A STRUCTURE THEORY OF MODULES WITH A SPECIAL REFERENCE TO THOSE DEFINED BY THE CAUCHY CONVERGENCE CRITERION

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INTRODUCTION

The familiar construction of the real numbers as equivalence classes of Cauchy sequences of rational numbers assumes a particularly elegant form when expressed in algebraic terms. A development of this type is given, for instance, by Van der Waerden [5, p. 211-217]. Thus let R be the rational number field, R_1 the set of Cauchy sequences of elements of R, and N the set of null sequences of elements of R. We define addition and multiplication of elements in R_1 by

$$\{a_n\} + \{b_n\} = \{a_n + b_n\},$$

$$\{a_n\} \cdot \{b_n\} = \{a_n b_n\}.$$

Then R_1 is a commutative ring with identity and N is a maximal ideal in R_1. Hence the residue class ring R_1/N is a field, called the real number field. By defining "positive" sequences of rational numbers, an ordering is introduced into R_1/N. The principal results concerning R_1/N are: (1) R_1/N is a field; (2) R is ring- and order-isomorphic to a subfield R' of R_1/N; (3) R' is dense in R_1/N; (4) R_1/N is complete, that is every Cauchy sequence of elements of R_1/N

converges; and (5) every bounded set of elements of R_1/N has a least upper bound.

These results may be extended without any great difficulty to the case where R is an arbitrary ordered field (Archimedean or not) [5, p. 211-217]. The aim of the present investigation is two-fold: first, to vary the above construction by considering rings of sequences of rational numbers other than R_1 and kernels other than N and to investigate the algebraic properties of the quotient rings so obtained; and, secondly, to generalize the construction and the above-mentioned results to other metric-algebraic varieties, in such a way as to include some results which are known, as well, possibly, as some which are not known.

In section 1 we state and prove some preliminary results which are used in later sections. Section 2 is intended as at least a partial solution of the second problem posed above, while sections 3 and 4, algebraic in nature and intent, treat some very special cases of the first problem.

Throughout this paper free use is made of such well-known results as the "fundamental theorem of homomorphism for groups with operators" [2, p. 133] and its specializations to rings and to modules. In addition are used the following set-theoretic and algebraic symbols, with the usual meaning: X, ε, $\subseteq$, $\cap$, $\cong$, $\{x_n\}$ (denoting a sequence). Finally, we adopt the abbreviation "iff" for "if and only if."

SECTION 1. DEFINITIONS AND PRELIMINARY RESULTS

We begin this section with some fundamental theorems from the theory of rings and ideals.

Theorem 1. Let P and Q be arbitrary rings such that $P \subseteq Q$ and let N be an ideal in Q. Then the intersection $N \cap P$ is an ideal in P.

Proof. $N \cap P$ is not empty, for $0 \in N \cap P$. If $N \cap P = (0)$, then $N \cap P$ is an ideal in P. If $x, y \in N \cap P$ and $p \in P$, then $x - y \in P$ and $x - y \in N$, so that $x - y \in N \cap P$. Also $x \in P$, $p \in P$ implies $xp \in P$, $px \in P$; and $x \in N$, $p \in P$ implies $p \in Q$, which implies $xp \in N$, $px \in N$. Thus $xp \in N \cap P$ and $px \in N \cap P$.

Corollary 1.1. If N, P, Q are rings such that $N \subseteq P \subseteq Q$ and N is an ideal in Q, then N is an ideal in P.

Theorem 2. If P and Q are rings such that $P \subseteq Q$ and if N is a prime ideal in Q, then $N \cap P$ is prime in P.

Proof. Suppose $x, y \in P$ and $xy \in N \cap P$. Then $x, y \in Q$ and $xy \in N$. Since N is prime in Q, either $x \in N$ or $y \in N$. Hence either $x \in N \cap P$ or $y \in N \cap P$, and $N \cap P$ is prime in P.

If A and B are subsets of a group R, we denote by $A + B$ the set of all elements $a + b$ of R, where $a \in A$ and $b \in B$. In general, $A + B$ will not be a ring whenever A and B are. However, we have the following theorem.

Theorem 3. If P and Q are rings such that $P \subseteq Q$ and if N is an ideal in Q, then $P + N$ is a subring of Q. Moreover, it is the smallest subring of Q containing P and N.

Proof. $P + N$ is certainly not empty. Suppose $p_1 + n_1$, $p_2 + n_2$ are arbitrary elements of $P + N$ with $p_1, p_2 \in P$ and $n_1, n_2 \in N$. Then

$$(p_1 + n_1) - (p_2 + n_2) = (p_1 - p_2) + (n_1 - n_2) \in P + N$$

and

$$(p_1 + n_1)(p_2 + n_2) = (p_1 p_2) + (p_1 n_2 + n_1 p_2 + n_1 n_2).$$

Since N is an ideal in Q, $p_1 n_2$ and $n_1 p_2 \in N$, so that $p_1 n_2 + n_1 p_2 + n_1 n_2$ belongs to N. Since $p_1 p_2 \in P$, this proves that $P + N$ is a ring. Now let R be a subring of Q containing both P and N. If $p \in P$ and $n \in N$, then $p, n \in R$. Hence $p + n \in R$ and $P + N \subseteq R$. This completes the proof.

If N is an ideal in R, we write R/N for the residue-class (quotient) ring of R modulo N. Now in the above theorem we have, by corollary 1.1, that N is an ideal in $P + N$. Moreover, if $q \in P + N$ and we denote by $\bar{q}$ and $\tilde{q}$, respectively, the residue classes in Q and in $P + N$ which contain q, then quite obviously $\tilde{q} \subseteq \bar{q}$. However, since any element in $\bar{q}$ may be written in the form $q + n$ where $n \in N$, we have $\tilde{q} = \bar{q}$. Thus we arrive at the following inportant isomorphism theorem.

<u>Theorem</u> 4. If P and Q are rings such that $P \subseteq Q$ and N is an ideal in Q, then $P/(P \cap N) \cong (P + N)/N \subseteq Q/N$.

Proof. For each $x \in Q$, let $\bar{x} \in Q/N$ be the residue class containing x and consider the mapping $x \to \bar{x}$ of P into $(P + N)/N$. Clearly this mapping is a homomorphism. Moreover, it is onto, for any element of $P + N$ may be written in the form $p + n$ with $p \varepsilon P$ and $n \varepsilon N$, and $\overline{p+n} = \overline{p}$. If $x \varepsilon P$ and $\overline{x} = \overline{0}$, then $x \varepsilon N$ and hence $x \varepsilon P \cap N$. Thus the kernel of the mapping is $P \cap N$ and by the fundamental theorem of homomorphism

$$P/(P \cap N) \cong (P + N)/N.$$

We have already seen that $(P + N)/N \subseteq Q/N$, and so the theorem is proved.

