AN ABSTRACT OF THE DISSERTATION OF

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Margaret L. Niess

The purpose of the study was to investigate students' learning of the function concept and the role of the graphing calculator in a College Algebra course. Differences between students with high symbolic manipulation skills and students with low symbolic manipulation skills were also examined. On the basis of an algebraic skills test administered by the instructor (high/low) and students' academic majors (math & science, business, and liberal arts), 25 students from one College Algebra class were placed into six categories.

To gather data on students' understanding of functions, a pretest and posttest were administered. The Function Test consisted of four identification questions given in each of the representations, three questions asking for the definition, an example, and a nonexample of functions, and 15 questions consisting of three problem situations given in the numerical, graphical, and symbolic representations. To gather data on the role of the graphing calculator, daily classroom observations were conducted. To verify students' responses and classroom observations, formal interviews with students and informal interviews with the instructor were conducted.

Students' personal definition progressed towards the formal definition of functions. Yet, students had difficulties with the univalence requirement in three areas: (a) order of domain and range, (b) preference for simple algorithms, and (c) the restriction that functions were one-to-one. Compared to students with low symbolic manipulation skills, students with high symbolic manipulation skills were more flexible working between representations of functions. Half of the interviewed students with low

symbolic manipulation skills perceived a single function given in numerical, graphical, and symbolic representations as separate entities.

The graphing calculator played a role in all phases of the solution process. During the initial phases, students used calculators to develop a symbolic approach. The prime motivation for using graphing calculators during the solution-execution phase was to avoid careless errors. The most common use of graphing calculators was to check answers during the solution-monitoring phase.

However, graphing calculators created difficulties for students who accepted graphs at face value. Interpreting the truncated graph shown by the calculator, students determined that exponential functions possessed a bounded domain because they did not explore the graph.

Student Understanding of Functions and the Use of the Graphing Calculator in a College Algebra Course

by

Patrick J. Averbeck

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Major Professor, representing Mathematics Education

Redacted for Privacy

Chair of Department of Science and Mathematics Education

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Dean of Graduate School

I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my dissertation to any reader upon request.

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Patrick J. Averbeck, Author

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CHAPTER I
THE PROBLEM

Introduction

The undergraduate mathematics curriculum has served as a filter for college students' progress in their lower division curriculum. Many of these students do not develop an understanding of mathematics and are unsuccessful in completing their required mathematics courses (Ellis, 1995; National Science Foundation [NSF], 1996). Over 50% of the students taking a mathematics course in college take a pre-calculus course to meet prerequisites and gain necessary skills to enroll in a mathematics course required by the student's major (American Mathematical Association of Two Year Colleges [AMATYC], 1995). To make matters more pronounced, the changing demographics of college students indicates that more students do not have the skills needed to survive a mathematics course focusing on algebraic manipulation (National Research Council [NRC], 1989). Because many students are unable to succeed in college pre-calculus courses that have a singular focus on symbolic algebraic skill development, lower-level college mathematics courses have become a barrier for many students in reaching their academic goals.

In response to the difficulties undergraduate students have for understanding mathematical concepts in a setting that emphasizes symbolic algebraic skills, one goal of the current reform movement in mathematics education is to change the focus in pre-calculus courses from algebraic skill development to conceptual understanding of key mathematical ideas (AMATYC, 1995; Leitzel, 1991; National Council of Teachers of Mathematics [NCTM], 2000). Changing the focus of pre-calculus courses may make mathematical concepts more accessible to students who lack sufficient symbolic algebraic skill and whose mental strengths may lie in areas other than algebraic

manipulation (Romberg, Fennema, & Carpenter, 1993). The new recommended emphasis in the curriculum, therefore, represents a shift from the memorization of isolated facts and procedures that are poorly retained by many students to the understanding of concepts and relationships within mathematics and of connections between mathematics and other disciplines (AMATYC, 1995; NCTM, 2000).

The primary topic covered in college pre-calculus courses is the function concept, one of the most important topics in both lower- and higher-level undergraduate collegiate mathematics courses (Romberg, Carpenter, & Fennema, 1993). Much of mathematics is either directly related to or is an extension of the concept of function (Kaput, 1997; Leitzel, 1991; Selden & Selden, 1992). Within mathematics, the function concept is a fundamental idea in the study of many other mathematical topics. For example, functions are used in the study of many calculus concepts such as limits, continuity, derivatives, and integration (Ferrini-Mundy & Lauten, 1994). Also, functions are studied in discrete mathematics courses that are sometimes required for business, computer science, and liberal arts majors. Outside the mathematical field, the concept of function is more than a formula used by students to enter numbers and calculate results. Functions can be used to model scientific phenomena, such as the period of a pendulum, or real-life situations such as the stock market. Thus, because functions are used not only in mathematics but also in other scientific and nonscientific fields, functions play a pivotal role in the mathematics courses required for majors.

Unfortunately, students have not been successful learning functions. Vinner and Dreyfus (1989) indicated that students from different major groups had varying degrees of an understanding of functions. Students in all major groups exhibited a discrepancy between the definition they provided and the justifications they gave for classifying functions. Carlson (1998) reported students completing a precalculus course with high marks were not successful on numerous mathematical tasks related to functions. Selden and Selden (1992) suggested students were developing a limited understanding of functions. For example, students used the vertical line test almost exclusively to classify functions. From interviews with students, DeMarois and McGowen (1996) found that students did not view functions as an object that can be acted upon and had difficulty differentiating between $3f(2)$ and $2f(3)$.

Numerous misconceptions have been documented by researchers. Administering a survey to students in various levels of college mathematics courses, Becker (1991) and Slavit (1994) found that students held the misconceptions that functions are not constant, that functions are only linear, and that functions are only continuous and smooth. Vinner (1983) and Selden and Selden (1992) identified two misconceptions about constant functions: college students did not consider constant functions as functions because constant functions were many-to-one rather than one-to-one and the symbolic expression of constant functions did not contain a x-variable. Also, high school and college students considered functions to be rules with regularities (Tall & Bakar, 1992). Functions that had exceptions such as a piecewise functions were not considered to be functions. Hitt (1998) found that prospective mathematics teachers had difficulties identifying functions for graphs of irregular curves. Thus, research has indicated that students do not have a good grasp of the function concept after completing lower level college mathematics courses.

The most widely accepted view on student learning is constructivism (Selden & Selden, 1992). This perspective holds that students construct their own knowledge. Students are perceived as active agents rather than passive receptacles of knowledge. Students construct mathematical knowledge as they strive to make sense of their world (Cobb, Yackel, & Wood, 1992). Similar to Skemp (1971), Hiebert and Carpenter (1992) described students' knowledge structure as a spider's web, where related topics are connected by strands. An intricate web represents a more complete understanding of a concept than a web with only one strand connecting two pieces of information. Thus, rather than attempting to use only one strand by presenting functions algebraically, other representations are recommended so that students can use alternative approaches to develop an intricate knowledge structure representing their conceptual understanding of functions.

Another aspect of constructivism is that students construct their knowledge on the basis of their experiences. Therefore, the experiences that students encounter influence their knowledge structures. According to von Glasersfeld (1987), students' knowledge structures are built from the actions they have taken in the world. From an experience, students draw conclusions and modify their knowledge structure. Thus, on the basis of

the mathematical activities encountered, students develop their understanding of functions. Modification occurs if some context forces them to do so. Thus, the mathematical activities students encounter need to be varied in order to provide experiences that force students to modify their knowledge structures.

In essence, learning occurs when new knowledge is assimilated into an existing knowledge structure or when new connections are made (Strike & Posner, 1985). Meaningful learning takes place when students can relate new material in a substantive and non-arbitrary way to their already existing knowledge structure (Begle, 1979). Students do not readily incorporate abstract definitions into their knowledge structure (Sfard, 1992; Sierpinska, 1992; Vinner & Dreyfus, 1989). Thus, mathematical encounters for students need to be connected to existing knowledge structures.

Since the average student has been unable to succeed in courses that focus on abstract symbolic algebraic approaches to study functions (Kaput, 1997; Malik, 1980; Mathews, 1996; Sierpinska, 1992), one suggested method for increasing student understanding of functions is to use the graphing calculator to circumvent the symbolic algebraic barrier (AMATYC, 1995; Leitzel, 1991; NCTM, 1989). Graphing calculators allow students to focus on the concept rather than the prerequisite skills related to the concept (Yerushalmy & Schwartz, 1993). Graphing calculators can be used to bypass unmotivated skill development (Philipp, Martin, & Richgels, 1993). The use of the graphing calculator with alternative approaches to mathematical problems is an attempt to create a learning environment that is potentially more effective for many students, but particularly for students with low algebraic ability.

Because the function concept is a complex topic, understanding of functions is a multifaceted accomplishment (Williams, 1993). Graphing calculators are tools that can provide students numerous examples of functions in multiple representations (AMATYC, 1995). Graphing calculators allow students to encounter a larger number of graphs and computations that otherwise would not be possible without its use (Demana & Waits, 1990). Students can use multiple representations through the use of graphing calculators to make connections between properties of functions, and these connections can help students develop their conceptual understanding of functions (Janvier, 1987; NCTM, 1989). For example, graphing calculators can produce accurate and easily manipulated

graphs used by students to find the solutions of $2x^2 - 5x - 3 = 0$. Also, many graphing calculators have capabilities of providing students with tables of data. These tables can also be used to find zeros of functions.

Besides providing alternative methods, graphing calculators can help make abstract ideas more concrete (Mortensen, 1992). Real-life examples afforded by the use of graphing technology can be used to make abstract symbols relevant to a variety of students (Leinhardt, Zaslavsky, & Stein, 1990). The calculating capabilities of technology allow more flexibility in the types of application problems that may be accessible to students in a pre-calculus course (AMATYC, 1995). Because many of the application problems presented to students in traditional courses were contrived to have functions easily manipulated symbolically, students using graphing calculators have an opportunity to explore problem situations with sets of data that are relevant to their interests and majors rather than sets of data that are simplified for ease of symbolic manipulation (American Association for the Advancement of Science, 1993; NCTM, 1989; NRC, 1989; Yerushalmy & Schwartz, 1993).

Graphing calculators can be used to change not only how students learn functions but also what properties of functions they learn. With the use of graphing calculators, some topics of functions from higher level courses can be made accessible to students who may not have the prerequisite symbolic algebraic skills (Philipp, Martin & Richgels, 1993; Slavit, 1996; Williams, 1993). One example of this accessibility is the topic of local maximum and minimum points of a graph. The algebraic skills, primarily calculating the derivative, required to find the maximum and minimum points of a graph, are not typically taught in pre-calculus courses but are taught in calculus courses. For students who do not possess the necessary algebraic skills, technology provides graphical or numerical avenues to maximum and minimum concepts. Alternatively, graphing calculators have influenced the removal of some topics from the curriculum. For example, Descartes' rule of signs for finding zeros of functions has been replaced by a graphical approach incorporating the fundamental theorem of algebra. Thus with graphing calculators, relatively complex mathematical ideas usually restricted to a calculus course can be made accessible to students in pre-calculus courses and symbolical techniques replaced with alternative methods.

