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TERRANCE LEE SEETHOFF for the DOCTOR OF PHILOSOPHY
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Title: ZERO-ENTROPY AUTOMORPHISMS OF A COMPACT ABELIAN GROUP

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J. R. Brown

In this paper we are concerned primarily with characterizing the epimorphisms of a compact metrizable abelian group which have zero entropy. Our main result is

Theorem 1. Let G be a compact metrizable abelian group and φ be a continuous homomorphism of G onto G. Define subsets $P_n(T_\varphi)$ $(n \geq 0)$ and $P(T_\varphi)$ of the additively written character group $\widehat{G}$ of G by

$$P_0(T_\varphi) = \{0\},$$

$$P_{n+1}(T_\varphi) = \bigcup_{k>0} (T_\varphi^k - I)^{-1}[P_n(T_\varphi)] \quad (n \geq 0)$$

and

$$P(T_\varphi) = \bigcup_{n \geq 0} P_n(T_\varphi),$$

where T_φ is the adjoint of φ in $\widehat{G}$; i.e., $T_\varphi \hat{x} = \hat{x} \circ \varphi$ $(\hat{x} \in \widehat{G})$

and I is the identity map in $\hat{G}$. Then $h(\varphi) = 0$ iff $P(T_\varphi) = \hat{G}$.

Theorem 1 provides the basis and motivation for several other investigations. As a consequence of Theorem 1 we prove

Theorem 2. If G and φ are as in Theorem 1, then there exists a unique closed subgroup $E \subset G$ such that

i.) $\varphi E = E$,

ii.) $\varphi | E$ is ergodic,

iii.) $h(\varphi_E) = 0$, where φ_E is the epimorphism induced by φ in G/E and

iv.) $h(\varphi) = h(\varphi | E)$.

Motivated by Theorem 2, we say that a continuous epimorphism φ of a compact metrizable abelian group has E-Z decomposition property if G is the direct sum of two closed subgroups G_1 and G_2 such that $\varphi G_i = G_i$ $(i = 1, 2)$ and $\varphi | G_1$ is ergodic while $\varphi | G_2$ has zero entropy. We prove

Theorem 3. Let G be a compact metrizable abelian group and φ be a continuous homomorphism of G onto G. Then φ has the E-Z decomposition property iff there exists a subgroup $H \subset \hat{G}$ such that $T_\varphi H \subset H$ and $\hat{G} = H \oplus P(T_\varphi)$ where $P(T_\varphi)$ is as in Theorem 1.

We also investigate the possibility of extending Theorem 1 to cover arbitrary measure-preserving transformations.

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Terrance Lee Seethoff

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Acting Chairman of Department of Mathematics

Redacted for privacy

Dean of Graduate School

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Typed by Clover Redfern for Terrance Lee Seethoff

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ZERO-ENTROPY AUTOMORPHISMS OF A COMPACT ABELIAN GROUP

I. INTRODUCTION

Let (Ω, Σ, μ) be a measure space. Then by a μ-measure-preserving transformation is understood a mapping $\varphi : \Omega \to \Omega$ such that for each $A \in \Sigma$, $\varphi^{-1}A \in \Sigma$ and $\mu(\varphi^{-1}A) = \mu(A)$. If, in addition, φ is bijective and for each $A \in \Sigma$, $\varphi A \in \Sigma$, then φ is said to be an invertible μ-measure-preserving transformation. When there is no possibility of confusion we shall usually omit the prefix "μ-".

The theory of topological groups furnishes us with the class of measure-preserving transformations with which we shall be primarily concerned. Specifically, if G is a compact metrizable abelian group with normalized Haar measure m, then owing to the uniqueness of m, it follows that every continuous homorphism of G onto G is an m-measure-preserving transformation. We shall denote this class of measure-preserving transformations by $\mathfrak{E}$.

The class $\mathfrak{E}$ of epimorphisms of a compact metrizable abelian group does not contain examples of every metric possibility for a measure-preserving transformation. For example, this class contains no ergodic transformation with discrete spectrum defined on a nontrivial space. Indeed, every ergodic epimorphism of a nontrivial compact metrizable abelian group has a denumerably multiple Lebesgue spectrum [5, p. 51]. However, we point out that Foais has proven [2] that if T is an invertible measure-preserving transformation of a probability space (Ω, Σ, μ), then there exists a compact abelian group G, a continuous automorphism φ of G and a Borel probability measure λ on G which is preserved by φ such that T and φ are conjugate. For the meaning of the terminology introduced in this paragraph, see for example [5].

1. Entropy

One of the principle aims of this paper is to characterize the elements of $\mathfrak{E}$ which have zero entropy. For the reader's convenience we review briefly here the notion of entropy. For a detailed discussion, see [1, p. 60].

Let (Ω, Σ, μ) be a probability space and $\varphi : \Omega \to \Omega$ be a measure-preserving transformation. For a finite subalgebra $\mathcal{A} \subset \Sigma$ set

$$(1.1) \qquad H(\mathcal{A}) = -\sum_{A} \mu(A) \log \mu(A)$$

where the summation extends over the atoms of $\mathcal{A}$. By convention, $0 \log 0 = 0$.

Next, for any finite subalgebra $\mathcal{A} \subset \Sigma$, set

$$(1.2) \qquad H(\mathcal{A}, \varphi) = \limsup_{n\to\infty} \frac{1}{n} H\Big(\bigvee_{k=0}^{n-1} \varphi^{-k}\mathcal{A}\Big).$$

(If $\mathcal{C}_\lambda$ $(\lambda \in \Lambda)$ are collections of subsets of Ω, then $\bigvee\{\mathcal{C}_\lambda : \lambda \in \Lambda\}$ denotes the σ-algebra generated by $\cup\{\mathcal{C}_\lambda : \lambda \in \Lambda\}$. The variation of this notation used above is self-explanatory.)

Finally, the entropy of φ is defined to be

(1.3) $$h(\varphi) = \sup\{H(\mathcal{A}, \varphi) : \mathcal{A} \text{ is a finite subalgebra of } \Sigma\}.$$

Thus the entropy of φ is a nonnegative, extended realvalued number.

Now the quantity (1.2) can be computed in another manner. Denote by η the function defined in $[0, 1]$ by

$$\eta(t) = \begin{cases} -t \log t & 0 < t \leq 1 \\ 0 & t = 0 \end{cases} .$$

For a finite subalgebra $\mathcal{A}$ of Σ and a σ-subalgebra $\mathcal{C} \subset \Sigma$ set

(1.4) $$H(\mathcal{A} \mid \mathcal{C}) = \int \{\sum_A \eta[\mu(A \mid \mathcal{C})]\} d\mu$$

where the summation extends over the atoms of $\mathcal{A}$. Here $\mu(A \mid \mathcal{C})$ is the conditional expectation of A given $\mathcal{C}$.

Then for any finite subalgebra $\mathcal{A} \subset \Sigma$ [1, p. 126]

(1.5) $$H(\mathcal{A}, \varphi) = H(\mathcal{A} \mid \bigvee_{k=1}^{\infty} \varphi^{-k} \mathcal{A}) .$$

We shall make use of the following observation: if for some finite subalgebra $\mathcal{A} \subset \Sigma$ we have

$$(1.6) \qquad \mathcal{A} \subset \bigvee_{k=1}^{\infty} \varphi^{-k}\mathcal{A} ,$$

then $H(\mathcal{A}, \varphi) = 0$. Indeed, if (1.6) holds, then for $A \in \mathcal{A}$

$$\mu(A | \bigvee_{k=1}^{\infty} \varphi^{-k}\mathcal{A}) = \chi_A \qquad \text{a.e.}$$

(χ_A is the characteristic function of the set A). Thus

$$\eta[\mu(A | \bigvee_{k=1}^{\infty} \varphi^{-k}\mathcal{A})] = 0 \qquad \text{a.e.}$$

and the assertion follows from (1.4) and (1.5).

We shall also need an approximation theorem. If $\mathcal{F}_0$ is an algebra of subsets of Σ which generates Σ, then [1, p. 89]

$$(1.7) \qquad h(\varphi) = \sup \{H(\mathcal{A}, \varphi) : \mathcal{A} \text{ is a finite subalgebra of } \mathcal{F}_0\}$$

Finally, if G is a compact metrizable group, φ is a continuous homorphism of G onto G and H is a closed normal subgroup of G such that $\varphi H = H$, then [12]

$$(1.8) \qquad h(\varphi) = h(\varphi | H) + h(\varphi_H) ,$$

where φ_H is the epimorphism induced in G/H by φ. Here the entropy of φ, $\varphi|H$ and φ_H is computed with respect to the normalized Haar measure in G, H and G/H respectively.

2. Measure-Preserving Transformations with Zero Entropy

Let I denote the unit interval equipped with the usual Lebesgue structures. Denote by $\mathcal{a}$ the group of invertible measure-preserving transformations of I. If $\mathcal{a}$ is given the weak topology (the topology of "set-wise convergence") [5, p. 62], then Rohlin has shown [6] that the elements of $\mathcal{a}$ with zero entropy form an everywhere dense G_δ set. These transformations can be described in the following manner:

Let (Ω, Σ, μ) be a nonatomic Lebesgue space (i.e., isomorphic to I) and $\varphi : \Omega \to \Omega$ be an invertible measure-preserving transformation. Denote by $\mathfrak{A}$ the collection of all finite subalgebras of Σ. For $\mathcal{A} \in \mathfrak{A}$ let

$$\mathcal{A}^- = \bigvee_{k \leq 0} \varphi^k \mathcal{A}$$

and set

(1.9) $$\mathfrak{M}_\varphi = \{\bigcap_k \varphi^k \mathcal{A}^- : \mathcal{A} \in \mathfrak{A}\}.$$

Then [11] $h(\varphi) = 0$ iff $\mathfrak{M}_\varphi = \Sigma$.

Although this characterization has a number of technical applications (for example it guarantees the possibility of "separating off" from φ the "part" which has zero entroy [10]), it is usually very difficult to appraise the σ-algebra $\mathcal{M}_\varphi$ and consequently it is difficult to use this characterization to determine whether or not φ has zero entropy.

3. Summary of Main Results

For the class $\mathcal{E}$ we give a simpler, more constructive characterization. Specifically, we shall prove

Theorem A. Let G be a compact metrizable abelian group and φ be a continuous homomorphism of G onto G. Define subsets $P_n(T_\varphi)$, $(n \geq 0)$ and $P(T_\varphi)$ of the additively written character group $\widehat{G}$ of G by

$$P_0(T_\varphi) = \{0\},$$

$$P_{n+1}(T_\varphi) = \bigcup_{k>0} (T_\varphi^k - I)^{-1}[P_n(T_\varphi)] \quad (n \geq 0)$$

and

$$P(T_\varphi) = \bigcup_{n\geq 0} P_n(T_\varphi),$$

where T_φ is the adjoint of φ in $\widehat{G}$; i.e., $T_\varphi \hat{x} = \hat{x} \circ \varphi (\hat{x} \in \widehat{G})$ and I is the identity map on $\widehat{G}$. Then $h(\varphi) = 0$ iff $P(T_\varphi) = \widehat{G}$.