<u>Corollary</u> 4.1. If P and Q are rings and N is an ideal in Q, such that $N \subseteq P \subseteq Q$, then $P/N \subseteq Q/N$.

<u>Corollary</u> 4.2. If N is an ideal in the ring Q, and if P is a subring of Q such that $P \cap N = (0)$, then Q/N is an extension of P. $P \cong Q/N$ if and only if P intersects every residue class of Q modulo N.

Next we introduce the concept of a module and give a brief résumé of the elementary theory.

<u>Definition</u> 1. Let A be a ring. A set R is an <u>A-module</u> iff R is a commutative group (operation +) and there exists a function $\cdot$ on $A \times R$ to R such that for all $a, b \varepsilon A$ and all $x, y \varepsilon R$,

i) $a \cdot (x + y) = a \cdot x + a \cdot y$,

ii) $(a + b) \cdot x = a \cdot x + b \cdot x$,

iii) $(ab) \cdot x = a \cdot (b \cdot x)$.

For the "product" $a \cdot x$ we shall write briefly ax.

<u>Definition</u> 2. Let R be an A-module. A subset N of R is a <u>submodule</u> of R iff for all $x, y \varepsilon N$ and all $a \varepsilon A$,

i) $x - y \varepsilon N$,

ii) $ax \in N$.

Now let R be an A-module and N a submodule of R. We define $x \sim y$ iff $x - y \in N$. Then $\sim$ is an equivalence relation and partitions R into equivalence classes $\bar{x}$. Furthermore, if $x \sim y$ and $u \sim v$, we have $x + u \sim y + v$ and $ax \sim ay$ for all $a \in A$. We are thus led to make the following definition.

Definition 3. Let R be an A-module and N a submodule of R. The A-module consisting of the set of equivalence classes defined by N and the compositions given by

i) $\bar{x} + \bar{y} = \overline{x + y}$ for all $x, y \in R$,

ii) $a\bar{x} = \overline{ax}$ for all $x \in R$, $a \in A$, where

$\bar{x}$ is the equivalence class containing x, is called the quotient module of R modulo N, and is denoted by R/N.

That R/N is indeed an A-module is seen by the following equations:

$$a(\bar{x} + \bar{y}) = a(\overline{x + y}) = \overline{a(x + y)} = \overline{ax + ay} = \overline{ax} + \overline{ay} = a\bar{x} + a\bar{y},$$

$$(a + b)\bar{x} = \overline{(a + b)x} = \overline{ax + bx} = \overline{ax} + \overline{bx} = a\bar{x} + b\bar{x},$$

$$(ab)\bar{x} = \overline{(ab)x} = \overline{a(bx)} = a(\overline{bx}) = a(b\bar{x}).$$

Definition 4. Let P, Q be A-modules. A single-valued mapping $x \to y$ of P onto Q is called a homomorphism iff

i) $x_1 \to y_1$, $x_2 \to y_2$ implies $x_1 + x_2 \to y_1 + y_2$,

ii) $x \to y$ implies $ax \to ay$ for all $a \in A$.

Q is called the homomorphic image of P. The kernel of the homomorphism is the set of all $x \in P$ such that $x \to 0$. If the mapping is 1-1, it is called an isomorphism and we write $P \cong Q$.

A module is only a special case of the more general concept of a group with operators (see Jacobson [2, p. 128-186]). Thus the entire theory for the latter carries over to the former. In particular, we have the

<u>Fundamental Theorem of Homomorphism</u>. If P is an A-module and Q is a submodule, then P is homomorphic to P/Q; and, conversely, if P is homomorphic to R and N is the kernel of the homomorphism, then N is a submodule of P and $R \cong P/N$. [2, p. 133].

Quite obviously if $P \subseteq Q \subseteq R$ are A-modules and P is a submodule of R, then P is a submodule of Q. We may now state the analogue of theorem 4 for modules.

<u>Theorem</u> 5. Let Q be an A-module and let P and N be submodules of Q. Then $P \cap N$ and $P + N$ are submodules of Q and

$$P/(P \cap N) \cong (P + N)/N \subseteq Q/N.$$

Proof. Suppose $x, y \in P \cap N$ and $a \in A$. Then $x - y \in P \cap N$. Also $x \in P$, $x \in N$ implies $ax \in P$, $ax \in N$ and hence $ax \in P \cap N$. Thus $P \cap N$ is a submodule. Suppose $x_1 + y_1$ and $x_2 + y_2$ are arbitrary members of $P + N$ with $x_1, x_2 \in P$ and $y_1, y_2 \in N$ and let $a \in A$. Then

$$(x_1 + y_1) - (x_2 + y_2) = (x_1 - x_2) + (y_1 - y_2) \in P + N$$

and

$$a(x_1 + y_1) = ax_1 + ay_1 \in P + N.$$

Thus $P + N$ is a submodule.

Again let $\bar{x}$ be the equivalence class of $x \in Q$ modulo N and consider the mapping $x \to \bar{x}$ of P onto P + N. This mapping is a module homomorphism since $ax \to \overline{ax} = a\bar{x}$ for all $a \in A$. The proof follows then exactly as for theorem 4.

<u>Corollary</u> 5.1. If N, P and Q are A-modules such that $N \subseteq P \subseteq Q$, then $P/N \subseteq Q/N$.

<u>Corollary</u> 5.2. If Q is an A-module and if N and P are submodules of Q such that $N \cap P = (0)$, then Q/N is an extension of P. $P \cong Q/N$ iff P intersects every equivalence class of Q modulo N.

Theorems 4 and 5 may both be deduced from the "second isomorphism theorem" for groups with operators as stated and proved in Jacobson [2, p. 136]. In a similar way, the following theorem is a special case of the "first isomorphism theorem" [2, p. 135].