Few students are able to develop a process and object perspective of functions because the emphasis on the symbolic representation impedes development (Brenner et al., 1997; Sierpinska, 1992). Students use the symbolic form to calculate values or perform algebraic manipulations. This emphasis encourages the action understanding of functions. However, graphing calculators have features that relieve the students of the burden of computing values of functions. Using the numerical features of the calculators, students are provided an environment that allows them to develop a process perspective of functions. Additionally, it is thought that the graphical representation encourages students to develop an object understanding of functions (Slavit, 1994; Yerushalmy & Schwartz, 1993). Thus, with the combination of symbolic and graphical representations of functions, students would be able to develop an intricate knowledge structure of functions (Hiebert & Lefevre, 1986; Kieran, 1993; Yerushalmy & Schwartz,1993). Graphing calculators are tools that assist students with developing the process and object perspective of functions, essential for learning functions (Moschkovich, Schoenfeld, & Arcavi, 1993; Sfard, 1992).

Since 1989 many of the mathematical groups and organization have recommended the introduction of graphing calculators into the mathematics classroom in order to allow students to develop thinking skills and a rich conceptual understanding of functions. The schedules for many national and regional conferences are filled with meetings and workshops related to the use of graphing calculators. Schools are spending millions of dollars and investing much time and energy incorporating graphing calculators into the curriculum. Yet, the initial recommendations for the use of graphing calculators and computer graphics were not made from a strong research foundation, because recommendations by the NCTM (1989) were made before much research was conducted on the effectiveness of graphing calculators in teaching and learning mathematics. Following the recommendations of the Mathematical Association of America [MAA] (Leitzel, 1991) and the NCTM, mathematics departments at many colleges have required their pre-calculus students to have access to graphing calculators. Whether the students with different expectations and mathematical abilities have benefited from using graphing calculators in a pre-calculus course has yet to be determined.

The research conducted since the recommendations (NCTM, 1989) has not supported that the use of graphing calculators has aided students in developing a conceptual understanding of functions in college pre-calculus courses (Adams, 1994; Alexander, 1993; Becker, 1991; Rich, 1990; Smith & Shotsberger, 1997; Thomasson, 1993). Besides a small amount of research conducted on student understanding of functions as compared to other mathematical topics (Leinhardt, Zaslavsky, & Stein, 1990; Romberg, Carpenter, & Fennema, 1993; Penglase & Arnold, 1996), research has not provided evidence that students' use of graphing calculators has had a positive effect on their understanding of functions.

One difficulty with the existing research is the dilemma addressed by Dunham and Dick (1994) about creating artificial testing situations when students are not allowed to use their graphing calculators on tests even though they used the calculator to learn the concepts. This dilemma was especially pronounced for tests that focused on symbolic manipulation skills. Because the use of graphing calculators provided methods other than symbolic manipulation for students to solve mathematical problems, not allowing students to use their graphing calculators on tests restricted student use of their nonsymbolic methods to solve problems. Alternatively, the use of the graphing calculator on tests provided an advantage to the treatment group by lowering the difficulty level of many questions. This problem with the validity of tests created difficulties for some researchers (Rich, 1990; Ruthven, 1990; Thomasson, 1993). Thus, research is needed that does not utilize testing instruments that create the testing dilemma, especially since the use of graphing calculators is so widespread. Research is needed that goes beyond the comparison of two groups' achievement with and without access to graphing calculators, especially when the objectives of the groups differ (Hiebert, 1999).

Two studies that bypassed the dilemma described above were by Slavit (1994) and Becker (1991). Slavit conducted a study only with students who had access to graphing calculators. On the basis of student interviews, access to graphing calculators was found to facilitate students' translational abilities between representations of functions but not assist students' development of the process perspective of functions. However, the sample consisting of advanced high school students and an exceptional

teacher may not be representative of students in a typical college level pre-calculus course. Additionally, student learning in a year-long high school pre-calculus course is significantly different from learning in a term-long college course. The faster pace of instruction in a college course may create a different learning environment than in high school.

Using a testing instrument similar to Vinner and Dreyfus (1989), Becker (1991) reported the vertical line test was the predominant method for identifying functions by students in a College Algebra course. Without classroom observations, this result can not be attributed to access to graphing calculators. Additionally, improper statistical analysis of the students' responses hindered results regarding gains for students grouped by majors. Thus, research is needed to investigate student understanding of functions in college pre-calculus courses requiring graphing calculators.

Statement of Problem

AMATYC (1995), MAA (Leitzel, 1991), and NCTM (1989) recommended circumventing symbolic algebraic skills as the predominant goal for pre-calculus courses in hopes that abstract mathematical concepts may be more accessible to the average student. Some people in the mathematics community are concerned that the use of graphing calculators in college pre-calculus courses produces students with a weak mathematical foundation for further study of mathematics. Kaput (1997) recommends alternative methods for presenting the mathematical material, because the traditional method focusing on symbolic algebraic skills has not been effective (Sierpinska, 1992). Yet, others argue that symbolic representation is necessary for understanding mathematical ideas (Andrews, 1996). Additionally, a recent report to the NSF (1996) advocated that facts and symbolic algebraic skills are essential, but that a balance between symbolic algebraic skills and conceptual understanding is crucial in order for students to be successful in higher level college mathematics courses. Research is needed to determine whether the use of graphing calculators, by providing alternative methods to explore mathematical topics and solve problems, allows students to develop a process

and object view of functions as well as an understanding of the properties of functions. With the recommendations to lessen the focus on symbolic algebraic approaches, the use of graphing calculators provides an effective means for students to gain increased conceptual understanding of functions. Otherwise, if students' use of graphing calculators does not enhance their conceptual understanding of functions, then their mathematical foundation is weakened.

The primary goal of this study is to investigate student knowledge of functions in a college pre-calculus course, specifically a College Algebra course. College Algebra is a course that students of various majors are required to complete. Since a motivation for the recommendations for the use of graphing calculators is to provide an alternative avenue to concepts for students without sufficient symbolic algebraic ability, a secondary goal is to investigate student knowledge of functions for students with different algebraic ability and expectations for the course. To explore these goals, this study seeks to answer the following questions:

1. What knowledge of functions do students gain in a College Algebra course requiring graphing calculators?
2. What knowledge of functions do students of different levels of algebraic skills and with different academic majors gain in a College Algebra course?
3. What role do graphing calculators play in the classroom as students develop an understanding of functions?

Significance of the Study

Schools and students spend millions of dollars to outfit classes with graphing calculators. The recommendations of the AMATYC (1995) and NCTM (2000) for the use of graphing calculators are made on the basis of the constructivist perspective of learning. Graphing calculators are perceived as tools aiding students in the construction of their understanding of functions. Graphing calculators allow an ease of exploring functions in multiple representations. Alternative methods provide students with sufficient algebraic skills an opportunity to further develop their understanding of

functions and students with low algebraic skills an opportunity to overcome the algebraic barrier. Changing the emphasis of College Algebra from symbolic manipulation skills to conceptual understanding of functions has generated an occasionally heated debate among college instructors as well as curriculum developers. This study provides information concerning these issues.

Based upon constructivist perspective of learning, information about student understanding is needed to develop practices that can be used in student-centered instruction. According to Selden and Selden (1993), few college instructors have knowledge of how students conceptualize functions. Since one aspect of student-centered instruction is knowing students' capabilities, knowledge of how students develop their understanding of function in a classroom environment using graphing calculators is needed. Investigating how students use graphing calculators in the classroom provides college instructors with information that can be used by college instructors to develop instructional practices.

One issue of debate is whether students developing an understanding of functions have been hindered or enhanced with the use of graphing calculators. Some people in the mathematics community (Andrews, 1996; Prichard, 1993) are concerned that students will not be able to learn mathematics without a strong focus on symbolic algebraic skills and that the use of graphing calculators will weaken the curriculum. This study provides information about whether college students from a variety of backgrounds and different expectations for a College Algebra course are able to develop an increased understanding of functions with the use of graphing calculators.

Another issue concerning the use of graphing calculators in College Algebra is the impact on curriculum. Confrey (1993) indicated that technology either is incorporated into an existing curriculum or transforms a curriculum. Some developers of textbooks have introduced the use of graphing calculators with little adjustment to the curriculum. Other developers propose that to fully utilize the potentials of technology, curriculum should be completely redesigned. This study provides information about whether graphing calculators can be used effectively in an existing curriculum. If students do not develop a sound understanding of functions, then information from the study can be used to make recommendations for the redesign of the curriculum.

Finally, the AMATYC (1995) recommended guidelines for different programs for students with different academic goals. The current college algebra curriculum may not be suitable for the diverse population enrolled in the course. For example, the AMATYC recommends that the focus on symbolic algebraic skills should be reduced for students in technical programs. The focus on modeling and applicable aspects of functions should be enlarged. If students from different major groups are found to develop various levels of understanding of functions, then the information gained in the study can be used to make recommendations for curriculum development of different College Algebra courses for the respective majors.

CHAPTER II
REVIEW OF THE LITERATURE

When graphing calculators were first introduced into the classroom, many studies compared students having access to graphing calculators to students not having access. However, this simplistic framework did not consider all of the facets of the classroom affected by the use of graphing calculators. When graphing calculators were introduced, the goals of a class and the instructional methods changed in order to utilize the features of graphing calculators. Normally, the focus of the class switched from developing algebraic skills to developing problem solving skills with real world situations. Thus, Hiebert (1999) argued that comparing classes with and without access to graphing calculators was not appropriate. A different framework is necessary to properly develop and interpret studies on the effectiveness of graphing calculators in precalculus classes.

Student Learning

Hiebert and Carpenter (1992) and Skemp (1971) defined understanding as a mental structure connecting information. A student having an understanding of an idea possesses a mental representation of it. Ideas are represented as symbols, real objects, and mental images (Janvier, 1987). A representation can be a combination of something written on paper, a physical object, or a mental construction of ideas, where the purely mental construction of an idea is called a concept (Skemp, 1971). The internal representations of an idea are considered to be mental structures describing external representations. Hiebert and Carpenter (1992) described the mental structures as a spider's web. With its numerous connected strands, webs illustrate the intricate structures that are used to model students' understanding of a concept.

