(t)\rangle = b_\alpha(t)|0\rangle + \sum_i b_{i\lambda}(t)|i\rangle + \sum_f b_f(t)|f\rangle \]

(2-4)

where \( |0\rangle \) represents the ground state; \( |i\rangle \), \( |j\rangle \) the excited states with a photon absorbed and \( |f\rangle \) the final states. All these states are the eigenstates of \( H_0 \). In terms of the atomic and photon's states, they are written as

\[ |0\rangle = |c\rangle |0\rangle_R \]

\[ |i\rangle = |\alpha\rangle |1-\kappa\rangle_R \]

\[ |f\rangle = |b\rangle |1-\kappa\rangle_R \]

and

\[ |f\rangle = |c\rangle |1-k\rangle_R |k\rangle_R . \]
The state vectors $|\omega\rangle_R$, $|\mathbf{k}_{-}\rangle_R$ and $|\mathbf{R}_{\mathbf{X}}\rangle_R$ represent the states of the radiation field with no photons present, an absorbed photon with wave vector $\mathbf{k}_{\mathbf{X}}$ and polarization $\hat{e}_{\mathbf{X}}$, and an emitted photon with wave vector $\mathbf{R}_{\mathbf{X}}$ and polarization $\hat{e}_{\mathbf{X}}$, respectively.

In Eq. (2-4) the summations over $i$ and $j$ are over frequency, direction and polarization of the incident radiation, and the summation over $f$ is over frequencies, directions and polarizations of the incident and the emitted radiations.

The state vector $|\Psi(t)\rangle$ can be written in a simplified form as

$$|\Psi(t)\rangle = \sum_{m} b_{m}(t) |m\rangle \quad (2-5)$$

where $m$ includes all the quantum numbers defining the eigen states of $H_0$ in the interaction representation.

By inserting Eq. (2-5) into Eq. (2-3), multiplying by $\langle n |$ and integrating over all space, one obtains the following set of equations

$$i \frac{d}{dt} b_{n}(t) = \sum_{m} H_{nm}^{'} b_{m}(t) e^{i(E_{n} - E_{m})t} \quad (2-6)$$

where $H_{nm}^{'}$ denotes the matrix elements of $H^{'}$.

$$H_{nm}^{'} = \langle n | H^{'} | m \rangle \quad (2-7)$$

Since at $t = 0$, the system is in the ground state $|\omega\rangle$, the initial conditions are:

$$b_{n}(0) = 1 \quad (2-8)$$

$$b_{n}(0) = 0 \quad \text{for} \quad n \neq 0$$
and

\[ b_n(t) = b_o(t) = 0 \quad \text{for}\quad t < 0. \]

In order to satisfy these conditions a term involving a Dirac delta function is added (8) in Eq.(2-6), which gives

\[ i \frac{d}{dt} b_n(t) = \sum_m H_n^{mm} b_m(t) e^{i(E_n-E_m)t} + i \delta_{n0} \delta(t). \quad (2-9) \]

To solve Eq.(2-9) the \( b_n(t) \) are transformed by using the following Fourier transformation:

\[ b_n(t) = -\frac{i}{2\pi} \int_{-\infty}^{\infty} dE G_n(E) e^{i(E_n-E)t}. \quad (2-10) \]

In a similar fashion the representation of the Dirac delta function is used:

\[ \delta(t) = -\frac{i}{2\pi} \int_{-\infty}^{\infty} dE e^{i(E_0-E)t}. \quad (2-11) \]

With these transformations, Eq.(2-9) reduces to

\[ \int_{-\infty}^{\infty} dE (E-E_n) G_n(E) e^{i(E_n-E)t} = \int_{-\infty}^{\infty} dE \left[ \sum_m H_n^{mm} G_m(E) + \delta_{n0} \right] e^{i(E_n-E)t}. \quad (2-12) \]
This equation is satisfied if

\[(E - E_n) G_n(E) = \sum_{m} H_{nm} G_{m}(E) + \delta_{n0} \]  

(2-13)

This gives a set of equations for the \( \alpha(E) \):

\[(E - E_{\alpha}) G_{\alpha}(E) = H_{0\alpha} G_{\alpha}(E) + \sum_{i} H_{i\alpha} G_{i}(E) + \sum_{j} H_{\alpha j} G_{j}(E) \]

\[+ \sum_{f} H_{0f}^{\prime} G_{f}(E) + 1 , \]  

(2-14)

\[(E - E_{\xi}) G_{\xi}(E) = H_{0\xi} G_{\alpha}(E) + \sum_{i} H_{i\xi} G_{i}(E) \]

\[+ \sum_{f} H_{0f}^{\prime} G_{f}(E) + \sum_{j} H_{\xi j}^{\prime} G_{j}(E) , \]  

(2-15)

\[(E - E_{j}) G_{j}(E) = H_{j0} G_{\alpha}(E) + \sum_{i} H_{i j} G_{i}(E) \]

\[+ \sum_{f} H_{jf}^{\prime} G_{f}(E) + \sum_{j} H_{j j}^{\prime} G_{j}(E) \]  

(2-16)

and

\[(E - E_{j}) G_{j}(E) = H_{j0} G_{\alpha}(E) + \sum_{i} H_{i j} G_{i}(E) \]

\[+ \sum_{f} H_{jf}^{\prime} G_{f}(E) + \sum_{j} H_{j j}^{\prime} G_{j}(E) \]  

(2-17)

where

\[E_{\alpha} = E_{c} \]

\[E_{\xi} = E_{a} - k_{\sigma} \]  

(2-18)

\[E_{j} = E_{b} - k_{\sigma} \]

\[E_{j} = E_{c} - k_{\sigma} + k_{\lambda} . \]
Since only processes involving single photon are considered, the following matrix elements of $H$ are zero;

\[ H_{f_0} = 0 \]
\[ H_{o_f} = 0 \]

and also one finds that

\[ H_{i',i'} = V_{aa} \delta_{k_r k'_r} \]
\[ H_{j',j'} = V_{bb} \delta_{k_r k'_r} \]
\[ H_{i',j'} = V_{ab} \delta_{k_r k'_r} \]
\[ H_{o',i'} = V_{cc} \delta_{k_r k'_r} \]

\[ H_{o_j} = \langle 01| H + V | i \rangle = \langle 01| H | i \rangle \]
\[ H_{i',j'} = H_{i,j} \delta_{k_r k'_r} \]
\[ H_{j',j'} = H_{j,j} \delta_{k_r k'_r} \]

where

\[ V_{\ell,m} = \langle \ell | V | m \rangle \quad ; \quad \ell, m = a, b, c. \quad (2-21) \]

Substituting these matrix elements into Eq. (2-14) to Eq. (2-17), one gets the following equations

\[ (E - E_{s'}) G_s(E) = \sum_i H_{o,i} G_i(E) + \sum_j H_{o,j} G_j(E) + 1 \quad (2-22) \]
\[(E - E_i') G_i'(E) = H_{i0} G_0(E) + V_{ab} G_j'(E) + \sum_f H_{if} G_f(E) \quad (2-23)\]

\[(E - E_j') G_j(E) = H_{j0} G_0(E) + V_{ba} G_i(E) + \sum_f H_{jf} G_f(E) \quad (2-24)\]

and

\[(E - E_f') G_f(E) = H_{fi} G_i(E) + H_{fj} G_j(E) \quad (2-25)\]

where

\[E_o' = E_c + V_{cc}\]
\[E_i' = E_i + V_{aa}\]
\[E_j' = E_j + V_{bb}\]
\[E_f' = E_f + V_{cc} \quad (2-26)\]

The summation over \( f \) in Eq.(2-23) and Eq.(2-24) is over frequency, direction and polarization of the emitted radiation only, because the matrix elements \( H_{ij} \) and \( H_{ff} \) are zero if the state \( |i> \) contains an absorbed photon which differs from the photon in \( |i> \) and \( |j> \), respectively.

Equation(2-25) does not have a unique solution for \( G_f(E) \) because division by \( (E - E_f') \) is not unique. A solution of the equation, which satisfies the initial condition, however, can be obtained in the following way \( ^8 \)

\[G_f(E) = \zeta(E - E_f') \left[ H_{fi} G_i(E) + H_{fj} G_j(E) \right] \quad (2-27)\]
where $\zeta(E-E_y)$ is the Zeta function which has the following properties:

$$
\zeta(x) = -i \int_0^\infty e^{i\pi t} dt = \lim_{\rho \to 0} \frac{1}{x + i\rho},
$$

$$
= \mathcal{P} \left( \frac{1}{x} \right) - i \pi \delta(x),
$$

$$
x \zeta(x) = 1,
$$

$$
\int_{-\infty}^{\infty} \zeta(x) e^{i\pi t} dx = \begin{cases} 
0 & \text{for } t > 0 \\
-2\pi i & \text{for } t < 0
\end{cases}
$$

and

$$
\lim_{\tau \to \infty} \zeta(x) e^{\pm i\pi t} = \begin{cases} 
0 & \\
-2\pi i \delta(x)
\end{cases}
$$

In Eq. (2-28), $\mathcal{P}(1/x)$ is the principal value of $1/x$, which behaves like $1/x$ everywhere where $x \neq 0$ and vanishes at $x = 0$.

By substituting $G_f(E)$ from Eq. (2-27) into Eq. (2-23) and Eq. (2-24), one obtains the following equations

$$(E-E_i + i\frac{1}{2} \gamma_{i;i}) G_i(E) = H_{i0} G_0(E) + V_{ab} G_j(E)$$

$$-i\frac{1}{2} \gamma_{i;j} G_j(E)$$

$$(E-E_j + i\frac{1}{2} \gamma_{j;j}) G_j(E) = H_{j0} G_0(E) + V_{ba} G_i(E)$$

$$-i\frac{1}{2} \gamma_{j;i} G_i(E)$$
where

\[-\frac{i}{2} \gamma_{ij}(E) = \sum_{f} H_{i,f} H_{f,j} (E-E') \]  

(2-33)

In Eq.(2-31) and Eq.(2-32), the state vectors for the absorbed radiation field are the same in the two states \( |i\rangle \) and \( |j\rangle \) because the matrix element \( V_{ab} \) vanishes if \( |i\rangle \) and \( |j\rangle \) have different photons.

It can be shown that the \( \gamma_{ij} \) for \( i \neq j \) vanish whenever \( |i\rangle \) and \( |j\rangle \) are states of good angular momentum \( (9) \). For a proof see Ref. (11).

This is the case here and thus \( \gamma_{ij} = 0 \) for \( i \neq j \).

From Eq.(2-31) and Eq.(2-32), the following expressions for \( G_{i}(E) \) and \( G_{j}(E) \) are obtained:

\[
G_{i}(E) = \frac{[H_{i0} (E-E_{0} + \frac{i}{2} \gamma_{ii}) + V_{ab} H_{jo}]}{(E-E_{0} + \frac{i}{2} \gamma_{ii}) (E-E_{0} + \frac{i}{2} \gamma_{jj}) - |V_{ab}|^2} G_{0}(E) 
\]

(2-34)

\[
G_{j}(E) = \frac{[H_{j0} (E-E_{0} + \frac{i}{2} \gamma_{jj}) + V_{ba} H_{io}]}{(E-E_{0} + \frac{i}{2} \gamma_{jj}) (E-E_{0} + \frac{i}{2} \gamma_{ii}) - |V_{ab}|^2} G_{0}(E) 
\]

(2-35)

These expressions are substituted in Eq.(2-22), which yields

\[
G_{0}(E) = \frac{1}{(E-E_{0} + \frac{i}{2} \gamma(E))} \]

where

\[-\frac{i}{2} \gamma(E) = \sum_{k} \left[ H_{0} V_{ab} H_{j} + H_{0} V_{ba} H_{i} \right] + \left| H_{0} \right|^2 (E-E_{0} + \frac{i}{2} \gamma_{ii}) \]

(2-34)

\[
\left[ (E-E_{0} + \frac{i}{2} \gamma_{ii}) (E-E_{0} + \frac{i}{2} \gamma_{jj}) - |V_{ab}|^2 \right]^{-1} . \]

(2-36)
The summation over $k_r$ is over frequency, direction and polarization of the incident radiation. The states $|i\rangle$ and $|j\rangle$ have the same absorbed photon $E$.

Combining Eq. (2-34) and Eq. (2-35), one obtains

$$G_i(E) = \frac{H_{io} \left( E - E_j' + \frac{i}{2} \gamma_{ji} \right) + V_{ab} H_{jo}}{\left[ (E - E_j' + \frac{i}{2} \gamma_{ji}) (E - E_j' + \frac{i}{2} \gamma_{ij}) - |V_{ab}|^2 \right] \left[ E - E_j + \frac{i}{2} \Gamma \right]}$$

(2-37)

and

$$G_j(E) = \frac{H_{jo} \left( E - E_i' + \frac{i}{2} \gamma_{ij} \right) + V_{ba} H_{io}}{\left[ (E - E_i' + \frac{i}{2} \gamma_{ij}) (E - E_i' + \frac{i}{2} \gamma_{ji}) - |V_{ab}|^2 \right] \left[ E - E_i + \frac{i}{2} \Gamma \right]}$$

(2-38)

B. Determination of the Poles in G(E)

The real parts of $\gamma_{ii}(E)$ and $\gamma_{jj}(E)$ have the following E dependence (see Appendix A-2):

$$\text{Re} \left( \gamma_{ii}(E) \right) = (E - E_i' + k_r) D_a$$

and

$$\text{Re} \left( \gamma_{jj}(E) \right) = (E - E_j' + k_r) D_b$$

(2-39)

where $D_a$ and $D_b$ are defined in Eq. (A-9) and Eq. (A-14).

In Appendix A-3, it is shown that the real part of $\Gamma$ does not depend on E. Thus, absorbing the imaginary parts of $\gamma_{ii}$, $\gamma_{jj}$ and $\Gamma$ in the energies of the atomic states (see Appendix A-2), one can write the denominator of G(E) as

$$\left[ \left( E - E_i' + \frac{i}{2} \text{Re} (\gamma_{ii}) \right) \left( E - E_j' + \frac{i}{2} \text{Re} (\gamma_{jj}) \right) - |V_{ab}|^2 \right] \left[ E - E_o + \frac{i}{2} \text{Re} (\Gamma) \right]$$

(2-40)
Substituting the real parts of $\gamma_{ii}$ and $\gamma_{jj}$ from Eq.(2-39) and replacing $\text{Re}(\Gamma')$ by $\Gamma'$, one gets

$$\left[ \{ E(1+iD_a/2)-E_i' + (k_r-E_c')iD_a/2 \} \{ E(1+iD_b/2)-E_j' + (k_r-E_c')iD_b/2 \} \right] - |V_{ab}|^2 \right] [E-E_{e'} + i\Gamma/2] . \tag{2-41}$$

Combining Eq.(2-18), Eq.(2-26) and Eq.(2-41) and rearranging the terms, one can write the denominator of $G(E)$ as

$$(1+iD_a/2)(1+iD_b/2) \left[ \{ E + k_r - \frac{E_a' (1-iD_a/2)}{1 + D_a^2/4} \} - \frac{E_c' D_b^2}{4(1 + D_b^2/4)} \right]$$

$$- |V_{ab}|^2 \left[ \{ 1 - D_aD_b/4 - i(D_a+D_b) \} / \{ (1+D_a^2/4)(1+D_b^2/4) \} \right] [E-E_{e'} + i\Gamma/2] .$$

Since $D_a$ and $D_b$ are very small (see Eq.(A-13)), one can neglect the terms $D_a^2/4$, $D_b^2/4$ and $D_aD_b/4$ compared to unity and one gets

$$(1+iD_a/2)(1+iD_b/2) \left[ \{ E - E_a' + k_r - E_c' D_a/4 + i\gamma_a/2 \} \times \{ E - E_b' + k_r - E_c' D_b/4 \} \right]$$

$$- \frac{E_c' D_b^2}{4} + i\gamma_b/2 \right] - |V_{ab}|^2 \left[ \{ 1 - i(D_a+D_b) \} \right] [E-E_{e'} + i\Gamma/2] . \tag{2-42}$$

where

$$\gamma_a = (E_a' - E_c') D_a \tag{2-43}$$

and

$$\gamma_b = (E_b' - E_c') D_b .$$
It is shown in Chapter III that $\gamma_a$ and $\gamma_b$, as defined in Eq. (2-43) are the decay constants of the unperturbed atomic states $|a\rangle$ and $|b\rangle$, respectively. It is worth mentioning here that $\gamma_a$ and $\gamma_b$ are the real parts of $\gamma_{ii}(E)$ and $\gamma_{jj}(E)$ evaluated at $E$ equal to $E'_i$ and $E'_j$, respectively.

Combining Eq. (2-37) to Eq. (2-39), Eq. (2-42) and Eq. (2-43), one gets

$$G_i(E) = \left[ H_{io} (E - E'_b - E'_c D^2_{b}/4 + R_{k}) + V_{ab} H_{jo} + i/2 (H_{io} \gamma_{b} - V_{ab} H_{jo} D_{b}) \right]$$

and

$$G_j(E) = \left[ H_{jo} (E - E'_a - E'_c D^2_{a}/4 + k_{r}) + V_{ba} H_{io} + i/2 (H_{jo} \gamma_{a} - V_{ba} H_{io} D_{a}) \right]$$

Since the matrix elements $H_{io}$ and $V_{ab}$ are of the order of $h_{10}$ and $\gamma$, respectively, and since $D_{b} \sim 10^{10}$ (See Appendix A-2), the second term in $(H_{io} \gamma_{b} - V_{ab} H_{jo} D_{b})$ can be neglected compared to the first one. Thus one can set

$$H_{io} \gamma_{b} = V_{ab} H_{jo} D_{b} \approx H_{io} \gamma_{b}$$

(2-46)
and by similar arguments

\[ H_{jo} \tau_a - V_{ba} H_{lo} D_a \approx H_{jo} \tau_a \quad (2-47) \]

Similarly in Eq. (2-44) and Eq. (2-45), the terms \( E_c' D_a^1/4 \) and \( E_c' D_b^1/4 \)
are also very small compared to the other terms and therefore can
be neglected without significant error. Thus one gets

\[
G_i(\varepsilon) = \left[ H_{io} (E - E_a' + k_r) + V_{ab} H_{lo} + i/2 H_{io} \gamma_a \right] / \left[ \{ (E - E_a' + k_r + i \gamma_a/2) - |V_{ab}|^2 (1 - i (D_a + D_b)/2) \} \right] 
\]

\[
x (1 + i \gamma_a/2) \right] . \quad (2-48)
\]

The Fourier coefficients \( G_i(\varepsilon) \) and \( G_j(\varepsilon) \) have three poles.
The first two poles are obtained from the roots of the equation

\[
(E - E_a' + k_r + i \gamma_a/2)(E - E_b' + k_r + i \gamma_b/2) - |V_{ab}|^2 (1 - i (D_a + D_b)/2) = 0 \quad (2-49)
\]

and the third pole is at

\[
E_3' = E_c' - i \gamma/2 \quad . \quad (2-50)
\]

The roots of Eq. (2-49) are obtained by the method discussed
in Appendix A-3. The result is

\[
\begin{align*}
\left\{ \begin{array}{c}
E'_1 \\ E'_2
\end{array} \right\} &= \frac{E'_a + E'_b}{2} - k_\infty + R - \imath (x_3 \pm 1)
\end{align*}
\] (2-51)

where \(x_3\), \(R\) and \(1\) are defined in Eq.(A-26) and Eq.(A-27).

C. Evaluation of the Probability Amplitudes

The probability amplitude \(b_n(t)\) is calculated by evaluating the integral

\[
b_n(t) = -\frac{1}{2\pi \imath} \int_{-\infty}^{\infty} G_n(E) \frac{\imath (E_n - E) t}{E - E'_n} dE
\]

by contour integration.

Thus, using the expressions for \(G_n(E)\) from Eq.(2-27), Eq.(2-35), Eq.(2-37), and Eq.(2-38), one gets the following integrals for the \(b_n(t)\):

\[
b_\circ(t) = -\frac{1}{2\pi \imath} \int_{-\infty}^{\infty} \frac{\imath (E_\circ - E) t}{E - E'_\circ} dE
\]

\[
b_\ell(t) = -\frac{1}{2\pi \imath} \int_{-\infty}^{\infty} \frac{H_{\ell 0} (E - E'_\ell + \imath/2 T_\ell) + V_{ab} H_{\ell 0}}{(1 + \imath D_{b/2}) (E - E'_\ell - E'_\ell') (E - E'_\ell)(E - E'_\ell')} \frac{\imath (E_\ell - E) t}{E - E'_\ell} dE
\]

\[
b_j(t) = -\frac{1}{2\pi \imath} \int_{-\infty}^{\infty} \frac{H_{j 0} (E - E'_j + \imath/2 T_j) + V_{ba} H_{j 0}}{(1 + \imath D_{b/2}) (E - E'_j - E'_j') (E - E'_j)(E - E'_j')} \frac{\imath (E_j - E) t}{E - E'_j} dE
\]

(2-52)
and

\[ b_j(t) = -\frac{i}{2\pi t} \int_{-\infty}^{\infty} dE \left( E - E_{f} \right) \left( H_{fj} \left( H_{ij} \left( E - E_{i} + i/2 \gamma_b \right) + V_{ab} H_{jo} \right) \right) \]

\[ \times \left( 1 + i \frac{D_{a}/2}{1 + i \frac{D_{b}/2}{1 + i \gamma_b/2}} \right) \left( E - E_{j} \right) \left( E - E_{f} \right) \left( E - E_{i} \right) \left( E - E_{j} \right) \left[ \left( E - E_{i} \right) \left( E - E_{j} \right) \right] \cdot e^{i(E_{j} - E)^{t}} \quad (2-53) \]

where \( E'_{i}, E'_{j} \) and \( E'_{j} \) are defined in Eq.(2-50) and Eq.(2-51).

The factors \( (1 + i \frac{D_{a}/2}{1 + i \frac{D_{b}/2}{1 + i \gamma_b/2}}) \) in the probability amplitudes \( b_{i}(t) \) and \( b_{j}(t) \) go to unity when their modulus square is taken \((D_{a} < 1 \text{ and } D_{b} < 1)\).

Since the imaginary parts of the poles are negative, the path of integration in the integrals in Eq.(2-52) and Eq.(2-53) is taken to be a clockwise infinite semicircular contour in the lower half of the complex plane. In the case of \( b_{o}(t) \), \( b_{c}(t) \) and \( b_{j}(t) \) all the poles lie in the lower half of the complex plane and thus, they all contribute to the integral. Thus using the method of residues, one gets

\[ b_{o}(t) = e^{i \gamma_{c} \cdot t - \gamma_{f} \cdot t/2} \quad (2-54) \]

\[ b_{c}(t) = \frac{H_{ic} \left( E_{i} - E_{j} + i/2 \gamma_b \right) + V_{ab} H_{jo}}{(E'_{i} - E'_{j})(E'_{j} - E'_{i})} e^{i(E_{i} - E_{j})t} \]

\[ + \frac{H_{ic} \left( E_{i} - E_{j} + i/2 \gamma_b \right) + V_{ab} H_{jo}}{(E'_{j} - E'_{i})(E'_{i} - E'_{j})} e^{i(E_{j} - E_{i})t} \]
\[ b_j(t) = \frac{H_{\text{ho}}(E_1' - E_c' + i\Gamma_{\text{a}}/2) + V_{b\alpha} H_{\text{ho}}}{(E_1' - E_2') (E_1' - E_3')} \cdot e^{i(E_j - E_1')t} \]

\[ + \frac{H_{\text{ho}}(E_2' - E_c' + i\Gamma_{\text{a}}/2) + V_{b\alpha} H_{\text{ho}}}{(E_2' - E_1') (E_2' - E_3')} \cdot e^{i(E_j - E_2')t} \]

\[ + \frac{H_{\text{ho}}(E_3' - E_c' + i\Gamma_{\text{a}}/2) + V_{b\alpha} H_{\text{ho}}}{(E_3' - E_1') (E_3' - E_2')} \cdot e^{i(E_j - E_3')t} \] (2-56)

The Zeta function \( \zeta(E-E_j') \) in the integral in Eq.(2-53) makes the path of integration along the real axis to go around a semicircle of infinitely small radius, in the upper half of the complex plane, centered at \( E = E_f' \) (see Fig.2-1). The remaining path of integration is the same as in the previous cases.

