A mixed initial and boundary value problem is considered for a partial differential equation of the form $M u_t(x, t) + L u(x, t) = 0$, where $M$ and $L$ are elliptic differential operators of orders $2m$ and $2l$, respectively, with $m \leq l$. The existence and uniqueness of a strong solution of this equation in $H^k_0(G)$ is proved by semigroup methods.

We are concerned here with a mixed initial boundary value problem for the equation

$$Mu_t + Lu = 0$$

in which $M$ and $L$ are elliptic differential operators. Equations of this type have been studied using various methods in [2, 3, 4, 6, 7, 10, 11, 13, 14, 15, 17, 18]. We will make use of the $L^s$-estimates and related results on elliptic operators to obtain a generalized solution to this problem similar to that obtained for the parabolic equation

$$u_t + Lu = 0$$

as in [7].

Let $G$ be a bounded open domain in $\mathbb{R}^n$ whose boundary $\partial G$ is an $(n - 1)$-dimensional manifold with $G$ lying on one side of $\partial G$. By $H^k(G) = H^k$ we mean the Hilbert space (of equivalence classes) of functions in $L^2(G)$ whose distributional derivatives through order $k$ belong to $L^2(G)$ with the inner product and norm given, respectively, by

$$(f, g)_k = \sum \int_G D^\alpha f D^\gamma g dx : |\alpha| \leq k$$

and

$$\|f\|_k = \sqrt{(f, f)_k}.$$  

$H^k_0 \equiv H^k_0(G)$ will denote the closure in $H^k$ of $C_\infty^\omega(G)$, the space of infinitely differentiable functions with compact support in $G$.

The operators are of the form

$$M = \sum \{(-1)^{\rho}|D^\rho m^\sigma(x)D^\sigma| : |\rho|, |\sigma| \leq m\}$$

and

$$L = \sum \{(-1)^{\rho}|D^\rho l^\sigma(x)D^\sigma| : |\rho|, |\sigma| \leq l\},$$

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and they are uniformly strongly elliptic in $G$. We shall investigate
the existence and uniqueness of solutions to (1) which coincide with the
initial function $u_0$ in $H^l_0$ where $t = 0$ and vanish on $\partial G$ together with
all derivatives of order less than or equal to $l - 1$.

If the order of $M$ is as high as that of $L (2m \geq 2l)$, then this
problem can be handled as in [10] by forming the exponential of the
bounded extension of $M^{-1}L$ on $H^m_0$ and thus obtaining a group of
operators on $H^m_0$ and a corresponding solution for all $t$ in $R$. The case
we shall consider is that of $m \leq l$, and this will include the parabolic
equation as a special case. We obtain a semi-group of operators on
$H^m_0$ and, hence, a solution for all $t \geq 0$.

2. In this section we shall formulate the problem. Assume tem-
porarily the following.

$P_1$: The coefficients $m^{\rho \sigma}$ in $M$ belong to $H^{m\rho}$, and
$D^\rho m^{\rho \sigma}$ is in $L^m(G)$ whenever $|\rho| \leq m$. A similar statement is true for the coefficients in $L$.

From $P_1$ it follows that the sesqui-linear forms defined on $C^m_0(G)$ by

$$B_M(\phi, \psi) = \sum \{(m^{\rho \sigma}D^\rho \phi, D^\rho \psi)_0 : |\rho|, |\sigma| \leq m\}$$

and

$$B_L(\phi, \psi) = \sum \{(l^{\rho \sigma}D^\rho \phi, D^\rho \psi)_0 : |\rho|, |\sigma| \leq l\}$$

satisfy the identities

$$(2) \quad B_M(\phi, \psi) = (M\phi, \psi)_0$$

and

$$(2') \quad B_L(\phi, \psi) = (L\phi, \psi)_0$$

for all $\phi, \psi$ in $C^m_0(G)$. Since $P_1$ implies that

$$K_m = \sup \{|m^{\rho \sigma}|_m : |\rho|, |\sigma| \leq m\}$$

and

$$K_l = \sup \{|l^{\rho \sigma}|_m : |\rho|, |\sigma| \leq l\}$$

are finite, we see that

$$|B_M(\phi, \psi)| \leq K_m \|\phi\|_m \|\psi\|_m$$

and

$$|B_L(\phi, \psi)| \leq K_l \|\phi\|_l \|\psi\|_l$$

for all $\phi, \psi$ in $C^m_0(G)$. Hence these sesqui-linear forms may be extended
by continuity to all of $H^m_0$ and $C^l$, respectively.
The final properties which we shall assume are the following. For any \( \varphi, \psi \) in \( C^0_0(G) \) we have

\[
P_2: \Re B_M(\varphi, \varphi) \geq k_m \| \varphi \|_m^2, \quad k_m > 0,
\]
and

\[
P_3: |B_M(\varphi, \psi)|^2 \leq (\Re B_M(\varphi, \varphi))(\Re B_M(\psi, \psi)).
\]