In Chapter IV we shall prove that with the notation of Theorem A, φ is ergodic iff $P(T_\varphi) = \{0\}$. Thus it appears that for the class $\mathcal{E}$, the properties of being ergodic $(P(T_\varphi) = \{0\})$ and having zero entropy $(P(T_\varphi) = \hat{G})$ are antithetical.

As an immediate consequence of these remarks, it follows that Theorem A contains as a special case the following theorem due to Rohlin [8]:

Theorem B. Every ergodic automorphism of a nontrivial compact metrizable abelian group has positive entropy.

As a consequence of Theorem A we also prove

Theorem C. Let G be a compact metrizable abelian group and φ be a continuous homomorphism of G onto G. Then there exists a unique closed subgroup E of G such that

i.) $\varphi E = E$,

ii.) $\varphi | E$ is ergodic,

iii.) $h(\varphi) = h(\varphi | E)$,

iv.) $h(\varphi_E) = 0$ when φ_E is the epimorphism induced by φ in G/E.

In Chapter V we will consider the possibility of improving the decomposition indicated in Theorem C. Namely, we are interested in determining the circumstances under which G is the direct sum of two closed subgroups, G_1 and G_2, each invariant under φ (i.e., $\varphi G_i = G_i$ $(i=1, 2)$) and such that $\varphi | G_1$ is ergodic while $\varphi | G_2$ has zero entropy. When φ admits such a decomposition we shall say that φ has the E - Z decomposition property. The problem of the existence of an E - Z decomposition for φ is seen to be a group-theoretic version of an unsolved conjecture of Pinsker's [6].

We will prove

Theorem D. Let G be a compact metrizable abelian group and φ be a continuous homomorphism of G onto G. Then φ has the E - Z decomposition property iff there exists a subgroup $E \subset G$ such that $T_\varphi E \subset E$ and $G = E \oplus P(T_\varphi)$ where $P(T_\varphi)$ is defined as in Theorem A.

In the first part of Chapter VII we list several open questions and discuss the relationship between them. In the final section we discuss the possibility of extending Theorem A to cover a wider class of measure-preserving transformations than $\mathcal{E}$.

4. Notation and Conventions

With the exception of Chapter VII, we shall write all groups additively.

If we speak of a group Γ without indicating a topology, then it always is to be understood that Γ is given the discrete topology.

The additive group of integers will be denoted by Z.

If G is a locally compact abelian group, then we shall denote by $\widehat{G}$ the character (= dual) group of G. We shall always suppose that $\widehat{G}$ is equipped with the compact-open topology. Thus $\widehat{G}$ is itself a locally compact abelian group (see for example [10, p. 7]). In particular then, if G is compact (discrete) then $\widehat{G}$ is discrete (compact). When G is compact $\widehat{G}$ is countable iff G is metrizable [10, p. 38].

If G is a topological group, then we shall denote

by

Hom(G, G) the set of all continuous homomorphisms of G into G,

by

Epi(G, G) the set of $\varphi \in \mathrm{Hom}(G, G)$ such that $\varphi G = G$,

by

Mon(G, G) the set of all injective $\varphi \in \mathrm{Hom}(G, G)$,

and by

Aut(G, G) the set of all $\varphi \in \mathrm{Epi}(G, G) \cap \mathrm{Mon}(G, G)$ such that $\varphi^{-1} \in \mathrm{Hom}(G, G)$.

If G is a locally compact abelian group and $\varphi \in \mathrm{Hom}(G, G)$, then we shall denote by T_φ the adjoint of φ in $\widehat{G}$; i.e., $T_\varphi \hat{x} = \hat{x} \circ \varphi$ $(\hat{x} \in \widehat{G})$. Notice that if $\varphi \in \mathrm{Epi}(G, G)$, then

$T_\varphi \in \mathrm{Mon}(\widehat{G}, \widehat{G})$.

From the theory of topological groups we shall need

Theorem E. Let G be a compact abelian group, $\varphi \in \mathrm{Hom}(G, G)$ and G_1 be a closed subgroup of G such that $\varphi G_1 \subset G_1$. Let $G_2 = G/G_1$, $\varphi_1 = \varphi | G$, and φ_2 be the homomorphism induced by φ in G_2. Set $H = \{\hat{x} \in \widehat{G} : \hat{x}|G_1 = 0\}$. Define $\psi : \widehat{G}_2 \to \widehat{G}$ by

$$(\psi \hat{a})(x) = \hat{a}(x+G_1) \quad (\hat{a} \in \widehat{G}_2, \ x \in G).$$

Define $\tau : \widehat{G}/H \to \widehat{G}_1$ by

$$[\tau(\hat{x}+H)](g_1) = \hat{x}(g_1) \quad (\hat{x} \in \widehat{G}, \ g_1 \in G_1) .$$

Then

i.) ψ is a continuous monomorphism

ii.) $\psi(\widehat{G}_2) = H$ (i.e., $\widehat{G/G_1} \approx H$)

iii.) τ is a continuous isomorphism (i.e., $\widehat{G}/H \approx \widehat{G}_1$)

iv.) $\psi T_{\varphi_2} = T_\varphi \psi$

v.) $\tau T_H = T_{\varphi_1} \tau$ where T_H is the homomorphism induced by T in $\widehat{G}/H$.

For the proofs of the first three assertions see for example [10, p. 35]. The proofs of the last two assertions are straight forward and will be omitted.

Let G_1 and G_2 be topological groups and $\varphi_i \in \mathrm{Hom}(G_i, G_i)$ $(i=1, 2)$. We shall write $(G_1, \varphi_1) \approx (G_2, \varphi_2)$ to mean there exists a continuous isomorphism τ of G_1 onto G_2 such that $\tau\varphi_1 = \varphi_2\tau$. Thus in Theorem E, conditions ii.) and v.) imply

$(G/H, T_H) \approx (G_1, T_{\varphi_1})$.

If G is a compact abelian group and $\varphi \in \mathrm{Epi}(G, G)$, then the entropy of φ, $h(\varphi)$, will always be understood to be computed with respect to the normalized Haar measure on G.

Finally, if f is a mapping of some set X into itself, we shall say that $A \subset X$ is invariant under f if $f(A) = A$.

II. MOTIVATION

Before proceeding to the proof of Theorem A we include here a few remarks intended to help clarify the statement of that theorem.

Let us see what happens when G is the n-dimensional torus. In this case we may identify $\widehat{G}$ with Z^n. Thus if $\varphi \in \mathrm{Aut}(G, G)$, then with respect to the euclidean basis of $\widehat{G}$, there is associated with T_φ a certain unimodular matrix A. Now if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the (not necessarily distinct) eigenvalues of A, then [4]

$$h(\varphi) = \sum_{|\lambda_i| \geq 1} \log |\lambda_i| .$$

Hence $h(\varphi) = 0$ iff $|\lambda_i| \leq 1$ $i = 1, 2, \ldots, n$. Because the λ_i are algebraic numbers and $\lambda_1 \lambda_2 \ldots \lambda_n = \det A = \pm 1$, we may conclude (see for example [8]) that $h(\varphi) = 0$ iff all the λ_i are roots of unity. Consequently, if $h(\varphi) = 0$, there exists a polynomial of the form

$$(2.1) \qquad u(t) = (t^{k_1} - 1)(t^{k_2} - 1) \ldots (t^{k_n} - 1) \quad k_i > 0, i = 1, 2, \ldots, n$$

such that $u(T_\varphi) = 0$ (recall that A, and hence T_φ, satisfies its characteristic polynomial).

Now the condition $u(T_\varphi) = 0$ makes sense for a homomorphism of an arbitrary group. Denote by $\mathcal{U}$ the set of all polynomials of the form (2.1). Thus if Γ is an abelian group it would appear from what we have just said that a characterization of the vanishing of entropy would involve those $T \in \mathrm{Mon}(\Gamma, \Gamma)$ for which $u(T) = 0$ for some $u(t) \in \mathcal{U}$. It turns out, however, that this condition is too strong. That is, there exists a compact metrizable abelian group G and $\varphi \in \mathrm{Aut}(G, G)$ such that $u(T_\varphi) = 0$ holds for no $u(t) \in \mathcal{U}$ but $h(\varphi) = 0$. Hence we weaken the condition by "localizing" it. In other words, for every $x \in \Gamma$ we suppose there exists a polynomial $u(t) \in \mathcal{U}$ (depending on x) such that $u(T)x = 0$. We can give this last condition a more compact form. Namely, we are now asking that

$$\Gamma = \cup \{\ker u(T) : u(t) \in \mathcal{U}\} . \tag{2.2}$$

The relationship between the condition (2.2) and that of Theorem A is given as

Proposition 2.1. Let Γ be an abelian group and $T \in \mathrm{Hom}(\Gamma, \Gamma)$. Define subsets P_n $(n \geq 0)$ and P of Γ by

$$P_0 = 0 ,$$

$$P_{n+1} = \bigcup_{k \geq 0} (T^k - I)^{-1}[P_n] \qquad (n \geq 0)$$

and

$$P = \bigcup_{n \geq 0} P_n .$$

Then

$$P = \cup \{\ker u(T) : u(t) \in \mathcal{U}\} .$$

Proof:

Let $k = \cup\{\ker u(T): u(t) \in \mathcal{U}\}$. Let us show first that $P \subset k$. For this it is sufficient to show that $P_n \subset k$ $(n \geq 0)$. Clearly $P_0 \subset k$. Suppose that $P_n \subset k$ and let $x \in P_{n+1}$. Then there exists an integer $k > 0$ such that $y = T^k x - x \in P_n$. By assumption there exists a polynomial

$$u(t) = (t^{k_1} - 1)(t^{k_2} - 1) \ldots (t^{k_n} - 1)$$

such that $u(T)y = (T^{k_1} - I)(T^{k_2} - I) \ldots (T^{k_n} - I)[T^k x - x] = 0$. Set $v(t) = (t^{k_1} - 1)(t^{k_2} - 1) \ldots (t^{k_n} - 1)(t^k - 1)$. Then $v(t) \in \mathcal{U}$. Moreover, $v(T)x = 0$. Thus $x \in k$.

To prove that $k \subset P$ it is sufficient to prove that if

(2.3) $$(T^{k_1} - I)(T^{k_2} - I) \ldots (T^{k_n} - I)x = 0 \quad k_i > 0, \ i = 1, 2, \ldots, n$$

for some $x \in \Gamma$ and $n \geq 1$, then $x \in P_n$. This statement is clearly true for $n = 1$. Suppose that every $y \in \Gamma$ satisfying (2.3) is an element of P_n and that

$$(T^{k_1} - I)(T^{k_2} - I) \ldots (T^{k_n} - I)(T^{k_{n+1}} - I)x = 0 .$$

Then by assumption, $y = (T^{k_{n+1}} - I)(x) \in P_n$. Consequently

$$x \in (T^{k_{n+1}} - I)^{-1}[P_n] \subset P_{n+1}$$

and the proposition is proved.

We proceed now to develop the machinery needed for the proof of Theorem A.

