ON A GENERALIZATION OF MEHLER'S INVERSION FORMULA AND SOME OF ITS APPLICATIONS

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ON A GENERALIZATION OF MEHLER'S INVERSION FORMULA AND SOME OF ITS APPLICATIONS

1. INTRODUCTION

It is proposed to generalize an inversion formula by Mehler (18) involving conical functions.

$$
\text{(a)} \qquad
\begin{aligned}
g(y) &= \int_0^\infty P_{-\frac{1}{2}+ix}(y)\, f(x)\,dx \\
f(x) &= x \tanh(\pi x) \int_1^\infty P_{-\frac{1}{2}+ix}(y)\, g(y)\,dy
\end{aligned}
$$

or, the equivalent pair

$$
\text{(a.i)} \qquad
\begin{aligned}
G(y) &= \int_1^\infty F(x)\, P_{-\frac{1}{2}+iy}(x)\,dx \\
F(x) &= \int_0^\infty y \tanh(\pi y)\, g(y)\, P_{-\frac{1}{2}+iy}(x)\,dy
\end{aligned}
$$

Here $P_{-\frac{1}{2}+ix}(y)$ denotes the first Legendre function of order $-\frac{1}{2}+ix$ with an argument $y > 1$ (8, Ch. 3). These "conical" function occur in certain boundary value problems involving configurations of a conical shape. The paper by Mehler referred to before deals with the distribution of electrostatic charges on an infinite conical shell. In a following paper, Neumann (19) extended Mehler's investigations to the case of the distribution of charges under the influence of an external field. The analysis for the derivation of (a) as used by Mehler is purely formal. The first reliable investigation as to the class of functions $f(x)$ for which (a) holds, seems to be due to Fock (13). In recent times considerable interest in inversion formulas of the type (a) has been displayed. For instance, Karp (4) gave the charge distribution on a finite conical shell (cup) employing the Wiener-Hopf technique; Carslaw (5), Buchholz (2, 3, 4), Felsen (11, 12), among others, investigates the propagation of (plane and spherical) electromagnetic and acoustic waves in a (infinitely extended) conical horm.

It will be shown in section 2 that an inversion formula similar to (a) can be given when, instead of $P_{-\frac{1}{2}+ix}(y)$, $P^{\mu}_{-\frac{1}{2}+ix}(y)$ is used. This inversion formula reduces to (a) for the special case $\mu = 0$. The proof of this inversion theorem is based on the analysis employed by Fock (13). It may be mentioned in this connection that a recent survey (14)[1] concerning integral transforms makes no mention of Mehler's formula.

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[1]Shortly after writing this thesis, the author discovered that a generalization of Mehler's inversion formula for the case $\mu = \frac{2-n}{n}$, where n = 1,2,3,--etc. had been treated before (21).

Section 3 will represent the general addition theorem for the modified Hankel function of the Gegenbauer type (9, p. 43) in the form of such a generalized Mehler transform (generalized spherical wave). Finally previously known results as integral expressions for the cylindrical and spherical wave (9, p. 55) and certain integral representations for the product of two modified Hankel functions (6, 7) can be derived as special cases.

2. INVERSION FORMULA FOR THE FUNCTION $P^{\mu}_{-\frac{1}{2}+ix}(y)$, $y \geq 1$

The integral expression for the conical function (8, p. 156)

$$(0.1) \quad P_{-\frac{1}{2}+ix}(y) = \sqrt{\frac{2}{\pi}} \cosh(\pi x) \int_0^{\infty} (y + \cosh t)^{-\frac{1}{2}} \cdot \cos(xt)dt, \; y > 0$$

$$(0.2) \quad P^{\mu}_{-\frac{1}{2}+ix}(y) = \sqrt{\frac{2}{\pi}} \Gamma(\tfrac{1}{2} - \mu) \left[\Gamma(\tfrac{1}{2}-\mu+ix)\Gamma(\tfrac{1}{2}-\mu-ix)\right]^{-1} \cdot (y^2 - 1)^{\frac{1}{2}\mu} \int_0^{\infty} (y + \cosh t)^{\mu-\frac{1}{2}} \cos(xt)dt, \; R_e\mu < \tfrac{1}{2}\; y > 1$$

suggest instead of (a) the inversion formula

$$(0.3) \quad \begin{aligned} g(y) &= \int_0^{\infty} P^{\mu}_{-\frac{1}{2}+ix}(y) f(x)dx, \\ f(x) &= \frac{x}{\pi} \sinh(\pi x) \Gamma(\tfrac{1}{2}-\mu+ix)\Gamma(\tfrac{1}{2}-\mu+ix) \cdot \int_1^{\infty} g(y) P^{\mu}_{-\frac{1}{2}+ix}(y)dy \end{aligned}$$

or the equivalent pair

$$G(y) = \int_1^{\infty} F(x) P^{\mu}_{-\frac{1}{2}+iy}(x)\,dx$$

(0.4)

$$F(x) = \pi^{-1} \int_0^{\infty} y \sinh(\pi y)\,\Gamma(\tfrac{1}{2}-\mu+iy)\cdot$$

$$\Gamma(\tfrac{1}{2}-\mu-iy)\cdot P^{\mu}_{-\frac{1}{2}+iy}(x)\,dy$$

Since by (8, p. 150)

$$P^{\frac{1}{2}}_{-\frac{1}{2}+ix}(\cosh a) = \sqrt{\frac{2}{\pi}}\,(\sinh a)^{\frac{1}{2}} \cos(xa)$$

(0.5)

$$P^{-\frac{1}{2}}_{-\frac{1}{2}+ix}(\cosh a) = \sqrt{\frac{2}{\pi}}\,(\sinh a)^{-\frac{1}{2}}\, x^{-1} \sin(xa)$$

one obtains from (0.3) for the special case $\mu = \pm \frac{1}{2}$ respectively putting $y = \cosh a$

$$(\sinh a)^{\frac{1}{2}} g(\cosh a) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos(xa)\,dx$$

(0.6)

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} (\sinh a)^{\frac{1}{2}} g(\cosh a) \cos(xa)\,da$$

and

$$(\sinh a)^{\frac{1}{2}} g(\cosh a) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} x^{-1} f(x) \sin(xa)\,dx$$

(0.7)

$$x^{-1}f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty (\sinh a)^{1/2} g(\cosh a)\sin(xa)\,da$$

These [(0.6) and (0.7)] are the Fourier cosine and the Fourier sine transform formula. Therefore Fourier's inversion formulas are a special case of the generalized Mehler transform.

