AN ABSTRACT OF THE THESIS OF

SOMMAI VONGSURI for the Doctor of Philosophy
(Name of Student) (Degree)

in Applied Mathematics presented on March 7, 1967
(Major) (Date)

Title A METHOD OF SEPARATION OF EXPONENTIALS AND ITS RELATIONSHIP TO TIME DOMAIN SYNTHESIS OF A FINITE LUMPED-PARAMETER RELAXATION SYSTEM

Abstract approved Redacted for Privacy
Harry E. Goheen

In this paper, the general mathematical theory of linear passive one-ports and the class of positive real functions are briefly reviewed as background material. Then a time domain method for synthesis of a finite lumped RC system is given, which involves breaking down the given system into n subsystems. Finally, it is shown that one method of separation of a finite number of exponentials from a given completely monotonic curve over a finite interval is related to the above mentioned method of time domain synthesis. Here the given curve is treated as the impulse response of a finite lumped RC system. It is shown that the parameters being sought are related to the eigenvalues of the system matrix and the first components of the orthonormal eigenvectors corresponding to those eigenvalues, and it is proved that the system matrix has distinct positive eigenvalues.

A Method of Separation of Exponentials and Its Relationship to Time Domain Synthesis of a Finite Lumped-Parameter Relaxation System

by

Sommai Vongsuri

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the

DOCTOR OF PHILOSOPHY

June 1967

APPROVED:

Redacted for Privacy

Professor of Mathematics

(In Charge of Major)

Redacted for Privacy

Chairman of Department of Mathematics

Redacted for Privacy

Dean of Graduate School

Date thesis is presented March 7, 1967

Typed by Carol Baker for Sommai Vongsuri

ACKNOWLEDGMENTS

The author wishes to express his sincere gratitude to Dr. H. Goheen and Dr. G. Bödvarsson for having read the manuscript and for having given valuable suggestions. He would like to thank Dr. D. Carlson and Dr. W. Stone for consultation. To Dr. A. T. Lonseth he wishes to express appreciation for the teaching assistantship. Finally, he would like to thank the National Science Foundation for financial support in the study of this thesis problem.

TABLE OF CONTENTS

LIST OF FIGURES

A METHOD OF SEPARATION OF EXPONENTIALS AND ITS RELATIONSHIP TO TIME DOMAIN SYNTHESIS OF A FINITE LUMPED-PARAMETER RELAXATION SYSTEM

I. INTRODUCTION

A systems scientist is always striving to get a good mathematical model of some physical situation. It is well known that to many a physical situation there corresponds a linear model from which analysis can be made to obtain certain information concerning system behavior or system parameters. In general, there are two types of problems which one encounters. The first one is the model problem. Here a system is given and one wishes to find the response due to a given excitation,or vice versa. This is known as systems analysis. The second one is the direct problem. Here an excitation is applied to a system whose parameters are unknown and the corresponding response is measured. If the system is linear, then it is well known that the excitation and the response are related by a correlation integral, whose kernel is called the impulse response of the system and from it the system parameters may sometimes be found [2]. The problem of finding the system and parameters is known as the synthesis problem. In many cases a system may by synthesized uniquely if additional information regarding the type of arrangement of its parameters is known. Also, a systems scientist is sometimes interested in a gross picture of a system. Here the actual distributed system is replaced by a lumped system, partly because it is more convenient to handle, and partly because a lumped system in many cases closely approximates its corresponding real distributed system. An example for this is the analysis of the distribution of a drug in the human body [15] in which the organs are lumped into compartments.

In the class of linear passive one-ports (Cf. Definition 2.1) there is a subclass which is characterized by the class of completely monotonic functions [14]. It is known as the class of relaxation systems which one often encounters in many areas of research. Examples of relaxation systems in geophysics are the semi-infinite horizontally stratified earth, which is characterized by the heat flow-temperature impulse response; and the hydrological system, which is characterized by the water inflow-water level impulse response [2]. In electrical engineering one has the RC and RL ladders with either current or voltage as excitation and voltage or current correspondingly as response [18]. One of the most important investigations in biology uses the tracer experiment with lumped compartmental systems. For example, a known quantity of traced material is suddenly injected into an irreversible catenary system. Then the independent disappearance of the traced material in each compartment with time is known to behave exponentially [1, 15]. Hence its impulse response takes the form

$$h(t) = \sum_{i=1}^{n} b_i e^{-\lambda_i t} \qquad (1)$$

where n is the number of compartments, λ_i, $i = 1, 2, \cdots, n$, are the decay constants, and b_i are related to the compartment sizes. Suppose that $h(t)$ has been obtained from an experiment on an irreversible catenary system as a continuous curve over a finite interval. It is desired to find its analytical expression as the right side of (1). The easiest method is the graphical approach [15]. But a direct mathematical approach to the problem possesses well-known pitfalls [10] due to the fact that the series (1) is non-linear and highly non-orthogonal. One may even resort to Prony's method [13]. Gardner et al. [8] have suggested a method which is based on the Fourier transform of (1). This is the only known method in which the number n falls out of the analysis. Among the important problems which involve the solution of (1), are the problems of radioactive disintegration, e.g., of delayed neutron activity [9], where each of the radioactive particles decays independently.

In this paper the following problems are investigated

(a) Given the current-voltage impulse response of a finite RC ladder. Find the system parameters corresponding to their exact location directly in the time domain.

(b) Given an accurate experimental curve which has its analytical expansion of the form (1). Find the parameters n, λ_i and b_i, $i = 1, 2, \ldots, n$.

The solution of (a) is based on the mathematical method of tearing the given system down into n subsystems, and analysis is performed on each subsystem individually in the time-domain, from which the system parameters corresponding to their exact locations can be uniquely determined. The solution of (a) is of great interest since many finite lumped systems, e.g., the irreversible catenary compartmental systems [17], have their network representation in the form of an RC ladder. Usually the synthesis of RC networks is done in the frequency-domain [18].

The solution of (b), which is of mathematical interest, is based on the fact that to every completely monotonic function of the form (1), there corresponds an RC ladder network. It will be shown that the numbers n and λ_i, $i = 1, 2, \cdots, n$, correspond to the number of network elements and the eigenvalues of the system matrix respectively, while the number b_i is related to the first component of the orthonormal eigenvector corresponding to λ_i, $i = 1, 2, \cdots, n$.

It may be pointed out here that the solutions to the above mentioned problems are purely formal. That is, no attention is given to numerical difficulties arising from their use. However, the solutions do throw added light on problems of this nature.

II. GENERAL THEORY OF LINEAR PASSIVE ONE-PORTS AND THE CLASS OF POSITIVE-REAL FUNCTIONS

In this chapter we shall give definitions and some of the important theorems concerning linear passive one-ports in general. Then we shall state some well-known theorems on the class of positive real functions which will lead to the special subclass corresponding to the class of relaxation systems. We start with

Definition 2.1: A system with only one pair of terminals into which an excitation $f(t)$ is applied and from which the response $g(t)$ is observed is called a one-port, or a driving-point system. In particular if the system is composed of a finite number of lumped elements, then it is called a finite lumped-parameter one-port.

Definition 2.2: [11] The unit pulse function is defined as

$$q_n(t) = \begin{cases} 0 & t < 0 \\ n & 0 \le t \le \frac{1}{n} \\ 0 & t > \frac{1}{n} \end{cases} \qquad (2.1)$$

Definition 2.3: The Dirac delta function $\delta(t)$ is defined as [1]

---

[1] Of course, the limit does not exist in a strict sense since ∞ is not a number. See [11].

$$\lim_{n \to \infty} q_n(t) = \delta(t) \qquad (2.2)$$

or,

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases} \qquad (2.2.1)$$

such that

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \qquad (2.2.2)$$

We note that since

$$\lim_{n \to \infty} q_n(t) = 0 \qquad t \neq 0 ,$$

it is clear that

$$\lim_{t \to 0^+} \delta(t) = \lim_{t \to 0^+} (\lim_{n \to \infty} q_n(t)) = 0 \qquad (2.3)$$

Definition 2.4: The impulse response of a one-port is defined to be the response due to a unit pulse excitation $f(t) = \delta(t)$.

The following theorems [20] deal with one-ports. The first shows that the relationship between an excitation and the corresponding response of a one-port is expressible as a convolution integral.

Theorem 2. 1: A one-port is a convolution operator if and only if it is (1) linear, (2) single-valued, (3) continuous and (4) time-invariant. If $g(t)$ is the response due to excitation $f(t)$, we then write

$$g(t) = \int_{-\infty}^{\infty} a(t-\tau)f(\tau)d\tau = \int_{-\infty}^{\infty} a(\tau)f(t-\tau)d\tau \qquad (2.4.1)$$

or simply

$$g(t) = a(t) * f(t) = f(t) * a(t) \qquad (2.4.2)$$

where $a(t)$ is the impulse response of the one-port.

In the sense of Theorem 2. 1, the impulse response given in Definition 2. 4 can be rewritten as follows

Definition 2. 5: Let the one-port satisfy the conditions of Theorem 2. 1. Then the impulse response is written as

$$g(t) = a(t) * \delta(t) = \delta(t) * a(t) \qquad (2.5)$$

where $g(t) = a(t)$ and $\delta(t)$ are respectively the impulse response and the unit pulse excitation for the given one-port.

We now give the definitions of passivity and causality.

Definition 2. 6: Let the one-port be a convolution operator. Then it is said to be passive if and only if

$$\mathrm{Re} \int_{-\infty}^{T} f(t)g(t)dt \geq 0 \quad \forall \text{ real } T > 0 \qquad (2.6)$$

where the complex-valued functions $f(t)$ and $g(t)$ are respectively the excitation and the response of the one-port.

Definition 2.7: A function $f(t)$ is said to be causal if it vanishes for $t < 0$.

The following theorems give the relationships among linearity, passivity and causality of a one-port.

Theorem 2.2: A passive one-port which has a convolution representation is also causal.

Theorem 2.3: Let a one-port have a convolution representation with impulse response $a(t)$. It is causal if and only if $a(t) = 0$ for $t < 0$.