<u>Theorem</u> 6. If M and N are ideals in the ring P, and if $M \subseteq N$, then N/M is an ideal in P/M and

$$(P/M)/(N/M) \cong P/N.$$

Proof. For each $x \in P$, let $\bar{x} \in P/M$ and $\tilde{x} \in P/N$ be the equivalence classes modulo M and N respectively which contain x. Consider the mapping $\bar{x} \to \tilde{x}$ of P/M onto P/N. The mapping is single-valued, for if $\bar{x} = \bar{y}$, then $x - y \in M$ and hence $x - y \in N$. Thus $\tilde{x} = \tilde{y}$. Moreover,

$$\bar{x} + \bar{y} = \overline{x + y} \to \tilde{x} + \tilde{y},$$

$$\bar{x} \cdot \bar{y} = \overline{xy} \to \widetilde{xy} = \tilde{x}\tilde{y},$$

so that the mapping is a homomorphism. Now

$$\tilde{z} = \tilde{0} \quad \text{iff} \quad z \in N \quad \text{iff} \quad \bar{z} \in N/M,$$

so that N/M is the kernel of the homomorphism. By the fundamental theorem of homomorphism then, N/M is an ideal and $(P/M)/(N/M) \cong P/N$.

This theorem obviously remains valid if rings and ideals are replaced throughout by modules and sub-modules. However, we shall not make use of this fact in the sequel.

<u>Theorem</u> 7. Let Q be a commutative ring with identity, and let MQ be the set of all elements of the form $\sum_{i=1}^{r} m_i q_i$ where $m_i \in M \subseteq Q$, $q_i \in Q$ $(i=1,2,\ldots,r)$. Then MQ is the smallest ideal in Q which contains M.

Proof. If $\sum_{i=1}^{r} m_i q_i \in MQ$ and $\sum_{i=1}^{s} m_i' q_i' \in MQ$, and if $q \in Q$, then

$$\sum_{i=1}^{r} m_i q_i - \sum_{i=1}^{s} m_i' q_i' = \sum_{i=1}^{r+s} m_i'' q_i'' \in MQ$$

and

$$q \sum_{i=1}^{r} m_i q_i = \sum_{i=1}^{r} m_i (q q_i) = \sum_{i=1}^{r} m_i \bar{q}_i \in MQ.$$

Thus MQ is an ideal in Q. Since Q has an identity e, $M \subseteq MQ$, for $m \in M$ implies $me = m \in MQ$. Finally, if P is an ideal in Q and $M \subseteq P$, then $MQ \subseteq PQ = P$.

SECTION 2. COMPLETE NORMED MODULES

As before we shall mean by a complete system or space one in which every Cauchy sequence converges. It may be shown that any metric space can be imbedded in a complete metric space, whose elements are equivalence classes of Cauchy sequences [4, p. 84-88]. The space which we shall consider is in some ways less general, but it includes as special cases two important applications of this construction: integral domains with valuation and normed linear spaces. Moreover, the present theory has the advantage of preserving the algebraic structure of these special cases. Thus an integral domain will be imbedded in an integral domain and a linear space in a linear space. We begin this section with some preliminary definitions.

Definition 5. A valuation for an integral domain A is a function φ on A such that for all $a, b \in A$,

i) $\varphi(a) \in P$, a complete ordered field,

ii) $\varphi(a) > 0$ for $a \neq 0$; $\varphi(0) = 0$,

iii) $\varphi(ab) = \varphi(a)\varphi(b)$,

iv) $\varphi(a+b) \leq \varphi(a) + \varphi(b)$.

Since any ordered field may be completed in the manner indicated in the introduction, it is no restriction to suppose that P is complete. We note that, if A has an identity e, then by ii) and iii) $\varphi(e) = \varphi(-e) = 1$. In any case $\varphi(-a) = \varphi(a)$.

Recalling the definition of an A-module from the previous section we note that by i) and ii) of definition 1, if $o \in A$ and $0 \in R$ are the respective "zero" elements, then $ox = a0 = 0$ and $(-a)x = a(-x) = -ax$ for all $a \in A$ and all $x \in R$. In case A has an identity e, we shall for the sake of simplicity insist on the additional restriction

iv) $ex = x$ for all $x \in R$.

We are now justified in making the following definition.

Definition 6. Let A be an integral domain with valuation φ to P an ordered field. Let R be an A-module satisfying the condition

o) $ax = 0$ implies $a = o$ or $x = 0$

for all $a \in A$, $x \in R$. A norm for R is a function μ on R such that for all $x,y \in R$ and all $a \in A$,

i) $\mu(x) \in P$,

ii) $\mu(x) > 0$ for $x \neq 0$: $\mu(0) = 0$,

iii) $\mu(ax) = \varphi(a)\mu(x)$,

iv) $\mu(x+y) \leq \mu(x) + \mu(y)$.

Since $\mu(ax) = \varphi(-a)\mu(-x) = \varphi(a)\mu(-x)$, we have $\mu(-x) = \mu(x)$ by condition iii). A module with a norm defined is called briefly a normed module. The ring A itself is a normed module if R is taken to be the additive group of A, multiplication is ring multiplication and $\mu = \varphi$. If A is the real (complex) number field and $\varphi(a) = |a|$, then R is a real (complex) normed linear space. In any case, since

$\mu(-x) = \mu(x)$, R is a metric space, with the metric $\bar{\mu}(x,y) = \mu(x - y)$.

We are now ready to define what is meant by a Cauchy sequence in a normed module and thus to introduce the question of completeness.

<u>Definition</u> 7. Let R be an A-module with norm μ having values in P and let $\{x_n\}$ be an infinite sequence of elements of R. Then $\{x_n\}$ is

i) convergent iff there exists an $x \in R$ and for each positive $\tau \in P$ an integer N such that

$$n > N \quad \text{implies} \quad \mu(x - x_n) < \tau;$$

ii) Cauchy iff for each positive $\tau \in P$ there exists an integer N such that

$$n,m > N \quad \text{implies} \quad \mu(x_m - x_n) < \tau; \quad \text{and}$$

iii) null iff for each positive $\tau \in P$ there exists an integer N such that

$$n > N \quad \text{implies} \quad \mu(x_n) < \tau.$$

In case i) we call x the <u>limit</u> of $\{x_n\}$ and write $x = \lim x_n$ (or $\lim_n x_n$).

Let R be an arbitrary normed A-module (norm in P); let N be the set of null sequences of elements of R; R_0 the set of convergent sequences of elements of R; and R_1 the set of Cauchy sequences of elements of R. If R is complete, $R_0 = R_1$. Otherwise, we shall see that R may be imbedded in a complete A-module. Let us define addition in R_1 and multiplication of elements of R_1 by elements of A by

$$\{x_n\} + \{y_n\} = \{x_n + y_n\},$$

$$a\{x_n\} = \{ax_n\}.$$

We shall show that N, R_0 and R_1 are A-modules and that $N \subseteq R_0 \subseteq R_1$.