Equation (0.3) will now be proved.

<u>Theorem 1.</u>

If a function $\psi(x)$ given in the interval $(1 \leq x < \infty)$ is such that $\phi(t) = (\sinh t/2)^{1-2\beta}\, \psi(\cosh t)$ has its first derivative integrable over an infinite interval $(0 \leq t < \infty)$ while its second derivative is absolutely integrable over any finite interval, and if $\lim\limits_{t\to 0} \phi(t)=0$, $\mathrm{Re}\beta \leq 0$, $\phi(t)_{t=0}$ be integrable for $\mathrm{Re}\beta \geq 0$, and $\lim\limits_{t\to\infty} \phi(t)=0$, then $\psi(x)$ is representable in the form of the integral

$$(1) \qquad \psi(\cosh\theta) = \int_0^\infty P^{\beta}_{i\mu-1/2}(\cosh\theta) f(\mu)\,d\mu$$

where $f(\mu)$ is

$$(2) \qquad f(\mu) = \pi^{-1}\mu \,\sinh\pi\mu\, \Gamma(1/2-\beta+i\mu)\,\Gamma(1/2-\beta-i\mu)\cdot$$

$$\int_0^\infty P^{\beta}_{i\mu-1/2}(\cosh\theta)\,\psi(\cosh\theta)\cdot\sinh\theta\, d\theta.$$

Theorem 2.

If a function $f(\mu)$ is absolutely integrable over any finite interval and $f(\mu) = O(\mu^{-\frac{1}{2}-\beta-\epsilon})$ for $\mu \to \infty$ has its derivative absolutely integrable over any finite interval, and if $f(0) = 0$, then $f(\mu)$ is representable in the form of the integral (2), where $\psi(\cosh \theta)$ is defined by (1). The proof of theorems 1 and 2 will now be given. The proof of theorems 1 and 2 consists in examining the course of computations which formally led to the form of the inversion integral (2) when given integral (1).

Starting with

$$(1.1) \quad \psi(\cosh \theta) = \int_0^\infty P^{\beta}_{-\frac{1}{2}+i\mu}(\cosh \theta) f(\mu)\, d\mu$$

where

$$(1.2) \quad P^{\mu-\frac{1}{2}}_{-\frac{1}{2}+iy}(\cosh a) = (\tfrac{1}{2}\pi)^{-\frac{1}{2}} \left[\Gamma(1-\mu)\right]^{-1} (\sinh a)^{\mu-\frac{1}{2}} \int_0^a \cos yt\, (\cosh a - \cosh x)^{-\mu}\, dt \quad (8, \text{p. } 156)$$

let $\mu - \frac{1}{2} = \beta$, $y = \mu$, $a = \theta$, $\cosh x = \cosh t$, $\cosh a = \cosh \theta$ so (1.2) becomes

$$(1.3) \quad P^{\beta}_{-\frac{1}{2}+i\mu}(\cosh\theta) = (\tfrac{1}{2}\pi)^{-\frac{1}{2}}(\sinh\theta)^{\beta}\left[\Gamma(\tfrac{1}{2}-\beta)\right]^{-1} \int_0^{\theta} \cos\mu t\,(\cosh\theta-\cosh t)^{-\frac{1}{2}-\beta}dt,$$

$$\mathrm{Re}\beta < \tfrac{1}{2}.$$

Replacing Rhs (right hand side) of (1.3) into (1.1) gives (1.4) which is

$$(1.4) \quad \psi(\cosh\theta) = \int_0^{\infty} (\pi/2)^{\frac{1}{2}}(\sinh\theta)^{\beta}\left[\Gamma(\tfrac{1}{2}-\beta)\right]^{-1}. \int_0^{\theta} \cos\mu t\,(\cosh\theta-\cosh t)^{-\frac{1}{2}-\beta}\,dt\,f(\mu)d\mu$$

Assume next $f(\mu)$ is such that order of integration can be interchanged in (1.4). But the conditions stated in theorem 2 allow this since this gives rise to absolute integrability of (1.4) so by Fubini's theorem interchange is allowed. So (1.4) now becomes

$$(1.5) \quad \psi(\cosh\theta) = \int_0^{\theta} (\pi/2)^{-\frac{1}{2}}(\sinh\theta)^{\beta}\left[\Gamma(\tfrac{1}{2}-\beta)\right]^{-1}. (\cosh\theta-\cosh t)^{-\frac{1}{2}-\beta} \int_0^{\infty} \cos\mu t\,f(\mu)d\mu dt$$

Next let $s = \cosh\theta$, $v = \cosh t$ so (1.5) now becomes

$$(1.6) \quad \psi(s) = \int_1^s (\pi/2)^{-1/2}(s^2-1)^{\beta/2}\left[\Gamma(\tfrac{1}{2}-\beta)\right]^{-1} \cdot (s-v)^{-1/2-\beta}\int_0^\infty \cos\left[(\cosh^{-1}v)\mu\right] f(\mu)\,d\mu \cdot (v^2-1)^{-1/2}dv,$$

so (1.6) becomes

$$(1.7) \quad \psi(s)(s^2-1)^{-\beta/2} = \int_1^s dv(s-v)^{-1/2-\beta}(\pi/2)^{1/2}\left[\Gamma(\tfrac{1}{2}-\beta)\right]^{-1} \int_0^\infty \cos\left[(\cosh^{-1}v)\mu\right] f(\mu)\,d\mu\,(v^2-1)^{-1/2}$$

But (1.7) is in the form of an Abel integral equation. The conditions of Theorem 1 are such that (1.7) can be solved and (1.7) becomes

$$(1.8) \quad (\tfrac{1}{2}\pi)^{-1/2}\left[\Gamma(\tfrac{1}{2}-\beta)\right]^{-1}\int_0^\infty \cos\left[(\cosh^{-1}s)\mu\right](s^2-1)^{-1/2}f(\mu)\cdot d\mu = \pi^{-1}\left[\sin\,(\tfrac{1}{2}+\beta)\pi\right]\frac{d}{ds}\int_1^s \psi(\sigma)\cdot(s-\sigma)^{-1/2+\beta}(\sigma^2-1)^{-\beta/2}\,d\sigma$$

where $-\tfrac{1}{2} < \beta < \tfrac{1}{2}$.

