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This study investigated exemplary middle school mathematics teachers' knowledge of students' thinking with algebraic word problem solving and how they used this knowledge and understanding in their planning and teaching of students to solve algebraic word problems.

Initially, a questionnaire was distributed to nine teachers to gather their interpretation of students' thinking with particular algebraic word problems. From the analysis, four teachers participated in follow-up interviews and classroom observation to identify two teachers with the clearest strength in understanding students' mathematical thinking with algebraic word problems and the ability to clearly describe how they used this knowledge for planning to teach units in this area. Upon identification, extended classroom observations led to the development of in-depth cases where teachers were asked to describe their thinking about their planning through stimulated recall interviews.

While these two teachers demonstrated a clear understanding of students' solution strategies for algebraic word problems, they each displayed unique ways of using their understandings in planning and teaching. In planning and teaching their lessons, they carefully chose problems amenable to multiple approaches and solutions. They relied on outlines from their previous teaching experience which they modified mentally in relying on students' mathematical thinking. They noted their attention to student strategies enabled them to plan their lessons effectively. They used questioning to extract students' thinking and worked to support students in taking risks. Their consistent strategies included: (1) extending time to help students unpack the algebraic word problems; (2) using students' ideas to describe solution strategies; (3) relying on questioning to gather students' ideas; (4) using students' strategies to summarize the lessons. While these points suggest a student-centered instructional approach, the two teachers differed in this respect with one teacher using primarily a teacher-leaded whole class questioning approach and the other teacher using a student-centered questioning in small group work. Both approaches appeared to engage students in successfully working with the problems.

Because knowledge of students' mathematical thinking helps teachers prepare and teach their lessons, teacher preparation programs are encouraged to provide multiple experiences in analyzing student thinking with word problems in preparation for designing their lessons.

# Teacher's Knowledge of Middle School Students' Mathematical Thinking in Algebra Word Problem Solving

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I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my dissertation to any reader upon request.

Kwangho Lee, Author

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## Chapter I

## The Problem

### Introduction

Knowledge of people's thinking provides opportunities to challenge and modify their ways of thinking. Knowledge of student thinking helps teachers construct lessons that engage students in comprehending new ideas. uit Beijerse (2000) states that knowledge is "the amount of information necessary to function and achieve goals; the capacity to make information from data and to transform it into useful and meaningful information; the capacity with which one thinks creatively, interprets and acts; and an attitude that makes people want to think, interpret and act." This definition of knowledge alone describes the importance of the role of guiding students' thinking toward comprehending new ideas.

In fact, a growing body of research explains that teachers' knowledge and teaching skills are essential for improving students' performance (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Loef & Kazemi, 2001; Staub & Stern, 2002). Naturally, what teachers know and can do directly affects the quality of students' learning (Sparks & Hirsh, 2000). Teachers who are knowledgeable of the behavior of their students have more flexibility, capacity and creativity in constructing lessons and tasks that meet student learning needs. Such actions, in turn, promote effectiveness and quality in education.

Teachers' knowledge of students' thinking plays an important role in planning effective lessons (Maher, Davis & Alston, 1992). Understanding students and their capabilities enables teachers to determine the lessons' levels of difficulty, as well as the nature of the instruction that is needed. Furthermore, the National Council of Teachers of Mathematics (NCTM, 1991, 2000) has proposed that knowledge of students' mathematical understanding and ways of thinking helps teachers construct worthwhile mathematical tasks. In support of NCTM, Carpenter, Fennema, and Franke (1996) also claimed that teachers who understand children's thinking are in a better position to craft more effective mathematics instruction.

Little information currently exists that describes the impact of teachers' knowledge of students' thinking in the development of their instruction. For example, Brophy and Good (1974) found that teachers use their knowledge of students' prior academic success, family background and personality traits to set realistic educational goals, plan appropriate learning activities and provide for individual student needs in the classroom. Such studies have focused on developing a generic description, rather than a subject specific description of teachers' knowledge of students' understandings. While Carpenter and his colleagues focused on teachers' understanding of students' mathematical understanding, their studies were limited to grades K-3 (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). Also, Ball's studies on content knowledge have led to the discovery that teaching can be more effective when applied with effective pedagogical knowledge (Ball, 2000; Ball, & Wilson, 1990; Ball, &

Wilcox, 1989). Moreover, most of Ball's studies have found that content knowledge alone is incomplete for teaching mathematics. While Ball's studies have relied on the framework of the importance of pedagogical content knowledge (PCK) for teaching, her studies have focused primarily on teachers' content knowledge for teaching mathematics.

## Pedagogical Content Knowledge

Knowledge of the subject matter has been traditionally a requirement in mathematical teaching, but the exploration of the need to consider the pedagogy for teaching mathematics has led to the recognition and expansion of pedagogical content knowledge (PCK). Past studies by Ball (1989, 1990, 2000) argued and presented the facts about the importance of content knowledge in teaching – the conceptual understanding of principles and meanings within the subject. Early in her research, Ball (1990) argued that the subject matter knowledge of teachers needed to be the focus of educational reforms. She recognized that teachers' content knowledge provided students with a clearer delivery, but she found that content knowledge alone was not enough. Teachers also needed to know about students' mathematical thinking in order to help them gain a better understanding of mathematics.

Shulman (1986) proposed the concept of pedagogical content knowledge, in which teachers apply a multiple dimensional knowledge to their teaching that can be described by:

- knowledge of how to structure and represent academic content for direct teaching of students;
- knowledge of the common conceptions, misconceptions, and difficulties that students encounter when learning particular content; and
- knowledge of the specific teaching strategies that can be used to address students' learning needs in particular classroom circumstances.

By focusing on different factors, researchers have enhanced the understanding of PCK. Ball and Wilson (1990) asked respondents how they would deal with pedagogical situations such as suggestions by students, their inquiry, and presentation of errors in a paper form. The respondents stated that they would respond directly to students (i.e., answering questions directly and showing them what to do). Based on their findings, Ball and Wilson concluded that teachers who themselves relied on a procedural knowledge of mathematics were not equipped to represent mathematical ideas to students in ways that connected students' prior and current knowledge and the mathematics they were to learn.

Similarly, Ball and Wilcox (1989) illustrated an approach for analyzing the content and contexts of inservice teacher education programs. Two inservice programs in mathematics for elementary teachers were compared. The context dealt with factors of organization, surroundings, time, and timing, each contributing to shaping the conditions for the teachers' learning; content focused on the programs' goal of teaching mathematics for understanding – basically what each program meant by this goal, and what they thought teachers needed to know in order to teach for understanding, as well as their assumptions about how teachers learned. The researchers found that both factors were important and needed further investigation. In addition to content and context, researchers have also emphasized teachers' knowledge of students' thinking.

As the research developed, researchers have recognized the importance of pedagogy in their investigations with teachers' PCK. However, they have extended their recognition of pedagogy beyond the work of generic pedagogical knowledge to subject specific pedagogical knowledge that supports teachers in guiding student thinking in mathematics. This research has been directed more specifically at teachers' knowledge of student thinking - one of the most important factors in PCK.

## Word Problem Solving in Algebra

For most students, problem solving has been identified as one of the most difficult processes to learn in mathematics (Manes, 1996; Schoenberger & Liming, 2001). Teachers have described mathematics word problem solving as the most frustrating and difficult type of problem solving (Kameenui & Griffin, 1989). Benko, et al. (1999) and Jitendra, et al. (1998) have stated that in general, student failures in word problem solving are caused by problems such as: the inability to read story problems adequately; low reading ability in general; improper reading strategies when faced with a story problem; incorrect use of problem solving strategies; lack of effort in understanding the mathematical logic of the problem; memorization and application of solutions without problem comprehension; and lack of sufficient time for solving the problem or executing the solution. They further indicated that all these factors are more likely to be addressed if the teachers know that such problems exist.

Furthermore, researchers have agreed that algebra is a major stumbling block for many students in middle and high school (Hart, 1981; Carpenter, Corbitt, Kepner, Linquist, & Reys, 1981). Kieran and Wagner (1989) identified two arguments for the difficulty of learning algebra: poor teaching and the students themselves. The first argument might involve the failure of providing proper instruction, assisting students to understand the variables, and understanding and knowing the students' weaknesses and strengths in problem solving. On the other hand, the second argument perhaps stemmed from the limitations imposed by the learning capabilities of the students. These problems have been viewed as intertwined because in order to improve the knowledge of the students or knowledge of their problems in learning algebra, the teacher must first know which area of students' comprehension must be targeted for improvement.

With the difficulties of learning algebra and solving word problems, particularly the complex learning and the barriers to learning, a need has existed to understand the students' thinking, specifically knowing their strengths and weakness in learning, and assuring that they are learning the right information. For this reason, the teacher's knowledge of student thinking is an important factor in algebraic learning.

Teaching and Knowledge of Student Thinking

Teaching and learning mathematics with understanding requires knowledge about the subject matter and involves fundamental mental activities such as: constructing relationships; extending and applying knowledge; reflecting about experiences; articulating what one knows; and making knowledge one's own (Carpenter & Lehler, 1999). Capraro, Kulm and Capraro (2002) identified that activities that build students' prior understanding of mathematics and promote students' thinking and reasoning about mathematical concepts are important for building their understanding. Students' misconceptions are included among the issues of understanding student thinking. Basically, some researchers found that misconceptions can arise if students have a narrow conception of the ideas or procedures that cannot be extended to other situations. According to this view, teachers who understand students' knowledge and thinking are able to use such knowledge to improve the quality of their instruction. Some of the strategies found in identifying and addressing students' thinking or prior knowledge are determined by how they perceive the difference between two solutions to an exercise or problem, and how students extend procedures to other contexts and situations. Knowing how students perceive the difference between two solutions to a problem provides insights into students' understanding, thus, knowing how they extend procedures to problems provides insights of student misconceptions or lack of understanding (Carparo, Kulm & Capraro, 2002).

## Research Problem

In the mathematics field, a teacher's knowledge may be characterized according to cognitively guided instruction (CGI), originating from the theories developed by Shulman about pedagogical content knowledge and pedagogical reasoning. Bright and Vacc (1994) stated: "CGI is an approach to teaching mathematics in which children's knowledge is central to instructional decision making" (p. 3). As with the pedagogical content knowledge and reasoning theory, teachers use research-based knowledge about students' mathematical thinking to help them learn specifics about an individual student's performance level. Teachers using CGI indicate that this strategy fits their own teaching styles, knowledge base, beliefs and students. Overall, Carpenter, Fennema, Franke, Levi, and Empson (2000) have indicated that CGI focuses on:

(1) the development of students' mathematical thinking and designing instruction to influence that development;

(2) teachers' knowledge and beliefs that influence their instructional practice; and

(3) the way teachers' knowledge, beliefs and practices are influenced by their understanding of students' mathematical thinking.

Furthermore, these researchers found that even though teachers had a great deal of intuitive knowledge about children's mathematical thinking, their knowledge was fragmented and did not play an important role in most of their decision-making. From the perspective of these researchers, a coherent basis is needed to be able to effectively plan instructions based on students' thinking.

Expanding on their results, the researchers have indicated that teachers' pedagogical content knowledge focuses not only on developing knowledge of the subject matter to improve their teaching, but also on their knowledge of students' interests, capabilities and other factors useful in the development of content to which students can relate and grasp according to their own capacity. This teacher knowledge is, however, considered to be a fragmented knowledgebase.

The purpose of this study was to explore this topic by developing a framework that hypothesized a relationship between teachers' knowledge of students' mathematical thinking in learning algebra and their active teaching. The specific research questions guiding this study are:

1. What is the nature of teachers' knowledge of middle school students' mathematical thinking about algebraic word problem solving?
2. How do teachers use their knowledge of students' mathematical thinking when planning lessons for guiding their students in solving algebraic word problems?
3. How do teachers use their knowledge of students' mathematical thinking when actively teaching students to solve algebraic word problems?

## Significance of the Study

This study sought to identify and elucidate middle school mathematics teachers' knowledge of the mathematical thinking of their students in solving algebraic word problems and how they used that knowledge in their planning and teaching of the students. The significance of this study was that these findings have the potential to contribute to the PCK research by demonstrating ways in which teachers combine their content knowledge with their knowledge of students' thinking. The study led to a description of the effort that middle school mathematics teachers make using knowledge about their students to improve their teaching and to ensure that learning is effective. Identifying teacher knowledge illuminates areas for teacher improvement, as well as the rationale for specific improvements. The results of this study also suggest that mathematics teachers should make instructional decisions based on their understanding of the students' mathematical thinking, and thus provides a rationale for teachers to align their mathematical instruction with student thinking.

This study proposed to investigate the research questions by focusing on the work of exemplary middle school mathematics teachers as they guided their students' learning with algebraic word problem solving. An exploration of exemplary teachers' knowledge of students' thinking contributes to the improvement of the preparation of preservice teachers. Several studies have focused on the importance of a teacher's knowledge of subject matter, but only a few have focused on the importance of teacher's knowledge of students' thinking. This study helps to identify specific ways to improve preservice teachers' teaching and learning experiences in the classroom.

Furthermore, this study provides insights to teachers who wish to improve their algebraic word problem instruction or those who wish to explore the effectiveness of gaining knowledge about students' mathematical thinking.

# Chapter II

# Review of the Literature

## Introduction

This study focused on understanding middle school mathematics teachers' knowledge of students' mathematical thinking about algebra word problem solving. The intent was to develop a framework and hypothesis that emphasized the importance of teachers' knowledge of students' mathematical thinking in learning algebra to their instruction of these students. The literature review is divided into two parts. The first part presents the theoretical framework and conceptual framework for the study. The theoretical framework is based on the concept of pedagogical content knowledge (PCK) (Shulman, 1986) and the conceptual framework is adapted from the study of An, Kulm, and Wu (2004). Using Shulman's work, the PCK discussion has focused on pedagogy to help the reader's understanding of the importance of pedagogical factors in PCK aside from the content explained through other research. In this discussion, pedagogical reasoning is also described to stress the process of considering specific pedagogies in teaching. Overall, what is stressed in the theoretical framework section is that understanding students' thinking is as important as understanding the subject of algebra in teaching. In addition, the conceptual framework explains how the theories are incorporated into the topic of this study. The conceptual design advanced by this study is based on the PCK and pedagogical reasoning to explain factors used in teaching algebraic word problem solving.

The second part of the literature review contains a discussion of teachers' and students' cognition. The literature discussed is related to several cognition topics, such as student thinking, constructivism, studies on pedagogical content knowledge and algebraic word problems.

## Framework for the Study

### Theory of Pedagogical Content Knowledge

The topic of this study combines multiple theoretical perspectives, particularly in learning, understanding, and cognition. The framework upon which the study is based is Shulman's pedagogical content knowledge. This framework provides opportunities to represent possible interactions as well as a series of events that lead to an understanding of the topic. Pedagogical content knowledge depicts an understanding of factors that lead to the development and practice of effective teaching. The focus of knowledge is on the subject matter, students, learning, and knowledge of curriculum, school context, and teaching (Shulman, 1986, 1987). Shulman (1986) stated that the theory also includes "…an understanding of what makes the learning of specific topics easy or difficult; the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons" (p. 9-10).

Grossman (1995) clarified the variables in the theory by stating four aspects of pedagogical knowledge. The aspects were stated through questions: What content is most important to know? What is the impact of students' prior knowledge and misunderstanding? In what order should the subject matter be presented? And what instructional strategies are more useful for teaching a given subject?

Both Gudmundsdottir (1987) and Shulman (1987) explained that this theory about pedagogical content knowledge was the synthesis or incorporation of teachers' subject matter knowledge with their pedagogical knowledge into an understanding of how specific topics, problems, or issues were prepared, represented and modified to meet the different interests and abilities of learners through instruction. In other words, Manouchehri (1996) suggested, pedagogical content knowledge is simply 'know-how'. In terms of mathematics, the 'know-how' refers to the ability to formulate, solve and critically reflect on problems (Manouchehri, 1996). In a sense, teachers need to know how students learn and what relevant motivation, readiness, reinforcement and other psychological processes have for learning in addition to knowing more about the teaching-learning process in education (Manouchehri, 1996). In addition, Fuller (1996) stated: "It is a form of knowledge that makes teachers 'teachers' rather than subject area experts, for teachers differ from biologists, historians, writers, or mathematicians, not necessarily in the quality or quantity of their knowledge, but on how that knowledge is organized and used" (p. 4). Teachers basically have the responsibility to transform their subject matter knowledge into actual teaching, particularly as counselors, not just as leaders and teachers. As mentioned, these three roles are essential for a teacher. But with respect to the pedagogical content knowledge, their role in understanding the factors that relate to effective student comprehension lies with their capability to counsel – because counseling enables them to observe their students' capability to learn and then develop approaches that would fit well with those limited capabilities. Thus, interpersonal communication is helpful for a teacher as it is the one medium that can be used in counseling. However, according to a supplemental theory constructed by Shulman (1986, 1987), teachers should also use their own observation and reasoning, adhering to the principles of pedagogical reasoning.

Pedagogical reasoning is an essential consideration in this study. Pedagogical reasoning implies examination and a critical interpretation of the instructional materials in terms of a teacher's own understanding of the subject matter (Shulman, 1987). Pedagogical reasoning entails thinking through the key ideas and identifying other methods of representing subject matter to students through analogies, metaphors, demonstrations, elaborations, simulations, and many more (Shulman, 1987). Pedagogical reasoning refers to adapting the instructional materials to students' characteristics such as gender, culture, language, ability, race, prior knowledge, conceptions, misconceptions, expectations, difficulties, strategies, limitations, biases, etc. (Shulman, 1987). The material needs to be modified for specific students in the classroom, such as those with special needs and behavior problems (Shulman, 1987). Overall, Shulman (1987) defines this theory as: "…that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding…the category most likely to distinguish the understanding of the content specialist from that of the pedagogue" (p. 8).

The first process in Shulman's (1987) pedagogical reasoning theory is *comprehension*. Comprehension means the understanding of the teacher and the learner about subject matter. This also involves the impression of the teacher or understanding of the teacher about the learning skills of his or her own students. For instance, a teacher may know that one student has a learning difficulty because of behavior problems. Furthermore, the teacher may also know that half of the class already knows the basics of algebra, while the others still cannot understand the ideas. The teacher must understand the factors that affect the students' learning (e.g., the arrangements of the chairs do not work for the class when several students sitting together are talkative).

Comprehension is followed by *transformation*. Transformation requires preparation, representation, selection, and adaptation and tailoring. Transformation refers to the changes that are made to the instructions for and approaches to students. Transformation must be addressed first with preparation. Preparation means choosing the necessary approach that is suitable for what the teacher identifies through *comprehension*. For instance, the teacher identifies that the class of students cannot easily learn because they are bored with monotonous lectures. The teacher then moves to another activity (such as special games or group activities) or location (such as in the field) to connect ideas to the lecture. Of course, the necessary tools for the activity need to be prepared. The same actions must be completed with other new instructions, assignments, activities, reward motivation policies, etc.

*Representation* requires explaining ideas clearly to students using effective instructions and approaches. Types of representation include analogies, metaphors, examples, illustrations, explanations, demonstrations and simulations (Shulman, 1987). The teacher must know what type of representation is necessary for introducing the lessons. For example, the teacher might consider using analogies or stories that help students better understand their mathematical readings or solve their mathematical problems more efficiently. On the other hand, instructional selection, according to Shulman (1987), occurs when teachers must shift from the reformulation of content through representations to the embodiment of representations in instructional forms or methods. For this action to happen, teachers draw upon their instructional repertoire of approaches and strategies for teaching, and ultimately decide which are best for the specific situations.

Finally, *adaptation* and *tailoring* refer to the adaptation of the class to the new strategies being implemented. Here transformation is considered a valid approach as that which has already been used within the class, or has already been accepted by the students.

Transformation is followed by *instruction*. Instruction simply means the process of instructing the students using the teaching strategies. Instructions must be clear and well-stated to ensure that the students can follow the implications. Moreover, teachers need to be attuned to the learning capabilities of the students. For instance, simple wordings are used with younger students who are unable to understand more complex words (Shulman, 1987).

The instruction is then evaluated to see if the change in instruction is effective. *Evaluation* considers the factors affecting the new instructions or teaching strategy. In evaluation, the teacher needs to list visible improvements of students after the new strategy has been implemented. This process also includes evaluating how the students react to the instructions, how they understand the instructions, how they study with these instructions, and how they improve based on these instructions (Shulman, 1987).

Evaluation is followed by *reflection* or looking for areas of improvement. Reflection refers to considering what has been achieved so far with the new strategy. Questions are asked at this point, such as: Are the strategies effective? Are they suitable with the learning capabilities of the students? Does it improve the thinking of the students regarding the subject matter? Is there any part of it that needs to be improved or revised? (Shulman, 1987).

After the teacher has evaluated and reflected on the transformations, new comprehensions are developed. New comprehensions set the path for further improvement in the instructions or strategies, raising them to a new level. The new comprehension can start anytime the teacher finishes reflecting and evaluating on the strategies (Shulman, 1987).

Wilkes (1994) stated that this model provides a key to understanding the development of pedagogical content knowledge and hence to understanding at least some aspects of the professional growth of teachers.

## Conceptual Perspective

The conceptual framework of this study is extended from Shulman's (1987) pedagogical content knowledge and pedagogical reasoning theory through the adaptations from the study of An, Kulm, and Wu (2004) and described in Figure 1.

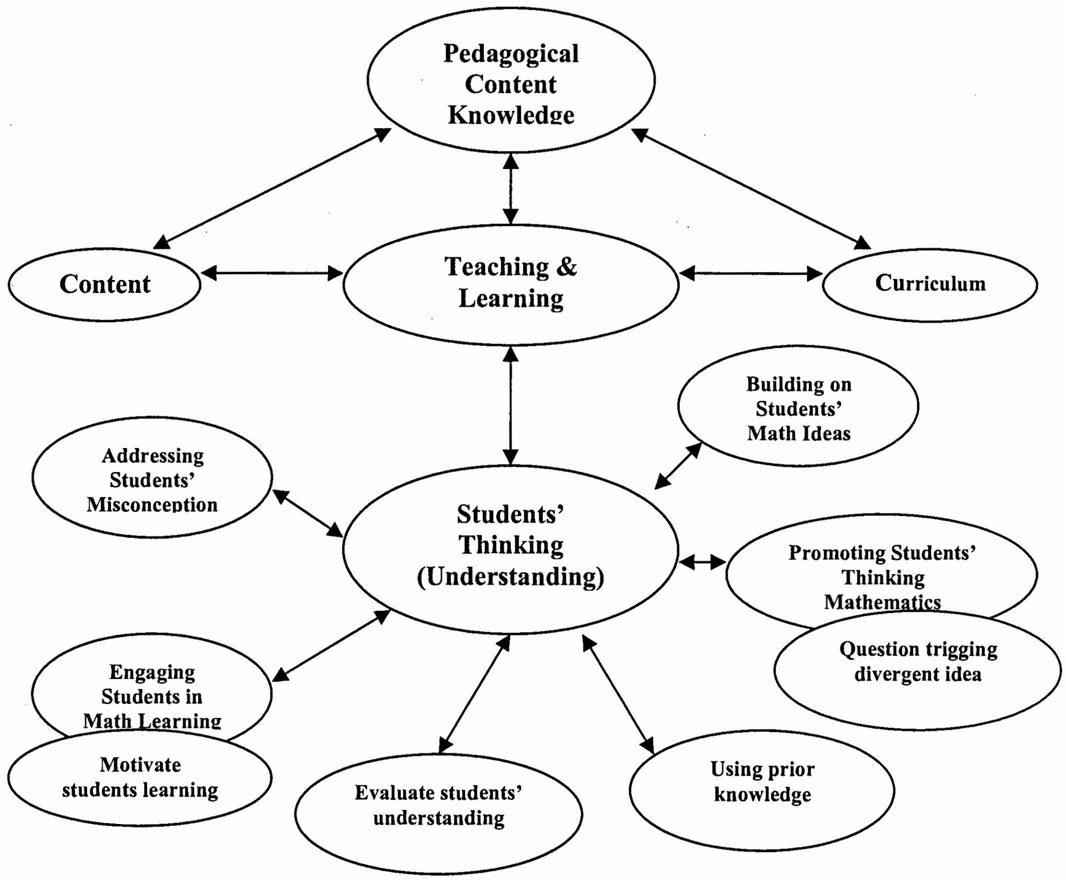



*Figure 1.* Conceptual framework.

The top two layers of circles in Figure 1 begin with pedagogical content knowledge, previously described as the teacher's knowledge of content, teaching and learning, and curriculum. However, more careful consideration of teaching must focus on student's understanding. Here, teachers need to consider important factors: addressing students' misconceptions; building on students' mathematical ideas; offering questions to trigger divergent ideas; promoting students' thinking in mathematics; engaging students in learning mathematics; motivating students' learning; evaluating student understanding; and using prior knowledge.

Part of understanding students' thinking requires an understanding of their misconceptions about topics (An, Kulm, &Wu, 2004). Misconceptions may arise on different issues and topics in any algebraic word problem. Teachers can attend to this concern by first ensuring that the students understand the problems. Furthermore, once the teacher confirms that a misconception on the part of the students exists, based on their answers, the misconception must be addressed. Misconceptions require immediate attention. Thus, the teachers must correct students with the right strategies to ensure that they understand the topic clearly and the techniques of related to how the problem may be approached. Misconceptions can be spotted when the teacher evaluates the effectiveness of the instruction. Reflection takes place, and then new comprehension is developed about why the students misconceive the idea. This process leads to a revision of the strategies to better meet the comprehension needs of the students.

Building on students' mathematical ideas involves considering students' interest in mathematics, including algebra, in particular. As previously mentioned, representation is an important part of transformation. Improving the interest of students can happen by using analogies or illustrations that more clearly describe the algebraic concern. For instance, an example of how algebra can be used might illustrate how students can balance the money their parents occasionally give them, or perhaps how they can use algebra in the future.

To promote students' mathematical thinking, students need to be encouraged to consider their own thinking. Perhaps the teacher can relate mathematical problems to everyday life situations that students experience and challenge them to consider a variety of ideas.

- How do you relate algebra to your everyday life?
- What are the situations you experienced that can be solved with the methods of algebra?
- How can algebra be fun and effective for students?

Engaging students in mathematical learning simply means, according to An, Kulm, and Wu (2004), conducting mathematical lessons that challenge students to analyze their strengths and weakness so that instructions and strategies can be evaluated and reflected upon. The process leads to better student mathematical learning and understanding. Engaging students in mathematical learning also requires motivating them to learn. Specifically, this consideration involves activities such as praising students or giving them motivational advice when they struggle or fail to solve word problems. This activity might involve the use of manipulatives (such as visual manipulatives in computer games) so that the students may be motivated to work on the task.

Evaluating students' understanding is a necessary tool for pedagogical reasoning (An, Kulm, & Wu, 2004). Students' understanding can be evaluated during instruction and during lessons. Students' understanding can be evaluated based on how they understand the instructions, how they learned, and how they performed.

Finally, using prior knowledge about students is also important in understanding them even better. Teachers' prior knowledge of students' understandings and barriers in algebraic problem solving can be used during reflection to reconsider how students can be approached. Prior knowledge of students considers how well they know the current subject as well as the next topics to be discussed with them.

## Constructivism

Constructivism is a theory of knowledge and cognition that has been proposed in describing student learning. Von Glasersfeld (1996) defined constructivism in the following principle: "knowledge is not passively received, either by sensing or by communicating, but is actively built up by the cognizing subject; and … the function of cognition is adaptive, and tries to increase fitness or viability - serves the organization of the experiential world of the subject, not the discovery of ontological reality." This principle explains that the perception and confirmation of a person's knowledge in social interaction plays a crucial role in that person's construction of his or her experiential reality. In other words, the learning process within the constructivist paradigm is based on subjectivity (Rueda & Vacas, 2001) meaning that the student must explore the system according to his or her own interest. Also, subjectivity describes a 'blind exploration' for the students, enhancing his or her knowledge through his or her own efforts and determination to learn. Constructivism suggests that learning is more likely achieved when instruction is student-centered and when it involves more active learning experiences, different interactions between teachers and students, and additional work in solving realistic problems using concrete materials (Jensen, 2001). More simply put, constructivism is a communication theory based on the assumption that people make sense of the world through their own systems of personal constructs.