Since we shall be dealing with real systems only, we give the following [3].

Definition 2.8: A linear passive one-port is said to be real if and only if real excitations give rise to real responses.

From now on we shall assume that all the functions under consideration are real, causal and Laplace transformable.

We now give the definition of a positive real function [14].

Definition 2.9: A function $F(s)$ of a complex variable s is said to be a positive real function if it is

(1) holomorphic in the right-half s - plane

(2) real whenever s is real

(3) positive whenever $\mathrm{Re}\, s > 0$

(4) not identically zero.

We are now ready to give the definition of the system function of a linear passive one-port and the relationship between the impedance and admittance functions [18].

Definition 2.10: The Laplace transform of the impulse response $a(t)$ of a given linear passive one-port, denoted by $A(s)$, is called a driving-point system function, or an immittance function which is either an impedance $Z(s)$ or an admittance $Y(s)$ satisfying

$$Z(s) \cdot Y(s) = Y(s) \cdot Z(s) = 1 \qquad (2.7)$$

Definition 2.11: A function $F(s)$ of a complex variable s is said to be realizable if it is the Laplace transform of the impulse response of a linear passive system [18].

We now give an important theorem concerning realizability.

<u>Theorem 2.4:</u> A function $F(s)$ of a complex variable s is positive real if and only if it is realizable by a continuous linear passive system. In particular, $F(s)$ is a positive real finite rational function if and only if it can be realized by a finite lumped-parameter linear passive system.

The first part of the above theorem is a special case of Meixner's theorem [14], and the second part is due to Brune [4].

We shall now give some theorems concerning relaxation systems. We first give the definition [14] of spectral functions and then Cauer's integral representation of a positive real function [5].

<u>Definition 2.12.1:</u> A function $k(\eta)$ is said to be a spectral function if

(1) $k(\eta)$ is defined in $-\infty < \eta < \infty$ and is real

(2) $k(\eta)$ is nowhere decreasing

(3) $k(\eta)$ is bounded, i.e., $k(+\infty) - k(-\infty) < \infty$

(4) $k(\eta)$ is continuous at zero and is normalized to zero there, i.e.,

$$k(0^+) = k(0^-) = 0$$

(5) $k(\eta) = \frac{1}{2}\left[k(\eta+0) + k(\eta-0)\right]$

Definition 2. 12. 2: A function $\psi(\eta)$ is a spectral function in $0 \leq \eta < \infty$ if by odd continuation to negative η it becomes a spectral function in the sense of Definition 2. 12. 1.

Theorem 2. 5(a): A function $Y(s)$ of a complex variable s is positive real for $\text{Re } s > 0$ if and only if

$$Y(s) = As + s \int_{-\infty}^{\infty} \frac{1+\eta^2}{s^2+\eta^2} dk(\eta) \qquad (2.\ 8.\ 1)$$

where $A \geq 0$ is real and $k(\eta)$ is a spectral function.

Cauer [5] also gives an equivalent alternative form of (2. 8. 1) as follows.

Theorem 2. 5(b): A function $Y(s)$ of a complex variable s is positive real for $\text{Re } s > 0$ if the following integral exists

$$Y(s) = As + s \int_{0}^{\infty} \frac{d\psi(\lambda)}{s^2+\lambda} \qquad (2.\ 8.\ 2)$$

where $\psi(\lambda)$ is a non-decreasing real function, and $A \geq 0$ is real.

For our present purpose, we shall call $Y(s)$ an admittance function. We now state a theorem [14] which will eventually lead to the relaxation systems.

<u>Theorem 2.6:</u> Let $Y(s)$ be a positive real function. Then

$$\text{(i)}\quad W_1(s) = s^{\frac{1}{2}}\, Y(s^{\frac{1}{2}}) \tag{2.9}$$

$$\text{(ii)}\quad W_2(s) = s^{-\frac{1}{2}}\, Y(s^{\frac{1}{2}}) \tag{2.10}$$

are also positive real if we choose the branch for which $\arg s^{\frac{1}{2}} = 0$ for $\operatorname{Re} s > 0$.

The proof is based on Brune's theorem [4] on positive real function $Y(s)$, which states that if $|\arg Y(s)| \leq |\arg s|$, for $|\arg s| < \frac{\pi}{2}$ then $Y(s)$ is a positive real function. Hence (2.9) and (2.10) follow immediately.

<u>Definition 2.13:</u> A linear passive one-port with admittance $W_1(s)$ or $W_2(s)$ is called a relaxation system of the first or second kind if $W_1(s)$ and $W_2(s)$ are related to $Y(s)$ in (2.9) and (2.10) respectively.

In order to gain further insight into the properties of relaxation systems, we again consider (2.8.1) which is rewritten in Meixner's form [14]

$$Y(s) = As + \frac{B}{s} + s\int_0^\infty \frac{1+\eta^2}{s^2+\eta^2}\, dk(\eta) \quad \operatorname{Re} s > 0 \tag{2.11}$$

where A, B are real and non-negative and $k(\eta)$ is a spectral function in the sense of Definition 2.12.2. Then (2.9) and (2.10) can be rewritten respectively as

$$W_1(s) = A_1 s + B_1 + s \int_0^\infty \frac{1+\nu}{s+\nu}\, d\,\phi_1(\nu) \qquad (2.12.1)$$

$$W_2(s) = A_2 + \frac{B_2}{s} + \int_0^\infty \frac{1+\nu}{s+\nu}\, d\,\phi_2(\nu) \qquad (2.12.2)$$

where η^2 is replaced by ν and $k(\eta)$ by $\phi_i(\nu)$ $(i = 1, 2)$ both defined in $0 \leq \nu < \infty$ and continuous at $\nu = 0$, A_i and B_i are real and non-negative. The Laplace–Stieltjes transforms of $\phi_i(\nu)$, $i = 1, 2$, are

$$h_i(t) = \int_0^\infty e^{-\nu t} d\phi_i(\nu), \qquad i = 1, 2, \qquad (2.13)$$

which are completely monotonic in $0 \leq t < \infty$. Now taking the Laplace transform of $h_i(t)$, we get

$$\int_0^\infty h_i(t) e^{-st} dt = \int_0^\infty \frac{1}{s+\nu} d\phi_i(\nu), \quad \text{Re } s > 0 \qquad (2.14)$$

We write

$$\int_0^\infty \frac{\nu}{s+\nu} d\phi_i(\nu) = s \int_0^\infty \frac{\nu}{s(s+\nu)} d\phi_i(\nu) = s\Big[\int_0^\infty \frac{1}{s} d\phi_i(\nu) - \int_0^\infty \frac{1}{s+\nu} d\phi_i(\nu)\Big]$$

$$(2.15)$$

Using (2.14) and (2.15), (2.12.1) and (2.12.2) can be rewritten as

$$W_1(s) = A_1 s + B_1 + s \int_0^{\infty} e^{-st} h_1(t) dt + s^2 \int_0^{\infty} e^{-st} [h_1(0) - h_1(t)]\, dt \qquad (2.16.1)$$

$$W_2(s) = A_2 + \frac{B_2}{s} + \int_0^{\infty} e^{-st} h_2(t) dt + s \int_0^{\infty} e^{-st} [h_2(0) - h_2(t)]\, dt \qquad (2.16.2)$$

It now becomes clear that the admittance functions $W_1(s)$ and $W_2(s)$ can be mapped into the set of real non-negative A_i and B_i and the completely monotonic function $h_i(t)$ in $0 \leq t < \infty$ with $h_i(\infty) = 0$. Meixner [14] also shows that the impedance functions $Z_1(s)$ and $Z_2(s)$ have the same properties as $W_1(s)$ and $W_2(s)$.

We now give the representations of (2.11),(2.12.1) and (2.12.2) in the finite-lumped parameter case [16]. They are quite important since we shall be dealing only with the finite lumped cases. We have

$$Y(s) = As + \frac{B}{s} + \sum_{i=1}^{n} \frac{2 k_i' s}{s^2 + w_i^2} \qquad (2.17.1)$$

Setting $2k_i' = k_i$, (2.17.1) becomes

$$Y(s) = As + \frac{B}{s} + \sum_{i=1}^{n} \frac{k_i s}{s^2 + w_i^2} \qquad (2.17.2)$$

Hence the finite cases of (2.12.1) and (2.12.2) have the respective representations,

$$W_1(s) = A_1 s + B_1 + \sum_{i=}^{n} \frac{k_i s}{s+\lambda_i} \qquad (2.18.1)$$

$$W_2(s) = A_2 + \frac{B_2}{s} + \sum_{i=}^{n} \frac{k_i}{s+\lambda_i} \qquad (2.18.2)$$

where we have written $\lambda_i = w_i^2$. It is clear that $W_1(s)/s$ has the same property as (2.18.2). This is well-known in Network Theory [19]. Since if $W_1(s)$ and $W_2(s)$ are thought of as driving-point admittances of an RC and RL networks respectively, then $W_2(s)$ and $W_1(s)$ can be thought of as the RC and RL driving-point impedances respectively. Since (2.14) has Stieltjes continued fraction expansion [7], it follows that (2.18.1) and (2.18.2) can also be expanded into continued fractions with positive coefficients, which correspond to Cauer's [18] finite ladder networks with RC or RL elements only.

In this paper we shall be interested in Cauer's finite ladders of relaxation type. We shall give two examples of finite lumped relaxation one-ports, each with its corresponding electrical network analog. They are depicted in Figure 1 and Figure 2 based upon reference [2]. The equations describing their motions will also be given.

Finally, we give a definition and theorem concerning it for the stability of linear passive one-port.

Definition 2.14: A linear passive one-port is said to be stable if a bounded excitation gives rise to a bounded response for all $t > 0$.

In the sense of the above definition we state the following well-known theorem [3].

Theorem 2.8: A linear passive one-port is said to be stable if and only if its impulse response $a_1(t)$ is in L_1, i.e.,

$$\int_0^\infty |a_1(t)|\, dt < \infty \tag{2.19}$$

It is clear that relaxation systems of both kinds are stable systems.