With $\cosh^{-1}s = \theta$ (1.8) becomes

$$(1.9)\qquad \int_0^\infty \cos(\theta\mu) f(\mu)\, d\mu = (2\pi)^{-\frac{1}{2}}\, \Gamma(\tfrac{1}{2}-\beta)\sin\left[(\tfrac{1}{2}+\beta)\pi\right]\cdot \frac{d}{d\theta}\int_1^{\cosh\theta} \psi(\sigma)(\cosh\theta-\sigma)^{-\frac{1}{2}+\beta}(\sigma^2-1)^{-\beta/2}\, d\sigma$$

Since the conditions of Theorem 1 and 2 are such that the inverse Fourier integral exists, it follows from (1.9)

$$(1.10)\qquad f(\mu) = (2/\pi)^{3/2}\int_0^\infty \cos\mu\theta\, \Gamma(\tfrac{1}{2}-\beta)\sin\left[(\tfrac{1}{2}+\beta)\pi\right]\cdot \frac{d}{d\theta}\int_1^{\cosh\theta} \psi(\sigma)(\cosh\theta-\sigma)^{-\frac{1}{2}+\beta}(\sigma^2-1)^{-\beta/2}\, d\sigma\, d\theta$$

Integrating (1.10) by parts (this is permitted since by Theorem 1 $\psi(\chi)$ is such that the operation is valid) gives

$$(1.11)\quad f(\mu) = (2/\pi)^{3/2}\,\Gamma(\tfrac{1}{2}-\beta)\sin\left[(\tfrac{1}{2}+\beta)\pi\right]\cdot \left[\cos \mu\theta \int_1^{\cosh\theta} \psi(\sigma)(\cosh\theta-\sigma)^{-\frac{1}{2}+\beta}\cdot (\sigma^2-1)^{-\beta/2}\right]_{\theta=0}^{\theta=\infty} + \mu\cdot \int_0^{\infty}\sin\mu\theta\int_1^{\cosh\theta}\psi(\sigma)(\cosh\theta-\sigma)^{-\frac{1}{2}+\beta}\cdot (\sigma^2-1)^{-\beta/2}\,d\sigma\,d\theta.$$

Let

$$(1.12)\quad g(\theta) = \int_1^{\cosh\theta}\psi(\sigma)(\cosh\theta-\sigma)^{-\frac{1}{2}+\beta}(\sigma^2-1)^{-\beta/2}\,d\sigma$$

Then (1.11) becomes by Theorem 1 and fact that lower limit is zero by definition of the improper integral

$$(1.14)\qquad f(\mu) = (2/\pi)^{3/2}\,\Gamma(\tfrac{1}{2}-\beta)\sin\left[(\tfrac{1}{2}+\beta)\pi\right]\mu\cdot \int_0^\infty \sin(\mu\theta) \int_1^{\cosh\theta} \psi(\sigma)(\cosh\theta-\sigma)^{\beta-\frac{1}{2}} \cdot (\sigma^2-1)^{-\frac{1}{2}\beta}\, d\sigma\, d\theta.$$

Integrating over a triangular domain and observing that the conditions of Theorem 1 are such that Fubini's theorem is valid, (1.14) becomes

$$(1.15)\qquad f(\mu) = (2/\pi)^{3/2}\,\Gamma(\tfrac{1}{2}-\beta)\sin\left[(\tfrac{1}{2}+\beta)\pi\right]\mu\cdot \int_1^\infty \psi(\sigma)(\sigma^2-1)^{-\beta/2} \int_{\cosh^{-1}}^{\infty} \sin\mu\theta \cdot (\cosh\theta-\sigma)^{-\frac{1}{2}+\beta}\, d\theta\, d\sigma.$$

With $\sigma = \cosh t$

$$(1.16)\qquad f(\mu) = (2/\pi)^{3/2}\,\Gamma(\tfrac{1}{2}-\beta)\sin\left[(\tfrac{1}{2}+\beta)\pi\right]\mu\cdot \int_1^\infty \psi(\cosh t)(\sinh t)^{-\beta+1} \int_t^\infty \sin\mu\theta \cdot (\cosh\theta - \cosh t)^{-\frac{1}{2}+\beta}\, dt.$$

But

$$(1.17)\qquad P^{\frac{1}{2}-\mu}_{\frac{1}{2}+iy}(\cosh a)\,\Gamma(\mu+iy)\,\Gamma(\mu-iy)\,\sinh \pi y \cdot (\sinh a)^{\frac{1}{2}-\mu}(2\pi)^{-\frac{1}{2}}\,\Gamma(1-\mu) = \int_a^\infty \sin xy\,(\cosh x - \cosh a)^{-\mu}\,dx.$$

$$0 < \operatorname{Re}\mu < 1 \quad (20,\ p.\ 165).$$

Let $a = t$, $y = \mu$, $\mu = \frac{1}{2}-\beta$, $x = \theta$.