The associative and commutative laws of addition follow immediately from the corresponding laws in R. Moreover,

$$\{x_n\} + \{0\} = \{x_n\},$$

$$\{x_n\} + \{-x_n\} = \{0\},$$

where $\{0\}$ is the sequence in which each element is 0. In addition we have

$$\begin{aligned}(a+b)\{x_n\} &= \{(a+b)x_n\} = \{ax_n + bx_n\} \\ &= \{ax_n\} + \{bx_n\} = a\{x_n\} + b\{x_n\},\end{aligned}$$

$$\begin{aligned}a(\{x_n\} + \{y_n\}) &= a\{x_n + y_n\} = \{ax_n + ay_n\} \\ &= \{ax_n\} + \{ay_n\} = a\{x_n\} + a\{y_n\},\end{aligned}$$

$$(ab)\{x_n\} = \{(ab)x_n\} = \{a(bx_n)\} = a\{bx_n\} = a(b\{x_n\}).$$

If A has an identity, then

$$1\cdot\{x_n\} = \{1\cdot x_n\} = \{x_n\}.$$

Now if $\{x_n\},\{y_n\} \in R_1$ and $a \in A$, then since

$$\mu((x_n - y_n) - (x_m - y_m)) \leq \mu(x_n - x_m) + \mu(y_n - y_m),$$

we have

$$\mu(x_n - x_m) < \tau/2, \ \mu(y_n - y_m) < \tau/2$$

implies

$$\mu((x_n - y_n)-(x_m - y_m))<\tau;$$

and

$$\mu(x_n - x_m) < (\varphi(a))^{-1}\tau$$

implies

$$\mu(ax_n - ax_m) = \varphi(a)\mu(x_n - x_m) < \tau,$$

provided $a \neq 0$. This shows that $\{x_n\} - \{y_n\} \in R_1$ and $a\{x_n\} \in R_1$. Thus R_1 is an A-module. Similarly, R_0 and N are A-modules.

Suppose $\{x_n\} \in R_0$. Then there exists $x \in R$ and for each positive $\tau \in P$ a positive integer N such that

$$n > N \quad \text{implies} \quad \mu(x - x_n) < \tau/2.$$

Thus

$$m,n > N \quad \text{implies} \quad \mu(x_n - x_m) \leq \mu(x - x_n) + \mu(x - x_m) < \tau$$

and $\{x_n\} \in R_1$. Therefore, R_0 is a submodule of R_1. If $\{x_n\} \in N$, then $\lim x_n = 0$ and $\{x_n\} \in R_0$. Thus N is a submodule of R_0 and hence of R_1.

We have seen in section 1 that R_0/N and R_1/N are also A-modules. They are related in the following manner.

Theorem 8. $R \cong R_0/N \subseteq R_1/N$.

Proof. Let $\overline{R}$ be the A-module of sequences $\{x_n\}$ for which $x_n = x$ for all n. Then $R \cong \overline{R} \subseteq R_0$, for $x \to \{x\}$ defines a 1-1 mapping of R onto $\overline{R}$ under which, for all $x,y \in R$ and all $a \in A$,

$$x + y \to \{x + y\} = \{x\} + \{y\},$$

$$ax \to \{ax\} = a\{x\}.$$

Since $\{x\}$ converges to x, $\overline{R} \subseteq R_0$.

Now $\overline{R} \cap N = \{0\}$, and for each $\{x_n\} \in R_0$ there exists $x \in R$ such that $\lim x_n = x$, that is $\{x_n\} - \{x\} \in N$. By corollary 5.2, we have $\overline{R} \cong R_0/N$. By corollary 5.1, R_0/N R_1/N. Thus $R \cong R_0/N \subseteq R_1/N$.

Corollary 8.1. If R is complete, then $R_1/N \cong R$.

Next we show that R_1/N is complete.

Lemma 1. If $\{x_n\} \in R_1$, then $\{\mu(x_n)\}$ converges.

Proof. From the properties of a norm,

$$\mu(x_n) \leq \mu(x_n - x_m) + \mu(x_m),$$

$$\mu(x_n) - \mu(x_m) \leq \mu(x_n - x_m).$$

Also

$$\mu(x_m) - \mu(x_n) \leq \mu(x_m - x_n) = \mu(x_n - x_m).$$

Thus

$$|\mu(x_n) - \mu(x_m)| \leq \mu(x_n - x_m).$$

Therefore, $\{\mu(x_n)\}$ is a Cauchy sequence of elements in P. Since P is assumed to be complete, the lemma follows.

Let

$$\mu^*(\{\overline{x_n}\}) = \lim \mu(x_n),$$

where $\{\overline{x_n}\}$ is the equivalence class modulo N which contains $\{x_n\}$.

<u>Lemma</u> 2. μ^* is a norm for R_1/N and agrees with μ on the image R_0/N of R.

Proof. First we show that R_1/N satisfies condition o) of definition 6. Thus if $a\{\overline{x_n}\} = 0$, then $\{ax_n\} \varepsilon N$ and either $a = 0$ or $\{x_n\} \varepsilon N$. Therefore, either $a = 0$ or $\{\overline{x_n}\} = 0$.

By definition and lemma 1, $\mu^*(\{\overline{x_n}\}) \varepsilon P$. If $\{x_n\} - \{y_n\} \varepsilon N$, then

$$\lim \mu(x_n - y_n) = 0.$$

But

$$|\mu(x_n) - \mu(y_n)| \leq \mu(x_n - y_n)$$

for all n, so that

$$\lim (\mu(x_n) - \mu(y_n)) = 0,$$

$$\lim \mu(x_n) = \lim \mu(y_n) = \mu^*(\{\overline{x_n}\})$$

and μ^* is single-valued. Moreover,

$$\mu^*(\{\overline{x_n}\}) = 0 \quad \text{iff} \quad \{x_n\} \varepsilon N \quad \text{iff} \quad \{\overline{x_n}\} = 0,$$

$$\mu^*(\{\overline{x_n}\}) \geq 0 \quad \text{since} \quad \mu(x_n) \geq 0 \quad \text{for all } n;$$

and

$$\mu^*(a\{\overline{x_n}\}) = \mu^*(\overline{a\{x_n\}}) = \mu^*(\{\overline{ax_n}\})$$
$$= \lim \mu(ax_n)$$
$$= \varphi(a) \lim \mu(x_n)$$
$$= \varphi(a)\mu^*(\{\overline{x_n}\}).$$

Finally

$$\mu(x_n + y_n) \leq \mu(x_n) + \mu(y_n), \quad (n = 1,2,\ldots),$$

so that

$$\mu^*(\{\overline{x_n + y_n}\}) \leq \mu^*(\{\overline{x_n}\}) + \mu^*(\{\overline{y_n}\}),$$

$$\mu^*(\{\overline{x_n}\} + \{\overline{y_n}\}) \leq \mu^*(\{\overline{x_n}\}) + \mu^*(\{\overline{y_n}\}).$$

This proves that μ^* is a norm.