Given this understanding, teachers need to identify instructions based on their previous knowledge of teaching and some information they have acquired about student thinking. Because constructivism involves many interactions, teachers have many opportunities to know or become familiar with their students' thinking about the subject. Basically, teaching and learning from a constructivist perspective gives the students and the teachers more freedom to learn from one another. Constructivism assumes that "knowledge and truth are constructed by people and do not exist outside the human mind" (Duffy & Jonassen, 1991).

Constructivism differs from objectivism in that it emphasizes the construction of knowledge while objectivism is concerned mainly with the object of knowing. The learning process of objectivism is based on traditional methods of learning and teaching where the teachers organize the curriculum and instruction from their perspectives about the way the material is to be taught. By contrast, in constructivism, learners may conceive of the external reality somewhat differently, based on their unique set of experiences with the world and their beliefs about these experiences. Thus, teachers with this perception frame instruction around the students' knowledge and understandings. This instruction can be accomplished by evaluating their understandings and then discussing these understandings with others in order to develop shared understandings (Tam, 2000).

From a constructivist perspective, then, good problems require students to make and test a prediction, can be solved with inexpensive equipment, are realistically complex, benefit from group effort, and are seen as relevant and interesting by students. Aside from the requirement to present a good problem, interaction with others is also strongly promoted. Learners work together as peers, applying their combined knowledge to the solution of the problem. Another constructivist characteristic is that teachers are transformed from being "intimidating experts to understanding guides." Teachers serve as models and guides, showing students how to reflect on their evolving knowledge and provide direction when the students are having difficulty. However, the amount of guidance provided by the teacher depends on the knowledge level and experience of students (Tam, 2000). Other traits that have been found to be effective include: using a wide variety of materials, including raw data, primary sources, and interactive materials and encouraging students to use them; inquiring about students' understandings of concepts before sharing his/her own understanding of those concepts; encouraging students to engage in dialogue with the teacher and with one another; encouraging student inquiry by asking thoughtful, open-ended questions and encouraging students to ask questions to each other and seeking elaboration of students' initial responses; engaging students in experiences that show contradictions to initial understandings and then encouraging discussion; providing time for students to construct relationships and create metaphors; and assessing students' understanding through application and performance of open-structured tasks (Brooks & Brooks, 1993; Tam, 2000).

Constructivism also has limitations. Jensen (2000) criticized the theory's lack of corresponding theories of instruction that describe optimal learning environments for teachers in their preparation. However, he found in his investigations that presenting cases to teachers helps them to enhance their skills on the tenets of constructivism, such as: prior knowledge serves as a basis for construction; meaning is negotiated through social interaction; and knowledge construction is an active process in terms of individual cognitions, and social and political process. Overall, Jensen (2000) stated that teachers can create an effective environment by asking themselves, "What background knowledge do my students share that can serve as a reference point?" The teacher should also think of effective experiential activities that would enhance students' common knowledge (Jensen, 2000).

## Student Thinking

Logically, thinking is a process of the mind that enables the individual to analyze, discover and learn new ideas or refresh old ones. Everyone with a healthy mind can think and develop knowledge. Thinking is a key to learning, involving interpretation and conveyance of what is being taught, experienced, read or written. People who are exceptionally skilled thinkers are often leaders and achievers in life. They have reasoning abilities and are skilled in both internal and external cognitions. Internal cognition refers to the mental strategies used to develop highly effective decisions, problem solutions, and creative thought (Zhang, 1991). On the other hand, external cognition considers the actions people take as a consequence of the thoughts that emanated from their internal cognition. Thus, thinking is a psychological process that is both complex and interesting. In simpler terms, thinking can be considered as the process of interpreting information.

Students are expected to think in school. Thinking is a psychological process that involves types of cognition. The student is the determinant of how he or she will think - or in other words, how he or she will interpret and understand the information being given. As individual persons, students have different ways of thinking or strategies of interpreting the ideas being shared with them. This phenomenon can be reflected in almost all academic subjects. Mathematics is an example of a subject that students find difficult to understand. Wu (2002) stated that the mathematics subject is often presented to school students as a 'mystifying mess' - a type of presentation where the receivers are 'mystified' with what is being taught to them, and find it difficult to comprehend because of several problems. This case is most obvious with younger students because their cognitive skills are not yet fully developed. For instance, Wu (2002) demonstrated that to children, the difference between whole numbers and fractions is that, while whole numbers can be understood in concrete terms, the understanding of fractions requires an element of abstraction. Wu also stated that children have difficulty in reasoning, particularly with proportional reasoning. They have a propensity to set up mindless proportions and solve the wrong missing-value problems. Similarly, in geometry, beginners do not learn well when they are forced to start from the basic axioms of an axiomatic system. He explained that "many intuitive ideas may have been hidden or thrown away." In summary, students depend highly on their early intuition and they can easily become uninterested or confused with what is being taught because of factors such as boredom, distractions, and other concerns. Essentially, students' learning in mathematics refers to the process of acquiring a 'mathematical disposition' or a 'mathematical point of view' (Schoenberger & Liming, 2001).

Manes (1996) stated that in many cases students do not have the mental structure to connect with abstract definitions; that can lead to problems of incorrect generalizations due to their limited mental structures. Basically, generalization and abstraction are important in mathematics, but as Manes (1996) described: "…students need time to think about things in the same way that mathematicians do" (p. 10). Young students still struggle with identifying and disregarding unconnected information in word problems, completing each part of the problem accurately and working with basic facts of computation (Schoenberger & Liming, 2001). Cornell (1999) suggested that students have problems identifying "specific mathematical operations when the problem involves more than simple computation."

Co-existing problems in reading and comprehension make word problem solving even more difficult for students (Jitendra, Griffin, McGoey, Gardill, Bhat, & Riley, 1998; Schoenberger & Liming, 2001). Normal students usually experience such problems, but those with disabilities are much farther behind. Nonetheless, both categories of students also experience limitations in their mental structures and some psychological limitations such as fear and failure (Cornell, 1999). Attitudes such as unwillingness to take personal risks, low self-efficacy, lack of persistence to find a solution and lack of engagement in higher level of reasoning are also barriers to effective student thinking (Henningsen & Stein, 1997; Lawson & Chippanan, 2000; Schoenberger & Liming, 2001). Furthermore, disconnected instructions not based on students' prior knowledge can also be barriers (Cornell, 1999). Finally, students' lack of experience and limited mathematical capability are additional barriers that may prevent them from extending their learning in mathematics. Their thinking is an important concern because they may have limited capabilities in conceptual thinking, reasoning and mathematical problem-solving (Henningsen & Stein, 1997).

## Pedagogical Content Knowledge and Mathematics

Past studies by Ball (1988, 1989, and 1990) argued the importance of content knowledge in teaching – that is, a conceptual understanding of principles and meanings within the subject. Ball (1990) argued that subject matter knowledge of teachers should be the focus of educational reforms. However, in the process, Ball found that content knowledge or knowledge of the subject matter cannot be effective alone and additional research needed to be conducted. Pedagogical knowledge has been viewed as a missing factor that needs to be integrated with the focus on content knowledge.

Pedagogical content knowledge exists at the intersection of content and pedagogy but does not refer to a simple consideration of content and pedagogy, together but in isolation; rather pedagogical content knowledge is an amalgam of content and pedagogy that enables the transformation of content into pedagogically powerful forms. Wu (2005) recently stated that the most difficult factor in becoming a teacher is to achieve a firm mastery of the mathematical content knowledge because without such knowledge, effective pedagogy is impossible. Benko, Loisa, Long, Sacharski, and Winkler (1999) stated that pedagogical content knowledge consisted of ways to represent specific topics and issues in ways that are appropriate to the diverse abilities and interests of learners. Grossman (1991) described four aspects of pedagogical content knowledge: focus on the subject matter; focus on the students; focus on the curriculum; and focus on pedagogy (Grossman, 1991). Perhaps the most difficult of these foci for teachers to attain is that of understanding the students or students' thinking. Grossman (1995) added that teachers must adapt their content knowledge to students' prior knowledge and skills, track students' misunderstanding, and guide them toward new conceptions.

## Algebraic Word Problem Solving

Algebra is the branch of mathematics that deals with symbolizing general numerical relationships and mathematical structures and with operating on those structures (Reed, 1999). Algebra is the branch of mathematics that symbolizes general number relations and properties and includes such topics as signs of operations, solution of equations, and polynomials (Kieran, 1992). Problem solving is an important part of work in algebra. Problem solving is defined as "a process of devising and implementing a strategy for finding a solution or for transforming a less desirable condition into a more desirable one" (Coy, 2001, p. 2). Finally, in the study of algebra, students are confronted with word problems often described through questions "written in paragraph or sentence form that contains a mathematical concept that needs to be considered, solved or answered" (Coy, 2001, p.3). In essence, then, algebra word problems can be described as questions written in paragraph or sentence form that rely on algebraic concepts for solution. Interestingly, Reed (1999) stated that algebraic word problem solving places cognitive demands. Whereas on the one hand, the problem solver uses symbolic representations with little or no semantic content as mathematical objects and operates on these objects with processes that usually do not yield numerical solutions; on the other hand, algebraic word problem solving requires that problem solvers modify their former interpretations of certain symbols and begin to represent the relationships of word problem situations with operations that are often the inverses of those that are used almost automatically for solving similar problems in arithmetic. Representing algebraic word problems requires mental construction of a problem-solving situation and the integration of information from a word problem into an algebraic representation via symbols to replace unknown quantities (Maccini & Hughes, 2000). Finally, the solution consists in deriving the value of unknown variables by applying appropriate operations (i.e., arithmetic or algebraic) to solve the sub-goals and goal of the problem (Mayer, 1985; Maccini & Hughes, 2000).

## Students' Difficulties in Algebraic Word Problem Solving

Algebra word problems traditionally involve multiple content areas, making them difficult to solve without extended knowledge and understanding. The knowledge requirements include linguistic, factual, schematic, algorithmic, and strategic aspects. An important consideration is that the student learner needs linguistic knowledge of the English language before fully grasping algebra word problems (Mayer, 1982, 1987). Aside from the various knowledge areas required, solving word problems typically involves four activity phases: understanding the problem; making a plan; carrying out the plan; and reviewing the solution (Low, Over, Doolan & Michell, 1994).

Blessman and Myszczak (2001) emphasized that mathematical vocabulary is one of the problems for students in solving any mathematical problem. Poor mathematical vocabulary is documented through teacher and student surveys and questionnaires, student vocabulary checklists, and teacher observation of students' daily work. Students have varied mathematical backgrounds, suffer from mathematics anxiety, and have poor reading comprehension, each impacting their ability to solve word problems. Changes in the standards for mathematics curriculum and instruction are additional reasons for students suffering from poor mathematical vocabulary; changing standards create changes in the expectations for students. Further, most problems focus on computational facts rather than mathematical vocabulary, compounding students' difficulties with word problem solving (Blessman & Myszczak, 2001). Croteau, Heffernan and Koedinger (2004) also stated that many documents suggest that students have problems with algebra word problems because they have trouble comprehending the words in an algebraic word problem. In yet another view, studies of Heffernan and Koedinger (1997, 1998) showed that students comprehend words but had problems with the symbolization of the words.

According to the rule-based theory of learning called ACT theory (Anderson, 1983), in which human cognition arises as an interaction between declarative and procedural knowledge structures and human cognitive skill requires learning thousands of rules that relate task goals to actions and consequence of those actions, learning to perform complex cognitive tasks such as reading or solving mathematics problems usually requires the successful integration of many component skills. In this theory, comprehension depends on understanding the ideas in the text, but understanding depends on the successful execution of many skills (Reed, 1999). Chi, Bassok, Lewis, Reimann and Glaser (1989) found that good problem solvers are quite accurate at evaluating their own comprehension, whereas poor problem solvers usually thought that they understood. Another difference was that the good problem solvers used examples as a reference source, but poor problem solvers consistently reread the examples in search of exact solution procedures (Chi, Bassok, Lewis, Reimann & Glaser, 1989). Reed (1999) similarly stated that good problem solvers provided many more self-explanations than poor problem solvers.

Aside from linguistics and comprehension, students with underdeveloped schemata are also known to be poor in mathematical problem solving. A schema provides a framework for organizing both objects and procedures by providing general structures that can be filled in with the detailed properties of a particular instance (Reed, 1999). A fundamental assumption of schema theory is that all new information interacts with old information contained in an activated schema. In contrast to stimulus-response theory, which is based on small units of knowledge, a schema links knowledge connected to form larger clusters of related information (Reed, 1999), known as chunks. Chunking is perceived to be intertwined with problem familiarity. Problem schemata are basically chunked information (Chi, Glaser & Rees, 1982). De Jong and Ferguson-Hessler (1986) aimed to test the importance of problem familiarity with chunking by testing the hypothesis that "good novice problem solvers have their knowledge organized according to problem schemata, as opposed to poor novice problem solvers, who were expected to lack this kind of organization" (De Jong & Ferguson-Hessler, 1986, p. 280). Using a card-sorting task with three of the four elements of schemata (declarative knowledge of principles, formulae, and concepts; characteristics of problem situations to make connections between the actual problem and problem schemata; and procedural knowledge for solving problems), they found that good problem solvers sorted by problem-type, where as the poor problem solvers tended to sort by surface characteristics of the elements.

In another study, Finegold and Mass (1985) found that the good problem solvers relied more on planning. Specifically, they found that the good problem solvers tended to plan their solutions more fully and that the poor problem solvers rarely planned. Good problem solvers also spent more time on translation than poor problem solvers.

## Teachers' Difficulties

The lack of agreement on the curriculum may be one of the reasons why some teachers find it difficult to teach algebraic word problem solving. Stetcher and Mitchell (1995) found that portfolio assessment programs enhance teachers' understanding of mathematical problem solving and broadened their instructional practices, but that teachers encountered difficulty in understanding certain components of the reform and making the relevant changes. They also found that teachers did not share a common understanding of mathematical problem solving or agree on skills that students should master.

Skemp (1978) defined understanding in school mathematics through two words: instrumental and relational. Instrumental understanding is to know "rules without reasons" (Skemp, 1978) and is not related to promoting sense-making in mathematics. For example, memorizing the steps in an algorithm to learn how to manipulate numbers and symbols without making connections to the underlying mathematical concepts develops instrumental understanding. Teachers who teach isolated rules and procedures in mathematics are encouraging the development of instrumental understanding in their classroom teaching. On the contrary, relational understanding in mathematics is identified as knowing "what to do and why" (Skemp, 1978). Relational understanding involves developing a network of connections and relationships. The more students connect the mathematical ideas, procedures, and skills in the network, the deeper they understand. In this way, developing understanding is ongoing and builds on previous knowledge; in other words, it is generative in nature (Hiebert & Carpenter, 1992). Teachers need a relational view of teaching mathematics for understanding in order to assess students' mathematical understanding.

Silver (1985) stated that teachers may have difficulty if they are not clear about their goals and that teachers should be clear about what sorts of problems they want students to be able to solve. If the goal is for students to learn to solve relatively complex mathematical problems that do not fall into standard categories, then the teacher must be willing to invest substantial instructional time in demonstrating and giving practice in heuristic procedures.

The diversity of students' individual thinking about mathematics may also contribute to teachers' difficulty in teaching algebraic word problems. In a classroom setting where student thinking is the focus, classroom practice becomes much harder to manage and much less predictable (Hammer, 1996). Students' ideas can be oblique, but sometimes they are also inventive and insightful (Schifter, 1997). For this reason, teachers also need to consider pedagogical content instead of just focusing on instructions (Schifter, 1997).

## Conclusions

The literature review in this proposal explains the theory of pedagogical content knowledge and reasoning, as well as the nature of student thinking and algebraic word problem. Since solving algebraic word problems requires more than just one area of knowledge, the background of the students needs to include many knowledge bases. Student thinking involves both internal and external cognitions. Thus, the inability to recognize students' cognitions contributes to their learning difficulties. Teaching is also difficult when focused only on content knowledge. In this case, the interaction between the teacher and the students is more likely to be passive rather than active. Pedagogical content knowledge explains the importance of mastering content, but also considers other important pedagogies. The theory thus aligns easily with the subject of this study and its development of a conceptual framework which aims to explain how teachers rely on their understandings of student thinking in planning for and guiding students' work with algebraic word problem solving.

# Chapter III

## Design and Methodology

Although many characterizations of effective teachers exist, most attend to generic pedagogy such as a teacher's management of classroom behaviors. Prior to Shulman's (1986) introduction of the notion of pedagogical content knowledge (PCK), most researchers considered content and more generic pedagogies. With the introduction of the idea of PCK, several researchers have explored specific aspects of clarifying teachers' use of their knowledge for teaching (Ball, 1988; Fennema & Franke, 1992; Ma, 1999). In particular, Ball (1988, 1989, 1990, 2000) has focused on the importance of the content knowledge or knowledge of mathematics in PCK for teaching mathematics. In other words, researchers have emphasized content knowledge for teaching rather than particular pedagogies for teaching specific concepts.

However, to teach mathematics effectively, teachers need to understand students' mathematical thinking. Researchers need to rethink the pedagogy involved with students' thinking and understanding with the content of mathematics. Several researchers have studied the use of knowledge of children's mathematics thinking in classroom teaching (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; An, Kulm & Wu, 2004). Also Stylianides and Ball (2005) suggested the importance of other factors as being critical for the development of PCK, such as teachers' knowledge of students' thinking in mathematics.

This study was focused on an investigation of the impact of teachers' knowledge of students' thinking for designing and presenting effective classroom lessons dealing with algebraic word problem solving. The direction of the study was to develop a framework that hypothesized a relationship between expert middle school mathematics teachers' knowledge of students' mathematical thinking in learning algebra and their active teaching. The primary research questions for this study were:

1. What is the nature of the participating teachers' knowledge of middle school students' mathematical thinking in algebraic word problem solving?
2. How do these teachers use their knowledge of students' mathematical thinking when planning lessons for guiding their students in solving algebraic word problems?
3. How do these teachers use their knowledge of students' mathematical thinking when actively teaching students to solve algebraic word problems?

## Perspective and Design

This research study used a phenomenological multiple case study approach to investigate participating teachers' knowledge of students' mathematical thinking in teaching students algebraic word problem solving. This qualitative inquiry proposed an investigation of 2 individual teachers' teaching actions. The purpose of this study was to describe and interpret the nature of these selected middle school mathematics teachers' knowledge of students' mathematical thinking and how they use this knowledge in their classroom teaching of algebraic word problem solving. Cases were purposely selected for building an understanding that supported an increased comprehension of the impact of teachers' knowledge of students' thinking in designing and implementing their instruction. These case studies were used in building hypotheses with respect to the complexities of this phenomenon for further investigation.

This phenomenological analysis was principally concerned with understanding how the participating teachers use their knowledge about students' mathematical thinking in their classroom teaching (Schwandt, 2000). The aim was to grasp how they interpreted their students' actions and reconstructed the impact of this knowledge in their instructional planning and teaching. From an interpretivist's view, the teachers' actions in their classroom teaching were meaningful and useful in building an understanding for further investigation.

## Research Process

The research process of the study can be best illustrated through the use of the "Research Process Onion" (see Figure 2). The onion refers to the central issue of collecting the data needed to answer the research questions. Important layers of the onion must be peeled away (Saunders, Lewis, & Thornhill, 2003). Figure 2 describes the researcher's conceptualization of this research approach to identify the pertinent data needed for answering the research questions.

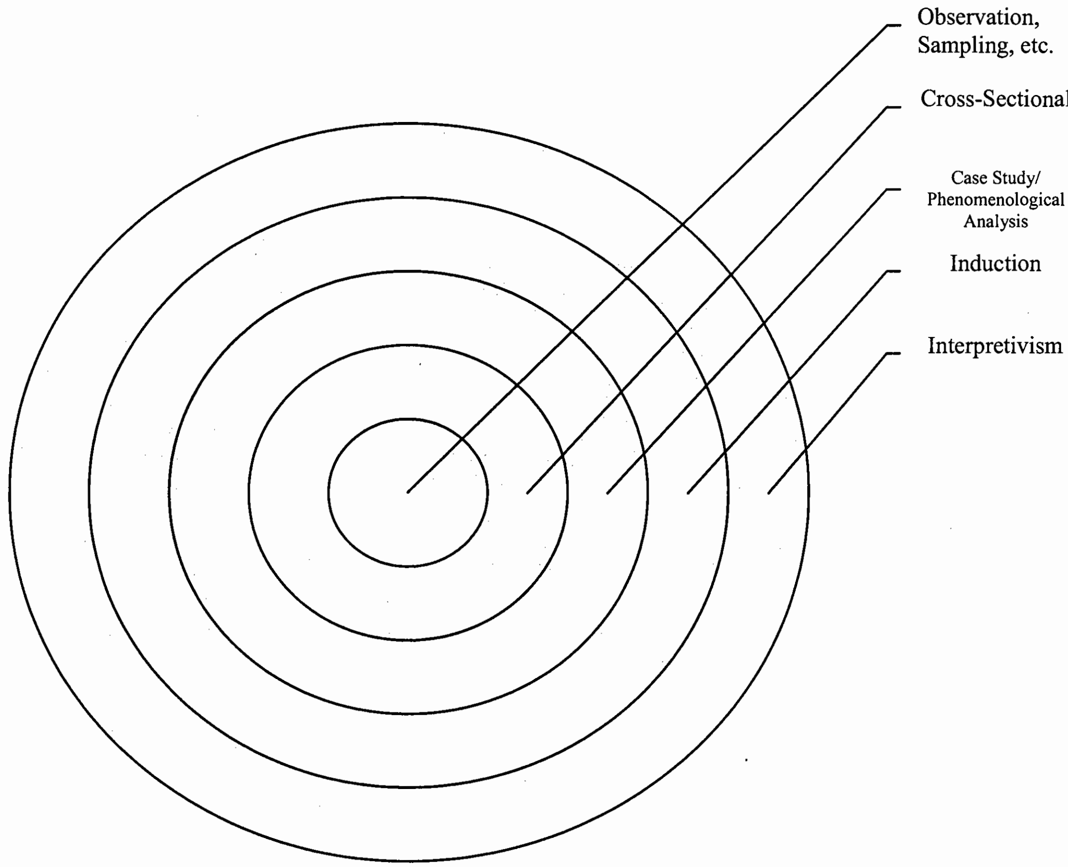



*Figure 2*. Research process onion.

The first layer, the outer layer of the onion, refers to the *research philosophy* of the study, *interpretivism*. As previously mentioned, the aim of this study was to investigate new situations and propose a hypothesis, not to prove it. Interpretivism was the appropriate research philosophy for this study because it directed the search for the details of the situation, to understand the reality of teachers working with middle school mathematics students (Remenyi, Williams, Money, & Swartz, 1998). From the interpretivist perspective, it was necessary to explore the subjective meanings that motivated the teachers' actions in order to understand their actions. In other words, the researcher's aim, from an interpretivist's perspective, was to understand the teachers' actions and present plausible and acceptable accounts of the experiences (Varey, Wood-Harper & Wood, 2002).

The second layer of the onion refers to the research approach of the study. Because the study was based on intepretivism, the research approach was inductive rather than deductive. The deductive approach involves deducing hypotheses or expressing and testing hypotheses in operational terms. By contrast, the inductive approach permitted the researcher to get a feel for what was happening, in order to better understand the nature of the problem (Mertens, 1998) and enabled the researcher to focus on making sense of the interview data.

The research strategy involved the development of *case studies*. In this strategy, considerable effort was directed toward generating answers to the research questions of why, what and how. Data collection methods that were used include questionnaires, interviews, observations, and documentary analysis (Robson, 2002; Mertens, 1998). Phenomenological analysis of the study enabled the researcher to explain the phenomenon of how teachers understood the mathematical thinking of their students, specifically in teaching algebraic word problem solving, and how this activity was or was not reflected both in the documents they produced and in their classroom actions.

The fourth layer of the onion presents the time horizons of the study, the *cross-sectional*. Cross-sectional ideas, however, pertained only indirectly to this study because the study only investigated middle school teachers' knowledge and their teaching using their knowledge of students' mathematical thinking in a unit in their curriculum that focused on guiding student learning that included algebraic word problem solving. However, the cases of the teachers was linked with data from teachers responding to the questionnaire as well as those who were interviewed and observed prior to the identification of the teachers for the more in-dept cases.

Finally, the fifth and last layer of the research process onion was the data collection of the study. The data collection procedures included sampling, interviews, observations along with secondary data (from the questionnaire and interviews) and have been described in more detail in the succeeding sections of this chapter.

## Participants

This study was concerned with the general phenomenon of teaching using knowledge of students' mathematical thinking. The study included and simultaneously compiled multiple cases, where each case study was a concentrated inquiry into an exemplary middle school teacher's knowledge and actions in incorporating students' thinking when teaching algebraic word problem solving (Stake, 2000). The goal of the participant selection was to identify two exemplary middle school mathematics teachers for in-depth multiple case studies.

The participant selection process began with the identification of exemplary middle school teachers. Based on recommendations from principals and other mathematics education leaders, 17 teachers were identified. From the 17 teachers, nine exemplary teachers matching the selection criteria agreed to participate in the study. These nine exemplary teachers demonstrated concerns about assisting students in learning with understanding and the key to their teaching with understanding was often through verbal interactions that enabled them to monitor their students' mathematical thinking. The nine exemplary middle school teachers were asked to complete a questionnaire (see Appendix A for the survey questionnaire) in which they were asked to provide their interpretation of students' thinking with particular algebraic word problems.

The nine teachers were selected from the northwest United States, within an area of close proximity to the researcher's location. To be considered, teachers had to have taught more than five years in a middle school and spent at least three years teaching the same grade. The criteria for the selection relied upon important research results. The time period was identified as the time when most teachers have gained the knowledge and skills needed to improve the scores of their students (Berliner, 2005). Moreover, Henry (1994) indicated that expert teachers had 16 or more years of teaching experience.

From the analysis of the teachers' responses to the questionnaire, four teachers were identified for follow-up interviews. These teachers provided clear and detailed information of students' mathematical thinking in their responses to the questionnaire items. Follow-up interviews were designed to obtain additional information for identifying the two key informants for the case studies. The goal was to find two teachers who displayed the clearest strength in understanding students' mathematical thinking while solving word problems in algebra and who were able to identify and describe how they used this knowledge for planning to teach units in this area. In addition to follow-up interviews, the researcher observed the four teachers in order to gain additional information about their use of students' thinking during classroom activities as well as their willingness to discuss their planning and implementation of their plans.

Through this purposeful identification process the researcher was more likely to identify respondents who were well-suited for the study - two exemplary middle school mathematics teachers who demonstrated knowledge of students' thinking when they were teaching. Two teachers were targeted so that the researcher had adequate opportunities for in depth daily observations along with the purposeful interviews. Of the four teachers, Teacher *A* demonstrated the knowledge of students' thinking and had student-centered teaching but she did not plan to teach a specific algebra unit during Winter term. Teacher *I* answered the questionnaire explaining the strategies that she teaches in the classroom step-by-step. She described students' answers using equation, guess-and-check, and charts that students were supposed to draw. She attempted to draw out students' ideas but the questions did not lead to gathering their ideas. She also tried to use students' discussion and their communication. However, in the classroom she had a difficulty in managing the classroom with problems that did not interest her students. Teacher *C* had a conflicting schedule with the other possible teachers. Therefore, she was not selected as a case.

The two teachers who were selected as the key informants demonstrated the clearest strength in understanding students' mathematical thinking while solving word problems in algebra and described how they used this knowledge for planning to teach units in this area. These two teachers exhibited the kinds of teacher behaviors that appeared to be necessary for observing student thinking.