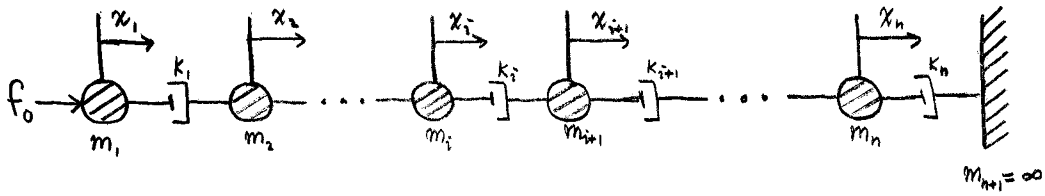


Figure 1a. Mass-Resistance Ladder.

m_i = mass; k_i = dashpot conductance;

x_i = displacement ; f_0 = excitation force

$$m_i \frac{du_i}{dt} = f_{i-1} - f_i \tag{2.20}$$

$$k_i f_i = (u_i - u_{i+1})$$

$$i = 1, 2, 3, \cdots, n$$

where $u_i = \dfrac{dx_i}{dt}$, $x_{n+1} = 0$, $m_{n+1} = \infty$

$$x_1(t) = a_1(t) * f_0(t)$$

$a_1(t)$ is the impulse response .

$A_1(s)$ is the admittance function.

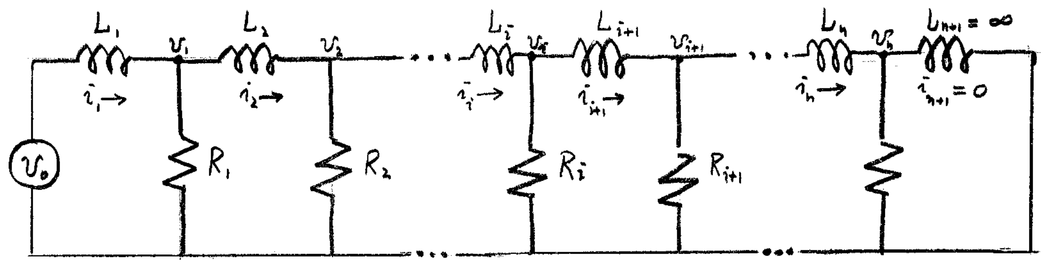


Figure 1b. Inductance-Resistance Ladder.

ν_i = voltage; i_i = current; L_i = inductance;

R_i = resistance

$$L_i \frac{di_i}{dt} = \nu_{i-1} - \nu_i \qquad (2.21)$$

$$\nu_i = R_i(i_i - i_{i+1})$$

$$i = 1, 2, 3, \cdots, n$$

where $L_{n+1} = \infty$ and $i_{n+1} = 0$

$$i_1(t) = b_1(t) * \nu_0(t)$$

$b_1(t)$ is the impulse response.

$B_1(s)$ is the admittance function.

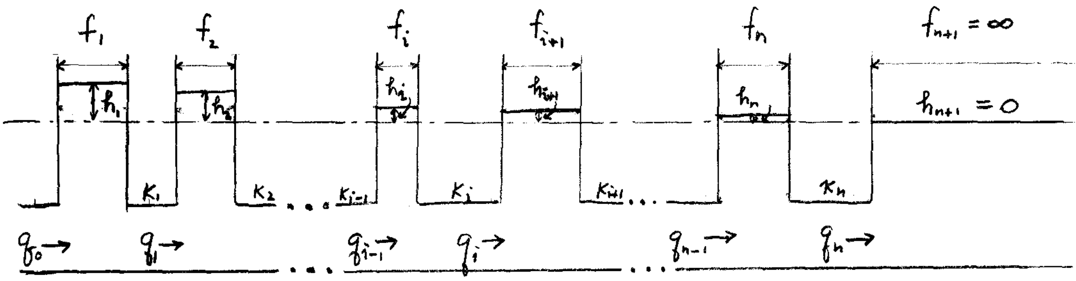


Figure 2a. Hydraulic-Capacitance-Resistance Ladder.

q_i = flow; k_i = water conductance; h_i = water level displacement; f_i = container cross-section

$$f_i \frac{dh_i}{dt} = q_{i-1} - q_i$$

$$k_i q_i = (h_i - h_{i+1}) \qquad (2.22)$$

$$i = 1, 2, \cdots, n$$

where $f_{n+1} = \infty$ and $h_{n+1} = 0$

$$h_1(t) = \gamma_1(t) * q_0(t)$$

$\gamma_1(t)$ is the impulse response.

$\Gamma_1(s)$ is the impedance function.

Note that the flow is laminar.

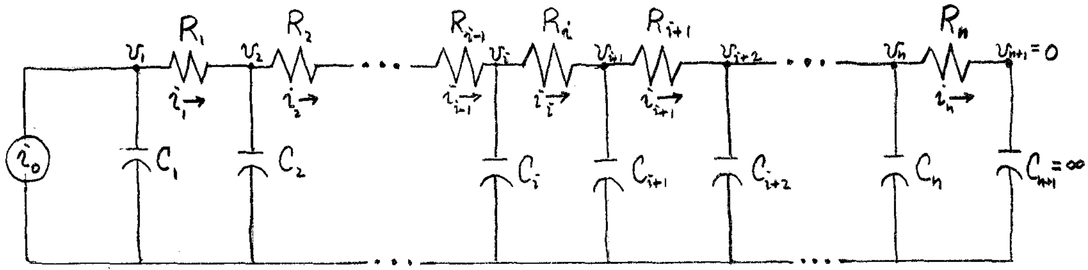


Figure 2b. Capacitance-Resistance Ladder.

i_i = current; ν_i = voltage; R_i = resistance;

C_i = capacitance

$$C_i \frac{d\nu_i}{dt} = i_{i-1} - i_i \tag{2.23}$$

$$i_i = \frac{1}{R}(\nu_i - \nu_{i+1})$$

$$i = 1, 2, \cdots, n$$

where $C_{n+1} = \infty$ and $\nu_{n+1} = 0$

$$\nu_1(t) = s_1(t) * i_0(t)$$

$s_1(t)$ is the impulse response.

$S_1(s)$ is the impedance function.

III. FORMULATION OF PROBLEMS AND METHOD OF SOLUTIONS

Statement of Problem 1 and Solution

Given the finite lumped-parameter RC one-port network shown in Figure 3. Assuming that all the elements of the network are concealed in a black box, find the system parameters $C_i^{(1)}$ and $R_i^{(1)}$ and the number of elements n based on the impulse response of the system obtained from the relationship between the given current excitation $q_0^{(1)}(t)$ and the measured voltage response $\nu_1^{(1)}(t)$ at the left-hand terminal pair in the time domain.

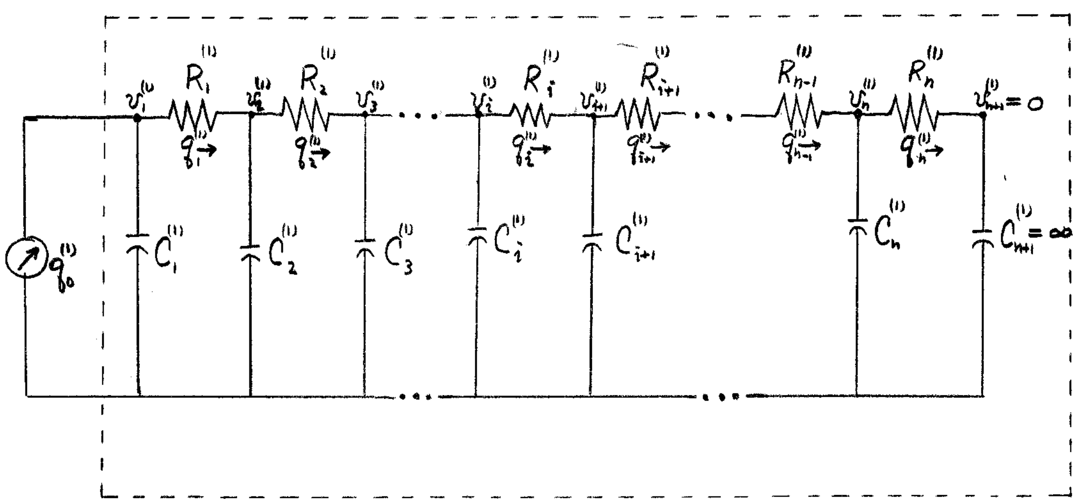


Figure 3. RC One-port Network .

Let $q_0^{(1)}(t) = \delta(t)$ be the current excitation applied to the left end of the given system, and let $\nu_i^{(1)}$ denote the corresponding voltage response of the i^{th} element, where the superscript (1) denotes the location at which the excitation is applied. Then the behavior of the system is governed by the following equations

$$C_i^{(1)} \frac{d\nu_i^{(1)}}{dt} = q_{i-1}^{(1)} - q_i^{(1)}$$

$$q_i^{(1)} = k_i^{(1)} (\nu_i^{(1)} - \nu_{i+1}^{(1)}) \qquad t \geq 0 \tag{3.1}$$

$$i = 1, 2, \cdots, n$$

where we have written $k_i^{(1)}$ for $\frac{1}{R_i^{(1)}}$, and the end conditions are $C_{n+1}^{(1)} = \infty$ and $\nu_{n+1}^{(1)} = 0$. Then by Definition 2.5 in the last section, the impulse response $a_1(t)$ of the given RC one-port is given by

$$a_1(t) = \nu_1^{(1)}(t) \tag{3.2}$$

Physically it is impossible for us to measure the voltages at certain desired points inside the black box. However, we can consider the given system as being broken down into n subsystems based on the given current excitation $q_0^{(1)}(t)$ and the measured voltage response $\nu_1^{(1)}(t)$. Hence we can analyze each subsystem independently in the time domain in order to determine the number of elements n in the black box and the system parameters $C_i^{(1)}$ and $R_i^{(1)}$, $i = 1, 2, 3, \cdots, n$. We start with the definition of the j^{th} subsystem as follows.