In order to be understood, the idea needs to be incorporated into the mental structure (Kieren, 1994). The level for understanding a concept is determined by the number and strength of connections to other concepts. Understanding involves recognizing relationships between pieces of information. For example, understanding the

discriminant of the quadratic formula involves relating $b^2 - 4ac$ to at least the domain of the square root and to real zeros of a quadratic function. Without possessing these and other relationships, students would not be considered to have an understanding of the concept of the discriminant. Therefore, a useful method for modeling student understanding of mathematics is that students possess an internal network of representations connecting related mathematical ideas.

Within the mathematics education community, constructivism is the widely accepted view of how students develop their mental networks (Selden & Selden, 1992). From this framework, students are active members in the construction of their existing knowledge structures. By assimilating external representations of mathematical ideas, students adjust or accommodate their knowledge structures. Rather than being filled with knowledge, students are considered to develop their own knowledge structures. Through thinking and talking about similarities and differences of mathematical ideas, students can construct relationships between the ideas (Hiebert & Carpenter, 1992).

Learning is described as a change to a knowledge structure as a result of some experience (Kieren, 1994). According to Strike and Posner (1985), learning consists of students relating new encounters or experiences to their existing ideas. When a mathematical topic is initially presented, students try to relate this new information to their knowledge structure. According to Mansfield (1985), some outcomes for this attempt of students to relate the new material to their knowledge structure include: (1) no change to the students' knowledge structure occurs, (2) the new material and their knowledge structure coexist and may be self-contradictory, (3) the new material is incorporated into their knowledge structure in a way that produces a misconception, and (4) the new material is incorporated into their knowledge structure and both are enhanced. Of course, the first outcome is considered as no learning and the fourth outcome is a goal of mathematics educators. The accumulation of pieces of information or isolated facts does not necessarily imply a good understanding. Discovering connections and making relationships between the pieces of information is the main determining factor of understanding a concept.

The most important single factor influencing learning is what the learner already knows (Fosnot, 1996). Students make sense of their worlds as they actively interpret

what the teacher says and does, and they can only do so in terms of their current ways of knowing (Kieren, 1994). Students are ready to learn only if they have existing internal representations to which the new information can be connected (Hiebert & Carpenter, 1992). Student learning of a mathematical idea can be assisted by providing examples that are meaningful to students, when ideas are closely related to students' existing knowledge structures (Begle, 1979). Examples of mathematical ideas that are too far removed from students' understanding provide little opportunity for the students to assimilate the idea into their knowledge structures. "The closer the match between salient features of the materials and the mathematical relationships, the more contextual support there is for students to construct the intended connections" (Hiebert & Carpenter, 1992, p. 71). Students can only learn what they are prepared and able to perceive (Sfard, 1992). When new information cannot be connected within a knowledge structure, then the information is easily forgotten (Mansfield, 1985; Skemp, 1971). If students have developed misconceptions or have an inadequate understanding of an idea, then new experiences need to be provided to the students, so that students will be forced to adjust their knowledge structure in order to overcome discrepancies between their internal representation and the external representation (Kieren, 1994). For example, if students are found to be familiar with graphs of quadratic functions but lack an understanding of the discriminant, then graphical examples of quadratic functions may assist students to develop the relationships between the value of the discriminant and the number of zeros.

The Function Concept

Mathematical knowledge has been categorized as two types: procedural and conceptual (Hiebert & Carpenter, 1992). Procedural knowledge is defined as a sequence of actions or series of operations. Conceptual knowledge is defined as the connected networks or knowledge structures. Conceptual knowledge is rich in relationships. Similarly, Skemp (1971) uses the terms instrumental and relational knowledge, where instrumental knowledge represents rules without reason and relational knowledge consists of knowing both what to do and why. To illustrate the two types of knowledge,

refer to the discriminant example. A student possessing procedural knowledge would be able to evaluate the discriminant and determine whether the quadratic function has one, two, or no real zeros. However, an understanding about the relationship between the square root within the quadratic formula and zeros of quadratic functions would be considered conceptual knowledge.

According to Hiebert and Leferve (1986), conceptual knowledge can extend the range of applying procedural knowledge. For example, certain types of fourth degree polynomials can be solved using the quadratic formula. Without this conceptual knowledge, students would not know how to extend the use of the quadratic formula to other polynomials. Thus, both kinds of knowledge are required for mathematical expertise.

The function concept possesses a dual nature evident in its historical development. An early definition of functions was that they were mathematical expressions containing variables used for evaluation (Malik, 1980). Through the centuries, the definition of functions has evolved to include arbitrary correspondences between two sets. However abstract the modern definition is, it still encompasses the earlier definitions. Because functions can be considered as an expression used for evaluating or as a correspondence that can be acted upon with higher mathematics, researchers (Kieran, 1993; Selden & Selden, 1992) have proposed an operational-structural perspective of functions as a theoretical framework to use for studying students' understanding of functions.

Students with an *operational* perspective of functions perceive them as mathematical recipes that are used to calculate values for a given input (Sfard & Linchevski, 1994). For example, evaluating the function, $f(x) = x^2 + 3$, would be viewed as the procedures of squaring the input value and adding 3. A student with this viewpoint would be considered to possess a procedural understanding of functions (Hiebert & Lefevre, 1986). Additionally, other researchers would label this the *action* perspective of functions (Kieran, 1993; Selden & Selden, 1992)

Students possessing a *structural* perspective of functions view them as entities that can be acted upon (Selden & Selden, 1992; Sfard & Linchevski, 1994). In order to compose two functions symbolically, students need to view functions as objects similar

to how numbers are perceived in the evaluation of a function. When graphs of functions are transformed, the graphs are considered as objects moved or manipulated on the coordinate plane. Additionally, the structural view of functions is needed for higher level concepts. In order to perform derivation or integration techniques functions in a calculus course, students need to view functions as objects on which to perform these procedures. Other researchers have labeled this view as the *object* perspective of functions (Kieran, 1993; Selden & Selden, 1992).

Some researchers include a middle perspective called the *process* view of functions. With this perspective, a student views functions as a relationship between elements of the domain with elements of the range without any particular algorithm (Kieran, 1993; Selden & Selden, 1992). The process view of functions can include either explicit or arbitrary correspondences between two concepts or sets. With the process perspective, students' attention is directed to the relationship between the x and y values of an equation or graph and not towards the actions required to evaluate the output for each input or to the actions performed upon functions (Moschkovich, Schoenfeld, & Arcavi, 1993).

The growth of understanding within these three perspectives does not occur linearly (Sfard, 1992). Having the process perspective is not a necessary condition in order to possess the object perspective. However, flexibility between these perspectives is needed in order to be mathematically competent (Moschkovich, Schoenfeld, & Arcavi, 1993; Sfard & Linchevski, 1994; Slavit, 1995). For example, when modeling a real world situation, students can use transformations to develop the modeling function and then evaluate the function to make predictions. Initially, to develop the function, students need a structural perspective for performing the transformations to obtain the graph fitting the data. Furthermore, utilizing the function into new a situation encompassing the original situation requires the object perspective of functions to combine two functions into one function that represents the new situation. Then, for evaluating the function to make predictions, the operational perspective is used. Thus, the dual nature of functions, especially in real-world applications, is reflected in the situational perspective required to be successful in a precalculus or calculus course.

Slavit (1995) included another perspective called the growth conception of functions. Where set-theory was the foundation for the process perspective, the growth conception of functions looks at characteristics of a family of functions found in modeling real-world situations. Students with this perspective are able to view functions as objects that have particular properties such as symmetry or periodicity. The properties of functions can have either a global or local aspect to them. Global properties can pertain to the overall shapes of graphs of functions such as parabolas or rational functions. Local properties include intercepts or extrema. This perspective fits well with courses that include modeling or graphical explorations of families of functions.

The work of Vinner (1992) provides a suitable framework to use for student understanding of functions. Based upon constructivist views of learning, Vinner describes a cognitive structure in which students adapt their understanding on the basis of their mathematical experiences. *Concept image* was defined by Tall and Vinner (1981) as the total knowledge structure including all the mental pictures and associated properties and procedures. For example, students' concept image of quadratic functions could include items such as the parabola with the line of symmetry and vertex highlighted, a table of data listing the x-values and y-values, the standard form of the quadratic function, and the quadratic formula. Also, the concept image could contain any nonmathematical idea that students have associated with the quadratic function such as the shape of a sound dish seen on sidelines of sporting events or a satellite communication dish. A student's concept image would contain all of the functional properties, representations, and classes of functions necessary for a student to develop the growth conception of functions (Slavit, 1995).

Not guaranteed to be incorporated completely into students' concept image is the formal mathematical definition, called the *concept definition*. The concept definition is the mathematically accepted definition that accurately explains the concept in a non-circular way (Vinner, 1983). For example, the concept definition of quadratic functions would be: let a, b, and c be real numbers with a not equal to zero. The function of x given by $f(x) = ax^2 + bx + c$ is called a quadratic function. However, students do not initially incorporate the formal definition in its pure mathematical form (Sfard, 1992; Skemp, 1971; Vinner, 1983; Vinner & Dreyfus, 1989). Students construct their own

definition on the basis of their concept image. For example, some students could construct the definition of quadratic functions as a formula that has degree two and no imaginary numbers for coefficients. Vinner defines student-constructed mathematical definitions as *personal definitions* that are students' verbal descriptions of their concept image.

In order for students to accept the formal definition as their personal definition, the formal definition must be able to assist students to interpret or solve problems more efficiently than other aspects of their concept image (Strike & Posner, 1985). For example, if students are only required to evaluate functions for given values, then their personal definition may consist of functions being a recipe for calculating values or being an expression with variables and constants. When students are confronted with a case where their personal definition cannot be used for a mathematical task, then the formal definition may be considered useful. Otherwise, if the concept definition is not incorporated, as in Manfield's (1985) second outcome of learning, then it remains separated and remains inactive or easily forgotten (Skemp, 1971). Memorizing definitions for tests is an example of students not attempting to incorporate the formal definition into their concept image.

Research on students' learning of the function concept have not been positive. Much of the recommendations for mathematical reform have been based on the results describing the difficulties students had for learning functions. Tall and Bakar (1992) administered to college students a questionnaire consisting of 19 questions assessing students' ability to identify functions given graphically and symbolically. The results of the test indicated that students perceived graphs of functions to be smooth, regular, not piecewise, and not constant. With regards to the symbolic representation, a mathematical expression was considered a function if it could be solved for y.