Teacher *J* planned to teach algebra during the term of the research. She provided explicit ideas about how students solve algebraic word problems. Most of the teachers assumed that their students would identify 4b+6r=90 but she added one more that some of her students might identify b=22.5-3/2r. She elaborated on her ideas regarding students' problem solving during the interview. She explained, "I was hoping they would use systems of equations to solve that problem at the beginning, and they went a completely different direction which was technically still systems of equations. They did that work, but they want to come about it without the algebra, without any variables." In the first lesson the researcher observed, she used a whole classroom discussion strategy to engage the class in solving algebraic problems. The classroom climate was comfortable, allowing student to take risks that they might be wrong. Although the lesson was not student-centered, she worked at gathering students' mathematical ideas for solving the mathematical problems. The lesson was directed by students' answers.

Teacher S was the second teacher selected for the study who planned to continue teaching algebra during the term of the study. His answers to the questionnaire emphasized preparing lessons and considerations about students' algebraic word problem solving. During the observation, the lesson matched his responses to the questionnaire. He placed students in teams where each member had a role in the solving the problem. Students were given about 20 minutes to work through the problem to come up with the solution. His role was to facilitate their discussion and question their reasoning for the answer. He presented the problem and as students were solving the problem he visited each team to ask questions that provoked student thinking. At the end of the problem students shared their ideas on how they found the solution with the class. The lesson was student-centered.

## Data Sources

Multiple sources of evidence were used for triangulation to ensure the validity of the case studies (Patton, 2002;Yin, 1994). The primary sources of data for the two key informants included an initial open-ended questionnaire (Appendix A) with a follow-up interview (Appendix B), classroom observations including pre- and post-observation interviews, and stimulated recall interviews where teachers were encouraged to discuss the classroom actions and their use of students' thinking. The observations and interviews used an instrument developed by the researcher in previous research. See Appendix C for the Observation Guide, Appendix D for the Pre-observation Teacher Interview Protocol, Appendix E for the Post-observation Interview Protocol and Appendix F for the Stimulated Recall Interview Protocol. Videotapes of the two selected teachers' lessons and the pre-observation interviews were used to engage the teachers in more in-depth interviews to provide primary evidence in support of the teachers' lesson plans, the researcher's journal and additional literature for providing a context for the teachers' knowledge of their students' mathematical thinking in algebraic word problem solving. Field notes, students' work and transcripts from classroom observations also were used to clarify the analysis of the classroom observations.

Selection of Exemplary Middle School Teachers

Selection of the exemplary middle school teachers used multiple data sources.

Teacher Open-ended Survey Questionnaire

A Teacher Open-ended Survey Questionnaire (Appendix A) was designed to gather information from the initial group of nine exemplary teachers, and initiated the process for identifying the two key informants. The open-ended questionnaire was composed of two algebraic word problems for middle level algebra classes. The problems were identified from research and relevant literature and from the recommendation of mathematics educators. The teachers were asked to choose one problem in which they described students' mathematical thinking; this strategy provided a way for identifying each teacher's knowledge of students' mathematical thinking.

For validation purposes, the researcher pre-tested a sample of the questionnaire problems by gathering responses from graduate mathematics teaching preservice students as well as responses from mathematics educators. The researcher revised the questionnaire based on the clarifications and other communication suggestions.

The questionnaire was distributed to the nine exemplary middle school mathematics teachers by mail. After five days, a follow-up email was sent to confirm that they received the questionnaire and to encourage attention to completing their responses. Additional follow-up emails were used to encourage the non-respondents within two weeks. The nine teachers responding to the open-ended questionnaire were asked to complete the informed consent form indicating their willingness to continue with the research and to volunteering for continuing in the research.

### Teacher Follow-up Interview and Observation

After analysis of the teachers' responses to the open-ended questionnaire, the pool of potential key informants was reduced to four teachers. These teachers were interviewed and observed to provide more detail of their understanding and application of students' mathematical thinking. The interview was face-to-face in each teacher's school. The follow-up interview time was flexible, providing the teachers sufficient time to express their ideas. The length of the interview was approximately 30 minutes.

A semi-structured, open-ended interview was designed for follow-up interviews to gather more detailed information about the teachers' knowledge and awareness of their students' capabilities and skills in algebra as well as their familiarity with students' strengths and weaknesses when working with algebraic word problems. (See Appendix B for a sample of this interview protocol.) Additional questions were used to probe for their understanding of students' thinking when solving algebraic word problems.

A classroom observation was conducted to gather more practical information about each teacher's understanding of students' mathematical thinking and use of students' understanding. Through the observation, the researcher assessed each teacher's interactions with their students with a specific focus on students' mathematical thinking and how they used their students' ideas in their classroom teaching.

The observation and interviews were used to select the two teachers who: 1) were more student-centered in their instruction rather than teacher-centered; 2) focused on more interaction with their students and included more activities; 3) developed a classroom atmosphere that provided students with opportunities to learn, challenge and take risks through a variety of strategies.

## Sources for Multiple Case Studies

Once the two exemplary teachers had been identified, the remaining data sources were directed at gathering detailed data for a careful and detailed construction of the cases in order to address the research questions.

## Pre-Observational Interview

Prior to observing the teacher's classroom instruction, the researcher conducted a short pre-observation interview using a protocol as described in Appendix D. This interview was conducted in a semi-structured interview format with the teacher in the classroom before the observation. The interviews lasted approximately 10 to 15 minutes, were audiotaped and later transcribed. The purpose of these interviews was to identify the teacher's expectations and plans for the lesson. The researcher also questioned the teachers as to how they planned to incorporate students' mathematical thinking in algebraic word problem solving and how they planned to instruct their students in solving algebraic word problems.

## Unit Observations

The primary purpose of this study was to develop multiple in-depth case studies of the two carefully-identified exemplary middle school teachers. The researcher worked with each teacher to identify a specific algebra middle school class session and algebra unit to be observed. The observation period occurred daily during the instruction of an entire algebra unit. The unit selected for the study was recommended by the teachers because they were able to identify units that focused on algebraic word problem solving. The focus of the observation was to identify and describe how the teachers actually used their knowledge of students' thinking in guiding algebraic word problem solving and how they incorporated instructions and strategies that fit with the students' algebraic word problem thinking. Another aim of this observation was to compare the teachers' interview responses with their actual practice and to provide clips to use in the stimulated recall interview.

The observations took place over a two-month period beginning in February 2006. Each teacher's algebra unit was approximately 10 weeks. All observations were video-taped to provide an accurate record of the teachers' teaching using their knowledge of students' thinking in algebraic word problem solving. During the first week, the video was operated to provide students and their teacher an opportunity to get used to the additional equipment and minimize the bias in collecting these data. For specific lessons, the researcher stood at the back or side of the classrooms next to the video camera and served as the person guiding the video taping. Also, the researcher carried the video camera around the classroom to tape details of the interactions between the teacher and students. Teachers were asked to wear a microphone to more accurately gather the verbal interactions in the class.

Researcher field notes were used to gather classroom actions. The researcher used an observation protocol based on the Annenberg Institute of School Reform (2004) to provide additional clarity in recording (see Appendix C). The researcher's notes for the entire class were captured through a global scanning process throughout the class period.

After each observation was completed, the videotapes that involved student and teacher interactions were transcribed daily, compared to the observation notes, and filled in more completely if necessary. As a result of this process, the researcher was able to form an impression of the reality of the teacher's classroom from the observations. Thus the researcher did not have to rely solely on the teachers' interview responses about how they said they taught. By comparing the teachers' actual classroom practices and their responses in the interviews about their practices, the researcher developed a more accurate picture of the teachers' utilization of students' thinking with algebraic word problem solving.

Any class discussions in which the teacher asked the students questions and solicited their responses were given close attention. Special attention was paid to the questions posed by students to the teacher and the questions that the teacher asked the students, both as a group and individually. The questions asked in the classroom, the responses and the conversations between teacher and student that portrayed the teacher's knowledge of students' mathematical thinking were of utmost interest. For example, a teacher asked this question: "Are there any other situations in which we might be able to use this problem solution?" This type of question allowed a teacher to encourage students to make conjectures from their real life experiences. These questions and the students' responses were carefully analyzed. When the teacher asked these types of questions, struggled at a particular point in the lesson in asking more questions, or wanted to listen more carefully to students' responses in the video clips, the researcher gathered more detailed information about the teacher's use of students' thinking. During the stimulated recall interviews with the teachers, the researcher asked the teachers about the knowledge they used about students' mathematical thinking in responding to students' understandings at the particular points from the lesson that were represented in the video.

Post-Observation Interview and Stimulated Recall Interview

The post-observation interviews were conducted with the teachers using an interview protocol as described in Appendix E. The teachers were asked to reflect on the lessons they just completed, how they used their knowledge of students' thinking to deliver the lessons and any unexpected student mathematical thinking they encountered during the lesson. The interview was audio-recorded. The interview was conducted at the school site during regular school hours for an average of about 30 minutes. The researcher analyzed each interview immediately by listening repeatedly for ideas to be used in follow-up observations.

At least weekly over the course of the unit, the researcher engaged the teachers in a stimulated recall interview where the teachers were asked to view short video segments of their teaching and reflect upon their thinking and teaching during those segments. The selection of the clips was designed to include active discourse among the teacher and students. These interviews were used to engage the teacher in reflecting on his/her actions during the lessons to determine the impact of their knowledge of student thinking on actions taken during the class. The video clips chosen for these interviews covered situations where teacher and student discussion took place. Segments were selected if they involved the teacher asking students about their mathematical ideas, prior knowledge, or when the teacher was attempting to understand students' misconceptions among their work of algebraic word problem solving. Non-examples were presented to determine if the teacher recognized students' mathematical thinking in these cases. Through this interview procedure, a more accurate idea of the teacher's knowledge of students' mathematical thinking behind their actions was gathered. The researcher was able to compare the teachers' responses during the stimulated recall interview to their actual teaching practices observed during classroom observations. The protocol for the stimulated recall interviews is provided in Appendix F.

The Researcher and Researcher's Journal

Since personal experience reflects the flow of thoughts and meanings that persons have in their immediate situations (Denzin & Lincoln, 2000), the researcher's background is an important aspect of this study. The researcher has 10 years teaching experience in elementary school including 1st through 6th grade. The researcher is in his second year of teaching developmental and undergraduate mathematics in pre-algebra for the Educational Opportunity Program (EOP). The researcher has experience in examining middle mathematics teachers' beliefs in problems solving. The researcher has conducted and published research on the *Instructional Design in All (K-3) Students' Mathematical Achievement in Solving Word Problems* (Lee & Niess, 2005). Through another professional development program using spreadsheets to teach mathematics in middle schools, the researcher has studied the teachers' development of technological pedagogy content knowledge (TPCK). This experience has helped to broaden the researcher's views of teachers' knowledge of teachers' actions and responses.

All the researcher's experiences and background had potential for interjecting a bias that might be inherently problematic to the study (Denzin & Lincoln, 2000). The researcher took care to reduce this bias by using a journal where he described his thoughts and any developing conjectures throughout the study. Because the researcher uses English as a second language, to reduce the researcher bias, the researcher also used audio recordings of the interviews and videotapes of classroom teaching observations to verify the analysis.

## Data Analysis

This study was qualitative in nature. In this case, interpretations were conducted in the analysis of data. These interpretations were organized in two main stages, namely: Identification of the key informants and development of multiple cases.

### Identification of Key Informants

The questionnaire data collected from the first 9 exemplary teachers were analyzed through coding and analytic induction (Patton, 2002). Individual teachers' responses were categorized according to the described conceptual framework of students' thinking (Figure 1). The resulting conceptual framework of students' thinking categories could be revised to fit emerging interpretations of the data during the analysis. The codes from the questionnaire were assigned prior to the administration of the teacher follow-up interviews. For example when teachers were given an algebraic word problem, they were asked to predict 'possible students' ideas' for solving the problem. For example, consider this problem:

> A large company employs 372 people. There are four times as many laborers as clerks and 18 more clerks than managers. How many laborers, clerks and managers are there in the company (Dooren, Verschaffel, & Onghena, 2003)?

Possible students' ideas that teachers describe might include an *algebraic* solution, manipulating the structure, guess and check, and misconceptions (see Table 1).

*Table 1*

Example of Coding from Questionnaire

| Code | | Teacher Descriptions |
|---|---|---|
| Proposed Student Solution | Algebraic | $x+(x+18)+4(x+18)=372$ |
| | Manipulating the Structure | Suppose there are 18 managers more then the total of people = 390. This total consists of 6 equal parts.<br>4 parts of laborers<br>1 part of clerks<br>1 part of managers |
| | Guess and Check | Clerks Managers Laborers Total<br>80 62 320 462 → Too much<br>60 42 240 342 → Too few |
| Misconceptions | Arithmetic Error | Order of operation<br>Distributive law<br>Addition<br>Division on equation |
| | Symbolic Error | Equal sign<br>(x+x+18=2x+18 +4(x+18)=6x+90=372) |
| Barriers | Comprehension | Lack of knowledge of how to map the problem onto linguistic statements.<br>Linguistic deficiencies (difficulties in recall text)<br>Lack of understanding of variables |

| | | |
|---|---|---|
| | Translation | Difficulties on translation of problem situations into equations |
| | Representing | Difficulties of representing relations among variables that students may try to avoid using algebra to solve problems<br>C=18+M → C-18=M<br>L = 4*C → 4*L = C |
| | Attitude & Experience | Fear of math<br>Lack of experience with Algebra<br>Idiosyncratic ways of thinking about money |

The teacher follow-up interviews were transcribed and coded for the analysis of the categories and emerging interpretations that needed to be confirmed. The transcripts were matched with the teacher open-ended questionnaire response data.

Development of Multiple Cases

From an interpretist perspective, the researcher considered the teachers' knowledge of students' understanding by observing what they did (Schwandt, 2000). The qualitative analytic induction approach was used in the following manner. First, the definition of the subject topic and the phenomenon were extracted from the literature and the data from the teachers' responses to the open-ended questionnaire. From the literature review, the researcher identified and clarified the topic of algebraic word problem solving. The teachers' responses were validated by the literature review and helped the researcher to identify the nature of teachers' knowledge of the student thinking involved in solving algebraic word problems easily. Second, redefinition of the phenomenon was considered from observation, pre- and post-observation interviews and stimulated recall interviews. The observations, pre- and post-observation interviews, and the stimulated recall interviews added to the reliability of the data resources in developing the analysis. The researcher checked the consistency of all the data from a variety of sources. Third, subjective interpretation of the data was developed from the individual case studies. Then a cross-case pattern analysis of the individual cases was conducted.

The analysis of data began with the initial data collection and continued throughout the data collection process; all the data were reviewed numerous times. The transcripts from all open-ended questionnaire responses, follow-up interviews, field notes from all observations, data from classroom observations, and data from pre- and post-observation interviews were sorted. The researcher read and reread the notes in an attempt to recognize any patterns, topics or regularities. During the first reading, the researcher identified potential ideas, noting them on 3X5 cards as a beginning look at the data. During the second reading, (using a different color of 3X5 card), the researcher again noted any topics or patterns that seemed to exist. After this action was completed, the researcher reviewed the note cards, removing cards that seemed to be extraneous, reorganizing the cards for consistency and searching for patterns. Then the researcher reread all the data, looking for consistency in the patterns, adding new cards when necessary. The extraneous pile was also reviewed to assure that the information continued to be considered as extraneous. Thus the researcher used an iterative process in developing the patterns, topics, or regularities.

In this process, 'unitising' data was also performed by attaching relevant bits of data to the appropriate categories that were devised within the study (e.g.

knowledge of teacher, types of student thinking, cognitive effects, etc.). Through these categories, relationships of variables within the study were identified for better analysis. For example, separating the strategies of the teachers in teaching from their knowledge about students' thinking allowed for more detailed organization of the data, which was helpful for relating the data with the variables of the conceptual framework.

The literature review also played an important role in this process, for this information was helpful in creating better results and interpretations. For instance, a quotation from the literature was cited to explain a particular answer from a respondent to a certain type of question. This information was placed on 3X5 cards and added to the evolving categorization as a means of connecting the literature to the analysis.

## Analysis of Unit Observations and Interviews

The videotaped recordings of each classroom session were reviewed. After each classroom observation for lessons covering algebraic word problem solving, the researcher reviewed the videotapes, searching for specific classroom segments in which the teacher and students were engaged in discourse, in which the teacher provided information to guide the students, and for any other action that jogged the teacher's memory of the specific activity. These segments were used to guide a more extended interview with the teacher. These extended interviews were videotaped to enable the researcher to specifically challenge the teacher to explain his/her thinking and knowledge about students' thinking that were guiding the actions. The videotapes of the extended interviews were analyzed in the same manner as previously described. Analysis of the observation was made using the observation checklist from the Annenberg Institute of School Reform (2004) classroom observation protocol (Appendix C). Here, the classroom learning environment was coded to determine whether teachers really knew how their students think. This coding was compared with the coding from the extended interviews where the teacher was challenged to explain his/her thinking and decision making with respect to students' knowledge and understandings. The consistency of the observations was checked through pre-observation interviews, post-observation interviews and the extended interviews.

Researcher's Journal

The researcher's journal was analyzed in an ongoing basis. The researcher used color-coding to highlight the evolving ideas and conjectures about the developing knowledge of each teacher's knowledge of student thinking. Weekly, throughout the study, the researcher's journal was specifically reviewed to identify any possible bias emerging from the data collection of teachers' responses and observations.

Ethical Considerations

Ethical considerations were observed in the research process to protect the rights of the respondents. First, respondents were told that personal information was confidential and would not be shared with anyone outside the study. Furthermore, respondents were consulted to ensure that the schedule of interviews and observations accommodated them and did not inhibit their other endeavors. The purposes of the study were explained briefly to the respondents to encourage their level of participation in the study. Finally, results were not manipulated in any way. The subjective interpretation of results was designed to avoid confusing the reader in a way that manipulates the actual results. The Internal Review Board (IRB) policies of the university where the researcher was located were incorporated in the study as an ethical consideration. Nothing that violated the IRB policies was implemented in the study.

# Chapter IV
## Results

Mathematics education research has demonstrated a unique history as it has evolved over the decades. The literature supporting this research has shown how teachers assist their students in comprehending the variety of applications found within the mathematical area known as *algebra*. The study of this form of mathematics can be extremely difficult for some students, and middle school teachers must therefore utilize multiple strategies to assist their students in understanding and analyzing the concepts and procedures. A key to teachers' support of these students is their understanding of what the students are thinking when solving algebraic word problems.

This research was completed in three stages. The first stage sought to identify expert middle school mathematics teachers' knowledge of middle school students' mathematical thinking about algebraic word problem solving. The research tool was an open-ended questionnaire (Appendix A) completed by nine expert middle school mathematics teachers (identified as teachers A, B, C, D, E, I, J, K, and S). The questionnaire was divided into three main sections. The first part of the questionnaire described those teachers who willingly responded to the questions. The second part focused on teachers' views and dealt with their perceptions of students' problem solving capabilities. In essence, this part of the questionnaire sought to elucidate the teachers' personal views regarding both their students' thinking about algebra and their students' capabilities for solving algebraic word problems. It also sought to uncover the teachers' views about how they prepared lessons to support the students in working with algebraic word problems. The third section focused on the students' thinking. A word problem was presented and the teachers were asked to describe how the students typically approached this problem. The barriers to successfully solving the word problem were described as well as student weaknesses in working toward solutions of word problems.

Of the nine expert teachers considered in the study of teachers' knowledge of middle school students' mathematical thinking about algebraic word problem solving, five of the teachers were in the 40-49 age bracket. The other three respondents were at least 50 years old, and only one teacher was in the 20-29 age group. Eight out of nine respondents already had Master's degrees, while only one teacher was 15 hours more advanced than those with a Bachelor's degree. With regard to the number of years of teaching experience, the average was found to be 21 years, with a range from 7 to 30 years, a mode of 25 years (where two out of the nine teachers had been in the profession for 25 years) and a median of teaching years as 18.

In this first stage of the research, the teachers described different conditions that affected their understandings of student knowledge in working with algebraic word problems. One condition was the curriculum, followed by the students and the subject being taught during the study. One teacher said that she was teaching Geometry II, which was also integrated with Algebra. Another teacher followed the curriculum in the *Connected Mathematics* series. An additional complication for teachers in relying on student understanding was the student mix. The teachers described their students' knowledge and capabilities by relating their abilities as low/high, and as Math I or Math C students. These descriptors were used in their individual school settings. Moreover, their classes also had an ethnic mix of whites and Hispanics, which accounted for another difference in the students' understanding of English and mathematical language. Generally, Hispanic students do not have a good grasp of the English language. They also have difficulty dealing with algebraic word problems because these problems are typically expressed in the English language. First, many of the words are not familiar to Hispanic students, particularly when they are new to schools that cater mostly to American students. Also, the literacy rate of Hispanics is often lower than that of their American counterparts. These students have difficulty not only in understanding the English words that they encounter but also in understanding the concepts embedded in the problems. This fact particularly adds to the burden that the teachers encounter during their mathematics classes. Understandably, these middle school mathematics teachers had to deal repeatedly with these issues when teaching algebra. Thus their responses reflected the impact of these diversities.

In the second stage of the research, the number of respondents was reduced to four teachers (Teacher A, Teacher I, Teacher J and Teacher S) who were willing to participate in extended interviews. Unlike the survey questionnaire process, the interview process provided more details on a number of questions directed toward gathering data extended beyond the questionnaire data as a means of clarifying the responses with respect to the objectives of the study. The entire interview process was audio recorded using a digital recorder to allow extended review of the teachers' responses. The interview questions first established rapport and ease with the interview process. Thus, the respondents were not hesitant to provide important details adding to the quality of the interview data. After the interview, the recorded responses were transcribed in detail.

The interview sought to extend the teachers' thinking about the question of how they used their knowledge of students' mathematical thinking when planning lessons for guiding their students in solving algebraic word problems. In the questionnaire, the nine respondents were asked for their perceptions of their students' level of thinking in algebra, with the teachers providing varied points of view. During the interview, the researcher asked the teachers how their knowledge of students' mathematical thinking was best used for the benefit of the students' learning. Responses to this question helped to determine how the teachers made use of their knowledge for guiding middle school students in overcoming difficulties in algebraic word problem solving. The four teachers expressed their views clearly and responded confidently as they spoke from their personal experiences in teaching algebraic word problem solving. The number of years that they spent teaching guided their understanding of middle school students' approaches in solving algebraic word problems. The teachers continually relied on their experiences and their knowledge about the students as a guide in planning their daily algebra lessons. They determined the activities for the class based on their understanding of how the students thought algebraically.

The third stage of the research involved observations, video recording of two teachers (Teacher J and Teacher S) in actual classes taught by expert middle level mathematics teachers and pre- and post-observation interviews. This stage was designed to assess whether the teachers actually operated as they indicated in the questionnaire and the interviews as well as to provide a more in depth understanding of teachers' use of student thinking. This stage was designed to specifically focus on the third research question of how expert middle level mathematics teachers used their knowledge of students' mathematical thinking when actively teaching students to solve algebraic word problems. The observations were repeated daily in order to establish each teacher's patterns in guiding their students' work with algebraic word problems.

## Teachers' Understanding of Students' Algebraic Problem Solving

The nine expert middle school mathematics teachers were asked to consider a specific problem designed to elicit an initial description of the nature of the mathematics teachers' knowledge about middle school students' algebraic thinking.

> Blue pencils cost 4 cents each and red pencils cost 6 cents each. I buy some blue pencils and some red pencils and the total cost is 90 cents. If *b* is the number of blue pencils bought, and *r* is the number of red pencils bought, what can you write down about *b* and *r*?

The teachers' described their ideas about how typical middle school students might work as they sought to answer this problem.

These nine teachers felt the most common method used to solve problems was guess-and-check. They expected low level students to use this strategy almost continuously as they solved problems. One of the teachers indicated that the average students used guess-and-check to answer the problems. Although the problem in the questionnaire required identification of an algebraic equation using variables, teachers described the students as relying on the guess-and-check strategy to find numeric solutions. The teachers indicated that students did not initially direct their attention towards using variables and identifying equations to show the relationships among variables.

> … My lower students possibly would just take 90/4 ≈22 blue, while others 90/6 ≈15 red. Some would combine those two into one answer. Others would guess-and-check [and] come up with 10 blue and 8 red, rounding to get 90. (Teacher A, survey questionnaire)

> The average middle school student would probably use guess-and-check to answer this problem. (Teacher E, survey questionnaire)

The teachers indicated that students made systematic lists, organized charts for each possibility (combinations), or relied on graphs to solve the problem. Teacher I successfully described the chart that some students might draw while others might identify specific answers or set up equations. Some students were able to find relationships among variables by graphing or charting the solution using both dependent and independent variables. The use of graphs and charts made it easier for students to visualize the problems. Students who represented the situations in the problem in terms of pictures and diagrams found the problems interesting and much easier to understand.

> I think some might make an organized systematic list and get most answers. (Teacher K, survey questionnaire)

> Students might chart

| Blue | Red | |
|---|---|---|
| 3*0.04 | 13*0.06 | =.90 |
| 6 | 11 | |
| 9 | 9 | |
| 12 | 7 | |
| 15 | 5 | |
| 18 | 3 | |
| 21 | 1 | |

> From the chart they could say as *b* increases *r* decreases. If b=0 then r=15.

> If b=3 then r=13. (Teacher I, survey questionnaire)

> Some students will make a chart, guess-and-check, make an equation and graph it, or organized chart for each possibility… (Teacher D, survey questionnaire)

Eight teachers figured their higher-level students would be able to set up equations for problems like the pencil problem. The teachers described students looking for patterns from possible solution sets. Only one teacher insisted that students would not try to set it up algebraically. In this view, the students used guess-and-check in order to find some possible numeric answers, and then they simply stopped because they got tired. At the beginning of the year in the classes, most students would not solve the problem algebraically. The teachers recognized that although the problem did not ask for a specific answer, the students were inclined to look quickly for a numerical value. Some teachers indicated that the students used variables in the equation while others included the money unit (¢) when their understanding was basically of whole numbers.

> Some may write b*4¢ + 6¢= 90¢. (Teacher I, survey questionnaire)

> My students would respond with things like you could get 4b+6r=90. (Teacher B, survey questionnaire)

> I don't think they would try to set it up algebraically. (Teacher K, survey questionnaire)

The teachers indicated that high-level students primarily used deductive strategies for solving algebraic word problems. The students first set up equations and used the guess-and-check strategy to check whether their equations were correct.

> My Math 1 students would set up the equation 4b+6r = 90. They'd try to solve it and then through guess-and-check, come up with 15 blue and 5 red. (Teacher A, survey questionnaire)

The teachers understood that students solved the problems in their own unique ways. One teacher suggested students might use decimals for setting up the equation as .04b+.06r=.9 while another teacher indicated the students' equations for the problem were represented differently. Often the students used the standard form of a linear function and some students even provided a legend to indicate what the variables meant. Even though they were setting up an equation that was similar, the students differed in their use of equations.

> 4b+6r=90
> b= number of blue pencils
> r= number of red pencils
> Some algebra students might come up with b=22.5-3/2r and graph it with their calculators. (Teacher J, survey questionnaire)

The teachers indicated that realistic problems were easier for students to understand and solve. If they had information that seems "realistic" to them (students), the problem was usually "easier" to attack and solve. Students found realistic problems easier because these problems resembled those instances that they encountered in their day-to-day living. Because of this experience, the situations were more familiar so they usually recalled how they reacted or solved these situations in real life, trying to apply the same logic to the word problems given to them. On the other hand, if the information was more "abstract," they struggled. Students normally did not picture the solution for an imaginary problem or a problem that they had never encountered in the real world. Also, if the information in the problem was related to other subjects they had learned, they were more engaged. The teachers recognized the importance of the students' prior knowledge in developing their strategies for working with algebraic word problems. They indicated that realistic mathematics problems were easier for students to understand and solve with precision.