III. QUASI-PERIODIC MAPPINGS

Let Γ be an abelian group and $T \in \text{Hom}(\Gamma, \Gamma)$. Define subsets $P_n(T)$ $(n \geq 0)$ and $P(T)$ of Γ by

(3.1) $$P_0(T) = \{0\},$$

(3.2) $$P_{n+1}(T) = \bigcup_{k>0} (T^k - I)^{-1}[P_n(T)] \quad (n \geq 0)$$

and

(3.3) $$P(T) = \bigcup_{n \geq 0} P_n(T),$$

where I denotes the identity map on Γ.

Then let us agree to say that

(3.4) T is <u>periodic</u> if $P_1(T) = \Gamma$,

(3.5) T is <u>quasi-periodic</u> if $P(T) = \Gamma$ and

(3.6) T is <u>aperiodic</u> if $P_1(T) = \{0\}$.

Notice that $P_1(T)$ consists precisely of those $x \in \Gamma$ for which there exists an integer $k > 0$ such that $T^k x = x$. Thus the terminology of (3.4) and (3.6) is clear. We will return to that of (3.5) a little later on.

Since $P_1(T) \subset P(T)$, it follows that every periodic mapping is quasi-periodic. It is not difficult to find examples of quasi-periodic, but not periodic, mappings. Consider

Example 3.1. Let $\Gamma = Z \times Z$ and define $T \in \mathrm{Aut}(\Gamma, \Gamma)$ by

$$T(m, n) = (m+n, n) \quad (m, n) \in \Gamma .$$

Then $T^k(m, n) = (m+kn, n)$ $(k > 0, (m, n) \in \Gamma)$. Thus

$$P_1(T) = \{x \in \Gamma : T^k x = x \text{ for some } k > 0\} = \{(m, 0) : m \in Z\} .$$

Thus T is not periodic. But since for every $(m, n) \in T$ we have

$$T(m, n) - (m, n) = (n, 0) \in P_1(T)$$

or in other words, $(T-I)^{-1}[P_1(T)] = \Gamma$, it follows that

$$P_2(T) = \bigcup_{k>0} (T^k - I)^{-1}[P_1(T)] = \Gamma .$$

Thus T is quasi-periodic.

Now the notions of periodicity and aperiodicity are antithetical. In view of the fact that the quasi-periodic mappings comprise a strictly larger class than the periodic mappings, it is perhaps surprising that the naturally defined notions of aquasi-periodicity coincides with that of aperiodicity. Indeed, we have

Proposition 3.1. Let Γ be an abelian group and

$T \in Hom(\Gamma, \Gamma)$. Then $P(T) = \{0\}$ iff $P_1(T) = \{0\}$.

Proof:

Observe first that if $P_n(T) = 0$ for some $n \geq 0$, then

$$P_{n+1}(T) = \bigcup_{k>0} (T^k - I)^{-1}[P_n(T)] = \bigcup_{k>0} (T^k - I)^{-1}[\{0\}] = P_1(T) .$$

Thus if $P_1(T) = \{0\}$, an easy induction argument shows that $P_n(T) = \{0\}$ $(n \geq 0)$. In this case we then have $P(T) = \bigcup_{n \geq 0} P_n(T) = \{0\}$.

Evidently if $P(T) = \{0\}$, then $P_1(T) = \{0\}$.

For the class of monomorphisms the notion of aperiodicity can be recast as

Proposition 3. 2. Let Γ be an abelian group and $T \in Mon(\Gamma, \Gamma)$. Then T is aperiodic iff for each nonzero $x \in \Gamma$, the set $\{T^k x : k \geq 0\}$ is infinite.

Proof:

Suppose that T is not aperiodic. Then there exists a nonzero $x \in \Gamma$ and $j > 0$ such that $T^j x = x$. Clearly then, $\{T^k x : k \geq 0\}$ is a finite set.

Suppose that for some nonzero $x \in \Gamma$ the set $\{T^k x : k \geq 0\}$ is finite. Then, since T is a monomorphism, it follows that $T^j x = x$ for some $j > 0$. Thus T is not aperiodic.

We now prove a rather technical result which justifies most of the group-theoretic constructions that we shall need.

Theorem 3.1. Let Γ be an abelian group, $T \in \mathrm{Hom}(\Gamma, \Gamma)$ and H_0 be a subgroup of Γ invariant under T. Set

$$H_1 = \bigcup_{k>0} (T^k - I)^{-1}[H_0] .$$

Then H_1 is a subgroup invariant under T. Moreover, $H_0 \subset H_1$.

Proof:

Let us first observe that if for some integer $j > 0$ and $h_0 \in H_0$

(3.7)
$$T^j x = x + h_0 ,$$

then for any integer $m > 0$

(3.8)
$$T^{mj} x = x + \sum_{\ell=0}^{m-1} T^{\ell j} h_0 .$$

Indeed, if $m=1$, then (3.8) reduces to (3.7). Operating in both sides of (3.8) by T^j and applying (3.7) yields

$$T^{(m+1)j}x = T^j x + T^j\left(\sum_{\ell=0}^{m-1} T^{\ell j} h_0\right)$$

$$= x + h_0 + \sum_{\ell=0}^{m-1} T^{(\ell+1)j} h_0$$

$$= x + \sum_{\ell=0}^{m} T^{\ell j} h_0$$

and the assertion follows.

To prove that H_1 is a subgroup, let $x, y \in H_1$. Then there exists integers $j, k > 0$ and elements $g_0, h_0 \in H_0$ such that

$$T^j x = x + h_0 \quad \text{and} \quad T^k y = y + g_0 .$$

Hence from (3.8) it follows that

$$T^{jk}(x-y) = T^{jk}x - T^{jk}y$$

$$= x + \sum_{\ell=0}^{k-1} T^{\ell j} h_0 - y - \sum_{\ell=0}^{j-1} T^{\ell k} g_0$$

$$= (x-y) + \left\{ \sum_{\ell=0}^{k-1} T^{\ell j} h_0 - \sum_{\ell=0}^{j-1} T^{\ell k} g_0 \right\} .$$

Since $TH_0 = H_0$ and H_0 is a subgroup, it follows that the term in brackets is an element of H_0. Thus

$$(x-y) \in (T^{jk}-I)^{-1}[H_0] \subset H_1$$

Hence H_1 is a subgroup.

We show now that $TH_1 = H_1$. To prove that $TH_1 \subset H_1$ let $x \in H_1$. Then there exists an integer $j > 0$ and $h_0 \in H_0$ such that

$$T^j x = x + h_0 \tag{3.9}$$

Operating on both sides of (3.9) by T we have

$$T^j(Tx) = Tx + Th_0 .$$

Since $TH_0 = H_0$ it follows that

$$Tx \in (T^j-I)^{-1}[H_0] \subset H_1$$

and we conclude that $TH_1 \subset H_1$.

To prove that $TH_1 \supset H_1$, suppose that $x \in \Gamma$ satisfies (3.9) (i.e., $x \in H_1$). Then since $TH_0 = H_0$ there exists $h_0' \in H_0$ such that $Th_0' = h_0$. Thus we may rewrite (3.9) as

$$T^j x = x + Th_0' \tag{3.10}$$

and so

$$x = T[T^{j-1}(x)-h_0']\ .$$

Hence to show that $x \in TH_1$ it is sufficient to show that

$$T^{j-1}(x) - h_0' \in H_1$$

We have, applying (3. 10),

$$\begin{aligned} T^j[T^{j-1}(x)-h_0'] &= T^{j-1}T^j(x) - T^jh_0' \\ &= T^{j-1}[x+Th_0'] - T^jh_0' \\ &= T^{j-1}x + T^jh_0' - T^jh_0' \\ &= (T^{j-1}(x)-h_0') + h_0'\ . \end{aligned}$$

Thus

$$T^{j-1}(x) - h_0' \in (T^j-I)^{-1}[H_0] \subset H_1$$

and so $TH_1 \supset H_1$.

Finally, let us show that $H_0 \subset H_1$. Let $h_0 \in H_0$. Then since H_0 is a subgroup invariant under T we clearly have

$$Th_0 - h_0 \in H_0$$

or in other words $h_0 \in (T-I)^{-1}[H_0] \subset H_1$.

The next theorem contains the basic facts concerning $P_n(T)$ and $P(T)$.

Theorem 3.2. Let Γ be an abelian group and $T \in \mathrm{Hom}(\Gamma, \Gamma)$. Then the sets $P_n(T)$ $(n \geq 0)$ form an increasing sequence of subgroups each invariant under T. Thus $P(T) = \bigcup_{n \geq 0} P_n(T)$ in a subgroup invariant under T. Moreover, the endomorphism T_p induced by T in $\Gamma / P(T)$ is aperiodic.

Proof:

The first assertion is proved by induction by noting that in Theorem 3.1 if $H_0 = P_n(T)$, then $H_1 = P_{n+1}(T)$.

Let us now show that T_p is aperiodic. Suppose that there exists an integer $j > 0$ and $x \in \Gamma$ such that

$$T_p^j(x+P(T)) = (T^j x + P(T)) = (x+P(T)) .$$

Then we must have

$$T^j(x) - x \in P(T) .$$

But if $T^j(x) - x \in P_n(T)$, then $x \in P_{n+1}(T) \subset P(T)$. Consequently $(x+P(T)) = 0$. Thus $P_1(T_p) = \{0\}$; i.e., T_p is aperiodic.

Let us remark here that in Theorem 3.2, if T is a monomorphism, then since $TP(T) = P(T)$ it follows that T_p is also a monomorphism. In this case it follows from Proposition 3.2 that for each nonzero $x \in \Gamma / P(T)$, $\{T_p^k x : k \geq 0\}$ is an infinite set.

As we shall see in Chapter IV, the fact that T_p is aperiodic is the crucial bit of information needed to prove the necessity of the condition introduced in Theorem A. We now prove that $P(T)$ is the smallest of the subgroups P such that $TP \subset P$ and T_p is aperiodic. We shall make use of this observation in Chapter V.

Proposition 3.3. Let Γ be an abelian group and $T \in \mathrm{Hom}(\Gamma, \Gamma)$. If P is a subgroup of Γ such that $TP \subset P$ and $P_1(T_p) = \{0\}$, where T_p is the endomorphism induced in Γ/P by T, then $P(T) \subset P$.

Proof:

Suppose that P satisfies the conditions of the proposition. We prove by induction that $P_n(T) \subset P$ $(n \geq 0)$. Clearly $P_0(T) = \{0\} \subset P$. Suppose that $P_n(T) \subset P$ and let $x \in P_{n+1}(T)$. Then there exists an integer $k > 0$ and $p_n \in P_n(T)$ such that

$$T^k x = x + p_n$$

Consequently,

$$T_p^k(x+P) = (T^k x+P) = (x+p_n+P) = (x+P) .$$

But then $(x+P) \in P_1(T_p) = \{0\}$. Thus $x \in P$.

Now

$$P(T) = \bigcup_{n \geq 0} P_n(T) \subset P$$

and the proposition is proved.

The term "quasi-periodic" is suggested by

Proposition 3.4. Let Γ be an abelian group and $T \in \mathrm{Hom}(\Gamma, \Gamma)$. Then for each integer $n \geq 0$, the homomorphism T_n induced by T in $P_{n+1}(T)/P_n(T)$ is periodic.