Definition 3.1: The new RC one-port obtained by taking away the first $(j-1)$ elements of the given RC one-port is called the j^{th} subsystem, $j = 1, \cdots, n$. In particular, we define the first subsystem, i.e., for $j = 1$, to be the given RC one-port.

A typical j^{th} substytem is depicted in Figure 4.

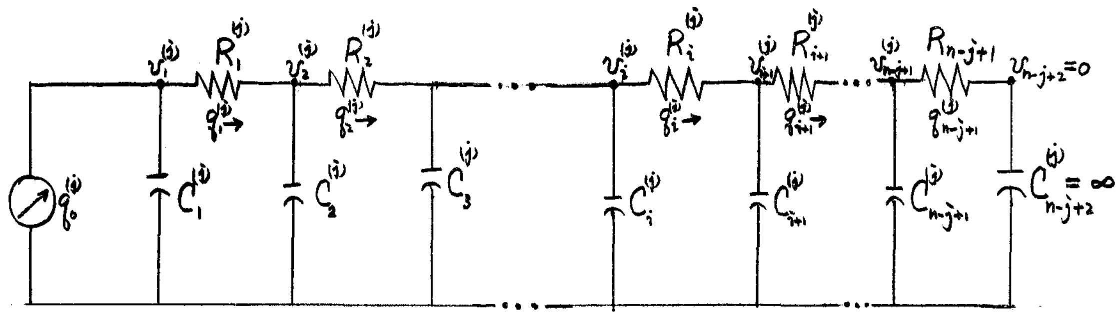


Figure 4. The j^{th} Subsystem.

We now give the definition of the impulse response of the j^{th} subsystem as follows.

Definition 3.2: Let $q_0^{(j)}(t) = \delta(t)$ and $\nu_i^{(j)}(t)$ be the excitation current at the left end and the voltage response of the i^{th} element respectively, $i = 1, 2, \cdots, n-j+1$. Then the impulse response $a_j(t)$ of the j^{th} subsystem is given, according to Definition 2.5, by

$$a_j(t) = \nu_1^{(j)}(t) \tag{3.3}$$

for each j, $j = 1, 2, 3, \cdots, n$.

The behavior of the j^{th} subsystem is governed by the following equations.

$$\begin{aligned} C_i^{(j)} \frac{d\nu_i^{(j)}}{dt} &= q_{i-1}^{(j)} - q_i^{(j)} \\ q_i^{(j)} &= k_i^{(j)}(\nu_i^{(j)} - \nu_{i+1}^{(j)}) \qquad t \geq 0 \end{aligned} \tag{3.4}$$

$$i = 1, 2, 3, \cdots, n-j+1$$

where $k_i^{(j)} = \frac{1}{R_i^{(j)}}$ with end conditions $C_{n-j+2}^{(j)} = \infty$ and $\nu_{n-j+2}^{(j)} = 0$.

We now show that once the impulse response of the j^{th} subsystem $a_j(t)$ is known then the impulse response $a_{j+1}(t)$ of the $(j+1)^{st}$ subsystem can be calculated. We have the following

<u>Theorem 3.1:</u> Let the j^{th} subsystem be excited by $q_0^{(j)} = \delta(t)$ whose impulse response is given by (3.3). Then the impulse response $a_{j+1}(t)$ of the $(j+1)^{st}$ subsystem is related to $\nu_2^{(j)}(t)$ and $q_1^{(j)}(t) \neq \delta(t)$ by

$$\nu_2^{(j)}(t) = a_{j+1}(t) * q_1^{(j)}(t) \tag{3.5}$$

where $q_1^{(j)}(t)$ and $\nu_2^{(j)}(t)$ are the excitation current and the voltage response of the second element of the j^{th} subsystem respectively.

Proof: From (3.4) for $i = 1$ we have

$$\begin{aligned} C_1^{(j)} \frac{d\nu_1^{(j)}(t)}{dt} &= \delta(t) - q_1^{(j)}(t) \\ \nu_2^{(j)}(t) &= \nu_1^{(j)}(t) - \frac{q_1^{(j)}(t)}{k_1^{(j)}} \quad , t \geq 0 \end{aligned} \tag{3.6}$$

Since $\lim_{t \to 0^+} \delta(t) = 0$, (3.6) becomes

$$\begin{aligned} C_1^{(j)} \frac{d\nu_1^{(j)}(t)}{dt} &= - q_1^{(j)}(t) \\ \nu_2^{(j)}(t) &= \nu_1^{(j)}(t) - \frac{q_1^{(j)}(t)}{k_1^{(j)}} \quad , t > 0 \end{aligned} \tag{3.7}$$

In general $q_1^{(j)}(t) \neq \delta(t)$. However, from Theorem 2.1 about the impulse response $a_{j+1}(t)$ of a one-port with arbitrary excitation $q_1^{(j)}(t) \neq \delta(t)$ and the corresponding voltage response $\nu_2^{(j)}(t)$ we have

$$\nu_2^{(j)}(t) = a_{j+1}(t) * q_1^{(j)}(t)$$

where $q_1^{(j)}(t)$ and $\nu_2^{(j)}(t)$ are given by (3.7).

Q. E. D.

We have the following obvious

Corollary 3.1.1: Let the $(j+1)^{st}$ subsystem be excited by $q_0^{(j+1)}(t) = \delta(t)$ giving rise to the response $\nu_1^{(j+1)}(t)$. Then the impulse response $a_{j+1}(t)$ is given by

$$a_{j+1}(t) = \nu_1^{(j+1)}(t) \tag{3.8}$$

and is identical with that obtained from (3.5).

The proof follows directly from the impulse response definition of a linear passive system.

We use a well-known definition due to S. Bernstein [14].

Definition 3.3: A function $f(t)$ is said to be completely monotonic in $0 \leq t < \infty$ if it has finite derivatives of every order in $0 < t < \infty$ and if

$$(-1)^k f^{(k)}(t) \geq 0 \qquad k = 0, 1, 2, \cdots, n, \cdots$$

are satisfied there and $f(0^+) < \infty$.

<u>Theorem 3.2:</u> The impulse response $a_1(t)$ of the first subsystem is completely monotonic in $0 \leq t < \infty$.

Proof: By eliminating $q_i^{(1)}(t)$ between the equations given in (3.1) for every i and writing the results in matrix form, we have

$$F^{(1)}DH^{(1)} + K^{(1)}H^{(1)} = E^{(1)} \tag{3.9}$$

where

$$F^{(1)} = \begin{bmatrix} C_1^{(1)} & & & \bigcirc \\ & C_2^{(1)} & & \\ & & \ddots & \\ \bigcirc & & & C_n^{(1)} \end{bmatrix} \tag{3.9.1}$$

$$D = \frac{d}{dt} \tag{3.9.2}$$

$$H^{(1)} = \begin{bmatrix} \nu_1^{(1)}(t) \\ \nu_2^{(1)}(t) \\ \vdots \\ \nu_n^{(1)}(t) \end{bmatrix} \tag{3.9.3}$$

$$E^{(1)} = \begin{bmatrix} \delta(t) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3.9.4}$$

and the system matrix

$$K^{(1)} = \begin{pmatrix} k_1^{(1)} & -k_1^{(1)} & & & & & & \bigcirc \\ -k_1^{(1)} & (k_1^{(1)} + k_2^{(1)}) & -k_2^{(1)} & & & & & \\ & -k_2^{(1)} & (k_2^{(1)} + k_3^{(1)}) & -k_3^{(1)} & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & -k_{i-1}^{(1)} & (k_{i-1}^{(1)} + k_i^{(1)}) & -k_i^{(1)} & & \\ & & & & -k_i^{(1)} & (k_i^{(1)} + k_{i+1}^{(1)}) & -k_{i+1}^{(1)} & \\ & & & & & \ddots & \ddots & \ddots \\ & & & & & -k_{n-2}^{(1)} & (k_{n-2}^{(1)} + k_{n-1}^{(1)}) & -k_{n-1}^{(1)} \\ \bigcirc & & & & & & -k_{n-1}^{(1)} & (k_{n-1}^{(1)} + k_n^{(1)}) \end{pmatrix} \quad (3.9.5)$$

which is real symmetric and positive definite (see Lemma 3. 1).

Taking the Laplace transform of (3. 9), we have

$$sF^{(1)}\widetilde{H}^{(1)} + K^{(1)}\widetilde{H}^{(1)} = \widetilde{E}^{(1)} \quad (3.10)$$

where $s\widetilde{H}^{(1)}$, $\widetilde{H}^{(1)}$ and $\widetilde{E}^{(1)}$ are the Laplace transforms of $DH^{(1)}$, $H^{(1)}$ and $E^{(1)}$ respectively. The matrix $\widetilde{E}^{(1)}$ has only one non-zero first component which is equal to 1. We now define a matrix

$$B^{(1)} = \begin{bmatrix} \frac{1}{\sqrt{C_1^{(1)}}} & & & & \\ & \frac{1}{\sqrt{C_2^{(1)}}} & & \bigcirc & \\ & & \frac{1}{\sqrt{C_3^{(1)}}} & & \\ & \bigcirc & & \ddots & \\ & & & & \frac{1}{\sqrt{C_n^{(1)}}} \end{bmatrix} \tag{3.11}$$

Clearly $B^{(1)} = B^{(1)T}$, where $B^{(1)T}$ is the transpose of $B^{(1)}$. Now let $\tilde{H}^{(1)} = B^{(1)}\tilde{Y}^{(1)}$, then (3.10) becomes

$$sF^{(1)}B^{(1)}\tilde{Y}^{(1)} + K^{(1)}B^{(1)}\tilde{Y}^{(1)} = \tilde{E}^{(1)} \tag{3.12}$$

Pre-multiplying (3.12) by $B^{(1)T}$ and, noting that $B^{(1)T}F^{(1)}B^{(1)} = I$, we get

$$s\tilde{Y}^{(1)} + G^{(1)}\tilde{Y}^{(1)} = \tilde{Q}^{(1)} \tag{3.13}$$

where $\tilde{Q}^{(1)} = B^{(1)}\tilde{E}^{(1)}$, and $G^{(1)} = B^{(1)T}K^{(1)}B^{(1)}$ is also a real symmetric and positive definite matrix. Hence its n eigenvalues