Fig.2-1. Position of poles in the complex plane
Thus, using the method of residues, Eq.(2-53) gives

\[ b_f(t) = \left[ H_{fi} \{ H_{ic} (E_f - E'_j + i \gamma_b / 2) + V_{ab} H_{jo} \} \right. \]

\[ + H_{fi} \left\{ H_{jo} (E_c - E'_j + i \gamma_a / 2) + V_{ba} H_{co} \} \right] \]

\[ \times \left[ (E_f' - E'_j)(E_f' - E'_2)(E_f' - E_3') \right]^{-1} e^{-i V_{cc} t} \]

\[ + [H_{fi} \{ H_{ic} (E_i - E'_j + i \gamma_b / 2) + V_{ab} H_{jo} \} \]

\[ + H_{jo} \{ H_{ic} (E_c - E'_j + i \gamma_a / 2) + V_{ba} H_{co} \}] \]

\[ \times \left[ (E_i' - E'_j)(E_i' - E'_2)(E_i' - E_3') \right]^{-1} e^{i (E_f' - E_i') t} \]

\[ + [H_{ji} \{ H_{ic} (E_i - E'_j + i \gamma_b / 2) + V_{ab} H_{jo} \} \]

\[ + H_{ji} \{ H_{ic} (E_i - E'_j + i \gamma_a / 2) + V_{ba} H_{co} \}] \]

\[ \times \left[ (E_i' - E'_j)(E_i' - E'_2)(E_i' - E_3') \right]^{-1} e^{-i (E_f' - E_i') t} \]

\[ + [H_{fi} \{ H_{ic} (E_i - E'_j + i \gamma_b / 2) + V_{ab} H_{jo} \} \]

\[ + H_{jc} \{ H_{ic} (E_c - E'_j + i \gamma_a / 2) + V_{ba} H_{co} \}] \]

\[ \times \left[ (E_c - E'_j)(E_c - E'_2)(E_c - E_3') \right]^{-1} e^{i (E_f' - E_c') t} \]

\[ + [H_{ji} \{ H_{ic} (E_i - E'_j + i \gamma_b / 2) + V_{ab} H_{jo} \} \]

\[ + H_{ij} \{ H_{ic} (E_i - E'_j + i \gamma_a / 2) + V_{ba} H_{co} \}] \]

\[ \times \left[ (E_i' - E'_j)(E_i' - E'_2)(E_i' - E_3') \right]^{-1} e^{-i (E_f' - E_i') t} \]

\[ + [H_{ji} \{ H_{ic} (E_i - E'_j + i \gamma_b / 2) + V_{ab} H_{jo} \} \]

\[ + H_{ij} \{ H_{ic} (E_i - E'_j + i \gamma_a / 2) + V_{ba} H_{co} \}] \]

\[ \times \left[ (E_i' - E'_j)(E_i' - E'_2)(E_i' - E_3') \right]^{-1} e^{i (E_f' - E_i') t} \].

\[ (2-57) \]
From Eq. (2-54), it is seen that the probability \( b_0(t) \) decays monotonically with a single decay constant \( r \).

The probability amplitudes \( b_i(t) \) and \( b_j(t) \) have three terms. Each term has two time dependent factors, one is oscillatory and the other is exponentially decaying. The oscillatory behaviour depends on the frequencies of emitted and absorbed radiations, the coupling strength of the external perturbation and the energy difference between the two excited states. The coupling strength and the energy difference also affect the decay constants of \( b_i(t) \) and \( b_j(t) \).

The amplitude of the final states, \( b_f(t) \) in Eq. (2-57) has four terms. The first term is pure oscillatory in time and the other three terms have an oscillatory and an exponentially decaying factors with different oscillatory frequency and decay constant. The decay constants are the same as the ones in \( b_i(t) \) and \( b_j(t) \).

As \( t \to \infty \), only the first term in \( b_f(t) \) survives and all other probability amplitudes decay to zero. Thus

\[
\begin{align*}
\mathcal{B}_f(t \to \infty) &= [H_{fi} + \sum_{ij} \delta_{ij} \{ H_{ij} (E_j - E_j') + \mathcal{L}_{ij} \} + V_{ab} H_{0j} ] + H_{fj} \{ H_{0j} (E_j - E_j' + \mathcal{L}_{ij} / 2) \\
&\quad + V_{ba} H_{i0} \} \times [(E_j' - E_i)(E_j' - E_j'')(E_j - E_j')^{-1} - \mathcal{V}_{cc} t] - j \mathcal{V}_{cc} t 
\end{align*}
\]

This gives the probability amplitude for the process where the atom absorbs a photon with wave vector \( \mathbf{k}_\nu \) and polarization \( \mathbf{e}_\nu \) and emits a photon with wave vector \( \mathbf{k}_\lambda \) and polarization \( \mathbf{e}_\lambda \).

The probability of emission of a photon with wave vector \( \mathbf{k}_\lambda \) and polarization \( \mathbf{e}_\lambda \) is obtained by summing over the incident frequencies:
\[ P(\hat{\mathbf{e}}_r, k_\lambda) = \sum_{\mathbf{k}_r} |b_j(\omega)|^2 \]  

(2-59)

The summation over \( k_r \) is over the frequency of the absorbed photons only. The symbol \( k_\lambda \) in the parenthesis stands for the frequency, direction and polarization of the scattered beam and \( \hat{\mathbf{e}}_r \) stands for the direction and polarization of the incident beam.

For a continuous distribution of incident frequencies, the \( P(\hat{\mathbf{e}}_r, k_\lambda) \) is given by (see Appendix B-1)

\[
P(\hat{\mathbf{e}}_r, k_\lambda) = e^{i\mathbf{k}_r \cdot \mathbf{r}} / (L^2 \pi^{3/2} k_0^3) \cdot \left[ \{ (k+\Delta/2)^2 + \gamma_b^2/4 \} |P_{\alpha\lambda}^\wedge|^2 |P_{\beta\lambda}^\wedge|^2 \\
+ \{ (k-\Delta/2)^2 + \gamma_a^2/4 \} |P_{\alpha\lambda}^\wedge|^2 |P_{\beta\lambda}^\wedge|^2 + V^2 \{ |P_{\alpha\lambda}^\wedge|^2 |P_{\beta\lambda}^\wedge|^2 + |P_{\alpha\lambda}^\wedge|^2 |P_{\beta\lambda}^\wedge|^2 \\
+ 4 \text{Re} (P_{\alpha\lambda}^\wedge P_{\beta\lambda}^\wedge) \text{Re} (P_{\alpha\lambda}^\wedge P_{\beta\lambda}^\wedge) - 2 \text{Re} (P_{\alpha\lambda}^\wedge P_{\beta\lambda}^\wedge) \\
+ \{ (k+\Delta/2) |P_{\alpha\lambda}^\wedge|^2 + (k-\Delta/2) |P_{\beta\lambda}^\wedge|^2 \} \text{Re} (P_{\alpha\lambda}^\wedge P_{\beta\lambda}^\wedge) \\
+ (\gamma_a^2 |P_{\beta\lambda}^\wedge|^2 - \gamma_b^2 |P_{\alpha\lambda}^\wedge|^2) \frac{1}{2} \text{Im} (P_{\alpha\lambda}^\wedge P_{\beta\lambda}^\wedge) \\
+ (\gamma_b^2 |P_{\alpha\lambda}^\wedge|^2 - \gamma_a^2 |P_{\beta\lambda}^\wedge|^2) \frac{1}{2} \text{Im} (P_{\alpha\lambda}^\wedge P_{\beta\lambda}^\wedge) \\
+ \{ \gamma_a (k+\Delta/2) - \gamma_b (k-\Delta/2) \} \text{Im} (P_{\alpha\lambda}^\wedge P_{\beta\lambda}^\wedge P_{\alpha\lambda}^\wedge P_{\beta\lambda}^\wedge) \\
x \left[ \{ (k+\Delta/2)^2 + \gamma_b^2/4 \} + (k+\Delta/2)^2 \right] \frac{1}{2} \text{Re} (P_{\alpha\lambda}^\wedge P_{\beta\lambda}^\wedge P_{\alpha\lambda}^\wedge P_{\beta\lambda}^\wedge) \\
- \frac{2V^2 (k+\Delta/2)(k+\Delta/2) + (V^2 + \gamma_a \gamma_b/4)^2}{2} \right]^{-1} \quad (2-60)
\]
This expression has in general two peaks. The lineshapes and line-widths depend on the external coupling strength, the energy separation of the excited states and the decay constants of the excited states. In the case, where one of the decay constants is zero, a hole is observed in the frequency spectrum of \( P(\hat{e}_c, k_\lambda) \) at the frequency equal to the frequency difference of the ground state and non-decaying unperturbed excited state. A detail analysis of \( P(\hat{e}_c, k_\lambda) \) is presented in Chapter III for some special cases.

**D. Intensity of Scattered Radiation in a Given Direction with Definite Polarization**

For a beam of incident radiation with polarization \( o^- \) and direction \( (\Omega_a, \phi_a) \), the intensity of the scattered radiation, \( I(\Omega_a, \Omega_e, \lambda) \), with polarization \( \lambda \) and direction \( (\Omega_e, \phi_e) \) is obtained by integrating the probability \( P(\hat{e}_c, k_\lambda) \) over the frequency \( k_\lambda \):

\[
I(\Omega_a, \Omega_e, \lambda) = \frac{L^3}{(2\pi)^3} \int_{0}^{\infty} P(\hat{e}_c, k_\lambda) k_\lambda^2 dk_\lambda
\]

(2-61)

where \( \Omega_a \) and \( \Omega_e \) indicate the directions; and \( o^- \) and \( \lambda \) the polarizations of the incident and the emitted radiation fields, respectively. This integral is evaluated in Appendix B-2. The result is

\[
I(\Omega_a, \Omega_e, \lambda) = \frac{e^4 I_0}{8\pi^2 \hbar \omega} \left[ \frac{1}{\chi_a} \left( \frac{|p_{ca}^\lambda|^2 |p_{ac}^\omega|^2}{|p_{ca}^\lambda|} + \frac{|p_{cb}^\lambda|^2 |p_{bc}^\omega|^2}{|p_{cb}^\lambda|} \right) \right]
\]
\[
\begin{align*}
&+ \left[ V^2 (x_a + x_b) x_a x_b \left\{ \frac{4 \text{Re}(P_{ca}^{\lambda} P_{cb}^{\lambda}) \text{Re}(P_{ac}^{\eta} P_{bc}^{\eta})}{x_a x_b} \right\} \\
&- (|P_{ca}^{\lambda}|^2 / x_a - |P_{cb}^{\lambda}|^2 / x_b) (|P_{ac}^{\eta}|^2 / x_a - |P_{bc}^{\eta}|^2 / x_b) \right]
\end{align*}
\]

\[ + V (x_a + x_b) \left\{ \Delta \left\{ (|P_{ca}^{\lambda}|^2 - |P_{cb}^{\lambda}|^2) \text{Re}(P_{ac}^{\eta} P_{bc}^{\eta}) + (|P_{ac}^{\eta}|^2 - |P_{bc}^{\eta}|^2) \text{Re}(P_{ca}^{\lambda} P_{cb}^{\lambda}) \right\} \\
- 2 \left\{ (|P_{ca}^{\lambda}|^2 x_b - |P_{cb}^{\lambda}|^2 x_a) \text{Im}(P_{ac}^{\eta} P_{bc}^{\eta}) + (|P_{ac}^{\eta}|^2 x_b - |P_{bc}^{\eta}|^2 x_a) \text{Im}(P_{ca}^{\lambda} P_{cb}^{\lambda}) \right\} \right\} \\
+ 4 x_a x_b (x_a + x_b) \text{Re}(P_{ca}^{\lambda} P_{cb}^{\lambda} P_{ac}^{\eta} P_{bc}^{\eta}) \right\} \right]
\]

\[ + 4 x_a x_b \Delta \left\{ \text{Im}(P_{ca}^{\lambda} P_{cb}^{\lambda} P_{ac}^{\eta} P_{bc}^{\eta}) \right\} \]

\[ \left( x_a + x_b \right)^2 \left( V^2 + x_a x_b \right) + \Delta^2 \]  \hspace{5cm} (2-62)
separation $\Delta$, the crossing signal has two terms; one is Lorentzian and the other is dispersion. The crossing signal will be pure Lorentzian or dispersion depending on whether $\rho_c^{\lambda^{\pm}} \rho_{ac}^{\lambda^{\pm}} \rho_{bc}^{\lambda^{\pm}}$ is a real or imaginary quantity, respectively.

As an application, the intensity $I(\nu_a, \nu_c, \nu_x)$ for the $2^2P$ states of $\text{Li}^7$ is calculated in Chapter IV.
III. RESONANCE FLUORESCENCE WITH ONE DECAYING AND ONE NON-DECAYING STATES

The theory developed in Chapter II is used here to discuss the resonance scattering of radiation from an atomic system where only one of the excited states is decaying. The atomic state \( |b\rangle \) is considered to be the non-decaying state (\( \gamma_b = 0 \)).

In general, the energies of the excited states obtained from the real parts of the poles of \( G(E) \) are different from the energies obtained without considering the radiation interaction (16). A detail discussion of these energies is presented in the next section. In the special case where the two excited states decay with the same rate, the energies obtained by the two different methods are the same. Hence, here the radiation interaction has no effect on the energy levels.

A. Discussion of \( E_1 \) and \( E_2 \)

The real parts of the poles \( E_1 \) and \( E_2 \) of \( G(E) \) (see Chapter II-B) give the energies of the system when the atom is in an excited state. From Eq.(2-51), these energies are

\[
\begin{align*}
\text{Re}(E'_1) &= \frac{(E_a' + E_b')}{2} + \left[ \left( \Delta^2 - \chi^2 + 4\nu^2 \right)^2 + 4\chi^2 \Delta^2 \right]^{1/2} \\
&+ \Delta^2 - \chi^2 + 4\nu^2 \right]^{1/2} - k_r \\
\end{align*}
\]

where \( \Delta, \chi \) and \( \nu \) are defined in Eq.(A-23) and Eq.(A-27).
The energy $E_p$ is the energy of the absorbed photon. The energies of the excited states of the atom are obtained by dropping the term $-\mathbf{R}_r$ in Eq. (3-1). Thus,

$$\begin{align*}
\frac{E_1}{E_2} &= \left( E_3' + E_6' \right)/2 \pm \left[ \left( A^2 - x^2 + 4V^2 \right)^2 + 4x^2A^2 \right]^{1/2} \\
&\quad + \frac{A^2 - x^2 + 4V^2}{2}\sqrt{2}.
\end{align*}$$

(3-2)

If one does not take into account the radiation interaction, then the energies of the excited states become

$$\begin{align*}
\frac{E_1^0}{E_2^0} &= \left( E_{a} + E_{b} \right)/2 \pm \frac{\sqrt{(A^2 + 4V^2)}}{2}.
\end{align*}$$

(3-3)

This result has been obtained by diagonalising the Hamiltonian of the atom (see Appendix C).

Comparing Eq. (3-2) with Eq. (3-3), one finds that the energies $E_1$ and $E_2$ are different from the energies $E_{1}^0$ and $E_{2}^0$ provided $X \neq 0$. But for $X = 0$, Eq. (3-2) reduces to Eq. (3-3). This means that the radiation interaction has no effect on the energy levels if the two excited states decay with the same decay rates.

For the case where the excited states are degenerate ($\Delta = 0$) and $X \neq 0$, then Eq. (3-2) reduces to

$$\begin{align*}
\frac{E_1}{E_2} &= \left( E_{a} + E_{b} \right)/2 \pm \sqrt{4V^2 - x^2}/2
\end{align*}$$

(3-4)

and Eq. (3-3) becomes

$$\begin{align*}
\frac{E_1^0}{E_2^0} &= \left( E_{a} + E_{b} \right)/2 \pm V
\end{align*}$$

(3-5)
Fig. 3-1. Energies $E_1$ and $E_2$ as a function of the energy separation $(E_a' - E_b')$. The origin on the ordinate is set at $(E_a' + E_b')/2$. 
Fig. 3-2. Energies $E_1$ and $E_1^0$ as a function of the energy separation $(E_a' - E_b')$.

The origin on the ordinate is set at $(E_a' + E_b')/2$. 

$V = 0.25\tau_a$

$X = 0.5\tau_a$
Fig. 3-3. Energies $E_1$ and $E_1^0$ as a function of the energy separation $(E_a' - E_b')$. The origin on the ordinate is set at $(E_a' + E_b')/2$. 

$V = 0.3 r_a$  
$X = 0.5 r_a$
Equation (3-4) shows that the energies $E_1$ and $E_2$ are degenerate if $4V^2 < x^2$, because in this case the term within the square root sign is either zero or imaginary (The imaginary part of the energies contribute to the decay rates of the excited states). For $4V^2 > x^2$, the two energies are non-degenerate and separated by an energy difference of $\sqrt{4V^2 - x^2}$. On the other hand if one neglects the radiation interaction, one finds that the external perturbation always removes the degeneracy and the two levels are separated by an energy difference of $2V$ [see Eq. (3-5)].

The energies $E_1$ and $E_2$ are plotted in Fig. 3-1 as a function of the energy difference $\Delta$ for different $V$. The value of $X$ is taken to be $\pi a/2$. For $V \leq 0.25a$ ($4V^2 \leq x^2$), one observes that the two energies approach each other and become degenerate for $\Delta = 0$. But for $V > 0.25a$, the two energies are separated.

In Fig. 3-2, the energies $E_1$ and $E_1^\circ$ are plotted as a function of $\Delta$ for $V = 0.25a$ and $X = 0.5a$ ($4V^2 = x^2$). The energy $E_1$ differs from $E_1^\circ$ significantly in the vicinity of $\Delta = 0$. At large values of $\Delta$, $E_1$ approaches $E_1^\circ$ asymptotically. In Fig. 3-3, similar graphs are plotted for $V = 0.3a$ and $X = 0.5a$ ($4V^2 > x^2$). In this case, the energy difference, $(E_1^\circ - E_1)$ decreases as $V$ increases.
B. Probability Amplitudes in Time and Frequency

For this special case where the state $|b\rangle$ is not decaying, the probability amplitudes are obtained from Eq.(2-54) to Eq.(2-57) by setting $\bar{t}_b$, $H_{00}$ and $H_{ij}$ equal to zero. The results are:

\begin{align*}
    b_0(t) &= e^{-iV_{cc}t - \frac{\lambda t}{2}}, \\
    b_i(t) &= \frac{H_{i0} (E_i' - E_j') e^{i(E_i - E_i')t}}{(E_i' - E_3')(E_i' - E_2')} + \frac{H_{i0} (E_2' - E_j') e^{i(E_i - E_2')t}}{(E_2' - E_3')(E_2' - E_i')}, \\
    b_j(t) &= \frac{V_{ab} H_{i0}}{(E_i' - E_2')} \left[ \frac{e^{i(E_j - E_i')t}}{(E_i' - E_3')} - \frac{e^{i(E_j - E_2')t}}{(E_2' - E_3')} \right] + \frac{V_{ba} H_{i0}}{(E_3' - E_i') (E_3' - E_2')} e^{i(E_j - E_3')t} \\
    &+ \frac{H_{f0} H_{i0} (E_j' - E_f')}{(E_j' - E_j')(E_j' - E_3')(E_j' - E_2')} e^{-iV_{cc}t}, \\
    b_j(t) &= \frac{H_{f0} e^{i(E_f - E_j')t}}{(E_j' - E_j')(E_j' - E_3')(E_j' - E_2')} e^{-iV_{cc}t},
\end{align*}

and
Substituting the expressions for $E_f$, $E_f'$, $E_{1}'$, $E_{2}'$ and $E_{3}'$ {see Eq.(2-18), Eq.(2-26) and Eq.(2-51)} into Eq.(3-7) to Eq.(3-9), one obtains:

$$b_i(t) = H_{10} \left[ \frac{\Delta/2 - i (\gamma_{a}/4 + I)}{2(R-i I) [R - \kappa_{\alpha} - i (\gamma_{a}/4 + I - \Pi/2)]} \exp \{i(\Delta/2 + R)t - (\gamma_{a}/4 + I)t\} \right]$$

$$+ \frac{\Delta/2 - R - i (\gamma_{a}/4 - I)}{2(R-i I) [R + \kappa_{\alpha} + i (\gamma_{a}/4 - I - \Pi/2)]} \exp \{i(\Delta/2 + R)t - (\gamma_{a}/4 - I)t\}$$

$$+ \frac{\kappa_{\alpha} + \Delta/2 - i \Pi/2}{[R + \kappa_{\alpha} + i (\gamma_{a}/4 - I - \Pi/2)]} \exp \{-i(\Delta/2)t - (\gamma_{a}/4 + I)t\}$$

$$+ \frac{\exp\{i(R + \Delta/2)t - (\gamma_{a}/4 + I)t\}}{K_{\alpha} - R + i (\gamma_{a}/4 + I - \Pi/2)}$$

$$+ \frac{V_{ba} H_{10} \exp\{-i(K_{\alpha} + \Delta/2)t - \Pi t/2\}}{[K_{\alpha} - R + i (\gamma_{a}/4 + I - \Pi/2)]}$$

(3-10)
where

\[ b_j(t) = \frac{H_{ij} H_{io}}{[K_{\alpha} - K_{\lambda} + i(\gamma a/4 + i)][K_{\alpha} - R + i(\gamma a/4 + i)][K_{\alpha} + R - i(\gamma a/4 - i)]} \]

and

\[ b_j(t) = \frac{H_{ij} H_{io}}{[K_{\lambda} - K_{\sigma} + i(\gamma a/4 + i)][K_{\lambda} - R + i(\gamma a/4 + i)][K_{\lambda} + R - i(\gamma a/4 - i)]} \]

and

\[ b_j(t) = \frac{H_{ij} H_{io}}{[K_{\sigma} - K_{\lambda} + i(\gamma a/4 - i)][K_{\sigma} - R + i(\gamma a/4 - i)][K_{\sigma} + R + i(\gamma a/4 - i)]} \]

and

\[ K_{\alpha} = k_{\alpha} - \frac{E_a' + E_b'}{2} + E_c' \]

and

\[ K_{\lambda} = k_{\lambda} - \frac{E_a' + E_b'}{2} + E_c' \]

and

\[ K_{\sigma} = k_{\sigma} - \frac{E_a' + E_b'}{2} + E_c' \]

and

\[ R \text{ and } I \text{ are defined in Eq.}(A-27). \]

The diagonal matrix elements

\[ V_{aa}, V_{bb} \text{ and } V_{cc} \text{ are assumed to be zero in the expression for the} \]

probability amplitudes in Eq.(3-10) to Eq.(3-12). This assumption

is true in most of the cases of practical interest.

From Eq.(3-6), one finds that the probability \(|b_0(t)|^2\) of the ground state decays exponentially with a decay constant \(\Gamma\).

The probability \(|b_i(t)|^2\) gives the probability of the atom

being in the excited state \(\{|a\}\) with an absorbed photon with wave

vector \(\vec{k}_a\). This probability has three pure decaying terms with decay

constants \((\gamma a/2 + 2I)\), \((\gamma a/2 - 2I)\) and \(\Gamma\) and three oscillatory terms
with decaying amplitudes. These oscillation frequencies are $2R$, $K_0 + R$ and $K_0 - R$ and the decay constants of their amplitudes are $rac{\gamma_0}{2}$, $(\frac{\gamma_0}{4} - \frac{1}{2})$ and $(\frac{\gamma_0}{4} + \frac{1}{2})$, respectively.