Proof:

Let $n \geq 0$ and $x \in P_{n+1}(T)$ be given. Then there exists an integer $k > 0$ and $p_n \in P_n(T)$ such that

$$T^k x = x + p_n .$$

Thus

$$T_n^k(x+P_n(T)) = (T^k x+P_n(T)) = (x+P_n(T)) .$$

Hence T_n is periodic.

We conclude this chapter with two more technical results.

Proposition 3.5. Let Γ_i be abelian groups, $T_i \in \mathrm{Hom}(\Gamma_i, \Gamma_i)$ $(i = 1, 2)$ and $\psi : \Gamma_1 \to \Gamma_2$ be a homomorphism such that $\psi T_1 = T_2 \psi$. Then $\psi P(T_1) \subset P(T_2)$. Consequently if $\psi\Gamma_1 = \Gamma_2$ and $P(T_1) = \Gamma_1$, then $P(T_2) = \Gamma_2$.

Proof:

We prove by induction that $\psi P_n(T_1) \subset P_n(T_2)$ $(n \geq 0)$. Since

$P_0(T_1) = \{0\}$, the assertion holds trivially for $n = 0$. Suppose that $\psi P_n(T_1) \subset P_n(T_2)$ and let $x \in P_{n+1}(T_1)$. Then there exists an integer $k > 0$ and $p_n \in P_n(T_1)$ such that

$$T_1^k x = x + p_n .$$

Thus we have

$$T_2^k(\psi x) - \psi x = \psi(T_1^k(x) - x) = \psi p_n \in P_n(T_2) .$$

It follows from the definition of $P_{n+1}(T_2)$ that $\psi x \in P_{n+1}(T_2)$. In other words, $\psi P_{n+1}(T_1) \subset P_{n+1}(T_2)$. The proposition now follows from

$$\psi P(T_1) = \psi(\bigcup_{n \geq 0} P_n(T_1)) = \bigcup_{n \geq 0} \psi P_n(T_1) \subset \bigcup_{n \geq 0} P_n(T_2) = P(T_2) .$$

<u>Proposition 3.6.</u> Let Γ be an abelian group and $T \in \mathrm{Hom}(\Gamma, \Gamma)$. If H is a subgroup of Γ such that $TH \subset H$, then $P(T|H) = H \cap P(T)$.

Proof:

We show first that

$$H \cap P_n(T) \subset P_n(T|H) \quad (n \geq 0) . \tag{3.11}$$

Clearly (3.11) holds for $n = 0$. Suppose that (3.11) holds and let

$x \in H \cap P_{n+1}(T)$. Then there exists an integer $k > 0$ and $p_n \in P_n(T)$ such that

$$T^k x = x + p_n$$

Since H is a subgroup, $TH \subset H$ and $x \in H$ it follows that $p_n \in H$. Thus $p_n \in H \cap P_n(T) \subset P_n(T|H)$ and we conclude that $x \in H \cap P_{n+1}(T|H)$.

Since clearly $H \cap P_n(T) \supset P_n(T|H)$ it follows that

$$H \cap P_n(T) = P_n(T|H) \quad (n \geq 0) . \tag{3.12}$$

Thus

$$H \cap P(T) = \bigcup_{n \geq 0} H \cap P_n(T) = \bigcup_{n \geq 0} P_n(T|H) = P(T|H)$$

establishing the proposition.

IV. EPIMORPHISMS WITH ZERO ENTROPY

The principle concern of this chapter is the proof of Theorem A. With its aid we prove a decomposition theorem which we will investigate in greater detail in Chapter V.

We prove first

<u>Lemma 4.1.</u> Let G be a compact abelian group and $\varphi \in \mathrm{Epi}(G, G)$. Then φ is ergodic iff $P(T_\varphi) = \{0\} = P_1(T_\varphi)$.

Proof:

It is known (see for example [3, p. 53]) that φ is ergodic iff for every nonzero $\hat{x} \in \hat{G}$

$$\{T_\varphi \hat{x} : k \geq 0\}$$

is an infinite set. Thus according to Proposition 3.2, φ is ergodic iff $P_1(T_\varphi) = \{0\}$. The lemma follows then from Proposition 3.1.

We are now prepared to prove

<u>Theorem 4.1.</u> Let G be a compact metrizable abelian group and $\varphi \in \mathrm{Epi}(G, G)$. If $P(T_\varphi) \neq \hat{G}$, then $h(\varphi) > 0$.

Proof:

Suppose the conditions of the theorem are satisfied. Let E be the annihilator of $P(T_\varphi)$ in G. In other words

$$E = \cap \{\ker(\hat{x}) : \hat{x} \in P(T_\varphi)\} .$$

It is clear that E is a closed and hence compact subgroup of G. Since $T_\varphi P(T_\varphi) = P(T_\varphi)$ it follows that $\varphi E = E$. Let $\varphi_1 = \varphi|E$. Then from formula (1.8) of Chapter I it follows that $h(\varphi) \geq h(\varphi_1)$. Thus the theorem will follow if we show that $h(\varphi_1) > 0$.

We will show that the hypotheses of Theorem B of Chapter I are satisfied. Since $P(T_\varphi) \neq \hat{G}$, it follows that E is not the trivial group. Now E is isomorphic to $\hat{G}/P(T_\varphi)$. If $\overline{T}$ denotes the monomorphism induced in $\hat{G}/P(T_\varphi)$ by T_φ, then (Theorem 3. 2), $\overline{T}$ is aperiodic. Thus T_{φ_1} is aperiodic and so by Lemma 4. 1, φ_1 is ergodic. Consequently $h(\varphi_1) > 0$ follows from Theorem B.

The converse of Theorem 4. 1 is proved in three steps. The first is

Theorem 4. 2. Let Γ be a countable abelian group, $T \in \mathrm{Mon}(\Gamma, \Gamma)$, $G = \hat{\Gamma}$ and φ be the adjoint of T in G. If $P_1(T) = \Gamma$, then $h(\varphi) = 0$.

Proof:

Recall that

$$P_1(T) = \bigcup_{k>0} (T^k - I)^{-1}[\{0\}] .$$

Thus $x \in P_1(T)$ iff there exists an integer $k > 0$ such that

$T^k x = x$. It follows that if $x_1, x_2, \ldots, x_\ell \in P_1(T)$, then there exists an integer $k > 0$ such that $T^k x_j = x_j$ $(1 \leq j \leq \ell)$. Indeed, if $T^{k_j} x_j = x_j$ $(1 \leq j \leq \ell)$, then we may take

$$k = \prod_{j=1}^{\ell} k_j$$

Now the topology on $G = \hat{\Gamma}$ is the Tichonov product topology and Γ is countable. Thus it is easily verified that sets of the form

$$(4.1) \quad C(x_1, x_2, \ldots, x_\ell; A) = \{g \in G : (g(x_1), g(x_2), \ldots, g(x_\ell) \in A\}$$

where A is a Borel subset of the ℓ-dimensional torus S^ℓ form a field of subsets which generates the Borel subsets of G. Denote by $\mathcal{F}_0$ the collection of all sets of the form (4.1).

Let $\mathcal{A}$ be a finite subalgebra of $\mathcal{F}_0$. Then (formula (1.5))

$$H(\mathcal{A}, \varphi) = H(\mathcal{A} \mid \bigvee_{k=1}^{\infty} \varphi^{-k} \mathcal{A})$$

If $C = \{g \in G : (g(x_1), g(x_2), \ldots, g(x_\ell)) \in A\}$

is an element of $\mathcal{A}$, then choose $k > 0$ such that

$$T^k x_j = x_j \quad (1 \leq j \leq \ell).$$

Then

$$\varphi^{-k}C = \{g \in G : (g(T^k x_1), g(T^k x_2), \ldots, g(T^k x_\ell)) \in A\}$$

$$= \{g \in G : (g(x_1), g(x_2), \ldots, g(x_\ell)) \in A\}$$

$$= C.$$

Thus $C \in \varphi^{-k}\mathcal{A} \subset \bigvee_{j=1}^{\infty} \varphi^{-j}\mathcal{A}$. Hence $\mathcal{A} \subset \bigvee_{k=1}^{\infty} \varphi^{-k}\mathcal{A}$ and so $H(\mathcal{A}, \varphi) = 0$. Now (formula (1.7))

$$h(\varphi) = \sup \{H(\mathcal{A}, \varphi) : \text{ is a finite subalgebra of } \mathcal{F}_0\} = 0$$

and the theorem is proved.

An application of the Pontrjagin Duality Theorem yields

<u>Theorem 4.2.</u> Let G be a compact metrizable abelian group and $\varphi \in \mathrm{Epi}(G, G)$. If $P_1(T_\varphi) = \widehat{G}$, then $h(\varphi) = 0$.

We now prove that Theorem 4.2 remains valid if we replace $P_1(T_\varphi)$ by $P_n(T_\varphi)$ $(n \geq 0)$.

<u>Theorem 4.3.</u> Let G be a compact metrizable abelian group and $\varphi \in \mathrm{Epi}(G, G)$. If $P_n(T_\varphi) = \widehat{G}$ for some integer $n \geq 0$, then $h(\varphi) = 0$.

Proof:

The proof is by induction. If $P_0(T_\varphi) = \widehat{G}$, then (since $P_0(T_\varphi) = \{0\}$) $G = \{0\}$. Evidently then, $h(\varphi) = 0$. Suppose that

$h(\varphi) = 0$ for all compact metrizable abelian groups G and all $\varphi \in \mathrm{Epi}(G, G)$ for which $P_n(T_\varphi) = \widehat{G}$. Let G be a compact metrizable abelian group and $\varphi \in \mathrm{Epi}(G, G)$ be such that $P_{n+1}(T_\varphi) = \widehat{G}$. Let

$$G_1 = \cap \{\ker(\widehat{x}) : \widehat{x} \in P_n(T_\varphi)\} .$$

Since $T_\varphi P_n(T_\varphi) = P_n(T_\varphi)$ we have $\varphi G_1 = G_1$. Let $G_2 = G/G_1$, $\varphi_1 = \varphi | G$, and φ_2 be the epimorphism induced in G_2 by φ. Then from formula (1.8)

$$h(\varphi) = h(\varphi_1) + h(\varphi_2) .$$

We will show in turn that $h(\varphi_1)$ and $h(\varphi_2)$ are zero. Now

$$\widehat{G}_1 \approx \widehat{G}/P_n(T_\varphi) = P_{n+1}(T_\varphi)/P_n(T_\varphi)$$

and according to Proposition 3.4, the transformation T_n induced in $P_{n+1}(T_\varphi)/P_n(T_\varphi)$ by T_φ is periodic; i.e., $P_1(T_n) = \widehat{G}/P_n(T_\varphi)$. It follows that T_{φ_1} is periodic and so by Theorem 4.2, $h(\varphi_1) = 0$.

On the other hand, $\widehat{G}_2$ is isomorphic to $P_n(T_\varphi)$ if $T^1 = T_\varphi | P_n(T_\varphi)$, then (Proposition 3.6) $P_n(T^1) = P_n(T_\varphi)$. Consequently $P_n(T_{\varphi_2}) = \widehat{G}_2$ and by our induction hypothesis $h(\varphi_2) = 0$. The theorem is thus proved.

Let us now recall the following proposition (see for example

[1, p. 64]).