Carlson (1998) found similar results as Tall and Bakar (1992). A 25-item test covering many topics of functions taught in a college algebra course, typically the first course in a two course sequence for precalculus, was administered to 30 students who finished the course with an A. Interviews were conducted to investigate students' responses on the test. For a grading scale of five points for each question, the range of means for the students' scores was 0.7 to 3.3. Carlson concluded that students had little

understanding of the language of functions and were unable to use function notation to represent real-life relationships. Based on results of the tests and interviews, Carlson concluded that students demonstrated little persistence in solving mathematical problems because they showed little confidence in their mathematical abilities. Additionally, students did not like to figure out problems on their own and liked it when the teacher showed the solution procedures. Also, students were concerned about the rapid pace of the course because they considered their learning was reduced to memorizing formulas resulting in a superficial understanding of the concepts of functions.

DeMarois and McGowen (1996) investigated students' understanding of the definition of functions by administering a pretest and posttest and by conducting interviews to students in four sections of a beginning algebra course at a community college. The results indicated that students had little understanding of symbolic function notation. Only a small percentage of the students were able to distinguish between $3f(2)$ and $2f(3)$. Also, students did not view $f(x)$ as an object even though much of the course consisted of the skills for manipulating functions.

Even (1998) investigated students' flexibility for working between representations of functions. The sample for this study consisted of 152 prospective secondary mathematics teachers at eight different universities in the last stage for obtaining their degrees. A questionnaire on functions in multiple representations was administered. Qualitative analysis indicated that students were reluctant to work between representations for finding the solution. When given values for some of the parameters of a quadratic function, only 20% of the students used the graphical representation to determine the number of zeros for the function. Also, Even concluded that the context of the problem had an important role in determining approaches for the solution. When a problem assessed a function concept in general, the use of a particular type of function created difficulties for some students because the characteristics of the given function created obstacles for developing a correct solution.

The above studies illustrate that students have not been successful developing a good understanding of functions. The results of Carlson (1998) and Even (1998) were especially noteworthy because the sample consisted of students with high grades and with

prospective mathematics teachers. These studies offer little hope for students with average or lower than average mathematical abilities to develop a good understanding of functions.

Research on Graphing Calculators

Graphing calculators have been recommended for use in mathematics courses because their features correspond to many of the perspectives on student learning. Graphing calculators provide students with quick and accurate graphs and tables of values (Demana & Waits, 1990). Students can encounter many examples of functions with relative ease, providing experiences on which students develop their concept image of functions.

Graphing calculators are tools that students can use to explore functions in multiple representations enabling them to make connections between forms of functions in order to develop a fuller understanding of a concept. Also, graphing calculators allow students to solve mathematical problems using alternative methods. These alternative methods provide students with opportunities to learn concepts that they would otherwise not be ready to learn. Traditionally, function topics are presented symbolically. Students without proficient algebraic skills are not able to learn because they lack the concept image to which the new material is connected. Romberg, Carpenter, and Fennema (1993) suggest that emphasizing multiple representations make functions easier to learn for most students, especially students lacking proficient algebraic skills. Graphing calculators can be a tool to aid at-risk students in developing an understanding of functions.

Design Issues

When reviewing studies on the influence of graphing calculators on students' learning of functions, many issues arose regarding the complexity for designing a study investigating the impact of graphing calculators. One issue involved the differences

between the goals of a traditional course and of a reformed course incorporating graphing calculators. A motivation for the use of graphing calculators is that they provide students with alternative methods to develop a conceptual understanding of functions. Graphing calculators are implemented in order to overcome the algebraic barrier that exists for some students. Because alternative methods become feasible through the use of graphing calculators, a course with graphing calculators does not solely focus on the development of symbolic algebraic skills as in a traditional course. The goals of the traditional and reformed courses differ. Therefore, studies that compared traditional courses with courses using graphing calculators were, in a sense, comparing apples and oranges (Hiebert, 1999). Many studies such as McLain (1992), Thomasson (1993), Quesada and Maxwell (1994), Rich (1990), Smith and Shotsberger (1997), Alexander (1993), Ruthven (1990), Adams (1994), and Hollar and Norwood (1999) made this type of comparison.

The different goals of the courses used in studies comparing groups with access to graphing calculators and groups without access to graphing calculators created difficulties for researchers with respect to the testing instrument used to assess students' understanding of functions. First, content validity of the tests was rarely addressed. Though the classes were similar, students with access to graphing calculators were usually exposed to different instructional techniques and content that was allowed through the use of graphing calculators. Obtaining content validity for the exams was a necessity and could not have been implied.

A second difficulty in comparing traditional courses with courses using graphing calculators was whether students were allowed access to their graphing calculators during tests (Dunham & Dick, 1994). Rich (1990) and Ruthven (1990) allowed students to use their graphing calculators on the tests. However, no discussion was given about whether access to graphing calculators provided an advantage. Questions requiring a graph of a function were biased towards students who had access to graphing calculators since results could be easily obtained without students having a sound understanding of functions. Quesada and Maxwell (1994) addressed this issue by allowing the control group access to scientific calculators. However, the difference in capabilities of calculators in terms of graphing abilities produced a bias for specific questions that could be obtainable through a graphical approach. Thomasson (1993) also addressed this issue

by using two groups with various degrees of access to graphing calculators. One group was allowed to use graphing calculators on the tests and another group was not. However, due to improper statistical analysis, the results of the study were questionable.

Another issue concerned the design of some studies that did not incorporate pretests in order to ascertain gains in achievement or understanding. Quesada and Maxwell (1994), Rich (1990) and Ruthven (1990) investigated the impact of graphing calculators on students' gain in understanding of functions but did not use a pretest in the study in order to determine students' gain in understanding functions. The effect on analysis for the lack of a pretest can be reduced if the groups are randomly chosen. Yet, Rich's selection process appeared to be biased because honors classes were assigned only to the treatment group. Using classes from an existing project, Ruthven did not have the random selection necessary when no pretest is used. Quesada and Maxwell used existing classes that did not fully ensure a random selection for each class. Thus, gain in understanding of functions can not be determined in these studies.

Another obstacle in determining gain in understanding was the improper use of statistical tests by McLain (1992), Thomasson (1993), Alexander (1993), Rich (1990), Hollar and Norwood (1999) and Becker (1991). Having designed the use of a pretest and posttest into their studies, they should have used an ANCOVA rather than ANOVA or *t*-tests to analyze gains in understanding by students. Becker used numerous *t*-tests and Hollar and Norwood applied numerous analysis of variance tests, producing a high probability of making a Type I error. Thus, the results produced from these analytical methods were questionable.

Another aspect of analysis that was missing from studies was determining whether the results had practical significance. Previous studies that investigated student understanding of functions regardless of technology (Selden & Selden, 1994; Brenner et al, 1997; Carlson, 1998, Even, 1998; Hitt, 1998; Tall & Bakar, 1992; Zaslavsky, 1997) have indicated that students have difficulties in developing a good understanding. Thus, if 70% is considered a passing score, then only the scores from the study by Quesada and Maxwell (1994) had practical significance. Even if the means of the treatment and comparison groups were found to be statistically significant, without practical

significance the statistics would indicate that the treatment was simply a better failure than the comparison method.

The last issue concerning studies on the impact of graphing calculators on students' learning of functions was the limited classroom observations conducted by the researchers. Classroom observations were needed to provide valuable information about the use of graphing calculators. Without classroom observations the results of studies could not be directly attributed to graphing calculators because other classroom factors could have been a factor, such as discovery teaching methods or group activities. Only Alexander (1993), Rich (1990), Smith and Shotsberger (1997), and Slavit (1994) reported conducting classroom observations. McLain (1992) reported anecdotal information. Since no attempt to reduce potential biases from being the instructor was reported, the anecdotal information needs to be viewed with caution. However, consistent results obtained anecdotally together with data from observations provide areas that can be considered for research.

Relevant Studies

Due to the concerns addressed by Hiebert (1999), the results of many studies comparing students with access to graphing calculators to students without access to graphing calculators can not be readily applied to developing a research study. Additionally, many studies possessed design flaws that limited the validity of the results. The studies used to develop this study included Becker (1991), Quesada and Maxwell (1994), Ruthven (1990), and Slavit (1994).

One of the early studies on the impact of graphing calculators (Ruthven, 1990) investigated students' abilities to translate between graphic and symbolic forms of functions. The study was a static group comparison using four classes from two high schools involved in a two-year project for the introduction of graphing calculators. Because of the difference in goals for the two groups due to the access to graphing calculators, the results obtained by comparing scores on the testing instrument were not used as a basis for this study.

However, the researcher's observations of methods used by students provided information about students' approaches for translating between representations of functions. When translating functions from graphical to symbolic form, students from the treatment group having access to graphing calculators used graphical techniques to check answers and students from the comparison group used numerical techniques. Ruthven (1990) identified two stages that students used to translate functions from graphical to symbolic form: identification and refinement stages. Within the *identification stage*, students classified graphs with respect to which family of functions the graph belonged, such as linear, quadratic, cubic, or exponential. During the *refinement stage*, the parameters of the symbolic expression were adjusted to conform to the given graph. Ruthven observed students from the treatment group used graphical methods to refine their functions and students from the comparison group used numerical methods.

Ruthven's (1990) observations indicated a difference in students' approaches used to translate functions from graphical to symbolic form. However, the impact of the graphing calculator on students' learning was not readily determined because no observations were conducted of the classroom. The difference in approaches for the groups could have been attributed to the techniques taught in class. The comparison group may have been taught only the numerical method for refining parameters of functions. Additionally, the success that was attributed to the treatment group may have been due to the efficiency of the technology rather than ability of the students. Students could check answers easier with the graphing calculator than with a scientific calculator. To check their refinements, students in the comparison group using a scientific calculator would have to spend more time than the treatment group. Thus, the success attributed to the treatment group and the technique that they used to refine their answers could have been attributed to the technology rather than an increase in understanding or ability.

The results of the study by Quesada and Maxwell (1994) had similar limitations to the results of Ruthven (1990). Because the study was a static-group comparison spanning three terms, many confounding factors were not addressed such as the class term, instructors, class size, instructional approach, content, correction procedures for final exam, incorrect statistical test, and biases of test questions towards the use of

graphing calculators. Even though the researchers analyzed the scores of the final exam with regard to class size and treatment, the statistical analysis did not address the other confounding variables. Thus, results of comparing the scores on the final exam between treatment and comparison groups could not be readily accepted.

