PACIFIC JOURNAL OF MATHEMATICS
Vol. 31, No. 3, 1969

PARTIAL DIFFERENTIAL EQUATIONS OF SOBOLEV-GALPERN TYPE

R. E. SHOWALTER

A mixed initial and boundary value problem is considered for a partial differential equation of the form $Mu_t(x, t)+Lu(x, t)=0$, where M and L are elliptic differential operators of orders $2m$ and $2l$, respectively, with $m \leqq l$. The existence and uniqueness of a strong solution of this equation in $H_0^l(G)$ is proved by semigroup methods.

We are concerned here with a mixed initial boundary value problem for the equation

$$Mu_t + Lu = 0 \tag{1}$$

in which M and L are elliptic differential operators. Equations of this type have been studied using various methods in [2, 3, 4, 6, 7, 10, 11, 13, 14, 15, 17, 18]. We will make use of the L^2-estimates and related results on elliptic operators to obtain a generalized solution to this problem similar to that obtained for the parabolic equation

$$u_t + Lu = 0$$

as in [7].

Let G be a bounded open domain in R^n whose boundary ∂G is an $(n-1)$-dimensional manifold with G lying on one side of ∂G. By $H^k(G) \equiv H^k$ we mean the Hilbert space (of equivalence classes) of functions in $L^2(G)$ whose distributional derivatives through order k belong to $L^2(G)$ with the inner product and norm given, respectively, by

$$(f, g)_k = \sum\left\{\int_G D^\alpha f \overline{D^\alpha g} dx : |\alpha| \leqq k\right\}$$

and

$$\| f \|_k = \sqrt{(f, f)_k} \,.$$

$H_0^k \equiv H_0^k(G)$ will denote the closure in H^k of $C_0^\infty(G)$, the space of infinitely differentiable functions with compact support in G.

The operators are of the form

$$M = \sum \{(-1)^{|\rho|} D^\rho m^{\rho\sigma}(x) D^\sigma : |\rho|, |\sigma| \leqq m\}$$

and

$$L = \sum \{(-1)^{|\rho|} D^\rho l^{\rho\sigma}(x) D^\sigma : |\rho|, |\sigma| \leqq l\} \,,$$

and they are uniformly strongly elliptic in G. We shall investigate the existence and uniqueness of solutions to (1) which coincide with the initial function u_0 in H_0^l where $t = 0$ and vanish on ∂G together with all derivatives of order less than or equal to $l - 1$.

If the order of M is as high as that of $L\,(2m \geqq 2l)$, then this problem can be handled as in [10] by forming the exponential of the bounded extension of $M^{-1}L$ on H_0^m and thus obtaining a group of operators on H_0^m and a corresponding solution for all t in $\boldsymbol{R}$. The case we shall consider is that of $m \leqq l$, and this will include the parabolic equation as a special case. We obtain a semi-group of operators on H_0^m and, hence, a solution for all $t \geqq 0$.

2. In this section we shall formulate the problem. Assume temporarily the following.

P_1': The coefficients $m^{\rho\sigma}$ in M belong to $H^{|\rho|}$, and $D^\rho m^{\rho\sigma}$ is in $L^\infty(G)$ whenever $|\rho| \leqq m$. A similar statement is true for the coefficients in L. From P_1' it follows that the sesqui-linear forms defined on $C_0^\infty(G)$ by

$$B_M(\varphi, \psi) = \sum \{(m^{\rho\sigma}D^\sigma\varphi, D^\rho\psi)_0 : |\rho|, |\sigma| \leqq m\}$$

and

$$B_L(\varphi, \psi) = \sum \{(l^{\rho\sigma}D^\sigma\varphi, D^\rho\psi)_0 : |\rho|, |\sigma| \leqq l\}$$

satisfy the identities

$$B_M(\varphi, \psi) = (M\varphi, \psi)_0 \tag{2}$$

and

$$B_L(\varphi, \psi) = (L\varphi, \psi)_0 \tag{2'}$$

for all φ, ψ in $C_0^\infty(G)$. Since P_1' implies that

$$K_m = \sup \{\| m^{\rho\sigma} \|_\infty : |\rho|, |\sigma| \leqq m\}$$

and

$$K_l = \sup \{\| l^{\rho\sigma} \|_\infty : |\rho|, |\sigma| \leqq l\}$$

are finite, we see that

$$|B_M(\varphi, \psi)| \leqq K_m \|\varphi\|_m \|\psi\|_m$$

and

$$|B_L(\varphi, \psi)| \leqq K_l \|\varphi\|_l \|\psi\|_l$$

for all φ, ψ in $C_0^\infty(G)$. Hence these sesqui-linear forms may be extended by continuity to all of H_0^m and H_0^l, respectively.