Now if $\{\overline{x_n}\}$ is the image of $x \varepsilon R$, then $\{x_n\} - \{x\} \varepsilon N$ and as above

$$\lim (\mu(x_n) - \mu(x)) = 0,$$

$$\lim \mu(x_n) = \mu^*(\{\overline{x_n}\}) = \mu(x).$$

This completes the proof of lemma 2.

<u>Lemma</u> 3. R_0/N is dense in R_1/N with respect to the topology imposed by the metric $\overline{\mu}(x,y) = \mu^*(x - y)$.

Proof. Let $\{\overline{x_n}\}$ be any element of R_1/N and τ any positive element of P. Then there exists a positive integer n_0 such that

$$n > n_0 \quad \text{implies} \quad \mu(x_n - x_{n_0}) < \tau/2.$$

Thus if $\{y_n\}$ is the constant sequence defined by $y_n = x_{n_0}$ for all n, then

$$n > n_0 \quad \text{implies} \quad \mu(x_n - y_n) < \tau/2,$$

or, passing to the limit

$$\mu^*(\{\overline{x_n - y_n}\}) \leq \tau/2 < \tau,$$

$$\mu^*(\overline{\{x_n\} - \{y_n\}}) < \tau,$$

$$\mu^*(\{\overline{x_n}\} - \{\overline{y_n}\}) < \tau.$$

But $\{\overline{y_n}\} \varepsilon R_0/N$, since $\{y_n\}$ converges to x_{n_0}. Thus every τ-neighborhood of $\{\overline{x_n}\}$ contains an element of R_0/N.

Lemma 4. Every element of R_1/N is the limit of a convergent sequence of elements of R_0/N.

Proof. Let $\{\tau_n\}$ be a monotone, null sequence of positive elements of P. By lemma 3, there exists for each $\alpha \varepsilon R_1/N$, a sequence $\{\beta_n\}$ with $\beta_n \varepsilon R_0/N$ such that $\mu^*(\alpha - \beta_n) < \tau_n$. Thus, for each $\tau > 0$ in P, there exists a positive integer n_0 such that

$$n > n_0 \quad \text{implies} \quad \mu^*(\alpha - \beta_n) < \tau_n < \tau,$$

and hence

$$\alpha = \lim \beta_n.$$

Theorem 9. R_1/N is complete with respect to the norm μ^*.

Proof. Let $\{\tau_n\}$ be a null sequence of elements of P. (In the Archimedean case we may take, for instance, $\tau_n = 1/n$.) Let $\{\alpha_n\}$ be a Cauchy sequence of elements of R_1/N and let $\{\beta_n\}$ be a sequence of elements of R_0/N such that

$$\mu^*(\alpha_n - \beta_n) < \tau_n.$$

Now for any positive $\tau \varepsilon$ P there exist positive integers n_1 and n_2 such that

$$n, m > n_1 \quad \text{implies} \quad \mu^*(\alpha_n - \alpha_m) < \tau/3$$

and

$$\tau_{n_2} < \tau/3.$$

Then for $n,m > n_0 = \max(n_1, n_2)$, we have

$$\mu^*(\beta_n - \beta_m) \leq \mu^*(\beta_n - \alpha_n) + \mu^*(\alpha_n - \alpha_m) + \mu^*(\alpha_m - \beta_m)$$
$$\leq \tau_n + \mu^*(\alpha_n - \alpha_m) + \tau_m$$
$$< \tau/3 + \tau/3 + \tau/3 = \tau$$

and hence $\{\beta_n\}$ is a Cauchy sequence. By theorem 8, $R_0/N \cong R$. Let $\{x_n\}$ be a sequence of elements of R such that $x_n \to \beta_n$ under this isomorphism. By lemma 2, $\{x_n\}$ is a Cauchy sequence. Let $\alpha \in R_1/N$ be the equivalence class containing $\{x_n\}$. Then for each n

$$\mu^*(\alpha - \alpha_n) \leq \mu^*(\alpha - \beta_n) + \mu^*(\beta_n - \alpha_n)$$
$$\leq \lim_k \mu(x_k - x_n) + \tau_n.$$

Now there exists an integer n_0 such that

$$n,k > n_0 \quad \text{implies} \quad \mu(x_k - x_n) < \tau/2$$

and

$$n > n_0 \quad \text{implies} \quad \tau_n < \tau/2.$$

Thus

$$n > n_0 \quad \text{implies} \quad \lim_k \mu(x_k - x_n) \leq \tau/2$$
$$\text{implies} \quad \mu^*(\alpha - \alpha_n) < \tau.$$

Thus $\{\alpha_n\}$ converges.

We have shown that any normed module can be imbedded in a complete normed module. In particular, any real (complex) normed linear space can be imbedded in a real (complex) Banach space. The remainder of this section will be devoted to applying this result to integral domains.

<u>Theorem</u> 10. Any integral domain with valuation can be imbedded in a complete integral domain with valuation.

Proof. Let R be an integral domain with valuation φ having values in P. Then we have seen that R is an R-module with norm φ. By the above construction we obtain the complete normed R-module R_1/N. Now, since $R \cong R_0/N$, we have, in an obvious way, that R_1/N is an R_0/N-module, in the sense of isomorphism. Moreover, $\varphi^* = \mu^*$ is a valuation for R_0/N and φ^* is a norm for R_1/N with respect to this valuation. Let α, β be arbitrary elements of R_1/N. By lemma 4, there exists a sequence $\{\alpha_n\}$ of elements of R_0/N such that

$$\lim \alpha_n = \alpha.$$

The sequence $\{\alpha_n\beta\}$ is a Cauchy sequence, for, if $\beta \neq 0$, then given any positive $\tau \in P$, there exists an integer n_0 such that

$$\begin{aligned} m,n > n_0 \quad \text{implies} \quad & \varphi^*(\alpha_n - \alpha_m) < \tau(\varphi^*(\beta))^{-1} \\ \text{implies} \quad & \varphi^*(\alpha_n\beta - \alpha_m\beta) = \varphi^*(\alpha_n - \alpha_m)\varphi^*(\beta) < \tau. \end{aligned}$$

If $\beta = 0$, then $\{\alpha_n\beta\} = \{0\}$ is a Cauchy sequence.