Then

$$(1.18)\qquad P^{\beta}_{-\frac{1}{2}+i\mu}(\cosh t)\,\Gamma(\tfrac{1}{2}-\beta+i\mu)\,\Gamma(\tfrac{1}{2}-\beta-i\mu)\,\sinh \pi\mu \cdot (\sinh \theta)^{\beta}(2\pi)^{\frac{1}{2}}\,\Gamma(\tfrac{1}{2}+\beta) = \int_t^\infty \sin \mu\theta\,(\cosh\theta - \cosh t)^{-\frac{1}{2}+\beta}\,d\theta.$$

Also

$$(1.19)\qquad \pi^{-1}\sin\left[(\tfrac{1}{2}+\beta)\pi\right] = \left[\Gamma(\tfrac{1}{2}+\beta)\,\Gamma(\tfrac{1}{2}-\beta)\right]^{-1}$$

By (1.18) and (1.19), (1.16) becomes

$$(1.20)\qquad f(\mu) = \pi^{-1}\mu \sinh(\pi\mu)\Gamma(\tfrac{1}{2}-\beta+i\mu)\cdot$$

$$\Gamma(\tfrac{1}{2}-\beta+i\mu)\int_0^\infty P^{\beta}_{i\mu-\frac{1}{2}}(\cosh\theta)\psi(\cosh\theta)\cdot$$

$$\sinh\theta\, d\theta.$$

Here β can be any complex number such that Rhs of (1.20) is well defined.

It will be next shown how the conditions imposed on ψ of Theorem 1 came about.

From (1.12) the following lemma is used to meet the condition that $g(\theta) \to 0$ as $\theta \to \infty$.

If $\lim_{\theta\to\infty} \phi(\cosh\theta) = 0$, where $\phi(\cosh\theta) = \psi(\cosh\theta)\sinh^{1-2\beta}\theta/2$, then $\lim_{t\to\infty} h(t) = 0$, where

$$(1.21)\qquad h(t) = \int_0^t \psi(\cosh\theta)(\sinh\theta)^{1-\beta}(\cosh t -$$

$$\cosh\theta)^{-\frac{1}{2}+\beta}\, d\theta.$$

Proof: By the following transformations

1. $\cosh\theta = \sigma$, $\cosh t = s$
2. $\sigma = (s-1)w + 1$

and use of hyperbolic identifies (1.21) becomes

$$(1.22) \qquad h(t) = 2^{1-2\beta}(s-1)^{\beta} \int_0^1 \phi\left[(s-1(w+1)\right] \cdot$$
$$\left[(s-1)w+2\right]^{-\beta}(1-w)^{-\frac{1}{2}+\beta}\ w^{-\frac{1}{2}}\ dw.$$

using next a mean value theorem for improper integrals (1.22) becomes

$$(1.23) \qquad h(t) = 2^{1-2\beta}(s-1)^{\beta}\ \phi\left[(s-1)\rho+1\right] \cdot$$
$$\int_0^1 \left[(s-1)w+2\right]^{-\beta}(1-w)^{-\frac{1}{2}+\beta}\ w^{-\frac{1}{2}}\ dw,$$

where $0 < \rho < 1$. For (1.23) to exist $\frac{1}{2} > \beta > -\frac{1}{2}$ also if $0 < \beta < \frac{1}{2}$.

$1\ (s-1)w+2\ ^{-\beta} < 1(s-1)^{-\beta}w^{-\beta}$ where $s > 1$ so (1.23) becomes

$$(1.24) \qquad /h(t)/ < 2^{1-2\beta}\ /\psi[(s-1)\rho+1]/ \int_0^1 w^{-\frac{1}{2}-\beta}(1-w)^{-\frac{1}{2}+\beta}\ dw.$$

But $\int_0^1 w^{\frac{1}{2}-\beta}(1-w)^{\frac{1}{2}-\beta}\ dw$ is the beta function $B(+\frac{1}{2}-\beta,\frac{1}{2}+\beta)$

and therefore the above is valid for all β such that $B(+\frac{1}{2}-\beta,\frac{1}{2}+\beta)$ exists.

Thus $h(t) \to 0$ as $t \to \infty$ by given in lemma.

This gives rise to one of the conditions required of is Theorem 1. Going from steps (1.9) to (1.10) required that ψ be such that this is permitted. This requires that $d\mathfrak{h}/dt$ is absolutely integrable in $(0,\infty)$ and $d^2\mathfrak{h}/dt^2$ absolutely integrable in $(0,a)$, $0 < a < \infty$.

Equation (1.23) can be written as follows:

$$(1.25)\quad \mathfrak{h}(s) = 2^{1-2\beta}(s-1)^{\beta}\,\phi[(s-1)\rho+1]\cdot$$

$$[(s-1)\rho+2]^{-\beta}\int_0^1 (1-w)^{-\frac{1}{2}+\beta}\,w^{-\frac{1}{2}}\,dw,\quad 1>\rho>0$$

where again a mean value theorem for improper integrals is used, or

$$\mathfrak{h}(s) = 2^{1-2\beta}\,\phi[(s-1)\rho+1][(s-1)\rho+2]^{-\beta}(s-1)^{\beta}B,$$
$$B \equiv B(\tfrac{1}{2}, \tfrac{1}{2}+\beta)$$

So from (1.25) dh/dt, d^2h/dt^2 depends directly on $d\phi/dt$ and $d^2\phi/dt^2$ respectively. Hence assume $d\phi/dt$ is absolutely integrable in $\langle 0,\infty\rangle$ and $d^2\phi/dt^2$ is absolutely integrable in $\langle 0,a\rangle$ $a>0$, and arbitrary. Going from steps (1.9) to (1.10) is now valid.

The condition that $\lim_{t\to 0}\phi(t) = 0$, $\mathrm{Re}\,B \leq 0$ results from the following lemma:

$$\psi(\cosh \theta)(\sinh \theta/2)^{1-\beta} = O(\theta), \quad \theta \text{ near } 0$$

where

$$\psi(\cosh \theta) = \int_0^{\infty} P^{\beta}_{-\frac{1}{2}+i\mu}(\cosh \theta) f(\mu)\, d\mu.$$

Using (1.2) and facts

$$(\cosh \theta - \cosh t)^{-\frac{1}{2}-\beta} \leq (\cosh \theta - \cosh t)^{-\frac{1}{2}}$$

if $\beta \leq 0$.