However, the results of a questionnaire given to the treatment group did provide information about students' impression of the use of graphing calculators. The results of the questionnaire were supported by interviews to clarify and expand students' responses. The four most common positive aspects of graphing calculators given by students in the treatment group were that the use of graphing calculators (a) facilitated understanding of the mathematical material, (b) provided the ability to check answers, (c) helped determine a solution strategy, and (d) saved time on calculations. Two negative concerns of the students were that they would (a) become dependent on the use of graphing calculators and (b) not be properly prepared for calculus. Even though the results of the statistical tests were not appropriate for determining the impact of the graphing calculator on students' learning of functions, results of the questionnaire and interviews provided information about students' impressions about using graphing calculators. Yet, observations of the classes were not conducted and, therefore, information concerning how students were exposed to graphing calculators within the classroom was not gained to support the students' comments on the survey.

One of the few studies to include the use of classroom observations as well as testing instruments to gather data on students' understanding of functions with access to graphing calculators was a study by Slavit (1994). Additionally, Slavit did not make the comparison between groups of students with and without access to graphing calculators. This study was one of the few studies that avoided the concerns addressed by Hiebert (1999) concerning the difficulties of comparing classes with different objectives when graphing calculators are introduced.

Slavit (1994) collected data through three unit tests, student interviews of three students, and two student questionnaires given near the beginning and end of the study. Incorporated into the unit tests were the questions, "give a precise definition of function" and "give an example of a function." The remaining questions on the unit test reflected the content in the unit covered in class.

Since no statistical analysis was conducted, the results of the tests and change in students' responses to the function questions were not shown to be statistically significant. However, observations and interviews provided the researcher information about students' understanding of functions and the influence of graphing calculators. With regard to students' definitions of functions and their use of functions, Slavit (1994) observed results corresponding to Mansfield second outcome of learning. Although students provided definitions categorized within the process perspective, students were observed to discuss functions in terms of the operational perspective. The students' personal definition of functions did not match the concept definition that they memorized. The concept definition coexisted as a separate entity outside of their concept image of functions. Therefore, the researcher concluded that the graphing calculator was not effective for assisting students to develop a robust understanding of the process perspective of functions, even though the graphing calculator was observed to assist students in applying multiple representational approaches to solving problems and in translating functions between representations.

Slavit (1994) observed three main misconceptions developed by students: functions were required to be solved for y, be continuous, and possess infinite domain. The use of graphing calculators may have supported the development of these misconceptions. In order to graph functions, the symbolic expressions need to be solved for y in order to input the functions into most graphing calculators. Because the CONNECT feature of the graphing calculator provides a visually appealing graph, graphs shown on the view screen appear to be continuous. Students need to be proficient with their calculators in order to have a discontinuous function appear appropriately on the view screen. Many of the functions used in precalculus courses are polynomials or exponential functions. Once again, students need to be proficient with their calculators in order to graph a polynomial function with a restricted domain. The features and methods for inputting functions into graphing calculators may assist students in developing misconceptions of functions.

With regard to generalizing the results of this study, the sample provided a limitation for a college-level precalculus course because the sample consisted of an exceptional teacher with gifted high school students. Further limiting the ability to

generalize the results was the pace of the course. Because the high school course lasted one academic year, the quicker pace of a college course may not correspond readily. Additionally, the caliber of students is much different because most of the gifted high school students would be enrolled in the college calculus course rather than a precalculus course.

The previous studies did not address diversity issues of mathematical reform. The AMATYC (1995) made different recommendations for students on the basis of their academic goals. Success for students with diverse mathematical abilities was not addressed in the previous studies because the statistical analysis did not include symbolic manipulation skill level as a variable. Much of the analysis was performed on the mean of the groups, obscuring changes in understanding or ability based on skill level. For example, a higher standard deviation on a posttest than on the pretest indicates various levels of change in scores for members within individual sample groups. Thus, a major motivation for the introduction of graphing calculators, overcoming the algebraic barrier for students without sufficient algebra skills, was not addressed in the previous studies.

Goldenberg (1988) was concerned that only the talented students would benefit more from access to graphing calculators than students with less manipulation. Becker (1991) was one of the few researchers to address this concern. Similar to the study by Vinner and Dreyfus (1989), Becker categorized students by their academic majors, because different academic major programs required different mathematics courses. Students with a mathematics or science major would be required to take higher levels of mathematics courses than the mathematics courses taken by students with a business or liberal arts major. However, this method for categorizing students can not be readily accepted as a method for categorizing students with respect to algebraic ability. Thus, the results of Becker can only provide background information for comparing the influence of graphing calculators on students of various symbolic manipulation skill level.

Becker's (1991) study consisted of two stages: (a) determining student misconceptions with the intent to develop curriculum material to assist students in overcoming or avoiding the development of the misconceptions and (b) investigating the influence of the graphing calculator. Similar to the instrument used by Vinner and Dreyfus (1989), the questionnaire used in the study consisted of 13 questions presented in

one of the representations of functions together with the question, "What is a function?" However, due to inappropriate statistical analysis the results were questionable because the application of a *t*-test on individual questions leads to a high probability of a Type I error.

Observations of the responses by the students on the questionnaire provided some insight to students' approaches and misconceptions. The common method for determining whether the relationships given in the questionnaire were functions was the application of the vertical line test. Students translated the relationships given in numerical, written, or symbolic form into graphical form so that the vertical line test could be applied. Also, students had difficulties classifying piecewise relationships as functions. Some of the students' misconceptions about functions were that functions were required to be one-to-one, linear, smooth, and continuous.

Conclusions and Recommendations

As indicated from the studies by Carlson (1998), DeMarois and McGowen (1996), Even (1998), Selden and Selden (1992), and Sfard (1992), students have not been successful developing an understanding of functions. Addressing this lack of success, AMATYC (1995) and NCTM (2000) have recommended the use of graphing calculators to circumvent the algebraic barrier that exists for some students and to assist students making connections between the different representations of functions.

A number of researchers investigated the impact of access to graphing calculators on students' learning functions. This review of literature indicated that many of the studies had designs that did not appropriately address the change in goals of courses that allow access to graphing calculators. Other researchers did not conduct classroom observations to ensure that the intended instruction actually occurred. Without classroom observations determining the mathematics and instruction to which students were exposed, testing instruments were not assured to possess content validity. Because graphing calculators provide alternative methods for students, testing instruments may not have matched the objectives of both the treatment and control groups. Without

classroom observations, possible confounding variables such as changes in instructional practices were not recognized.

Besides not ensuring that the treatment was performed, lack of classroom observations failed to address an issue associated with the constructivist perspective of learning. A key aspect of constructivism is that students develop their understanding based upon the experiences they encounter (von Glasersfeld, 1987). Thus, the manner in which students used graphing calculators was an important factor not investigated in many studies. Separating students into groups according to whether they had access to graphing calculators ignored students' encounters with graphing calculators. According to the contructivist perspective, these encounters are used by students to develop their understanding of functions.

A second key component for learning is connecting new material to what the learner already knows (Fosnot, 1996; Hiebert & Carpenter, 1992; Kieren, 1994). Begle (1979) indicated that student learning of mathematical ideas are assisted by providing examples that are closely related to students' existing understanding. The mathematical abilities of students need to be considered by teachers when interacting with students. Students with various mathematical backgrounds respond differently to classroom instruction. Thus, students with different symbolic manipulation skill level would respond differently to examples or problems presented in a mathematics course. With respect to the various types of students, only Becker (1991) attempted to investigate the effect of graphing calculators on students' understanding of functions. No significant difference was found between groups of students with different academic majors. However, the analysis of data was questionable. Thus, the influence of using graphing calculators to enhance learning by students of different levels of algebraic ability and in various academic majors is still undecided.

Much discussion in the mathematics community has been focused on the impact of incorporating technology into pre-calculus courses. Because of the prerequisite nature of these courses, concern is warranted about the impact of the use of graphing calculators on the students' understanding of conceptual and procedural concepts needed for higher level courses such as calculus. With the aid of technology that can perform complex calculations and manipulations, college students enrolling in remedial mathematics

courses may learn relevant mathematics without the emphasis on the symbolic representation. Thus, the focus of mathematics has changed to the structural perspective of functions since graphing calculators can perform much of the operational aspects of functions. Whether students can develop a conceptual understanding of functions without the procedural skills still needs to be determined.

The recommendations of the mathematics community to change the curriculum can potentially motivate the introduction of graphing technology into the classroom. Yet, as with any approach or tool, these recommendations need to be introduced to the students in an effective manner. The studies that were reviewed did not conclusively determine whether graphing calculators could be used effectively to assist students' acquisition of key mathematical concepts presented in pre-calculus courses.

In summary, researchers often made inappropriate comparisons between classes that possessed different goals and performed improper statistical analysis on tests that did not have validity with respect to access to graphing calculators. Additionally, few researchers conducted classroom observations and student interviews to gather data to support and explain students' responses on testing instruments. Furthermore, few researchers addressed a major motivation for the introduction of graphing calculators that of providing access to the function concept through alternative methods for students with diverse algebraic skills. Therefore, there is a need for research studies on the influence of graphing calculators on students' learning of functions that avoids pitfalls when using the agriculture model for research (Hiebert, 1999).

CHAPTER III
DESIGN AND METHOD

Introduction

Graphing calculators in pre-calculus courses have been recommended by American Mathematical Association of Two-Year Colleges [AMATYC] (1995) and National Council of Teachers of Mathematics [NCTM] (2000) to enhance students' development of their understanding of functions and to overcome the symbolic algebraic skill obstacle that exists for many students. The graphing calculator has been recommended to be used as a tool to provide students with mathematical experiences to aid them in the construction of an understanding of functions presented in pre-calculus courses (Kieran, 1993). Multiple representations with graphing calculators attempt to provide a variety of mathematical experiences for a diverse student population with different levels of mathematical skills and different academic goals. However, research has not provided conclusive results with respect to the impact of the use of graphing calculators on students' development of their understanding of functions (Penglase & Arnold, 1996).

The purpose of this study was to investigate student understanding of functions and the use of graphing calculators in a College Algebra course. This study focused on three primary questions:

1. What knowledge of functions do students gain in a College Algebra course requiring graphing calculators?

2. What knowledge of functions do students of different levels of algebraic skills and with different academic majors gain in a College Algebra course?

3. What role do graphing calculators play in the classroom as students develop an understanding of functions?

Given the nature of the research questions, a mix of qualitative and quantitative methods were used. Since the primary intent of the study was to investigate student understanding of functions, two methods of data collection were used to answer the first

and second research questions: (a) a pretest and posttest on knowledge of functions was administered to students in a College Algebra course and (b) interviews with students about their results on the pretest and posttest on knowledge of functions and about their use of graphing calculators in the class were conducted with a cross section of the student population. The term 'role' used in the third research questions possessed two aspects to its meaning. The first was how graphing calculators were used in the classroom. The second was how graphing calculators facilitated learning. Data about the first aspect were gathered in the classroom observations and data about the second aspect were gathered in the student interviews.