The probability $|b_1(t)|^2$ is plotted in Figs. 3-4 to 3-7 as a function of $K_\omega \{K_\omega = k_\omega - (E_a + E_b)/2 + E_c\}$ for different times. At short times, this probability is small and quite broad but as the time increases, the probability narrows up into two peaks one near $K_\omega = R$ and other near $K_\omega = -R$. The peak near $K_\omega = -R$ is weaker than the peak near $K_\omega = R$. There are some wiggles on either sides of the peaks. These wiggles increase in number and become weaker as time increases. At short times, the height of the peaks increases for some time and then starts decreasing at larger times and becomes zero $t = \infty$. An increase in $V$ decreases the height of the peak near $K_\omega = R$ and increases the height of the peak near $K_\omega = -R$ (compare Fig. 3-6 and Fig. 3-8). For $V = 0$, only one peak at $K_\omega = R$ is obtained. A physical reason of these changes in the peak heights can be attributed to the fact that the decaying state $|a\rangle$ is coupled to the non-decaying state $|b\rangle$ through the coupling matrix element $V$ and this coupling mixes the two states unequally for $\Delta \neq 0$. This means that the probability of the atom in state $|a\rangle$ is larger at the energy $E_1$ (the perturbed energy corresponding to $E_a$) than the probability at the energy $E_2$ (the perturbed energy corresponding to $E_b$). This explains that the peak near $K_\omega = R$ (corresponding to $E_1$) is higher than the peak near $K_\omega = -R$ (corresponding to $E_2$). As $V$ increases, the probability of the atom in state $|a\rangle$ decreases at the energy $E_1$ and increases at energy $E_2$. This explains the change in the heights of the
peaks with the change in $V$. For $\Delta = 0$, the two states are mixed equally and thus the two peaks in Fig. 3-10 have equal heights.

In Fig. 3-10, the probability $|b_i(t)|^2$ is plotted as a function of time for different frequencies of the absorbed photons. At short times, all the three oscillation frequencies ($2R, K_+ R$ and $K_- R$) contribute to the oscillations in $|b_i(t)|^2$ but at large times $K_+ R$ is the dominant frequency because the term with this frequency dies out slowly compared to the other terms. These frequencies depend on the relative positions of the two excited states and the frequency of the absorbed photon. A physical reason for these frequencies can be attributed to the radiation reaction on the basis of the following classical picture of the system. The atom which consists of two oscillators of frequencies $\text{Re}(E_1) - E_c$ and $\text{Re}(E_2) - E_c$ interacts with the radiation field of frequency $k_\nu$ through radiative coupling. This coupling produces three oscillation frequencies which are $k_\nu - \text{Re}(E_1) + E_c, K_\nu - \text{Re}(E_2) + E_c$ and $\text{Re}(E_1 - E_2)$. These are the same frequencies which appear in the probability $|b_i(t)|^2 \{ k_\nu - \text{Re}(E_1) + E_c = K - R; K_\nu - \text{Re}(E_2) + E_c = K + R; \text{Re}(E_1 - E_2) = 2R \}$. The decrease in the amplitudes of the oscillations can be attributed to the fact that the probability of the incident photon to exist without being absorbed decreases as time increases and it tends to zero as $t \to \infty$. Thus, there are no photons left to interact with the atom and therefore no wiggles appear in $|b_i(t)|^2$ at large times.

From Eq. (3-11), one finds that $|b_j(t)|^2$, the probability of the atom in the excited state $|b\rangle$ with an absorbed photon with wave
vector $\mathbf{k}_\omega$, has also three pure decaying terms and three oscillatory terms with decaying amplitudes. These oscillation frequencies and the decay constants are the same as those obtained in the case of $|b_i(t)|^2$. The probability $|b_j(t)|^2$ is plotted as a function of $K_\omega$ and time $t$ in Fig. 3-11 and Fig. 3-12, respectively. These plots have, in general, similar features as those of $|b_c(t)|^2$ as discussed earlier.

The probability $|b_j(t)|^2$ gives the probability of the atom in the state $|c\rangle$ with a photon of wave vector $\mathbf{K}_\omega$ absorbed and a photon of wave vector $\mathbf{K}_\lambda$ emitted. This probability is plotted in Figs. 3-13 to 3-16 as a function of the absorbed frequency $K_\omega$ for a given emitted frequency $K_\lambda$ and time $t$. One principal maximum is observed at $k_\omega = k_\lambda$ with many secondary maxima. The principal maximum is wide for short times and narrows up at large times. This means that the off channel ($k_\omega \neq k_\lambda$) scattering is quite significant at small times but becomes negligible at large times. As $t \to \infty$, the plot of $|b_j(t)|^2$ as a function of $K_\omega$ becomes a Dirac delta function $\delta(k_\omega - k_\lambda)$ [see Fig. 3-16].

As time increases, the number of secondary maxima increase and their amplitudes decrease. A physical reason for these secondary maxima can be attributed to the same reasoning as discussed in the case of $|b_c(t)|^2$, previously.
Fig. 3-4. The probability $|b_1|^2$ as a function of $K_r = k_r - (E_a' + E_b')/2 + E_c'$

at time $t = 1/\tau_a$. 

- $\gamma_b = 0$
- $\gamma = 2\gamma_a$
- $\gamma = 3\gamma_a$
- $\gamma = \gamma_a$

$$K_r/\gamma_a$$
Fig. 3-5. The probability $|b_c|^2$ as a function of $K_{\varphi} = k_{\varphi} - (E_a + E_A^b) / 2 + E_c^0$ at time $t = 5 / \tau_a$. 

- $\tau_b = 0$
- $\Delta = 2 \tau_a$
- $\Gamma = 0.05 \tau_a$
- $\nu = \tau_a$
Fig. 3-6. The probability $|b_c|^2$ as a function of $K_{\rho} = k_{\rho} - (E_a' + E_b')/2 + E_c'$ at time $t = 10/\tau_a$. 

- $\gamma_b = 0$
- $\Delta = 2\tau_a$
- $r = 0.05\tau_a$
- $v = \tau_a$
Fig. 3-7. The probability $|b_t(t)|^2$ as a function of $K_r = k_r - (E_a' + E_b')/2 + E_c'$ at time $t = 40/\tau_a$. 

- $\tau_b = 0$
- $\Delta = 2\tau_a$
- $\Gamma = 0.05\tau_a$
- $V = \tau_a$
Fig. 3-8. The probability $|b_1|^2$ as a function of $K_r (K_r = k^-(E_a + E_b)/2 + E_c)$ at time $t = 10/r_a$ for $V = 1.511/r_a$. 

$\tau_b = 0$

$\Delta = 2\tau_a$

$\Gamma = 0.05\tau_a$
Fig. 3-9. The probability $|b_z(t)|^2$ as a function of $K/	au_\alpha$ at time $t = 10/\tau_\alpha$ for $\Delta = 0$. 

$\tau_b = 0$
$\tau = 0.05 \tau_\alpha$
$V = 0.5 \tau_\alpha$
Fig. 3-10 The probability $|b_c|^2$ as a function of time $t$. 

$\tau_0 = 0$

$\Delta = 2\tau_a$

$p = 0.05\tau_a$

$V = \tau_a$
Fig. 3-11. The probability $|b_j|^2$ as a function of $K_r$ for $\Delta = 2F_0$ and $V = \gamma_0$. 

$\gamma_b = 0$

$\Gamma = 0.05 \gamma_0$
Fig. 3-12. The probability $|b_j|^2$ as a function of time $t$.

Conditions:
- $\gamma_b = 0$
- $\Delta = 2\gamma_a$
- $\gamma' = 0.05\gamma_a$
- $\nu = \gamma_a$
Fig. 3-13. The probability $|b_f|^2$ as a function of $K_\varphi$ for $K_\Lambda = \gamma_a$ at time $t = 1/\tau_a$ 

$K = k_\varphi - (E'_a + E'_b)/2 + E'_c$, $K_\Lambda = k_\varphi - (E'_a + E'_b)/2 + E'_c$. 

$\tau_b = 0$

$\Delta = 2\gamma_a$

$\Gamma = 0.05\gamma_a$

$\nu = \gamma_a$
Fig. 3-14. The probability $|b_f|^2$ as a function of $K_x$ for $K_x = \gamma_a$ at time $t = 5/\gamma_a$. The ordinate is reduced by a factor of 1/20 compared to the ordinate in Fig. 3-13.
Fig. 3-15. The probability $|b_f|^2$ as a function of $K_\alpha/r_a$ for $K_\alpha = Y_\alpha$ at time $t = 20/r_a$. The ordinate is reduced by a factor of $1/20$ compared to the ordinate in Fig. 3-14.
Fig. 3-16. The probability $|b_f|^2$ as a function of $K$, for $K_x = \gamma_{\alpha}$ at time $t = 50/\tau_{\alpha}$.

The ordinate is reduced by a factor of 0.36 compared to the ordinate in Fig. 3-15.
C. Lineshape and Linewidth of Emitted Radiation

A study of the lineshape and linewidth of the emitted radiation for the case where $\tau_b = 0$ (state $|b\rangle$ is non-decaying) is presented in this section. An expression for $P(\hat{e}_\omega, k_\lambda)$, the probability that a photon with wave vector $\vec{k}_\lambda$ and polarization $\hat{e}_\lambda$ has been emitted between time $t = 0$ to $t \to \infty$, is obtained by setting $\hat{p}_{cb}$ and $\hat{p}_{bc}$ in Eq. (2-60) equal to zero:

$$P(\hat{e}_\omega, k_\lambda) = \frac{[\mathcal{E}^4 I_0 / (L^3 \mu k_\lambda^3)] \mathcal{E}_c^\lambda |P_{c\alpha}|^2 |P_{d\beta}|^2 (k + \Delta/2)^2}{[(K - \Delta/2)(K + \Delta/2) - V^2]^2 + (k + \Delta/2)^2 \gamma_a^2 / 4}$$  \hspace{1cm} (3-13)

This probability has two maxima, one at $K = \sqrt{\Delta^2 + 4V^2} / 2$ and the other at $K = -\Delta / 2$. These maxima have equal heights but unequal linewidths. These linewidths are obtained as follows. The values of $K$ for which $P(\hat{e}_\omega, k_\lambda)$ is equal to half of its maximum value, are determined from the equation:

$$\frac{(k + \Delta/2)^2}{[(K - \Delta/2)(K + \Delta/2) - V^2]^2 + (k + \Delta/2)^2 \gamma_a^2 / 4} = \frac{2}{\gamma_a^2}.$$  \hspace{1cm} \text{After simplifying the above equation, one gets:}

$$\left(\frac{K^2 - \Delta^2 / 4 - V^2}{(K - \Delta/2)^2} - \frac{K^2}{(K + \Delta/2)^2} \gamma_a^2 / 4\right) \frac{\gamma_a^2}{4} = 0.$$  \hspace{1cm} \text{This equation gives four values of $K$ for which $P(\hat{e}_\omega, k_\lambda)$ is half its maximum value. The results are:}

$$k_1, k_2 = \frac{\gamma_a^2 / 4 + \Delta^2 + 4V^2 + \gamma_a^2 \Delta / 2}{2}$$
and

\[
\begin{align*}
K_3, K_4 \} &= - \frac{\gamma_a}{4} \pm \frac{1}{2} \left( \frac{\gamma_a^2}{4} + \Delta^2 + 4 \nu^2 - \gamma_a \Delta \right)^{\frac{1}{2}}
\end{align*}
\]

The probability \( P(\epsilon, k_{\Delta}) \) is zero at \( K = -\Delta/2 \). Thus the width of the line at \( K = \frac{1}{2}(\Delta^2 + 4\nu^2)^{\frac{1}{2}} \) is

\[
\omega_1 = \frac{\tau_a}{2} + \frac{1}{2} \left[ \left( \frac{\tau_a^2}{4} + \Delta^2 + 4 \nu^2 + \gamma_a \Delta \right)^{\frac{1}{2}} - \left( \frac{\tau_a^2}{4} + \Delta^2 + 4 \nu^2 - \gamma_a \Delta \right)^{\frac{1}{2}} \right]
\]

(3-14)

and the width of the line at \( K = -\frac{1}{2}(\Delta^2 + 4\nu^2)^{\frac{1}{2}} \) is

\[
\omega_2 = \frac{\tau_a}{2} - \frac{1}{2} \left[ \left( \frac{\tau_a^2}{4} + \Delta^2 + 4 \nu^2 + \gamma_a \Delta \right)^{\frac{1}{2}} - \left( \frac{\tau_a^2}{4} + \Delta^2 + 4 \nu^2 - \gamma_a \Delta \right)^{\frac{1}{2}} \right]
\]

(3-15)

The two linewidths add up to \( \tau_a \) (the linewidth of the single line when no coupling is considered). As \( \nu \) increases, the linewidth \( \omega_1 \) decreases and \( \omega_2 \) increases. But as \( \Delta \) increases, \( \omega_1 \) increases and \( \omega_2 \) decreases. Both linewidths approach \( \tau_a/2 \) when \( V \to \infty \). For \( \Delta = 0 \), the two linewidths reduce to the same value \( \tau_a/2 \), and thus are independent of \( V \).

For \( \Delta = 0.5 \tau_a \), the probability \( P(\epsilon_{\Delta}, k_{\Delta}) \) is plotted in Figs. 3-17 to 3-19 as a function of \( K \) for different \( V \). Figure 3-17 shows that only one emission line of linewidth \( \tau_a \) is observed when \( V = 0 \). In
Fig. 3-18 where $V = .2 \gamma_a$, a "hole" is observed in the emission line at $K = - .25 \gamma_a (x = E'_b - E'_c)$. This "hole" becomes wider with increasing coupling strength $V$. The change in $V$ and $\Delta$ also affects the linewidths as discussed earlier (compare Fig. 3-18 and Fig. 3-19 for the effect of $V$ and Fig. 3-19 and Fig. 3-20 for the effect of $\Delta$).

For $\Delta = 0$, the probability $P(\beta, k)$ is plotted in Figs. 3-21 to 3-23 as a function of $K$ for different values of $V$. Comparing Fig. 3-21 and Fig. 3-22, one finds that one Lorentzian line centered at $K = 0$ for $V = 0$ splits up into two lines for $V = .2 \gamma_a$ with a "hole" at $K = 0$. Figure 3-22 and Fig. 3-23 show that as $V$ increases the two lines move farther away from $K = 0$ without changing their linewidths. Thus the linewidth in this case is independent of $V$ and is equal to $\gamma_a / 2$ (half of the linewidth of the line when $V = 0$).

A physical explanation of the linewidth and lineshape of the emitted radiation can be presented in the following way. The external coupling $V$ mixes the two excited states unequally for $\Delta \neq 0$. Thus state $|a\rangle$ has some probability at energy $E_1$ (perturbed energy corresponding to state $|a\rangle$) and a lower probability at $E_2$ (perturbed energy corresponding to state $|b\rangle$). As $V$ increases, the probability of the atom in the state $|a\rangle$ at the energy $E_2$ increases and the probability at $E_1$ decreases. This increase in probability at $E_2$ and decrease in probability at $E_1$ explains the increase in the linewidth of the line at $E_2$ and decrease in the linewidth of the line at $E_1$. For $\Delta = 0$, the two states are mixed equally, thus the probability of the decaying state $|a\rangle$ at the energies $E_1$ and $E_2$ are equal for any value of $V$. 
Fig. 3-17. The probability $P(\hat{E}_x, R_\lambda)$ as a function of $K_\lambda (K_\lambda = k_\lambda - (E'_a + E'_b)/2 + E'_c)$ for $V = 0$. The maximum is at $K_\lambda = 0.25 R_\alpha$. 

$\gamma_b = 0$

$\Delta = 0.5 \gamma_\alpha$
Fig. 3-18. The probability $P(\xi, K_x)$ as a function of $K_x$ for $V = 0.2 \tau_a$. A "hole" appears at $K_x = -0.25 \tau_a$. 

$\Delta = 0.5 \tau_a$
Fig. 3-19. The probability $P(e_k, k_\lambda)$ as a function of $K_\lambda$ for $\Delta = 0.5 \gamma_a$ and $\gamma_b = 0$. 
Fig. 3-20. The probability \( P(\hat{e}_r, K_{\lambda}) \) as a function of \( K_{\lambda} \) for \( \Delta = 2 \gamma_{\alpha}, V = \gamma_{\alpha} \) and \( \gamma_{b} = 0 \).
Fig. 3-21. The probability $P(\hat{e}_\alpha, K_\alpha)$ as a function of $K_\alpha$ for $\Delta = 0$, $V = 0$ and $f_b = 0$. 
Fig. 3-22. The probability $P(\hat{e}_x, k_\lambda)$ as a function of $K_\lambda/N$ for $\Delta = 0$, $V = 0.2 \gamma_a$ and $\gamma_b = 0$. 
Fig. 3-23. The probability $P(\hat{e}_r, k_\lambda)$ as a function of $k_\lambda$ for $\Delta = 0$ and $r_b = 0$. 

$V = r_a$

$V = 0.5 r_a$
Therefore the linewidths are independent of $V$.

The "hole" in the emission line can be attributed to an interference phenomenon. If one solves for the eigenstates of the Hamiltonian of the atom plus the external perturbation, in terms of the states $|a\rangle$ and $|b\rangle$, then one gets two eigenstates which are not coupled any more by the external static field. If the radiation field is now allowed to interact with the atom then the two eigenstates decay by emitting photons. The amplitudes of these eigenstates at the energy $E_b^\prime$ are such that the photons emitted at that frequency interfere destructively and produce a "hole" in the emission line. A similar "hole" was observed by Lamb and Retherford (17) in atomic Hydrogen for $2^2S_{1/2}$, $2^2S_{1/2-1/2}$ and $2^2P_{1/2}$ states. An external rf field couples the states.
The theory developed in Chapter II is used here to discuss the resonance scattering of radiation from an atomic system where both the excited states decay with equal decay constants \((\gamma_a = \gamma_b)\).

The probabilities \(|b_i|^2\), \(|b_j|^2\) and \(|b_f|^2\) as a function of time \(t\) have similar features as those of the probabilities discussed in Chapter III and thus they are not discussed here.

A study of linewidth and lineshape of the scattered radiation is presented in the next section. In section B of this chapter an expression for the total intensity of the scattered radiation is derived for a given direction and polarization of the incident and the scattered beam. In section C, this intensity is calculated for the \(2^2\) state in Li\(^7\) as a function of an external magnetic field.

### A. Lineshape and Linewidth of Emitted Radiation

The expression for the probability \(P(\hat{e}_\gamma, \mathbf{k}_\lambda)\) that a photon has been emitted between time \(t = 0\) and \(t = \infty\) with wave vector \(\mathbf{k}_\lambda\) and polarization \(\hat{e}_\gamma\) is obtained from Eq. (2-60) by setting \(\gamma_a = \gamma_b = \gamma\):

\[
P(\hat{e}_\gamma, \mathbf{k}_\lambda) = \frac{e\mathbf{I}_0}{\mathbf{J}^\lambda \gamma \mathbf{k}_\lambda} \left[ \{ (k + \Delta/2)^2 + \gamma^2/4 \} |P^\lambda_{cb}|^2 |P^\lambda_{bc}|^2 + \beta^2 |P^\lambda_{ac}|^2 |P^\lambda_{bc}|^2 \right]
\]

\[+ |P^\lambda_{cb}|^2 |P^\lambda_{ac}|^2 + 4 \text{Re} (P^\lambda_{ca} \bar{P}^\lambda_{cb}) \text{Re} (P^\lambda_{ac} \bar{P}^\lambda_{bc}) - 2 \text{Re} (P^\lambda_{ca} \bar{P}^\lambda_{cb} \bar{P}^\lambda_{ac} P^\lambda_{bc})\]
\[ + 2V \left\{ (K+\Delta/2)|P_\text{ca}^\lambda|^2 + (K-\Delta/2)|P_\text{cb}^\lambda|^2 \right\} \text{Re} \left( P_\text{ac}^\lambda P_\text{bc}^{\lambda*} \right) \]
\[ + \left\{ (K+\Delta/2)|P_\text{ac}^{\lambda*}|^2 + (K-\Delta/2)|P_\text{bc}^{\lambda*}|^2 \right\} \text{Re} \left( P_\text{ca}^\lambda P_\text{cb}^{\lambda*} \right) \]
\[ + \frac{\Gamma}{2} \left\{ (1|P_\text{cb}^\lambda|^2-1|P_\text{ca}^\lambda|^2) \text{Im} \left( P_\text{ac}^\lambda P_\text{bc}^{\lambda*} \right) + (1|P_\text{bc}^{\lambda*}|^2-1|P_\text{ac}^{\lambda*}|^2) \text{Im} \left( P_\text{ca}^\lambda P_\text{cb}^{\lambda*} \right) \right\} \]
\[ + \left( K^2 - \frac{\Delta^2}{4} + \frac{\Gamma^2}{4} \right) \frac{1}{2} \text{Re} \left( P_\text{ca}^\lambda P_\text{cb}^{\lambda*} P_\text{ac}^{\lambda*} P_\text{bc}^\lambda \right) \text{Im} \left( P_\text{ac}^\lambda P_\text{bc}^{\lambda*} \right) \]
\[ + \frac{\Gamma}{2} \Delta \frac{1}{2} \text{Im} \left( P_\text{ca}^\lambda P_\text{cb}^{\lambda*} P_\text{ac}^{\lambda*} P_\text{bc}^\lambda \right) \]
\[ \left[ \left\{ (K-R)^2 + \frac{\Gamma^2}{4} \right\} \left\{ (K+R)^2 + \frac{\Gamma^2}{4} \right\} \right]^{-1} \]  
\[ (4.1) \]

where

\[ \Delta = \frac{1}{2} \left( \Delta^2 + 4V^2 \right)^{1/2}, \]

and

\[ \Delta = E_a' - E_b'. \]

The matrix elements \( p_{ij}^\ell \) for \( i,j = a, b \) and \( \ell = \lambda, \) are defined in Eq. (A-4).

From Eq. (4.1), one finds that this probability has two maxima, one near \( K = R \) and other near \( K = -R. \) The probability \( P(\ell, R, \Delta) \) is plotted as a function of \( K \) in Figure 4-1 for different \( V_{ab}. \) For these graphs, the matrix elements are chosen such that \( p_{ca}^\lambda = p_{cb}^{\lambda*} \) and \( p_{ac}^\lambda = p_{bc}^{\lambda*} \) and are real.

Figure 4-1 shows that the increase in \( V \) suppresses the maximum near \( K = -R \) and enhances the maximum near \( K = R. \) This effect is easily understood if one thinks in terms of the eigenstates of the perturbed Hamiltonian \( (H_0 + V) \) \{see Appendix C\}. The two
Fig. 4-1. The probability $P(\hat{e}_r, k_\lambda)$ as a function of $k_\lambda$ for $\Delta = 2\gamma_a$ and $\gamma_a = \gamma_b \neq 0$. 
eigenstates of this Hamiltonian are \( c_0 |a\rangle + D_0 |b\rangle \) and \(-D_0 |a\rangle + c_0 |b\rangle\) and have energies equal to \((E_a' + E_b')/2 + R\) and \((E_a' + E_b')/2 - R\), respectively. The coefficients \(c_0\) and \(D_0\) are defined in Eq.\((C-10)\). As \(V\) increases \(c_0\) decreases and \(D_0\) increases. Thus due to the sign difference in the eigenstates, the probability of photon emission from the state \(c_0 |a\rangle + D_0 |b\rangle\) is greater than the probability from the state \(-D_0 |a\rangle + c_0 |b\rangle\).

B. Intensity of Scattered Radiation

The intensity of the scattered radiation for a given direction and polarization of the incident and the scattered beams is obtained by setting \(X_a = X_b = \tau/2\) in Eq.\((2-62)\):

\[
I(\tau, \epsilon, \sigma, \lambda) = \frac{e^{i\Delta \lambda N}}{4\pi^2 \eta \kappa_0} \left[ |P_{ca}^\lambda|^2 |P_{ac}^{\lambda^*}|^2 + |P_{cb}^\lambda|^2 |P_{bc}^{\lambda^*}|^2 \right] \\
+ 2 \sqrt{2} \left\{ 4 \text{Re}(P_{ca}^\lambda \bar{P}_{cb}^{\lambda^*}) \text{Re}(P_{ac}^{\lambda^*} P_{bc}^\lambda) - (|P_{ca}^\lambda|^2 - |P_{cb}^\lambda|^2) (|P_{ac}^{\lambda^*}|^2 - |P_{bc}^\lambda|^2) \right\} \\
+ 2 \sqrt{2} \left\{ \Delta \left\{ (|P_{ca}^\lambda|^2 - |P_{cb}^\lambda|^2) \text{Re}(P_{ac}^{\lambda^*} P_{bc}^\lambda) + (|P_{ac}^{\lambda^*}|^2 - |P_{bc}^\lambda|^2) \text{Re}(P_{ca}^\lambda P_{cb}^{\lambda^*}) \right\} \right\} \\
\left( \Delta^2 + 4 \nu^2 + \tau^2 \right)^{-1}
\]  
\((4-2)\)
As discussed earlier in Chapter II section D, the first two terms in Eq. (4-2) are constant. These terms arise from the direct resonance fluorescence process. The next six terms arise from the presence of the non-diagonal matrix element $V_{ab}$. These terms are called anti-crossing signals. The last two terms are crossing signals. The name anti-crossing and crossing are used because the energy levels repel and cross each other, respectively, when plotted as a function of the external perturbation.