$\lambda_i^{(1)}$, $i = 1, 2, \cdots, n$, are positive real and distinct (see Lemma 3.1). It follows that there exist n orthonormal eigenvectors

$$u_i^{(1)} = \begin{pmatrix} b_{1i}^{(1)} \\ b_{2i}^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ b_{ni}^{(1)} \end{pmatrix} \qquad i = 1, 2, \cdots, n \tag{3.14.1}$$

satisfying

$$G^{(1)} u_i^{(1)} = \lambda_i^{(1)} u_i^{(1)} \tag{3.14.2}$$

We now expand $\tilde{Y}^{(1)}$, $G^{(1)}\tilde{Y}^{(1)}$ and $\tilde{Q}^{(1)}$ in a series of $u_i^{(1)}$. Then (3.13) is rewritten as

$$s\sum_{i=1}^{n} \alpha_i u_i^{(1)} + \sum_{i=1}^{n} \alpha_i \lambda_i^{(1)} u_i^{(1)} = \sum_{i=1}^{n} \beta_i u_i^{(1)} \tag{3.15}$$

where α_i and β_i are coefficients to be determined. For β_i we have

$$\beta_i = (\tilde{Q}^{(1)}, u_i^{(1)}) = \frac{1}{\sqrt{C_1^{(1)}}} \, b_{1i}^{(1)} \tag{3.15.1}$$

From (3. 15) and (3. 15. 1) we get

$$\alpha_i = \frac{1}{\sqrt{C_1^{(1)}}} \frac{b_{1i}^{(1)}}{s+\lambda_i^{(1)}} \tag{3. 15. 2}$$

Hence

$$\tilde{Y}^{(1)} = \sum_{i=1}^{n} \alpha_i u_i^{(1)}$$

$$= \frac{1}{\sqrt{C_1^{(1)}}} \sum_{i=1}^{n} \frac{b_{1i}^{(1)} u_i^{(1)}}{s+\lambda_i^{(1)}} \tag{3. 16}$$

Since we are interested in the first component of $\tilde{Y}^{(1)}$, namely $\tilde{y}_1^{(1)}(s)$, (3. 14. 1) and (3. 16) give

$$\tilde{y}_1^{(1)}(s) = \frac{1}{\sqrt{C_1^{(1)}}} \sum_{i=1}^{n} \frac{(b_{1i}^{(1)})^2}{s+\lambda_i^{(1)}} \tag{3. 17}$$

Again since $\tilde{H}^{(1)} = B^{(1)}\tilde{Y}^{(1)}$, then the first component of $\tilde{H}^{(1)}$, namely $\tilde{\nu}_1^{(1)}(s)$, is given by

$$\tilde{\nu}_1^{(1)}(s) = \frac{1}{C_1^{(1)}} \sum_{i=1}^{n} \frac{(b_{1i}^{(1)})^2}{s+\lambda_i^{(1)}} \tag{3. 18}$$

The inverse Laplace transform of (3. 18) yields immediately

$$\nu_1^{(1)}(t) = \frac{1}{C_1^{(1)}} \sum_{i=1}^{n} (b_{1i}^{(1)})^2 e^{-\lambda_i^{(1)} t}, \quad t \geq 0 \qquad (3.19)$$

It is clear that $\nu_1^{(1)}(t)$ satisfies the conditions of Definition 3.3 and must be completely monotonic in $0 \leq t < \infty$. It follows that the impulse response $a_1(t) = \nu_1^{(1)}(t)$ is completely monotonic in $0 \leq t < \infty$.

Q. E. D.

We now propose to show that the impulse response $a_j(t)$ of the j^{th} subsystem is also completely monotonic in $0 \leq t < \infty$, and we wish to derive a formula for $a_j(t)$ which is true for every j. We note that by eliminating $q_i^{(j)}(t)$ in (3.4) we can rewrite (3.4) in matrix form similar to (3.9), with superscript (1) being replaced by superscript (j). We first show that the system matrix $K^{(j)}$ is positive definite and that its eigenvalues are distinct. We have the following

<u>Lemma 3.1:</u> Let $K^{(j)}$ denote the system matrix of the j^{th} subsystem, where

$$K^{(j)} = \begin{bmatrix} k_1^{(j)} & -k_1^{(j)} & & & & & & & \text{\huge 0} \\ -k_1^{(j)} & (k_1^{(j)}+k_2^{(j)}) & -k_2^{(j)} & & & & & & \\ & -k_2^{(j)} & (k_2^{(j)}+k_3^{(j)}) & -k_3^{(j)} & & & & & \\ & & & \ddots & & & & & \\ & & & -k_{i-1}^{(j)} & (k_{i-1}^{(j)}+k_i^{(j)}) & -k_i^{(j)} & & & \\ & & & & -k_i^{(j)} & (k_i^{(j)}+k_{i+1}^{(j)}) & -k_{i+1}^{(j)} & & \\ & & & & & & \ddots & & \\ & \text{\huge 0} & & & & -k_{n-j-1}^{(j)} & (k_{n-j-1}^{(j)}+k_{n-j}^{(j)}) & -k_{n-j}^{(j)} & \\ & & & & & & -k_{n-j}^{(j)} & (k_{n-j}^{(j)}+k_{n-j+1}^{(j)}) & \end{bmatrix} \tag{3.20}$$

Then (i) $K^{(j)}$ is positive definite.

(ii) the $n-j+1$ characteristic roots of $K^{(j)}$ are distinct.

Proof: To prove the first statement we note that by performing a finite number of row operations on $K^{(j)}$, we get

$$K^{(j)'} = \begin{pmatrix} k_1^{(j)} - k_1^{(j)} & & & & & & 0 \\ & k_2^{(j)} - k_2^{(j)} & & & & & \\ & & k_3^{(j)} - k_3^{(j)} & & & & \\ & & & \ddots & & & \\ & & & & k_i^{(j)} - k_i^{(j)} & & \\ & & & & & \ddots & \\ & & & & & k_{n-j}^{(j)} - k_{n-j}^{(j)} & \\ 0 & & & & & & k_{n-j+1}^{(j)} \end{pmatrix} \qquad (3.21)$$

Since all the k's are positive, it is clear that $\det K^{(j)} = k_1^{(j)} \cdot k_2^{(j)} \cdot k_3^{(j)} \cdots k_{n-j}^{(j)} \cdot k_{n-j+1}^{(j)} > 0$. It follows that $K^{(j)'}$ has rank $(n-j+1)$. But $K^{(j)}$ and $K^{(j)'}$ are row equivalent, and they therefore must have the same rank. This implies that there are $n-j+1$ eigenvalues $\lambda_i'^{(j)}$ which are non-zero. But $K^{(j)}$ is real symmetric, hence $\lambda_i'^{(j)}$ are real and non-zero. It remains to be shown that $\lambda_i'^{(j)} > 0$.

By Geršgorin's theorem (see Theorem B, Appendix 1), it follows that $\lambda_i'^{(j)}$ are non-negative for $i = 1, 2, 3, \cdots, n-j+1$.

Since we have already shown that $\lambda_i'^{(j)}$ are non-zero, it follows immediately that $\lambda_i'^{(j)} > 0$ for every i. Hence $K^{(j)}$ is positive definite. This proves (i).

(ii) To prove the second statement we first note that $K^{(j)}$ is a real $(n-j+1) \times (n-j+1)$ Jacobi matrix. Since $k_i^{(j)}$ is real and positive for each i, then $(-k_i^{(j)})(-k_i^{(j)}) > 0$ for $i = 1, 2, \cdots, n-j$. It follows from Theorem C in Appendix 1 that $\lambda_i'^{(j)}$ are distinct, $i = 1, 2, 3, \cdots, n-j+1$.

Q. E. D.

We are now ready to prove the following

<u>Theorem 3.3:</u> Let the impulse response of the first subsystem be given by (3.19). Then the impulse response $a_j(t)$ of the j^{th} subsystem in the sense of Definition 3.2 is also completely monotonic in $0 \leq t < \infty$. In fact, for each j we have

$$a_j(t) = \frac{1}{C_1^{(j)}} \sum_{i=1}^{n-j+1} (b_{1i}^{(j)})^2 e^{-\lambda_i^{(j)} t} \tag{3.22}$$

Proof: Equation (3.4) can be written in matrix form as

$$F^{(j)} DH^{(j)} + K^{(j)} H^{(j)} = E^{(j)} \tag{3.23}$$

which is of the same form as (3.9), except superscript (1) being replaced by superscript (j) and n by n-j+1. Proceeding along the same line as the proof of Theorem 3.2, we note that the matrix $G^{(j)} = B^{(j)T}K^{(j)}B^{(j)}$ is real symmetric and positive definite since $K^{(j)}$ is positive definite by Lemma 3.1. Hence $G^{(j)}$ has positive real and distinct eigenvalues $\lambda_i^{(j)}$, $i = 1, 2, \cdots, n-j+1$. Then there exist $n-j+1$ orthonormal eigenvectors

$$u_i^{(j)} = \begin{pmatrix} b_{1i}^{(j)} \\ b_{2i}^{(j)} \\ \cdot \\ \cdot \\ \cdot \\ b_{(n-j+1)i}^{(j)} \end{pmatrix} \qquad (3.24.1)$$

satisfying

$$G^{(j)}u_i^{(j)} = \lambda_i^{(j)}u_i^{(j)} \qquad i = 1, 2, 3, \cdots, n-j+1 . \qquad (3.24.2)$$

The remainder of the proof is similar to that of Theorem 3.2. We finally arrive at

$$\nu_1^{(j)}(t) = \frac{1}{C_1^{(j)}} \sum_{i=1}^{n-j+1} (b_{1i}^{(j)})^2 e^{-\lambda_i^{(j)}t} \qquad t \geq 0 \qquad (3.25)$$

which is completely monotonic in $0 \leq t < \infty$. Since $a_j(t) = \nu_1^{(j)}t$, it must have the representation (3.22).