These inequalities are valid for the respective extensions to \( H_0^m \) and \( H_1^l \). The assumptions of \( P_2 \) are inequalities of the Garding type which imply that \( M \) and \( L \) are uniformly strongly elliptic. Only the first of these is essential in applications, for the usual change of dependent variable \( u = ve^{it} \) changes our equation to one with \( L \) replaced by \( L + \lambda M \), and the Garding inequality is true for \( B_{L + \lambda M} \) if \( \lambda \) is sufficiently large and if the coefficients \( L^{\rho \sigma}(x), |\rho| = |\sigma| = l \) are uniformly continuous in \( G \). See [3, 8] for sufficient conditions that \( P_2 \) be true.

The assumption \( P_3 \) is a Cauchy-Schwarz inequality for the form \( B_M \). In view of the positivity of \( B_M \), a necessary and sufficient condition for \( P_3 \) is that \( M \) be symmetric, that is, \( m^{\rho \sigma} = m^{\sigma \rho} \) for all \( \rho, \sigma \). Such is the case for the examples

(i) \( ku_t - \Delta u = 0 \) (\( m = 0 \)) and
(ii) \( -\gamma \Delta u_t + ku_t - \Delta u = 0 \),

where \( \Delta \) is the Laplacian and \( \gamma \) and \( k \) are positive. Example (i) is a parabolic equation, and examples like (ii) appear in various problems of fluid mechanics and soil mechanics, where a solution is sought which satisfies an initial condition \( u(x, 0) = u_0(x) \) on \( G \) and the Dirichlet condition \( u(x, t) = 0 \) on the boundary of \( G \). See [1, 11, 12, 13].

We shall not need the full strength of \( P_3 \) so we replace it with the following weaker assumption.

\( P_3': \) The coefficients \( m^{\rho \sigma} \) and \( l^{\rho \sigma} \) belong to \( L^\infty(G) \) for all \( \rho, \sigma \).

We shall proceed under the assumptions \( P_1 \), \( P_2 \) and \( P_3 \) and remark that \( P_3' \) is needed only when we wish to interpret our weak solutions by means of (2) and (2').

Under the hypotheses above there is by the theorem of Lax and Milgram [7] a closed linear operator \( M_0 \) with domain \( D_m \) dense in \( H_0^m \) and range equal to \( H^m = L^2(G) \) such that

\[
B_M(\varphi, \psi) = (M_0 \varphi, \psi),
\]
whenever \( \varphi \) belongs to \( D_m \) and \( \psi \) to \( H_0^m \). Furthermore, \( M_0^{-1} \) is a bounded operator from \( H^m \) into \( H_0^m \). Similarly, there is a closed linear operator \( L_0 \) with domain \( D_l \) dense in \( H^l \) and range equal to \( H^l \) with

\[
B_M(\varphi, \psi) = (L_0 \varphi, \psi),
\]
whenever \( \phi \) belongs to \( D_t \) and \( \psi \) to \( H^t_0 \). Also, \( L_0^{-1} \) is bounded from \( H^0 \) into \( H^t_0 \).

Consider the bijection \( A = -M_0^{-1}L_0 \) from \( D_t \) onto \( D_m \). For any \( \phi \) in \( D_m \) we have

\[
\begin{align*}
  k_1 \| A^{-1} \phi \|_m & = k_1 \| L_0^{-1}M_0 \phi \|_m \\
  & \leq \text{Re} B(u(L_0^{-1}M_0 \phi, L_0^{-1}M_0 \phi)) = \text{Re} (M_0 \phi, L_0^{-1}M_0 \phi) \\
  & = \text{Re} B(u(\phi, L_0^{-1}M_0 \phi)) \leq K_m \| \phi \|_m \| A^{-1} \phi \|_m \\
  & \leq K_m \| \phi \|_m \| A^{-1} \phi \|_m
\end{align*}
\]

which yields

\[
\| A^{-1} \phi \|_t \leq (K_m/k_1) \| \phi \|_m
\]

for all \( \phi \) in \( D_m \). But \( D_m \) is dense in \( H_0^m \) so \( A^{-1} \) has a unique extension by continuity from \( H_0^m \) onto the set \( D = A^{-1}(H_0^m) \) in \( H_t^t \), the domain of the closed extension of \( A \). The continuity of the injection of \( H_t^t \) into \( H_0^m \) implies that \( A^{-1} \) is a bounded operator on \( H_0^m \), and this is the space in which we formulate the Generalized Problem:

For a given initial function \( u_0 \) in \( D \), find a differentiable map \( u(t) \) of \( \mathbb{R}^+ \) into \( H_0^m \) for which \( u(t) \) belongs to \( H_t^t \) for all \( t \geq 0 \), \( u(0) = u_0 \), and

\[
B_M(u'(t), \phi) + B_L(u(t), \phi) = 0
\]

for all \( \phi \) in \( C_0^\infty(G) \) and \( t \geq 0 \).