The crossing signal is pure Lorentzian or dispersion as a function of the energy difference $\Delta$ depending on whether $P_{a \rightarrow b} \langle a | \hat{v} | b \rangle P_{b \rightarrow a}$ is real or imaginary, respectively. If it is a pure Lorentzian then it has a half width equal to $(4\nu^2 + r^2)^{1/2}$.

Separating the crossing and anti-crossing signals from Eq. (4-2), one gets:

$$S_c \propto \frac{2\nu \text{Re} (P_{a \rightarrow b} \langle a | \hat{v} | b \rangle P_{b \rightarrow a} + 2\Delta \text{Im} (P_{a \rightarrow b} \langle a | \hat{v} | b \rangle P_{b \rightarrow a})}{\Delta^2 + 4\nu^2 + r^2} \quad (4-3)$$

and

$$S_A \propto \left[ 2\nu^2 \{4 \text{Re} (P_{a \rightarrow b} \langle a | \hat{v} | b \rangle P_{b \rightarrow a} - \nu (|P_{a \rightarrow b}|^2 - |P_{b \rightarrow a}|^2) \} \right.$$  
$$+ \nu \{ \Delta (|P_{a \rightarrow b}|^2 - |P_{b \rightarrow a}|^2) \} - r \{ |P_{a \rightarrow b}|^2 - |P_{b \rightarrow a}|^2 \} \text{Im} (P_{a \rightarrow b} \langle a | \hat{v} | b \rangle P_{b \rightarrow a}) \right] \times$$

$$\frac{1}{\Delta^2 + 4\nu^2 + r^2} \quad (4-4)$$
where $S_C$ and $S_A$ stand for the crossing and anti-crossing signals, respectively. The above signals depend on the direction and polarization of the incident and the scattered radiation.

For an atom which has many coupled excited states with equal decay constants $\tau$ (coupled by an external static perturbation) and many ground states, the intensity of the scattered beam is calculated in the following way.

The total Hamiltonian of the atom (excluding the radiation interaction) is first diagonalised and then the probability $|b_3(t \rightarrow \infty)|^2$ is calculated (11):

$$|b_3(t \rightarrow \infty)|^2 = \sum_{\mu} \frac{H_{f \mu} H_{\Lambda \mu} H_{f \mu}^* H_{\Lambda \mu}^*}{[(k_\mu - k_\Lambda)^2 + \frac{\tau^2}{4}] [k_\mu - E_\mu + E_c + \frac{\mu \tau}{2}] [k_\Lambda - E_\mu + E_c - \frac{\mu \tau}{2}]}$$

(4-5)

where $H$ stands for the radiation interaction (see Appendix A-1).

Summation over $\mu$ and $\mu'$ are over the excited states (the eigenstates of the total Hamiltonian) and $E_\mu$ are the energy eigenvalues of the excited states $|\mu\rangle$. The probability $|b_3(t \rightarrow \infty)|^2$ is the probability that the atom has absorbed a photon with wave vector $\vec{k}_\mu$ and polarization $\hat{e}_\mu$ during a transition from a ground state $|c\rangle$ to the excited states $|\mu\rangle$ and has emitted a photon with wave vector $\vec{k}_\Lambda$ and polarization $\hat{e}_\Lambda$ during the transition from the excited states $|\mu\rangle$ to a ground state $|c\rangle$. In this calculation (Eq. (4-5)), the ground states are considered to be degenerate. The final state $|f\rangle$ is defined as:

$$|f\rangle = |c\rangle |-\vec{k}_\mu\rangle_R|+\vec{k}_\Lambda\rangle_R$$
If all the ground states are equally probable at $t = 0$, then the total probability that the atom has absorbed a photon with wave vector $\vec{k}_a$ and polarization $\vec{e}_a$ and has emitted a photon with wave vector $\vec{k}_e$ and polarization $\vec{e}_e$ at $t = \infty$ is given by:

$$\sum_{c,c'} \left| b_{fe}(t=\infty) \right|^2 = \sum_{\mu,\mu'} \frac{H_{f,\mu} H_{\mu,c} H_{f',\mu'}^* H_{\mu',c}^*}{\left( (k_{\lambda'} - k_{\nu})^2 + \frac{\pi^2}{4} \right) \left[ k_{\lambda'}^2 + k_{\lambda}^2 + \frac{\pi^2}{4} \right] \left[ k_{\nu'}^2 + k_{\nu}^2 + \frac{\pi^2}{4} \right]}$$

(4-6)

where summations over $c$ and $c'$ are over the ground states.

In order to get the total intensity $I(-\nu_a, \nu_e, r, \lambda)$ of the scattered beam for a given direction and polarization of the incident and the scattered radiation, the probability in Eq. (4-6) is integrated over all frequencies of the absorbed and the emitted photons.

Using the fact that $1/\{(k_{\lambda} - k_{\nu})^2 + \pi^2/4\}$ reduces to $2\pi/\nu \delta(k_{\lambda} - k_{\nu})$ for very small $\Gamma$ and that the product $H_{f,\mu} H_{\mu,c} H_{f',\mu'}^* H_{\mu',c}^*$ can be considered to be constant near the poles of $|b_{fe}(\infty)|^2$, this integration gives:

$$I(-\nu_a, \nu_e, r, \lambda) \propto \sum_{\mu,\mu'} \frac{P_{\mu,c}^c P_{\mu,c}^{c*} P_{e,c}^{-e} P_{e,c}^e*}{\gamma + i \left( E_{\mu} - E_{\mu'} \right)}$$

(4-7)

where $P_{\mu,c}^c$ is the matrix element of $\vec{p} \cdot e_{\lambda}$ between states $|c'\rangle$ and $|\mu\rangle$ (see Appendix A-1) and $\vec{p}$ is the momentum of the electron. This is the well known Breit equation (9).

The intensity $I(-\nu_a, \nu_e, r, \lambda)$ can be written as:

$$I(-\nu_a, \nu_e, r, \lambda) \propto \sum_{\mu,\mu'} \frac{\Re \left( P_{\mu,c}^c P_{\mu,c}^{c*} P_{e,c}^{-e} P_{e,c}^e* \right) + \left( E_{\mu} - E_{\mu'} \right) \Im \left( P_{\mu,c}^c P_{\mu,c}^{c*} P_{e,c}^{-e} P_{e,c}^e* \right)}{(E_{\mu} - E_{\mu'})^2 + \gamma^2}$$

(4-8)
As an application of the theory developed in the previous section, the intensity of the scattered radiation from a Li\(^7\) atom in an external magnetic field, is calculated. The frequency range of the incident beam is so chosen that only the \(1^2S\) and \(2^2P\) states are involved in the scattering process.

Ignoring the hyperfine structure for the moment, the Hamiltonian of the Lithium atom in the external magnetic field is written as:

\[
\mathcal{H} = \mathcal{H}_0 + A \vec{L} \cdot \vec{S} + g_\text{s} \mu_0 \vec{S} \cdot \vec{H} + g_\text{e} \mu_0 \vec{L} \cdot \vec{H} \quad (4-9)
\]

where \(\mathcal{H}_0\) is the Hamiltonian of the atom excluding the fine structure and the magnetic field interactions. The symbols \(g_\text{s}\) and \(g_\text{e}\) stand for the electronic spin and orbital Lande g-factors, respectively, \(A\) stands for the Bohr magneton, \(A\) for the fine structure coupling constant, \(\vec{H}\) for the magnetic field, \(\vec{L}\) for the orbital angular momentum operator and \(\vec{S}\) for the spin angular momentum operator.

The experimental value of the coupling constant \(A\) is approximately 6.75 KMC/sec. (12, p. 13).

If the magnetic field is chosen to be along the Z-direction, then the Hamiltonian \(\mathcal{H}\) in Eq. (4-9) reduces to

\[
\mathcal{H} = \mathcal{H}_0 + A \vec{L} \cdot \vec{S} + g_\text{s} \mu_0 S_z \vec{H} + g_\text{e} \mu_0 L_z \vec{H} \quad (4-9a)
\]
Due to the presence of the magnetic field interaction terms in this Hamiltonian, the total angular momentum $J$ is no more a good quantum number. However, the magnetic quantum number $m_j$ is a good quantum number.

A representation involving $\ell$, $s$, $m_\ell$, and $m_j (|\ell s m_\ell m_j\rangle)$ is used to diagonalize the Hamiltonian in Eq. (4-9a). The following eigenstates are obtained as a result of this diagonalization:

\[
|\Phi_1\rangle = |1\ell/2, \ell/2\rangle,
\]
\[
|\Phi_2\rangle = A_0|1\ell/2, -\ell/2\rangle + B_0|1\ell/2, -1\ell/2\rangle,
\]
\[
|\Phi_3\rangle = C_0|1\ell/2, -1\ell/2\rangle + D_0|1\ell/2, 0\rangle,
\]
\[
|\Phi_4\rangle = |1\ell/2, 1\ell/2\rangle,
\]
\[
|\Phi_5\rangle = -B_0|1\ell/2, 1\ell/2\rangle + A_0|1\ell/2, 0\rangle,
\]

and their corresponding energy eigenvalues are:

\[
E_1 = E^0 + A/2 + (g_\ell + g_\ell s/2) \mu_0 H,
\]
\[
E_2 = E^0 - A/4 + 1/2 g_\ell \mu_0 H + 1/2 [ (g_\ell s - g_\ell)^2 \mu_0^2 H^2 + (g_\ell s - g_\ell)^2 \mu_0^2 H^2 + \frac{1}{2} \frac{1}{2} \frac{1}{2} + A^2/4],
\]
\[
E_3 = E^0 - A/4 - 1/2 g_\ell \mu_0 H + 1/2 [ (g_\ell s - g_\ell)^2 \mu_0^2 H^2 - (g_\ell s - g_\ell)^2 \mu_0^2 H^2 + A^2/4],
\]
\[
E_4 = E^0 + A/4 - (g_\ell + g_\ell s/2) \mu_0 H.
\]
\[ E_5 = E^0 - A/4 + \frac{1}{2} g_e \mu_0 H - \frac{1}{2} \left[ (g_s - g_e)^2 \mu_0^2 H^2 + (g_s - g_e) \mu_0 H A + q A^2/4 \right]^{1/2} \]
\[ E_6 = E^0 - A/4 - \frac{1}{2} g_e \mu_0 H - \frac{1}{2} \left[ (g_s - g_e)^2 \mu_0^2 H^2 - (g_s - g_e) \mu_0 H A + q A^2/4 \right]^{1/2} \] (4-11)

where \( E^0 \) is the energy of the atom without the fine structure and the magnetic field interactions. The coefficient \( A_0, B_0, C_0 \) and \( D_0 \) are defined as:

\[
\begin{align*}
A_0 &= \frac{1}{\sqrt{2}} \left( 1 + \frac{\mu_0 H + A/2}{\sqrt{(\mu_0 H + A/2)^2 + 2 A^2}} \right)^{1/2} \\
B_0 &= \frac{1}{\sqrt{2}} \left( 1 + \frac{\mu_0 H - A/2}{\sqrt{(\mu_0 H - A/2)^2 + 2 A^2}} \right)^{1/2} \\
C_0 &= \frac{1}{\sqrt{2}} \left( 1 + \frac{\mu_0 H + A/2}{\sqrt{(\mu_0 H + A/2)^2 + 2 A^2}} \right)^{1/2} \\
D_0 &= \frac{1}{\sqrt{2}} \left( 1 + \frac{\mu_0 H - A/2}{\sqrt{(\mu_0 H - A/2)^2 + 2 A^2}} \right)^{1/2}.
\end{align*}
\] (4-12)

When the energies in Eq. (4-11) are plotted as a function of the external magnetic field \( H \), one finds that energy \( E_5 \) crosses \( E_6 \) at \( H = 2 A/(3 \mu_0) \) and \( E_6 \) at \( H = A/\mu_0 \).

The hyperfine interaction \( H_D \) and the electrostatic nuclear quadrupole interaction \( H_Q \) are (13):

\[ H_D = \alpha \left[ \frac{I - S}{\gamma^3} + 3 \frac{(3\gamma)(I)}{\gamma^5} \right] \cdot I + \xi \hbar \cdot \mathbf{S}, \] (4-13)
and

\[ H_Q = -e^2 Q \left[ \frac{3I_x^2 - I_z^2}{4I(2I-1)} \right] \left( \frac{3\cos^2 \theta - 1}{\gamma^3} \right), \]  

(4-14)

where \( Q \) stands for the nuclear electric quadrupole moment, \( \theta \) and \( r \) are the coordinates of the \( P \) electron and \( \alpha \) and \( \xi \) are constants and have the following experimental values (13).

\[ \alpha \langle \frac{1}{T^3} \rangle = 13.37 \pm 0.05 \text{ MC/sec} \]  

(4-15)

\[ \xi = -31.6 \pm 0.7 \text{ MC/sec} \]

where \( \langle 1/T^3 \rangle \) is the average value of \( 1/T^3 \) in the 2P state in Li\textsuperscript{7}.

The total Hamiltonian \( \mathcal{H}_T \) of the atom including the hyperfine and the nuclear quadrupole interaction, is:

\[ \mathcal{H}_T = \mathcal{H}_0 + A \vec{L} \cdot \vec{S} + (g_S \vec{S} + g_L \vec{L}) \cdot \vec{H} \chi + H_D + H_Q \]  

(4-16)

The presence of the hyperfine interaction in this Hamiltonian, couples the hyperfine states of the same total magnetic quantum number \( m_j \) (\( m_j = m_L + m_S + m_I \)). Thus, \( m_j \) is no more a good quantum number. Near the high field crossing of the energy levels \( E_4 \) and \( E_6 \), the coupling between the hyperfine states of \( |\Phi_4\rangle \) and \( |\Phi_6\rangle \) is stronger than the coupling with the other hyperfine states. Thus, one can just consider the coupling between the hyperfine states of \( |\Phi_4\rangle \) and \( |\Phi_6\rangle \) and can neglect the other couplings without significant error. The intensity of the scattered beam is calculated near the high field crossing of \( E_4 \) and \( E_6 \).
The following representation is used to diagonalize the total Hamiltonian $\mathcal{H}_T$ in Eq. (4-16):

$$|\Phi_{i}', m_I\rangle = |\Phi_{i}\rangle |3/2\rangle |m_I\rangle,$$

where the wave vector $|\Phi_{i}\rangle$ are defined in Eq. (4-10). For Li$^7$ the nuclear spin quantum number $I$ is $3/2$.

Considering only the coupling between the hyperfine states of $|\Phi_{4}\rangle$ and $|\Phi_{6}\rangle$ near the high field crossing, the following eigenstates result.

$$|\psi_1\rangle = c_1 |\phi_{4}, 3/2\rangle + d_1 |\phi_{6}, 1/2\rangle,$$

$$|\psi_2\rangle = -d_1 |\phi_{4}, 3/2\rangle + c_1 |\phi_{6}, 1/2\rangle,$$

$$|\psi_3\rangle = c_2 |\phi_{4}, 1/2\rangle + d_2 |\phi_{6}, -1/2\rangle,$$

$$|\psi_4\rangle = -d_2 |\phi_{4}, 1/2\rangle + c_2 |\phi_{6}, -1/2\rangle,$$

$$|\psi_5\rangle = c_3 |\phi_{-}, 1/2\rangle + d_3 |\phi_{6}, -3/2\rangle,$$

$$|\psi_6\rangle = -d_3 |\phi_{-}, 1/2\rangle + c_3 |\phi_{6}, -3/2\rangle,$$

$$|\psi_7\rangle = |\phi_{4}, -3/2\rangle,$$

$$|\psi_8\rangle = |\phi_{6}, 3/2\rangle.$$

The coefficients in Eq. (4-17) are defined as

$$\begin{bmatrix} C_i \\ D_i \end{bmatrix} = \frac{1}{\sqrt{2}} \left(1 \pm \frac{\Delta \xi}{\sqrt{\Delta \xi^2 + 4v_i^2}} \right)^{1/2} i = 1, 2, 3.$$

(4-18)
The energy differences $\Delta_i$ and the off diagonal matrix elements $V_i$ of the hyperfine interaction $H_D$, are defined as:

$$\Delta_i = \langle \Phi_{4i}, m_I | H_T | \Phi_{4i}, m_I \rangle - \langle \Phi_{6i}, m_I | H_T | \Phi_{6i}, m_I \rangle \quad (4-19)$$

and

$$V_i = \langle \Phi_{4i}, m_I | H_D | \Phi_{6i}, m_I \rangle - \langle \Phi_{6i}, m_I | H_D | \Phi_{4i}, m_I \rangle \quad (4-20)$$

where $i = 1, 2, 3$ for $m_I = 3/2, 1/2$ and $-1/2$, respectively.

The energy eigenvalues of the states $|\Psi_i\rangle$ are:

$$E_1^,' = \frac{\langle \Phi_{41}, 3/2 | H_T | \Phi_{41}, 3/2 \rangle + \langle \Phi_{62}, 1/2 | H_T | \Phi_{62}, 1/2 \rangle}{2} \pm \sqrt{\Delta_1^2 + 4V_1^2} \quad (4-21)$$

$$E_3^,' = \frac{\langle \Phi_{42}, 1/2 | H_T | \Phi_{42}, 1/2 \rangle + \langle \Phi_{63}, -1/2 | H_T | \Phi_{63}, -1/2 \rangle}{2} \pm \sqrt{\Delta_2^2 + 4V_2^2} \quad (4-22)$$

$$E_5^,' = \frac{\langle \Phi_{43}, -1/2 | H_T | \Phi_{43}, -1/2 \rangle + \langle \Phi_{64}, 3/2 | H_T | \Phi_{64}, 3/2 \rangle}{2} \pm \sqrt{\Delta_3^2 + 4V_3^2} \quad (4-23)$$

$$E_7^,' = \langle \Phi_{45}, -3/2 | H_T | \Phi_{45}, -3/2 \rangle \quad (4-24)$$

$$E_8^,' = \langle \Phi_{64}, 3/2 | H_T | \Phi_{64}, 3/2 \rangle \quad (4-24)$$
where the subscript on \( E' \) corresponds to the subscript of the eigenstate \( |\Psi\rangle \). Using the matrix elements of \( H_D \) and \( H_Q \) as calculated in Appendix D, the following expressions are obtained for the diagonal matrix elements of the total Hamiltonian:

\[
\langle \Phi_4, m_I | H_T | \Phi_4, m_I \rangle = E^o + A/2 - 2 \kappa_o H
- m_I \left( \frac{4}{5} \alpha < \frac{1}{T^2} > + \frac{1}{2} \xi \right) + \frac{b}{4} (m_I^2 - \frac{3}{4})
\]

and

\[
\langle \Phi_6, m_I | H_T | \Phi_6, m_I \rangle = E^o - (D_0^2/2 + \sqrt{2} C \xi D_0)\gamma
- m_I \alpha < \frac{1}{T^3} > (6 D_0^2 + 2 C^2 + \sqrt{2} 3 C \xi D_0) / 5 + \frac{b}{4} (D_0^2 - 2 C^2) \xi m_I
- C^2 \kappa_o H + b (D_0^2 - 2 C^2) / 4 \cdot (m_I^2 - 5/4) .
\]

The off-diagonal matrix elements of \( H_D \) as calculated in Appendix D, are:

\[
V^i = \langle \Phi_4, m_I | H_D | \Phi_6, m_I \rangle = \frac{1}{2} \left[ (\frac{7}{5} + \frac{3}{2} C \xi \xi D_0) \alpha < \frac{1}{T^2} \right]
- D_0 \xi \cdot \sqrt{(5/2 - m_I)(3/2 + m_I)}
\]

where \( i = 1,2,3 \) for \( m_I = 3/2, 1/2 \) and \(-1/2\) respectively.

If the off-diagonal matrix elements of \( H_D \) are assumed to be zero, then there is no coupling between the states of the same total magnetic quantum number \( m_I \), and the states \( |\Phi_4, m_I\rangle \) and \( |\Phi_6, m_I\rangle \) are the eigenstates of \( H_T \). For this case \( (V^i = 0) \), the energies of the states \( |\Phi_4, m_I\rangle \) and \( |\Phi_6, m_I\rangle \) are plotted in Fig. 4-2 as a function of the external magnetic field \( H \). There are two sets of levels, one originating from the \( ^2P_{3/2-3/2} \) state and the
Fig. 4-2. The hyperfine levels of $^2P_{3/2-3/2}$ and $^2P_{1/2-1/2}$ in Li$^7$ as a function of an external magnetic field $H$, without considering the coupling between the hyperfine states.
Fig. 4-3. The hyperfine levels of $^2P_{3/2-3/2}$ and $^2P_{1/2-1/2}$ in Li$^7$ as a function of an external magnetic field $H$. The coupling between the different hyperfine states is considered here.
other from the $^{2p}_{1/2-1/2}$ state. These two sets of states cross each other near $H = 4825$ gausses. In order to get a "crossing" signal in the scattered beam, the states which cross each other must decay to a common ground state. There are four such crossings of interest where the two levels have the same $m_I$.

If the off-diagonal matrix elements of $H_D$ are considered to be non-zero, the two states of the same $m$ are coupled and the crossings mentioned earlier no longer exist. The energy eigenvalues for this case ($V_i \neq 0$) are plotted in Fig. 4-3 as a function of the external magnetic field $H$. There are three pairs of states which anticross and there are six crossings that contribute to the signal. These crossings are far apart. The curves in Fig. 4-3 are quite different from the ones presented by Wieder and Eck (7). According to their graph only two crossings contribute to the signal.

From Eq. (4-8), the intensity of the scattered beam for a given direction and polarization of the incident and the scattered beam, is

$$I(I_n, n_e, \omega, \lambda) \propto \sum_{c' c} \left[ \sum_{\lambda} P_{c' \lambda}^{\lambda} P_{c \lambda}^{\lambda} P_{c' c}^{\lambda} P_{c c}^{\lambda} / \gamma \right] + \frac{1}{2} \sum_{\lambda \neq \lambda'} \frac{2 \tau \text{Re}(P_{c' \lambda}^{\lambda} P_{c \lambda}^{\lambda} P_{c' c}^{\lambda} P_{c c}^{\lambda}) + 2(E_{\lambda'} - E_{\lambda}) \text{Im}(P_{c' \lambda}^{\lambda} P_{c \lambda}^{\lambda} P_{c' c}^{\lambda} P_{c c}^{\lambda})}{(E_{\lambda'} - E_{\lambda})^2 + \gamma^2}$$  \hspace{1cm} (4-27)

where the summations over $\lambda$ and $\lambda'$ are over the excited states as defined in Eq. (4-17) and the summation over $c$ and $c'$ are over the ground states. The following states are the ground states of Li$^7$:

$$|C_j\rangle = |0 \frac{1}{2} 0 \frac{1}{2} \rangle | \frac{3}{2} m_I \rangle$$
where \( j = 1, 2, 3, 4 \) for \( m_\perp = 3/2, 1/2, -1/2 \) and \(-3/2\), respectively,
and

\[
|C_j\rangle = |0 \ 1/2 \ 0 - 1/2\rangle |3/2 \ m_\perp\rangle ,
\]

where \( j = 5, 6, 7, 8 \) for \( m_\perp = 3/2, 1/2, -1/2 \) and \(-3/2\), respectively. It is assumed that there is no coupling between the different hyperfine states of the ground states.