Q.E.D.

We are now in a position to do the time domain synthesis of the given RC system. However, we need the following

Lemma 3.2: Let

$$u_i^{(j)} = \begin{pmatrix} b_{1i}^{(j)} \\ b_{2i}^{(j)} \\ \cdot \\ \cdot \\ \cdot \\ b_{(n-j+1)i}^{(j)} \end{pmatrix}$$

be the i^{th} orthonormal eigenvector corresponding to each eigenvalue $\lambda_i^{(j)}$ of the matrix $G^{(j)}$, $i = 1, 2, 3, \cdots, n-j+1$. Then for each fixed k and j, we have

$$\sum_{i=1}^{n-j+1} (b_{ki}^{(j)})^2 = 1 \qquad j = 1, 2, 3, \cdots, n \qquad k = 1, 2, 3, \cdots, n-j+1 \tag{3.26}$$

Proof: Let R denote the orthogonal matrix with $R = (u_1^{(j)}, u_2^{(j)}, \cdots, u_{n-j+1}^{(j)})$. Then $RR^T = I$ where R^T is the transpose of R. Since $\det R \neq 0$, then R has a unique inverse.

But R^T is the right inverse of R, then we can write

$$RR^T = R^TR = I$$

But $R = (R^T)^T$, then we have

$$R^T(R^T)^T = I$$

i. e., R^T is orthogonal, and its column vectors are mutually orthogonal. Hence its length given by (3. 26) must be equal to 1 for each k.

Q. E. D.

<u>Theorem 3. 4:</u> Let the impulse response of the j^{th} subsystem be given by (3. 22). Then for each j,

$$C_1^{(j)} = \frac{1}{a_j(0^+)} \qquad j = 1, 2, 3, \cdots, n \tag{3. 27}$$

Proof: Since by (3. 22)

$$a_j(t) = \frac{1}{C_1^{(j)}} \sum_{i=1}^{n-j+1} (b_{1i}^{(j)})^2 e^{-\lambda_i^{(j)} t} \qquad j = 1, 2, 3, \cdots, n,$$

then

$$\lim_{t \to 0^+} a_j(t) = \frac{1}{C_1^{(j)}} \sum_{i=1}^{n-j+1} (b_{1i}^{(j)})^2$$

But $\sum_{i=1}^{n-j+1} (b_{1i}^{(j)})^2 = 1$ for each j, by Lemma 3.2 with $k = 1$.
It follows immediately that

$$C_1^{(j)} = \frac{1}{\lim_{t \to 0^+} a_j(t)} = \frac{1}{a_j(0^+)}$$

Q. E. D.

<u>Theorem 3.5:</u> Let the behavior of the j^{th} subsystem be described by (3.4). Then

$$k_1^{(j)} = \frac{q_1^{(j)}(0^+)}{\nu_1^{(j)}(0^+)} \tag{3.28.1}$$

Also,

$$k_1^{(j)} = C_1^{(j)} q_1^{(j)}(0^+) \tag{3.28.2}$$

Proof: For $i = 1$ (3.4) becomes (3.7) when $t > 0$. That is,

$$C_1^{(j)} \frac{d\nu_1^{(j)}(t)}{dt} = - q_1^{(j)}(t) \tag{3.29.1}$$

$$\nu_2^{(j)}(t) = \nu_1^{(j)}(t) - \frac{q_1^{(j)}(t)}{k_1^{(j)}} \tag{3.29.2}$$

Consider (3.29.2). We note that

$$\lim_{t \to 0^+} \nu_2^{(j)}(t) = 0. \tag{3.30}$$

Since $\nu_i^{(j)}(t)$, $i > 1$, is continuous at all values of t and by causality 0 for all $t < 0$. Hence taking limits at $t \to 0^+$ on both sides of Equation (3.29.2), we get

$$k_1^{(j)} = \frac{\lim\limits_{t \to 0^+} q_1^{(j)}(t)}{\lim\limits_{t \to 0^+} \nu_1^{(j)}(t)} = \frac{q_1^{(j)}(0^+)}{\nu_1^{(j)}(0^+)} ,$$

which is (3.28.1). Also, $\nu_1^{(j)}(0^+) = a_j(0^+)$. By Theorem 3.4

$$k_1^{(j)} = C_1^{(j)} q_1^{(j)}(0^+),$$

which is (3.28.2).

Q.E.D.

We now come to our final

Theorem 3.6: Let $a_j(t)$ be the impulse response of the j^{th} subsystem. Then

$$a_{n+1}(t) = 0 \quad \forall\ t.$$

Proof: Since the end condition of (3.4) gives

$$\nu_{n-j+2}^{(j)}(t) = 0$$

and since by Theorem 3.1

$$\nu_2^{(j)}(t) = a_{j+1}(t) * q_1^{(j)}(t),$$

then for $j = n$, the above two equations become

$$0 = a_{n+1}(t) * q_1^{(n)}(t) .$$

Since $q_1^{(n)}(t) \neq 0$, by continuity of current flow, then $a_{n+1}(t) = 0 \ \forall \ t \geq 0$.

Q. E. D.

We note that $C_1^{(j)}$ and $k_1^{(j)}$ obtained from Theorems 3.4 and 3.5 respectively are the capacitance and the conductance of the first RC element of the j^{th} subsystem respectively. Hence they are related to the system parameters of the first subsystem. This motivates the following

<u>Definition 3.4:</u> Let $C_1^{(j)}$ and $k_1^{(j)}$ be the first capacitance and conductance of the j^{th} subsystem respectively. Then we define

$$C_1^{(j)} = C_i^{(1)}$$

and

$$k_1^{(j)} = k_i^{(1)}$$

for $i = j$, $i, j = 1, 2, 3, \cdots, n$, where $C_i^{(1)}$ and $k_i^{(1)}$ are i^{th} capacitance and conductance of the first subsystem respectively.

We now summarize our solution to problem 1 as follows.

Given a passive RC one-port concealed in a black box shown in Figure 3. We wish to synthesize the system based on its impulse response function $a_1(t)$.

By definition the impulse response $a(t)$ of a linear system is the response due to a unit impulse excitation. In practice, it is impractical trying to generate an impulse current excitation $\delta(t)$. However, we can apply any arbitrary current excitation $q(t)$ to a system and observe its voltage response $\nu(t)$. Then

$$\nu(t) = a(t) * q(t)$$

from which $a(t)$ can be computed by evaluating the above convolution integral. Since $a(t)$ characterizes the system behavior, it does not change with any excitation-response pair.

Once the function $a_1(t)$ has been obtained by applying an arbitrary current excitation to our system, we can treat it as if it were obtained by having applied an impulse function $q_0^{(1)}(t) = \delta(t)$ to the system, since the function $a_1(t)$ is characteristic and independent of any excitation. Then the corresponding voltage response $\nu_1^{(1)}(t)$ will be equal to $a_1(t)$. Hence from (3.27) and (3.28.2) we can obtain $C_1^{(1)}$ and $k_1^{(1)} = \frac{1}{R_1^{(1)}}$ respectively. We now imagine that the first element is removed and that we are now working with a new system, namely the second subsystem. We compute $a_2(t)$ from Equation (3.5) for $j = 1$. Than $a_2(t) = \nu_1^{(2)}(t)$.

Again, using (3.27) and (3.28.2), we obtain $C_1^{(2)}$ and $k_1^{(2)}$ respectively. This process of analyzing the system piecemeal is repeated until we come to a particular j for which $\nu_{n-j+2}^{(j)} = 0$. Then Theorem 3.6 ensures the termination of the process. Hence the number of elements n is determined. Using Definition 3.4 and putting the $C_1^{(j)}$ and $k_1^{(j)}$ at their appropriate locations, the synthesis of the given RC one-port is complete.

A simple example illustrating our time domain method is given in Appendix 2. The results are then compared with those obtained by the standard continued fraction expansion method in the frequency-domain.

Statement of Problem 2 and Solution

Given a continuous function $h(t)$ in $0 \leq t \leq T$, which has been obtained accurately from an experiment whose analytical expression in the form of a finite sum of exponentials is unknown. Determine n, the number of components, and for each i from 1 through n, A_i and λ_i which are real and positive when $\lambda_i \neq \lambda_j$ for $i \neq j$ such that

$$h(t) = \begin{cases} \sum_{i=1}^{n} A_i e^{-\lambda_i t} & 0 \leq t \leq T \\ 0 & t < 0 \end{cases} \tag{3.31}$$

We first note that (3.31) satisfies the conditions of Definition 3.3 in $0 \leq t \leq T$ and hence must be completely monotonic there. Furthermore if the parameters n, A_i and λ_i, $i = 1, 2, \cdots, n$, can be found, then $h(t)$ must be completely monotonic in $0 \leq t < \infty$ with $h(\infty) = 0$. By Meixner's theorem [14] $h(t)$ can be thought of as the impulse response of a finite lumped relaxation system. The solution of (3.31) then reduces to reconstruction of a finite lumped-parameter RC one-port based on the given function $h(t)$.