> Their previous knowledge is the biggest factor when assessing word problems. If they have information that seems "realistic" to them (students), the problem is usually "easier" to attack and solve. If information is "abstract" in thinking, they struggle. (Teacher D, survey questionnaire)

> Some problems can engage students more than other problems. It also depends on how the problems connect to other learning. (Teacher I, survey questionnaire)

The teachers agreed that the students' abilities for appropriately reading the wording in algebraic problems were essential. All of these teachers perceived their students attitudes were a mixture of fear and apprehension when they first encountered the idea of algebra and solving algebraic word problems. Students' reading comprehension was a primary concern for these teachers who asked students to engage in solving algebraic word problems. The teachers indicated that since algebraic word problems often appeared foreign to their students, they were concerned about how well the students might do in such situations. The teachers perceived problems with reading comprehension to be one of the main reasons for this form of distress and confusion. However, once the students realized that the application of algebra involved letters representing missing values or variables, they relaxed and were more open-minded toward the whole idea of problem solving.

Most of the teachers insisted that the concept of solving algebraic word problems was related to the students' capabilities in reading and comprehending what they read. If they did well in this area then the mathematics as presented was even easier for them. However, if they struggled simply to understand the word placement then they had a difficult time with the fundamental algebraic processes. The teachers indicated that students who were good readers liked word problems. Some students did not read word problems at all because before they deciphered a word problem they felt that such problems were tedious. As two teachers indicated:

> Many are not good readers to begin with; they often do not read it before they say "I don't get it." Once they do read it (also if necessary) and underline key points, they can often attack it. (Teacher B, survey questionnaire)

> Those students who are low readers generally do not like word problems because they do not understand what they are being asked to do. (Teacher A, survey questionnaire)

Considering reading and comprehension, the teachers indicated that some students did not understand the mathematics vocabulary in the problem. The teachers thought that some students were stuck on the vocabulary instead of focusing on the problem asked. Because of this, students just used the time for understanding what the word or words in the problem meant rather than spending time searching for a solution to the problem.

> Many of my students are Hispanic and do not have a good grasp of English. (Teacher S, survey questionnaire)

These middle school teachers claimed that students' dislike of word problems was due to the lack of confidence that they exhibited every time they worked with an algebraic problem. They believed that their lack of confidence resulted from their reading and comprehension abilities in a generalized context. Some teachers stated that past experiences with mathematics was the issue that resulted in the development of a fear of working with algebraic problems.

> Most students who have little algebra experience are scared by even the word itself… However even those who are quite experienced at solving equations freeze when presented with a word problem. Again I believe it goes back to

> their lack of confidence with their reading/comprehension abilities. (Teacher A, survey questionnaire)

The teachers highlighted many barriers for students attempting to solve algebraic word problems. Students did not understand what the questions asked. Their comprehension of the questions was a barrier for them in solving algebraic word problems. The teachers indicated that the question itself stressed their students, confusing them as to what the problem was actually asking.

> The question itself would be a barrier (they wouldn't understand what it's asking them to find)… Some students wouldn't know where to start. (Teacher A, survey questionnaire)

> Not really knowing what is expected. (Teacher B, survey questionnaire)

The order of the sentences in the problems and the placement of the questions within the sentences of the problems presented another barrier that frustrated students, because they often did not know where to begin. They tried starting from the explicit questions in the problems instead of dealing with important vocabulary comprehension of the problems. This process created incorrect answers also.

> … They may not know where they could begin. (Teacher I, survey questionnaire)

Another barrier resulted from using letters to represent the relationships. Students usually associated word problems with numbers. Even before solving word problems, they were already oriented to seeing numbers being used to solve specific problems. Conflicts arose, however, when variables were introduced into the problems. Some of the students found problems with variables and minimal numbers to be difficult. They did not like the idea of having numbers represented by a letter when in the end there really was only one value to represent the variable. Also, in many of the problems they were not sure how to describe the relationships among the variables.

> Students may not know how to use the *b* and *r* to represent the relationship. (Teacher I, survey questionnaire)

> They would ignore the variables and answering the actual problem. Or they would be nervous, upset and frustrated by the variables. (Teacher J, follow-up interview)

The teachers claimed that students were not good at representing situations with algebraic equations and using variables because they did not have enough opportunities to use variables in symbolic representations; they were simply scared because they were used to using numbers when solving problems. The teachers felt that the students were not able to see the relationships among the variables and focused on just finding possible solutions. The symbolic representations seemed to make the problems more difficult for some students.

> They aren't that good at representing situations with algebraic equations and using variables. (Teacher J, survey questionnaire)

For some students, working with decimals intensified the difficulty of the problem as a result of their inexperience with decimal placement. This issue often resulted in incorrect answers. The teachers indicated that some middle school students were weak in basic computational skills with multiplication and division due to their overuse of calculators.

> The only other weakness I can think of is their basic computation skills (multiplying, adding). This is a weakness for many students but not realized due to the over-use of calculators. (Teacher A, survey questionnaire)

> We are finding that our students are coming to us with lower and lower basic skills due to lack of basic conceptual understandings and over use of calculators before basic idea are clear. (Teacher K, survey questionnaire)

One barrier involved students' hesitation or refusal to check or verify their solutions. Students were expected to try to solve problems in different ways at this stage. Even when the problems required students to determine the relationship between variables, they looked for specific solutions and, when they found them, they considered the problem solution completed rather than verifying their solutions. Hence, they were not able to detect other means or strategies of solving a particular problem.

> The weaknesses might really be in checking or verifying their solutions. (Teacher C, survey questionnaire)

Students' misconceptions about algebraic word problems included thinking that number rules and variable rules were different. Numbers and variables appeared physically different and the students generally did not see that variables were representations of numbers and that the treatments were at some point the same.

> … And the ones that are not, they still understand that N is the number of students. Their problem is putting the N in with numbers and making the rules apply to both. They think since one is a letter and one is a number the rules shouldn't apply the same way. (Teacher A, follow-up interview)

These teachers described middle school students' solution strategies when working with algebraic word problems. They understood that students had their own unique solution strategies and different approaches. They also indicated students' barriers and misconceptions. They felt they knew what their students knew and what the students needed. In summary, the questionnaire results of the nine teachers' understanding of students' algebraic problem solving indicated that:

1. Students' solution strategies to algebraic word problems included:

    1) guess-and-check

2) graph and chart
3) equations

2. Strategies for enhancing students' comprehension and identification of solutions relied on:
    1) realistic problems
    2) readability and comprehension of the problems requiring:
        - good reading abilities
        - appropriate vocabulary
        - adequate student confidence
3. Students' barriers in solving algebraic problems arose from:
    1) comprehension of the problem itself
    2) use of variables
    3) adequate basic computational skills
    4) review and verification of solutions

## Teachers' Plans Based on Their Understanding of Student Thinking

The expert middle school teachers were asked about their planning in both the questionnaire and the interview. Specifically, they were asked to consider how their understanding of students' thinking informed their lesson planning processes.

The teachers did make yearlong plans involved with preparing students for word problem solving. They relied on their preparation by the Oregon Department of Education to focus on problem solving skills and strategies.

> Train students each year with the math problem solving training provided by ODE (Oregon Department Education) years ago. Teach them problem solving skills and strategies and the state scoring guide more than they used to. I show them how it gives them opportunities to do math in many different approaches yet come up with the "right" answer. (Teacher K, survey questionnaire)

The teachers explained that their general planning for teaching algebraic word problem solving for this age level started with a good plan. However, they indicated that they did not prepare specific, directed lessons on word problem solving. Instead they incorporated word problems in each of their lessons, using word problems to launch the lessons. They mentally outlined their lessons and carefully thought about the following lessons. They thought about questions that might enhance students' thinking along with questions about their approaches and solutions. They imagined how the lessons needed to go rather than preparing detailed written plans. These thoughts formed the framework for their lessons and were consistently on their minds as they taught the classes.

> I don't prepare specific lessons for word problems. Most of the curriculum I use begins with a word problem. (Teacher I, survey questionnaire)

> Most of the time I don't (plan). I've been teaching algebra for 15 years. What I have done is because this book is a very traditional type book as I get better at [teaching] *Connected Math* and work at the idea how that supposed to be taught. (Teacher S, follow-up interview)

The teachers focused on creating a classroom climate where students were comfortable taking risks, clarifying their misconceptions and describing their strategies in solving problems without fear. They focused on consistency in supporting students in learning mathematics through long-term preparation with high expectations. They knew that giving specific directions made students uncomfortable and also helped them focus on the problem solving. They were confident that their classes supported students in learning to solve algebraic word problems.

> Creating a climate where students are comfortable taking risks is important. I do not give them a specific format to follow (first do this, then do this etc.). I found that when I did that the focus was not the problem itself. (Teacher C, survey questionnaire)

> I think it starts from day one and it's consistency. You have to have very high expectations and then you maintain them throughout the year. You can have expectations and then apply them. It's a lot easier to just start out with very strict expectations and then let off a little as they become comfortable and know what you expect. I think – yeah, just consistency. (Teacher A, follow-up interview)

The teachers indicated that they used routine instructional strategies that placed them in the role of a facilitator. They gave the problems and organized the students in groups to work on the problems with their groups. The students discussed the problem solutions and the teacher facilitated their identification of the solutions using a questioning strategy to evoke students' mathematical thinking without providing the teacher's ideas for solving the problems. After identifying the solutions, the students shared their ideas on how they found the solutions. Routine instructional strategies prepared students for what they had to do and how they had to approach solving algebraic word problems.

> I used *Connected Mathematics* (CMP) as the core curriculum. … Most of the time, I place students into teams where each member of the team has a role to play in the solving the problem. Students are given about 20 minutes to work through the problem to come up with the solution. At the end of the problem we summarize or learn by letting the students share their ideas on how they found the solution. (Teacher S, survey questionnaire)

> My role is to facilitate. I will launch the problem and while students are working I will walk from team to team and ask genuine questions. Genuine questions are questions that provoke responses and thoughts from the students without giving them any of my ideas on how to solve the problem. I use a tone of voice that does not disclose whether there is a good one or not. I want my

> questioning strategy to evoke thinking that will let them determine if their problem solving strategy is appropriate or not. (Teacher S, survey questionnaire)

### Planning Focused on Students' Solution Strategies

The teachers described their students' various problem-solving strategies. The students used guess-and-check, working backwards, drawing a picture, making tables to look for patterns, and setting up equations to solve algebraic word problems. When the teachers incorporated the problems in their instruction, they expected students to use different types of solution strategies and expressly encouraged students to solve algebraic word problems using multiple strategies. However, to help the students solve problems, the teachers provided them with enough time for them to use guess-and-check strategies. They also recognized the need to be patient for students to solve the problems no matter which strategies they choose.

> I start the year with a problem solving unit where we go over just tons of different strategies to solve problems. Guess and check, work backwards, draw a picture, make a table, you know, look for a pattern. And so, since we start the year with that I get to use it all year long. (Teacher A, follow-up interview)

> Given enough time, they may find the answer while others would just get "close enough." If this were a problem solving task, I would expect most of my students to find the answer. (Teacher A, survey questionnaire)

> So not only is that supportive piece there, but there's patience that you know that it's not you know, it's not quick, not quick to solve this problem. It will take us a while before we are able to do this. (Teacher J, follow-up interview)

The teachers spent most of their time choosing good problems with many different possible solution strategies to show the advantages of using different problem solving strategies. As a result, the students were able to solve the problems using guess-and-check as well as other strategies. Still others said that they let the students answer the word problems but required multiple solution paths. If the students were able to answer the problem using varying solution paths that resulted in the same answer, then the teacher was able to get a glimpse of the students' thinking. The teachers let the students work the problems by themselves because they felt that too much intervention did not help these students learn to solve problems independently. In addition the teachers provided the students with a variety of problems to solve so that they were able to experience different ways of solving a problem and to develop different solution strategies. Also, problem-solving exercises helped the students become at ease with finding solutions to these problems and lessened their fears of taking risks in solving.

> I try to select problems that can be solved several different ways and work through them myself. (Teacher B, survey questionnaire)

> I show them how it gives them opportunities to do math in many different approaches yet come up with the "right" answer. (Teacher K, survey questionnaire)

> It is important for students to have lots of experiences with many types of problems. I tried to give them a variety of varied and interesting topics… (Teacher C, survey questionnaire)

The teachers knew that the guess-and-check strategy was not the most efficient strategy, yet some problems were more easily considered using the guess-and-check strategy. Usually, these problems were not as complicated and did not require an equation for solution. However, the teachers did not discount the fact that at one point in their class discussion, the students needed to shift to other more efficient and systematic strategies than guess-and-check. For this reason, they adapted their instruction to include work with different strategies in addition to guess-and-check.

The teachers suggested that they begin this work with ideas focused around students' interests in developing student understanding. The teachers encouraged finding different and easier ways derived from other students' strategies.

> Can I use guess-and-check? In last period when he wanted to solve that problem by guess-and-check I wanted to point out to him that guess-and-check wasn't so efficient, and so I was trying to point out that, you know, your method here is slow and ours is quick. So that caught my attention. I wanted to talk to him about it because it caught my attention that he wanted to do it different from everyone else. And that's actually what I tend to do, is when someone wants to do it different than everyone else. (Teacher J, follow-up interview)

The teachers knew that the students thought that they were finished with the problem when they reached a solution. Therefore, they guided students to find more solutions through questioning and adapting their instruction to include work with different strategies in addition to guess-and-check.

> … and I said what's the most basic easiest fall back strategy to use and three of the students right away said guess-and-check and that's because that's how I start my year is with different strategies. (Teacher J, follow-up interview)

> Many students would guess-and-check 4¢*# plus 6¢# until it equals 90¢. They would probably get a solution and be done. Then you could say, "Can you find more than one possible way?" Then they could continue guessing and checking to find more solutions. Some may be systematic in their guesses. They may look for patterns in the solutions that work. (Teacher I, survey questionnaire)

> …Yeah, I start the year with a problem solving unit where we go over just tons of different strategies to solve problems -- guess-and-check, work backwards, draw a picture, make a table, you know, look for a pattern. (Teacher A, follow-up interview)

**Strategies for Enhancing Students' Comprehension in Solving Problems**

Students were easily bored with solving abstract problems, especially algebraic word problems. If they were given realistic problems or problems related to their lives, they were more likely to attack the problems. Also their level of anxiety increased when working on these algebraic word problems. In order to reduce their anxiety, the teachers prepared varied and realistic problems. They also relied on the textbook, in an effort to help students connect the problems to real world situations.

> I tried to give them interesting topics. (Teacher C, survey questionnaire)
>
> To prepare this lesson what I actually did is search this book and whole traditional books I have and couple of books another textbooks company gave me. Then I can find something new. (Teacher S, follow-up interview)
>
> They don't ever see algebra as interconnected or that I might use something here down the road or that I might use it in the real world. You know, any of that there, they have a difficult time making that connection. (Teacher J, follow-up interview)

Their lessons were planned around their students' unique abilities to read and interpret words correctly. The teachers used their knowledge of the students' level of algebraic thinking in preparing their lesson plans for a particular class. Student concerns, especially in regards to solving algebraic word problems, were addressed. With regard to how students viewed algebra and how they solved algebraic word problems, teachers also expressed that they expected the improvements to continue when the lessons that the teachers prepared actually addressed students' needs. Although they felt that many of their students enjoyed the word problems they gave, they were concerned about the students' ability to decipher the proper step-by-step mechanics of the algebraic word problem solving themselves. Some students simply did not know how to solve word problems and the concepts were confusing since they were used to thinking that algebra was difficult.

They stated that some of their students had problems with reading and others did not favor doing algebraic word problems because they could not understand what was being asked. For this reason, the teachers also focused on reading habits. They established individual work routines for their students. They based this activity on where each of the students stood in his or her academic reading capabilities. The teachers also believed that the best way to reach their students and ensure that they comprehended the material was to have them read the problems aloud. If they were able to read it correctly, the teachers felt that their students could work with the word problem accurately as well. They felt that it would not be fair to give the same type of word problem to students who were not as advanced as others, or who excelled and were highly proficient in reading.

The students' skills in solving algebraic word problems were associated with the level of difficulty of the algebraic word problems that they solved. Word choice played an important part in algebraic word problem solving. The first step that the teachers took was to consider the students' level of understanding of the word problem itself as well as the vocabulary.

> First I take into account reading levels and vocabulary. Depending on the level of students, I adjust the problems to their level (less words, smaller answers, etc). We'd start with a problem that is simple, and work our way up to more complex problems. All problems, no matter the level, would be read aloud. (Teacher A, survey questionnaire)

For the reading and vocabulary difficulties, the teachers defined the words initially to get them started rather than let them read words with which they were unfamiliar. Defining the words helped them to connect all the elements of the problem and make one coherent story. They were able to see the interconnection between words and concepts, making it easier for them to understand what the problem was asking. Without this attention, students consumed the time intended to be spent looking for the solution to the problem by attempting to understand the word or words in the paragraph. Using this strategy, the teachers gained a clearer understanding of the vocabulary students knew and did not know and this process made it easier for the students to understand what the problem really wanted from them.

> I often read the problems and define words to get them started. (Teacher S, survey questionnaire)

> We work on understanding math vocabulary (i.e. perimeter, sum, product, area, difference etc) throughout the year. We practice turning problems into symbolic representations or finding key information and organizing a table to solve the problem. After reading the problem I ask students to tell me what is being asked. (Teacher I, survey questionnaire)

Some teachers required the students to restate the problems in their own words as they found sentences that they did not understand.

> I think clarifying the last sentence is the key to asking students "What do you think that means?" I'm not sure myself what is being asked for exactly. I tell students when they aren't sure about the question to decide what they think it means and say that on their paper – "I think this is asking me to find all the different solutions" or "this is not asking me for the solution, it's asking me what I know about how b and r are related." (Teacher C, survey questionnaire)

To prevent anxiety, the teachers created a classroom climate where students were comfortable taking risks and clarifying their misconceptions and describing their strategies in solving problems without fear. Also they focused on consistency when supporting students in learning mathematics through their yearlong preparation and high expectations. They knew that giving specific directions made students comfortable and also helped them focus on problem solving. They had confidence that their classes support students in learning to solve algebraic word problems.

> Creating a climate where students are comfortable taking risks is important. I do not give them a specific format to follow (First do this, then do this etc.). I found that when I did that because the focus and not the problem itself. (Teacher C, survey questionnaire)

> I think it starts from day one and it's consistency. You have to have very high expectations and then you maintain them throughout the year. You can have expectations and then apply them. It's a lot easier to just start out with very strict expectations and then let off a little as they become comfortable and know what you expect. I think – yeah, just consistency. (Teacher A, follow-up interview)

**Planning for Dealing with Students' Barriers in Solving Problems**

When the expert teachers prepared their lessons, they considered the students' level of understanding in solving algebraic word problems. Before the teachers presented difficult problems to students, they carefully chose easy problems and layered problems in order to increase the difficulty over time.

> 1. Starting with whole small numbers no decimals – relate to everyday activity like going to movies (for example, five persons at two dollars a piece, how much would it cost for the group to see the movie),
> 2. Choosing between decimals and fractions – students are more comfortable with decimals because they are like working with money,
>
> 3. Transferring from number to symbol, start with a question mark then replace that with letter or symbol for the missing number. (Teacher J, follow-up interview)

The teachers mentioned that students often simply said "I don't get it" without reading when they did not know how to begin solving the problem. They felt that they could remedy these problems by asking the students to read the problems aloud and describe their understanding of the problems. After reading aloud, the students were asked to discuss their comprehension of the problems indicating possible strategies for solving the problems.

> Being able to read – read it out loud to the group – discuss possible things they noticed besides just getting the answer. (Teacher E, survey questionnaire)

With knowledge of the background of the problem, they gained a better perspective. They were then able to reason and think of other strategies that they could use to solve the problem.

> Usually I try to give students the context for the problem. If we solve motion problems about water currents, we discuss how currents would slow you down or speed you up. (Teacher J, survey questionnaire)

The most difficult aspect of algebraic word problem solving was transferring from words to symbolic expressions that also used variables. Students were confused using variables and they did not know how to represent situations symbolically. The teachers thought that demonstrations provided one way to help students more easily represent the numbers using symbols.

> I may first ask students to find possible numbers of blue and red pencils. I could show how to represent a particular solution using variable. They could find other solutions and represent them using variables. (Teacher I, survey questionnaire)

Teachers also used pictures that helped them to translate the problems from words to symbolic expressions.

> You know the other thing that we do often times is represent it pictorially. So we'll look at it as a picture instead of as a variable. Let's try to draw a picture of what's going on here and that is another way that we can sort of get around because pictures are like variables. (Teacher J, follow-up interview)

Some teachers indicated that they used variables in all mathematics units and integrated variables throughout the year to familiarize students with using variables in algebraic problem solving. Once students were accustomed to using variables, they were not reluctant to use them for solving algebraic word problems.

> … throughout the year you have to introduce variables everywhere else. Everywhere, in every unit, in every thing there is always a variable. So that by the time they get to the unit it's nothing … Like before we do our algebra unit we do some problem solving we do integers, we do and there's variables all through there … They want to use a variable to solve it. (Teacher I, survey questionnaire)

While the teachers did not mention the term 'transfer' in the survey questionnaire or the interview, they did indicate that transferring (words to variables) was a difficult task for students. For this reason, they planned lessons focused on the translation of words to variables and expressions at the beginning of the academic year. Students were proficient in using variables and knew how to transfer mathematics vocabulary to mathematics symbol.

> Well, I think one of the things that we began with at the beginning of the year was looking at how vocabulary shapes the equations that you write. How can you interpret what you write? … We spent some time just talking about how, what words mean. Like addition could be represented by the words "the sum of," "increased by," "plus," "more than," "added to," "the total of," just looking at different things that expressed that operation. (Teacher S, March 13th Post-Observation-Interview)

Although some teachers indicated that students' low basic computational skills were due to the overuse of calculators, they did allow the students to use calculators in solving word problems because calculators helped student solve them.

> Some student a lack multiplication facts so they will be dependent on calculators. (Teacher S, survey questionnaire)

The teachers indicated that in order to teach algebraic word problem solving they needed to be good listeners. They listened to the students carefully. This process gave them more of a sense of students' thinking about algebraic word problem solving. The teachers used questioning strategies in communicating with the students. The questions needed to provoke students' ideas and thinking about solving algebraic word problems.

> I'm a pretty good listener and probably one of my strengths in terms of teaching algebraic story problems is that I listen to the kids and I'm willing to go where they go and either try to make sense of it or not … I certainly believe that listening to them talk and listening to their methods of solving problems – gives you a much better idea of how they are thinking about it. I don't know if I promote thinking, but at least I get a better idea of what they're thinking about. (Teacher J, follow-up interview)

To promote student reasoning, the teachers let the students discuss the problems with other students to give them various ideas. In these discussions the students learned how their classmates actually thought. They learned new approaches and they were able to recognize that the problems presented to the class actually did not have unique solutions, but that they could actually solve them in various ways and, moreover, that each had its own strengths and weaknesses. In the end, the teachers discussed the most appropriate solution. After working with multiple strategies, they made comparisons so that students learned to solve problems using different approaches.

> The classroom situation where one kid starts to tell you how to solve it and then another says, "No," you do it this way, then I think that strengthens each other's thinking because they are a bit competitive and at the same time they're able to learn from each other where, I mean for me it's pretty boring and should be avoided at all cost…letting them talk more, and think out loud, and think – you know, talk to each other, and hearing responses from each other is really the best way to let them do the thinking and give myself a break. (Teacher J, follow-up interview)

According to the expert teachers, teachers should be more patient and supportive of the students even if they were asking practically the same questions over and over. Some students easily grasped what a problem asked of them. However, some students just could not understand the problem as well as how to work on the problem after much explanation. During this instance, the teacher needed to be able to project an encouraging image. They did not feel it was good to let the students feel that they would never be able to learn how to solve algebra word problems. Students need to get used to seeing variables in word problems and they need to practice solving these types of problems. Problems that represented diagrams and pictures in addition to variables helped. While students faced barriers in solving word problems, they also demonstrated strengths. Some students easily grasped the concept of using variables in solving equations and were challenged to answer the problem. Others had good reasoning skills.

To promote student reasoning, the teachers let the students communicate with other students. This method provided students with multiple ideas. Class discussions supported the students in learning how their classmates actually thought about the problems. They learned new approaches to solve word problems and were able to recognize that the problems presented to the class actually were only one approach to the solution. Teachers presented different solving strategies and compared them so that students learned multiple solving approaches. This practice was used to enrich the learning and helping students recognize that they could effectively approach a particular problem in a number of ways.

## Classroom Teaching of Algebraic Word Problem Solving

In response to the third research question, two exemplary middle school classroom teachers were observed teaching. These observations related the teachers' in depth understanding of students' thinking about algebraic word problem solving and how they planned and thought while putting these plans into action. The results were analyzed to further explain how middle school teachers put their understanding of students' thinking into their plans and follow these plans in their teaching. The selected teachers were Teacher S and Teacher J; both teachers had participated in the earlier questionnaire and the more detailed interviews.

## Teacher S's Classroom

Teacher S had 29.5 years in teaching mathematics and science with a Bachelor's degree plus an additional 30 graduate hours of coursework. The observation period occurred in Teachers S's algebra class for $7^{th}$-$8^{th}$ grade students. He arranged the classroom with tables facing the front of the room where he had posted mathematics concepts on the walls. The class consisted of 24 students, 14 boys and 10 girls. Most of the students were Hispanic; thus a language barrier existed in this class. The students consistently worked in pairs to solve problems. The classes began each morning at 8:00 and lasted for one hour. After nine class observations and multiple pre- and post-interviews using the stimulated-recall protocol with videos of the classes during February and March 2006, the researcher determined that the evidence had become repetitious, not adding new information about how Teacher S planned and taught with his understanding of students' thinking about working with algebraic word problem solving

Teacher S explained his view of mathematics as an accounting system.

> To me mathematics is a system to account for things. I go back to the original development of mathematics. If you're a caveman and you bring home a deer that you've hunted, and another cave-family is growing corn —or picking wild berries, and you didn't have time to pick wild berries because you were hunting and they didn't have time to hunt because they were picking wild berries, you would have to find a way to make a fair trade. Then and there some kind of system of counting of accountability toward making things fair and even in a trade has to be developed. You're going to trade this much for this much, so systems were developed. That's what math is: systems to account for everything there is. We also create math to explain things. Physics. The universe. Many things were calculated long before they were discovered. (Teacher S, March 15th Post-Observation Interview)

Teacher S used the *Connected Mathematics* series as the core curriculum for the class. In this curriculum, the mathematics was taught using a framework of real world problems. The *Connected Mathematics* series was used in the sixth grade, and ultimately, over a period of three years, 25 booklets were used. In this curriculum the students collected data, built tables and made graphs. This practice prepared the students for higher-level mathematics and focused on solving problems that became progressively more difficult. Teacher S indicated that the advantage of *Connected Mathematics* as compared with other curricula was in detecting misconceptions like that of a line on a graph of a parabola. Students often misconstrue that the graphical representation of a parabola suggested that the parabola had both length and width rather than simply a representation of a line with only the dimension of length. He indicated that they would point to where the line representing the parabola crossed the X-axis (X-intercept), indicating that the line no longer was a rectangle because it had no more width. One boy commented "Well, really it is a rectangle because it does have a little bit of width, but then is it a line?" The students concluded that it was just a representation of a line at that point.

Teacher S carefully organized the activities interchanging the textbook and problem-solving strategies, mathematical vocabulary, and translating words to expressions including symbolic representations with variables. Preparing students to work in the mathematical classroom was important to Teacher S.

Before they changed to the next textbook in the *Connected Mathematics* series, Teacher S inserted new problems to focus more directly on algebraic word problems that were similar to those in the previous books that they had completed. Teacher S thought that if students got the new textbook suddenly in the middle of the academic year, they would reject the material.