We define two compositions + and $\cdot$ in the set R_1/N in the following way. + is the group addition in the module R_1/N. For any $\alpha, \beta \in R_1/N$

$$\alpha \cdot \beta = \lim \alpha_n\beta$$

where

$$\alpha = \lim \alpha_n, \quad \alpha_n \in R_0/N.$$

[Note that if $\alpha \in R_0/N$, then $\alpha\cdot\beta = \lim\limits_n \alpha\beta = \alpha\beta$.]

This composition is single-valued, for if $\{\alpha_n'\}$ also converges to α, then $\{\alpha_n - \alpha_n'\}$ and consequently $\{\alpha_n\beta - \alpha_n'\beta\}$ is a null sequence. Thus

$$\lim \alpha_n\beta = \lim \alpha_n'\beta.$$

We shall show that R_1/N with these compositions is an integral domain.

The theory of double sequences of real numbers [3, p. 247-292] is easily carried over to R_1/N. In particular, if $\{\alpha_n\}$ and $\{\beta_n\}$ are convergent sequences, then

$$\lim (\alpha_n + \beta_n) = \lim \alpha_n + \lim \beta_n,$$

$$\lim \alpha_n\beta_n = \lim_n \alpha_n \lim_k \beta_k = \alpha\cdot\beta,$$

where $\alpha = \lim \alpha_n$, $\beta = \lim \beta_n$. The first of these is a consequence of

$$\varphi^*(\alpha_n + \beta_n - \alpha - \beta) \leq \varphi^*(\alpha_n - \alpha) + \varphi^*(\beta_n - \beta).$$

To prove the other we note that for some $M \in P$ and all n

$$\varphi^*(\alpha_n) < M.$$

Thus

$$\varphi^*(\alpha_n\beta - \alpha_n\beta_n) = \varphi^*(\alpha_n)\varphi^*(\beta - \beta_n) < M\, \varphi^*(\beta - \beta_n)$$

so that $\{\alpha_n\beta - \alpha_n\beta_n\}$ is a null sequence. Thus

$$\lim (\alpha_n\beta - \alpha_n\beta_n) = 0$$

or

$$\lim \alpha_n\beta_n = \lim \alpha_n\beta = \lim_n \alpha_n \lim_k \beta_k.$$

Now let $\{\alpha_n\}$, $\{\beta_n\}$ be Cauchy sequences of elements of R_0/N converging respectively to the arbitrary elements α and β of R_1/N, and let γ be any other element of R_1/N. Then

$$\begin{aligned}\alpha\cdot(\beta\cdot\gamma) &= \lim_n \alpha_n(\lim_k \beta_k\gamma)\\ &= \lim_n \lim_k \alpha_n(\beta_k\gamma)\\ &= \lim_n \lim_k (\alpha_n\beta_k)\gamma\\ &= \lim_n (\alpha_n\beta_n)\gamma\\ &= (\alpha\cdot\beta)\cdot\gamma.\end{aligned}$$

Also

$$\begin{aligned}\alpha\cdot(\beta+\gamma) &= \lim \alpha_n(\beta+\gamma)\\ &= \lim (\alpha_n\beta+\alpha_n\gamma)\\ &= \lim \alpha_n\beta + \lim \alpha_n\gamma\\ &= \alpha\cdot\beta + \alpha\cdot\gamma\end{aligned}$$

and

$$\begin{aligned}(\alpha+\beta)\cdot\gamma &= \lim (\alpha_n+\beta_n)\gamma\\ &= \lim (\alpha_n\gamma+\beta_n\gamma)\\ &= \lim \alpha_n\gamma + \lim \beta_n\gamma\\ &= \alpha\cdot\gamma + \beta\cdot\gamma.\end{aligned}$$

Finally, suppose $\alpha\cdot\beta = 0$ and $\beta \neq 0$. Then $\lim \alpha_n\beta = 0$ and for each positive $\tau \varepsilon P$ there exists an integer n_0 such that

$$n > n_0 \text{ implies } \varphi^*(\alpha_n\beta) < \tau\ \varphi^*(\beta).$$

Since $\varphi^*(\beta) \neq 0$, this implies

$$\varphi^*(\alpha_n) = \varphi^*(\alpha_n\beta)(\varphi^*(\beta))^{-1} < \tau$$

and hence

$$\alpha = \lim \alpha_n = 0.$$

Now R_1/N is automatically a commutative group with respect to addition by the definition of $+$. Thus R_1/N is an integral domain and theorem 10 is proved.

If R has an identity e, then R_0/N has an identity $\bar{e}$ and

$$\bar{e}\cdot\alpha = \bar{e}\alpha = \alpha,$$

$$\alpha\cdot\bar{e} = \lim \alpha_n\bar{e} = \lim \alpha_n = \alpha,$$

so that $\bar{e}$ is an identity for R_1/N. If R is commutative, then

$$\beta\cdot\alpha = \lim \beta_n\alpha_n = \lim \alpha_n\beta_n = \alpha\cdot\beta.$$

Thus R_1/N is also commutative.

Finally, if R is a division ring, then

$$(\lim \alpha_n^{-1})\cdot\alpha = \lim \alpha_n^{-1}\alpha = \lim_n \lim_k \alpha_n^{-1}\alpha_k = \lim \alpha_n^{-1}\alpha_n = \bar{e}$$

so that

$$\alpha^{-1} = \lim \alpha_n^{-1},$$

and we have proved the following corollary.

<u>Corollary</u> 10.1. Any field with valuation can be imbedded in a complete field with valuation; any division ring

with valuation can be imbedded in a complete division ring with valuation.

SECTION 3. RINGS AND IDEALS OF BOUNDED SEQUENCES

We return now to the investigation of sequences of rational numbers. Let R be the rational number field, let N, R_0, R_1 have the same meanings as in the previous section, and let R_2 be the set of all bounded sequences of rational numbers. Thus $N \subset R_0 \subset R_1 \subset R_2$. We define addition and multiplication in R_2 just as we did for R_1 in the introduction; thus,

$$\{a_n\} + \{b_n\} = \{a_n + b_n\}$$

$$\{a_n\}\cdot\{b_n\} = \{a_n b_n\}.$$

<u>Theorem</u> 11. R_2 is a commutative ring with identity.