The final properties which we shall assume are the following. For any φ, ψ in $C_0^\infty(G)$ we have

$$P_2\colon \operatorname{Re} B_M(\varphi, \varphi) \geqq k_m \|\varphi\|_m^2,\ k_m > 0\ ,$$
$$\operatorname{Re} B_L(\varphi, \varphi) \geqq k_l \|\varphi\|_l^2,\ k_l > 0\ ,$$

and

$$P_3\colon | B_M(\varphi, \psi)|^2 \leqq (\operatorname{Re} B_M(\varphi, \varphi))(\operatorname{Re} B_M(\psi, \psi))\ .$$

These inequalities are valid for the respective extensions to H_0^m and H_0^l. The assumptions of P_2 are inequalities of the Garding type which imply that M and L are uniformly strongly elliptic. Only the first of these is essential in applications, for the usual change of dependent variable $u = ve^{\lambda t}$ changes our equation to one with L replaced by $L + \lambda M$, and the Garding inequality is true for $B_{L+\lambda M}$ if λ is sufficiently large and if the coefficients $l^{\rho\sigma}(x)$, $|\rho| = |\sigma| = l$ are uniformly continuous in G. See [3, 8] for sufficient conditions that P_2 be true.

The assumption P_3 is a Cauchy-Schwarz inequality for the form B_M. In view of the positivity of B_M, a necessary and sufficient condition for P_3 is that M be symmetric, that is, $m^{\rho\sigma} = \overline{m^{\sigma\rho}}$ for all ρ, σ. Such is the case for the examples

(i) $ku_t - \Delta u = 0$ $(m = 0)$ and

(ii) $-\gamma \Delta u_t + ku_t - \Delta u = 0$,

where Δ is the Laplacian and γ and k are positive. Example (i) is a parabolic equation, and examples like (ii) appear in various problems of fluid mechanics and soil mechanics, where a solution is sought which satisfies an initial condition $u(x, 0) = u_0(x)$ on G and the Dirichlet condition $u(x, t) = 0$ on the boundary of G. See [1, 11, 12, 13].

We shall not need the full strength of P_1' so we replace it with the following weaker assumption.

P_1: The coefficients $m^{\rho\sigma}$ and $l^{\rho\sigma}$ belong to $L^\infty(G)$ for all ρ, σ.

We shall proceed under the assumptions P_1, P_2 and P_3 and remark that P_1' is needed only when we wish to interpret our weak solutions by means of (2) and (2′).

Under the hypotheses above there is by the theorem of Lax and Milgram [7] a closed linear operator M_0 with domain D_m dense in H_0^m and range equal to $H^0 = L^2(G)$ such that

$$B_M(\varphi, \psi) = (M_0\varphi, \psi)_0 \tag{3}$$

whenever φ belongs to D_m and ψ to H_0^m. Furthermore, M_0^{-1} is a bounded operator from H^0 into H_0^m. Similarly, there is a closed linear operator L_0 with domain D_l dense in H_0^l and range equal to H^0 with

$$B_L(\varphi, \psi) = (L_0\varphi, \psi)_0 \tag{3′}$$

whenever φ belongs to D_l and ψ to H_0^l. Also, L_0^{-1} is bounded from H^0 into H_0^l.