> These kids were last year in the *Connected Mathematics* series, and so they already knew about where the independent and dependent variable would go on the graph. They also have [...] extensive use of tables, and so they quickly went back to what they had been doing when they were in the series using tables and graphs to help support their work. (Teacher S, March 13th Post-Observation Interview)

Although Teacher S organized the classroom management, working groups and routine instructions, he did not plan specific lessons. He planned his classroom teaching mentally. From his past experiences in teaching this class, he used a questioning strategy that both engaged the students in the ideas intended for the lessons and allowed their ideas to determine the direction of the class.

> I thought I didn't have it really well-planned out. I just know I wanted to get into this a little bit with the kids, and so exploring the path of a flare that's moving one hundred and initial velocity of 144 feet per second seemed like a good way to do it. (Teacher S, March 2nd Post-Observation Interview)

Teacher S consistently began each lesson with warm up exercises to strengthen students' basic skills while also providing opportunities to return student papers. Figure 3 is a typical example of these warm up exercises. This exercise focused on having the students work on the order of operations to find the answer of 3 and to learn that the parenthetical expression (d) was to be completed first.

$\underline{(6+2)}$ -3 ÷3 = 2
What do you do first?
a. subtraction
b. divide by 3
c. divide by 2
d. parenthesis (brackets)

*Figure 3*. Warm up exercise example. (Teacher S, March 2$^{nd}$ Observation)

Given the student diversity (Caucasian and Hispanic), he had to spend extra time explaining the English words as well as the background needed for the algebraic word problems. When students knew the background of the problems, they more willingly attacked the problems. Teacher S introduced problems that came from other areas or subjects. The students solved a variety of problems as shown in Figure 4 using different problem solving strategies. By solving various problems, students not only learned mathematics concepts but also acquired knowledge they could apply to new areas. He believed students needed to experience different solving strategies to increase their reasoning and logical thinking.

1. Photo Frame Problem

   In order to fit a square photograph into a rectangular frame, each side had been trimmed off. One inch was trimmed off of two opposite sides of the photo, and two inches were trimmed off of the other two sides. Altogether, the trimmed section added up to 64 square inches. What was the original dimension of the photograph? (Teacher S, Feb 23$^{rd}$ Observation)

2. Greatest Area of a Rectangle Problem

   Consider rectangles with a perimeter of 20 meters.

a) Draw a rectangle to represent this situation. Label one side L and label the other side in terms of L.
b) Write an equation for the area, A in terms of L
c) Make a table for your equation. Then, use your table to estimate the greatest area possible for a rectangle with a perimeter of 20 meters. Give the side lengths of this rectangle.
d) Use a calculator or data from your table to help you sketch a graph of the relationship between the length of a side and the area
e) How can you use your graph to find the maximum area? How does your graph show the side length that corresponds to the maximum area?(Teacher S, March 15th Observation)

3. Rectangular Playground Problem

   The length of a rectangular playground is 10 yards less than 3 times its width. In order to add a baseball diamond next to it the width of the playground is decreased by 15 yards and the length is decreased by 35 yards. The area of this reduced playground is 675 square yards. What are the dimensions of the original playground? (Teacher S, March 4th Observation)

4. Velocity Problem

   A flare is launched from a life raft with an initial upward velocity of 144 feet per second. How many seconds will it take for the flare to return to the sea? (Teacher S, March 2nd Observation)

*Figure 4.* Teacher S's classroom Algebraic Word problems.

For example, he introduced students to physics through the **Velocity Problem** shown in Figure 4 by tossing a ball into the air. Some of his students were taking a physical science class from him; therefore, Teacher S could elicit some ideas from them. Also, when they worked in pairs, the students could give one another some direction.

| | |
|---|---|
| Teacher S: | …How high is it after one second? ... |
| C1: | [quiet discussion] Doesn't it depend on the arc of how you shoot it? |
| Teacher S: | Oh, that's a good question. Does it depend on the arc? …This one's like shot more straight up, this one's shot at an angle. |

| | |
|---|---|
| | Flare goes up in the air. But let's say both of them reach the same height. What can you say about how long it takes for both of them to hit the ground? .... At what time will they both reach the peak of the height? |
| C2: | At the same time |
| Teacher S: | At the same time. Now let's talk about why is that so. And this has to do with something that I'm not sure I want to get into heavily today, but we call it "vectors." "Vector" means the object has more than one direction of travel. ... And so it's combined upward motion and sideward motion means the flare has probably a greater initial velocity, because it's not just going up and down, it's also going sideways. ... it doesn't matter how much it's moving sideways. We're just measuring this one vector only. Ok? We're ignoring any sideward vectors in this problem. Yes? |
| C3: | If you throw something that is curved, wouldn't it like never stop? Wouldn't it just keep going slower? |
| Teacher S: | You mean the upward? ...If I throw a ball I can't throw it more than like out there somewhere, right? Because I'm not really good at throwing and it will come back to the earth. It will curve back to the earth. |

(Teacher S, March 2nd Observation)

The effects of gravity were discussed together with trajectories, quadratic equations, and factoring trinomials. With gravity and acceleration, he started the discussion with the simple act of jumping and asked the question: How high can one jump as compared with a popular basketball star, Michael Jordan who can jump 50 inches. By giving familiar examples Teacher S motivated students to understand the situation and to attack the problem.

| | |
|---|---|
| Teacher S: | .... Who here can tell me at what accelerating rate does an object fall? |
| C1: | About ten meters per second. |
| Teacher S: | Per second. Who can explain what that means? |
| C2: | For every second... per second is how fast it's going...faster. |
| Teacher S: | Ah. How fast it's going faster. What do you think of that, is that a good way to describe it? So if we drop a ball off of a tall building, after one second how fast is it going? ... How much is it gaining speed each second? Ten meters per second. So that's what the acceleration of gravity does. You can go |

anywhere on the planet earth and it will gain the same speed everywhere …What keeps you—what are the factors involved in how high you jump?

C3: Gravity and strength.

Teacher S: In basketball players, one of the things that we learn about a player is that—is their vertical leap.… For an average person it's about twenty or thirty inches is all. A human beings vertical leap is not very high. It would be like this. And the older you get, the less it is. Now there are some players that are famous that had a huge vertical leap. Remember Michael Jordan? His was over like fifty inches. That means he could jump up and the bottom of his feet were like this high, Ok? That's a long ways.

(Teacher S, March 2nd Observation)

He then drew the trajectory of a ball tossed into the air (as in Figure 5), indicating the initial speed, the speed going up after one second, two seconds, and continuing until its speed reaches zero at the top before descending. Once students had imagined and visualized the trajectory from the tossed ball, he questioned the students to make sure they understood the problems clearly. Teacher S started with simple problems and then introduced a complicated one. The progression helped students understand more easily.

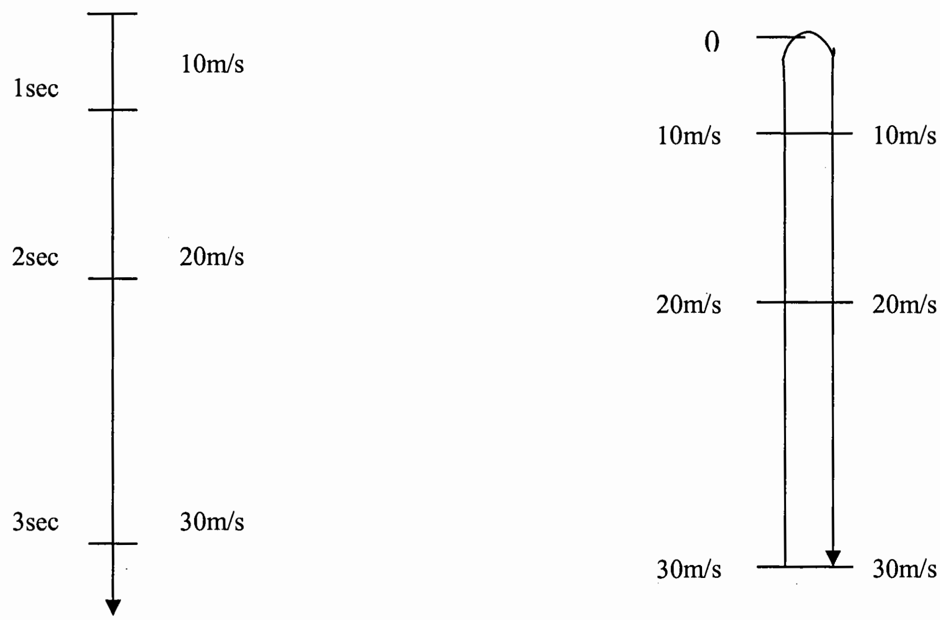



*Figure 5*. Falling objective and shooting objective.

| | |
|---|---|
| Teacher S: | Well, if it falls for one second, it's moving 10 meters per second. In two seconds it's moving at 20 meters per second. After three seconds, it's falling at a rate of 40 meters per second. |
| C2: | Isn't it 30? |
| Teacher S: | Thirty, thank you. Now let's say we've got an object going up and then it's going to come down, and I start it upwards at 30 meters per second. How does gravity act on all objects? … Ok, so let's throw a ball up in the air at 30 meters per second. In one second, how fast is it now moving as it goes upward? |
| C3: | Twenty? |
| Teacher S: | Ten meters per second. And in one more second, how fast is it moving? |
| C4: | Zero. |
| Teacher S: | Zero. It can't go up anymore. It has now slowed down. And now what it has to do because gravity doesn't stop working. Now it's got to go down. Now this is interesting. After one more second, how fast is it moving? |
| C5: | Ten. |
| Teacher S: | In one more second? |
| C6: | Twenty. |
| Teacher S: | And one more second? |
| C7: | Thirty. |

(Teacher S, March $2^{nd}$ Observation)

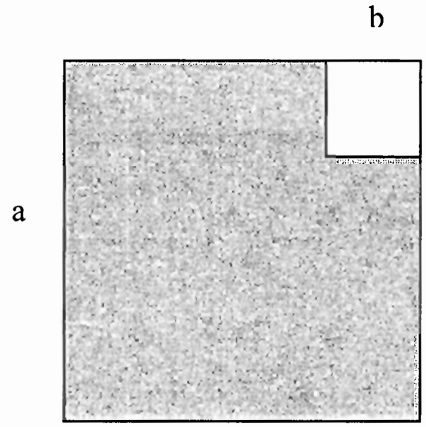
As he watched the replay video of the lesson, he commented:

> Ok, [I'm] trying to give them just a little bit of power and knowledge about gravity. We didn't do any experiments on gravity for students to determine it, so I'm trying to give them a little bit of background so they can get an idea of what gravity's doing to an object that goes up into the air so that they can reason out the next problem. (Teacher S, March $2^{nd}$ Stimulated Recall Interview)

When students encountered algebraic word problems with which they were unfamiliar, he explained that they would give up solving the problems because they thought it was too difficult. Teacher S used the students' prior knowledge in beginning the algebraic word problem solving. He spent a lot of time helping students understand the problems clearly and what they had to do to solve them. He wanted them to understand what was happening before they tried to apply any algorithms or formulas. Sometimes he helped students visualize the problems by drawing graphs, pictures, tables, charts, or diagrams. With those strategies he broke the problems into parts to improve overall understanding of the picture.

To engage students to solve the **Photo Frame Problem** in Figure 4, Teacher S introduced the difference of squares that they had previously studied. He helped them to visualize the difference of squares and gave some ideas to help them understand the story problem. He tried to help students see the relationship between cutting out squares and the new rectangular shapes. After this introduction, the students worked on and solved this problem as shown in Figure 6.

He sketched the ideas as in Figure 6 and asked students to find the area of the shaded area. The students found two answers which he demonstrated were the same. The students had already completed this problem on their own.




$$a^2 - b^2 = (a+b)(a-b)$$

*Figure 6*. Difference of square. (Teacher S, February 23rd Observation)

After the students found the factors, he asked them to redraw the rectangle for (a+b)(a-b). He thought if they could redraw the rectangle for (a+b)(a-b), they could solve the algebraic word problem easily. He asked them what (a+b)(a-b) represented with the new rectangle. After the students drew the rectangle, he inserted a dotted line (as in Figure 7) that helped the students figure out how that affected the problem.

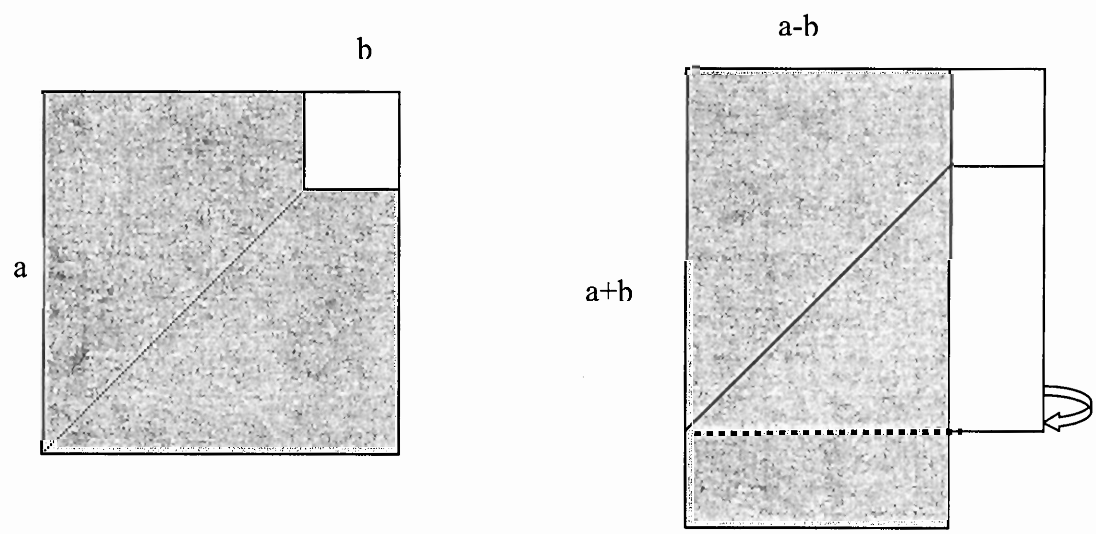



*Figure 7*. Redrawn rectangle for (a+b)(a-b). (Teacher S, February 23rd Observation)

When Teacher S thought the students understood the background of the algebraic word problems clearly, he wrote word problems on the board or assigned problems from the textbook. Teacher S selected the algebraic word problems from the textbook to assure that they would work with a diverse set of problems and that the problems truly challenged them. He felt the problems in Figure 4 represented this diversity.

After the students were assigned the problems, teacher-student interactions occurred briefly and involved Teacher S asking the students questions, seeking clarification, and having them share their ideas and suggestions. His brief interactions included directions to the students to start solving the problem. They also enabled him to assess which students might be having difficulties so that he could spend more time with those students.

| | |
|---|---|
| Teacher S: | Alright, so let's take a look at each one of those one more time. Make sure you understand what you're going to do. Part A says: you are going to sketch several rectangles with a perimeter of 20 meters. You are going to label the dimensions: some with small areas, some with large areas. Then, you're going to make a graph. What are the axes of the graph going to be? Take a look what it says in C. What are the two axes? |
| C: | [multiple answers] |
| Teacher S: | What are they going to be? One of them is going to be area and one of them is going to be the length of a side. Which one goes where? You have your X-axis and your Y-axis. Which one goes on the X-axis? |
| C1: | Length of a side? |
| Teacher S: | Length of a side. That would be correct. And your Y-axis would be...? |
| C2: | Area. |

(Teacher S, March 13th Observation)

If the students had misconceptions, he helped them clarify their understandings by debating and discussing the problem. By debating and discussing with each other, students appeared to understand better than they had after the Teacher S' initial explanation. In one case, the students were confused about the line and area of rectangles. He asked questions consistently to extract their knowledge about these concepts. He did not force his knowledge on them. By guiding them in understanding the representation of a value, he helped them understand the algebraic concepts because he was concerned that their misunderstandings would interfere with their processes in solving algebraic problems.

| | |
|---|---|
| Teacher S: | You said it can't be a line because the width is zero. What do you think about that statement, class? |
| C: | I think it's wrong. |
| Teacher S: | Ok. But he stated that because it has no area, it said no width, it's not a line. |
| C: | It is a line. |
| Teacher S: | It is a line. Now that is directly in counter to what he said. That's the exact opposite. If it has no width, he said it can't be a line, and he said that has no width, that makes it a line. Now, what's true? What do you think? ...Let me ask you a question. ...Can I actually draw what's represented at 40, zero? |
| C: | You cannot physically draw it. |
| Teacher S: | I could not physically draw it, he says. What do you think of that? ... It still would have no thickness. How many sheets of paper could I put on here without it building up any thickness? |
| C: | Infinite ... So, a line does not exist? Like 40 over 40 by itself? |
| Teacher S: | All right, so you have this side of the line which is 40, and you have this side of the line which is 40, what's the total perimeter of this rectangle, if it was a rectangle? |
| C: | 80. |
| Teacher S: | If it has no thickness, does it exist? |
| C: | No. |
| C: | Theoretically. Yes, because you can't see molecules and atoms and they exist. |
| C: | Oh, yeah. |
| Teacher S: | So we invented a symbol... |
| C: | It doesn't exist. |

Teacher S: Right? To represent a value. …
C: Well, shouldn't it exist, because, like gravity is...we know it's there, but you can't explain it. Like, in like numbers.
(Teacher S March 15th Observation)

Once students understood what the question asked, he let them work in pairs because when they worked in pairs, each student's ideas were included in the conversation and at the same time they helped each other. He believed that when they talked with each other, they could organize their thoughts.

Teacher S: (To the researcher) When two students talk to each other, those that have an idea in their conversation, they're also reflecting upon what they're saying at the same time, and the other person is also reflecting on what they're saying to see if it makes sense. They're kind of – talking means that they're organizing their thoughts. Because now, see if it's the first time they're doing it. Their thoughts are kind of all mixed up and they're kind of getting it, but once they can start sharing their thoughts and organizing it in a coherent manner, then it comes together. (Teacher S, March 2nd Observation)

His students were accustomed to working in groups. He often used small groups of two students from a group of four for working on problems in the textbook. In the groups of two students, the students talked to each other, assuring that both students were involved in the discussion. When the groups were larger, he typically assigned specific roles, like timekeeper, facilitator, checker and presenter, but he only used the larger group sizes at the beginning of the academic year.

After watching the small groups work for a while, Teacher S moved among the different groups, observing how the students were approaching the problems. After briefly looking at their strategies throughout the classroom, he then worked with each group, asking for them to explain the approach they were following. One of the groups started solving the **Greatest Rectangle Problem,** but they were diverted in a wrong direction. Therefore, he asked them some questions to redirect their attention toward a more effective path. The group initially solved the problem as in Figure 8. The students were able to find the numbers but they had trouble transferring the ideas into variables. During the conversation he tried not to give too much information. When they got too much information from the teacher, he explained, they did not think any more and they relied solely on the teacher to solve the problem.

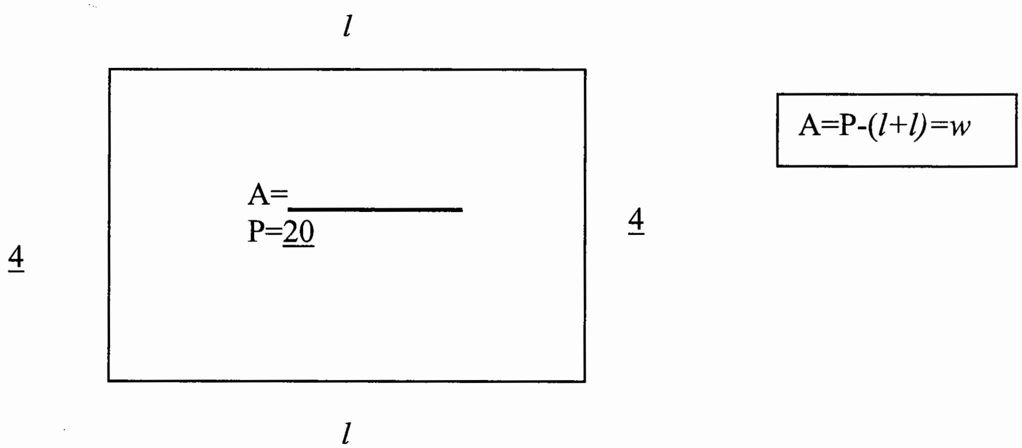



*Figure 8.* A solution strategy by one group of students for Greatest Area of a Rectangular Problem. (Teacher S, March 15th Observation)

| | |
|---|---|
| Teacher S: | So what is—what is the width, then? Tell me what the width is in terms of L. |
| C: | Perimeter minus L plus L. |
| Teacher S: | So, how much is the width? |
| C: | 4. |
| Teacher S: | So, how do we get 4? This equation gets 8. |
| C: | Well, yeah. This is width squared. |
| Teacher S: | Is that right? Is that the width squared? |
| C: | No. It's width plus width. |
| Teacher S: | Ok, so how do you find one of them? |
| C: | Divided by two? |
| Teacher S: | Oh, divided by two. |
| C: | Yeah! We had that. |

(Teacher S, March 15th Observation)

Then he visited other groups. He tried to talk with every group in the class because he wanted to collect all of the students' ideas. Also, he was able to discern the logic students applied in their solutions. He did not want students to go in too many different directions.

| | |
|---|---|
| Teacher S: | How'd you guys do over here? Where's X going to be on here? |
| C1: | On the bottom part of the graph. |
| Teacher S: | X is going to be the length or the width or the area? |
| C1: | I think it's going to be... |
| C2: | I think it's the width. |
| Teacher S: | All right. And then what would Y be? |
| C1: | Length. |
| C2: | the length. |

(Teacher S, March 15th Observation)

Teacher S considered his role as that of a facilitator who interacted with each of the groups by asking questions to provoke their responses and thinking, while taking care not to give them clues and specific answers as to how to solve the problem. When a group was stalled, he helped them by guiding their thinking about the problem through his questions. Although he generally allotted about 30 minutes for this problem solving experience, often the students exceeded this allotment. In fact, this activity was the most typical and consuming part of the lessons. He thought this activity was the most important in solving algebraic problems because through this activity students struggled, reasoned, and developed their logic.

He employed these questioning strategies by first reviewing the whole classroom, looking for students who were struggling more than the others. Whenever he visited a group, he asked what they were doing to solve the problem and where they were in that process. "How are you doing over here?" When he could not understand their ideas, or the explanations did not make sense to him, he consistently asked how they knew what they knew, why they thought like that, or pointed to specific sections in their work as he tried to understand their solving strategies. By giving students the opportunity to convince the teacher about the effectiveness of their approach, the teacher enabled them to clarify their ideas about their strategies.

| | |
|---|---|
| C1: | Start there and go to ten and up. |
| Teacher S: | I'm not sure what you mean by that. |
| C1: | We did zero, 5,6,7,8,9,10. |
| Teacher S: | Why would you start at five? |
| C1: | Because we don't have any data for 1 through 4. |
| Teacher S: | Show me your rectangles. You have no rectangles there, say, with a side length of one? |
| C2: | No, we do, just not the largest side. Because you have to do one...you said... |
| Teacher S: | Oh, you just did the length of the largest side. Why did you pick that? |
| C1: | Instead of the smaller one. |
| Teacher S: | Can you—see, make a table showing the length of a side, |
| C2: | It said any side and we chose the largest one. |
| Teacher S: | So you chose the largest side. Ok. And, so what would the graph look like if you did that? Here's a kind of I wonder. I wonder what would happen if you used the smaller lengths and how that might effect the graph. |
| C2: | Can we do both? |
| Teacher S: | Why don't you try it? |

(Teacher S, March 13th Observation)

Giving the groups adequate time allowed them to approach the problems in multiple ways in teacher S's class. His students did not focus on guess-and-check but used graphs, charts, pictures, and equations to solve algebraic problems.

> And so for them to think on their own and come up with their own solution – which is exactly what's happening… and when you give them the time, they do find other ways to solve problems that you weren't thinking of yourself as a teacher. (Teachers S, February 27th Post-Observation Interview)

Three student examples (Figure 9, Figure 10, and Figure 11) demonstrate the various approaches that the students used in solving different problems: using pictures, charts, graphs, and equations. In Figure 9, the student drew the rectangle for describing the problem and solved the problem by using equations to find the width and length. After finding the answer, the student checked the solution by drawing the rectangle. In Figure 10 the student used the graph and table as well as equations. In Figure 11 the student used a rectangle to solve the problem, indicating his understanding of what the teacher initially explained about the difference of squares by redrawing the rectangle the teacher had used earlier. In each of these cases, Teacher S gave them about 25 to 30 minutes to work in their groups of two to solve the problem by using these strategies. Whenever students wanted more time, he allowed them to work more when he thought they struggled. This approach made them think more and find various solving strategies as shown Figure 9, Figure 10, and Figure 11.