Setting

The subjects for this study were students in a College Algebra course offered at a community college located in the Northwest United States. College Algebra was the first half of a two-term pre-calculus sequence offered by the Mathematics Department and was the first course that covered the function concept in depth. Five 50-minute lectures were given per week during the 10-week term.

The Mathematics Department required students enrolled in College Algebra to have access to either a TI-85, TI-83, or TI-82 graphing calculator. The instructors had access to overhead devices for the TI-83 from the Mathematics Department. Videotapes introducing the TI-83 graphing calculator were available to students at the campus library.

The textbook used for the course was *Precalculus: Graphs and Models* by Bittinger, Beecher, Ellenbogen, and Penna (1996). The textbook was used both in the College Algebra course and the Trigonometry course, the second term of the pre-calculus sequence. The textbook was written with the assumption that students had access to graphing calculators while enrolled in the course. Thus, graphing technology was incorporated throughout the text. For example, many sections had questions that asked students to solve problems by using graphing calculators. Many examples and questions

provided students with the appropriate viewscreen in which to view the graphs of functions.

The text used multiple representations in describing many of the mathematical concepts. For example, the definition of function was introduced using an application about determining the distance of lightning based on the time interval between when lightning was seen and when thunder was heard. The example was given in both graphical and numerical form. At the end of the section, applications were used to provide real-life situations of functions. These applications were presented graphically as well as symbolically. Throughout the textbook, explanations, examples, and exercises of the mathematical topics were given in multiple representations using real-life examples.

Since the study investigated student understanding of functions, only the first course in the pre-calculus sequence was included in the study. The function concept was introduced in the College Algebra course rather than the Trigonometry course. Topics for College Algebra included linear, quadratic, polynomial, rational, exponential, and logarithmic functions. The course was separated into three units. The first unit, the focus of this study, consisted of the review chapter and the first chapter of the textbook. The review chapter in the textbook consisted mainly of a review of symbolic algebraic skills. Chapter One introduced graphs and functions. The definition of functions was introduced in this section. Also, included in the first chapter were transformations and combinations of functions and inverse functions. Chapter Three, covering Logarithmic and Exponential functions, was the next chapter incorporated into unit two. Then, the second chapter, covering intercepts, zeros and solutions to equations, was incorporated into unit three. The switch was made because the department wanted to incorporate inverse functions into unit one. The section for inverse functions was the first section in Chapter Three. Thus, to have a smooth flow to the lessons, the department decided to cover the remaining sections of Chapter Three before Chapter Two. Since the focus of the study was the on the concept of functions rather than types of functions, the study was conducted only during the first unit, a period lasting five weeks.

Subjects

The subjects for this study were the students enrolled in one section of College Algebra during the Fall Term. Of the 29 students completing the course, 25 students participated in the study. Students participating in the study were volunteers who had not previously enrolled in the course. Seventeen freshmen, five sophomores, one junior, and two Running Start high school students participated in the study. Running Start students were high school students who were qualified for college level courses and earned college credits before graduating from high school.

On the basis of the results of the Skills Test, a ten-question test assessing the symbolic manipulation skills covered in the prerequisite course (Appendix A), the students were categorized with regard to symbolic manipulation skill ability. The categorization of the students was performed to develop two distinct groups with regard to symbolic manipulation skills. Students who answered eight or more questions correctly were categorized at the high symbolic manipulation skill level. Students who answered five or fewer questions correctly were categorized at the low symbolic manipulation skill level. Students who answered six or seven questions correctly were not placed into either high or low symbolic manipulation skill level. The omission of these students was an attempt to increase the distinction between the two groups. The distribution of the students is given in Table 1.

Table 1. Distribution of Students with Respect to Grade and Symbolic Manipulation Skill Level

Symbolic Manipulation Skill Level	Grades				
	Freshman	Sophomore	Junior	Running Start	Total
High	10	1	1	1	13
Between	2	1	0	1	4
Low	6	2	0	0	8

Another factor distinguishing the students was the mathematical requirements for the students' majors. Similar to the categories used by Vinner and Dreyfus (1989) and Becker (1991), the students who enrolled in College Algebra at this community college were divided into three general categories according to the mathematics requirement of their majors: (a) liberal arts, (b) business, and (c) mathematics and science. Liberal arts majors fulfilled their mathematics requirement either solely with this course or with the combination of the College Algebra course and Trigonometry course. For liberal arts majors, the course was not used as a prerequisite for another higher-level mathematics course. For these students, the pre-calculus sequence was terminal. Business majors enrolled in the course in order to prepare themselves for Business Calculus. Science and mathematics majors enrolled in the course in order to prepare themselves for the calculus sequence for science and mathematics majors. Therefore, the students enrolled in the course varied due to the mathematics requirement for their majors.

Thus, students enrolled in a typical section of College Algebra varied with regard to the symbolic manipulation skill levels, High Symbolic Manipulation Skill Level and Low Symbolic Manipulation Skill Level, and with regard to mathematical requirements of the students' academic majors, Mathematics and Science, Business, and Liberal Arts. The students were placed into six categories on the basis of symbolic manipulation skill level and major requirement: High Symbolic Manipulation Level-Mathematics & Science [HSM-MS], High Symbolic Manipulation Level-Business [HSM-B], High Symbolic Manipulation Level-Liberal Arts [HSM-LA], Low Symbolic Manipulation Level-Mathematics & Science [LSM-MS], Low Symbolic Manipulation Level-Business [LSM-B], and Low Symbolic Manipulation Level-Liberal Arts [LSM-LA]. Table 2 shows the distribution of the students within each of the six categories.

Two volunteers from each of the six categories were interviewed. Each of the interviewed students took the pretest and posttest. For consideration of their time, these students were given two movie tickets. One additional student from the HSM-MS group was interviewed before the other students as a trial run of the interview.

Table 2. Distribution of Students with Respect to Symbolic Manipulation Skill Level and Major's Mathematics Requirement

Symbolic Manipulation Skill Level	Academic Major		
	Mathematics & Science	Business	Liberal Arts
High	7	2	4
Between	2	1	1
Low	4	2	2

Methods

On the first and second day of the term, the Skills Test and student consent forms were administered to students in four sections of College Algebra. On the consent form, students indicated whether they were willing to participate in the study and willing to be interviewed. For only the students who volunteered to be in the study, the results of the Skills Test for these students were obtained from instructors. Only one section had the minimum of at least two students in each of the six categories differentiated by symbolic manipulation skill level and major requirement that volunteered to be interviewed. This section was selected for the study.

To access student understanding of functions, as recommended by Schuell (1985), a combination of methods was used: (a) a pretest and posttest on knowledge of function, (b) classroom observations, and (c) student interviews. On Friday of the first week of the term and before the unit on functions began, the instructor administered the pretest on Knowledge of Functions to the students. (Henceforth, the Knowledge of Functions test will be referred to as the Function Test.) To entice students, the instructor offered five bonus points for taking the pretest. The instructor collected the completed pretests and provided the researcher with the pretests of only students who consented to be in the study. The researcher did not have access to pretests for students who did not consent to be in the study.

The researcher postponed grading all of the pretests. The delay in grading the pretests was intended to reduce biases in conducting and analyzing the classroom observations and student interviews. Prior to conducting the student interviews, the pretests for only the students being interviewed were graded to develop questions for the student interviews. The remaining pretests were graded after the interviews had been analyzed. All of the pretests were stored in files, while the researcher conducted, transcribed, and analyzed the classroom observations.

For the first unit of the course, the researcher conducted daily classroom observations. During each week of the term, the class met daily for 50-minute periods. Unit one of the course, the unit introducing functions, consisted of 19 class periods. The first five class periods were used to discuss the syllabus, administer the Skills Test, review algebra skills, and introduce the graphing calculator. The lessons were videotaped with the camera located in the back of the room and the instructor wore a cordless microphone.

All actions in the classroom such as instruction, group activities, tests, and quizzes were observed. The observations focused on the students' use and instructor's use of graphing calculators as well as how and what material was covered in the lessons. All materials provided to the students were collected. At the end of each class period, the researcher completed an observation summary and, when necessary, a document summary.

On the next class period after the completion of the first unit, the instructor administered the posttest of the Function Test to the class. The instructor gave five bonus points for taking the posttest. Similar to the pretest, the instructor transferred to the researcher the posttests for only students who had consented to be in the study. The researcher did not have access to any of the posttests for students who did not volunteer to be in the study.

All of the collected posttests were graded after the classroom observations had been analyzed to reduce potential biases. The posttests for students participating in the interviews were graded before each of the student interviews were conducted. The posttests for the remaining students were graded after the interviews had been analyzed.

A class profile was developed from the observation summaries. From the class profile, the researcher identified ways that graphing calculators were used in the classroom and situations that the researcher considered relevant to learning functions with graphing calculators. With this information, the researcher was able to follow and probe students' train of though in the interviews. Then, questions about use of graphing calculators were developed for the interviews.

Student interviews were conducted within four weeks after the completion of the first unit and after the classroom profile had been developed. From the group of students who had volunteered to participate in student interviews, the students who took the pretest and posttest of the Function Test were identified. In order to get a representative sample from each of the groups, interviews with two volunteers from all of the student categories were scheduled. Three students from the HSM-MS group volunteered to be interviewed. One of the three students was randomly selected to be the first student interviewed providing a test run of the interview. Thus two students from each category were contacted by the researcher and scheduled a meeting time. To provide incentive for the students to volunteer for the interviews, the researcher offered the students two tickets to a local movie theater.

Before each interview, the researcher graded the pretest and posttest for only the students participating in the interviews. The remaining pretest and posttest were graded after the student interviews had been analyzed. The researcher identified changes in students' responses on the pretest and posttest and these changes were investigated during the interviews.

All of the interviews were conducted within four weeks of the posttest. All of the interviews were conducted on the campus in an office of a mathematics instructor at the college, who was not the instructor in the study. Additionally, the interviews were transcribed after all of the classroom observations were transcribed.

After all of the interviews were conducted, detailed analysis of the interviews was performed. The remaining pretest and posttest of the Function Test were graded after the analysis of the interviews. Then, the results of the pretest and posttest for all of the students were analyzed. The results from the classroom observations and the student

interviews were used to support and explain the results of the pretest and posttest of the Function Test.

Due to the nature of qualitative studies, biases of the researcher can have a profound effect on the analysis of the data. Throughout the study, the researcher kept a journal in an attempt to reduce potential biases. Entries in the journal were made to help make evident biases that may have developed from the researcher's prior experiences with teaching the course. Another benefit of keeping the journal was that it provided a framework for the researcher to retain ideas. By recording thoughts and ideas, the researcher did not lose an idea emanating from the study. The journal allowed the researcher to retrace lines of thought, so that the researcher could remain focused.