Substituting in Eq. (4-27), the matrix elements \( P_{c_i, m_i}^\lambda, P_{c_i, m_i}^{\pm} \), \( P_{c_i, c_i}^\alpha \) and \( P_{c_i, c_i}^\beta \) in terms of the states \( |\phi_4, m_\perp\rangle \) and \( |\phi_4, m_\perp\rangle \), using the selection rule \( \Delta m_\perp = 0 \) (\( P_{c_i, m_i}^\lambda = 0 \) if \( \langle c \rangle \) and \( \langle m_\perp \rangle \) have different \( m_\perp \)) and regrouping the terms within the numerator, one obtains the following expression for the anticrossing signals (after some tedious algebraic manipulations):

\[
S_A \propto (-i) \sum_{c=1,2} 2V_i^2 \left( \frac{\Sigma_c |P_{c_a, c_i}^\lambda|^2 - \Sigma_c |P_{c_b, c_i}^\pm|^2}{\Delta_i^2 + 4V_i^2 + \gamma^2} \right) \tag{4-28}
\]

where the states \( |a_i\rangle \) and \( |b_i\rangle \) stand for a set of two states which are coupled by \( H_D \). In terms of \( |\phi_4, m_\perp\rangle \) and \( |\phi_6, m_\perp\rangle \), the states \( |a_i\rangle \) and \( |b_i\rangle \) are:

\[
|a_i\rangle = |\phi_4, m_\perp\rangle ,
\]

\[
|b_i\rangle = |\phi_6, m_\perp-1\rangle ,
\]
where \( i = 1, 2, 3 \) for \( m_i = 3/2, 1/2, \) and \(-1/2, \) respectively.

There are four terms of the form \( |\langle c| \hat{\rho}^i | \phi_i, m_i \rangle|^2 \) and four of the form \( |\langle c| \hat{\rho}^i | \phi_i, m_i \rangle|^2 \) in the total intensity \( I (\alpha, \beta, \gamma, \lambda) \). These terms provide a very slowly varying signal as a function of the external magnetic field. They are not discussed here and therefore are absorbed in the background signal.

The remaining terms in Eq. (4-27) give the crossing signal:

\[
S^c = \sum_\mu \sum_{\mu'} \sum_{cc'} \frac{\tau \text{Re}(\hat{E}_{\mu} \hat{E}_{\mu'} \hat{P}_{cc' \mu} \hat{P}_{cc' \mu'}) + (E_{\mu'} - E_{\mu}) \text{Im}(\hat{E}_{\mu} \hat{E}_{\mu'} \hat{P}_{cc' \mu} \hat{P}_{cc' \mu'})}{(E_{\mu} - E_{\mu'})^2 + \gamma^2/4}
\]

(4-29)

where

\( \mu = 1 \) for \( \mu' = 3, 4, 5, 6, 7, 8 \)

\( \mu = 2 \) for \( \mu' = 3, 4, 5, 6, 7, 8 \)

\( \mu = 3 \) for \( \mu' = 5, 6, 7, 8 \)

\( \mu = 4 \) for \( \mu' = 5, 6, 7, 8 \)

\( \mu = 5 \) for \( \mu' = 7, 8 \)

\( \mu = 6 \) for \( \mu' = 7, 8 \)

\( \mu = 7 \) for \( \mu' = 8 \).

These values of \( \mu \) and \( \mu' \) are the subscript of the eigenstates \( |\psi\rangle \) defined in Eq. (4-17).

Calculating the product \( \hat{P}_{cc' \mu} \hat{P}_{cc' \mu'} \) in terms of the orbital states \( |e \, m_e \rangle \) (see Appendix E) and substituting the result into Eq. (4-29), one obtains:

\[
S^c \propto C_0 \left[ \frac{2 \tau \text{Re}(f) + (E_{\epsilon} - E_{\epsilon'}) \text{Im}(f)}{2 (E_{\epsilon} - E_{\epsilon'})^2 + \gamma^2} + D_1^2 D_2^2 \frac{\tau \text{Re}(f') + (E_{\epsilon'} - E_{\epsilon''}) \text{Im}(f')}{(E_{\epsilon'} - E_{\epsilon''})^2 + \gamma^2} \right]
\]
\[
\begin{align*}
&+ C_1^2 \frac{\tau^2 Re(\epsilon) - (E_1 - E_3) Im(\epsilon)}{(E_1 - E_3)^2 + \gamma^2} + C_1^2 C_2^2 \frac{\tau Re(\epsilon) + (E_2 - E_3) Im(\epsilon)}{(E_2 - E_3)^2 + \gamma^2} \\
&+ C_1^2 D_2 \frac{\tau Re(\epsilon) + (E_2 - E_4) Im(\epsilon)}{(E_2 - E_4)^2 + \gamma^2} + D_1^2 \frac{\tau Re(\epsilon) - (E_2 - E_4) Im(\epsilon)}{(E_2 - E_4)^2 + \gamma^2} \\
&+ C_3^2 D_2^2 \frac{\tau Re(\epsilon) + (E_2 - E_5) Im(\epsilon)}{(E_2 - E_5)^2 + \gamma^2} + D_3^2 D_2^2 \frac{\tau Re(\epsilon) + (E_3 - E_4) Im(\epsilon)}{(E_3 - E_4)^2 + \gamma^2} \\
&+ D_3^2 \frac{\tau Re(\epsilon) + (E_3 - E_5) Im(\epsilon)}{(E_3 - E_5)^2 + \gamma^2} + C_3^2 \frac{\tau Re(\epsilon) + (E_3 - E_6) Im(\epsilon)}{(E_3 - E_6)^2 + \gamma^2} \\
&\text{(4.30)}
\end{align*}
\]

where

\[
\epsilon = \langle 001| \hat{p} \cdot \hat{e}_\lambda | 10 \rangle \langle 1-1| \hat{p} \cdot \hat{e}_\lambda | 00 \rangle \langle 10| \hat{p} \cdot \hat{e}_\mu | 00 \rangle \langle 00| \hat{p} \cdot \hat{e}_\mu | 1-1 \rangle
\]

(4.31)

The energies \(E_j\) are defined in Eq.(4.21) to Eq.(4.24) and the coefficients \(C_i\) and \(D_i\) in Eq.(4.18). The vector \(\hat{p}\) is the momentum of the valence electron in \((\\lambda, \\mu)\) and \(\hat{e}_\lambda\) and \(\hat{e}_\mu\) define the direction and polarization of the incident and the scattered beams, respectively.

From Eq.(4.28) and Eq.(4.30), the anticrossing and crossing signals are calculated for the following experimental situation.
The incident beam is plane polarized and is along the x-direction. The scattered beam is also plane polarized and is along the y-direction. The plane of polarization of the incident beam makes an angle $\theta_a$ and the plane of polarization of the scattered beam makes an angle $\theta_e$ with the z-axis, respectively. The results are:

$$S_A \propto (-1) C_o^4 \left(1 - 3 \cos^2 \theta_e \right) \left(1 - 3 \cos^2 \theta_a \right) \sum_{\xi = 1, 2, 3} V_{\xi}^2 / (\Delta^{2}_\xi + 4V^2_{\xi} + \gamma^2)$$

and

$$S_C \propto (-1) C_o^2 \sin 2 \theta_e \sin 2 \theta_a \left[ \frac{D_1^2 C_2^2 (E_1' - E_3')}{(E_1' - E_3')^2 + \gamma^2} + \frac{C_1^2 D_1^2 (E_2' - E_1')}{(E_2' - E_1')^2 + \gamma^2} + \frac{C_2^2 D_2^2 (E_3' - E_2')}{(E_3' - E_2')^2 + \gamma^2} \right] \quad (4-32)$$

$$+ D_1^2 D_2^2 \frac{(E_1' - E_4')}{(E_1' - E_4')^2 + \gamma^2} + \frac{C_1^2 D_1^2 (E_2' - E_4')}{(E_2' - E_4')^2 + \gamma^2} + \frac{C_2^2 D_2^2 (E_3' - E_5')}{(E_3' - E_5')^2 + \gamma^2}$$

$$+ D_3^2 D_2^2 \frac{(E_3' - E_6')}{(E_3' - E_6')^2 + \gamma^2} + \frac{C_3^2 D_3^2 (E_4' - E_5')}{(E_4' - E_5')^2 + \gamma^2} + \frac{C_4^2 D_4^2 (E_5' - E_6')}{(E_5' - E_6')^2 + \gamma^2}$$

$$+ C_2^2 D_2^2 \frac{(E_2' - E_3')}{(E_2' - E_3')^2 + \gamma^2} + \frac{C_3^2 D_3^2 (E_3' - E_4')}{(E_3' - E_4')^2 + \gamma^2} + \frac{C_4^2 D_4^2 (E_4' - E_5')}{(E_4' - E_5')^2 + \gamma^2}$$

$$+ \frac{C_1^2 (E_1' - E_2')}{(E_1' - E_2')^2 + \gamma^2} + \frac{C_2^2 (E_2' - E_3')}{(E_2' - E_3')^2 + \gamma^2} + \frac{C_3^2 (E_3' - E_4')}{(E_3' - E_4')^2 + \gamma^2} + \frac{C_4^2 (E_4' - E_5')}{(E_4' - E_5')^2 + \gamma^2} + \frac{C_5^2 (E_5' - E_6')}{(E_5' - E_6')^2 + \gamma^2}$$

$$+ \frac{C_1^2 (E_1' - E_2')}{(E_1' - E_2')^2 + \gamma^2} + \frac{C_2^2 (E_2' - E_3')}{(E_2' - E_3')^2 + \gamma^2} + \frac{C_3^2 (E_3' - E_4')}{(E_3' - E_4')^2 + \gamma^2} + \frac{C_4^2 (E_4' - E_5')}{(E_4' - E_5')^2 + \gamma^2} + \frac{C_5^2 (E_5' - E_6')}{(E_5' - E_6')^2 + \gamma^2}$$

$$+ \frac{C_1^2 (E_1' - E_2')}{(E_1' - E_2')^2 + \gamma^2} + \frac{C_2^2 (E_2' - E_3')}{(E_2' - E_3')^2 + \gamma^2} + \frac{C_3^2 (E_3' - E_4')}{(E_3' - E_4')^2 + \gamma^2} + \frac{C_4^2 (E_4' - E_5')}{(E_4' - E_5')^2 + \gamma^2} + \frac{C_5^2 (E_5' - E_6')}{(E_5' - E_6')^2 + \gamma^2}$$

$$+ \frac{C_1^2 (E_1' - E_2')}{(E_1' - E_2')^2 + \gamma^2} + \frac{C_2^2 (E_2' - E_3')}{(E_2' - E_3')^2 + \gamma^2} + \frac{C_3^2 (E_3' - E_4')}{(E_3' - E_4')^2 + \gamma^2} + \frac{C_4^2 (E_4' - E_5')}{(E_4' - E_5')^2 + \gamma^2} + \frac{C_5^2 (E_5' - E_6')}{(E_5' - E_6')^2 + \gamma^2}$$

$$+ \frac{C_1^2 (E_1' - E_2')}{(E_1' - E_2')^2 + \gamma^2} + \frac{C_2^2 (E_2' - E_3')}{(E_2' - E_3')^2 + \gamma^2} + \frac{C_3^2 (E_3' - E_4')}{(E_3' - E_4')^2 + \gamma^2} + \frac{C_4^2 (E_4' - E_5')}{(E_4' - E_5')^2 + \gamma^2} + \frac{C_5^2 (E_5' - E_6')}{(E_5' - E_6')^2 + \gamma^2}$$

$$+ \frac{C_1^2 (E_1' - E_2')}{(E_1' - E_2')^2 + \gamma^2} + \frac{C_2^2 (E_2' - E_3')}{(E_2' - E_3')^2 + \gamma^2} + \frac{C_3^2 (E_3' - E_4')}{(E_3' - E_4')^2 + \gamma^2} + \frac{C_4^2 (E_4' - E_5')}{(E_4' - E_5')^2 + \gamma^2} + \frac{C_5^2 (E_5' - E_6')}{(E_5' - E_6')^2 + \gamma^2}$$

$$+ \frac{C_1^2 (E_1' - E_2')}{(E_1' - E_2')^2 + \gamma^2} + \frac{C_2^2 (E_2' - E_3')}{(E_2' - E_3')^2 + \gamma^2} + \frac{C_3^2 (E_3' - E_4')}{(E_3' - E_4')^2 + \gamma^2} + \frac{C_4^2 (E_4' - E_5')}{(E_4' - E_5')^2 + \gamma^2} + \frac{C_5^2 (E_5' - E_6')}{(E_5' - E_6')^2 + \gamma^2}$$

$$+ \frac{C_1^2 (E_1' - E_2')}{(E_1' - E_2')^2 + \gamma^2} + \frac{C_2^2 (E_2' - E_3')}{(E_2' - E_3')^2 + \gamma^2} + \frac{C_3^2 (E_3' - E_4')}{(E_3' - E_4')^2 + \gamma^2} + \frac{C_4^2 (E_4' - E_5')}{(E_4' - E_5')^2 + \gamma^2} + \frac{C_5^2 (E_5' - E_6')}{(E_5' - E_6')^2 + \gamma^2}$$

$$+ \frac{C_1^2 (E_1' - E_2')}{(E_1' - E_2')^2 + \gamma^2} + \frac{C_2^2 (E_2' - E_3')}{(E_2' - E_3')^2 + \gamma^2} + \frac{C_3^2 (E_3' - E_4')}{(E_3' - E_4')^2 + \gamma^2} + \frac{C_4^2 (E_4' - E_5')}{(E_4' - E_5')^2 + \gamma^2} + \frac{C_5^2 (E_5' - E_6')}{(E_5' - E_6')^2 + \gamma^2}$$

$$+ \frac{C_1^2 (E_1' - E_2')}{(E_1' - E_2')^2 + \gamma^2} + \frac{C_2^2 (E_2' - E_3')}{(E_2' - E_3')^2 + \gamma^2} + \frac{C_3^2 (E_3' - E_4')}{(E_3' - E_4')^2 + \gamma^2} + \frac{C_4^2 (E_4' - E_5')}{(E_4' - E_5')^2 + \gamma^2} + \frac{C_5^2 (E_5' - E_6')}{(E_5' - E_6')^2 + \gamma^2}.$$
From Eq. (4-32), one finds that the anticrossing signal vanishes for either $\theta_a = \cos^2(1/\sqrt{3})$ or $\theta_e = \cos^2(1/\sqrt{3})$. Similarly from Eq. (4-33), the crossing signal is found to vanish if $\theta_a = \pi n/2$ or $\theta_e = m \pi/2$, where $n$ and $m$ are integers. Thus, there is no crossing signal if the polaroids are oriented parallel, perpendicular or anti-parallel to the magnetic field. These signals are plotted in Fig. 4-4 and Fig. 4-5 as a function of the external magnetic field. The anticrossing signal is Lorentizian and the crossing signal is dispersion type. In order to compare these signals with the experimental signals obtained by Wieder and Eck (7), the derivative of the signals are plotted in Fig. 4-6 and Fig. 4-7 as a function of $H$. It is observed that the two signals are in good agreement with the experimental signals (Compare Fig. 4-6 with Fig. 4-8 and Fig. 4-7 with Fig. 4-9). In Fig. 4-6, the separation between the maximum and minimum of the derivative signal $\frac{dS_A}{dH}$ is approximately 35 gauss which is the same as that obtained by Eck, Foldy and Wieder (2).

The form of the derivative signal $\frac{dS_e}{dH}$ in Fig. 4-7 is similar to that of the experimental signal in Fig. 4-9. The position of the central minimum is near the center of the anticrossing signal as pointed out by Wieder and Eck (7). The separation between the prominent maximum and the minimum is of the order of 36 gauss. Wieder and Eck have not quoted the experimental value for this separation.

It is interesting to note that the crossing and anticrossing signals are observed even if the incident and the scattered beams are unpolarized. The following experimental geometry is considered...
Fig. 4-4. Anticrossing signal in $^{7}\text{Li}$ as a function of an external magnetic field $H$. The incident and the scattered beams are along the $x$- and $y$-axes and their planes of polarization make angles $\theta_\alpha$ and $\theta_\epsilon$ with the $z$-axis, respectively.
Fig. 4-5. Crossing signal in Li$^7$ as a function of an external magnetic field $H$. The incident and the scattered beams are along the $x$- and $y$-axes and their planes of polarization make angles $\theta_a$ and $\theta_c$ with the $z$-axis, respectively.

\[ \theta_a = \cos^{-1} \left( \frac{1}{\sqrt{3}} \right) \]
\[ \theta_c = \cos^{-1} \left( \frac{1}{1/3} \right) \]
Fig. 4-6. Derivative of the anticrossing signal in Li$^7$ as a function of the external magnetic field $H$. The incident and the scattered beams are along the $x$- and $y$-axes and their planes of polarization make angles $\theta_a$ and $\theta_e$ with the $z$-axis, respectively.
Fig. 4-7. Derivative of the crossing signal in Li$^7$ as a function of the external magnetic field $H$. The incident and the scattered beams are along the $x$- and $y$-axes and their planes of polarization make angles $\theta_\alpha$ and $\theta_e$ with the $z$-axis, respectively.

$e^{-} = \cos^{-1}(\frac{1}{\sqrt{3}})$

$\theta_\alpha = \cos^{-1}(\frac{1}{\sqrt{3}})$

$\theta_e = \cos^{-1}(\frac{1}{\sqrt{3}})$
Fig. 4-8. The derivative of the anticrossing signal in the $2^2\text{P}$ term of Li$^7$ as obtained by Wieder and Eck (7) experimentally.

Fig. 4-9. The derivative of the crossing signal in the $2^2\text{P}$ term of Li$^7$ as obtained by Wieder and Eck (7) experimentally.
for this case. The incident and the scattered beams are unpolarized and they make angles $\theta_1$ and $\theta_2$ with the z-axis, respectively (see Fig. 4-10).

The anticrossing signal in this case has the same functional dependence in $H$ and the same angular dependence in $\theta_1$ and $\theta_2$ as discussed earlier for the polarized beam, except that $\theta_q$ and $\theta_r$ are now replaced by $\theta_1$ and $\theta_2$, respectively. This signal is independent of the azimuthal angle of the incident and the scattered beams.

The following expression is obtained for the crossing signal

$$S_c \propto C_0^2 \sin^2 \theta_1 \sin^2 \theta_2 \left[ D_1^2 C_2^2 \gamma P + Q \left( \frac{E_1 - E_3}{E_1 - E_3} \right)^2 + \tau^2 \right]$$

$$+ D_1^2 D_2^2 \frac{\gamma P + Q \left( \frac{E_2 - E_4}{E_2 - E_4} \right)^2 + \tau^2}{(E_2 - E_4)^2 + \tau^2} + C_1^2 D_2^2 \frac{\gamma P + Q \left( \frac{E_2 - E_4}{E_2 - E_4} \right)^2 + \tau^2}{(E_2 - E_4)^2 + \tau^2}$$

$$+ C_1^2 C_2^2 \frac{\gamma P + Q \left( \frac{E_2 - E_4}{E_2 - E_4} \right)^2 + \tau^2}{(E_2 - E_4)^2 + \tau^2} + C_1^2 D_2^2 \frac{\gamma P + Q \left( \frac{E_2 - E_4}{E_2 - E_4} \right)^2 + \tau^2}{(E_2 - E_4)^2 + \tau^2}$$

$$+ D_3^2 D_2^2 \frac{\gamma P + Q \left( \frac{E_3 - E_6}{E_3 - E_6} \right)^2 + \tau^2}{(E_3 - E_6)^2 + \tau^2} + C_3^2 D_2^2 \frac{\gamma P + Q \left( \frac{E_3 - E_6}{E_3 - E_6} \right)^2 + \tau^2}{(E_3 - E_6)^2 + \tau^2}$$

$$+ C_3^2 D_2^2 \frac{\gamma P + Q \left( \frac{E_3 - E_6}{E_3 - E_6} \right)^2 + \tau^2}{(E_3 - E_6)^2 + \tau^2} + D_3^2 \frac{\gamma P + Q \left( \frac{E_3 - E_6}{E_3 - E_6} \right)^2 + \tau^2}{(E_3 - E_6)^2 + \tau^2}$$

$$+ C_3^2 \frac{\gamma P + Q \left( \frac{E_3 - E_6}{E_3 - E_6} \right)^2 + \tau^2}{(E_3 - E_6)^2 + \tau^2} \right] \quad (4-34)$$
Fig. 4-10. The experimental geometry.
Fig. 4-11. Crossing signal in Li$^7$ as a function of the external magnetic field $H$. The incident and the scattered beams are unpolarized and are in the x-z plane.
Fig. 4-12. Derivative of the crossing signal in Li\textsuperscript{7} as a function of the external magnetic field H. The incident and the scattered beams are unpolarized and are in the x-z plane.
where

$$\mathbf{p} + i \mathbf{q} = e^{i(\phi_2 - \phi_1)}.$$

This signal depends on the difference of the azimuthal angles of the incident and the scattered beams. If $\phi_2 - \phi_1$ is $\pi/2$ or $3\pi/2$, the signal is pure dispersion. If $\phi_2 - \phi_1$ is 0 or $\pi$ then the signal is pure lorentizian. The signal vanishes for $\theta_1 = m\pi/2$ or $\theta_2 = n\pi/2$ where $n$ and $m$ are integers. The signal $S_c$ in Eq.(4-34) is plotted in Fig. 4-11 as a function of the external magnetic field. The incident and the scattered beams are in the x-z plane ($\phi_1 = \phi_2 = 0$). This signal is pure lorentzian and has a maximum near 4826 gauss. The derivative signal $\frac{dS_c}{dH}$ is also plotted in Fig. 4-12. The separation between the maximum and the minimum in this signal is approximately 28 gauss.


APPENDICES
APPENDIX A

1. **Matrix Elements of H**

The perturbation involving the interaction of the radiation field with the atom is

$$H = e \sum_{k_r} \left( \frac{2\pi}{L^3} \right)^{1/2} \left[ \alpha(k_r) e^{i \vec{k}_r \cdot \vec{r}} + \alpha^+(k_r) e^{-i \vec{k}_r \cdot \vec{r}} \right] \hat{e}_{\sigma} \cdot \vec{p}$$

(A-1)

where the summation over $k_r$ is over frequency, direction and polarization of the radiation field. The operators $\alpha(k_r)$ and $\alpha^+(k_r)$ are the anihilation and creation operators, respectively, for a photon with wave vector $\vec{k}_r$ and polarization $\hat{e}_{\sigma}$. The vectors $\vec{r}$ and $\vec{p}$ are the position and momentum vectors, respectively, of the electron with respect to the center of mass of the atom. The radiation field is considered to be enclosed in a box of volume $L^3$.

In the dipole approximation ($|k_r \cdot \vec{r}| \ll 1$),

$$e^{i \vec{k}_r \cdot \vec{r}} \approx 1$$

and the matrix elements $H_{f,i}$, $H_{f,j}$, $H_{o,i}$ and $H_{o,j}$ reduce to

$$H_{f,i} = \langle f | H | i \rangle = \langle c | \langle \vec{k}_\lambda | (-i \vec{k}_r \cdot \hat{H}) | H \rangle | k \rangle \langle k | \alpha \rangle$$

$$= e \left( \frac{2\pi}{L^3} \right)^{1/2} \left\{ \frac{\pi}{\alpha} \right\} \left\{ \frac{\pi}{\alpha} \right\} \left\{ \frac{\pi}{\alpha} \right\} \left\{ \frac{\pi}{\alpha} \right\}$$

(A-2)

$$H_{f,j} = e \left( \frac{2\pi}{L^3} \right)^{1/2} \left\{ \frac{\pi}{\alpha} \right\} \left\{ \frac{\pi}{\alpha} \right\} \left\{ \frac{\pi}{\alpha} \right\} \left\{ \frac{\pi}{\alpha} \right\}$$
and

\[ H_{0i} = \epsilon (2\pi / (L^3 k_r))^{1/2} \frac{p_{ca}^\epsilon}{\sqrt{n(k_r)}} \]

\[ H_{0j} = \epsilon (2\pi / (L^3 k_r))^{1/2} \frac{p_{cb}^\epsilon}{\sqrt{n(k_r)}} \]

where \( n(k) \) and \( n(k_r) \) are the occupation numbers of the emitted and the absorbed photons, respectively, and

\[ \tilde{\rho}_{C}^{m} = \langle \tilde{c} | \hat{\mathbf{e}}_m \cdot \hat{\mathbf{p}} | \tilde{c} \rangle ; \quad m = \lambda, \alpha ; \quad \ell = a, b. \]

Usually, the spontaneous emission process dominates over the induced one, which means that \( n(k_\lambda) \ll 1 \). Thus, one can neglect the occupation number \( n(k_\lambda) \) compared to one in the matrix elements \( H_{ij} \) and \( H_{ji} \); and they therefore reduce to

\[ H_{ji} = \epsilon \sqrt{\frac{2\pi}{L^3 k_\lambda}} \frac{p_{ca}^\lambda}{\sqrt{n(k_\lambda)}} \]

and

\[ H_{jj} = \epsilon \sqrt{\frac{2\pi}{L^3 k_\lambda}} \frac{p_{cb}^\lambda}{\sqrt{n(k_\lambda)}} \]
2. Evaluation of $\gamma_{i\ell}$

In Eq. (2-33), $\gamma_{i\ell}$ is defined as

$$-\frac{i}{2} \gamma_{i\ell}(E) = \sum_{f} H_{if} H_{if}^* \mathcal{S}(E-E_f').$$

using Eq. (2-28), the real part of $\gamma_{i\ell}$ is written as

$$\text{Re}(\gamma_{i\ell}(E)) = 2\pi \sum_{f} H_{if} H_{if}^* \delta(E-E_f'). \quad (A-6)$$

for a continuous distribution of frequencies, the summation is replaced by an integration:

$$\sum_{f} \rightarrow \left(\frac{1}{2\pi}\right)^3 \sum_{\lambda} \int_{0}^{\infty} k_{\lambda}^2 dk_{\lambda} \int_{0}^{\pi} \sin \theta_{e} d\theta_{e} \int_{0}^{2\pi} d\phi_{e} (A-7)$$

where $\theta_{e}$ and $\phi_{e}$ are the polar and azimuthal angles of the wave vector $\vec{k}_{\lambda}$. The summation over $\lambda$ is over the polarization of the emitted radiation. Thus, using Eq. (A-2) and Eq. (A-7), one gets

$$\text{Re}(\gamma_{i\ell}(E)) = \frac{e^2}{2\pi} \sum_{\lambda} \left[ \int_{0}^{2\pi} |P_{\lambda} \alpha|^{2} \sin \theta_{e} d\theta_{e} d\phi_{e} \right] \quad (A-8)$$

If one defines

$$D_{\alpha} = \frac{e^2}{2\pi} \sum_{\lambda} \int_{0}^{2\pi} |P_{\lambda} \alpha|^{2} \sin \theta_{e} d\theta_{e} d\phi_{e} \quad (A-9)$$

then the real part of $\gamma_{i\ell}$ can be written as

$$\text{Re}(\gamma_{i\ell}(E)) = (E-E_c'+k_{\alpha}) D_{\alpha} \quad (A-10)$$
This expression shows a linear $E$ dependence.