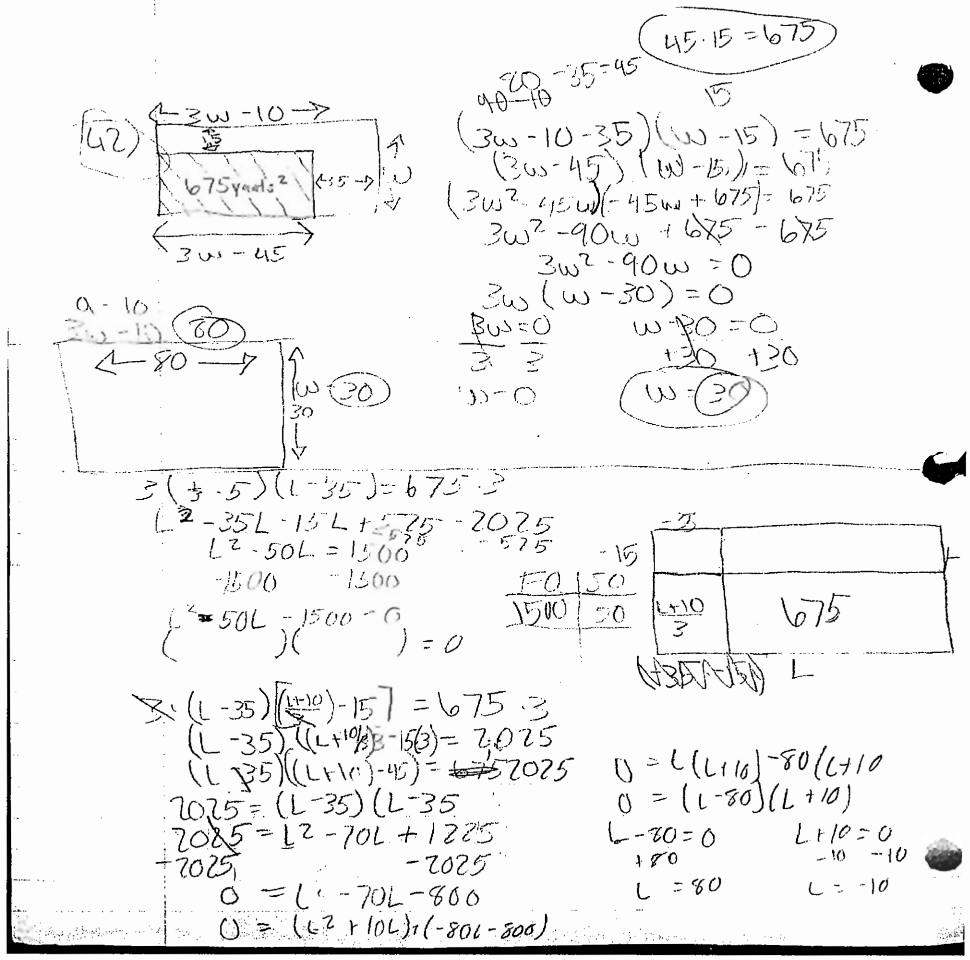


*Figure 9.* A student solution for the Rectangular Playground Problem. (Teacher S, March 4th Observation)

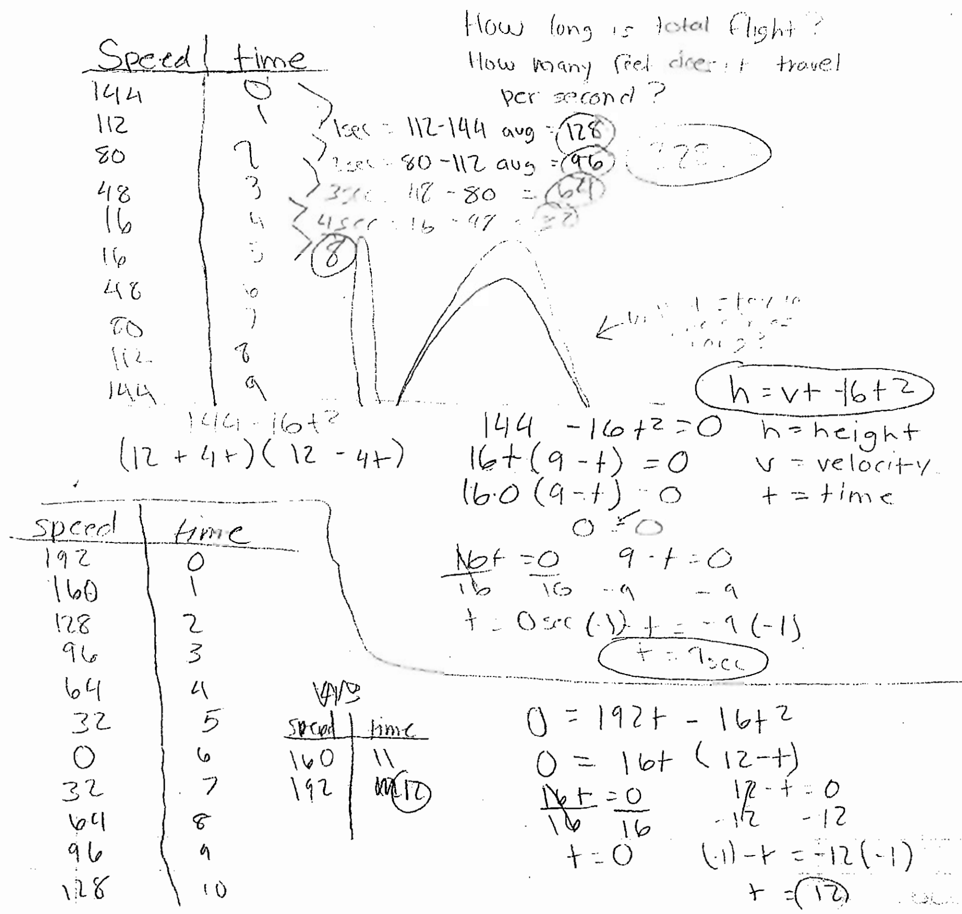



*Figure 10.* A student solution for the **Velocity Problem**. (Teacher X, March 2nd observation)

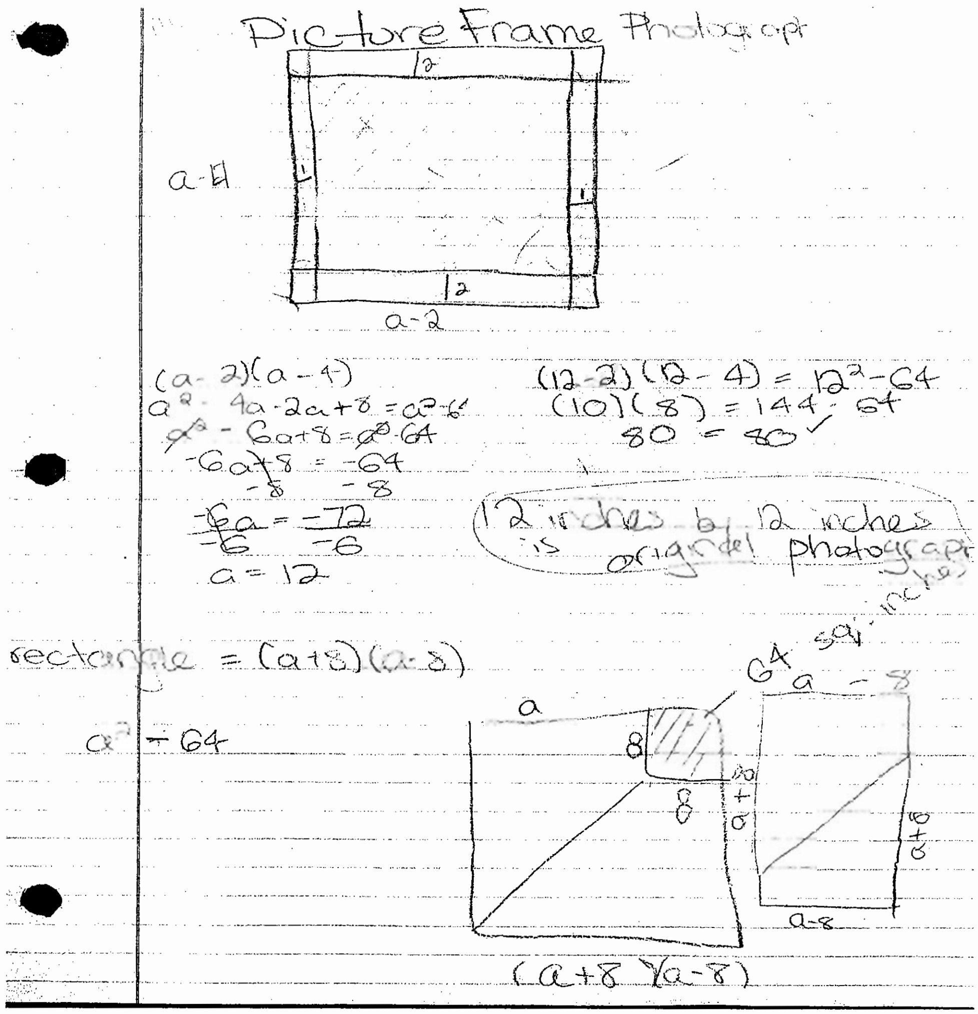



*Figure 11*. A student solution for **Photo Frame Problem**. (Teacher S February 23rd Observation)

After the small group problem solving sessions, he assigned students who used unique strategies for solving the problem to write their solutions on the board. He had discerned which students had different solving strategies while looking around and listening to their strategies.

> Teacher S: Ok. How much did you get? You did a table, right? Would you (Nick) come and put the table on the board? Who did it by diagram? Would you put your diagram on the board?

(Teacher S, March 2nd Observation)

When everyone had finished writing out their strategies, he asked specific students to explain their strategies and solutions to the whole class. Students' explanations enabled other students to understand the solution strategies and solutions better because they used their level of language. The class then collaboratively discussed and criticized the solution strategies and solutions until they understood the strategy. Their strategies were verified by their classmates. Teacher S thought that by communicating with each other, students gained a stronger grasp of mathematics concepts and developed clear solution strategies and solutions. In order to avoid overlooking any additional unique strategies, he asked whether anyone had any different ideas.

Teacher S: Does anybody have any different data? Nobody has different data. Yeah?
C: I got the same data, just I did it a little differently...I started with the top at zero then I started adding 32 on both sides, and at the end I had two—the difference between 128 and 144. Sixteen, 32...
Teacher S: Wait, wait, wait. I'm already lost. What did you do?
C: I started at zero and then I went down...I started adding 32 to...
Teacher S: Oh, you started at the top when it was moving zero.
C: Yeah.
Teacher S: And you kind of like did two paths downward?
C: I had 32 and 64, 96, and then got 128, and then I had 144 at the bottom, but that wasn't 32 so I had—let's say I had 16, because I got sixteen—oh sorry.
Teacher S: I'm just trying to picture how you did this.
(Teacher S, March 2nd Observation)

The solution strategies were presented in class so that the students were not confined to just one type of solution strategy to the problem. Presentation and discussion of the solution strategies in class provided the venue for these students to learn new ideas from their class-mates. As a benefit of this exercise, they learned which of the solution strategies were the most appropriate and the easiest.

> When you do that, it allows the other students who have solved it one way to say: oh, this person solved it this way, I get it. Or oh, I like that one. You know? It gives kids the opportunity to maybe find a method that they didn't come up with themselves. (Teacher S, March 2nd Stimulated Recall Interview)

At the conclusion of the discussions, Teacher S summarized the work by referring to the students' solution strategies. At the end of the class, Teacher S provided follow up questions and assignments for the next class. If he had not had a chance to talk to particular students in the classroom, he emailed them with tasks for them to consider.

Teacher S reviewed the lesson based on students' mathematical thinking and understanding. He already knew much about students' mathematical thinking or understanding for solving algebraic word problems because he had much experience, but he still was not able to always understand their ideas and thinking. Therefore, he used questions to identify what the students were thinking and how they understood algebraic word problems. He listened to students' strategies often and summarized their thinking by asking them to show their different strategies.

Teacher J's Classroom

In the classroom of Teacher J, individual desks were arranged toward the front with pictures, graphs, charts, and assignments posted on the wall. Her classroom was more organized with a more relaxed classroom atmosphere. The students, 17 girls and 10 boys, were all white. Her students had no problems in reading. In fact, she stated that her students were pretty good readers. She indicated in a post-observation interview that she did not use any reading aloud strategy. Teacher J and the students worked together as a whole group. The class was held in a period of 45 minutes in the afternoon at 2:00 PM. She dimmed her classroom light to project the overhead more clearly and to calm the students. Eleven classes were observed from February to March 2006.

Mathematics is about making sense of the world to Teacher J. She thinks that the power of mathematics is that it gives sense to what is happening in the real world.

> You know, from me, math is about making sense of the world, trying to make sense of things. You know it really isn't much beyond that to me. If I can use math to make sense of something, what's going on around me, that's the power from me. That's why I think it's important. (Teacher J, March 15th Post-Observation Interview)

Teacher J purposefully avoided the preparation of detailed lesson plans. If she planned her lessons specifically, the lessons would be scripted and not open to her students' understanding. If she planned her students' understanding of solving algebraic problems, the lesson would go in different directions than she planned. Many times she indicated that her lessons went in unexpected directions. However, she enjoyed the different directions when teaching word problem solving.

> It [a lesson] turned into-like most of my lesson it turned out different than I expected. You know, you kind of plan on one thing and as something comes up you go with it because you see it's going someplace. (Teacher J, February 22nd Post-Observation Interview)

This example shows one lesson that veered in a direction different than she expected. She encouraged the students to solve their own questions and they enjoyed doing so. When they designed their own questions, they were more engaged and they enjoyed solving the problems.

Teacher J: Ann asked a really great question, and she said: What if you lose five percent a year, what if Bill Gates started with 40 billion dollars and he loses five percent a year, how long would he have to go before he wasn't rich anymore...We go back before he was born, now this is way before he was born and he still has—what does he have?

C1: 807 million.

Teacher J: Yeah, I've got to make this into billions.

C2: 807 million.

Teacher J: Yeah, 807 million, doesn't he? Sheesh, and he's not even born yet! … So, that was a great question actually, but I'm going to go back. If Bill Gates decided to put $40 billion in the bank and earn compound interest of 5 % on his money, how long would he have to leave it there to double his money?

(Teacher J, March 16th Observation)

The **Number of Stars and Sand Problem** also went in unexpected directions. After giving information for this problem (Figure 13, Problem 2), Teacher J wanted the students to create a question because she thought it could be really interesting for them. She planned for them to learn more about division, but the students wanted to look at a different problem with exponents. She continued the lesson by using the students' questions and work on Problem 2 in Figure 13.

Teacher J: I'm looking for a question you could ask maybe about all the data [in this problem].

C1: So, you want us to use both of those facts?

C2: If the number of stars in our galaxy is around ten to the eleventh and the number of sand grains is around ten to the twentieth, what is the difference between the two?

C3: Yeah.

Teacher J: Ok. Here's a good question. What's the difference between the number of stars in the galaxy and the number of grains of sand on earth? … And then show me how you got your answer.

(Teacher J, March 7th Observation)

Actually, I planned for them to do something different. I planned for them to divide. I wanted them to divide the sand among all the stars and then find out that they were going to have to cut the sand into pieces and try to introduce the idea of negative exponents. So I had no plan at all to go the direction that we went. (Teacher J, March 7th Post-Observation Interview)

When Teacher J planned her lessons, she relied on her teaching experience from the previous students' responses. She reflected on her lessons and the students' responses from previous years. She tried different ways to teach algebraic word problem solving until she determined ways to guide students in better understanding the ideas. Her experience and practice were the main factors in her lesson preparation which in turn prepared her mentally for her lessons.

> It's a lot of experience. I taught Algebra for 17 years. So, that's most what I rely on it because it [students' different strategies) is very difficult to understand. (Teacher J, Mar 2nd Post-Observation Interview)

> I'm going to try them again a different way next year if I remember, and um – You know, until I can figure out how to make them work, and then I'm going to want a new problem, or a new idea, or a new something. So, I don't like to be bored. (Teacher J, March 8th Post-Observation Interview)

Teacher J planned her lessons by asking herself about the kinds of questions that would lead her students and what kind of questions might be good to facilitate students' mathematical thinking. She thought often about questions she might use to provoke students' thinking because questions were the most important element in the lessons and finding adequate questions was the most difficult job.

> What's the question that I ask next [when students are stuck on a problem]. I don't plan that well. I don't plan well enough to know that I have really thought through … that this is the best question that I ask … I think that really is a piece that as a teacher you're asking yourself how can I ask them the leading question that will lead them to their solution instead of give them their solution, which is where I am feeling. (Teacher J, Feb 23rd Post-Observation-Interview)

When the students had problems with the mathematical language, she explained the language by distinguishing its use in mathematics from its application in other areas because mathematics vocabulary differed from that used in other areas.

When the students did not understand the mathematical word 'difference', for example, Teacher J was concerned that they would apply those same errors when solving the problem.

> Teacher J: This is when you get into the subtleties of language, because difference when you get into another class means something different than it does mathematically. If you're talking about mathematics the difference between two numbers is going to be how far apart they are. How far apart are these two numbers? That's what we're talking about difference mathematically. When you're talking difference, you know the word different has many meanings, many definitions.
>
> (Teacher J, March 7th Observation)

Teacher J started each lesson by reviewing previous lessons and went over questions from homework using the overhead projector to present the lessons. Reviewing the previous lessons, Teacher J checked whether students understood the previous lesson clearly or not. Checking students' prior knowledge was important to structuring her subsequent lessons.

> Teacher J: All right, so my tax problem is this. We have these two equations and we have sort of a sketch of the graph. We started to—I want to know how can I basically zero out my tax curve, and I owe $242 to Oregon, but I get $69 back from the federal government. How much do I need to put into my IRA so that the amount that I get from the federal government is offset by the amount that I have to pay the state government. And because I got a different placement every class, I'm unsure how much further than this we are.
>
> C: This is exactly where we were.
>
> (Teacher J, February 23rd Observation)

Students were comfortable asking their questions and stating their observations while she went over the questions. Teacher J created a comfortable atmosphere in which students felt safe to say anything as long as they were not especially disruptive to the class. In this comfortable situation, students shared their ideas without fear or hesitation, even if their ideas were wrong. She thought students learned a lot from the conversation. The students even understood other students' thinking. Sometimes students found really good ideas by following to other students' tips.

Teacher J: Remember, this was the question you asked yesterday...we said, if you take the number of stars in the universe and you subtract out the grains of sand on earth, what's the difference between the two of them? And when we started looking at those questions where you're subtracting really large numbers that you have to write in exponential form, that there was sort of an interesting pattern that the base value was ten for all of them. So what happened here, ten to the fourth minus ten to the second?
C: 9.9 times ten to the third.
Teacher J: How did you come about that?
C: You take the difference between the two exponents, and it tells you how many *nines* you have, how many digits of nine, and then you take one less than the largest exponent, that tells you ten to what power.
Teacher J: Okay, Is there another way to remember this?
C: It was something like, um, what are the numbers times ten and, uh, the largest exponent minus one.
J: And why? Ann?
C: So when you minus ten thousand…or when you're min…bleh, that thing on the board that you're doing, you get nine-point-nine, or nine-nine-zero-zero-zero. And so when you put that in scientific notation, you get nine-point-nine times ten to the third.
C: Well, I was just going to elaborate on what she said, about the formula. About it is being ten to the *m* minus ten to the *n* equals ten, well, nine, bleh bleh, equals however many nines is the difference between *m* minus *n*…
(Teacher J, March 8th Observation)

For the lesson on exponents she wrote on the board and asked the students to simplify $X^3X^5$. Some answers were: $2X^{14}$, $X^8$, and $X^{15}$. Because Teacher J knew one student could solve this problem easily, she asked this student to explain his answers. The student wrote on the board, decomposing $X^3$ and $X^5$ as in Figure 12.

$$\begin{array}{ccccc} (X\bullet X\bullet X) & \bullet & (X\bullet X\bullet X\bullet X\bullet X) & - & \text{there are 8 X's} \\ \downarrow & & \downarrow & & \\ X^3 & \bullet & X^5 & = & X^8 \end{array}$$

*Figure 12.* A student's thinking on operation of exponent. (Teacher J, February 24th Observation)

She explained this solution further using the overhead, and even used this solution in later classes to introduce exponentials and their applications in compound interests (as in investments) and exponential growth. This interactive activity usually took less than five minutes. Before she advanced to more difficult problems, she introduced easy mathematics concepts.

Teacher J showed a word problem using the overhead after going over review or homework questions. She presented real-world problems to keep students interested in solving algebraic word problems. The problems that she used included those of her own creation (Figure 13, Problem 1), a problem from the Internet (Figure 13, Problem 2), and problems from textbooks or other resources (Figure 13, Problem 3). In the Pre-Algebra class she switched between solving equations and doing story problems. She stated that she was more of story problem-based teacher. The story problem-based teaching made students struggle more. She believed when students struggled more, they learned more in mathematics.

1. Tax Problem

> I just finished my taxes and found out that I will be getting $69 back from the federal government but I will have to pay $242 to Oregon for state taxes. I refigured my taxes with a $1500 contribution to my IRA. If I make that contribution, I will get $294 back from the federal government and

$127 back from the state government. How much should I invest in my IRA so that I break even? In other words, when will my return from the federal government be the same as my payment to the state government? (Teacher J, February 22nd Observation)

2. The Number of Stars and Sand Problem

The number of stars in our galaxy is around $10^{11}$.
The number of sand grains on earth is probably somewhere around $10^{20}$.
(Teacher J, March 7th, Observation)

3. Bill Gate Living Cost Problem

Bill Gates decides that he can survive, he's going to rough it, living on a million dollars a year. You know, plus as the years go by he's going to want the cost of living. So he's going to want a raise each year, not just a million dollars. What will a million dollars in today's money be worth in one year, two years, three years, five years, ten years, and twenty years if the cost of living goes up three percent each year? (Teacher J, March 16th Observation)

*Figure 13*. Teacher J's classroom algebraic word problems.

Teacher J spent some time to make sure the students understood the problems. She explained the background of the problems to help students understand the problem better beginning with their investigations and particularly when students did not have any experience with the problems. Students had more good ideas and approached the problems with a greater number of strategies when they understood the background of the algebraic problems.

Teacher J: When you go to work for company, you're gonna need to pay taxes, you have to figure out how much you need to pay, The taxes system, they take a certain part for the federal government state and this much for social security in the end of the year the school district you pay this taxes what I do I made a table, at the end of the year, and it turns out that I'm good this year, so I have $69 dollars back, but usually I ask my what I have to pay, and usually I got hundred dollars off, the government have that. However, It is pretty disappointing here is I underestimated I have to pay $259, I have their money and I'll gonna keep it till the end of the year. (Teacher J, February 22nd, Observation)

Sometimes Teacher J asked other students to explain to one another how they understood the problem because she felt that students understood each other's ideas better when they could ask their individual questions of each other and they used their own words as they understood them. For example, in one class Teacher J asked that students explain the concept of cost of living to each other in order that they could begin to understand the idea of the problem. In response, one student discussed the **Bill Gates Living Cost Problem** (Figure 13).

Teacher J: Well you explain to her?
C1: Okay. You're finding out how much one million dollars is worth in one year, two years….
C2: …but if the cost of living goes up…
C1: That's what you're finding out.
Teacher J: Cost of living is three percent.
C2: It's three percent of a million.
C1: It says he can survive on one million *plus* the cost of living.
C2: Oooooh!
C1: Next year it's going to be three percent more than the last year.
C2: I got it!
(Teacher J, March 16th Observation)

As soon as students understood what the problem asked, they attacked the problem, working individually or with their neighbors. Teacher J did not make specific groups. Teacher J thought that by learning about other students' thinking, students learned more and found more various solving strategies. While the students worked to solve the problem, she moved around, observed how the students solved the problem, modified or re-stated the problem changing one variable at a time, challenged the students to adjust their solutions, and considered revising the problem situation.

Teacher J advocated for the importance of encouraging students to work on problems collaboratively as if they were a team. She believed that students understood better when they listened to other students' ideas. After moving around the room as the students worked on **Tax Problem** (Figure 13), she wrote the equation 1500 – X – 69 =242 on the board. That equation was from the students. Then she asked what X told them. The students wrote on the board: 1500/294. Sensing that the students were at a loss, she rephrased the question. She decomposed the problem:

> Teacher J: What I'd like you to do is…how much money could I get from the government for each dollar. If I put in 1 dollar how much the government reward me? (Teacher J, February 22nd, Observation)

She felt that visualizing the problem would more effectively enable students to solve the problem more easily. She wrote a table on a transparency as in Figure 14 and then explained:

> Teacher J : If I put zero dollars I got 69 dollars back from the government and 242 dollars I'll have to pay to the state ,and so if I put money in the future the federal will reward and give me more back right now and also the state more. So I'm thinking there must be breakeven point some point where the amount, the federal will give me, and the amount to the state, what I have to pay , combine them they will give zero, the amount I give to the government will be the same I get to from the state how much the government pays you with each dollar, if I put in one dollar what will the federal give you? (Teacher J, February 22nd, Observation)

| IRA | Fed Tax | State Tax |
|---|---|---|
| 0 | 69 | -242 |
| 1500 | 294 | 127 |

*Figure 14*. The table for the **Tax Problem**. (Teacher J, February 22nd, Observation)

When they struggled but had a vague idea about solving the problem, she let them work in smaller groups because when they worked with smaller groups, they could use their own ideas more directly. The students worked in their smaller groups and after a few minutes, they gave different answers: 0.28, 0.15, and 0.03 dollars for each tax dollar paid. Then, she asked them to explain their answers:

> C: Ok, 225 divide by 1500 equals 0.15; 1500 divide by 15 equals100; 421 divide by 15 equals 28, 1500 divide by 421 equals.
>
> Teacher J: When did you get 421?
>
> (Teacher J, February 22nd, Observation)

The student explanations were not clear. After much discussion, the teacher explained how they got 225 (294-69), then the ratio 225/1500 to get the answer 0.15, and continued:

> Teacher J: For every dollar I could give to myself for the future, they're going to give me 0.15 right now. Good deal? Pretty good deal, however I could not get it for 19 years, and the government said the limit after $4000 I could not get any benefit. The state will also give $0.246. (Teacher J, February 22nd Observation)

Teacher J believed that students understood mathematics more clearly when they solved problems by using multiple methods. She asked the class to graph IRA on the X axis and taxes on Y axis, using the equations as shown Figure 15. To solve this problem she used equation, graph and chart with students by using a graphing calculator.

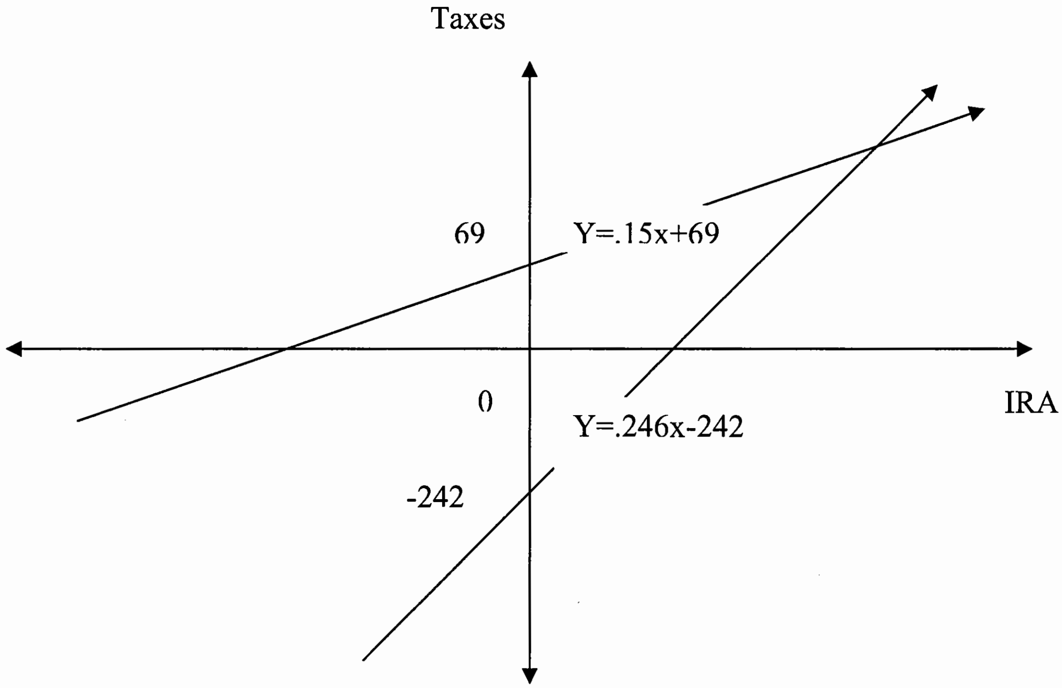



Figure 15. The solution for **Tax Problem** (Teacher J, Feb 23rd Observation)

This lesson was continued on the following day with questions regarding the breakeven point, combining the two equations and the (0,0) point. She planned to explain each solution of the graphs by setting up the equation as shown Figure 16.

| | |
|---|---|
| Y=.15x+69 | Y=.396x-173 |
| Y=.246x-242 | 0=.396x-173 |

*Figure 16*. The equations for solving **Tax Problem.** (Teacher J, February 23rd Observation)

She wanted students to consider different solution strategies that the students in the other class suggested. The students started thinking about the meaning of the equation with variables. She helped the students find the solution to the problem by suggesting a more effective perspective. Teacher J did not provide the students with the specific solution path; rather she only assisted them at the start of the problem. Afterward the students were on their own to work on the problem. Teacher J made students use deduction in this lesson to encourage them to consider another strategy to solve algebraic word problems.

Teacher J: Here's what they said. (She wrote an equation on the board [ 0=.396*x*-173]) I know you know how to solve this. Don't tell me how to solve this. Explain to me why a seventh grade kid came in and said this is how you solve the problem.
C1: They added them together.
Teacher J: Why do you think they add them together?
C2: Well, this problem is to find out where we break even, isn't it?
C3: They added them together and took out Y for some reason.
Teacher J: Why is the zero there?
C1: Wait, see because the zero is there for the break even point. Because you're trying to find the break even point of the two. And if you are getting fifteen cents per dollar and .246 per dollar...
Teacher J: So altogether I'm getting what?
C2: You're getting .396
Teacher J: For every dollar...
Teacher J: So why did they want to subtract the 173?
C3: Um, because I think it has to do with how much money you put into your retirement account you still have to pay.
(Teacher J, February 23rd Observation)

Teacher J knew the students had some psychological barriers when approaching algebraic word problems: fear of math, nervousness and frustrations. She assured her students, "My math is hard, too, but if you stay with me we'll work it out and we'll do it together." In this way, she emphasized the need for teamwork. However, she stated that she was not a "group person." Instead of breaking up the class into small groups, she preferred the whole class interacting and let the students talk among themselves as they worked. She believed in communication in order to understand students' mathematical thinking as well as students' development of mathematical thinking.