Data Sources

The purpose of the study was to investigate student learning of functions and the role of the graphing calculator in this learning. To gather data addressing these issues, four instruments were used: the Skills Test focusing on students' symbolic manipulation skills ability, the pretest and posttest of the Function Test assessing students' knowledge of functions, student interviews, and classroom observations. Due to the qualitative nature of parts of the study, the researcher was considered part of the data collection and analysis.

Skills Test

The Skills Test consisted of ten questions assessing symbolic manipulation skills and was developed by an instructor in the Mathematics Department at the college (Appendix A). The purpose of the test was to assess whether students possessed the symbolic manipulation skills needed for the pre-calculus course. In order for a question to be valid, it had to assess a symbolic manipulation skill that students were expected to possess entering into the pre-calculus course. Because of the sequential nature of courses offered at the college, the skills needed for entry into the pre-calculus course were taught

in the prerequisite course. An indicator of the symbolic manipulation ability of a student entering the pre-calculus course was performance on questions assessing skills taught in the prerequisite course. In order for a question to be valid, it needed to assess a skill that was taught in the prerequisite course or lower level course. A question assessing a skill that was to be taught in the pre-calculus course was not considered a valid question. A committee of mathematics educators familiar with the pre-calculus course and the prerequisite course were asked to determine whether the questions assessed a skill that was taught in the prerequisite course. Based on an agreement rate of 80%, the validation committee determined that the questions on the Skills Test possessed content validity with regard to assessing symbolic manipulation skills taught in the prerequisite course (Appendix B). The committee determined that the questions assessed skills that were included in the outcome goals of the prerequisite course. However, it should be noted that the Skills Test was not an exhaustive test with respect to assessing all of the outcome goals of the prerequisite course. In order to posses content validity, each question had to be included in the outcome goals, but each outcome goal did not have to be represented in the Skills Test.

The reliability of the Skills Test was found to be 0.686 with using Kuder Richardson-21. The low value for the reliability indicator was not unexpected. Each of the questions on the Skills Test assessed a different symbolic manipulation skill. And, for closely related questions, the questions differed enough to require different skills or knowledge.

To lend support for the reliability of the Skills Test for categorizing students with respect to symbolic manipulation skills, scores on the placement test for some of the students were analyzed. The placement test was a symbolic based test that was used by the Mathematics Department to decide which course students should enroll. Because the consistency between grading of sections of the prerequisite course could not be verified, scores on the placement test was considered the only comparable score concerning symbolic manipulation skill that was readily available.

The instructor obtained scores on the placement test for nine student volunteers. The scores of four students from the LSM groups and five students from the HSM groups

were ranked and analyzed using the Mann-Whitney test. Significant results ($p < 0.025$) were found for the ranking of the placement scores.

Since the reliability of the Skills Test was not above 0.80, another source concerning students' symbolic manipulation skill ability was needed. The scores on the placement test for students entering the pre-calculus course without taking the prerequisite course lent support that the categorization was reliable. The ranking of the students correlated with the categorization of the students. Thus, using the results of the Skills Test was deemed a reliable method for categorizing the students with respect to symbolic manipulation skill level.

Pretest and Posttest of the Function Test

A pretest and posttest on knowledge of functions, the Function Test, was administered to the students to gather the bulk of the data about student understanding of functions. The pretest was given to the students during the first week of the term before functions were presented. The posttest was given after the unit on functions was completed.

The posttest was a parallel form of the pretest. With five weeks between the administration of the pretest and the posttest, the researcher considered that the students most likely would not be able to recall the content of the tests. However, to reduce the possibility that students would recall particular questions, a parallel form of the pretest was developed as the posttest. Only numbers and graphs were changed to develop the posttest. The wording of the questions and the format of the posttest were identical to the pretest.

In the development of the pretest and posttest Function Test, the researcher followed the procedures presented by Borg and Gall (1989): (1) define the target population, (2) define the objectives, (3) review related measures, (4) develop an item pool, (5) prepare a prototype, and (6) evaluate the prototype. Specifically, the target population for this study was the group of students enrolled in a College Algebra course.

In defining the objectives for the pretest and posttest, multiple sources were consulted: the recommendations of the AMATYC (1995) and NCTM (1989) on reform, Becerra, Sirisaengtaksin, and Waller (1997) on the use of graphing technology, and the course objectives provided by the Mathematics Department. These objectives matched factors of student understanding of functions described by Markovits, Eylon, and Bruckheimer (1986), Vinner and Dreyfus (1989), and Selden and Selden (1992). To be included as an objective, the objective had to be cited in each source. The following objectives were thus adopted because they appeared in all of the sources:

Students will be able to:

1. identify whether a relationship is a function when represented either graphically, numerically, verbally, or symbolically;
2. provide examples and nonexamples of functions;
3. define functions;
4. determine domain and range of functions;
5. analyze functions for increasing/decreasing;
6. determine maximum/minimum of functions from graphs;
7. use functions to model real-world relationships;
8. evaluate functions given symbolically;
9. read and interpret graphs of real data; and
10. read and interpret charts of real data.

The objectives were separated into two main categories: function identification, included objectives one, two, and three and application of functions included the remaining objectives. The questions corresponding to the categories of objectives were grouped into separate sections. As a result, the Function Test (see Appendix C for both pretest and posttest versions) consisted of two sections reflecting the objectives: the Identification section and the Application section.

Each of the sections of the Function Test was modeled after tests used in research by Bell and Janvier (1981), Vinner and Dreyfus (1989), and Becker (1991). For the Identification section, students were asked to identify functions presented in various representations of functions. The Application section asked that students apply their knowledge of functions for situations presented numerically, graphically, and

symbolically. The test items were identified on the basis of matching the objectives and representing the content covered in the course.

A prototype of the test was developed and administered to five students enrolled in College Algebra to assess the clarity of the questions and the length of time for taking the test. Two students were science majors, two students were business majors, and one student was a liberal arts major. The students' algebraic skills were not identified. From the feedback of the students, the wording of some questions were changed. Also, redundant questions were discarded in order to reduce the amount of time needed to administer the test.

Because the Function Test was a domain-referenced test, content validity of the test was required. Using the validation form, method, and results founding Appendix D, a committee of five mathematics educators experienced with teaching College Algebra at the community college assessed the content validity of the test for students enrolled in the course. The responses of he committee members were tallied for each question on the Function Test. At least 80% of the committee matched every question to a course objective. Thus, the questions on the Function Test were deemed to possess content validity with respect to the questions assessing a topic covered in the course. Additionally, the committee determined that a graphing calculator was not required to solve any of the questions. All of the questions were solvable without access to a graphing calculator.

Reliability was determined for the Application section of the Function Test and not for the Identification section. Since many of the questions in the Identification section of the test were unique to each other with respect to function representation or topic, internal reliability for this section of the test was not applicable. Additionally, research by Becker (1991), Bell and Janvier (1981), and Vinner and Dreyfus (1989) indicated that students performed differently for identifying functions between representations. Thus, prior research lent support that reliability was not applicable to questions between representations of functions.

Determining the reliability within the Application section of the Function Test (Appendix C) was needed within each function representation. Since there were no repeat questions with respect to representation of functions, reliability between

representations was not applicable. However, for questions within the same representation of functions, reliability was applicable. The first two questions in each representation assessed students' abilities to read graphs, read tables, and evaluate functions. The skills required for the first two questions in each representation were required for the subsequent questions. For example, in order to answer questions 9, 10, and 11 of the numerical part of the Application section correctly, students needed the skills required to answer questions 7 and 8.

However, due to the small number of questions for which reliability can be computed, applying typical reliability tests such as Kuder Richardson-20 was not appropriate (Hopkins, Stanley, & Hopkins, 1990). Thus, an alternative method for determining reliability was developed. Since reliability is usually reported in terms of percentage of agreement for domain-referenced tests (Borg & Gall, 1989), the percentage of agreement between the questions in each part of the Application section was determined. The reliability indicator for the Application section was calculated as the percentage of students who answered any of the subsequent questions correctly and answered both initial questions correctly. For each part distinguished by representation of function, this percentage was calculated. The average for the three parts of the Application section was determined as the reliability indicator.

For example, within the numerical part of the Application section, if students answered question 9 correctly and did not answer questions 7 and 8 correctly, then these students did not answer consistently. Their responses did not contribute to the reliability indicator. Students' responses that did contribute to the reliability indicator were when question 9 was answered correctly and both questions 7 and 8 were answered correctly.

Additionally, determining pretest-to-posttest reliability was not applicable. Because discovering changes in student understanding of functions was goal for the use of the pretest and posttest, consistency for student responses on the pretest and posttest was not expected. Thus, reliability was determined for the Function Test only on the basis of responses within the representations of functions.

Based on the above method, the reliability of the Function Test was determined with scores obtained from students in a second section of the College Algebra course who took the pretest form of the Function Test. The Function Test was administered to

another section of the course in order to restrict the researcher in gaining information about the students in the study. Because the students in the second section were not interviewed and the scores were not analyzed for patterns in responses, knowledge gained from grading and scoring the tests was considered not to significantly bias the researcher when the classroom observations and student interviews were analyzed. The reliability indicator for the Function Test, as determined from this second group, was found to be 0.86.

Student Interviews

Since tests alone do not provide a complete picture of students' understanding, student interviews were needed to investigate the reasoning of the students. To obtain a more complete picture of students' understanding of functions, interviews of 12 students, two students from each category, were conducted within four weeks after the posttest of the Function Test was administered. Data from the student interviews provided explanations for students' responses on the Function Test. The protocol for the student interviews was developed following the recommendations of Borg and Gall (1989).

The goals of the student interviews were to gather information about students' responses on the Function Test, students' use of graphing calculators, and students' abilities to apply their understanding on a new mathematical task. The first goal was to clarify confusing or omitted responses, to investigate students' preferences of representations of functions, and to investigate factors for changes in responses from pretest to posttest. Even though investigating student misconceptions of functions was not a primary goal of the interviews, misconceptions were investigated when they arose. The second goal of the interviews was to investigate students' use of graphing calculators in the classroom, for homework, and on exams. The third goal was to investigate whether students were able to apply their understanding of functions to a new mathematical task. The students were presented with two mathematical tasks that applied their understanding of functions to solve the problem. The representation of functions

that the students chose to solve the problem and their use of the graphing calculator on the tasks were investigated.

The protocol (see Appendix E) for the student interviews was modeled after the interviews conducted by Rich (1990) and Becker (1991). The interviews were semistructured with open-ended questions to gather information about students' understanding of functions. Due to the variability in students' responses on the Function Test and in the interview, the protocol was adjusted when found to be necessary.