Consider the bijection $A = -M_0^{-1}L_0$ from D_l onto D_m. For any φ in D_m we have

$$\begin{aligned} k_l \| A^{-1}\varphi \|_l^2 &= k_l \| L_0^{-1}M_0\varphi \|_l^2 \\ &\leqq \operatorname{Re} B_L(L_0^{-1}M_0\varphi, L_0^{-1}M_0\varphi) = \operatorname{Re}(M_0\varphi, L_0^{-1}M_0\varphi)_0 \\ &= \operatorname{Re} B_M(\varphi, L_0^{-1}M_0\varphi) \leqq K_m \| \varphi \|_m \| A^{-1}\varphi \|_m \\ &\leqq K_m \| \varphi \|_m \| A^{-1}\varphi \|_l \end{aligned}$$

which yields

$$\| A^{-1}\varphi \|_l \leqq (K_m/k_l) \| \varphi \|_m \tag{4}$$

for all φ in D_m. But D_m is dense in H_0^m so A^{-1} has a unique extension by continuity from H_0^m onto the set $D = A^{-1}(H_0^m)$ in H_0^l, the domain of the closed extension of A. The continuity of the injection of H_0^l into H_0^m implies that A^{-1} is a bounded operator on H_0^m, and this is the space in which we formulate the Generalized Problem:

For a given initial function u_0 in D, find a differentiable map $u(t)$ of R^+ into H_0^m for which $u(t)$ belongs to H_0^l for all $t \geqq 0$, $u(0) = u_0$, and

$$B_M(u'(t), \varphi) + B_L(u(t), \varphi) = 0 \tag{5}$$

for all φ in $C_0^\infty(G)$ and $t \geqq 0$.

Sufficient conditions for a solution of this generalized problem to be a classical solution will be discussed in [9].

3. The objective of this section is to prove the following results.

THEOREM. *There exists a unique solution of the generalized problem. If $u(t)$ is in D_l then $u'(t)$ is in D_m and*

$$M_0u'(t) + L_0u(t) = 0 \tag{6}$$

in H^0. The mapping of u_0 to $u(t)$ is continuous from H_0^m into itself for each $t \geqq 0$ and furthermore satisfies

$$\| u(t) \|_m \leqq \sqrt{(K_m/k_m)} \, \| u_0 \|_m \exp(-k_l t/K_m) \, . \tag{7}$$

We first show that the operator A is the infinitesimal generator of a semi-group of bounded operators on H_0^m; this semi-group will provide a means of constructing a solution to the problem. From the assumptions on B_M, it follows that the function defined by

$$| \varphi |_M = \sqrt{(\operatorname{Re} B_M(\varphi, \varphi))}$$

is a norm on H_0^m that is equivalent to the norm $\| \cdot \|_m$. In the following we shall use $| \cdot |_M$ as the norm on H_0^m, noting further that

$$k_m^{1/2} \| \varphi \|_m \leqq | \varphi |_M \leqq K_m^{1/2} \| \varphi \|_m \tag{8}$$

for φ in H_0^m.

To obtain the necessary estimates we let λ be a nonnegative number and consider the operator $\lambda M_0 + L_0 = N$ from the domain $D_m \cap D_l$ into H^0. We can define a sesqui-linear form on $D_m \cap D_l$ by

$$B_N(\varphi, \psi) = ((\lambda M_0 + L_0)\varphi, \psi)_0 = \lambda B_M(\varphi, \psi) + B_L(\varphi, \psi)$$

and then note that B_N is bounded as well as positive-definite with respect to the norm of H_0^l. We extend B_N by continuity to all of H_0^l, and then by the theorem of Lax and Milgram there is a closed linear operator N_0 from a domain D_n in H_0^l onto H^0 for which

$$B_N(\varphi, \psi) = (N_0\varphi, \psi)_0$$

whenever φ is in D_n and ψ in H_0^l. Clearly N_0 is an extension of N whose domain is $D_m \cap D_l$.