For word problems with very large numbers she used real objects such as the number of stars in the universe, number of grains of sand in a cube, and the number of plant/animal cells on earth. Teacher J knew that her students used the large numbers already; however, she wondered whether they could solve problems with the numbers. In that case, during the introduction to the exponential operation problems, she encouraged the students to use graphing calculators for expressing large numbers in scientific notations. She stated the number of cells problem as follows:

> Teacher J: Did you know that $10^{10}$ plant or animal cells can fit into a cubic inch? The surface area of the earth is $7.87 \times 10^{17}$ square inches. If cell life can survive up to 15 feet under ground, what is the maximum number of cells the earth can hold? (Teacher J, March 14th Observation)

Also, she used a graphing calculator to ask what the intersection of two lines meant in the **Tax Problem** as in Figure 15. Students used calculators to solve complicated problems and calculate large numbers that otherwise hindered them in solving algebraic word problems. She believed that using a calculator freed the students from dealing with tedious numbers so that they could concentrate on problem solving.

She was interested in exploring students' wrong answers. Teacher J understood students' mathematical thinking from the exploration of the wrong answers. By exploring the wrong answers, Teacher J discovered the students' mathematical errors or misconceptions. In solving the **Number of Stars and Sand Problem** (Figure 13), Teacher J did not correct students' wrong answers directly. To answer their questions for the **Number of Stars and Sand Problem**, students divided two numbers and they got$10^9$, which was not the correct answer. Teacher J did not tell them whether the solution was right or wrong. She asked about their strategies in order to understand their students' thinking by listening to their solution strategies. The students used a few different strategies, but they all got the same answer.

C: (Showing graphing calculator) Is this right?
Teacher J: How did you get ten to the ninth?
C: I put ten to the twentieth on my calculator and said one E to the twentieth. I put ten to the eleventh and it says one E to the eleventh.
Teacher J: What did you do with those two numbers?
C: I minused them and I got ten to the ninth.
Teacher J: So you just did twenty minus eleven because you didn't figure the tens mattered? So you got nine, so it must me ten to the ninth?
(Teacher J, March 7th Observation)

Teacher J extracted answers from the students' thinking. She asked how they subtracted two numbers using vertical subtraction (as in Figure 17) from a student. Another student corrected by saying "borrowing." However, some students still insisted ten to the ninth was the correct answer. Teacher J did not insist on the correct answer because they needed to explore more about those kinds of problems until they understood clearly from their peers or by themselves.

$$\begin{array}{rl} & 100000000000000000000 \\ - & 100000000000 \\ \hline & 99999999900000000000 \quad \Rightarrow 9.99999999 \times 10^{19} \end{array}$$

*Figure 17*. Subtraction of big numbers. (Teacher J, March 7th Observation)

Another student asked different questions but arrived at similar answers to that of previous students. Teacher J accepted the problem because some students were still confused about solving the problem. One student said, "Assume that all galaxies have ten to the eleventh stars. How many more stars than grains of sand?" After posting the problem on the board, Teacher J looked around and asked them how they got the answers. Some students produced the same kinds of errors. They got 100 because they subtracted exponents. She clarified their understandings about division of large numbers by selecting a specific student to discuss existing errors.

| | |
|---|---|
| Teacher J: | I want to know how many stars total, and he's saying that ten to the eleventh times ten to the eleventh, and why would you do that? |
| C: | One of those ten to the elevenths is the galaxies and one of them is the stars per galaxy. |
| Teacher J: | I've got ten to the twenty-second stars… We know the grains of sand is ten to the twentieth. So what do I do with the two numbers? |
| C: | You would subtract ten to the twentieth from ten to the twenty-second and I believe that's nine point nine times ten to the twenty-first. |
| Teacher J: | Why didn't I get ten to the twenty-second or 100? Because ten to the twenty second or one hundred is the answer to what question? |
| C: | If you divided by two numbers you get 100 but it is subtraction. |
| Teacher J: | Ok, so Jack, when you're subtracting you're still dividing. You're not the only one, but you're the one I'm picking on because other people get embarrassed and you don't. |

(Teacher J, March 7th Observation)

To make sure that students understood the subtraction of large numbers, Teacher J gave another example which was "$10^8 - 10^4 =$." She asked them to do this problem mentally. She thought that if students could solve the problem mentally, then they clearly understood the problem. After solving that problem, they found that there were patterns and they made a formula. Because the students solved the same kind of problems repeatedly, they discerned patterns in the problems. Teacher J pretended that she did not know what happened when the base was not 10. Pretending not to know what would happen made students curious to explore the problem and encourage them to find a more general formula.

C: It's nine point nine, nine, nine, times ten to the seventh.
Teacher J: Why is it to the seventh?
C: To start out with they're all multiples of ten, the two numbers, and so if you're going to borrow from the top one then you lose one place. So instead of there being eight places there are seven.
C: But also I would say what Ann was just saying about having the eight be one less than the number of places, I don't think that would work for anything other than tens.
Teacher J: If the base number isn't—
C: If the base number isn't ten, I don't think there's a rule about how many places you have compared to...
Teacher J: I think that's an interesting question. Like, if I subtracted four to the six from—Or, subtracted four to the second from four to the sixth I know I wouldn't get four to the fourth, but would the number I get have four decimal places or would it have five decimal—You're saying that there's no magic math thing that tells you. That's interesting. She said that she doesn't think—and she's pretty smart, so I'm kind of leaning in her direction—She doesn't think that if you change the base to a different number not tens that we're going to have some sort of nifty little trick for subtracting. That, in fact, it's not going to work so nicely. But I don't know, because I have never explored that.

(Teacher J, March 7th Observation)

Teacher J summarized the lesson by asking students to make formulas such as those shown in Figure 18. She purposefully said that *m* was great than *n* because the lesson continued the next day. However, when another student critiqued her formula, she pretended that she did not know about other ways to solve the problem. Her pretending not to know encouraged the students, helping them feel capable of solving the problem.

Teacher J: *m* is greater than *n*…If we take ten to the power of *m* minus ten to the power of *n*, how do we know what our answers going to be? In terms of we were trying to create like a formulaic type of idea of.
C: We know that *m* minus *n* is going to be the number of nines we get.
Teacher J: So, we're going to get nine point nine dot, dot, dot, and this number of nines is going to be *m* minus *n*.

C: And we know that X times ten is going to be the placement of *m* minus *n* ...I said that the exponent was one less.

Teacher J: Oh, the exponent thing was always one less because you were borrowing, and so your answer was one less place.

C: What do you do if you don't know that *m* is greater than *n*?

J: I'm wondering if—I said this because every problem we looked at this power was bigger than this power, right? So I don't know. If this one's less than this one, what do you think would happen?

(Teacher J, March 7th Observation)

$$m > n \quad 10^m - 10^n = \underbrace{9.9}_{(m-n)9\text{'s}} \times 10^{m-1}$$

*Figure 18*. The students' rule for difference for large numbers. (Teacher J, March 7th Observation)

Teacher J introduced the rule by eliciting ideas from the students. Students were very proud of themselves for creating the formula. She felt that the questioning strategy was not easy, but that by following students' ideas, teachers helped their students figure out meaningful mathematics concepts.

The next day Teacher J verified the rule by subtracting specific numbers. The students realized that their rule was working when *m* was greater than *n* only. Teacher J then guided them to make their own rule and verified the rule by carefully giving a pop quiz.

$$10^4 - 10^2 =$$
$$10^{100} - 10^{95} =$$
$$10^{40} - 10^{42} =$$

*Figure 19*. Pop quiz problem. (Teacher J, March 8th Observation)

Teacher J finished her classes by giving homework or explaining plans for lessons that might occur in the following week for which students should be prepared.

> Teacher J: Okay. Um, first of all, as far as the unit goes, we're pretty much done with this, so probably tomorrow we'll do a review kind of thing. On Thur—on Monday is that play day, and so the play's going to be going on all day, so I'm gonna skip that day. On Tuesday we'll take a test on this stuff. So, kind of looking at where we're at, probably Tuesday we'll finish this unit up with a test. Get it all done! And, on the test, you are going to need that, that formula for the amount of money you have with compound interest, so make sure that's in your notes.
>
> (Teacher J, March 8th Observation)

## Commonalities and Differences of the Two Teachers

A number of commonalities were apparent between the two teachers with respect to their knowledge about students' mathematical thinking in solving algebraic word problems and their use of their knowledge in planning their lessons. The two teachers' teaching process was consistent. Having taught for 25 years or more, they had developed their individual teaching styles. Therefore, they did not feel the need to create new strategies as they planned their lessons. Neither teacher prepared detailed lesson plans; rather they only used mental outlines because they wanted the lessons to be guided by the students' thinking rather than the teacher's more advanced ideas for solving the problems. As a result of the reliance on their students' thinking, their lessons often veered in different directions than they had imagined the lesson might go.

Since the teachers maintained the consistency in how they taught their lessons, the students were familiar with the process and teaching style that framed the classes.

The students knew that they would be expected to think through the problems either in whole class discussions or in pairs. The students knew that they would be expected to explain their ideas. Both teachers consistently reminded the students of the prior information or previous lesson ideas before students were encouraged to work on the problems. They began with easy problems before moving to the more difficult problems so that they would be able to relate one problem to the other.

Both teachers felt the need to consciously prepare the kinds of problems or the kinds of questions that would best guide the lessons. This effort took the majority of their planning time. And, as the teachers worked with the students in the lessons, they willingly extended the class time to help students unpack the algebraic word problems and this willingness helped their students in working with the word problems.

The questionnaire results indicated that teachers needed to spend time helping students understand the word problems. These two teachers focused on guiding the students in understanding the background for the problems at the level of the students' understanding. They re-phrased the problem using simpler language or framed analogies to the problems using similar real world situations. Explaining the background of the algebraic word problems seemed to help the students read the problems and enabled them to identify and understand the question embedded in the problem. These two teachers had an abundance of knowledge that helped them in framing the background for the algebraic word problems. This result showed that one of the important aspects in teaching how to solve algebraic word problems was to have enough knowledge about the problems and to be able to provide additional information about the problems when guiding the students' understanding of the problem. Teachers need to be able to provide enough information to establish a background for the problems when guiding the students in clarifying, interpreting, and attempting to construct one or more solution processes.

Teacher S and Teacher J emphasized the importance of solving algebraic word problems using various strategies because this process enhanced students' reasoning skills. Students could openly explore solutions to the problems and develop multiple solution strategies. The two teachers consistently tried to encourage students to identify different solution strategies, making use of charts, graphs or pictures to visualize their thinking and understandings about the problems. Through these visualization techniques, they were more comfortable applying variables in solving the problems. Since the two classes were more advanced mathematically, the students did not tend to resort to guess-and-check strategies.

Both teachers worked at guiding the students in working with strategies other than guess-and-check through their consistent questioning of the students' ideas for solving the problems. Teacher S consistently questioned students to find out what they knew and understood about the algebraic word problems. He had the students working in pairs to develop their own strategies and then he questioned them to have them explain their ideas and strategies. Through this process, he felt he had a better understanding of his students' understanding of the mathematics concepts embedded in the problems. Teacher J used questioning in a whole group format as a way of guiding her students' thinking. She encouraged her students to explore alternative methods that might further enhance their conceptual understanding of the ideas embedded in the problems. In both cases, the teachers used questioning to engage their students in describing their personal solution strategies in working with the algebraic word problems.

Both teachers emphasized communication among students and between teacher-students. They listened to students' comments, questions and ideas. They also expected students to listen to their classmates. Both of the teachers believed that through listening to each other, word problems became easier for the students to understand the problems and to find solutions for the problems. Both teachers felt that the time spent in promoting students' reasoning and higher order thinking was critical for exchanging their ideas.

In both classes, the students had difficulties in transferring from numbers to letters to represent variables. In Teacher S's class the students demonstrated this problem in the **Greatest Area of a Rectangle Problem** whereas in Teacher J's class, the students demonstrated this difficulty in determining the meaning of the equation when they tried to find a solution for the **Tax Problem**. Again both teachers relied on their questioning strategies to help students overcome the difficulties in transforming from numbers to variables.

Both teachers worked to assure that the ideas for the solutions came from the students rather than providing "the" answers to the questions. They never were observed to give direct answers to the students. Through multiple questions, the teachers gathered students' solving strategies and their understandings and then had the class determine the best direction for working toward a solution.

As was indicated in the questionnaire, most students in both classes failed to check or verify their solutions. They simply thought the problems were finished when they had an answer, whether it was correct or not. Both teachers challenged their students to think about the problems and their solutions. They did not give the correct solutions immediately, rather they asked the students to explain their work without giving an indication of correctness or incorrectness. The teachers felt their students potentially had more ideas and learned more from their peers. This strategy promoted independence among the students in answering the problems. Also, even when the students explained incorrect solutions, the teachers felt they gained an understanding of how the students actually thought through a problem and this understanding would be useful in guiding students in rethinking the problem solution.

Both of the teachers summarized their lesson similarly. They asked students to present and explain their solution strategies for the whole class. This process helped all the students see how different strategies may have been useful in solving the algebraic word problems. The teachers asked the students to discuss the different solution strategies which were the best from their perspectives at finding a solution to the problem.

There were also differences between Teacher S's and Teacher J's classroom teaching strategies with algebraic word problem solving. Probably the clearest difference between the two expert middle school mathematics teachers was the focal point in the classroom. Teacher S worked to have a student-centered classroom while Teacher J maintained a teacher-leaded classroom. Teacher S used pair discussion in solving algebraic word problems. The class of Teacher S was more organized and quiet. When the students needed to discuss with others in the classroom, they needed to use low voices in order not to hinder other students' thinking and reasoning.

However, Teacher J used whole class discussion more than small group discussion. Teacher J's class was more relaxed and had more interactions among students as well as between teacher and students than the class of Teacher S. As long as the students did not bother other students she allowed them to talk because she could ensure that they stayed on task. With this classroom atmosphere, students did not hesitate to express their thinking or their strategies to solve algebraic word problems even when they were wrong.

However, despite this basic difference, both classes were marked by student discussions. Students in both classes did not hesitate to express their thinking about the problems. Each classroom atmosphere encouraged the students to take risks, to express their ideas about the problem solutions to the teachers and to the whole class.

Although both teachers were careful about planning the problems that the students used, their strategies for finding the problems were different. Teacher S used *Connected Mathematics* as a textbook because it had many algebraic word problems. Most of his lesson problems were from the textbook because the problems were well organized to facilitate students' thinking and reasoning. From his previous experience with students, he had some knowledge about how students solved the problems and what kind of barriers they would have. This understanding made it easy for him to prepare the lessons. However, Teacher J used a traditional textbook. She spent a lot of time finding good algebraic word problems. She found problems on the Internet or through other resources because the problems were more realistic to her as well as to her students. She knew that students were more engaged in solving realistic problems and more willingly attacked such problems.

The basic organization of the lessons was different for each teacher. Teacher S consistently began lessons with warm-up exercises to strengthen students' basic mathematics skills. Once they solved the problems, the students were ready to begin the main activity. Teacher J discussed the homework or previous lessons to check whether her students understood. Depending on the students' understanding she presented new problems or reviewed previous lessons. In either case, both the classes continued with student discussions, either in pairs or in a group as a whole directed by the teacher. Finally, both teachers used the students' thinking and ideas for the closure to the problem solutions that were developed through the discussion.

## Summary of the Research Results

An, Kulm, and Wu, (2004) proposed student thinking and understanding as an important factor in mathematic teachers' pedagogical content knowledge and more specifically in their knowledge of teaching and learning. In their model, they proposed that teachers' concerns about student thinking and understanding relied on addressing students' misconceptions, engaging students in mathematical learning, evaluating students' understandings, using students' prior knowledge, promoting student's mathematical thinking and building on students' mathematical ideas.

The study for this research drew from the An, Kulm, and Wu model focusing on how middle level mathematics teachers use student thinking and understanding in the teaching of algebraic word problem solving. To address this problem, the study purposefully selected expert middle level mathematics teachers to address the nature of their knowledge of students' mathematical thinking when working with algebraic word problem solving, their use of this knowledge in planning their lessons and finally their implementation of those plans when actively teaching. The results of this study are summarized through key attributes (Table 2) of knowledge, planning and teaching for middle school mathematics teachers teaching algebraic word problem solving.

Table 2

*Key Attributes of Knowledge, Planning and Teaching in Teaching Algebraic Word Problem Solving*

| Knowledge | Planning | Teaching |
|---|---|---|
| 1. Students' solution strategies to algebraic word problems include:<br>o guess-and-check<br>o graph and chart<br>o equations<br><br>2. Strategies for enhancing students' comprehension and identification of solutions relied on:<br>o realistic problems<br>o readability and comprehension of the problems requiring:<br>▪ good reading abilities<br>▪ appropriate vocabulary<br>▪ adequate student confidence<br><br>3. Students' barriers in solving algebraic problems arose from:<br>o comprehension of the problem itself<br>o use of variables<br>o adequate basic computational skills<br>o review and verification of solutions | o Plan for consistent classroom routines<br>o Plan broad outlines that encourage flexibility in the direction of the lesson<br>o Careful selection of problems to find problems that<br>▪ Support multiple strategies and approaches<br>▪ Are realistic<br>▪ Are comprehensible to student levels<br>o Plan important questions to engage students in thinking about strategies<br>o Begin the year guiding students in using variables in their solutions; continue throughout the year | o Encourage students to use multiple strategies, particularly strategies other than guess-and-check.<br>o Encourage students to rely on the use of variables in their strategies<br>o Provide as much time as needed for student work in solving the problems.<br>o Use student ideas to direct class discussions about the problems.<br>o Encourage students to take risks in sharing their ideas.<br>o Use student ideas in summarizing the solutions to the problems. |

This selection of the exemplary middle school mathematics teachers for this study was purposeful because of their expertise and effectiveness in teaching about algebraic word problem solving. Numerous data sources resulted in the identification of their knowledge of students' mathematical thinking about algebraic word problem solving. Table 2 describes the knowledge these teachers relied on and more specifically their actualization of their knowledge of student thinking in planning and teaching. Careful review of the items in each of these columns shows that the exemplary middle school mathematics teachers did rely on their knowledge of students' thinking about solving algebraic word problem in planning their lessons. They purposefully planned lessons to be flexible because of their determinations to rely on student ideas. Thus, the lessons were more directed by student ideas and thinking. The extended observations of the two teachers revealed diverse teaching strategies. One teacher maintained a teacher-leaded classroom where the teacher led the whole class through the problem solving processes. The other teacher relied on a student-centered instruction where the students worked in pairs engaged in the word problem solving processes. These small group discussions were followed by whole class discussions where the students shared their thinking about the problems. Although the teachers maintained different instructional strategies, both were successful in teaching algebraic word problem solving in their classes as evidenced through their identification as expert middle school mathematics teachers who were able to guide students in positive achievement in algebraic word problem solving. The key is that despite their instructional strategy differences, their goals were identical for engaging students, using student ideas, urging students to take risks in sharing their approaches, and using students' ideas to frame the discussions as well as to conclude the discussions.

# Chapter V

# Discussion and Implications

## Introduction

The main goal of this study was to identify a framework for hypothesizing the relationship between the teachers' knowledge of their students' mathematical thinking and understanding in solving algebraic word problems and their planning and teaching using this knowledge. Each research question was answered through carefully planned data gathering techniques. These methods included a survey questionnaire, a follow-up interview plus one classroom observation, and multiple observations of actual classroom lessons supported with pre- and post-observation interviews and stimulated recall interviews. Nine expert middle school mathematics teachers participated in the survey questionnaire, four of these nine teachers took part in the interview and single classroom observation, and, finally, two teachers were observed teaching their classes and interviewed to gather how they used their knowledge of their students' thinking and understanding in teaching about algebraic word problem solving.

The teachers' knowledge of students' thinking when solving algebraic word problems focused on the students' solution strategies, strategies for enhancing their comprehension and identification of solutions, and students' barriers. The teachers planned their lessons from this knowledge of students' thinking by carefully selecting problems, making broaden lesson outlines, thinking about important questions, integrating instruction with variables continuously throughout the year, and maintaining consistent classroom routines. In their classroom teaching, the teachers used their knowledge of students' thinking as a guide in helping and encouraging students to solve algebraic word problems. Even though the teachers relied on different basic frameworks for their classes (teacher-leaded versus student-centered), they were consistent in having the classroom discussions about the problems led by students ideas, student presentations of their strategies, and students taking risks to challenge and share their ideas. Both teachers used students' ideas to conclude the discussions and both encouraged students to solve the problems in multiple ways. This chapter discusses the relationships between teachers' knowledge and their planning and teaching. Implications for teacher education, limitations, and recommendations for future research are described in the conclusion of this chapter.

## Discussion

This discussion is presented in three sections that deal with the relationships between teachers' knowledge and their planning and teaching. The first section examines the relationship between the teachers' knowledge of students' solution strategies and their planning and teaching. The second section examines the relationship between the teachers' knowledge about strategies for enhancing students' comprehension and identifications. The third section discusses how teachers use their knowledge of students' barriers in planning and teaching.

Knowledge of Students' Solution Strategies

According to An, Kulm, and Wu, (2004) teachers' knowledge about students' mathematical thinking builds from the students' mathematical ideas. In the questionnaire results, the expert middle school teachers identified the broad knowledge of students' mathematical thinking when working with algebraic word problem solving. They recognized that the students brought diverse prior mathematical knowledge and diverse life experiences. Drawing from their multiple teaching experiences, they described a wide variety of solution strategies that students used when working with algebraic word problems.

Broad literature support suggests that as the students perceive or confirm their knowledge, they rely on social interactions (Rueda & Vacas, 2001). These teachers knew that their students needed social interactions in their classroom when working with the problems in order to improve their mathematical reasoning. And with their constructivist epistemological beliefs (Von Glasersfeld, 1996), they realized the need for both making connections between the students' previous experiences with the problems and the social interactions between students to improve their strategies for solving algebraic word problems. These teachers also displayed extensive knowledge of students' mathematical thinking before the teachers came to the class; but they still had extended their knowledge of students' mathematical thinking by listening to the students as they were explaining their thinking about various solving strategies. Thus, as they planned their lessons, they spent a great deal of time selecting the problems for their particular students and planned to encourage student discussions about their solution strategies.

The teachers were predisposed to encouraging students to use multiple strategies. They recognized that students' solution strategies often began with guess-and-check but they wanted the students to try additional strategies, strategies that included graphs in representing the problem and potential solution and symbolic equations that included identification of important variables in the problems. Reed (1999) concurred indicating that when students solved algebraic word problems in different ways, they typically had better access to a variety of strategies. He added that the reliance on flexible strategies enhanced students' abilities to solve algebraic word problems correctly. Tabachneck, Koedinger and Nathan (1994) also viewed a flexible strategy (or alternative strategy) helped students solve algebraic word problems. And, according to Weidemann (2005), middle school students are able to solve problems that do not have unique solutions or problems that can be solved in multiple ways through a wide variety of strategies. He found that the use of different strategies to solve the problems allowed the students freedom to express their individuality by looking for solution methods reflecting their own learning style.

Algebraic word problem solving requires the ability to express algebraic relationships. The difficulty in expressing these algebraic relationships, however, typically leads students to consider non-algebraic methods, especially guess-and-check. The teachers in this study recognized this problem and encouraged their students to use multiple strategies. Reed (1999) found that the use of non-algebraic methods can be avoided by including problems that are difficult to solve through a non-algebraic approach. The problems that both teachers presented to their students in this study were difficult to approach using guess-and-check and ultimately encouraged students to use charts and graphs before they moved to systematic equations. With these problems, the students realized that they needed to engage in a different mathematical thinking. This requirement, though, helped them see the necessity of engaging in discussion with their peers. The teachers purposefully drew from the theory of social constructivism in planning their lessons. Teacher S had the students share their different solutions and ideas to the whole class and encouraged the whole class to engage in a discussion of the validity of these solutions to help each other in understanding the approaches. He encouraged his students to find unique ways as he moved from group to group, questioning them as a means of supporting them in identifying different approaches. His strategy for summarizing the ideas was to rely on the student presentations of the various solution strategies. These activities encouraged the students to discuss their ideas with their peers, listen to their peers' ideas, build from the ideas of their peers and work to identify different ways to approach problems. When linking the teachers' previous knowledge of their previous students' solution strategies and the current students' solution strategies, the teachers further extended their own knowledge of students' mathematical thinking.

In recognition of the value of encouraging alternative solution strategies for the problems, these expert middle school teachers' planning focused on a careful selection of the problems that encouraged multiple strategies and approaches. They also searched for problems that involved real world situations and were written within the students' comprehension abilities. To prepare good problems, the teachers searched the Internet, drew from their own experiences and from other resources. The good problems were realistic and complex, thus benefiting from group discussions. Moreover, the problems for the most part were relevant and interesting to students.

The teachers were careful to design questions that engaged students in thinking about various solution strategies to algebraic word problems. From a constructivist perspective, as the teachers designed the questions, they often reflected on their previous experiences in listening to the students' ways of approaching the problems in the process of preparing their lessons. They framed their questions with the potential of eliciting students' thinking along with questions about their approaches and solutions. Then during the lesson by asking questions, they worked to elicit the students' mathematical thinking in the process of helping them develop formal solution strategies for working on the algebraic word problems. Through this type of planning these expert teachers became more adept with their questioning strategies. These questions were designed to trigger students' divergent ideas and to clarify their strategies. In addition, the experienced teachers' knowledge of the diverse thinking of their students helped them incorporate the questioning.

Another technique the teachers used for encouraging students to focus on multiples strategies came from their maintenance of consistent classroom routines. The focus, then, for the lessons was on working with the problems, rather than reorganizing the class differently. Students knew the classroom routines and were able to begin immediately on the problems. As a result, more class time was devoted to students identifying multiple solution strategies.

Hammer (1996) indicated that a classroom setting where student thinking was the focus was much harder to manage and much less predictable. The teachers in this study, prepared for this unpredictability. Reflecting on the advantages of social interactions, they allowed the students to work together and to divert the instruction to investigate their ideas because they believed that when students developed their own methods, they had a more in-depth understanding of the concepts. The teachers were experts at managing the discussions; they were patient with the students; they allowed the students to work longer with their problems than they planned; they used their questioning to guide the students in their discussions; and they planned for the lesson to be diverted. To support students' variety solution strategies, the teachers provided enough time as needed for students to solve algebraic word problems.

Dellarosa, Kintsch, Reusser, and Weimer (1988) concurred with the expert teachers strategies. They indicated that given enough time, students are able to organize their knowledge of the problems and form an initial representation that includes the information to be discovered as well as the information considered relevant to the goal. Stecher and Mitchell (1995) added that the students then proceed to apply operators to modify the initial representation of the problem by substituting given values into formulas, transforming equations, and introducing new equations. If the teachers wait for students' description of solution methods, they are encouraging elaboration of the students' responses.

The outcome of learning can be described in terms of the acquisition of new representational and operational abilities. This kind of psychological analysis is particularly useful in circumscribed domains like algebraic word problems, where problem solving can be understood as a set of transformations to a problem representation in order to achieve desired answers (Kieran, 1992; Reed, 1999). In this process students need time and if the teachers provide enough time for the students to be successful in solving algebraic word problems, they are better able to apply their unique solution strategies to the problems.

The teachers in this study supported students' various solution strategies in solving the algebraic word problems. They were sure that the students' understanding was based on the discussion and the questions contributed to their understanding of the problems as well as the teachers' understanding of the students' mathematical thinking. The teachers' knowledge of students' understanding was expanded by the combination of previous and current students' mathematical thinking and the social interactions among the students and the teacher-students. Teacher J orchestrated classroom discussions using students' explanations for solution strategies and monitoring students' engagement in solving the problems. Teacher S interacted with the students in their smaller groups using his questioning strategies to understand the solution strategies. Both teachers incorporated the students' mathematical understandings into their questions. This process helped them determine the strengths and weaknesses of each student when confronted with algebraic word problems.