Sufficient conditions for a solution of this generalized problem to be a classical solution will be discussed in [9].

3. The objective of this section is to prove the following results.

**Theorem.** There exists a unique solution of the generalized problem. If \( u(t) \) is in \( D_t \) then \( u'(t) \) is in \( D_m \) and

\[
M_0 u'(t) + L_0 u(t) = 0
\]

in \( H^0 \). The mapping of \( u_0 \) to \( u(t) \) is continuous from \( H_0^m \) into itself for each \( t \geq 0 \) and furthermore satisfies

\[
\| u(t) \|_m \leq \sqrt{(K_m/k_m)} \| u_0 \|_m \exp (-k_1 t/K_m).
\]

We first show that the operator \( A \) is the infinitesimal generator of a semi-group of bounded operators on \( H_0^m \); this semi-group will provide a means of constructing a solution to the problem. From the assumptions on \( B_M \), it follows that the function defined by

\[
| \phi |_M = \sqrt{\text{Re} B_M(\phi, \phi)}
\]

is a norm on \( H_0^m \) that is equivalent to the norm \( \| \cdot \|_m \). In the following we shall use \( | \cdot |_M \) as the norm on \( H_0^m \), noting further that
To obtain the necessary estimates we let $\lambda$ be a nonnegative number
and consider the operator $\lambda M_0 + L_0 = N$ from the domain $D_m \cap D_l$ into $H^\omega$. We can define a sesqui-linear form on $D_m \cap D_l$ by

$$B_N(\varphi, \psi) = ((\lambda M_0 + L_0)\varphi, \psi) = \lambda B_M(\varphi, \psi) + B_L(\varphi, \psi)$$

and then note that $B_N$ is bounded as well as positive-definite with respect to the norm of $H^\omega$. We extend $B_N$ by continuity to all of $H^\omega$, and then by the theorem of Lax and Milgram there is a closed linear operator $N_0$ from a domain $D_n$ in $H^\omega$ onto $H^\omega$ for which

$$B_N(\varphi, \psi) = (N_0\varphi, \psi)_0$$

whenever $\varphi$ is in $D_n$ and $\psi$ in $H^\omega$. Clearly $N_0$ is an extension of $N$ whose domain is $D_m \cap D_l$.

For all $\varphi$ in $D_n$ we have

$$\Re (N_0\varphi, \varphi)_0 = \lambda \Re B_M(\varphi, \varphi) + \Re B_L(\varphi, \varphi) \geq (\lambda + k_1/K_m) \Re B_M(\varphi, \varphi) = (\lambda + k_1/K_m) |\varphi|_H^2.$$ 

Thus, for any $\psi$ in $D_n$ we see that $N^{-1}_0 M_0 \psi$ belongs to $D_n$ and from above

$$(\lambda + k_1/K_m) |N^{-1}_0 M_0 \psi|_H^{-1} \leq \Re (M_0 \psi, N^{-1}_0 M_0 \psi)_0$$

$$= \Re B_M(\psi, N^{-1}_0 M_0 \psi) \leq |\psi|_M |(N^{-1}_0 M_0 \psi)|_M$$

by $P_2$, so we have obtained the estimate

$$|N^{-1}_0 M_0 \psi|_M \leq (\lambda + k_1/K_m)^{-1} |\psi|_M$$

for all $\psi$ in $D_n$.

Letting $\varphi$ be an element of $D_l \cap D_n$ we see

$$(N^{-1}_0 M_0)(\lambda + M^{-1}_0 L_0)\varphi = N^{-1}_0 (\lambda M_0 \varphi + L_0 \varphi) = N^{-1}_0 N \varphi = \varphi,$$

so $\lambda + M^{-1}_0 L_0$ is injective and satisfies

$$(\lambda + M^{-1}_0 L_0)^{-1} = N^{-1}_0 M_0$$

on $D_m \cap D_l$. Combining this with the estimate above we see that

$$| (\lambda + M^{-1}_0 L_0)^{-1} \psi |_M \leq (\lambda + k_1/K_m)^{-1} |\psi|_M$$

for all $\psi$ in $D_l \cap D_m$. It follows by continuity that $\lambda - A$ is invertible on $H^\omega_m$ and satisfies the estimate

$$| (\lambda - A)^{-1} |_M \leq (\lambda + k_1/K_m)^{-1}.$$
By the theorem of Hille and Yoshida [5, 16] on the characterization of the infinitesimal generators of semi-groups of class \( C_0 \), we have the following results: there exists a unique family of bounded operators \( \{ S(t): t \geq 0 \} \) on \( H_0^m \) for which