θ is such that $\cosh \theta < 2$, $\theta > 0$, using also

$$(1.28) \qquad P_{-\frac{1}{2}}(\cosh \theta) = (2/\pi)^{\frac{1}{2}} \int_0^{\theta} (\cosh \theta - \cosh t)^{-\frac{1}{2}} dt =$$

$$(\tfrac{1}{2}\pi)^{-1} \cosh(\theta/2)^{-1} K(\tanh \theta/2)$$

(8, p. 173)

$K(z)$ complete elliptic integral of first kind and

$(\cosh \theta/2)^{-1} < 2^{-1}\theta(\sinh \theta/2)^{-1}$ gives

$$\left| P^{\beta}_{-\frac{1}{2}+i\mu}(\cosh \theta) \right| < \left| \Gamma(\tfrac{1}{2}-\beta)\, \theta\, (\sinh \theta/2)^{\beta-1} \right|$$

This proves the lemma.

The condition that $f(\mu) = 0$, $\mu = 0$ results from observing formula (2). Reversing the steps to obtain (1) requires that $f(\mu)$ meet the conditions of Fourier's inversion theorem (see steps (1.10) and (1.9)).

The integrability condition for $f(\mu)$ is due to asymptotic behavior of the kernel (the Legendre Function of the first kind) This completes the proof of Theorems 1 and 2.

As an example for the transforms (1) and (2), the transform of a ψ will be computed to give a f. Then the transform of this f will be computed to recover ψ.

Let

$$(2.1)\quad \psi(\cosh\theta) = \sin a\,(\cosh\theta + \cos a)^{-3/2+\beta}(\sinh\theta)^{-\beta}$$

where $-\pi < a < \pi$, Re $\beta < \frac{1}{2}$. (2.1) meets the conditions of Theorem 1.

Inserting (2.1) into (1) and using (1.3) gives

$$(2.3)\quad f(\mu) = \pi^{-1}\mu\,\sinh(\pi\mu)\,\Gamma(\tfrac{1}{2}-\beta+i\mu)\,\Gamma(\tfrac{1}{2}-\beta-i\mu)\cdot$$

$$\int_0^\infty (\pi/2)^{-\frac{1}{2}}[\Gamma(\tfrac{1}{2}-\beta)]^{-1}\int_0^\infty \cos\mu t\,\cdot$$

$$(\cosh\theta - \cosh t)^{-\frac{1}{2}-\beta}\sin a\,\cdot$$

$$(\cosh\theta + \cos a)^{-3/2+\beta}\sin\theta\,(\sinh\theta)^{-\beta}\,d\theta.$$

Rewriting (2.3)

$$(2.4)\quad f(\mu) = \pi^{-1}\mu \ \sinh(\pi\mu)\,\Gamma(\tfrac{1}{2}-\beta+i\mu)\,\Gamma(\tfrac{1}{2}-\beta-i\mu)\cdot$$

$$\int_0^\infty (\pi/2)^{-1/2}\,[\Gamma(\tfrac{1}{2}-\beta)]^{-1} \cos\mu t \ \cdot$$

$$\int_t^\infty \sin a \sinh\theta\,(\cosh\theta - \cosh t)^{-1/2-\beta}\ \cdot$$

$$(\cosh\theta + \cos a)^{-3/2+\beta}\, d\theta\, dt.$$

Let $\cosh\theta - \cosh t = (\cosh t + \cos a)\,z$, (2.4) becomes

$$f(\mu) = \pi^{-1} \ \sinh(\pi\mu)\,\Gamma(\tfrac{1}{2}-\beta+i\mu)\,\Gamma(\tfrac{1}{2}-\beta-i\mu)\cdot$$

$$(\pi/2)^{-\frac{1}{2}}\int_0^\infty \cos(\mu t) \sin a\,(\cosh t + \cos a)^{-1}\, dt\ \cdot$$

$$\int_0^\infty z^{-1/2-\beta}\,(1+z)^{-3/2+\beta}\, dz.$$

Both integrals are known.

$$(2.6)\quad f(\mu) = 2^{1/2}\pi^{-3/2}\pi\,[\Gamma(\tfrac{1}{2}-\beta)]^{-1}\,\Gamma(\tfrac{1}{2}-\beta+i\mu)\ \cdot$$

$$\Gamma(\tfrac{1}{2}-\beta-i\mu)\,B(\tfrac{1}{2}-\beta,1)\sinh a\mu$$

(2.6) is the transform of (2.1). For $\mu \gg 1$

$$f(\mu) \sim \mu\, e^{a\mu}\ \mu^{1-2\beta} e^{-\pi\mu} \quad \text{, but since}$$

$-\pi < a < \pi$

$$\int_0^\infty |f(\mu)|\, d\mu < \infty \quad , \quad \int_0^c |f'(\mu)|\, d\mu < \infty \qquad \text{for all } c$$

such that $0 < c < \infty$. So $f(\mu)$ meets requirements of Theorem 2. Now taking (2.6) and computing the inverse transform of (2.6), (2.1) will be obtained.

Using (1.1) and (1.17) gives

$$(2.7)\quad \psi(\cosh\theta) = \int_0^\infty (2\pi)^{1/2}\left[\Gamma(\tfrac{1}{2}+\beta)\,\Gamma(\tfrac{1}{2}-\beta+i\mu)\,\cdot\right.$$

$$\left.\Gamma(\tfrac{1}{2}-\beta-i\mu)\,\sinh(\pi\mu)(\sinh\theta)^{\beta}\right]^{-1} \cdot$$

$$\int_\theta^\infty \sin(\mu t)(\cosh t - \cosh\theta)^{-1/2+\beta}\, dt \,\cdot$$

$$\pi^{-1}\pi\mu\,\Gamma(\tfrac{1}{2}-\beta+i\mu)\,\Gamma(\tfrac{1}{2}-\beta-i\mu)(\pi/2)^{-1/2} \cdot$$

$$\left[\Gamma(\tfrac{1}{2}-\beta)\right]^{-1} B(\tfrac{1}{2}-\beta,1)\sinh a\mu \ \ d\mu.$$