Proposition 4.1. Let $(\Omega_i, \Sigma_i, \mu_i)$ $(i = 1, 2)$ be probability spaces and $\varphi_i : \Omega_i \to \Omega_i$ $(i = 1, 2)$ be measure-preserving transformations. Suppose that $\psi : \Omega_1 \to \Omega_2$ is a measure preserving-transformation such that $\psi\varphi_1 = \varphi_2\psi$ (in this case φ_2 is said to be a factor endomorphism of φ_1). Let $\mathcal{A}_2$ be a finite subalgebra of Σ_2 and $\mathcal{A}_1 = \psi^{-1}\mathcal{A}_2$. Then $H(\mathcal{A}_1, \varphi_1) = H(\mathcal{A}_2, \varphi_2)$.

We now give the third and final step in the proof of the converse of Theorem 4.1.

Theorem 4.4. Let Γ be a countable abelian group, $T \in \mathrm{Mon}(\Gamma, \Gamma)$ and φ be the adjoint of T in $G = \hat{\Gamma}$. If $P(T) = \Gamma$, then $h(\varphi) = 0$.

Proof:

Recall that $P(T) = \bigcup_{n \geq 0} P_n(T)$ and $TP_n(T) = P_n(T)$ $(n \geq 0)$. Set $G_n = \widehat{P_n(T)}$ and define the maps $\pi_n : G \to G_n$ $(n \geq 0)$ by

$$\pi_n g = g \,|\, P_n(T) \qquad (g \in G)$$

Then the π_n are continuous epimorphisms (see for example [10, p. 36]) and are thus normalized Haar measure preserving transformations.

Let $\varphi_n : G_n \to G_n$ be defined by

$$\varphi_n g_n = g_n \circ (T | P_n(T_\varphi)) \quad (g_n \in G_n,\ n \geq 0).$$

Thus φ_n is the adjoint of $T | P_n(T_\varphi)$ in $\widehat{P_n(T)} = G_n$. Since $TP_n(T) = P_n(T)$ and T is a monomorphism it follows that the φ_n are continuous automorphisms.

Let us show that $\pi_n \varphi = \varphi_n \pi_n$ $(n \geq 0)$. Let $g \in G$ and $x \in P_n(T)$. The assertion follows from

$$\begin{aligned}
[\pi_n \circ \varphi(g)](x) &= [g \circ T | P_n(T)](x) \\
&= (g \circ T)(x) \\
&= [g | P_n(T)](Tx) \\
&= [g | P_n(T)] \circ [T | P_n(T)](x) \\
&= [\varphi_n \circ \pi_n(g)](x) .
\end{aligned}$$

Now observe that $h(\varphi_n) = 0$ $(n \geq 0)$. Indeed, it follows from the Pontrjagin Duality Theorem that $(\widehat{G}_n, T_{\varphi_n}) \approx (P_n(T), T | P_n(T))$. Thus $P_n(T_{\varphi_n}) = \widehat{G}_n$ and the assertion follows from Theorem 4. 3.

Let Σ and Σ_n denote the Borel subsets of G and G_n respectively. Set $\Sigma'_n = \pi_n^{-1} \Sigma_n$ $(n \geq 0)$. Keeping in mind that G is a subset of

$$(S^1)^\Gamma \quad (S^1 \text{ denotes the 1-dimensional torus})$$

it follows easily from the definition of π_n and the fact that $P_n(T) \subset P_{n+1}(T)$ that $\Sigma'_n \subset \Sigma'_{n+1}$ $(n \geq 0)$. Let

$$\mathcal{F}_0 = \bigcup_{n \geq 0} \Sigma'_n .$$

Then $\mathcal{F}_0$ is a field of subsets of Σ. Since

$$\Gamma = P(T) = \bigcup_{n \geq 0} P_n(T)$$

it follows that $\mathcal{F}_0$ generates Σ.

Let $\mathcal{A}'$ be a finite subalgebra of $\mathcal{F}_0$. Since $\Sigma'_n \uparrow \mathcal{F}_0$ and $\mathcal{A}'$ is finite, it follows that there exists an integer $n \geq 0$ such that

$$\mathcal{A}' \subset \Sigma'_n = \pi_n^{-1} \Sigma_n$$

Thus there exists a finite subalgebra $\mathcal{A} \subset \Sigma_n$ such that

$$\mathcal{A}' = \pi_n^{-1} \mathcal{A} .$$

Hence according to Proposition 4. 1

$$H(\mathcal{A}', \varphi) = H(\pi_n^{-1} \mathcal{A}, \varphi) = H(\mathcal{A}, \varphi_n) = 0$$

Finally since (see formula (1. 8))

$$h(\varphi) = \sup \{H(\mathcal{A}, \varphi) : \mathcal{A} \text{ is a finite subalgebra of } \mathcal{F}_0\}$$

we conclude that $h(\varphi) = 0$ and the theorem is proved.

Theorem 4.4 is evidently equivalent to

Theorem 4.4'. Let G be a compact metrizable abelian group and $\varphi \in \mathrm{Epi}(G, G)$. If T_φ is quasi-periodic, then $h(\varphi) = 0$.

Combining Theorems 4.1 and 4.4' we have finally

Theorem A. Let G be a compact metrizable group and $\varphi \in \mathrm{Epi}(G, G)$. Then $h(\varphi) = 0$ iff $P(T_\varphi) = \widehat{G}$.

Theorem A and Lemma 4.1 give an interesting relationship between the properties of being ergodic and having zero entropy for $\varphi \in \mathrm{Epi}(G, G)$, where G is a compact metrizable abelian group. Specifically, φ is ergodic iff $P(T_\varphi) = \{0\}$ while φ has zero entropy iff $P(T_\varphi) = \widehat{G}$. The two notions are in this sense antithetical.

Let Γ be an abelian group and $T \in \mathrm{Mon}(\Gamma, \Gamma)$. Then according to Theorem 3.2, it is possible to separate off the quasi-periodic part of T leaving an aperiodic factor monomorphism. As we have just seen, quasi-periodic monomorphisms correspond to epimorphisms with zero entropy, while (Lemma 4.1) aperiodic monomorphisms correspond to ergodic epimorphisms. Thus

Theorem 4.5. Let G be a compact metrizable abelian group and $\varphi \in \mathrm{Epi}(G, G)$. Then there exists a closed subgroup $E \subset G$ such that

i.) $\varphi E = E$

ii.) $\varphi | E$ is ergodic

iii.) $h(\varphi_E) = 0$ where φ_E is the epimorphism induced in G/E by φ.

iv.) $h(\varphi) = h(\varphi | E)$.

Proof:

We claim that

$$E = \cap \{\ker(\hat{x}) : \hat{x} \in P(T_\varphi)\} \tag{4.2}$$

satisfies all of the conclusions of the theorem. From $T_\varphi P(T_\varphi) = P(T_\varphi)$ it follows that $\varphi E = E$. Since $\hat{E}$ is isomorphic to $\hat{G}/P(T_\varphi)$, ii.) follows from a standard argument and Theorem 3.2. The third conclusion is inferred from the fact that $\widehat{G/E} \simeq P(T_\varphi)$ and Theorem A. Finally iv.) follows from iii.) and formula (1.8).

V. A DECOMPOSITION THEOREM

In the last chapter we saw (Theorem 4.5) that for an epimorphism of a compact metrizable abelian group we can always separate off the ergodic part leaving a factor automorphism with zero entropy. Now Pinsker [6] has conjectured that possibly every ergodic (measure-theoretic) automorphism, φ, of a Lebesgue space factors into the direct product of an automorphism with completely positive entropy (i.e., $H(\mathcal{A}, \varphi) > 0$ for every nontrivial finite subalgebra $\mathcal{A}$) and an automorphism with zero entropy. Rohlin [9] has shown that for (group-theoretic) automorphisms of a nontrivial compact metrizable abelian group, the notions of ergodicity and completely positive entropy coincide. Thus Pinsker's conjecture holds for the group-theoretic case.

1. E-Z Decompositions

In this chapter we consider a group-theoretic version of Pinsker's conjecture without the assumption of ergodicity. That is, if G is a compact metrizable abelian group and $\varphi \in \text{Epi}(G, G)$, we ask under what circumstances is G the direct product of two closed subgroups G_1 and G_2 such that $\varphi G_i = G_i$ $(i = 1, 2)$ and $\varphi | G_1$ is ergodic while $\varphi | G_2$ has zero entropy? Is such a decomposition always possible? For brevity we shall say that φ has the E-Z decomposition property if φ admits a decomposition of the sort just described.

We shall give an answer, though not an entirely satisfactory one, to the first question. We do not know the answer to the second question.

We begin by proving that the subgroup described in Theorem 4.6 is actually unique. Let G be a compact metrizable abelian group and $\varphi \in \mathrm{Epi}(G, G)$. Let us agree to denote by E_φ the subgroup of G given by

$$E_\varphi = \cap \{\ker(\hat{x}) : \hat{x} \in P(T_\varphi)\} .$$

(See (4.2). Thus E_φ is the annihilator of $P(T_\varphi)$ in G.

<u>Lemma 5.1.</u> Let G be a compact metrizable abelian group and $\varphi \in \mathrm{Epi}(G, G)$. If E is a closed subgroup of G invariant under φ such that $\varphi|E$ is ergodic, then $E \subset E_\varphi$.

Proof:

Suppose that $E \subset G$ satisfies the conditions of the theorem. Set

(5.1) $$H = \{\hat{x} \in \hat{G} : \hat{x}|E = 0\}$$

Now [10, p. 35]

(5.2) $$E = \cap \{\ker(\hat{x}) : \hat{x} \in H\} .$$

Thus it suffices to show that $P(T_\varphi) \subset H$.

It is clear that $T_\varphi H \subset H$. Let T_H denote the endomorphism induced by T_φ in G/H and set $\varphi_1 = \varphi|E$. Then (Theorem E)

$$(\widehat{E}, T_{\varphi_1}) \simeq (\widehat{G/H}, T_H)$$

Since φ_1 is ergodic it follows from Lemma 4.1 that T_{φ_1} and hence T_H is aperiodic. It thus follows from Proposition 3.3 that $P(T_\varphi) \subset H$ as desired.

We are now prepared to prove

Theorem 5.1. Let G be a compact metrizable abelian group and $\varphi \in \mathrm{Epi}(G, G)$. Then there exists a unique closed subgroup $E \subset G$ such that

i.) $\varphi E = E$

ii.) $\varphi|E$ is ergodic

iii.) $h(\varphi_E) = 0$ where φ_E is the epimorphism induced in G/E by φ

iv.) $h(\varphi) = h(\varphi|E)$.

Proof:

The existence of a closed subgroup $E \subset G$ satisfying i.)-iv.) was proved in Theorem 4.5. Indeed, we showed that

$$(5.3) \qquad E_\varphi = \cap \{\ker(\hat{x}) : \hat{x} \in P(T_\varphi)\}$$

satisfies these conditions. We will show that E_φ is the only such subgroup.

Suppose that $E \subset G$ satisfies the conclusions of the theorem. Let $H \subset \widehat{G}$ be defined by (5.1). Because E is uniquely determined by H (see (5.2)) it suffices to show that $H = P(T_\varphi)$.