Proof. The associative, commutative and distributive laws follow directly from the corresponding laws for the rational number field. Let $\{a_n\}$ be any element of R_2. Then

$$\{a_n\} + \{0\} = \{a_n\},$$

$$\{a_n\} + \{-a_n\} = \{0\},$$

$$\{a_n\}\cdot\{1\} = \{a_n\}.$$

Thus $\{0\}$ is the additive identity, $\{-a_n\}$ is the additive inverse or "negative" of $\{a_n\}$, and $\{1\}$ is the ring identity. Suppose now that $\{a_n\}$, $\{b_n\} \varepsilon R_2$. Then there exist rational numbers A and B such that

$$|a_n| < A, \ |b_n| < B \quad \text{for all } n.$$

Then

$$|a_n + b_n| \le |a_n| + |b_n| < A + B,$$

$$|a_n b_n| = |a_n||b_n| < AB$$

for all n. Thus $\{a_n\} + \{b_n\}$, $\{a_n\}\{b_n\} \varepsilon R_2$. This completes the proof.

<u>Theorem</u> 12. N is an ideal in R_2.

Proof. Suppose $\{a_n\} \varepsilon R_2$; $\{b_n\}$, $\{c_n\} \varepsilon N$. Then there exists a rational number A such that

$$|a_n| < A \quad \text{for all n.}$$

By the definition of a null sequence, there exists for each $\tau > 0$ a positive integer n_0 such that

$$n > n_0 \text{ implies } |b_n| < \tau/A$$

$$\text{implies } |a_n b_n| < \tau.$$

Likewise, there exists a positive integer n_1 such that

$$n > n_1 \text{ implies } |b_n| < \tau/2, \ |c_n| < \tau/2$$

$$\text{implies } |b_n - c_n| \le |b_n| + |c_n| < \tau.$$

Thus $\{a_n\} \cdot \{b_n\}$, $\{b_n\} - \{c_n\} \varepsilon N$, and N is an ideal in R_2.

<u>Theorem</u> 13. N is not prime in R_2.

Proof. Consider the sequences

$$a = \{0,1,0,1,\ldots\}$$

$$b = \{1,0,1,0,\ldots\}.$$

Now $a, b \ \varepsilon \ R_2 - N$ and $ab = \{0\} \ \varepsilon \ N$.

<u>Corollary</u> 13.1. R_2/N is not an integral domain.

<u>Corollary</u> 13.2. N is not maximal in R_2.

We are thus led to the conclusion that there exists at least one ring P such that

i) $N \subset P \subset R_2 \quad (N \neq P \neq R_2)$,

ii) P is an ideal in R_2.

Let Π be the class of all rings P satisfying i) and ii). By theorem 1, $P \cap R_1$ is an ideal in R_1 for all $P \varepsilon \Pi$. Moreover, $N \subseteq P \cap R_1$, and since $\{1\} \varepsilon R_1$ and $\{1\} \not\varepsilon P$, we have $P \cap R_1 \neq R_1$. Since N is maximal in R_1, this proves the following theorem.

<u>Theorem</u> 14. $P \cap R_1 = N$ for all $P \varepsilon \Pi$.

<u>Theorem</u> 15. R_2/N is a proper extension of the real number field R_1/N.

Proof. By corollary 4.1, $R_1/N \subseteq R_2/N$. By corollary 13.1, $R_1/N \neq R_2/N$.

<u>Theorem</u> 16. For each $P \varepsilon \Pi$, R_2/P is an extension of R_1/N.

Proof. By theorem 4 and theorem 14,

$$R_1/(P \cap R_1) = R_1/N \cong (R_1 + P)/P \subseteq R_2/P.$$

Obviously, R_2/P will be a proper extension of R_1/N provided P is not maximal in R_2. However, we are interested in the case where P is maximal in R_2, for then, and only then, R_2/P is a field. First of all, we must determine whether any $P \varepsilon \Pi$ is maximal in R_2. In order to do this we make use of the

<u>Maximum</u> <u>Principle</u>. Let Ω be a class of sets such that

the union of any linear subclass of Ω is an element of Ω. Then Ω has maximal elements.

By a linear subclass Λ of Ω we mean a class $\Lambda \subseteq \Omega$ such that, if $A, B \in \Lambda$, then either $A \subseteq B$ or $B \subseteq A$.

Theorem 17. Π contains maximal ideals.

Proof. By the definition of Π we have

$$P \in \Pi, \; Q \supset P, \; Q \neq P \quad \text{implies} \quad Q \in \Pi.$$

Thus we need only to show that there exist maximal elements in Π. Let Λ be any linear subclass of Π. Then

$$U(\Lambda) = \bigcup\{A : A \in \Lambda\}$$

is an element of Π. For suppose $a, b \in U(\Lambda)$ and $c \in R_2$. Then $a \in A$, $b \in B$ for some $A, B \in \Lambda$. Since Λ is linear, either $A \subseteq B$ or $B \subseteq A$. Suppose $A \subseteq B$. Then $a, b \in B$, and hence $a - b$ and ac belong to B. Thus $a - b$, $ac \in U(\Lambda)$ and $U(\Lambda)$ is an ideal in R_2. Since $N \subseteq A$ for each $A \in \Lambda$, we have also $N \subseteq U(\Lambda)$. Moreover, $U(\Lambda) \neq N$, for if it were, we would have $A \subseteq U(\Lambda) = N$ contrary to the definition of Π. Finally, $\{1\} \not\subseteq A$ for any $A \in \Lambda$, so that $\{1\} \not\subseteq U(\Lambda)$. Thus $U(\Lambda) \neq R_2$ and $U(\Lambda) \in \Pi$. By the maximum principle Π contains maximal elements.

Thus there exists an ideal $\overline{P}$ such that $R_2/\overline{P}$ is an extension field of R_1/N. Moreover, by theorem 6, $R_2/\overline{P}$ is a homomorphic image of R_2/N.

SECTION 4. OTHER RINGS RELATED TO THE REAL NUMBER FIELD

One might ask whether something similar to what we have done for R_1 and R_2 might not be done for other rings of sequences of rational numbers. This is indeed possible. Let S be the set of all sequences of rational numbers. We see immediately that S is a commutative ring with identity, for the sum or product of two sequences (as defined in the introduction) is certainly a sequence. We shall be concerned, in general, with subrings of S, ideals in these subrings and the corresponding quotient rings.

In order to make our results meaningful, we should like to be able to compare the resulting quotient rings with the real number field R_1/N. Thus the following theorem is not without interest.

<u>Theorem</u> 18. R_2 is the largest subring of S in which N is an ideal.