Rewriting (2.7)

$$(2.8)\quad \psi(\cosh\theta) = \pi(\pi/2)^{-1/2}B(1/2-\beta,1)\cdot$$

$$\pi^{-1}[(\sinh\theta)^{\beta}(2\pi)^{-1/2}\Gamma(1/2+\beta)\Gamma(1/2-\beta)]^{-1}\cdot$$

$$\int_{\theta}^{\infty} 1(\cosh t - \cosh\theta)^{-1/2+\beta}\cdot$$

$$\int_{0}^{\infty}\sin(\mu t)\mu\ \sinh(a\mu)\ (\sinh\pi\mu)^{-1}\,d\mu\ dt.$$

But

$$(2.9)\quad \int_{0}^{\infty}\sin(\mu t)\sinh(a\mu)\mu(\sinh\pi\mu)^{-1}\,d\mu =$$

$$1/2 \sin a \sinh t\ (\cos a + \cosh t)^{-2}$$

Proof of (2.9) thus gives

$$(2.10)\quad \psi(\cosh\theta) = \pi\pi^{-1}(\pi/2)^{-1/2}B(1/2-\beta,1)\cdot$$

$$[(\sinh\theta)^{\beta}(2\pi)^{-1/2}\Gamma(1/2+\beta)\Gamma(1/2-\beta)]^{-1}\cdot$$

$$\int_{\theta}^{\infty}(\cosh t - \cosh\theta)^{-1/2+\beta}\cdot$$

$$1/2 \sin a \sinh t\ (\cos a + \cosh t)^{-2}\,dt$$

With the substitution $\cosh t - \cosh\theta = (\cos a + \cosh\theta)z$ (2.10) becomes

$$(2.11)\qquad \psi(\cosh\theta) = \pi^{-1}(\pi/2)^{-1/2}B(1/2-\beta,1)\pi \cdot$$

$$\left[(\sinh\theta)^{\beta}(2\pi)^{-1/2}\Gamma(1/2+\beta)\Gamma(1/2-\beta)\right]^{-1} \cdot$$

$$1/2\int_{0}^{\infty} z^{1/2-1-\beta}(\cos a + \cosh\theta)^{-1/2+\beta} \cdot$$

$$\sin a\,(\cos a + \cosh\theta)(\cos a + \cosh\theta)^{-2} \cdot$$

$$(1+z)^{-2}\,dz.$$

(2.11) may also be

$$(2.12)\qquad \psi(\cosh\theta) = \pi^{-1}(\pi/2)^{-1/2}B(1/2-\beta,1)\pi \cdot$$

$$\left[(\sinh\theta)^{\beta}(2\pi)^{-1/2}\Gamma(1/2+\beta)\Gamma(1/2-\beta)\right]^{-1} \cdot$$

$$\sin a\,(\cos a + \cosh\theta)^{-3/2+\beta} \cdot$$

$$1/2\int_{0}^{\infty} z^{-1/2+\beta}\,(1+z)^{-2}\,dz.$$

But

$$(2.13)\qquad \int_{0}^{\infty} z^{1/2-1+\beta}\,(1+z)^{-2}\,dz = B(1/2+\beta,3/2-\beta)$$

and

$$(2.14)\qquad B(1/2+\beta,3/2-\beta) = \left[\Gamma(2)^{-1}\right]\Gamma(1/2+\beta)\,\Gamma(3/2-\beta)$$

$$(2.15)\qquad B(1/2-\beta,1) = \Gamma(1)\,\Gamma(1/2-\beta)\,\left[\Gamma(3/2-\beta)\right]^{-1}$$

Then (2.12) becomes

$$(2.16) \qquad \psi(\cosh \theta) = \sin a \, (\cos a + \cosh \theta)^{-3/2 + \beta} \cdot (\sinh \theta)^{-\beta}$$

(2.16) is (2.1) as was to be shown.

3. THE GENERALIZED SPHERICAL WAVE REPRESENTED AS A GENERALIZED MEHLER TRANSFORM

A basic solution of the modified wave equation $\Delta f - c^2 f = 0$ in $2n + 2$ dimensions is (17)

$$(3.1)\quad f = R^{-n} K_n(cR)$$

Here $K_n(cR)$ is the modified Hankel function of order n (9, Ch. 7) where R is defined

$$(3.2)\quad R = \sqrt{a^2 + b^2 - 2ab\cos A}$$

For the two or three dimensional space ($n = 0$ or $n = ½$ respectively) f reduces to a cylindrical wave $K_o(cR)$ or to a spherical wave $R^{-½} \exp(-cR)$ respectively. An expression for (3.1) has been given by Gegenbauer (9, p. 43) in the form of a series representation known as the (Gegenbauer) addition theorem of the modified Hankel function

$$(3.3)\quad R^{-v}K_v(R) = (½ab)^{-v}\,\Gamma(v)\cdot\sum_{n=0}^{\infty}(v+n)C_n^v(\cos A)\cdot I_{v+n}(a)K_{v+n}(b)$$

$$a < b$$

and for $a > b$ the same formula with a and b interchanged.

It will be shown later that instead of (3.3) the following relation can be proved.

$$(3.4) \quad (a^2 + b^2 - 2ab \cos A)^{-\frac{1}{2}v} K_v\left[(a^2 + b^2 - 2ab \cos A)^{\frac{1}{2}}\right] =$$

$$2^{\frac{1}{2}}\pi^{-3/2}(ab)^{-v} (\sin A)^{\frac{1}{2}-v} \int_0^\infty x \sinh (\pi x) \cdot$$

$$\Gamma(v+ix)\Gamma(v-ix) K_{ix}(a) K_{ix}(b) \cdot$$

$$\underline{P}^{\frac{1}{2}-v}_{-\frac{1}{2}+ix} (-\cos A) \, dx,$$

$$\text{Re } v \geqq 0, \; 0 < \text{Re } A < 2\pi.$$

Here $\underline{P}^{\frac{1}{2}-v}_{-\frac{1}{2}+ix}(-\cos A)$ is the Legendre function "on the cut" (see Appendix c) (8, p. 143) (note that (3.4) is symmetrical in a and b which (3.3) is not). Put $A = \pi - B$ and get