According to Lemma 5.1, $E \subset E_\varphi$ and so

$$P(T_\varphi) = \{\hat{x} \in G : \hat{x} | E_\varphi = 0\} \subset \{\hat{x} \in \widehat{G} : \hat{x} | E = 0\} = H$$

Let $\varphi_1 = \varphi | E$ and $G_0 = G/E$. Since $h(\varphi_E) = 0$ it follows that

$$P(T_{\varphi_E}) = \widehat{G}_0.$$

According to Theorem E, the mapping $\psi : \widehat{G}_0 \rightarrow \widehat{G}$ defined by

$$(\psi\hat{a})x = \hat{a}(x+E) \qquad (\hat{a} \in \widehat{G}_0, \ x \in G)$$

is a monomorphism satisfying $\psi T_{\varphi_1} = T_\varphi \psi$. Moreover, $\psi\widehat{G}_0 = H$. Thus by Proposition 3.5

$$H = \psi\widehat{G}_0 = \psi P(T_{\varphi_E}) \subset P(T_\varphi) .$$

Consequently, $H = P(T_\varphi)$ and the theorem is proved.

As will soon become apparent, Theorem 5.1 contains what information we have about the E-Z decomposition problem. However, the following reformulation seems justified.

Theorem 5. 2. Let G be a compact metrizable abelian group and $\varphi \in \mathrm{Epi}(G, G)$. Then φ has the E-Z decomposition property iff there exists a subgroup $H \subset \hat{G}$ such that $T_\varphi H \subset H$ and $\hat{G} = H \oplus P(T_\varphi)$.

Proof:

Suppose that G is the direct sum of two closed subgroups G_1 and G_2 such that $\varphi G_j = G_j$ $(j = 1, 2)$ and $\varphi | G_1$ is ergodic while $\varphi | G_2$ has zero entropy. Evidently then, the epimorphism $\bar{\varphi}$ induced in G/G_1 by φ has zero entropy. Thus the uniqueness assertion of Theorem 5. 1 gives

$$G_1 = E_\varphi = \cap \{\ker(\hat{x}) : \hat{x} \in P(T_\varphi)\}$$

Consequently, $\hat{G_1} = P(T_\varphi)$. If we set $H = \hat{G_2}$, then $T_\varphi H \subset H$ and $\hat{G} = H \oplus P(T_\varphi)$.

On the other hand, if there exists a subgroup $H \subset \hat{G}$ such that $T_\varphi H \subset H$ and $\hat{G} = H \oplus P(T_\varphi)$, then, according to Theorem 3. 2, $T_\varphi | H$ is aperiodic. Thus the adjoint of $T_\varphi | H$ in $\hat{H}$ is ergodic and the existence of an E-Z decomposition for φ now follws from Theorem 4. 5.

VI. QUASI-PERIODIC MAPPINGS (II)

As we saw in Chapter V, the E-Z decomposition problem admits a "discrete" reformulation. In this chapter we investigate the problem from this point of view. We also include here a discussion of the parallels that exist between our characterization of the vanishing of entropy (Theorem 4.5) and that given in Chapter I (formula (1.5)).

1. The E-Z Decomposition Property

Let Γ be an abelian group and $T \in \mathrm{Hom}(\Gamma, \Gamma)$. Then we are interested in determining whether or not there exists a subgroup $H \subset \Gamma$ such that $TH \subset H$ and $\Gamma = H \oplus P(T)$. If such a decomposition exists we will say that T has the E-Z decomposition property.

If Γ is torsion free, then a necessary condition for $P(T)$ to be a direct summand is that $\Gamma / P(T)$ also be torsion free. The following theorem shows that the non-existence of an E-Z decomposition cannot be asserted on these grounds:

Theorem 6.1. Let Γ be an abelian group and $T \in \mathrm{Hom}(\Gamma, \Gamma)$. If Γ is torsion-free, then $\Gamma / P(T)$ is torsion free.

Proof:

Suppose that $mx \in P(T)$ for some integer $m > 0$ and $x \in \Gamma$

(i.e., $m(x+P(T)) = 0$). Then

$$mx \in P_n(T) \tag{6.1}$$

for some $n \geq 0$. We prove by induction that from (6.1) it follows that $x \in P_n(T)$. If $n = 0$, then, since $P_0(T) = \{0\}$ and Γ is torsion free, it follows that $x \in P_0(T)$. Suppose that if $mx \in P_n(T)$ for some $m > 0$ and $x \in \Gamma$, then $x \in P_n(T)$. If $mx \in P_{n+1}(T)$, then there exists an integer $k > 0$ such that

$$T^k(mx) - mx = m(T^k(x)-x) \in P_n(T).$$

Thus $T^k(x) - x \in P_n(T)$ and so $x \in P_{n+1}(T)$. This completes the proof of the theorem.

Let us recall here that a subgroup Γ' of an abelian group Γ such that Γ/Γ' is torsion free is necessarily a pure subgroup of Γ (see [3, p. 76] for a discussion of this notion). In view of this fact, Theorem 6.1 guarantees that for an interesting class of abelian groups, $P(T)$ is at least a direct summand. Indeed, according to [3, p. 166] we have

Theorem 6.2. A torsion free abelian group Γ has the property that all its pure subgroups are direct summands iff $\Gamma = D \oplus H$ where D is a divisible group and $H = H_1 \oplus H \oplus \ldots \oplus H_r$ (r a nonnegative integer). The H_i being isomorphic torsion free abelian groups of rank 1.

We conclude this section by showing that every linear endomorphism of a finite dimensional vector space has the E-Z decomposition property.

For the remainder of this discussion we shall suppose that V is a finite dimensional vector space over some field and that $T : V \to V$ is linear.

The following proposition is immediate:

Proposition 6.1. Each of the following is a subspace of V:

i.) $P_n(T)$ $(n \geq 0)$

ii.) $(T^p - I)^{-1}[P_n(T)]$ $(p > 0, n \geq 0)$

iii.) $P(T)$.

Let us now prove

Proposition 6.2. For each integer $n > 0$, there exists an integer $\ell > 0$ such that

$$P_n(T) = (T^\ell - I)^{-1}[P_{n-1}(T)]$$

Proof:

Let $n > 0$ be given. Recall that

$$P_n(T) = \bigcup_{k>0} (T^k - I)^{-1}[P_{n-1}(T)]$$

Set

$$K_p = (T^p - I)^{-1}[P_{n-1}(T)] \quad (p>0) .$$

Now

(6. 2) $$\bigcup_{p=1}^{m} K_p \subset K_{m!} .$$

Indeed, if $x \in K_p$ for some $1 \leq p \leq m$, then

$$T^p x = x + y$$

where $y \in P_{n-1}(T)$. Let $\overline{p} = m!/p$. Then (see (3. 8))

$$T^{m!}x = T^{\overline{p}\,p}x = x + \sum_{\ell=0}^{\overline{p}-1} T^{\ell p} y .$$

Since $P_{n-1}(T)$ is a subspace and $TP_{n-1}(T) = P_{n-1}(T)$, it follows that

$$x \in (T^{m!} - I)^{-1}[P_{n-1}(T)] = K_{m!}$$

For each $m \geq 1$ let V_m be the subspace spanned by $\bigcup_{p=1}^{m} K_p$. Since $K_{m!}$ is a subspace it follows from (6. 2) that $V_m \subset K_{m!}$. Now since $\{V_m : m \geq 1\}$ is an increasing sequence and V is finite dimensional, there is an integer $s > 0$ such that

$V_{r+s} = V_s$ for all $r \geq 0$. Thus we have

$$P_n(T) = \bigcup_{p>0} (T^p - I)^{-1}[P_{n-1}(T)] = \bigcup_{p>0} K_p = \bigcup_{m>0} \bigcup_{p=1}^{m} K_p$$

$$\subset \bigcup_{m>0} V_m \subset \bigcup_{m>0} K_{m!} \subset P_n(T) .$$

Hence $P_n(T) = \bigcup_{m>0} V_m = V_s \subset K_{s!} \subset P_n(T)$

and so finally

$$P_n(T) = K_{s!}$$

Thus with $\ell = s!$ we have

$$P_n(T) = K_\ell = (T^\ell - I)^{-1}[P_{n-1}(T)]$$

and the proposition is proven.

<u>Corollary 6.1.</u> For each $n > 0$ there exist integers $k_1, k_2, \ldots, k_n > 0$ such that

$$P_n(T) = \ker\, [(T^{k_1} - I)(T^{k_2} - I) \ldots (T^{k_n} - I)]$$

The following elementary proposition plays a fundamental role in our discussions:

<u>Proposition 6. 3.</u> Let $k_1, k_2, \ldots, k_n$ be any positive integers and set

$$H_n := (T^{k_1}-I)(T^{k_2}-I)\ldots(T^{k_n}-I) .$$

If $H_n x = 0$, then $x \in P_n(T)$.

Proof:

The proof is by induction on n. From $(T^{k_1}-I)x = 0$ it follows that $x \in P_1(T)$. Suppose it is true that if

$$(T^{k_1}-I)(T^{k_2}-I)\ldots(T^{k_n}-I)x = 0 ,$$

then $x \in P_n(T)$. Thus if

$$(T^{k_1}-I)(T^{k_2}-I)\ldots(T^{k_n}-I)(T^{k_{n+1}}-I)x = 0 ,$$

then $T^{k_{n+1}}x - x \in P_n(T)$ and so $x \in P_{n+1}(T)$.

We prove finally

<u>Theorem 6. 3.</u> Let V be a finite dimensional vector space over some field K and T be a linear self-map of V. Then there exists a subspace R such that $V = R \oplus P(T)$ and $TR \subset R$, i. e., T has E-Z decomposition property.

Proof:

Since V is finite dimensional and $\{P_n(T) : n \geq 0\}$ is an increasing sequence of subspaces, there exists an integer $n \geq 0$ such that $P_{n+p}(T) = P_n(T)$ $(p \geq 0)$. Let $k_1, k_2, \ldots, k_n$ be positive integers such that

$$P_n(T) = \ker [(T^{k_1}-I)(T^{k_2}-I)\ldots(T^{k_n}-I)] .$$

Set

$$H_n = (T^{k_1}-I)(T^{k_2}-I)\ldots(T^{k_n}-I)$$

and

$$R = H_n V = \mathrm{Im} H_n .$$

Since clearly $TH_n = H_n T$ we have $TR \subset R$. Let us show next that

$$R \cap P(T) = \{0\}$$

Recall that $P(T) = P_n(T) = \ker(H_n)$. Thus if $y = H_n x \in \ker(H_n)$, then $H_n^2 x = 0$ or what is the same thing

$$(T^{k_1}-I)\ldots(T^{k_n}-I)(T^{k_1}-I)\ldots(T^{k_n}-I)x = 0 .$$

According to Corollary 6.1, $x \in P_{2n}(T)$. But $P_{2n}(T) = P_n(T)$. Consequently $x \in P_n(T) = \ker(H_n)$. Thus $y = Hx = 0$, i.e., $R \cap P(T) = \{0\}$.