Proof. Let $\{a_n\}$ be an unbounded sequence. Thus $a_n \neq 0$ for an infinite number of subscripts n. Moreover, there exists a monotone, unbounded subsequence $\{a_{n_k}\}$ of $\{a_n\}$, with $a_{n_k} \neq 0$. Let $b_{n_k} = 1/a_{n_k}$ (k=1,2,3,...) and let $b_n = 0$ for $n \neq n_k$ (k=1,2,3,...). Then $\{b_n\}$ is a null sequence, but $\{a_n\}\{b_n\} = \{a_n b_n\}$ has a subsequence converging to 1 and hence is not null.

In addition, we may show that there is no proper ideal in S which contains N, for, by theorem 7, NS is the smallest ideal with this property. But $NS = S$, for $\{1/n\} \varepsilon N$ and $\{n\} \varepsilon S$. Thus, if $\{a_n\}$ is any element of S, then $\{a_n\} = (\{1/n\}\{n\})\{a_n\} = \{1/n\}\{na_n\} \varepsilon NS$.

On the other hand, S is not simple, for if $\overline{N}$ is the set of all sequences having a finite number of non-zero terms, then $\overline{N}$ is an ideal in S. By a trivial application of the maximum principle S must contain maximal ideals. If M is a maximal ideal in S, then S/M is a field and (by theorem 2) $M \cap R_1$ is a prime ideal in R_1. Then S/M is an extension field of the integral domain $R_1/(M \cap R_1)$. The latter may or may not be comparable to the real number field.

A ring which is in many ways more interesting is the set C_1^{-1} of all sequences $\{a_n\}$ such that $\{(n+1)a_n - na_{n-1}\} \varepsilon R_1$. Now

$$\frac{1}{n+1} \sum_{k=0}^{n} [(k+1)a_k - ka_{k-1}] = a_n$$

so that, by a well-known theorem on the regularity of Cesaro sums [1, p. 101], we have that $C_1^{-1} \subset R_1$. Moreover, we have the following theorem.

<u>Theorem</u> 19. C_1^{-1} is a commutative ring with identity.

Proof. First of all, $\{1\} = \{(n+1)\cdot 1 - n\cdot 1\} \varepsilon R_1$. Thus $\{1\} \varepsilon C_1^{-1}$, and we need only show that C_1^{-1} is a subring of S. Let $\{a_n\}$ and $\{b_n\}$ be arbitrary elements of C_1^{-1}. Then $\{(n+1)a_n - na_{n-1}\} \varepsilon R_1$ and $\{(n+1)b_n - nb_{n-1}\} \varepsilon R_1$. Thus

$$\{(n+1)a_n - na_{n-1}\} - \{(n+1)b_n - nb_{n-1}\} =$$

$$= \{(n+1)(a_n - b_n) - n(a_{n-1} - b_{n-1})\}$$

belongs to R_1, and hence

$$\{a_n\} - \{b_n\} = \{a_n - b_n\}$$

belongs to C_1^{-1}. Also

$$x = \{(n+1)a_n - na_{n-1}\}\{b_n\} = \{(n+1)a_nb_n - na_{n-1}b_n\} \in R_1$$

and

$$y = \{a_{n-1}\}\{(n+1)b_n - nb_{n-1}\} = \{(n+1)a_{n-1}b_n - na_{n-1}b_{n-1}\}$$

belongs to R_1. Thus

$$x + y = \{(n+1)a_nb_n - na_{n-1}b_{n-1} + a_{n-1}b_n\}$$

$$= \{(n+1)a_nb_n - na_{n-1}b_{n-1}\} + \{a_{n-1}\}\{b_n\} \in R_1.$$

Finally, then

$$x + y - \{a_{n-1}\}\{b_n\} = \{(n+1)a_nb_n - na_{n-1}b_{n-1}\} \in R_1$$

and hence $\{a_nb_n\} = \{a_n\}\{b_n\} \in C_1^{-1}$. This completes the proof.

Now by theorem 2, $N \cap C_1^{-1}$ is a prime ideal in C_1^{-1}. Again calling upon the regularity of Cesaro summability, we see that $\{a_n\} \in N \cap C_1^{-1}$ if and only if $\{a_n\}$ converges to 0 and $\{(n+1)a_n - na_{n-1}\}$ converges, which in turn is possible if and only if $\{(n+1)a_n - na_{n-1}\}$ converges to 0. Thus

$$\{a_n\} \in N \cap C_1^{-1} \quad \text{iff} \quad \{(n+1)a_n - na_{n-1}\} \in N.$$

<u>Theorem</u> 20. $C_1^{-1}/(N \cap C_1^{-1})$ is a field.

Proof. It is certainly a commutative ring with the identity, e, which is the equivalence class containing $\{1\}$. Let $\{a_n\} \in C_1^{-1}$, and let

$$r_n = 1 \quad \text{if } a_n = 0,$$
$$= 0 \quad \text{otherwise.}$$

If $\{a_n\} \not\in N$, then $r_n = 0$ for n greater than some n_0. Thus $\{r_n\} \in N \cap C_1^{-1}$.

Let

$$\{b_n\} = \{a_n\} - \{r_n\}.$$

Then $b_n \neq 0$ for any n and $\{b_n\} \in C_1^{-1}$. Thus $\{b_n\}$, $\{b_{n-1}\}$, $\{1/b_n\}$, $\{1/b_{n-1}\}$ and $\{(n+1)b_n - nb_{n-1}\}$ all belong to R_1.

Hence

$$\{1/b_n\}\{1/b_{n-1}\}[\{b_n\} + \{b_{n-1}\} - \{(n+1)b_n - nb_{n-1}\}] =$$
$$= \{(n+1)/b_n - n/b_{n-1}\} \in R_1$$

and $\{1/b_n\} \in C_1^{-1}$. Now

$$\{a_n\}\{1/b_n\} = \{1\} - \{r_n\}$$

since

$$a_n(1/b_n) = 0 \quad \text{if } a_n = 0$$
$$= 1 \quad \text{otherwise.}$$

Since $\{r_n\} \in N \cap C_1^{-1}$, this completes the proof.

By theorems 20 and 4, we have the following theorem.

Theorem 21. $C_1^{-1}/(N \cap C_1^{-1})$ is a subfield of the real number field.

Finally, we remark, that the set C_1 of sequences summable by Cesaro means of the first order is not a subring. It is, however, an R-module (as is the summability field of any "proper" method of summability). We might thus construct modules comparable to the real number field (conceived as a module over the rational number field). However, in this case, the theorem corresponding to our lemma 1 fails to hold, so that we would be hard put to redefine the result as a ring.

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