The matrix element $\mathcal{F}^\lambda_{ca}$ can be written (10) as

$$\mathcal{F}^\lambda_{ca} = i (E_c - E_a) \langle c | \vec{r} | a \rangle \cdot \hat{e}_\lambda$$

(A-11)

where $E_a$ and $E_c$ are the energies of the atomic states $|a\rangle$ and $|c\rangle$, respectively.

Combining Eq.(A-9) and Eq.(A-11), one obtains

$$D_a = \frac{\alpha^2}{2\pi} (E_c - E_a)^2 \sum_{\lambda} \int_0^\pi \int_0^{2\pi} \langle c | \vec{r} | a \rangle \cdot \hat{e}_\lambda |^2 \sin \theta_a d\theta_a d\phi_a$$

$$= \varepsilon^2 (E_c - E_a)^2 \cdot \alpha_o^2$$

(A-12)

where $\alpha_o$ is the Bohr radius and $\sum_{\lambda} \int_0^\pi \int_0^{2\pi} \langle c | \vec{r} | a \rangle \cdot \hat{e}_\lambda |^2 \sin \theta_a d\theta_a d\phi_a$ is assumed to be of the order of $\alpha_o$. For an atomic system, this is a legitimate assumption.

In relativistic units ( $\hbar = m = c = 1$ ), the fine structure constant $\alpha$ is 1/137, the Bohr radius is

$$\alpha_o = \varepsilon^2 = 137$$

and the energy unit in wave numbers is

$$1 \text{ cm}^{-1} = 2.24 \times 10^{-10}.$$  

Thus the energy difference $(E_c - E_a)$ for an atomic system is of the order of $10^{-6}$ relativistic units and therefore

$$D_a \sim 10^{-10}.$$  

(A-13)

The real part of $\gamma_{ij}$ can be obtained in a similar fashion:
\[ \Re(\tau_{jj}(E)) = (E-E_c' + k_\nu) D_b \]

where \( D_b \) is defined by

\[ D_b = \frac{e^2}{2\pi} \sum \int_0^\infty \int_0^{2\pi} |<c| \vec{p} \cdot \vec{E}_\lambda |b>|^2 \sin \theta_e d\theta_e d\phi_e. \quad (A-14) \]

The magnitude of \( D_b \) is of the same order as that of \( D_a \).

The imaginary parts of \( \tau_{ic}(E) \) and \( \tau_{jj}(E) \) are

\[ \Im(\tau_{ic}(E)) = 2 \sum_f \mathcal{P} \left( \frac{H_{ic} H_{j\nu}^*}{E - E_f} \right) \]

and

\[ \Im(\tau_{jj}(E)) = 2 \sum_f \mathcal{P} \left( \frac{H_{jj} H_{j\nu}^*}{E - E_f} \right) \]

where the summation over \( f \) is over frequency, direction and polarization of the emitted radiation and \( \mathcal{P} \) is the principal value operator defined in Chapter II.

For a continuous distribution of frequencies, the summation over \( f \) is replaced by an integral of the form given in Eq. (A-7).

Thus one gets

\[ \Im(\tau_{ic}(E)) = \frac{2L^3}{(2\pi)^3} \sum \int_0^\infty \int_0^{2\pi} k_\lambda^2 dk_\lambda \mathcal{P} \left( \frac{|H_{ic}^*|^2}{E - E_f} \right) \sin \theta_e d\theta_e d\phi_e \]

\[ (A-15) \]
Combining Eq. (2-18), Eq. (A-2) and Eq. (A-15), the imaginary part of \( \tau_{ij}(E) \) is written as

\[
\text{Im}(\tau_{ij}(E)) = \frac{2L^3}{(2\pi)^3} \sum_{\lambda} \int_0^\infty \int_0^{\frac{2\pi}{r_\lambda}} \int_0^{2\pi} k_\lambda^2 dk_\lambda P \left( \frac{1}{E-E_f^2} \right) \sin \theta \sin \phi d\theta d\phi,
\]

The integral

\[
\int_0^\infty dk \frac{1}{k^2+K^2+E-E_c^2}
\]

in the previous equation is divergent which means that the imaginary part of \( \tau_{ii}(E) \) is infinite. Similarly, it can be shown that the imaginary part of \( \tau_{ij}(E) \) is infinite. These imaginary parts \{Im(\tau_{ij}(E))\} and \( \text{Im}(\tau_{ii}(E)) \) are added to the energies, \( E_a \) and \( E_b \) of the unperturbed atomic states, respectively. This results in a shift of the atomic levels. There is a similar kind of shift in the ground state, contributed by the imaginary part of \( \Gamma(E) \). Thus all the atomic states are shifted relative to each other.

It is important to note here that only the relative separation of the states are required in the discussion. Therefore one can use
the method of renormalization of the atomic states by absorbing these infinities in the atomic states \( \Rightarrow \). One can then set

\[
E_a + \text{Im}(\gamma_{ii}) \rightarrow E_a,
\]

\[
E_b + \text{Im}(\gamma_{jj}) \rightarrow E_b,
\]

and

\[
E_c + \text{Im}(\Gamma) \rightarrow E_c.
\]

3. Evaluation of \( \Gamma \)

The expression for \( \Gamma(E) \), obtained from Eq. (2-36) and Eq. (A-3), is

\[
-\frac{i}{2} \Gamma(E) = \epsilon^2 \left( \frac{2\pi}{L^3} \right) \sum_{k_r} \frac{n(k_r)}{k_r} \left[ \frac{V_{ab} P_{ca}^a P_{bc}^c + V_{ba} P_{ca}^b P_{bc}^c}{k_r} \right] \]

\[
+ |P_{ca}^c|^2 \left\{ E - E_i' + \frac{i}{2} \gamma_{ij}(E) \right\} + |P_{cb}^c|^2 \left\{ E - E_i' + \frac{i}{2} \gamma_{ij}(E) \right\} \]

\[
\times \left[ (E - E_i' + i \gamma_{ij}(E)/2)(E - E_i' + i \gamma_{ij}(E)/2) - |V_{ab}|^2 \right]^{-1}, \tag{A-16}
\]

For a continuous distribution of absorbed frequencies, the summation over \( k_r \) is replaced by an integration:

\[
\sum_{k_r} \rightarrow \left( \frac{L}{2\pi} \right)^3 \int_0^\infty \int_0^\pi \int_0^{2\pi} \rho(R_r) dk_r d\theta \alpha d\phi \alpha
\]

where \( \rho(R_r) \) represents the density of radiation oscillators in the incident beam with energy between \( k_r \) and \( k_r + dk_r \). The sum
over $\omega$ is over polarization. The angles $\sigma_\alpha$ and $\phi_\alpha$ are the polar and azimuthal angles, respectively, of the wave vector $\vec{k}_r$.

The density $\rho(k_r)$ is written in terms of $I_0(k_r)$, the intensity (energy per unit area) per unit frequency per unit solid angle as

$$\rho(k_r) = \frac{I_0(k_r)}{R_r n(k_r)}$$

where $n(k_r)$ is an average photon occupation number at the frequency $k_r$ of the incident radiation (6). Thus Eq. (A-16) can be written as

$$-i \Gamma(E)/2 = \frac{e^2}{(2\pi)^2} \sum_{\sigma} \int_{0}^{\infty} dk_r \sin e_\alpha d\sigma d\phi_\alpha \frac{I_0(k_r)}{k_r^2} \left[ V_{ab} P_{ca} P_{bc} + P_{cb}^{\sigma} P_{ac}^{\sigma} V_{ba} + |P_{ca}^{\sigma}|^2 \left\{ \frac{E-E_j'}{2} + i \Gamma_{ji}(E)/2 \right\} \right]$$

$$+ |P_{cb}^{\sigma}|^2 \left\{ \frac{E-E_i'}{2} + i \Gamma_{ii}(E)/2 \right\}$$

$$\left[ (E-E_i' + i \Gamma_{ii}(E)/2) (E-E_j' + i \Gamma_{jj}(E)/2) \right]^{-1}$$

$$- |V_{ab}|^2$$

(A-17)

As discussed in the previous section, the imaginary parts of $\Gamma_{ii}(E)$ and $\Gamma_{jj}(E)$ in Eq. (A-16) are absorbed in the energies $E_i$ and $E_j$, respectively. The real parts have the following $E$ dependence (see Appendix A-2):

$$\text{Re} (\Gamma_{ii}(E)) = (E-E_i' + k_r) D_a$$
and

\[ \text{Re} \left( \gamma_{ij} (E) \right) = (E - E'_c + R_{\nu}) D_b. \]

Thus, the denominator of \( \Gamma(E) \) reduces to

\[ \{E - E'_\alpha + R_{\nu} + i \left( E - E'_c + R_{\nu} \right) D_a/2 \} \left\{ E - E'_b + k_{\nu} + i \left( E - E'_c + R_{\nu} \right) D_b/2 \} - |V_{ab}|^2. \]

By regrouping the terms and factoring out \((1+iD_a/2)(1+iD_b/2)\), the above expression becomes

\[
\begin{align*}
&\left(1 + i \frac{D_a}{2} \right) \left(1 + i \frac{D_b}{2} \right) \left\{ \left\{ E - E'_\alpha/(1+ i D_a/2) + R_{\nu} - i E'_c D_a/(2+ i D_a) \right\} \times \left\{ E - E'_b/(1+ i D_b/2) + R_{\nu} - i E'_c D_b/(2+ i D_b) \right\} \right. \\
&\left. \left. - |V_{ab}|^2 \right\}/\left\{ (1+iD_a/2) \right\} \right\} \right. \\
&\left. \left(1 + i \frac{D_a}{2} \right) \left(1 + i \frac{D_b}{2} \right) \left\{ \left\{ E - E'_\alpha/(1+ D_a^2/4) \right\} \times \left\{ E - E'_b/(2+ D_a^2/4) - E'_c D_a^2/\{4 (1+ D_a^2/4) \} \right\} \right. \\
&\left. + R_{\nu} + i \left( E'_\alpha - E'_c \right) D_a/(2+ D_a^2/2) - E'_c D_a^2/\{4 (1+ D_a^2/4) \} \right\} \right. \\
&\left. \left. \left\{ E - \frac{E'_b}{1+ \frac{D_b}{2}} + R_{\nu} + i \frac{E'_b - E'_c}{1+ \frac{D_b}{2}} D_b \\
&\left. - \frac{E'_c D_b^2}{4 \{1+ \frac{D_b}{2}\}} \right\} - \frac{|V_{ab}|^2}{\left(1 + \frac{D_a^2}{4} \right) \left(1 + \frac{D_b^2}{4} \right)} \right. \\
&\left. + \frac{|V_{ab}|^2 \left\{ \frac{D_a D_b}{4} + i \left( D_a + D_b \right)/2 \right\}}{\left(1 + \frac{D_a^2}{4} \right) \left(1 + \frac{D_b^2}{4} \right)} \right. \\
&\left. \right) \right). \number{A-18}
\end{align*}
\]

Since, the magnitudes of \( D_a \) and \( D_b \) are very small (see Appendix A-2) compared to unity, the terms \( D_a^2/4, D_b^2/4 \) and \( D_a D_b/4 \) can be neglected in Eq.(A-18). Thus the denominator reduces to
\[(1 + i D_a/2)(1 + i D_b/2) \left\{ \frac{K_e}{E_a^2 - E_b^2} + E - E_c^2 D_a^2/4 + i \left( E_c^2 D_b^2/4 + i \left( E_c^2 D_a^2/4 \right) \right) \right\}
\]
\[-|V_{ab}|^2 \left\{ \left( 1 - i (D_a + D_b)/2 \right)^2 \right\} = \left(1 + i D_a/2 \right) \left(1 + i D_b/2 \right) \left\{ \frac{K_e}{E_a^2 - E_b^2} + i \right\}
\]
\[+ \left( \frac{K_e}{E_a^2 - E_b^2} \right) \left\{ \frac{K_e}{E_a^2 - E_b^2} + i \right\} - |V_{ab}|^2 \left\{ \left( 1 - i (D_a + D_b)/2 \right)^2 \right\} \]
\[(A-19)\]

where
\[E_a'' = E_a' + \frac{E_c D_a^2}{4} - E,\]
\[E_b'' = E_b' + \frac{E_c D_b^2}{4} - E,\]
\[(A-20)\]

and \(\gamma_a\) and \(\gamma_b\) are defined in Eq.(2-43).

In Eq.(A-20), the terms \(E_c D_a^2/4\) and \(E_c D_b^2/4\) are very small compared to other terms and therefore they can be neglected. Thus one gets
\[E_a'' = E_a' - E\]
and
\[E_b'' = E_b' - E\]

The expression in Eq.(A-19) can be written as
\[\left(1 + i (D_a + D_b/2) \right) \left( k_a - k_1 \right) \left( k_b - k_2 \right) \]
\[(A-21)\]

where \(k_1\) and \(k_2\) are the two roots of the quadratic equation
\[\left( k_a - E_a'' + i \gamma_a/2 \right) \left( k_b - E_b'' + i \gamma_b/2 \right) - |V_{ab}|^2 \left\{ \left( 1 - i (D_a + D_b)/2 \right)^2 \right\} = 0.\]
The roots are

\[
\begin{align*}
k_1 &= \frac{(E_a + E_b')}{2} - E - i (\gamma_a + \gamma_b) \sqrt{4 + \frac{1}{2} \left[ \Delta^2 - x^2 + 4|V_{ab}|^2 \right]} \\
k_2 &= \frac{(E_a + E_b')}{2} - E - i (\gamma_a + \gamma_b) \sqrt{4 + \frac{1}{2} \left[ \Delta^2 - x^2 + 4|V_{ab}|^2 \right]} \quad (A-22)
\end{align*}
\]

where

\[
\Delta = E_a - E_b' 
\]

and

\[
x = (\gamma_a - \gamma_b)/2 
\]

The complex quantity inside the square root sign in Eq.(A-22) can be separated into real and imaginary parts as follows:

If

\[
Y = [A + i B]^{\frac{1}{2}}
\]

then

\[
\text{Re}(Y) = \frac{1}{\sqrt{2}} \left[ (A^2 + B^2)^{\frac{1}{2}} + A \right]^{\frac{1}{2}}
\]

and

\[
\text{Im}(Y) = \frac{1}{\sqrt{2}} \left[ (A^2 + B^2)^{\frac{1}{2}} - A \right]^{\frac{1}{2}}. 
\]

Thus one gets

\[
\begin{align*}
k_1 &\quad = \frac{1}{2} \left\{ \left( \Delta^2 - x^2 + 4|V_{ab}|^2 \right)^2 + 4\{x \Delta + 1|V_{ab}|^2(D_a + D_b)^2 \} \right\}^{\frac{1}{2}} \\
\text{and} &\quad R = (E_a - E_b')/2 - E - i (\gamma_a + \gamma_b) \sqrt{4 + \frac{1}{2} \left[ \Delta^2 - x^2 + 4|V_{ab}|^2 \right]} \quad (A-24)
\end{align*}
\]

where

\[
\begin{align*}
R = &\quad \frac{1}{\sqrt{2}} \left\{ \left( \Delta^2 - x^2 + 4|V_{ab}|^2 \right)^2 + 4\{x \Delta + 1|V_{ab}|^2(D_a + D_b)^2 \} \right\}^{\frac{1}{2}} \\
&\quad + (\Delta^2 - x^2 + 4|V_{ab}|^2) \quad (A-25)
\end{align*}
\]
and

\[ x_3 = \frac{\mathcal{T}_a + \mathcal{T}_b}{4}. \tag{A-26} \]

Since \( D_a \) and \( D_b \) are much less than one, the term \( 4|V_{ab}|^2(D_a + D_b)^2 \) in the expression for \( R \) and \( I \), can be neglected compared to the other terms. Thus one obtains

\[
\begin{align*}
R \\ I 
\end{align*}
= \frac{1}{2} \left[ \frac{(\Delta^2 - x^2 + 4V^2)^2 + 4\Delta^2 x^2}{(\Delta^2 - x^2 + 4V^2)} \right]^{1/2}
\tag{A-27}
\]

where

\[ V^2 = |V_{ab}|^2 \]

Thus, substituting the real parts of \( \mathcal{R}_{ii} \) and \( \mathcal{R}_{ij} \) from Appendix A-2 into Eq.(A-17), combining Eq.(A-17) and Eq.(A-21) and rearranging, one gets

\[
-i \gamma(E)/2 = \frac{e^2}{(2\pi)^2} \sum_{\nu} \int_0^\infty \int_0^{2\pi} d\rho \sin \theta d\rho d\phi [I_0(\mathcal{R}_\nu^\ast \mathcal{R}_\nu)] \frac{I_0(\mathcal{R}_\nu^\ast \mathcal{R}_\nu)}{k_\nu^2}
\]

\[ + V_{ab} \mathcal{P}_{cb} \mathcal{P}_{ac} + |P_{ca}|^2 \left\{ R_{\nu^\prime} \left( 1 + i D_a / 2 \right) - E_{\nu^\prime} + E \left( 1 + i D_b / 2 \right) - \frac{i}{2} E_{\nu^\prime} D_b \right\} \]

\[ + \left| P_{cb}^\ast \right|^2 \left\{ \mathcal{R}_{\nu^\prime} \left( 1 + i D_b / 2 \right) - E_{\nu^\prime} + E \left( 1 + i D_a / 2 \right) - \frac{i}{2} E_{\nu^\prime} D_a \right\} \]

\[ \left\{ \left( 1 + i D_a / 2 \right) \left( 1 + i D_b / 2 \right) \right\} \left( \mathcal{R}_{\nu^\prime} \mathcal{R}_1 \right) \left( \mathcal{R}_{\nu^\prime} \mathcal{R}_2 \right) \]

If the incident beam is monochromatic then the intensity \( I_0(k_\nu) \) in Eq.(A-28) is replaced by \( I_0 \delta(k_{\nu^\prime} - k_\nu) \), where \( I_0 \) is a constant, which determines the intensity of the incident beam and \( \delta(k_{\nu^\prime} - k_\nu) \) is a Dirac delta function. The frequency \( k_\nu^\ast \) is the frequency of the incident beam. Thus, for this case, Eq.(A-28) reduces.
to:

\[-\frac{i}{2} \Gamma(E) = \frac{E^2}{(2\pi)^2} \sum_{\alpha=0}^{\pi} \int \sin \phi_{\alpha} d\phi_{\alpha} \frac{1}{k_0^2} \left[ V_{ba} P_{ca} P_{bc}^* \right] \]

\[+ V_{ba} P_{ac}^* P_{cb} + |P_{ca}|^2 \left\{ k_\alpha^0 (1 + i Db/2) - E_b' + E (1 + i Db/2) - i E_c D_b/2 \right\} \]

\[+ |P_{cb}|^2 \left\{ k_\alpha^0 (1 + i Da/2) - E_a' + (1 + i Da/2) E - i E_c Da/2 \right\} \]

\[X \left[ (1 + i Da/2) (1 + i Db/2) (k_\alpha^0 - k_1) (k_\alpha^0 - k_2) \right]^{-1} \]  

(A-29)

when the state \( |b\rangle \) is non-decaying \( (\gamma_b = 0) \), then Eq. (A-29) becomes

\[-\frac{i}{2} \Gamma(E) = \frac{D_a I_o}{k_\alpha^0} \frac{k_\alpha^0 - E_b/(1 + i Db/2) + E - i E_c D_b/(2 + i D_b)}{(1 + i Da/2) (k_\alpha^0 - k_1) (k_\alpha^0 - k_2)} \]

(A-30)

where \( D_a \) and \( D_b \) are defined in Eq. (A-9) and Eq. (A-14).

Multiplying the numerator and denominator of Eq. (A-30) by \((1 - i D_a/2)\), neglecting the term \( D_a^2/4 \) compared to unity and rearranging the denominator one gets,

\[-\frac{i}{2} \Gamma(E) = \frac{D_a I_o}{k_\alpha^0} \frac{k_\alpha^0 - E_b/(1 + i Db/2) + E - i E_c D_b/(2 + i D_b)}{(1 - i Da/2) (k_\alpha^0 - (E_a' + E_b')/2 + E - R + i (X_3 + I))} \]

\[X \left[ k_\alpha^0 - (E_a' + E_b')/2 + E + R + i (X_3 - I) \right]^{-1} \]

(A-31)

where \( R, I \) and \( X_3 \) are defined in Eq. (A-25) and Eq. (A-26).

For an incident monochromatic line of power \( 10^{-3} \) watts, the factor \( D_a I_o / k_\alpha^0 \) is of the order of \( 10^{-4} \) \( E_c \) \( D_a \sim 10^{-10} \); \( I_o \sim 10^{-11} \) and
\( E' \sim 10^{-5} \) in relativistic units). Thus, the magnitude of \( \Gamma(E') \) is very small compared to \( E'_c \). Therefore one finds that the pole of \( G(E) \) obtained from equation:

\[
E - E'_c + \frac{i}{2} \Gamma(E') = 0
\]

is not significantly different from the pole

\[
E = E'_c - \frac{i}{2} \Gamma(E'_c)
\]

where \( \Gamma(E) \) is evaluated at \( E = E'_c \).

The expression for \( \Gamma(E'_c) \) is:

\[
-\frac{i}{2} \Gamma(E'_c) = \frac{D_{a} \Gamma_{0}}{k_{r}^{2}} \left[ \left( k_{r}^{0} - E'_b + E'_c + i \left( E'_b - E'_c \right) D_{b}/2 \right)(1-iD_{a}/2) \right] \times \left[ k_{r}^{0} - \left( E'_a + E'_b \right)/2 + E'_c - R + i(\chi_3 + I) \right]^{-1}
\]

The real part of \( \Gamma(E'_c) \) gives the decay constant of the ground state \( |c> \) (see Eq.(2-54)) and the imaginary part is absorbed in the energy \( E_c \) of the ground state, giving rise to an energy shift. In this case, the decay constant \( \Re(\Gamma) \) depends on the frequency of the incident beam.