Finally since

$$\dim (R \oplus P(T)) = \dim (R) + \dim (P(T))$$
$$= \dim (\mathrm{Im} H_n) + \dim (\ker H_n)$$
$$= \dim (V)$$

and V is finite dimensional, it follows that

$$R \oplus P(T) = V.$$

2. Some Further Properties of P(T)

Let (Ω, Σ, μ) be a Lebesgue space and φ be an invertible measure-preserving transformation. Denote by $\mathfrak{A}$ the collection of all finite subalgebras of Σ and for $\mathcal{A} \in \mathfrak{A}$ set

$$\mathcal{A}^{-} = \bigvee_{k \leq 0} \varphi^{k} \mathcal{A} .$$

Finally let

$$\mathcal{M}_{\varphi} = \{ \bigcap_{k} \varphi^{k} \mathcal{A}^{-} : \mathcal{A} \in \mathfrak{A} \}.$$

Then [11]

(6.3) $$h(\varphi) = 0 \quad \text{iff} \quad \mathcal{M}_{\varphi} = \Sigma .$$

If in addition, Ω is a compact metrizable abelian group, Σ the Borel subsets of Ω, μ the normalized Haar measure in Ω and $\varphi \in \mathrm{Aut}(\Omega, \Omega)$, then (Theorem A)

$$(6.4) \qquad h(\varphi) = 0 \quad \text{iff} \quad P(T_\varphi) = \hat{\Omega}$$

Because of the similarity between the characterizations (6. 3) and (6. 4) it is natural to expect that $\mathcal{M}_\varphi$ and $P(T_\varphi)$ should share some common properties.

The σ-algebra $\mathcal{M}_\varphi$ can be characterized in another fashion. Namely [11], $\mathcal{M}_\varphi$ is the largest σ-subalgebra of Σ invariant under φ such that for any σ-subalgebra $\Sigma' \subset \mathcal{M}_\varphi$, the equality $\varphi\Sigma' = \Sigma'$ follows from $\varphi\Sigma' \supset \Sigma'$.

It is interesting to note therefore

Theorem 6. 4. Let Γ be an abelian group and $T \in \text{Hom}(\Gamma, \Gamma)$. If A is a subgroup of $P(T)$ such that $TA \subset A$, then $TA = A$.

Proof:

Let $y \in A$. Then we want to find an $x \in A$ such that $Tx = y$. Since $y \in A \subset P(T)$, there is according to Proposition 2. 1 a polynomial

$$(6.5) \qquad u(t) = (t^{k_1}-1)(t^{k_2}-1)\ldots(t^{k_n}-1) \quad k_j > 0 \quad 1 \leq j \leq n$$

such that $u(T)y = 0$. We may without loss of generality suppose that n is odd. Expanding the right-hand side of (6. 5) we obtain a polynomial of the form

(6.6) $$a_m t^m + a_{m-1} t^{m-1} + \ldots + a_1 t - 1$$

where $m > 0$ and the a_i are integers.

Thus from $u(T)y = 0$ it follows that

$$a_m T^m y + a_{m-1} T^{m-1} y + \ldots + a_1 T_y - y = 0$$

or

$$y = T[a_m T^{m-1} y + a_{m-1} T^{m-2} + \ldots + a_1 y] .$$

Since A is a subgroup, $TA \subset A$ and $y \in A$ we have

$$x = a_m T^{m-1} y + a_{m-1} T^{m-2} y + \ldots + y \in A$$

Clearly $Tx = y$ and the theorem is proved.

Let Γ be an abelian group and $T \in \mathrm{Hom}(\Gamma, \Gamma)$. Denote by $\mathcal{G}$ the set of all subgroups Γ' of Γ such that $T\Gamma' \subset \Gamma'$ and for every subgroup $A \subset \Gamma'$ such that $TA \subset A$ it follows that $TA = A$.

As we shall now proceed to show, $\mathcal{G}$ (ordered by inclusion) always contains a largest element which may be different from $P(T)$.

Let W denote the set of polynomials with integer coefficients and positive degree which are of the form

$$w(t) = a_m t^m + a_{m-1} t^{m-1} + \dots + a_1 t \pm 1 .$$

Set

$$K(W) = \cup \{\ker w(T) : w(t) \in W\}$$

We will prove that $K(W)$ is the unique maximal element of $\mathcal{S}$.

<u>Proposition 6.4.</u> $K(W)$ is a subgroup of Γ.

Proof:

Suppose $x, y \in K(W)$. Then there exists $w_1(t), w_2(t) \in W$ such that

$$w_1(T)x = w_2(T)y = 0 .$$

Set $w(t) = w_1(t)w_2(t)$. Then clearly $w(t) \in W$. Moreover,

$$\begin{aligned} w(T)(x-y) &= w(T)x - w(T)y \\ &= w_2(T)w_1(T)x - w_1(T)w_2(T)y = 0 . \end{aligned}$$

Thus $x - y \in K(W)$ and so $K(W)$ is a subgroup.

Now exactly as in the proof of Theorem 6.4 we can prove

<u>Proposition 6.5.</u> If A is a subgroup of $K(W)$ such that $TA \subset A$, then $TA = A$.

Clearly $TK(W) \subset K(W)$. Thus according to the preceding proposition, $TK(W) = K(W)$. We conclude therefore that $K(W) \in \mathcal{G}$. That $K(W)$ is the largest element of $\mathcal{G}$ follows finally from

Proposition 6.6. If $\Gamma' \in \mathcal{G}$, the $\Gamma' \subset K(W)$.

Proof:

Let $x \in \Gamma'$. We want to show that there exists an element $w(t) \in W$ such that $w(T)x = 0$. If $x = 0$, the assertion is clear. We suppose that $x \neq 0$. Let A be the subgroup generated by

$$\{x, Tx, T^2x \ldots\}.$$

Then surely $A \subset \Gamma'$ and $TA \subset A$. Thus $TA = A$. In particular, $x \in TA$. But TA is generated by

$$\{Tx, T^2x, \ldots\}.$$

Thus there exists integers $a_1, a_2, \ldots, a_m$ not all zero and $m > 0$ such that

$$x = a_m T^m x + a_{m-1} T^{m-1} x + \ldots + a_1 Tx.$$

Set

$$w(t) = a_m t^m + a_{m-1} t^{m-1} + \ldots + a_1 t - 1.$$

Then $w(t) \in W$ and $w(T)x = 0$.

Consider now

<u>Example 6.1.</u> Let $\Gamma = Z \times Z$ and define $T \in \mathrm{Aut}(\Gamma, \Gamma)$ by

$$T(m, n) = (m+n, m)$$

Then T is the adjoint of the <u>ergodic</u> automorphism $(z_1, z_2) \to (z_1 z_2, z_1)$ of the 2-dimensional torus (see for example [5, p. 55]). Thus according to Lemma 4.1

$$P(T) = \{0\}$$

But as the reader can easily verify

$$(T^2 - T - I)x = 0 \quad (x \in \Gamma) .$$

Since $w(t) = t^2 = t - 1 \in W$, it follows that

$$K(W) = \Gamma .$$

VII. SOME EXTENSIONS

Let G be a compact group with Borel sets Σ and $\varphi \in \mathrm{Epi}(G, G)$. Denote by $M(\varphi)$ the set of all probability measures defined on Σ which are preserved by φ. Thus the normalized Haar measure, m, belongs to $M(\varphi)$. Unless $G = 0$, $M(\varphi)$ always contains elements other than m (for example the probability measure with all its mass concentrated in the identity of G).

If we let $h(\varphi, \lambda)$ denote the entropy of φ computed with respect to $\lambda \in M(\varphi)$, then it is natural to ask what is

$$\sup \{h(\varphi, \lambda) : \lambda \in M(\varphi)\} \ ?$$

An interesting <u>conjecture</u> is that

$$h(\varphi, \lambda) \leq h(\varphi, m) \quad (\lambda \in M(\varphi)) \ . \tag{7.1}$$

In other words we conjecture that

$$\sup \{h(\varphi, \lambda) : \lambda \in M(\varphi)\} = h(\varphi, m) \ .$$

A consideration of the Bernoulli shift provides some motivation for this conjecture. Let $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ be a fixed finite group with n elements and set

$$G = \Gamma \times \Gamma \times \Gamma \times \ldots \ .$$

Then with the natural coordinate-wise operations and the Tichonov topology, G is a compact group.

If $P = (p_1, p_2, \ldots, p_n)$ is a probability measure defined on Γ by $P(\{\gamma_k\}) = p_k$ $(1 \leq k \leq n)$, we shall denote by $\overline{P}$ the induced product measure in G. The normalized Haar measure, $\overline{m}$, on G is obtained by taking $m = (1/n, 1/n, \ldots, 1/n)$.

The (one-sided) Bernoulli shift, τ, on G is defined by

$$\tau(w_0, w_1, w_2, \ldots) = (w_1, w_2, \ldots) \quad ((w_0, w_1, w_2, \ldots) \in G).$$

Clearly $\tau \in \mathrm{Epi}(G, G)$. In addition, it is easily shown that τ preserves each of the product measures described above. Moreover, these are the only product measures preserved by τ. That is, if μ_k $(k \geq 0)$ are probability measures on Γ such that the corresponding product measure is preserved by τ, then $\mu_0 = \mu_1 = \mu_2 = \ldots$.

Let $P = (p_1, p_2, \ldots, p_n$ be a probability measure on Γ. Then [1, p. 64]

$$h(\tau, \overline{P}) = -\sum_{k=1}^{n} p_k \log p_k \tag{7.2}$$

Now the right-hand side of (7.2) is maximized when $p_k = 1/n$ $(1 \leq k \leq n)$ (see e.g. [1, p. 61]. Thus

$$h(\tau, \overline{P}) \leq h(\tau, \overline{m})$$

holds for all the product measures preserved by T.

A rather less adventurous conjecture than (7.1) is the following:

(7.3) Let G be a compact metrizable abelian group with normalized Haar measure m and $\varphi \in Epi(G, G)$.

If $h(\varphi, m) = 0$, then $h(\varphi, \lambda) = 0$ for every $\lambda \in M(\varphi)$.

Evidently (7.3) would follow from (7.1). Although (7.3) is given a purely group theoretic formulation, its truth would lead to a natural generalization of Theorem (4.5) as we shall see.

Let (Ω, Σ, μ) be a probability space where Ω is a complete separable metric space and Σ is the collection of Borel subsets of Ω. We shall denote by $U = U(\Omega)$ the set of those complex valued $f \in L_2(\Omega)$ such that $|f| = 1$ a.e. If $\varphi : \Omega \to \Omega$ is an invertible measure preserving transformation, we shall call a set $\Gamma \subset U$ admissible for φ if

i.) Γ is a group under point-wise multiplication

ii.) $\overline{sp\Gamma} = L_2(\Omega)$ (that is the closure of the linear hull of is $L_2(\Omega)$).

iii.) $T_\varphi \Gamma \subset \Gamma$ where T_φ is the adjoint of φ in $L_2(\Omega)$

Thus for example U is admissible as is

$$U_0 = \{f \in u : f \text{ is a simple function}\}$$

If G is a compact metrizable group and $\varphi \in Aut(G, G)$, then $\widehat{G}$

is admissible for φ.