Consider again, the finite RC one-port shown in Figure 3 whose impulse response is given by (3.19), i.e.,

$$\nu_1^{(1)}(t) = \frac{1}{C_1^{(1)}} \sum_{i=1}^{n} (b_{1i}^{(1)})^2 e^{-\lambda_i^{(1)} t}, \qquad 0 \leq t < \infty$$

and we note that

$$\lim_{t \to 0^+} \nu_1^{(1)}(t) = \frac{1}{C_1^{(1)}} \neq \infty .$$

Substituting the expression on the right side of Equation (3.19) for the right side of Equation (3.31), we get

$$h(t) = \frac{1}{C_1^{(1)}} \sum_{i=1}^{n} (b_{1i}^{(1)})^2 e^{-\lambda_i^{(1)} t} \qquad 0 \leq t \leq T \qquad (3.32)$$

where

$$\lambda_i = \lambda_i^{(1)} \tag{3.33.1}$$

$$A_i = \frac{1}{C_1^{(1)}} (b_{1i}^{(1)})^2 \tag{3.33.2}$$

$$i = 1, 2, 3, \cdots, n .$$

Using the time domain synthesis method, we can get $C_i^{(1)}$ and $k_i^{(1)}$, $i = 1, 2, \cdots, n$, from (3.27) and (3.28.2) respectively, the integer n being determined by Theorem 3.6. Now putting $C_i^{(1)}$ and $k_i^{(1)}$ back into the matrix $G^{(1)} = B^{(1)T} K^{(1)} B^{(1)}$ in Theorem 3.2, we obtain the distinct eigenvalues $\lambda_i^{(1)} > 0$, $i = 1, 2, \cdots, n$. Corresponding to each $\lambda_i^{(1)}$ we compute the orthonormal eigenvector $u_i^{(1)}$ in the form (3.14.1), i.e.,

$$u_i^{(1)} = \begin{bmatrix} b_{1i}^{(1)} \\ b_{2i}^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ b_{ni}^{(1)} \end{bmatrix} \qquad i = 1, 2, 3, \cdots, n$$

Taking the first component of the above, knowing $C_1^{(1)}$, A_i can be obtained immediately. An example of the method is given in Appendix 2.

BIBLIOGRAPHY

1. Berman, M. and R. Schoenfeld. Invariants in experimental data on linear kinetics and the formulation of models. Journal of Applied Physics 27(11): 1361. 1956.

2. Bödvarsson, G. Direct interpretation methods in applied geophysics. Geoexploration 4: 133. 1966.

3. Brown, W. M. Analysis of linear time-invariant systems. New York, McGraw-Hill, 1963. 339 p.

4. Brune, O. Synthesis of a finite two-terminal network whose driving-point impedance is a prescribed function of frequency. Journal of Mathematics and Physics 10(3): 191. 1931.

5. Cauer, W. The Poisson integral for function with positive real part. Bulletin of the American Mathematical Society 38: 713. Oct., 1932.

6. Finkbeiner, D.T. Matrices and linear transformation. San Francisco, W.H. Freeman, 1960. 246 p.

7. Fry, T.C. The use of continued fractions in the design of electrical networks. Bulletin of the American Mathematical Society 35: 463. 1929.

8. Gardner, D.G., J.C. Gardner, G. Laush and W.W. Meinke. Methods for the analysis of multicomponent exponential decay curves. Journal of Chemical Physics 31(4): 978. 1959.

9. Keeping, G.R., T.F. Wimett and R.K. Zeigler. Delayed neutrons from fissionable isotopes of uranium, plutonium and thorium. Journal of Nuclear Energy 6: 1. 1957.

10. Lanczos, C. Applied analysis. Englewood Cliffs, New Jersey, Prentice Hall, 1956. 539 p.

11. Liverman, T.P.G. Generalized function and direct operational method. Vol. 1. Englewood Cliffs, New Jersey, Prentice Hall, 1964. 338 p.

12. Marcus, H. and H. Minc. A survey of matrix theory and inequalities. Boston, Allyn and Bacon, 1964. 180 p.

13. McDonough, R. N. and L. Weiss. Prony's method, Z-transforms, and Padé approximation. SIAM Review 5(2): 145. April, 1963.

14. Meixner, J. On the theory of linear passive systems. Archives for Rational Mechanics 17(4): 278. 1964.

15. Rescigno, A. and G. Serge. Drug and tracer kinetics. Toronto, Blaisdell, 1966. 207 p.

16. Richards, P. L. A special class of functions with positive real part in a half-plane. Duke Mathematical Journal 14: 777. 1949.

17. Sheppard, C. W. and A. S. Householder. The mathematical basis of the interpretation of tracer experiments in closed steady-state systems. Journal of Applied Physics 22(4): 510. 1951.

18. Van Velkenburgh, M. E. Introduction to modern network synthesis. New York, Wiley, 1960. 498 p.

19. ________________. Modern network analysis. Englewood Cliffs, New Jersey, Prentice Hall, 1959. 440 p.

20. Zemanian, A. H. Distribution theory and transform analysis. New York, McGraw Hill, 1965. 371 p.

APPENDICES

APPENDIX 1

In proving Lemma 3.1, we made use of the following [6]

<u>Definition:</u> An $n \times n$ matrix B is said to be row equivalent to an $n \times n$ matrix A if and only if B can be obtained by performing a finite number of elementary row operations on A.

<u>Theorem A:</u> Row equivalent matrices have the same rank.

<u>Theorem B (Geršgorin) [12] :</u> The characteristic roots of an $n \times n$ complex matrix A lie in the closed region of the z-plane consisting of all discs

$$|z - a_{ii}| \leq P_i \qquad i = 1, 2, \cdots, n$$

where P_i is the sum of the absolute values of the off-diagonal elements in the i^{th} row.

<u>Theorem C [12] :</u> Let

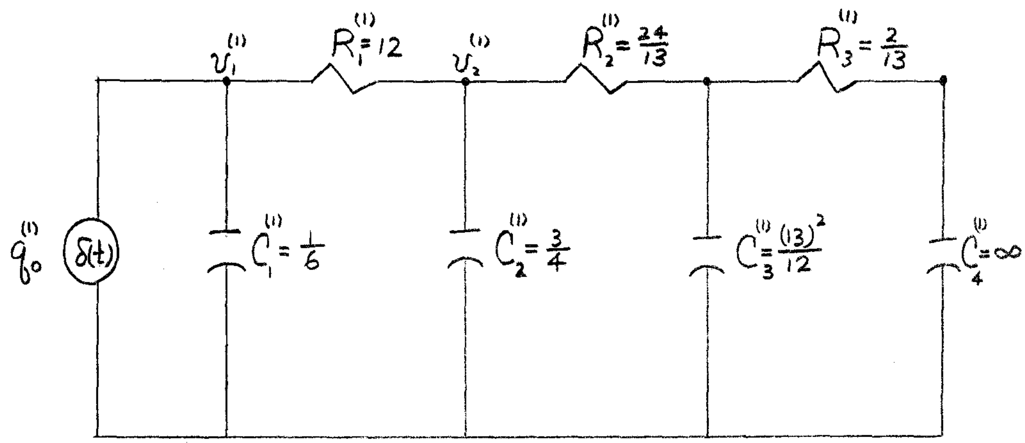
$$A = \begin{bmatrix} b_1 & c_1 & & & & \bigcirc \\ a_2 & b_2 & c_2 & & & \\ & a_3 & b_3 & c_3 & & \\ & & & \ddots & & \\ & & & a_{n-1} & b_{n-1} & c_{n-1} \\ \bigcirc & & & & a_n & b_n \end{bmatrix}$$

be an $n \times n$ real Jacobi matrix with $a_i c_{i-1} > 0$, $i = 1, 2, \cdots, n$.

Then the characteristic roots of A are simple.

APPENDIX 2

(A) Consider the RC one-port shown in the following figure.



We have

$$\nu_1^{(1)}(t) = a_1(t) * q_0^{(1)}(t) \tag{1}$$

Since $q_0^{(1)}(t) = \delta(t)$, therefore

$$a_1(t) = \nu_1^{(1)}(t) = e^{-t} + 2e^{-\frac{1}{2}t} + 3e^{-\frac{1}{3}t}, \quad t \geq 0 \tag{2}$$

Assuming that we know nothing about the system parameters, we wish to use our method of time domain synthesis on the given impulse response $a_1(t)$ in (2).

By (3. 27), we have

$$C_1^{(1)} = \frac{1}{a_1(0^+)} = \frac{1}{6} \tag{3}$$

By (3.7) for $j = 1$,

$$q_1^{(1)}(t) = -C_1^{(1)} \frac{d\nu_1^{(1)}(t)}{dt} = \frac{1}{6}(e^{-t} + e^{-\frac{1}{2}t} + e^{-\frac{1}{3}t}) \tag{4}$$

Now using (3.28.2) for $j = 1$, we get

$$k_1^{(1)} = C_1^{(1)} q_1^{(1)}(0^+) = \frac{1}{12} \quad \therefore \; R_1^{(1)} = \frac{1}{k_1^{(1)}} = 12 \tag{5}$$

Taking away the first RC element so that we are working now with the second subsystem, we make use of (3.7) $j = 1$. Then

$$\nu_2^{(1)}(t) = (\nu_1^{(1)}(t) - \frac{1}{k_1^{(1)}} q_1^{(1)}(t)) \tag{6}$$

$$= -e^{-t} + e^{-\frac{1}{3}t}$$

By (3.5) $j = 1$ we have

$$\nu_2^{(1)}(t) = a_2(t) * q_1^{(1)}(t) \tag{7}$$

from which

$$a_2(t) = \nu_1^{(2)}(t) = \frac{2}{3\sqrt{13}} [(\sqrt{13}+2)e^{-(\frac{11+\sqrt{13}}{18})t} + (\sqrt{13}-2)e^{-(\frac{11-\sqrt{13}}{18})t}] \tag{8}$$

which is the impulse response of the second subsystem. Using (3.27) with $j = 2$

$$C_1^{(2)} = \frac{1}{a_2(0^+)} = \frac{3}{4} \tag{9}$$

Again, using (3.7) for $j = 2$

$$q_1^{(2)}(t) = -C_1^{(2)} \frac{d\nu_1^{(2)}(t)}{dt}$$

$$= \frac{1}{36\sqrt{13}}\left[(35+\sqrt{13}) e^{-\left(\frac{11+\sqrt{13}}{18}\right)t} + (\sqrt{13}-35) e^{-\left(\frac{11-\sqrt{13}}{18}\right)t} \right] \quad (10)$$

By (3.28.2) $j = 2$

$$k_1^{(2)} = C_1^{(2)} q_1^{(2)}(0^+) = \frac{13}{24} \quad \therefore \; R_1^{(2)} = \frac{1}{k_1^{(2)}} = \frac{24}{13} \quad (11)$$