For all φ in D_n we have

$$\begin{aligned} \operatorname{Re}(N_0\varphi, \varphi)_0 &= \lambda \operatorname{Re} B_M(\varphi, \varphi) + \operatorname{Re} B_L(\varphi, \varphi) \\ &\geqq (\lambda + k_l/K_m) \operatorname{Re} B_M(\varphi, \varphi) \\ &= (\lambda + k_l/K_m) | \varphi |_M^2 . \end{aligned}$$

Thus, for any ψ in D_m we see that $N_0^{-1}M_0\psi$ belongs to D_n and from above

$$\begin{aligned} (\lambda + k_l/K_m) | N_0^{-1}M_0\psi |_M^2 &\leqq \operatorname{Re}(M_0\psi, N_0^{-1}M_0\psi)_0 \\ &= \operatorname{Re} B_M(\psi, N_0^{-1}M_0\psi) \leqq | \psi |_M |(N_0^{-1}M_0\psi)|_M \end{aligned}$$

by P_3, so we have obtained the estimate

$$| N_0^{-1}M_0\psi |_M \leqq (\lambda + k_l/K_m)^{-1} | \psi |_M$$

for all ψ in D_m.

Letting φ be an element of $D_l \cap D_m$ we see

$$\begin{aligned} (N_0^{-1}M_0)(\lambda + M_0^{-1}L_0)\varphi &= N_0^{-1}(\lambda M_0\varphi + L_0\varphi) \\ &= N_0^{-1} \cdot N\varphi = \varphi , \end{aligned}$$

so $\lambda + M_0^{-1}L_0$ is injective and satisfies

$$(\lambda + M_0^{-1}L_0)^{-1} = N_0^{-1}M_0$$

on $D_m \cap D_l$. Combining this with the estimate above we see that

$$| (\lambda + M_0^{-1}L_0)^{-1}\psi |_M \leqq (\lambda + k_l/K_m)^{-1} | \psi |_M$$

for all ψ in $D_l \cap D_m$. It follows by continuity that $\lambda - A$ is invertible on H_0^m and satisfies the estimate

$$| (\lambda - A)^{-1} |_M \leqq (\lambda + k_l/K_m)^{-1} .$$

By the theorem of Hille and Yoshida [5, 16] on the characterization of the infinitesimal generators of semi-groups of class C_0 we have the following results: there exists a unique family of bounded operators $\{S(t): t \geqq 0\}$ on H_0^m for which

(i) $S(t_1 + t_2) = S(t_1)S(t_2)$,

(ii) $S(t)x$ is strongly continuous for each x in H_0^m,

(iii) $S(0) = I$ and $|S(t)|_M \leqq \exp(-k_l t/K_m)$ for all $t \geqq 0$,

(iv) $\lim_{h\to 0} h^{-1}(S(h) - I)x_0 = Ax_0$ for each x_0 in D, and

(v) $S(t)$ commutes with $(\lambda - A)^{-1}$ for all $\lambda \geqq 0$.

The statement (v) implies in particular that D is invariant under each $S(t)$.

Having been given the initial function u_0 in D, we define

$$u(t) = S(t)u_0,\ t \geqq 0$$

and show that $u(t)$ is a solution of the generalized problem. Clearly we see $u(t)$ belongs to H_0^m and $u(0) = u_0$. Furthermore, since $S(t)$ leaves D invariant and u_0 is in D, it follows that $u(t)$ belongs to D and thus to H_0^l. The function $u(t)$ is differentiable with

$$u'(t) = Au(t) \tag{9}$$

for all $t \geqq 0$ by (i) and (iv), and hence $u'(t)$ is in H_0^m.

We shall verify that $u(t)$ satisfies the equation (5). Since D_m is dense in H_0^m there is a sequence $\{\varphi_n\}$ in D_m for which $\|\varphi_n - u'(t)\|_m \to 0$ as $n \to \infty$. Now $\{\varphi_n\}$ is a Cauchy sequence in H_0^m and it follows by (4) that $\psi_n = A^{-1}\varphi_n$ is a Cauchy sequence in the complete space H_0^l, so there is a ψ in H_0^l such that $\|\psi_n - \psi\|_l \to 0$ as $n \to \infty$. Since A^{-1} is continuous we have $\psi = u(t)$. Each ψ_n belongs to D_l, since φ_n is in D_m, and furthermore $M_0\varphi_n + L_0\psi_n = 0$. Now for each φ in $C_0^\infty(G)$ we have by the continuity of B_M and B_L

$$\begin{aligned} &B_M(u'(t), \varphi) + B_L(u(t), \varphi) \\ &\quad = \lim_{a\to\infty} B_M(\varphi_n, \varphi) + \lim_{n\to\infty} B_L(\psi_n, \varphi) \\ &\quad = \lim_{n\to\infty}\{B_M(\varphi_n, \varphi) + B_L(\psi_n, \varphi)\} = \lim_{n\to\infty}\{(M_0\varphi_n, \varphi)_0 + (L_0\psi_n, \varphi)_0\} \equiv 0\ , \end{aligned}$$

so the generalized problem does have a solution.