$$(3.5) \quad (a^2 + b^2 + 2ab \cos B)^{-\frac{1}{2}v} K_v\left[(a^2 + b^2 + 2ab \cos B)^{\frac{1}{2}}\right] =$$

$$2^{\frac{1}{2}}\pi^{-3/2}(ab)^{-v} (\sin B)^{\frac{1}{2}-v} \int_0^\infty x \sinh (\pi x) \cdot$$

$$\Gamma(v+ix)\Gamma(v-ix) K_{ix}(a) K_{ix}(b) \cdot$$

$$\underline{P}^{\frac{1}{2}-v}_{-\frac{1}{2}+ix} (\cos B) \, dx,$$

$$\text{Re } v > -1, \; -\pi \leqq \text{Re } B \leqq \pi.$$

It follows from (8, p. 122, equation (7)) that

$$(z^2-1)^{\frac{1}{2}\mu} \; P_v^{\mu}(z) = \frac{2^{\mu}}{\Gamma(1-\mu)} F(1-\mu+v,\ -\mu-v,\ 1-\mu,\ \tfrac{1}{2}-\tfrac{1}{2}z),$$

is a one valued function of z in the complex z plane cut along the real z axis from $+1$ to $-\infty$. If therefore we choose in (3.5) $\cos B = z$, we can write

$$(1-z^2)^{\frac{1}{2}\mu} \; \underline{P}_v^{\mu}(z) = (z^2-1)^{\frac{1}{2}\mu} \; P_v^{\mu}(z) \quad \text{in case } z > 1.$$

We obtain then instead of (3.5)

$$(3.6) \quad (a^2+b^2+2abz)^{-\frac{1}{2}v} \, K_v\left[(a^2+b^2+2abz)^{\frac{1}{2}}\right] =$$

$$2^{\frac{1}{2}}\pi^{-3/2} \, (ab)^{-v} \, (z^2-1)^{\frac{1}{4}-\frac{1}{2}v} \int_0^{\infty} x \sinh(\pi x) \cdot$$

$$\Gamma(v+ix)\,\Gamma(v-ix) \, K_{ix}(a) \, K_{ix}(b) \cdot$$

$$P_{-\frac{1}{2}+ix}^{\frac{1}{2}-v}(z) \, dx,$$

$$z > 1, \ \mathrm{Re}\ v > -1$$

Obviously the integral in (3.6) is of the type of a Mehler transform (0.3). But the integral in (3.6) can also be regarded as a Lebedev transform (9. p. 75).

Special Cases

The expression (3.4) can be used to obtain expressions for the cylindrical ($v = 0$) and spherical wave ($v = \frac{1}{2}$) in the form of an integral expression.

(a) Cylindrical wave ($v = 0$)

Since (8, p. 150)

$$P_v^{\frac{1}{2}}(\cos\theta) = (\tfrac{1}{2}\pi)^{-\frac{1}{2}}(\sin\theta)^{-\frac{1}{2}}\cos\left[(v+\tfrac{1}{2})\theta\right],$$

one obtains from (3.4)

$$(3.7)\quad K_0\left[(a^2+b^2-2ab\cos A)^{\frac{1}{2}}\right] = \frac{2}{\pi}\int_0^\infty K_{ix}(a)\,K_{ix}(b)\cosh\left[x(\pi-A)\right]dx.$$

(b) Spherical wave ($v = \frac{1}{2}$)

Since (9, p. 10)

$$K_{\frac{1}{2}}(z) = (\tfrac{1}{2}\pi/2)^{\frac{1}{2}}\,e^{-z},$$

one obtains

$$(3.8)\quad (a^2+b^2-2ab\cos A)^{-\frac{1}{2}}\,e^{-(a^2+b^2-2ab\cos A)^{\frac{1}{2}}} = \frac{2}{\pi}(ab)^{-\frac{1}{2}}\int_0^\infty x\,K_{ix}(a)\,K_{ix}(b)\tanh(\pi x)\cdot P_{-\frac{1}{2}+ix}(-\cos A)\,dx.$$

The expressions (3.7) and (3.8) were previously known (9, p. 55).

If we apply the inversion formula (0.3) for (3.6) we obtain

$$(3.9)\quad K_{ix}(a)\,K_{ix}(b) = (\pi/2)^{\frac{1}{2}}\,(ab)^{v}\int_{1}^{\infty}(y^2-1)^{\frac{1}{2}v-\frac{1}{4}}\cdot$$

$$(a^2+b^2+2aby)^{-\frac{1}{2}v}\,K_v\left[(a^2+b^2+2aby)^{\frac{1}{2}}\right]\cdot$$

$$P^{\frac{1}{2}-v}_{-\frac{1}{2}+ix}(y)\,dy$$

This formula is valid for Re $v > -\frac{1}{2}$ (8, p. 163), and seems to be new. (For formulas of this type see (6 and 7).)

The special case $v = 0$ gives

$$(3.10)\quad K_{ix}(a)\,K_{ix}(b) = \int_{0}^{\infty}K_{v=0}\left[(a^2+b^2+2ab\cosh t)^{\frac{1}{2}}\right]\cos(xt)\,dt$$

observing that (8, p. 150)

$$P^{\frac{1}{2}}_{v}(\cosh t) = (\tfrac{1}{2}\pi)^{-\frac{1}{2}}(\sinh t)^{-\frac{1}{2}}\cosh\left[(v+\tfrac{1}{2})t\right]$$