## Enhancing Students' Comprehension and Identification of Solutions

Several researchers have indicated that students' mathematical vocabulary and reading comprehension impact their ability to solve word problems (Benko, Loisa, Long, Sacharski, & Winkler, 1999; Jitendra, Griffin, McGoey, Gardill, Bhat, & Riley, 1998; Schoenberger & Liming, 2001). The expert teachers in this study indicated the students' reading comprehension and vocabulary hindered students in solving algebraic word problem solving and caused a lack of confidence. From a constructivist viewpoint, students' prior knowledge, including reading skills and knowledge of mathematical vocabulary, is important for solving algebraic word problems because it provides the conceptual and methodological foundation for this type of work. The teachers indicated that when the students had this information before they were given the problems to solve, they more easily found solutions to the problems because they had imagined possible solution steps for approaching the problems. To improve the students' reading comprehension and identification of algebraic word problem solution strategies, the teachers suggested the importance of realistic problems that incrementally expanded students' reading skills where they could interpret the vocabulary appropriately and enhance their self-confidence. Booth (1988) proposed that if students are to learn (and use) the more formal procedures, they must first see the need for them. Realistic problems better met students' needs. In this way, the teachers' knowledge about students' mathematical thinking was similar to what previous research has indicated. It is therefore possible that the teachers' knowledge of students' mathematical thinking was based both on their teaching experience and from their familiarity with such research studies, perhaps through their teacher preparation or professional development classes.

In planning the lessons, the teachers relied on their knowledge of students' mathematical understanding about algebraic word problem solving. They planned their lessons based on this understanding. Clark and Dunn (1991) defined planning as a psychological process of envisioning the future and considering goals and ways for achieving them. When the teachers planned their lessons, they created outlines based on their previous teaching experiences and mentally modified the outlines to match their students' mathematical thinking. Mental activities were the final stage among Panasuk and Todd's (2005) four stages of planning lessons. The mentally-planned lessons were developed to elicit and connect the learners' prior knowledge to new information, to consider appropriate questions for enhancing mathematical reasoning, and to provide a framework for new knowledge, as well as to efficiently review previous lessons. The mental preparation was easily changed according to current students' needs. But, no matter how much teaching experience the teachers had, the students seemed to continue to have different needs. The students were unpredictable. So the teachers needed backup plans and they needed to recognize that that the plans from previous years would not necessarily work for this year. While their teaching experience helped in guiding their lesson plans, the plans needed to attend to the diverse student needs. The teachers planned their lessons focusing on enhancing the current class students' understandings by finding problems that were realistic to their needs.

According to Panasuk and Todd (2005) the teachers' plans were initially based on identifying appropriate objectives. Indeed, they spent much time choosing good algebraic word problems that could be answered by using multiple solution strategies, realistic, and related with students' needs. Students could actually identify different solution strategies that still resulted in a single answer. The teachers also recognized that the various student backgrounds might even shift the way a problem is perceived and thus later solved. The solutions might vary as a result of their interpretations. The teachers' plans were more focused on encouraging student thinking and to allow multiple interpretations. Also, different types of problems were presented to the class to expose students to various kinds of algebra that they could encounter and to prepare them for solving problems of any kind. Fermiano (2003) suggested that algebraic problem solving is crucial for a child's development of mathematical skills and that the process of learning algebraic techniques must be built from the child's "discovery" and intuitive concepts of algebra. Thus, using word problems may provide a much more insightful view of the students' inherent understanding of algebra and its fundamental concepts since they cater to a child's innate mathematical responses.

In teaching teachers spend lots of time providing background information for the algebraic word problems. Students identify solutions that are connected with their existing schema or developing new schema (Skemp, 1978). Explaining the background information of the problems, the teachers expected students to solve the problems more easily by relating the information about the problem. The more students understood the background information of the problems, the better they were able to solve the problems. Providing background information of the problems has been shown to support students' mathematical thinking (Fraivillig, Murphy, & Fuson, 2002). The teachers in this study allowed their students to discuss the problems in a comfortable atmosphere that encouraged them to take risks in sharing their ideas for better understanding of the problems. Using students' ideas to describe solution strategies, the teachers encouraged the students to identify solutions more easily. The teachers mostly used students' ideas to illustrate solution strategies and solutions. They rarely added their own ideas. Rather their efforts and instructions emanated from the students' thinking and ideas.

## Students' Barriers in Solving Algebraic Word Problems

Misconceptions are a typical source of errors and resulting barriers in problem solving. Generally speaking, misconceptions, evidenced by persistent error patterns, have a psychological base. The students' existing cognitive structures are difficult to change (Herscovics, 1989). It has been suggested that for arithmetic word problems the student's mental representation of the problem specifies the operation to be conducted (Kintsch & Greeno, 1985). However, for algebraic word problems it is not the operation of solving an equation or expression but representing the problem situation initially. Because of the complexity of many algebraic word problems, the representation of the problem situation generally requires the use of some written form of representation, such as an algebraic equation. The biggest barriers in solving algebraic word problems involve the use of variables. Although students understand the vocabulary in many of the algebraic word problems, they still have problems with the symbolization of the words (Heffernan & Koedinger, 1997; 1998). Because representing algebraic word problems requires the integration of information into an algebraic representation via symbols to replace unknown quantities (Maccini & Hughes, 2000), students with prior mental constructions that support direct translation have no problems using symbols.

Another barrier for students from the teachers' perspectives was with the students not reviewing and verifying their solutions. Most students did not check their solutions and their strategies. Once they obtained a solution that they deemed reasonable, they felt they were finished with the problems. The teachers carefully selected problems that encouraged multiple solution strategies and approaches. They encouraged the students to check to see if their solutions made sense. The two teachers had slightly different approaches for students to check their solutions. Teachers J accepted all the students' ideas for solving the problems in the whole class environment. When they eventually got a possible solution, she asked the other students whether the solution made sense or not as they reflected on the original problems. Teacher S asked each student in the small groups to explain their solutions. This process encouraged them to review their thinking and to verify their solutions. Both of the teachers had the students present their solution strategies and explain to their classes. The rest of the class was expected to criticize the solution strategies and check whether the solution made sense. Through these activities both of the teachers made students review their solution strategies and verify their solutions. Because both of the teachers knew their students did not usually review the solution strategies and verify their solutions, they encouraged students to review and verify their solutions using students' cognitive aspects. Therefore in these two classes, the students rechecked their answers unconsciously because they knew the teachers might ask.

The teachers planned to incorporate variables in reviewing the problems from the beginning of academic year. With this emphasis, they guided the students in mentally considering the symbols to be used in their solutions. Through their prepared questioning processes and students' discussions of their strategies, the teachers believed that they were preventing students' barriers related to using variables. Encouraging students to discuss the interrelationship among concepts supported students' mathematical reflection. To assure this action, the teachers felt the need to be patient and listen to students' solution strategies, to understanding their thinking.

When Teacher S detected misconceptions, he questioned other students in the groups to discuss the misconceptions. He relied on students' ideas and their explanations of their ideas as a means of guiding the students in recognizing their own misconceptions; the peer explanations were more easily accepted by the students than an teacher's explanation. He made sure that student-to-student exchanges about ideas happened frequently. He tried to look closely at students' ideas and thinking about solving algebraic word problems in teaching. Teacher J used a slightly different strategy in working with students' misconceptions. She ascertained their misconceptions by asking how they arrived at the answers. She tried to unmask the students' misconceptions by doing similar problems and practicing with other examples. For the most part, the teachers used students' ideas to correct other students' misconceptions. Actually in this research, because the teachers were aware of the barriers, they often were able to prevent barriers by guiding students so that the students were confronted by their own misconceptions.

As the teachers taught, they consistently adapted the lessons to engage students' ideas. This approach supported the students in actively engaging in the lesson because they thought that the problems presented by teachers were not the teacher's problem but their problem. Regardless of whether the class was teacher-centered or student-centered, the students were more engaged and willingly attacked the problems as long as the teacher accepted their ideas in the process of discussing the algebraic word problem solutions. Even though the two teachers used different approaches, both were able to engage the students in solving the algebraic word problems. They spent a great amount of time choosing good problems and explaining the background information of the problems. They encouraged students to discuss and find various solution strategies giving enough time and establishing a comfortable and safe atmosphere. Both teachers asked a lot of questions to help all the students engage in the process of algebraic word problem solving.

## Limitations of the Study

This study investigated the nature of teachers' knowledge of students' thinking when engaged in algebraic word problem solving and their use of this knowledge in planning and teaching their plans. Generalization of study findings to other settings was not the purpose of this study. The study results and implications are important for understanding the teachers' knowledge about students' understandings and providing information for other teachers or preservice teachers in teaching algebraic word problem solving. The results of the study are proposed as a hypothesis for other researchers to conduct future investigation.

The study was limited in terms of the participants. The two expert middle school teachers' time was limited in ways that sometimes the pre-observation interview was skipped. They only had 10 minutes between classes and needed to focus on preparing for the next group of students. Although the two teachers agreed to participate in the research, they were different in terms of their willingness. One teacher was eager to participate from the beginning to the end of the research and another teacher was not as willing and eager at the beginning.

This study did not rely on the use of statistical measures to achieve an objective analysis of a research hypothesis. The method was designed specifically to gather in-depth data for an interpretation and analysis of a specific phenomenon. The goal of this research was not to generalize the results for a larger population but rather to provide an in-depth view of the two teachers' nature of knowledge about their students' mathematical thinking and their teaching practice. Generalizations in qualitative studies are ultimately developed from clusters of studies of which this study was one.

Although a variety of data collection techniques were used in this study, the researcher was the primary instrument for data collection and analysis. Although the researcher's background and past teaching experiences were helpful in interpreting the data, the researcher's background and experiences also had potential for biasing the data collection and analysis process. The researcher, however, took steps to protect against researcher bias by keeping a journal and reflecting on the thoughts, insights, and decisions made during the process of data collection and analysis. Triangulation also provided another source of protection as multiple sources of data were used to confirm the emerging patterns, themes, and conclusions. In particular, the stimulated recall extended interviews served as a means of gathering data about the teachers' thinking which could then be connected with the researcher's observations and the pre- and post-interviews.

The teacher' use of their knowledge of students' understanding about algebraic word problem solving could not be measured without knowing students' responses to the teachers' questions and the students' expressions of their ideas or thinking. Although students' understanding and thinking were not measured, certain inferences may be suggested from students' conversation with their teachers. In that case technical difficulties occurred during the analysis of data given all the discussion in the classroom. The audio recordings did not pick up students' voices in whole-class discussion or during the small-group interactions when the students spoke softly. The teachers' speaking was clear because they wore a recorder.

Finally, the study was limited by the Institutional Review Board (IRB) guidelines for research involving human participants. The teachers in the study were volunteers.

## Implications

Both teachers were successful in teaching algebraic word problem solving, yet they used different strategies– teacher-leaded and student-centered. The key was that each teacher engaged all the students in the class to discuss with their peers. The shift to cognitive science perspectives on student-centered instruction has been emphasized over teacher-leaded instruction, perhaps because of the focus on individual students' constructions of their personal understandings. On the other hand, what emerged from this research were key features of students engagement in solving algebraic word problems through: (1) engaging students in actively explaining the background information and sharing their ideas, strategies, and solutions; (2) establishing a comfortable atmosphere for students to take risks; (3) giving students time to think, and (4) asking students questions that help them think through the problems. The key features were not traditional ways of teaching algebraic word problem solving.

The results of this study have several implications for teacher preparation and teachers inservice programs. The researcher will suggest seven important aspects for inservice teachers and preservice teachers in teaching algebraic word problem solving.

First, teachers need to continually work to extend their knowledge of students' mathematical thinking. In this research even though the two teachers were expert, they continued to work to determining students' mathematical thinking by asking them multiple questions. Both of the teachers indicated that students were unpredictable in solving algebraic word problems. Thus, teachers need to work at enhancing their knowledge of students' mathematical thinking. They can work at this by thinking through results of published research in conjunction with their previous teaching experiences.

Second, teachers need to choose good problems as the beginning in planning their lessons. They need to identify problems that match students' comprehension levels and that are connected with students' interests and backgrounds. This requirement suggests that they need support in identifying students' background knowledge and in guiding students in unpacking the word and ideas embedded in word problems. The two teachers used different textbooks. One teacher used a traditional, mathematical-skill based textbook and the other teacher used a standards-based textbook. Both provided them with a good selection of problems.

Third, teachers need to encourage students to discuss actively where they share their ideas with their peers. The teachers emphasized communication in solving algebraic word problems. Through listening to their classmates and describing their ideas, the students seemed to gain more skill in solving algebraic word problems and the problems became easier for the students. However, the students needed extended times for exchanging their ideas if they were to engage in reasoning and higher order thinking.

Fourth, developing questioning strategies is key to their preparation for working with students with algebraic word problems. What kind of questions are good for enhancing, provoking, guiding, or promoting students' ideas and students' mathematical thinking? Again teachers need practice in questioning students individually, in small groups, and in whole class situations where the focus is on engaging students in a discussion, rather than simply answering questions. They may use different questioning strategies in different class situations.

Fifth, programs need to guide the teachers in thinking about and gathering various students' solution strategies. They need experiences where they can interact with students as they are working with algebraic word problems. Sharing what they learn among a group of preservice or inservice teachers is a good way for them to build their understanding of how students thinking about these problems and how they think with these problems.

Sixth, teachers need be able to explain the background information for various algebraic word problems in order to be prepared to engage students in attacking the problems and enhancing their comprehension and identification of the problem. This expectation requires a broad content knowledge about multiple content applications.

Finally, teachers need to consider ways of developing a comfortable classroom atmosphere where students can be encouraged to take risks, to share their ideas and to challenge other ideas. Such an atmosphere is important in enhancing student discussion where they are encouraged to express various ideas that may be linked to the particular problems. The question remains as to the level of comfort in the classroom that can still assure that teachers are able to manage the interactions in the classroom.

## Recommendations for Future Research

This research investigated the use of expert middle school teachers' knowledge of students' understanding about solving algebraic word problem in planning and teaching. Generalizations in qualitative studies are developed from cluster of studies with similar treatments, settings, populations, and times to those in the original research (Patton, 2002). It is recommended that other researchers can conduct similar studies as an approach toward generalizing what teachers need to know about students' mathematical thinking and understanding in order to help their students' gain a better understanding of algebraic word problem solving.

Novice teachers do not have the expertise of exemplary teachers with respect to an understanding of student thinking when engaged in the algebraic word problem solving. Additional research is needed to explore the novice teachers' knowledge of students' thinking about solving algebraic word problems and how they use their knowledge of students' understanding about algebraic word problem solving in planning and teaching. The results of the study are likely different for novice teachers. The comparison on the results for novice teachers with the results of exemplary teachers' will provide important knowledge about guiding novice teachers in planning and teaching in ways that rely on students' thinking.

The expert middle school teachers used a lot of questions in teaching algebraic word problem solving. Their questioning strategies were unique and different because their instructional strategies were different. In-depth study about what kinds of different questions encourage, provoke, and trigger students' mathematical understandings is needed. Research needs to explicate the exemplary teachers' rationale for their questions and how they generate motivational questions, idea-generating questions, clarification questions, and generalization questions.

This research study emphasized the strategies used by two expert middle school teachers in a limited geographical location. Other studies need to be conducted with more participants and geographically different areas to confirm or refute the proposed knowledge, planning and teaching attributes for using student thinking in teaching algebraic word problem solving. Studies need to determine the comprehensiveness of this set of attributes for teachers' use of students' thinking.

From the research the nine teachers indicated students' various solution strategies for solving algebraic word problems. The teachers indicated that some students solved algebraic word problems in an inductive way and other students used a deductive method. Another study is needed to determine which students use which solution strategy and how teachers can guide them toward using multiple strategies in solving algebraic word problems.

Data in this study suggested that students seemed to gain more skills in solving algebraic word problems by interacting with other students and listening to their peers' ideas and strategies. More study needs to confirm this strategy as a method for guiding students in algebraic word problem solving.

In this research the students had difficulties in transferring from numbers to letters for representing variables. One goal in solving algebraic word problems is to build students' abilities in using symbolic representations in a more efficient method of solving the problems. Students need to learn to use variables and symbolic expressions rather than relying solely on guess-and-check. More in-depth study is needed to investigate how teachers best guide students in transferring from numbers to letters in the process of solving algebraic word problems.

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# Appendix A

## Teacher Open-ended Survey Questionnaire

### A. Personal Information

*Instruction:* Please provide the information or place a check for the appropriate option.

1. Age bracket to which you belong
   ___ 20 but less than 30 yrs.
   ___ 30 but less than 40 yrs.
   ___ 40 but less than 50 yrs
   ___ 50 yrs. and above

2. Educational Attainment
   ___ Bachelor's Degree
   ___ Bachelor's Degree + 30 hours or more
   ___ Master's Degree
   ___ PhD

3. Major ____________________

4. Endorsement ______________________________

5. Years of Teaching __________________________

### B. Questionnaire with Word Problems

Please indicate your answers on the following open-ended questions.

1. How do your understanding of ways that students solve algebraic word problems when you are preparing lessons?

2. Do you think your students like algebraic word problems? Why or Why not?

3. Describe your level of knowledge about your students' thinking about algebra.

## C. Student thinking with algebra word problem solving

**Problem for students**

1. Blue pencils cost 4 cents each and red pencils cost 6 cents each. I buy some blue and some red pencils and altogether it costs me 90 cents. If b is the number of blue pencils bought and if r is the number of red pencils bought, what can you write down about b and r? (Hart, 1981, p108)

**Teachers:** **Assume that this problem is one with which your students are working with. Please indicate your response on the following questions about how students might think about this problem and how you would respond as their teacher.**

1. Please describe different correct solutions your students might come up with, as many as you can.

2. How would you think most children in middle school might answer the problem?

3. What barriers might your students encounter in understanding the problem? If there are any, how do you plan to address these barriers?

4. What possible weaknesses might your students have as they solve this problem?

**Possible teachers' responses to this problem**

1. 4b+6r = 90, Two correct pairs, of (0, 15), (3, 13), (6, 11), (9, 9), (12, 7), (15,5), (18,3), (21,1). b + r = 90, 6b + 10r = 90 or 12b + 5r = 90
2. The answer 'b + r = 90' could only seem to mean 'blue pencils and red pencils cost 90 cents' which though true, gives only limited information and uses the letters as objects. ('b + r = 90' could be read as 'the number of blue and red pencils bought cost 90 cents', but this is still very much tied to the concrete reality of the question, and is not a 'pure' statement about numbers. The number b + r does not equal the number 90).
3. Children who gave answer like '6b + 10r = 90' had found one correct pair of values for b and r (6, 10) but instead of expressing this in a form that showed that b and r are numbers (b=6, r=10) were essentially saying '6 blue pencils and 10 red pencils cost 90 cents' –again the letters were being used as objects.

**Alternate Problem for consideration for students**

2. A large company employed 372 people. There are 4 times as many laborers as clerks, and 18 more clerks than managers. How many laborers, clerks and managers are there in the company? (Dooren, Verschaffel & Onghena, 2003 for elementary and lower secondary preservice teachers)

**Possible teachers' responses**

1. Number of managers = x
   $x + (x + 18) + 4(x + 18) = 372$
   6x=372-90
   6x=282 ➔ x=47 = Number of managers
   → there were 65 clerks (47+18)
   → there were 260 laborers (4*65)
2. Let us suppose for a moment that there are 18 managers more (i.e. as many managers as clerks). Then the total of people = 390
   This total consists of 6 equal parts:
   4 parts of laborers
   1 part of clerks
   1 part of managers
   Each part consists of 390/6=65 people
   So there are 65 clerks, 260 laborers and 47 managers (65-18)
3. Suppose the number of clerks is 80, then…

| Clerks | Managers | Laborers | → Total |
|---|---|---|---|
| 80 | 62 | 320 | 462 = Too much |
| 60 | 42 | 240 | 342 = Too few |
| 70 | 62 | 280 | 412 = Too much |
| 65 | 47 | 260 | 372 = Correct! |

# Appendix B

## Follow-up Interview Protocol

I appreciate your responses for the questionnaire. I have some questions I'd like to ask you related to your responses in the questionnaire. Would you mind if I taped the interview? It will help me stay focused on our conversation and it will ensure I have an accurate record of what we discuss. You may turn it off at any point.

- **Would you explain more about your response in the questions?**
- **When are you planning to teach lessons for the algebra unit that includes solving word problems?**

**Problems for consideration by students**

A bus driver figured out that if he increased his average speed from 40 to 45 miles per hour, the time for the trip would be shortened from 7 ½ hours to 6 2/3 hours, a savings of 50 minutes. He then reasoned that increasing his average speed from 45 to 50 miles per hour would cut another 50 minutes off the trip. Do you agree with the bus driver's conclusions about the time he would save by driving faster? Explain your reasoning (CMP $8^{th}$ p.30; Thinking with Mathematical Models).

**Alternate Problem for consideration by students**

The astronomy club at King Middle School is planning a field trip to the science center to see a new 3-D science film. Renting a bus for the trip will cost $125. Admission to the film is $2.50 per person. Write an equation to show the relationship between the number of students who go on the trip, *n*, and the cost of the trip, *c*.

**Teachers:** **Assume that this problem is one with which your students are working. Please describe your thinking as much as you can.**

1. **How do you plan lessons that include word problems?**
2. **When do you plan to put the problem into your algebra unit?**
3. **What do you think about your students' understanding of word problems?**
4. **How do you plan to address your students' misconceptions or barriers for solving this problem?**
5. **What intuitions do students have that might help them solve this word problem?**

# Appendix C

## Observation Guide

**Observation Guidelines** (Kennedy, Ball & McDiarmid, 1993)

1. **Observing and taking notes:**
   a. Arrive half an hour before the scheduled observation time. Find a comfortable place to sit where you can see and hear well. Sketch a map of the physical arrangement of the classroom, labeling areas and displays.
   b. Study a checklist beforehand to help record features of the class.
   c. Video tape the classroom sessions
   d. When taking notes, try to get as many direct quotes as possible, especially when the teacher talks about students' understanding
   e. Assign students numbers as they are called on or speak in class.
   f. Note the time every few minutes or when something changes (task, activity)
   g. Write up notes as soon as possible after the observation while your memories are still fresh

2. **Writing field notes:**
   a. Teacher's code number and name
   b. Date of the observation
   c. Write description of the classroom and description of the tasks
   d. Use bracket for observer's interpretive comments
   e. Attach all the documents

**Observation**

Code ________________

Date ________________

Grade ________________

Observation # _________

Time: Begin _______ End ________ Total length _______hrs. _______min

**Classroom Description**

1. **The physical classroom**

   **A. Seating Arrangement:**

   ____ Students have assigned seats
   ____ Seating appears to be random
   ____ Desks arranged in rows and columns
   ____ Desks arranged in semi-circles
   ____ Desks arranged in clusters
   ____ Tables are used, rather than desks

   **B. Walls**

   ____ Student math assignments (nature, teacher's comments)
   ____ Rules of behavior posted
   ____ Rules of math posted
   ____ Illustrations of mathematical concepts posted
   ____ Pictures
   ____ Graphs or charts
   ____ Others

**Make notes of the things you see and describe them later in your narrative account.**

**2. The Students**

**A. Number of students _______**

**B. Ethnicity**

____ Mostly white
____ Mostly black
____ Mostly Hispanic
____ Mostly one other type ____________________
____ A mixture of: ___________________________

**C. Gender**

**____ About 50% girls, 50% boys**
**____ skewed (describe): _______________________**

**Observation Checklist**

The observation checklist is derived from the Annenberg Institute of School Reform (2004) classroom observation protocol. Here, the classroom learning environment will be coded to identify teachers' understanding of how their students think.

| Area | Content | Notes |
|---|---|---|
| A | **Building on students' math idea** | |
| | Teacher adapts students' ideas | |
| | Teacher considers students' interests about algebra | |
| | Teacher emphasizes clear communication of students' mathematical ideas | |
| | Teacher uses students' strengths in problem solving | |
| B | **Promoting students' thinking** | |
| | Teacher uses group work to promote students' discussion that encourages student thinking. | |
| | Teacher questioning triggers divergent ideas from students | |
| | Lesson employs a variety of instructional strategies to meet variety of students' needs | |
| | Teacher uses students' thinking in delivery of ideas | |
| | Teacher encourages diverse problem solving strategies | |
| | Teacher has high expectations for students | |
| | Teacher relates math problems to children's everyday life situations to challenge them to consider a variety of | |
| | Teacher promotes students' reasoning about mathematical concepts | |
| | Teacher uses analogies of stories to help student understanding | |
| | Teacher encourages students to challenge their own [illegible] | |
| | Students' work requires higher order thinking skills | |
| | Teacher encourages students to extend and apply problem | |
| C | **Using Prior knowledge** | |
| | Lessons connect mathematics content and the real world | |
| | Teacher activates prior and current student knowledge | |

| | | |
|---|---|---|
| | Teacher connects ideas within and outside the discipline | |
| D | **Evaluate students' understanding** | |
| | Teacher checks for understanding and reteaches | |
| | Classroom environment reflects/supports instruction | |
| | Teacher checks students' comprehension of the problem | |
| | Teacher checks students' application of the ideas | |
| | Teacher rephrases students ideas for clarity | |
| | Teacher examines the improvement in students' thinking regarding the ideas | |
| | Teachers examines students' understanding of the instructions | |
| | Teacher examines how the students react to the instructions | |
| E | **Engaging students in math learning** | |
| | Teacher uses manipulatives/technology to guide student thinking about ideas | |
| | Teacher motivates student learning | |
| | Teacher encourages student to ask questions about mathematics that they study | |
| | Teacher offers sufficient time for problem solving | |
| | Students have opportunities for independent practice | |
| | Teacher discussion demonstrates deep knowledge of subject | |
| | Teacher manages classroom behavior to support learning mathematics | |
| | Teacher considers classroom arrangement issues to support students' understanding | |
| | Teacher challenges students to analyze their strengths and weaknesses | |
| | Teacher creates an inviting thoughtful atmosphere | |
| F | **Addressing students' misconception** | |
| | Teacher monitors student misconceptions | |
| | Lessons extend procedures to other contexts and situations to guide students in challenging their own misconceptions | |
| | Lesson indicates possible students' misconception | |
| | Lesson adapts students' biases, difficulties, and limitations | |

# Appendix D

## Pre-Observation Teacher Interview Protocol

**Pre-Observation Teacher Interview Protocol**

Teacher Code ______________________

Date ______________________

1. Please give me a quick outline about what you are planning to do in this class.
2. What will the students be doing?
3. Why did you decide to organize the lesson in this manner?
4. How did you go about planning your lesson this way? (probe with pedagogical reasoning points)
5. What are your methods or strategies of gaining knowledge about students' thinking?
6. How did you attempt to identify the ideas students might have on this topic before teaching this unit? This lesson?
7. May I have a copy of the instructional plans for this lesson?

# Appendix E

## Post-Observation Interview Protocol

**Post-Observation Interview Protocol**

1. Describe the ability levels of your students for this lesson.
2. Please help me understand where this lesson fits in the sequence of the unit on which you are working. What have the students experienced prior to today's lesson?
3. How do you feel about how the lesson played out? What do you think the students gained from today's lesson?
4. What guided you in teaching the mathematics in this lesson?
5. What resources did you use to plan this lesson? Why (related with student mathematical thinking)? (Be sure to get details on sources of materials and activities.)
6. How did you incorporate your prior knowledge of students' thinking in this algebra lesson?
7. Did you encounter any unexpected student mathematical thinking during lesson? If so, what?

## Appendix F

## Stimulated Recall Interview General Protocol

1. Please describe what you are doing in this segment.
2. Why are you doing this?
3. What knowledge were you using about the students' mathematical thinking in solving word problems for this segment?
4. What were you thinking as you were involved in this segment?
5. What did you learn from this segment?
6. Was there any specific planning that you did prior to this segment?
7. Did you make any changes in your subsequent lesson plans after this segment, based on what happened in the classroom instruction?

## Additional Protocol Questions

1. What were you thinking at this point during the lesson? Why did you give that explanation? Why did you ask that particular question?
2. Were you addressing a particular mathematical concept at this point in the lesson? If so, why?
3. Why did you choose to use that representation (graph, table, diagram, etc)? Why did you select that solution strategy? Why did you select that example?
4. Were you addressing a particular potential student difficulty at this point? As a result, did you learn anything about what the students understood?