If $u(t)$ is in D_l then by (9) $u'(t)$ is in D_m. It follows from (5) that for every φ in $C_0^\infty(G)$

$$(M_0 u'(t) + L_0 u(t), \varphi)_0 = 0\ ,$$

and this implies (6). The estimate (7) is a consequence of (iii) and (8).

To show that the generalized problem has at most one solution, we let $u(t)$ be a solution of the problem with $u_0 = 0$. By linearity it suffices to show that $u(t) \equiv 0$. The differentiability of $u(t)$ in H_0^m

implies that the real valued function

$$\alpha(t) = \operatorname{Re} B_M(u(t), u(t))$$

is differentiable and

$$\alpha'(t) = 2 \operatorname{Re} B_M(u'(t), u(t)) .$$

Since (5) is true also for all φ in H_0^l by continuity, we have from P_2

$$\alpha'(t) = -2 \operatorname{Re} B_L(u(t), u(t)) \leqq 0 .$$

But $\alpha(0) = \operatorname{Re} B_M(u(0), u(0)) = 0$, so $\alpha(t) = 0$ for all $t \geqq 0$. From P_2 it follows that $u(t) = 0$ for $t \geqq 0$.

REFERENCES

1. G. Barenblat, I. Zheltov, and I. Kochiva, *Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks*, J. Appl. Math. **24** (1960), 1286-1303.
2. R. Carroll, *Partial differential equations* (Harper and Row, to appear)
3. A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall, 1964.
4. C. A. Galpern, Docklady Akad. Nauk. SSSR **119** (1958), 640-643.
5. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, A.M.S. Colloquium Publications, vol. 31, New York, 1957.
6. A. G. Kostachenko and G. I. Eskin, *Cauchy problem for an equation of Sobolev-Galpern type*, Trudy Moscow Math. Obshch. **10** (1961), 273-284.
7. P. D. Lax and A. Milgram, *Parabolic equations*, Contributions to the Theory of Partial Differential Equations, Ann. Math. Studies, no. 33 Princeton University Press, 1954, 167-190.
8. J. L. Lions, *Lectures on elliptic partial differential equations*, Tata Institute, Bombay, 1957.
9. R. E. Showalter, *Local regularity of solutions of pseudo-parabolic partial differential equations* (to appear).
10. R. E. Showalter and T. W. Ting, *Pseudo-parabolic partial differential equations*, (to appear, SIAM Journal)
11. S. Sobolev Izv. Akad. Nauk. SSSR Ser. Mat. **18** (1954), 3.
12. D. W. Taylor, *Research on consolidation of clays*, Massachusetts Institute of Technology, 1942.
13. T. W. Ting, *Certain non-steady flows of second order fluids*, Arch. Rat. Mech. Anal. **14** (1963), 1-26.
14. ———, *Parabolic and pseudo-parabolic partial differential equations* (to appear, J. Math. Soc. Japan).
15. T. W. Ting and R. E. Showalter, *Asymptotic behavior of solutions of pseudo-parabolic partial differential equations* (to appear, Proc. Amer. Math. Soc.)
16. K. Yoshida, *On the differentiability and representation of one-parameter semi-groups of linear operators*, J. Math. Soc. Japan **1** (1948), 15-21.
17. T. Zalenyak, *The behavior for $t \to \infty$ of solutions of a problem of S. L. Sobolev*, Soviet Math. **2** (1961), 956-958.
18. ———, *A problem of Sobolev*, Soviet Math. **3** (1962), 1756-1759.

Received February 14, 1969.

UNIVERSITY OF TEXAS
AUSTIN, TEXAS