(i) \( S(t_1 + t_2) = S(t_1)S(t_2) \),
(ii) \( S(t)x \) is strongly continuous for each \( x \) in \( H_0^m \),
(iii) \( S(0) = I \) and \( |S(t)| \leq \exp (-k_1 t / K_m) \) for all \( t \geq 0 \),
(iv) \( \lim_{h \to 0} h^{-1}(S(h) - I)x_0 = Ax_0 \) for each \( x_0 \) in \( D \), and
(v) \( S(t) \) commutes with \( (\lambda - A)^{-1} \) for all \( \lambda \geq 0 \).

The statement (v) implies in particular that \( D \) is invariant under each \( S(t) \).

Having been given the initial function \( u_0 \) in \( D \), we define

\[
  u(t) = S(t)u_0, \quad t \geq 0
\]

and show that \( u(t) \) is a solution of the generalized problem. Clearly we see \( u(t) \) belongs to \( H_0^m \) and \( u(0) = u_0 \). Furthermore, since \( S(t) \) leaves \( D \) invariant and \( u_0 \) is in \( D \), it follows that \( u(t) \) belongs to \( D \) and thus to \( H_0^m \). The function \( u(t) \) is differentiable with

\[
  (9) \quad u'(t) = Au(t)
\]

for all \( t \geq 0 \) by (i) and (iv), and hence \( u'(t) \) is in \( H_0^m \).

We shall verify that \( u(t) \) satisfies the equation (5). Since \( D_m \) is dense in \( H_0^m \) there is a sequence \( \{ \varphi_n \} \) in \( D_m \) for which \( \| \varphi_n - u'(t) \|_m \to 0 \) as \( n \to \infty \). Now \( \{ \varphi_n \} \) is a Cauchy sequence in \( H_0^m \) and it follows by (4) that \( \varphi_n = A^{-1}\varphi_n \) is a Cauchy sequence in the complete space \( H_0^m \), so there is a \( \varphi \) in \( H_0^m \) such that \( \| \varphi_n - \varphi \|_1 \to 0 \) as \( n \to \infty \). Since \( A^{-1} \) is continuous we have \( \varphi = u(t) \). Each \( \varphi_n \) belongs to \( D_1 \), and furthermore \( M_\varphi \varphi_n + L_0 \varphi_n = 0 \). Now for each \( \varphi \) in \( C_0^\infty (G) \) we have by the continuity of \( B_\varphi \) and \( B_L \)

\[
  B_\varphi (u'(t), \varphi) + B_L(u(t), \varphi) = \lim_{n \to \infty} B_\varphi (\varphi_n, \varphi) + \lim_{n \to \infty} B_L(\varphi_n, \varphi)
\]

\[
  = \lim_{n \to \infty} \{ B_\varphi (\varphi_n, \varphi) + B_L(\varphi_n, \varphi) \} = \lim_{n \to \infty} [(M_\varphi \varphi_n, \varphi)_0 + (L_0 \varphi_n, \varphi)_0] = 0 ,
\]

so the generalized problem does have a solution.

If \( u(t) \) is in \( D_1 \) then by (9) \( u'(t) \) is in \( D_m \). It follows from (5) that for every \( \varphi \) in \( C_0^\infty (G) \)

\[
  (M_\varphi u'(t) + L_0 u(t), \varphi)_0 = 0 ,
\]

and this implies (6). The estimate (7) is a consequence of (iii) and (8).

To show that the generalized problem has at most one solution, we let \( u(t) \) be a solution of the problem with \( u_0 = 0 \). By linearity it suffices to show that \( u(t) \equiv 0 \). The differentiability of \( u(t) \) in \( H_0^m \)
implies that the real valued function
\[ \alpha(t) = \text{Re} B_M(u(t), u(t)) \]
is differentiable and
\[ \alpha'(t) = 2 \text{Re} B_M(u'(t), u(t)). \]
Since (5) is true also for all \( \varphi \) in \( H_\delta^1 \) by continuity, we have from \( P_2 \)
\[ \alpha'(t) = -2 \text{Re} B_M(u(t), u(t)) \leq 0. \]
But \( \alpha(0) = \text{Re} B_M(u(0), u(0)) = 0 \), so \( \alpha(t) = 0 \) for all \( t \geq 0 \). From \( P_2 \) it follows that \( u(t) = 0 \) for \( t \geq 0 \).

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