The case $v = \frac{1}{2}$ gives

$$(4.0)\quad K_{ix}(a)\,K_{ix}(b) = \tfrac{1}{2}\pi\,(ab)^{\frac{1}{2}}\cdot$$

$$\int_{1}^{\infty}(a^2+b^2+2abt)^{-\frac{1}{2}}\exp\left[-(a^2+b^2+2abt)^{\frac{1}{2}}\right]\cdot$$

$$P_{-\frac{1}{2}+ix}(t)\,dt$$

Equivalent with (3.4) is

$$(3.11)\quad (a^2+b^2-2ab\cos A)^{-\frac{1}{2}v} K_v\left[(a^2+b^2-2ab\cos A)^{\frac{1}{2}}\right] =$$

$$-2^{v-1}\Gamma(v)(ab)^{-v}\int_{-\infty}^{\infty} x\cosh(\pi x - i\pi v)\cdot$$

$$C^{v}_{-v-ix}(-\cos A)\, I_{-ix}(a)\, K_{ix}(b)\, dx,$$

$$a < b,\ \mathrm{Re}\, v > 0.$$

Here $C^{v}_{-v-ix}(-\cos A)$ is the Gegenbauer function (8, p. 175)

$$(3.12)\quad C^{v}_{-v-ix}(-\cos A) = \pi^{\frac{1}{2}}[\Gamma(v)]^{-1}(\sin A)^{\frac{1}{2}-v}\cdot$$

$$\frac{\Gamma(v-ix)}{\Gamma(1-v-ix)} P^{\frac{1}{2}-v}_{-\frac{1}{2}-ix}(-\cos A)$$

Inserting this into (4.1) and using the relation (9, p. 5)

$$I_{-ix}(a) - I_{ix}(a) = i\, 2\pi^{-1}\sinh(\pi x)\, K_{ix}(a)$$

one finds (3.4).

<u>Proof of (3.11)</u>.

Consider the integral in (3.11) taken over a closed contour in the complex x plane consisting of the real axis between $-R$ and $+R$ and the semicircle of radius R in the upper half plane.

Choose R such, that the semicircle separates the consecutive poles $x_\mu = i(v+\mu)$ and $x_{\mu+1} = i(v+\mu+1)$ of the integrand. The residue of the integrand of (3.11) at a pole $x_\mu = i(v+\mu)$ is equal to

$$i(-1)^\mu \pi^{-1} (v+\mu) C_\mu^v(-\cos A) I_{v+\mu}(a) K_{v+\mu}(b)$$

$$= 0, 1, 2, .$$

The contribution of the integration along the semicircle tends to zero when $R \to \infty$, provided $a > b$. This follows from the behavior of the integrand in (3.11) on the semicircle for large values of R (see Appendix a). We obtain therefore by the residue theorem from (3.11)

$$(a^2+b^2-2ab\cos A)^{-\frac{1}{2}v} K_v\left[(a^2+b^2-2ab\cos A)^{\frac{1}{2}}\right] =$$

$$2^v (ab)^{-v} \Gamma(v) \cdot \sum_{\mu=0}^{\infty} (-1)^\mu (v+\mu) C_\mu^v(-\cos A) \cdot$$

$$I_{v+\mu}(a) K_{v+\mu}(b).$$

But (8, p. 176) $C_\mu^v(-\cos A) = (-1)^\mu C_\mu^v(\cos A)$. The Rhs (right hand side) of the last expression is equal to the Rhs of (3.3).

This proves (3.4) and (3.11).

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APPENDIX

(a) In order to show that the integral in (3.11) taken along a semicircle of radius R in the upper x-half plane tends to zero as $R \to \infty$, it is necessary to investigate the integrand of (3.11) for $x = R\, e^{i\varphi}$, $0 \leq \varphi \leq \pi$

From Stirling's formula (8, p. 47) and the definition of the modified Bessel functions (9, p. 5) one obtains for fixed z and large (complex) μ,

$$I_{\mu}(z) = (2\pi\mu)^{-\frac{1}{2}} (z/2)^{\mu}\, e^{-\mu(\log\mu - 1)} [1 + O(1/\mu)]$$

$$I_{-\mu}(z) = \sin(\pi\mu)(\tfrac{1}{2}\pi\mu)^{-\frac{1}{2}} (z/2)^{-\mu}\, e^{\mu(\log\mu - 1)} \cdot [1 + O(1/\mu)]$$

both expressions valid in $-\pi < \arg\mu < \pi$ and in the second expression μ not an integer. Since

$$K_{\mu}(z) = \frac{\pi}{2\sin(\pi\mu)} [I_{-\mu}(z) - I_{\mu}(z)],$$

it follows that (after substituting $x = i\mu$ in (3.11))

$$I_\mu(a)\, K_\mu(b) = O[\mu^{-1}\,(a/b)^{-\mu}]$$

for large μ in the right half plane (corresponding to the upper x half plane).

Furthermore, from (8, p. 147, formula (5)) together with Stirling's formula

$$P_\lambda^\beta(\cos\varphi) = (\tfrac{1}{2}\pi \sin\varphi)^{-\frac{1}{2}}\, \lambda^{\beta-\frac{1}{2}} \cdot$$

$$\sin[(\lambda + \tfrac{1}{2})\varphi + \tfrac{1}{2}\pi\beta + \tfrac{1}{4}\pi]\{1+O(1/\lambda)\}$$

for large λ in $-\pi < \arg\lambda < \pi$.

Therefore, because of (3.12) with $x = i\mu$

$$C^{v}_{-v+\mu}(-\cos A) = \frac{2^{1-v}}{\Gamma(v)}(\sin A)^{-v}\,\mu^{v-1} \cdot$$

$$\cos[\mu A + \tfrac{1}{2}\pi\mu]\;\{1+O(1/\mu)\}$$

for large μ in $-\pi < \arg\mu < \pi$.

The integrand in (3.11) is therefore equal to

$$O[\mu^{v-1}\,(a/b)^{-\mu}]$$

(b) Behavior of $P^{\mu}_{-\frac{1}{2}+ix}(z)$ for $x \gg 1$ and $z > 1$.

It follows from (8, p. 129, formula (26)) that

$$P^{\mu}_{-\frac{1}{2}+ix}(z) = O(x^{\mu-\frac{1}{2}})$$

(c) Definition of the Legendre function (8).

$$P_v^{\mu}(z) = \frac{1}{(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\frac{1}{2}} F\left(-v,\ v+1;\ 1-\mu;\ \frac{1-z}{2}\right)$$

z not a point on the real z axis between 1 and $-\infty$

$$\underline{P}_v^{\mu}(x) = \frac{1}{(1-\mu)} \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} F(-v,\ v+1;\ 1-\mu;\ \frac{1-x}{2}$$