Suppose that Γ is admissible for φ and let $T = T_\varphi | \Gamma$. We define subsets $P_n(T, \Gamma)$ $(n \geq 0)$ and $P(T, \Gamma)$ of Γ by

(7.4) $$P_0(T, \Gamma) = \{1\},$$

(7.5) $$P_{n+1}(T, \Gamma) = \{f \in \Gamma : T^k f = \theta f \text{ for some } k > 0, \theta \in P_n(T, \Gamma)\}$$

$$(n \geq 0)$$

and

(7.6) $$P(T, \Gamma) = \bigcup_{n \geq 0} P_n(T, \Gamma).$$

It is easily seen that (7.4)-(7.6) correspond precisely to (3.1)-(3.3) in multiplicative dress.

Now, _if_ (7.3) is true, then

(7.7) if $\overline{sp} P(T, \Gamma) = L_2(\Omega)$, then $h(\varphi) = 0$.

Indeed, if $\overline{sp} P(T, \Gamma) = L_2(\Omega)$, then since $L_2(\Omega)$ is separable and $P(T, \Gamma)$ is a group invariant under T, there is a _countable_ subgroup $H \subset P(T, \Gamma)$ which is admissible for φ. According to Proposition 3.6, $P(T_\varphi | H, H) = H$. Thus we may and shall suppose that Γ is countable and that $P(T, \Gamma) = \Gamma$. Give Γ the discrete topology and let $G = \widehat{\Gamma}$. If τ denotes the adjoint of T in G, then, as Foais has shown [4], there exists a Borel probability measure λ in G preserved by τ and such that φ and τ are conjugate in the sense of [5, p. 55]. Because the separable topology on G can be given by a complete metric and the same holds by assumption for Ω, it follows that φ and $\mathcal{T}$ are isomorphic in the sense of [1, p. 66]. Thus $h(\varphi, \mu) = h(\mathcal{T}, \lambda)$. Now since $P(T, \Gamma) = \Gamma$, it follows immediately from the Pontrjagin Duality Theorem that the adjoint, $T_{\mathcal{T}}$, of $\mathcal{T}$ in $G(\approx\Gamma)$ is quasi-periodic. Hence if m is the normalized Haar measure in G, it follows from Theorem (4.5) that $h(\mathcal{T}, m) = 0$. Thus (7.3) would imply that $h(\mathcal{T}, \lambda) = 0$ and (7.7) follows.

It is with this final conjecture that we shall concern ourselves for the remainder of this chapter. We shall prove that with the notation and hypotheses introduced above, if $\overline{sp} P(T, U_0) = L_2(\Omega)$, then $h(\varphi) = 0$. We also include an example showing the converse to be false.

Let (Ω, Σ, μ) be a probability space and $\varphi : \Omega \to \Omega$ be a measure-preserving transformation. For a set $\Phi \subset U(\Omega)$ let $\sigma - \Phi$ denote the least σ-algebra of Σ with respect to which all of the members of Φ are measureable. We shall denote by $<\Phi>$ the least of the subgroups Γ of $U(\Omega)$ such that $\Phi \subset \Gamma$ and $T\Gamma \subset \Gamma$.

<u>Proposition 7.1.</u> Let $\mathcal{A} = \sigma - \Phi$. Then $\sigma - <\Phi> \subset \bigvee_{i=0}^{\infty} \varphi^{-i}\mathcal{A}$.

Proof:

Let $\mathcal{A}^- = \bigvee_{i=0}^{\infty} \varphi^{-i}\mathcal{A}$. If $f, g \in \Phi$ and $\ell, m, n > 0$ let us show for example that $f^{\ell} T^{m} g^{n}$ is measurable with respect to $\mathcal{A}^{-}$.

Since $f, g \in \Phi \subset U(\Omega)$, there exists sequences

$$\psi_p = \sum_{k=1}^{r_p} a_{kp} \chi_{A_{kp}} \quad \text{and} \quad \gamma_p = \sum_{k=1}^{s_p} b_{kp} \chi_{B_{kp}} \qquad (p \geq 0)$$

of simple functions measurable with respect to $\mathcal{A}$ which converge point-wise to f and g respectively. Thus.

$$\psi_p^{\ell} T^{m} \gamma_p^{n} \to f\ T^{m} g^{n} \quad \text{(point-wise)} .$$

Now, assuming as we may that the expressions for ψ_p and γ_p are the canonical ones,

$$\text{(7.8)} \qquad \psi_p^{\ell} T^{m} \gamma_p^{n} = \left(\sum_{k=1}^{r_p} a_{kp}^{\ell} \chi_{A_{kp}} \right)\left(\sum_{k=1}^{s_p} b_{kp}^{n} \chi_{\varphi^{-m} B_{kp}} \right)$$

It is thus clear from (7.1) that for each $p \geq 0$, $\psi_p^{\ell} T^{m} \gamma_m^{n}$ is measurable with respect to

$$\mathcal{A} \vee \varphi^{-m} \mathcal{A} \subset \bigvee_{i=0}^{\infty} \varphi^{-i} \mathcal{A} = \mathcal{A}^{-} .$$

Consequently $f^{\ell} T^{m} g^{n}$ is measurable with respect to $\mathcal{A}^{-}$.

Since a typical element of $<\Phi>$ is of the form

$$T^{m_1} f_1^{\ell_1} T^{m_2} f_2^{\ell_2} \cdots T^{m_k} f_k^{\ell_k}$$

a rigorous proof based on these ideas is readily given.

Observing that with $\mathcal{A} = \sigma - \Phi$, we have $\sigma - T_\varphi^n \Phi = \varphi^{-n} \mathcal{A}$ $(n \geq 0)$, one easily establishes

<u>Proposition 7.2.</u> $\sigma - \bigcup_{n \geq 0} T_\varphi^n \Phi = \bigvee_{n=0}^{\infty} \varphi^{-n} \mathcal{A}$.

Thus we finally have

<u>Theorem 7.1.</u> $\sigma = \langle \Phi \rangle = \bigvee_{i=0}^{\infty} \varphi^{-i} \mathcal{A}$ where $\mathcal{A} = \sigma - \Phi$.

Proof:

Applying in turn Propositions 7.1 and 7.2 we have

$$\bigvee_{i=0}^{\infty} \varphi^{-i} \mathcal{A} \supset \sigma - \langle \Phi \rangle \supset \sigma - \bigcup_{i \geq 0} T^i \Phi = \bigvee_{i=0}^{\infty} \varphi^{-i} \mathcal{A}$$

and the theorem follows.

We are now prepared to prove

<u>Theorem 7.2.</u> Let (Ω, Σ, μ) be a separable probability space and $\varphi : \Omega \to \Omega$ be a measure-preserving transformation. If $\overline{\text{sp}} P(T, U_0) = L_2(\Omega)$, then $h(\varphi) = 0$.

Proof:

Recall that U_0 is the set of all complex valued simple functions $f \in L_2(X)$ such that $|f| = 1$.

Since $L_2(\Omega)$ is separable, there is a countable subset $\Gamma \subset P(T, U_0)$ such that $\overline{\mathrm{sp}\Gamma} = L_2(\Omega)$. Let $f_1, f_2, \ldots, f_n, \ldots$ be some enumeration of the elements of Γ. Set

$$\Phi_n = \{f_1, f_2, \ldots, f_n\} \quad (n \geq 1)$$

Then

$$\mathcal{F}_0 = \bigcup_{n \geq 1} \sigma - \Phi_n$$

is a field of subsets of Ω which generates Σ.

Notice that since each of the members of Φ_n is finitely valued and Φ_n is a finite set, $\sigma - \Phi_n$ is a finite subalgebra.

Now suppose that $\mathcal{A}$ is some finite subalgebra of $\mathcal{F}_0$. Then since $\sigma - \Phi_n \uparrow \mathcal{F}_0$ we must have $\mathcal{A} \subset \sigma - \Phi_n$ for some $n \geq 1$. Suppose first that

$$\mathcal{A} = \sigma - \Phi_n$$

Then (Theorem 7.1)

$$\bigvee_{i=0}^{\infty} \varphi^{-i} \mathcal{A} = \sigma - <\Phi_n> .$$

Now it is clear that $<\Phi_n> \subset P(T, U_0)$. Thus since $T <\Phi_n> \subset <\Phi_n>$ it follows from Theorem 6.4 that

$$T < \Phi_n> = < \Phi_n> . \tag{7.9}$$

But $T<\Phi_n> = <T\Phi_n>$ and it is readily verified that

$$\sigma - <T\Phi_n> = \bigvee_{i=1}^{\infty} \varphi^{-i}\mathcal{A} \qquad (\mathcal{A} = \sigma - \Phi_n) \tag{7.10}$$

Thus from (7.9) and (7.10) it follows that

$$\begin{aligned} \mathcal{A} &= \sigma - \Phi_n \subset \sigma - <\Phi_n> = \sigma - T<\Phi_n> \\ &= \sigma - <T\Phi_n> = \bigvee_{i=1}^{\infty} \varphi^{-i}\mathcal{A} . \end{aligned}$$

As we have observed before, the inclusion

$$\mathcal{A} \subset \bigvee_{i=1}^{\infty} \varphi^{-i}\mathcal{A}$$

implies $H(\mathcal{A}, \varphi) = 0$.

Now if $\mathcal{A} \subset \sigma - \Phi_n$ then [1, p. 78]

$$H(\mathcal{A}, \varphi) \leq h(\sigma - \Phi_n, \varphi) = 0.$$

Since

$$h(\varphi) = \sup \{H(\mathcal{A}, \varphi) : \mathcal{A} \text{ a finite subalgebra of } \mathcal{F}_0\}$$

we conclude that $h(\varphi) = 0$ as desired.

We conclude this chapter with an example for which the converse of Theorem 7.2 does not hold.

Let S' be the one-dimensional torus equipped with its normalized Haar measure. Consider a rotation $\varphi : z \to cz (z \in S')$

where c is not a root of unity. Then φ is an invertible measure-preserving transformation of X. Moreover, φ is totally ergodic (i.e., φ^n is ergodic for each $n \geq 0$) and $h(\varphi) = 0$ [6]. As with any totally ergodic transformation, we must have

$$P_1(T, U_0) \subset \text{constants}.$$

Since φ is totally ergodic we also have

<u>Proposition 7.3.</u> If $f \in L_2(S')$ is not a constant and satisfies $T_\varphi^k f = \lambda f$ for some $k > 0$ and scalar λ, then λ is not a root of unity.

Now with φ as above we prove that

(7.11) $$P_2(T, U_0) \subset \text{constants}.$$

If $f \in P_2(T, U_0)$, then, since $P_1(T, U_0) \subset$ constants

(7.12) $$T_\varphi^k f = \lambda f$$

for some $k > 0$ and scalar λ. Let $R(g)$ denote the range of $g \in U_0$. Then clearly

$$(R(T_\varphi^k g) = R(g) \quad \text{and} \quad R(\lambda g) = \lambda R(g)$$

Thus (7.11) implies

$$R(f) = R(T_{\varphi}^{k} f) = R(\lambda f) = \lambda R(f) .$$

But since $R(f)$ is a finite set and $z \to \lambda z$ is an injective mapping, it follows that λ must be a root of unity. Thus by Proposition 7.3, f must be a constant and (7.11) is established. It follows immediately that $P(T, U_0) \subset$ constants. Thus $h(\varphi) = 0$ but

$$\overline{sp} P(T, U_0) \neq L_2(S') .$$

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