If the incident beam is such that the energy per unit area per unit frequency, \( I_o \) is constant (white light beam) then Eq.(A-28) reduces to

\[
-\frac{i}{2} \Gamma(E) = \frac{e^2}{(2\pi)^2} I_o \sum_{\omega_o} \int \sum_{\phi_o} \left[ k_{r}(1+iD_{b}/2) - E'_b + E(1+iD_{a}/2) - iE'_c D_{b}/2 \right] \times \left[ \{ V_{ab} P_{ca} P_{bc} \}ight] + \left| P_{a} \right|^{2} \left[ k_{r}(1+iD_{a}/2) - E'_a + E(1+iD_{b}/2) - iE'_c D_{a}/2 \right]
\]
The term in the square bracket in Eq. (A-35) is a highly peaked function near the real parts of the poles. Thus one can replace

\( \frac{1}{k^2} \) in Eq. (A-35) by an average value \( \frac{1}{k^2_0} \). Thus one gets

\[
\frac{\xi}{2} \Gamma(E) = \frac{e^2}{(2\pi)^2} \frac{I_3}{k_0^2} \sum_{\alpha} \int_0^{2\pi} \frac{\sin \theta \, d\theta \, d\phi}{(1 + i \, Da/2) (1 + i \, Db/2)} \left\{ V_{ab} P_{\alpha b} P_{\alpha c} \right. \\
+ V_{ba} P_{\alpha c} P_{\alpha b} \left[ 2 \left( E \left( 1 + i \, Da/2 \right) - E_{b} - i \, E_{c} \, Da/2 \right) \right] I_1 \\
+ \left[ \left| P_{\alpha b} \right|^2 (1 + i \, Da/2) + \left| P_{\alpha c} \right|^2 (1 + i \, Da/2) \right] I_2 \}
\]  

(A-36)

where

\[
I_1 = \int_0^{\infty} \frac{dk}{(k - k_1)(k - k_2)}
\]

and

\[
I_2 = \int_{-\infty}^{\infty} \frac{dk \, k}{(k - k_1)(k - k_2)}
\]

(A-37)

The limits in the integrals of \( I_1 \) and \( I_2 \) can be extended into the non-physical region (\( k = -\infty \) to \( k = 0 \)) without significant error, because the integrands are negligible in this region. Thus one gets

\[
I_1 = \int_0^{\infty} \frac{dk}{(k - k_1)(k - k_2)}
\]

and

\[
I_2 = \int_{-\infty}^{\infty} \frac{dk \, k}{(k - k_1)(k - k_2)}
\]
The method of contour integration is used to evaluate $I_1$ and $I_2$. The results are:

$$I_1 = 0$$

$$I_2 = -i\pi$$

Thus, from Eq. (A-36) and Eq. (A-38) one obtains

$$\text{Re}(\Gamma(E)) = \frac{e^2}{(2\pi)^3} \frac{I_0}{k_0^2} \sum_0^\pi \int_0^{2\pi} \sin \theta_a d\theta_a d\phi_a \frac{\xi}{|P_{ca}|^2 + |P_{cb}|^2}.$$  \hspace{1cm} (A-38)

In the above equation, the terms $D_a^2/4$, $D_b^2/4$ and $D_a D_b/4$ are neglected compared to unity. This shows that $\text{Re}(\Gamma)$ does not depend on $E$. The average photon occupation number $n(k_0)$, in the incident beam is of the order of $I_0/k_0^3$. Thus Eq. (A-39) can be written as

$$\text{Re}(\Gamma) \sim n(k_0) k_0 \frac{e^2}{(2\pi)^2} \sum_0^\pi \int_0^{2\pi} \sin \theta_a d\theta_a d\phi_a \frac{\xi}{|P_{ca}|^2 + |P_{cb}|^2}$$

$$\sim n(k_0) (\gamma_a + \gamma_b)$$  \hspace{1cm} (A-40)

where $\gamma_a$ and $\gamma_b$ are defined in Eq. (2-43)

$$k_0 \sim (E_a' - E_c') \sim (E_b' - E_c')$$

Equation (A-40) shows that the magnitude of $\text{Re}(\Gamma)$ is very small compared to $\gamma_a$ or $\gamma_b$ provided the intensity of the incident beam is not too high.
APPENDIX B

1. Evaluation of $P(\hat{E}_e, k_\lambda)$

The probability of emission of a photon with a wave vector $\hat{k}_\lambda$ and polarization $\hat{e}_\lambda$ is

$$P(\hat{E}_e, k_\lambda) = \sum_{k_r} |b_{\gamma}(\infty)|^2$$

as defined in Eq.(2-59).

For a continuous distribution of frequencies in the incident radiation field, the summation over $k_r$ is replaced by an integration

$$\sum_{k_r} \rightarrow \left( \frac{L}{2\pi} \right)^3 \int_0^{\infty} \rho(k_r) \, dk_r$$

where

$$\rho(k_r) = \frac{I_0(k_r)}{n(k_r) k_r}.$$  

Thus, one can write

$$P(\hat{E}_e, k_\lambda) = \left( \frac{L}{2\pi} \right)^3 \int_0^{\infty} \frac{I_0(k_r)}{n(k_r) k_r} |b_{\gamma}(\infty)|^2 \, dk_r.$$  

(\text{B-1})

From Eq.(2-58), one finds that $|b_{\gamma}(\infty)|^2$ contains a factor $1/|E'_e - E'_3|^2$ which reduces to a Dirac delta function in the limit when $R \rightarrow 0$. This can be seen as follows:

$$|E'_e - E'_3|^2 = \left[ (k_\lambda - k_r)^2 + \eta^2/4 \right]^{-1} = \frac{2\pi}{\eta} \delta(k_\lambda - k_r).$$  

(\text{B-2})
where $\Gamma$ stands for the real part of $\Gamma$ and the imaginary part is absorbed in the atomic state $E_c$. In Eq. (B-2), the following representation of the Dirac delta function is used

$$\delta(x) = \frac{1}{\pi} \lim_{\omega \to \infty} \frac{\omega}{x^2 + \omega^2}$$

The assumption that $\Gamma$ is very small is appropriate provided the intensity of the incident beam is not too high ($n(k_\nu) \ll 1$) (see Eq. (A-40)).

From Eq. (2-51), Eq. (2-58), Eq. (A-3) and Eq. (A-5), $b_f(t \to \infty)$ is written as

$$b_f(t \to \infty) = \frac{2\pi e^2 \sqrt{n(k_\nu)}}{L^3 \sqrt{k_\nu k_\lambda}} \left[ \Gamma_{ab} p_{ca} p_{bc}^* + \Gamma_{ba} p_{cb} p_{ac}^* \right]$$

$$+ p_{ca}^2 (k_\lambda - E_b + i\gamma_b/2) + p_{cb}^2 (k_\lambda - E_a + i\gamma_a/2)$$

$$\times \left[ \{(k_\lambda - k_\nu) + i\gamma/2\} (k_\lambda - k_1) (k_\lambda - k_2) \right]^{-1} e^{-i\nu c t}$$

where

$$\begin{align*}
\begin{bmatrix}
E_a' \\
E_b'
\end{bmatrix} &= (E_a + E_b)/2 - E_c' \pm R - i(X_3 \mp 1),
\end{align*}$$

and $R$, $I$ and $X_3$ are defined in Eq. (A-25) and Eq. (A-26).

Thus, the probability of having a photon with wave vector $k_\lambda$ emitted, provided a photon with wave vector $k_\nu$ was absorbed is

$$|b_f(\infty)|^2 = \frac{2\pi^2 e^4 n(k_\nu)}{L^6 k_\nu k_\lambda} \cdot \frac{1}{(k_\lambda - k_\nu)^2 + \Gamma^2/4} \cdot \frac{N_i}{D_1}$$

(B-4)
\[ N_1 = |P_{cb}^\lambda|^2 |P_{ac}^\gamma|^2 [(k_\lambda - E_{b} + E_{c}')^2 + r_b^2/4] \]
\[ + |P_{cb}^\lambda|^2 |P_{bc}^\gamma|^2 [(k_\lambda - E_{a} + E_{c}')^2 + r_a^2/4] \]
\[ + V^2 \{ |P_{ca}^\lambda|^2 |P_{bc}^\gamma|^2 + |P_{cb}^\lambda|^2 |P_{ac}^\gamma|^2 + 2 \Re (P_{ca}^\lambda P_{bc}^- P_{cb} P_{ac}^+) \} \]
\[ + 2V \{ (k_\lambda - E_{a} + E_{c}') (k_\lambda - E_{b} + E_{c}') \} \]
\[ + \{ (k_\lambda - E_{a} + E_{c}') (k_\lambda - E_{b} + E_{c}') \} \frac{1}{2} \Im (P_{ac}^\gamma P_{bc}^-) \]
\[ + \{ |P_{cb}^\lambda|^2 - |P_{ca}^\gamma|^2 \} \{ |P_{ac}^\gamma|^2 - |P_{bc}^-|^2 \} \frac{1}{2} \Im (P_{cb} P_{ca}^+) \]
\[ + |P_{cb}^\lambda|^2 (k_\lambda - E_{b} + E_{c}') \gamma_b r_a / 4 \} \]
\[ + 2V \{ (k_\lambda - E_{a} + E_{c}') (k_\lambda - E_{b} + E_{c}') \} \]
\[ \frac{1}{2} \Im (P_{ac}^\gamma P_{bc}^-) \]
\[ + \frac{1}{2} \Re (P_{cb} P_{ca}^+) \]
\[ + \frac{1}{2} \Re (P_{cb} P_{ca}^+) \]
\[ + 2V \{ (k_\lambda - E_{a} + E_{c}') (k_\lambda - E_{b} + E_{c}') \} \]
\[ + \frac{1}{2} \Im (P_{ac}^\gamma P_{bc}^-) \]
\[ + \frac{1}{2} \Re (P_{cb} P_{ca}^+) \]
\[ + \frac{1}{2} \Re (P_{cb} P_{ca}^+) \]

and
\[ D_1 = \{ (k_\lambda - E_{a} + E_{c}')^2 + r_a^2/4 \} \{ (k_\lambda - E_{b} + E_{c}')^2 + r_b^2/4 \} + V^4 \]
\[ - 2V \{ (k_\lambda - E_{a} + E_{c}') (k_\lambda - E_{b} + E_{c}') \} \gamma_b r_a / 4 \}

Thus, combining Eq. (B-1), Eq. (B-2) and Eq. (B-4), one gets

\[ P(\hat{E}_{\sigma},k_\lambda) = \frac{C_4}{L^3 \Gamma R_\lambda} \left( \frac{\Gamma_0(k_F)}{k_F^2} \right) \frac{N_1}{D_1} \]
For an incident radiation beam with constant intensity \( I_0 \) (white light), the probability of having a photon with wave vector \( \mathbf{k}_\lambda \) is

\[
P(\hat{e}_\psi, \mathbf{k}_\lambda) = \frac{e^4 I_0 N_1}{L^3 \Gamma^3 k^3 \lambda D_1} \tag{B-5}
\]

This probability \( P(\hat{e}_\psi, \mathbf{k}_\lambda) \) depends on the direction and polarization of the incident radiation through the matrix elements \( p_{ac}^- \) and \( p_{bc}^- \).

It is shown in Chapter III that \( P(\hat{e}_\psi, \mathbf{k}_\lambda) \) is a highly peaked function near \( \mathbf{k}_\lambda \) equal to the frequency difference between the perturbed excited states and the ground state. Thus one can replace \( 1/k_\lambda^3 \) by an average value \( 1/k_0^3 \). Therefore \( P(\hat{e}_\psi, \mathbf{k}_\lambda) \) reduces to

\[
P(\hat{e}_\psi, \mathbf{k}_\lambda) = \frac{e^4 I_0 N_1}{L^3 \Gamma^3 k^3 \lambda D_1} \tag{B-6}
\]

In terms of a new variable

\[
k = k_\lambda - \frac{(E_a' + E_b')}{2} + E_c'
\]

one can write

\[
N_1 = \left\{ (k + \Delta/2)^2 + \frac{\gamma_6^2}{4} \right\} |p_{ac}^\lambda|^2 |p_{ac}'|^2 + \left\{ (k - \Delta/2)^2 + \frac{\gamma_6^2}{4} \right\} |p_{bc}^\lambda|^2 |p_{bc}'|^2
\]

\[+ V^2 \left\{ |p_{ac}^\lambda|^2 |p_{bc}'|^2 + |p_{ac}'|^2 |p_{bc}^\lambda|^2 + 4 \text{Re}(p_{ac}^\lambda p_{bc}'^*) \text{Re}(p_{ac}' p_{bc}^\lambda) \right\}
\]

\[- 2 \text{Re}(p_{ac}^\lambda p_{bc}^- p_{ac}' p_{bc}^-)
\]

\[+ 2 V \left\{ (k + \Delta/2) |p_{ac}^\lambda|^2 + (k - \Delta/2) |p_{bc}^\lambda|^2 \right\} \text{Re}(p_{ac}^\lambda p_{bc}^\lambda)
\]
\[
\{(K + \Delta/2) |P_{ac}^{*}|^2 + (K - \Delta/2) |P_{bc}^{*}|^2 \} \text{Re} \left( P_{ca}^{\lambda} P_{cb}^{\lambda*} \right) \\
+ \left( \gamma_a |P_{cb}^{\lambda}|^2 - \gamma_b |P_{ca}^{\lambda}|^2 \right) \frac{1}{2} \text{Im} \left( P_{ac}^{\lambda} P_{bc}^{\lambda*} \right) \\
+ \left( \gamma_a |P_{bc}^{\lambda}|^2 - \gamma_b |P_{ac}^{\lambda}|^2 \right) \frac{1}{2} \text{Im} \left( P_{ca}^{\lambda} P_{cb}^{\lambda*} \right) \\
+ \{ (K - \Delta/2)(K + \Delta/2) + \gamma_a \gamma_b / 4 \} 2 \text{Re} \left( P_{ca}^{\lambda} P_{cb}^{\lambda*} P_{bc}^{\lambda*} \right) \\
\{ \gamma_a (K + \Delta/2) - \gamma_b (K - \Delta/2) \} x \\
\text{Im} \left( P_{ca}^{\lambda} P_{cb}^{\lambda*} P_{bc}^{\lambda*} \right) \] 

and

\[
D_i = (K - \Delta/2)^2 (K + \Delta/2)^2 + (K + \Delta/2)^2 \gamma_a^2 / 4 \\
+ (K - \Delta/2)^2 \gamma_b^2 / 4 - 2 \sqrt{2} (K - \Delta/2)(K + \Delta/2) \\
+ (\sqrt{2} + \gamma_a \gamma_b / 4)^2 .
\]
2. Evaluation of $I(\Omega_{a}, \Omega_{e}, \omega, \lambda)$

The intensity of the scattered radiation with polarization $\lambda$ and direction $(\Theta_{e}, \Phi_{e})$, for a given incident radiation with polarization $\omega$ and direction $(\Theta_{a}, \Phi_{a})$, is defined in Eq. (2-61) as

$$I(\Omega_{a}, \Omega_{e}, \omega, \lambda) = \frac{3}{(2\pi)^3} \int_{0}^{\infty} P(\hat{e}_{\omega}, k_{\lambda}) k_{\lambda}^2 dk_{\lambda}.$$  

Combining Eq. (B-5) with the above equation, one obtains

$$I(\Omega_{a}, \Omega_{e}, \omega, \lambda) = \frac{e^4 I_{e}o}{(2\pi)^3 \Gamma} \int_{0}^{\infty} \frac{N_{1}}{R_{\lambda}D_{1}} dk_{\lambda}. \quad (B-10)$$

Since $N_{1}/D_{1}$ is a highly peaked function near the real parts of the poles of $N_{1}/D_{1}$, one can replace $1/k_{\lambda}$ by an average value $1/k_{0}$ near the poles. Therefore one can write

$$I(\Omega_{a}, \Omega_{e}, \omega, \lambda) = \frac{e^4 I_{e}o}{(2\pi)^3 \Gamma k_{0}} \int_{0}^{\infty} \frac{N_{1}}{D_{1}} dk_{\lambda}. \quad (B-11)$$

The limit in this integral can be extended into non-physical region ($k_{\lambda} = -\infty$ to $k_{\lambda} = 0$) without significant error, because the integrand is negligible in this region. Thus Eq. (B-11) reduces to

$$I(\Omega_{a}, \Omega_{e}, \omega, \lambda) = \frac{e^4 I_{e}o}{(2\pi)^3 \Gamma k_{0}} \int_{0}^{\infty} \frac{N_{1}}{D_{1}} dk_{\lambda}. \quad (B-12)$$
The numerator $N_1$ of the integrand can be written as

$$ N_1 = N'_0 + N'_1 k, + N'_2 R^2 $$

(B-13)

where

$$ N'_0 = \left\{ (E'_b - E'_c)^2 + \frac{\gamma_a^2}{4} \right\} |P_{ca}|^2 |P_{bc}|^2 + \left\{ (E'_a - E'_c)^2 + \frac{\gamma_b^2}{4} \right\} |P_{cb}|^2 |P_{bc}|^2 $$

$$ + V^2 \left[ |P_{ca}|^2 |P_{bc}|^2 + |P_{cb}|^2 |P_{bc}|^2 \right] + 4 \Re (P_{ca}^* P_{cb}^* P_{ac} P_{bc}^*) $$

$$ - 2 \Re (P_{ca}^* P_{cb}^* P_{ac} P_{bc}^* ) \right\} $$

$$ - 2 V \left\{ (E'_b - E'_c) |P_{ca}|^2 + (E'_a - E'_c) |P_{cb}|^2 \right\} \Re (P_{ac} P_{bc}^* ) $$

$$ + \left\{ (E'_a - E'_c) |P_{ca}|^2 + (E'_a - E'_c) |P_{cb}|^2 \right\} \Re (P_{ca}^* P_{cb}^* ) $$

$$ + \left\{ \gamma_a |P_{ca}|^2 - \gamma_b |P_{ac}|^2 \right\} \Im (P_{ac} P_{bc}^* ) $$

$$ + \left\{ \gamma_a |P_{ca}|^2 - \gamma_b |P_{ac}|^2 \right\} \Im (P_{ca}^* P_{cb}^* ) $$

$$ + \left\{ (E'_b - E'_c) |P_{ca}|^2 + (E'_a - E'_c) |P_{cb}|^2 \right\} \Re (P_{ac} P_{bc}^* ) $$

$$ - \left\{ E_b' + E'_a - 2 E'_c \right\} \Re (P_{ca}^* P_{cb} P_{ac} P_{bc}^* ) $$

$$ + (\gamma_a - \gamma_b) \Im (P_{ca}^* P_{cb}^* P_{ac} P_{bc}^* ) $$

(B-14)

$$ N'_1 = -2 \left\{ (E'_b - E'_c) |P_{ca}|^2 |P_{bc}|^2 + (E'_a - E'_c) |P_{cb}|^2 |P_{bc}|^2 \right\} $$

$$ + 2 V \left\{ |P_{ca}|^2 + |P_{cb}|^2 \right\} \Re (P_{ac} P_{bc}^* ) $$

$$ + |P_{ac}^2|^2 + |P_{bc}^2|^2 \left\{ \Re (P_{ca}^* P_{cb}^* ) $$

$$ - (E_b' + E'_a - 2 E'_c) \Re (P_{ca}^* P_{cb} P_{ac} P_{bc}^* ) $$

$$ + (\gamma_a - \gamma_b) \Im (P_{ca}^* P_{cb}^* P_{ac} P_{bc}^* ) $$

(B-15)
\[ N_2' = |p_{ca}^\lambda|^2|p_{ac}^\tau|^2 + |p_{cb}^\lambda|^2|p_{bc}^\tau|^2 \]
\[ + 2 \text{ Re}(p_{ca}^\lambda p_{cb}^\lambda p_{ac}^\tau p_{bc}^\tau) \]  \hspace{1cm} (B-16)

and the denominator, one can write
\[ D_1' = (k_\lambda - k_1')(k_\lambda - k_2')(k_\lambda - k_1^*)(k_\lambda - k_2^*) \] \hspace{1cm} (B-17)

where \( k_1' \) and \( k_2' \) are defined in Eq.(B-3).

Thus the integral in Eq.(B-12) reduces to
\[ \int_{-\infty}^{\infty} \frac{N_1'}{D_1'} dk = N_0' I_1' + N_1' I_2' + N_2' I_3' \] \hspace{1cm} (B-18)

where
\[ I_1' = \int_{-\infty}^{\infty} \frac{dk}{(k - k_1')(k - k_2')(k - k_1^*)(k - k_2^*)} \]
\[ I_2' = \int_{-\infty}^{\infty} \frac{R dk}{(k - k_1')(k - k_2')(k - k_1^*)(k - k_2^*)} \]

and
\[ I_3' = \int_{-\infty}^{\infty} \frac{R^2 dk}{(k - k_1')(k - k_2')(k - k_1^*)(k - k_2^*)} \]
The method of contour integration is used to evaluate these integrals. It can be shown that the integrands vanish on a semicircle of infinite radius on both sides (upper half and lower half) of the complex plane. Thus the contour of integration can be chosen either along \( C_1 \) or \( C_2 \) (see Fig. B-1).

Fig. B-1. Position of the poles in complex plane.

Taking one of the contours, one gets the following results for the integrals:

\[
I_1' = \pi \frac{I_m(k_1' + k_2')}{{I_m(k_1')} {I_m(k_2')} [2 \text{Re} (k_1' k_2') - |k_1'|^2 - |k_2'|^2]}
\]

\[
I_2' = \pi \frac{I_m(k_1' k_2')}{{I_m(k_1')} {I_m(k_2')} [2 \text{Re} (k_1' k_2') - |k_1'|^2 - |k_2'|^2]}
\]
Substituting the values of $k'_1$ and $k'_2$ (see Eq.(B-3)), the above expressions reduce to:

\[ I'_1 = 8\pi k'_2 x_3 / D \quad (B-19) \]

\[ I'_2 = 8\pi (x_3 \varepsilon' \pm \Delta)/D \quad (B-20) \]

and

\[ I'_3 = 8\pi \frac{\varepsilon'^2 x_3 + \varepsilon^2 + \varepsilon' \Delta x/2 + (4 \varepsilon^2 + \Delta^2 - x^2) x_3/4}{D} \quad (B-21) \]

where

\[ \varepsilon' = \frac{\varepsilon'_a + \varepsilon'_b}{2} - \varepsilon'_c \],

and

\[ D = 16 x_3^4 - x^2 \Delta^2 + 4 x_3^2 (4 \varepsilon^2 + \Delta^2 - x^2). \]

The definitions of $\Delta$, $x$, $x_3$ and $\varepsilon$ are given in Eq.(A-23), Eq.(A-26) and Eq.(A-27).

Combining Eq.(B-12), Eq.(B-14) to Eq.(B-16) and Eq.(B-18) to Eq.(B-21) one gets the following expression for the intensity of the scattered radiation:
\[ I(\Omega_a, \Omega_b, \gamma, \lambda) = \frac{e^4}{8 \pi^2} \frac{\Gamma \kappa}{\Gamma \kappa_o} \left[ \frac{|P_{ca}^\lambda|^2 |P_{ac}^-|^2}{X_a} + \frac{|P_{cb}^\lambda|^2 |P_{bc}^-|^2}{X_b} \right] \]

\[ + V^2(X_a + X_b) X_a X_b \left\{ \frac{4 \text{Re}(P_{ca}^\lambda P_{cb}^-*) \text{Re}(P_{ac}^- P_{bc}^-*)}{X_a X_b} \right\} \]

\[ - (|P_{ca}^\lambda|^2/X_a - |P_{cb}^\lambda|^2/X_b)(|P_{ac}^-|^2/X_a - |P_{bc}^-|^2/X_b) \}

\[ + V(X_a + X_b) \left[ \Delta \mathcal{E} (|P_{ca}^\lambda|^2 - |P_{cb}^\lambda|^2) \text{Re}(P_{ac}^- P_{bc}^-*) + (|P_{ac}^-|^2 - |P_{bc}^-|^2) \text{Re}(P_{ca}^\lambda P_{cb}^-*) \right] \]

\[ - 2 \left\{ (|P_{ca}^\lambda|^2 X_b - |P_{cb}^\lambda|^2 X_a) \text{Im}(P_{ca}^- P_{bc}^-*) \right\} \]

\[ + (|P_{ac}^-|^2 X_b - |P_{bc}^-|^2 X_a) \text{Im}(P_{ca}^\lambda P_{cb}^-*) \} \}

\[ + 4 X_a X_b \left\{ (X_a + X_b) \frac{\gamma}{2} (V^2 + X_a X_b) + \Delta^2 X_a X_b \right\} \]

\[ \left[ (X_a + X_b)^2 (V^2 + X_a X_b) + \Delta^2 X_a X_b \right] \]  \hfill (B-22)

where

\[ X_a = \gamma_a/2 \]

and

\[ X_b = \gamma_b/2 \]  \hfill (B-23)
APPENDIX C

Eigenvalues and Eigenvector of $\mathcal{H}$

In the Schrödinger representation, the Hamiltonian of the atom with an external perturbation $V$ is

$$\mathcal{H} = H_0 + V$$

where $H_0$ is the Hamiltonian of the unperturbed atom.