We now take away the second RC element. Using (3.7) for $j = 2$, we get

$$\nu_2^{(2)}(t) = \left[\nu_1^{(2)}(t) - \frac{1}{k_1^{(2)}} q_1^{(2)}(t) \right]$$

$$= \frac{6}{13\sqrt{13}} \left[-e^{-\left(\frac{11+\sqrt{13}}{18}\right)t} + e^{-\left(\frac{11-\sqrt{13}}{18}\right)t} \right] \quad (12)$$

Since

$$\nu_2^{(2)}(t) = a_3(t) * q_1^{(2)}(t) \quad (13)$$

$$\therefore \quad a_3(t) = \frac{12}{(13)^2} e^{-\frac{6}{13}t} \quad (14)$$

which is the impulse response of the third subsystem. Hence, again, by (3.27) for $j = 3$

$$C_1^{(3)} = \frac{1}{a_3(0^+)} = \frac{(13)^2}{12} \tag{15}$$

Now by (3.7)

$$q_1^{(3)}(t) = - C_1^{(3)} \frac{d\nu_1^{(3)}(t)}{dt}$$

$$= \frac{6}{13} e^{-\frac{6}{13}} \tag{16}$$

By (3.28.2) for $j = 3$

$$k_1^{(3)} = C_1^{(3)} q_1^{(2)}(0^+) = \frac{13}{2} \quad \therefore \quad R_1^{(3)} = \frac{1}{k_1^{(3)}} = \frac{2}{13} \tag{17}$$

By (3.7) with $j = 3$

$$\nu_2^{(3)}(t) = [\nu_1^{(3)}(t) - \frac{1}{k_1^{(3)}} q_1^{(3)}(t)]$$

$$= \frac{12}{(13)^2} e^{-\frac{6}{13}t} - (\frac{2}{13})(\frac{6}{13}) e^{-\frac{6}{13}t} = 0 \tag{18}$$

Now

$$\nu_2^{(3)}(t) = a_4(t) * q_1^{(3)}(t) \tag{19}$$

Since $\nu_2^{(3)}(t) = 0$ by (18), then by Theorem 3.6

$$a_4(t) \equiv 0 \quad \forall \; t \tag{20}$$

which is the end of the process.

By Definition 3.4

$$
\begin{aligned}
R_1^{(1)} &= 12 & ; \quad C_1^{(1)} &= \frac{1}{6} \\
R_1^{(2)} &= R_2^{(1)} = \frac{24}{13} & ; \quad C_1^{(2)} &= C_2^{(1)} = \frac{3}{4} \\
R_1^{(3)} &= R_3^{(1)} = \frac{2}{13} & ; \quad C_1^{(3)} &= C_3^{(1)} = \frac{(13)^2}{12}
\end{aligned}
\tag{21}
$$

We shall now use the continued fraction expansion in the frequency domain to show that the results are identical.

Taking the Laplace transform of (2), we get

$$
A_1(s) = \frac{6s^2 + 8s + \frac{7}{3}}{s^3 + \frac{11}{6} + s + \frac{1}{6}} \qquad \text{Re } s > 0. \tag{22}
$$

Now expand (22) into continued fraction

$$6s^2+8s+\frac{7}{3}\,\big)\,s^3+\frac{11}{6}s^2+s+\frac{1}{6}\,\big(\,\frac{s}{6}$$

$$s^3+\frac{8}{6}s^2+\frac{7}{18}s$$

$$\frac{1}{2}s^2+\frac{11}{18}s+\frac{1}{6}\,\big)\,6s^2+8s+\frac{7}{3}\,\big(\,12$$

$$6s^2+\frac{22s}{3}+2$$

$$\frac{2}{3}s+\frac{1}{3}\,\big)\,\frac{1}{2}s^2+\frac{11}{18}s+\frac{1}{6}\,\big(\,\frac{3}{4}s$$

$$\frac{1}{2}s^2+\frac{1}{4}s$$

$$\frac{13}{36}s+\frac{1}{6}\,\big)\,\frac{2}{3}s+\frac{1}{3}\,\big(\,\frac{24}{13}$$

$$\frac{2}{3}s+\frac{4}{13}$$

$$\frac{1}{13\times 3}\,\big)\,\frac{13}{36}s+\frac{1}{6}\,\big(\,\frac{(13)^2}{12}s$$

$$\frac{13s}{36}$$

$$\frac{1}{6}\,\big)\,\frac{1}{13\times 3}\,\big(\,\frac{2}{13}$$

$$\frac{1}{13\times 3}$$

$\therefore$

$$\frac{1}{A_1(s)} = \frac{1}{6}s+\cfrac{1}{12+\cfrac{1}{\frac{3}{4}s+\cfrac{1}{\frac{24}{13}+\cfrac{1}{\frac{(13)^2}{12}s+\cfrac{1}{\frac{2}{13}}}}}} \qquad (23)$$

,

from which

$$C_1^{(1)} = \frac{1}{6} \quad ; \qquad R_1^{(1)} = 12$$

$$C_2^{(1)} = \frac{3}{4} \quad ; \qquad R_2^{(1)} = \frac{24}{13}$$

$$C_3^{(1)} = \frac{(13)^2}{12} \quad ; \qquad R_3^{(1)} = \frac{2}{13}$$

(B) Suppose that $h(t)$ is continuous in $0 \leq t \leq T$ which is of the form

$$h(t) = \sum_{i=1}^{n} A_i e^{-\lambda_i t} \tag{24}$$

We wish to find n, A_i and λ_i in order to determine the analytic expression of $h(t)$. Since the verification of our method is formal, we make use of (2) and assume that we know nothing about the parameters being sought. We first write

$$h(t) = \sum_{i=1}^{n} A_i e^{-\lambda_i t} = \frac{1}{C^{(1)}} \sum_{i=1}^{n} (b_{1i}^{(1)})^2 e^{-\lambda_i^{(1)} t} \tag{25}$$

$$0 \leq t \leq T$$

Now we treat $h(t)$ in (25) as the impulse response $a_1(t)$ of some finite lumped RC one-port. Using the time domain synthesis method, we arrive at the system parameters given in (21), where n

has been determined to be equal to 3. It now remains to find A_i and λ_i, $i = 1, 2, 3$.

The system matrix $H^{(1)}$ for $n = 3$ is given by

$$H^{(1)} = \begin{bmatrix} k_1^{(1)} & -k_1^{(1)} & 0 \\ -k_1^{(1)} & (k_1^{(1)}+k_2^{(1)}) & -k_2^{(1)} \\ 0 & -k_2^{(1)} & k_2^{(1)}+k_3^{(1)} \end{bmatrix} = \begin{bmatrix} \frac{1}{12} & -\frac{1}{12} & 0 \\ -\frac{1}{12} & \frac{5}{8} & -\frac{13}{24} \\ 0 & -\frac{13}{24} & \frac{(13)^2}{24} \end{bmatrix} \tag{26}$$

and

$$B^{(1)} = B^{(1)T} = \begin{bmatrix} \frac{1}{\sqrt{C_1^{(1)}}} & & \bigcirc \\ & \frac{1}{\sqrt{C_2^{(1)}}} & \\ \bigcirc & & \frac{1}{\sqrt{C_3^{(1)}}} \end{bmatrix} = \begin{bmatrix} \sqrt{6} & & \bigcirc \\ & \frac{2}{\sqrt{3}} & \\ \bigcirc & & \frac{\sqrt{12}}{13} \end{bmatrix} \tag{27}$$

$$\therefore\ G^{(1)} = B^{(1)} H^{(1)} B^{(1)T} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{2}}{6} & 0 \\ -\frac{\sqrt{2}}{6} & \frac{5}{6} & -\frac{1}{6} \\ 0 & -\frac{1}{6} & \frac{1}{2} \end{bmatrix} \tag{28}$$

Now,

$$\begin{vmatrix} (\frac{1}{2} - \lambda) & -\frac{\sqrt{2}}{6} & 0 \\ -\frac{\sqrt{2}}{6} & (\frac{5}{6} - \lambda) & -\frac{1}{6} \\ 0 & -\frac{1}{6} & (\frac{1}{2} - \lambda) \end{vmatrix} = 0 \qquad (29)$$

from which

$$\lambda_1^{(1)} = 1, \quad \lambda_2^{(1)} = \frac{1}{2}, \quad \lambda_3^{(1)} = \frac{1}{3} \qquad (30)$$

Corresponding to each $\lambda_i^{(1)}$, $i = 1, 2, 3$, we get respectively the orthonormal eigenvectors as follows

$$\lambda_1^{(1)} = 1 : \quad \begin{bmatrix} b_{11}^{(1)} \\ b_{21}^{(1)} \\ b_{31}^{(1)} \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{2}{12}} \\ -\frac{3}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \end{bmatrix} \qquad (31)$$

$$\lambda_2^{(1)} = \frac{1}{2} : \quad \begin{bmatrix} b_{12}^{(1)} \\ b_{22}^{(1)} \\ b_{23}^{(1)} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ 0 \\ \sqrt{\frac{2}{3}} \end{bmatrix} \qquad (32)$$

$$\lambda_3^{(1)} = \frac{1}{3}: \quad \begin{bmatrix} b_{13}^{(1)} \\ b_{23}^{(1)} \\ b_{33}^{(1)} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \tag{33}$$

Hence,

$$h(t) = a_1(t) = \frac{1}{C_1^{(1)}} \sum_{i=1}^{n} (b_{1i}^{(1)})^2 e^{-\lambda_i^{(1)} t}$$

$$= \frac{1}{\frac{1}{6}} \left[\left(\frac{\sqrt{2}}{\sqrt{12}}\right)^2 e^{-t} + \left(\frac{-1}{\sqrt{3}}\right)^2 e^{-\frac{1}{2}t} + \left(\frac{\sqrt{2}}{2}\right)^2 e^{-\frac{1}{3}t} \right]$$

$$= e^{-t} + 2e^{-\frac{1}{2}t} + 3e^{-\frac{1}{3}t}$$

which is identical with (2). Hence we get

$$A_1 = 1 \qquad \lambda_1 = 1$$

$$A_2 = 2 \qquad \lambda_2 = \frac{1}{2}$$

$$A_3 = 3 \qquad \lambda_3 = \frac{1}{3} \; .$$