In order to obtain the eigenvalues and eigenstates of the Hamiltonian $\mathcal{H}$, one has to solve the time-independent Schrödinger equation:

$$\mathcal{H} |\psi\rangle = E |\psi\rangle \quad \text{(C-1)}$$

The state vector $|\psi\rangle$ can be written in terms of the eigenvectors $|a\rangle$ and $|b\rangle$ of $H_0$ as

$$|\psi\rangle = C_0 |a\rangle + D_0 |b\rangle \quad \text{(C-2)}$$

where $C_0$ and $D_0$ are the amplitudes of the states $|a\rangle$ and $|b\rangle$ respectively.

Substituting Eq.(C-2) into Eq.(C-1), one gets:

$$C_0 E_a |a\rangle + C_0 V |a\rangle + D_0 E_b |b\rangle + D_0 V |b\rangle = E \left( C_0 |a\rangle + D_0 |b\rangle \right) \quad \text{(C-3)}$$

where $E_a$ and $E_b$ are the eigenvalues of $H_0$ for the eigenstates $|a\rangle$ and $|b\rangle$ respectively.

Multiplying both sides of Eq.(C-3) by the state vector $\langle a|$ and
integrating over all space one gets

$$C_0 E_a' + D_0 V_{ab} = C_0 E \tag{C-4}$$

and similarly, if one multiplies Eq. (C-3) by \(<b|\) and integrates over all space, one obtains:

$$C_0 V_{ba} + D_0 E_b' = D_0 E \tag{C-5}$$

where

$$E'_a = E_a + \langle a|V|a\rangle$$
$$E'_b = E_b + \langle b|V|b\rangle$$

and

$$V_{ij} = \langle i|V|j\rangle; \quad i, j = a, b$$

Equation (C-4) and Eq. (C-5) can be written as a matrix equation:

$$
\begin{pmatrix}
E_a' - E & V_{ab} \\
V_{ba} & E_b' - E
\end{pmatrix}
\begin{pmatrix}
C_0 \\
D_0
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0
\end{pmatrix} \tag{C-6}
$$

A non-trivial solution of this equation exists, if the determinant of the matrix

$$(E_a' - E)(E_b' - E) - |V_{ab}|^2 = 0 \tag{C-7}$$

is zero. This gives
The two roots of this equation, are the energy eigenvalues of the Hamiltonian $\mathcal{H}$:

$$\begin{pmatrix} E_1^o \\ E_2^o \end{pmatrix} = \frac{E_a + E_b}{2} \pm \frac{1}{2} \left( \Delta^2 + 4\nu^2 \right)^{1/2}$$  \hspace{1cm} (C-8)

where

$$\Delta = E_a - E_b$$

and

$$\nu^2 = |\nu_{ab}|^2$$.

The eigenstates of the Hamiltonian $\mathcal{H}$ are obtained by substituting the energy eigenvalues of $\mathcal{H}$ from Eq.(C-8) into Eq.(C-6) and solving for the coefficients $C_0$ and $D_0$.

The normalised eigenstates of $\mathcal{H}$ are:

$$|1\rangle = C_0^{(1)} |a\rangle + D_0^{(1)} |b\rangle$$

and

$$|2\rangle = C_0^{(2)} |a\rangle + D_0^{(2)} |b\rangle$$

where

$$C_0^{(1)} = \frac{1}{\sqrt{2}} \left( 1 + \frac{\Delta}{\sqrt{\Delta^2 + 4\nu^2}} \right)^{1/2}$$

$$D_0^{(1)} = \frac{1}{\sqrt{2}} \left( 1 - \frac{\Delta}{\sqrt{\Delta^2 + 4\nu^2}} \right)^{1/2}$$

$$C_0^{(2)} = -D_0^{(1)}$$

and

$$D_0^{(2)} = C_0^{(1)}$$.
Matrix Elements of the Hyperfine Interaction $H_D$ and the Nuclear Quadrupole Interaction $H_Q$ in Li$^7$

The Hamiltonian for the hyperfine interaction is defined in Eq. (4-13) as:

$$H_D = \alpha \left[ \frac{\hat{\ell} - \frac{3}{5}}{\gamma_3} + 3 \frac{(\hat{S} \cdot \hat{T})}{\gamma_5} \right] \cdot \hat{I} + \xi \hat{S} \cdot \hat{I}$$

This Hamiltonian can be written (14) in the following way

$$H_D = \alpha \left[ \frac{\hat{\ell} - \frac{3}{5}}{\gamma_3} + \frac{24\pi}{5} \sum_{m_1, m_2} (-1)^{m_1 + m_2} C(112, m_1, m_2) Y_2(\hat{r}) \right] \cdot \hat{I} + \xi \hat{S} \cdot \hat{I}$$

where $C(112, m_1 m_2)$ are the Clebsch-Gordan coefficients, $Y_2^{m_1 + m_2}(\hat{r})$ are the spherical harmonics of rank two and $T_1$ is an irreducible tensor of rank one.

The matrix elements of $H_D$ between the two states $|\phi_4, m_I \rangle$ and $|\phi_\nu, m_I \rangle$ is obtained as follows:

$$<\frac{3}{2} m'_1 | \phi_4 \rangle | H_D | \phi_\nu \rangle | \frac{3}{2} m_I \rangle$$

$$= C_\alpha <\frac{3}{2} m'_1 | \phi_4 \rangle | \frac{1}{2} - \frac{1}{2} \rangle | H_D | \frac{1}{2} - \frac{1}{2} \rangle | \frac{3}{2} m_I \rangle$$

$$- D_\alpha <\frac{3}{2} m'_1 | \phi_4 \rangle | \frac{1}{2} - \frac{1}{2} \rangle | H_D | \frac{1}{2} - \frac{1}{2} \rangle | \frac{3}{2} m_I \rangle$$
\[= c_0 \alpha \left( \frac{1}{\sqrt{3}} \right) \left[ \langle \frac{3}{2}, m_x | 1 - \frac{1}{2} | \frac{1}{2}, \frac{1}{2} \rangle \langle \frac{1}{2}, \frac{1}{2} | \frac{1}{2}, \frac{1}{2} \rangle \langle \frac{1}{2}, \frac{1}{2} | \frac{1}{2}, \frac{1}{2} \rangle \right] \]
\[+ \sqrt{24 \pi / 5} \sum_{m_1, m_2} (-1)^{m_1 + m_2} c(11/2, m_1, m_2) \langle 1 - 1 | \gamma_2(\frac{3}{2}, m_2) | 10 \rangle \]
\[\times \langle \frac{1}{2} - \frac{1}{2} | \Gamma \frac{1}{2} - \frac{1}{2} \rangle \langle \frac{1}{2} - \frac{1}{2} | \Gamma \frac{1}{2} - \frac{1}{2} \rangle \langle \frac{1}{2} - \frac{1}{2} | \Gamma \frac{1}{2} - \frac{1}{2} \rangle \]
\[= -D_0 \alpha \left( \frac{1}{\sqrt{3}} \right) \sqrt{24 \pi / 5} \sum_{m_1, m_2} (-1)^{m_1 + m_2} c(11/2, m_1, m_2) \langle 1 - 1 | \gamma_2(\frac{3}{2}, m_2) | 1 - 1 \rangle \]
\[\times \langle \frac{1}{2} - \frac{1}{2} | \Gamma \frac{1}{2} - \frac{1}{2} \rangle \langle \frac{1}{2} - \frac{1}{2} | \Gamma \frac{1}{2} - \frac{1}{2} \rangle \langle \frac{1}{2} - \frac{1}{2} | \Gamma \frac{1}{2} - \frac{1}{2} \rangle \]
\[+ \frac{1}{2} \left( S_+ I_+ + S_- I_- \right) \langle \frac{1}{2} \rangle \langle \frac{1}{2} \rangle \langle \frac{1}{2} \rangle \langle \frac{1}{2} \rangle \]
\[= \frac{1}{2} \left[ \left( \frac{7}{12} c_0 - D_0 \right) \alpha \left( \frac{1}{\sqrt{3}} \right) \right] \sqrt{(5/2 - m_I)(3/2 + m_I)} \delta_{m_I} \delta_{m_-} \]
The diagonal matrix element of $H_D$ in state $|\Phi_4, m_I\rangle$ is:

$$\langle \Phi_4, m_I | H_D | \Phi_4, m_I \rangle = \alpha <\frac{1}{r_3} > <\frac{3}{2} m_I | <1/2-1/2 | <1/2-1/2 | l_z I_z^z$$

$$+ \frac{1}{2} (l_+ I_+ + l_- I_-) |1-1\rangle |1/2-1/2\rangle |3/2 m_I\rangle$$

$$+ \alpha <\frac{1}{r_3} > \sqrt{24/17/5} \sum_{m_1, m_2} (-1)^{m_1+m_2} C(1/2, m, m_2) <1-1 | Y_2 (r_3) | 1-1\rangle$$

$$\times <\frac{1}{2} - \frac{1}{2} | T_1^- m_1 | \frac{1}{2} - \frac{1}{2}\rangle <\frac{3}{2} m_1 | T_1^- m_2 | \frac{3}{2} m_2\rangle$$

$$+ \xi <1-1 | \frac{1}{2}-\frac{1}{2} | <3/2 m_1 | S_1 I_1^1 + \frac{1}{2} (S_1 I_1^1 + S_1 I_1^-) | 1-1\rangle |1/2,1/2\rangle |3/2 m_2\rangle$$

which, after substituting the different matrix elements of $Y_2$ and $T_1$ and Clebsch-Gordon coefficients, reduces to

$$<\Phi_4, m_I | H_D | \Phi_4, m_I > = -\frac{4}{5} m_I <\frac{1}{r_3} > -\frac{1}{2} m_I \xi$$  \hspace{1cm} (D-5)$$

Similarly, the following matrix elements of $H_D$ are obtained:

$$<\frac{3}{2} m_I | <\frac{1}{2}-\frac{1}{2} | H_D | 1-1\rangle |\frac{1}{2}, \frac{1}{2}\rangle |\frac{3}{2} m_I\rangle = -\frac{6}{5} m_I <\frac{1}{r_3} >$$

$$+ m_I \xi / 2$$

$$<\frac{3}{2} m_I | <\frac{1}{2}-\frac{1}{2} | H_D | 10\rangle |\frac{1}{2}, \frac{1}{2}\rangle |\frac{3}{2} m_I\rangle = -\frac{2}{5} m_I <\frac{1}{r_3} >$$

$$- m_I \xi / 2$$  \hspace{1cm} (D-6)$$

and

$$<\frac{3}{2} m_I | <\frac{1}{2}-\frac{1}{2} | H_D | 10\rangle |\frac{1}{2}, \frac{1}{2}\rangle |\frac{3}{2} m_I\rangle$$

$$= <\frac{3}{2} m_I | <\frac{1}{2}-\frac{1}{2} | 10 | H_D | 10\rangle |\frac{1}{2}, \frac{1}{2}\rangle |\frac{3}{2} m_I\rangle$$

$$= (3/5\sqrt{2}) m_I <\frac{1}{r_3} >$$  \hspace{1cm} (D-7)$$
Thus, using Eq. (D-6) and Eq. (D-7), the diagonal matrix element of $H_D$ in state $|\Phi_6, m_I \rangle$ is

$$
\langle \Phi_6, m_I | H_D | \Phi_6, m_I \rangle = - \frac{1}{5} (6 D_0^2 + 2 C_0^2 + 3.3 C_0 D_0) m_I \delta (\frac{1}{T_3}) + \frac{1}{2} (D_0^2 - C_0^2) m_I \xi .
$$

(D-8)

The nuclear electrostatic quadrupole interaction is

$$
H_Q = - e^2 Q \left[ \frac{3 I_x^2 - I^2}{4 I (2I-1)} \right] \left( \frac{3 \cos^2 \Theta - 1}{r^3} \right)
$$

$$
= - e^2 Q \left[ \frac{3 I_x^2 - I^2}{4 I (2I-1)} \right] \frac{4 \pi}{3} \frac{\gamma_{20}(r)}{\gamma^3} .
$$

Using Eq. (D-3), the matrix elements of $H_Q$ are written as:

$$
\langle \Phi_4, m_I | H_Q | \Phi_4, m'_I \rangle = \frac{b}{4} (m_I^2 - 5/4) \delta m_I m'_I
$$

(D-9)

and

$$
\langle \Phi_6, m_I | H_Q | \Phi_6, m'_I \rangle = \frac{b}{4} (D_0^2 - 2 C_0^2) (m_I^2 - 5/4) \delta m_I m'_I
$$

(D-10)

where

$$
b = \frac{2}{5} e^2 Q \langle 1/r^3 \rangle
$$

and $C_0$ and $D_0$ are defined in Eq. (4-12).
The experimental value of $b$ for $\text{Li}^7$ (13) is

$$b = -0.18 \pm 0.12 \text{ Mc/Sec}.$$  

(D-11)
APPENDIX E

Some Useful Matrix Elements of $\tilde{P}, \tilde{\epsilon}_\lambda$

The matrix elements of $\tilde{P}, \tilde{\epsilon}_\lambda$ for different excited states and ground states are calculated and the products $P_{c_3}^\lambda P_{c_3}^{\lambda *}, P_{c_3}^\omega P_{c_3}^{\omega *}$ are computed.

For $\lambda = 1$ and $\Lambda = 3$, one gets from Eq. (4-17)

$$P_{c_1}^\lambda = C_1 <c' | \tilde{P} \cdot \tilde{\epsilon}_\lambda | \phi_4, 3/2> + D_1 <c' | \tilde{P} \cdot \tilde{\epsilon}_\lambda | \phi_6, 1/2>$$

$$P_{c_3}^{\lambda *} = C_2 <\phi_4, 1/2 | \tilde{P} \cdot \tilde{\epsilon}_\lambda | c'> + D_2 <\phi_6, -1/2 | \tilde{P} \cdot \tilde{\epsilon}_\lambda | c'>.$$

The matrix element $P_{c_3}^\lambda$ is equal to zero if the states $|c'>$ and $|\psi_\lambda>$ do not have the same $m$. Thus the product $\sum_{c'} P_{c_1}^\lambda P_{c_3}^{\lambda *}$ reduces to

$$\sum_{c'} P_{c_1}^\lambda P_{c_3}^{\lambda *} = \sum_{c'} C_2 D_1 <c' | \tilde{P} \cdot \tilde{\epsilon}_\lambda | \phi_6, 1/2> <\phi_4, 1/2 | \tilde{P} \cdot \tilde{\epsilon}_\lambda | c'> \quad (E-1)$$

After writing the matrix elements of $\tilde{P} \tilde{\epsilon}_\lambda$ in terms of the orbital states $|l m_\epsilon>$ of the election, one gets

$$\sum_{c'} P_{c_1}^\lambda P_{c_3}^{\lambda *} = C_2 D_1 C_0 <001 | \tilde{P} \cdot \tilde{\epsilon}_\lambda | \phi_1, 10> <1-11 | \tilde{P} \cdot \tilde{\epsilon}_\lambda | 100> \quad (E-2)$$

and therefore

$$\sum_{c'} P_{c_1}^\lambda P_{c_3}^{\lambda *} P_{c_3}^\omega P_{c_3}^{\omega *} = C_2^2 D_1^2 C_0^2 <001 | \tilde{P} \cdot \tilde{\epsilon}_\lambda | \phi_1, 10> <1-11 | \tilde{P} \cdot \tilde{\epsilon}_\lambda | 100> \times <001 | \tilde{P} \cdot \tilde{\epsilon}_\omega | 1-1> <101 | \tilde{P} \cdot \tilde{\epsilon}_\omega | 100> \quad (E-3)$$
Similarly, one obtains the following set of nonvanishing matrix elements:

\[
\sum_{cc'} p_{c'1}^{\lambda} p_{c4}^{\lambda} p_{1c}^{\rho} p_{4c}^{\rho} = C_0^2 D_1^2 D_2 f,
\]
\[
\sum_{cc'} p_{c'1}^{\lambda} p_{c8}^{\lambda} p_{1c}^{\rho} p_{8c}^{\rho} = C_0^2 C_1^2 f^*,
\]
\[
\sum_{cc'} p_{c'2}^{\lambda} p_{c3}^{\lambda} p_{2c}^{\rho} p_{3c}^{\rho} = C_0^2 C_1^2 C_2^2 f,
\]
\[
\sum_{cc'} p_{c'2}^{\lambda} p_{c4}^{\lambda} p_{2c}^{\rho} p_{4c}^{\rho} = C_0^2 C_1^2 D_2 f,
\]
\[
\sum_{cc'} p_{c'2}^{\lambda} p_{c8}^{\lambda} p_{2c}^{\rho} p_{8c}^{\rho} = C_0^2 C_3^2 f^*,
\]
\[
\sum_{cc'} p_{c'3}^{\lambda} p_{c5}^{\lambda} p_{3c}^{\rho} p_{5c}^{\rho} = C_0^2 C_3^2 D_2 f,
\]
\[
\sum_{cc'} p_{c'3}^{\lambda} p_{c6}^{\lambda} p_{3c}^{\rho} p_{6c}^{\rho} = C_0^2 C_3^2 C_2^2 f,
\]
\[
\sum_{cc'} p_{c'4}^{\lambda} p_{c5}^{\lambda} p_{4c}^{\rho} p_{5c}^{\rho} = C_0^2 D_3^2 C_2^2 f,
\]
\[
\sum_{cc'} p_{c'5}^{\lambda} p_{c6}^{\lambda} p_{5c}^{\rho} p_{6c}^{\rho} = C_0^2 D_3^2 f,
\]
\[
\sum_{cc'} p_{c'6}^{\lambda} p_{c7}^{\lambda} p_{6c}^{\rho} p_{7c}^{\rho} = C_0^2 C_3^2 f.
\]

(E-4)

where

\[
f = \langle 00|\hat{p}.\hat{\varepsilon}_\lambda|00\rangle \langle 10|\hat{p}.\hat{\varepsilon}_\lambda|10\rangle (E-5)
\]

and \(f^*\) is the complex conjugate of \(f\).

For plane polarized beams of the incident and the scattered radiation, the polarization vectors \(\hat{\varepsilon}_\rho\) and \(\hat{\varepsilon}_\lambda\) are:
\[ \hat{e}_r = \hat{x} \cos \Theta_a - \hat{y} \sin \Theta_a \]  
\[ \hat{e}_\lambda = \hat{x} \cos \Theta_e + \hat{y} \sin \Theta_e \] (E-6)

where \( \Theta_a \) and \( \Theta_e \) are the angles between the plane of polarization and the \( z \)-axis for the incident and the scattered beams, respectively.

From Eq.(A-11), the matrix element \( \hat{P}_a \) is written as:

\[ \langle c | \bar{P} \cdot \hat{e}_\lambda | a \rangle = i \omega_0 \langle c | \bar{P} \cdot \hat{e}_\lambda | a \rangle \]

where \( \omega_0 \) is the frequency difference between the two states and \( \mathbf{r} \) is the position vector of the electron.

From Eq.(E-6), the scalar product \( \mathbf{r} \cdot \hat{e}_r \) and \( \mathbf{r} \cdot \hat{e}_\lambda \) can be written in terms of spherical harmonics of rank one as follows:

\[ \mathbf{r} \cdot \hat{e}_r = \frac{4 \pi}{3} r \sum_m (-1)^m Y_{1m}^m(\hat{r}) Y_{1m}^{-m}(\Theta_a, \pi/2) \]

and

\[ \mathbf{r} \cdot \hat{e}_\lambda = \frac{4 \pi}{3} r \sum_m (-1)^m Y_{1m}^m(\hat{r}) Y_{1m}^{-m}(\Theta_e, 0) . \] (E-7)

Thus the matrix element \( \langle 001| \bar{P} \cdot \hat{e}_\lambda | 10 \rangle \langle 1-1| \bar{P} \cdot \hat{e}_\lambda | 100 \rangle \) becomes:

\[ x \langle 001| Y_{1}^0 | 10 \rangle \langle 1-1| Y_{1}^{-1} | 100 \rangle Y_{1}^0(\Theta_e, 0) Y_{1}^{-1}(\Theta_e, 0) \]

\[ = \frac{\omega_0^2 <r^2>}{3} \sin \Theta_e \cos \Theta_e \] (E-8)
Similarly
\[
\langle 10| \hat{P} \hat{\mathbf{e}}_n | 100 \rangle \langle 001 | \hat{P} \hat{\mathbf{e}}_n | 11-1 \rangle = -i \frac{\omega_0^2 \langle r \rangle^2}{3} \sin \theta_a \cos \theta_a .
\]  
(E-9)

The quantity \( f \) defined in Eq.(E-5) reduces to
\[
f = \langle 001 | \hat{P} \hat{\mathbf{e}}_{\lambda} | 110 \rangle \langle 1-11 | \hat{P} \hat{\mathbf{e}}_{\lambda} | 100 \rangle \langle 101 | \hat{P} \hat{\mathbf{e}}_{\lambda} | 11-1 \rangle
= -i \frac{\omega_0^4 \langle r \rangle^4}{72} \sin 2 \theta_a \sin 2 \theta_e .
\]  
(E-10)

This quantity is pure imaginary that means the signal \( S_c \) as defined in Eq.(4-30) is pure dispersion as a function of the external magnetic field.

If the beams are unpolarised and have the incidence and scattering directions as shown in Fig. 4-10, then \( \hat{\mathbf{n}} \hat{\mathbf{e}}_{\perp} \) becomes:
\[
\sum_{n=1}^{\infty} \langle \mathbf{r} | \hat{\mathbf{e}}_{\perp} \rangle = \sqrt{\frac{4\pi}{3}} \sum_{\alpha=1,1-1}^{1} D^1_{m\alpha}(\Phi, \theta, 0) \tau Y^m_1(\hat{\mathbf{r}}).
\]
(E-11)

Therefore
\[
\sum_{\alpha=1}^{\infty} \langle 10| \hat{P} \hat{\mathbf{e}}_{\perp} | 100 \rangle \langle 001 | \hat{P} \hat{\mathbf{e}}_{\perp} | 11-1 \rangle = \frac{4\pi}{3} \omega_0^2 \langle r \rangle^2
\times \sum_{\alpha=1}^{\infty} D^1_{1\alpha}(\Phi, \theta, 0) \sigma D^{1\prime}_{0\alpha}(\Phi, \theta, 0) \langle 001 | Y^1_{11-1} \rangle \langle 101 | Y^0_{100} \rangle
= -\omega_0^2 \langle r \rangle^2/3 \sin \theta_a \cos \theta_a e^{-i \phi} ,
\]
and similarly
\[
\sum_{\lambda=1}^{\infty} \langle 001 | \hat{P} \hat{\mathbf{e}}_{\lambda} | 110 \rangle \langle 1-11 | \hat{P} \hat{\mathbf{e}}_{\perp} | 100 \rangle = -\omega_0^2 \langle r \rangle^2/3 \sin \theta_a \cos \theta_a e^{-i \phi} .
\]
Hence, for the unpolarized beams of the incident and the scattered light $f$ becomes:

$$f = \sum_{\lambda \sigma} \langle 00| \vec{p} \cdot \hat{e}_{\lambda 110} \rangle \langle 1-1| \vec{p} \cdot \hat{e}_{\lambda 100} \rangle \langle 10| \vec{p} \cdot \hat{e}_{\sigma 100} \rangle \langle 00| \vec{p} \cdot \hat{e}_{\sigma 11-} \rangle$$

$$= \frac{\omega_0^4 \langle r \rangle^4}{72} \sin 2\theta_1 \sin 2\theta_2 e^{i(\phi_2 - \phi_1)}. \quad (E-12)$$