In order to gather information about students' responses on the Function Test, the researcher reviewed the pretest and posttest of students being interviewed. Before each interview the researcher noted unclear responses of the student. Also, the researcher compared each student's responses of the pretest and posttest for each question. The information gained from the Function Test was used to develop questions particular to the responses of the individual students. These questions were typically either clarification questions or questions concerning the reasoning for change in responses.

During the interview, students had access to paper, pencil, and their graphing calculators. The interviews were videotaped with the focus on the students' work rather than on the students' faces. All papers used by the students were collected.

Because one of the purposes of the interviews was to investigate students' responses on the Function Test, piloting the interview was not possible before the study began. The interview was piloted during the study. After the administration of the posttest on knowledge of functions, one student from the HSM-MS group was used to pilot the interview. This student was randomly chosen from the three students who volunteered from this group. This group was used because it was the only group to have an extra volunteer for interviews. The results of the pilot interview were not used in the analysis of the interviews.

A committee of five mathematics educators determined the face validity of the interviews using the method found in Appendix F. With an agreement rate of at least 80%, the committee determined that the questions were not leading or biased. Additionally, the committee determined that the questions corresponded to the goals of the interview.

Classroom Observations

The purpose of the classroom observations was to collect data about the classroom environment to which students encounter functions. One focus of the observations was the students' use and instructor's use of graphing calculators. The other focus was what and how material was presented because the mathematical topics presented in class were the experiences on which students construct their understanding due to the abstract nature of mathematics (Skemp, 1971). The information gained from the observations was descriptive in nature to provide a more complete picture of factors influencing students than merely access to the graphing calculator.

The course was separated into three units and the function concept was introduced during the first unit that lasted 19 class periods during five weeks. Even though the first week of the term was used for review of symbolic manipulation skills and for administration of the Skills Test and Function Test, observations began the first week in order for the instructor and students to get acclimated to being observed. Daily observations continued until the first unit was completed and the posttest was administered.

The lessons were videotaped, and notes on the material presented were taken. The transcribing of the lessons began the first week and continued throughout the term. Handouts given by the instructors were collected. After each classroom observation, the researcher completed an observation and document summary shown in Appendix G (Miles & Huberman; 1994). These summaries were short forms that were used to develop questions for the student interviews and to help with data analysis. The observation summaries were made for the handouts given by the instructors such as homework assignments, worksheets, quizzes, and exams.

Description of the Researcher

Since data were collected and analyzed by the researcher, efforts were made in order to address possible biases of the researcher. The researcher kept a personal journal

containing personal impressions, opinions, and reflections in addition to completing the document, observation, and tutoring summaries. Entries were made after each classroom observation and student interview. This journal was used to become aware of potential biases that may affect data collection and analysis. The various types of data collection helped the researcher in overcoming any potential bias.

A brief description of the researcher is provided to assist the reader in determining the perspective from which data collection and analysis were made. The researcher obtained a Bachelor of Science degree majoring in mathematics with an emphasis in physics. While the researcher was obtaining a Masters of Science degree majoring in applied mathematics, the researcher discovered an affinity for teaching.

The researcher had taught at the college level for eight years (as a graduate student, instructor, mathematics consultant for a precollege program, and academic advisor). The researcher had been recognized by various groups for teaching accomplishments. The researcher was interim coordinator for the College Algebra course at his graduate institution and wrote the study guide used for the course. For four years the researcher had been one of two technical support personnel for the use of graphing calculators in the Mathematics Department; additionally, the researcher was an academic advisor/mentor for the Athletic Department.

The researcher was exposed to graphing technology as a graduate teaching assistant and viewed it as a tool that could be used to enhance student learning if properly used. The use of graphing calculators should be used to meet goals and objectives made by the instructor. It is the opinion of the researcher that the instructor's goals and objectives for a course should be the driving force and not technology. Yet, instructors should be aware of how technology can be used to assist them in attaining their goals for students.

The researcher views teachers of a college algebra course to be interpreters or ambassadors. In general, students enroll in College Algebra as a prerequisite for another mathematics course or as the fulfillment of their major requirement. Many of the students have not been successful in previous classes in which skills could have been developed to test out of the course, so they still need to develop an understanding of the content. Thus, the instructor needs to present material in ways that are accessible for

students who are not proficient with mathematical language. Furthermore, for the students who enroll in the course as their only mathematics requirement, the instructor provides to these students a brief and shallow glimpse of the mathematical field. Hence, the instructor can be viewed as a mathematical ambassador to these types of students.

Data Analysis

Data analysis was performed in four stages. First, the classroom observations were analyzed to develop a description of the classroom environment in which the students encountered functions. Developing the classroom profile at the beginning of the study was an attempt to reduce a potential bias. With knowledge of the results of the pretest and posttest of the Function Test, the researcher might have attempted to find justifications for students' responses where justifications did not exist.

The second stage consisted of grading the pretest and posttest of the Function Test for students participating in the interviews. Confusing or omitted responses and changes in responses between the pretest and posttest were identified. This information was used to develop questions for the interviews.

In the next stage, the twelve student interviews were analyzed. Patterns of students' responses were identified. Also, when and how students used their graphing calculator during the interview were investigated. Additionally, patterns to responses were identified with regard to types of students.

During the last stage of analysis, the responses to the pretest and posttest of the Function Test for the remaining students participating in the study were graded and analyzed to assess students' understanding of functions and to determine whether a change occurred for student responses. The goal of the analysis was not only to investigate students' understanding of functions but also to investigate the relationships between students' understanding of functions, the use of graphing calculators, and the classroom presentation of functions.

Classroom Observations

The videotape recordings of each class session were transcribed. After the classroom observations for the lessons covering functions concluded, the data were analyzed. Using the transcriptions, the notes, the videotape, and the material collected, the researcher developed a description of the students' use and instructor's use of graphing calculators and what and how the material on functions was presented in the classroom.

The researcher used a constant comparison method of analyzing the data (LeCompte & Preissle, 1993). The transcripts of the lessons were read for a general impression of the course, instruction, and the use of graphing calculators by students and instructor. During a second reading of each lesson, the researcher identified the topics covered in the lesson and verified this information with the observation summaries. Finally, use of graphing calculators by the students and instructor was identified.

On the basis of research by Slavit (1996) and Smith and Shotsberger (1997), three aspects of instructional use of graphing calculators were identified: (a) demonstration, (b) alternative methods, and (c) discovery. Demonstration represented when an instructor used the graphing calculator in order to provide information that could have been done by other means. For example, in the presentation of quadratic functions, an instructor displayed the graph of a quadratic function by using the graphing calculator rather than premade overheads or a drawing on the board. Using the graphing calculator to provide students with information and facts was a demonstration form of use of the graphing calculator.

The instructor used the graphing calculator to provide alternative methods to solving problems. For example, the instructor used the graphing calculator to find the relative maximum of a polynomial function of degree three when solving a real-world problem. Traditionally, derivatives are used to find the extrema. Rather than use this symbolic method which was not introduced in the course, the instructor used the graphing calculator to find the extrema. Alternative methods were used to solve a problem or derive a solution with the graphing calculator that could have been solved using a symbolic manipulation approach.

The last category for graphing calculator use by the instructor was discovery where the instructor uses the graphing calculator or allowed the students to use their graphing calculator to explore topics. For example, the instructor had students graph the functions $g(x)=\sqrt{x}$ and $g(x-3)=\sqrt{x-3}$ on their graphing calculators in order to discover the transformation caused by $y=g(x-c)$.

Grading the Pretest and Posttest of the Function Test

The second stage of the data analysis involved correcting the responses on the pretest and posttest of the Function Test for students participating in the interviews. For the questions having an explanation portion to them, only responses that included an explanation were graded. Blank responses or responses that did not include an explanation were not given credit. Only responses whose answer and explanation were correct and corresponded were marked correct. The researcher identified changes in responses between the pretest and the posttest for individual students.

Student Interviews

After the videotapes of the student interviews were transcribed, the researcher used a constant comparison method of analyzing the data (LeCompte & Preissle; 1993). The transcription of the interviews were read through the first time in order to get an overall impression of the students' responses. Next, each section of the interview was investigated for recurrent themes or patterns to the responses.

For the section of the interview covering the responses on the Function Test, the students' reasons for changing responses from the pretest to the posttest were identified. When the students' reasons for change were attributed to an action in the classroom, the reasons were verified from the transcription. If discrepancies existed between students' responses and the transcription, the researcher reviewed the videotape.

After the responses were verified, the researcher developed categories and sorted the responses. Patterns and differences were identified within the developed categories. Students' comments from the interview on the use of graphing calculators were verified from the transcripts of classroom observations. Categories derived from these comments were used to sort the students' comments in order to distinguish patterns and differences.

Analyzing the Pretest and Posttest of the Function Test

The final stage of the data analysis involved correcting the responses on the pretest and posttest of the Function Test for the remaining students participating in the study. For the questions having an explanation portion to them, only responses that included an explanation were graded. Blank responses or responses that did not include an explanation were not given credit. Only responses whose answer and explanation were correct and corresponded were marked correct. The percentage correct for each question was determined.

Analysis of the results of the Function Test was conducted in three phases: (a) analysis of change for the class as a whole, (b) comparison between the change for each group of students based on major and symbolic manipulation ability, and (c) analysis of definition of functions and the examples and nonexamples of functions. Because assigning a numerical value to yes, no, and I don't know responses had no inherent meaning, a chi-square test was used for comparing the numerical results of the test. Additionally, the use of a mean on the total score did not sufficiently represent the students' performance on the various sections. Therefore, a chi-square test was chosen to retain the information about the students' performance on the various sections.

First, the percentage of correct responses on each question of the test was compared between the pretest and the posttest for whole section. This comparison was made to determine whether a change in student understanding of functions had occurred. A 2 by 19 matrix was used, where the rows were the pretest and posttest scores and the columns were the question number and total percentage correct for the whole class.

Second, the percentage of correct responses for the groups based on majors and symbolic manipulation ability was compared using a chi-square test. In order to determine whether the gains were significant, three applications of the chi-square test were conducted. The first chi-square test compared the groups' percentage of correct responses for each question on the pretest. Then, the groups' percentages of correct responses for each question on the posttest was compared.

Because of the limitation of using just two tests to show gain of scores, a third comparison was made on the gains made by each group. For each group, the difference of the percentage of correct responses for each question between the pretest and posttest was calculated. The gain percentage for each question was compared between the groups. The combination of these three applications of the chi-square test provided evidence for differences for the groups in gain scores on the tests.

With the multiple use of the chi-square test for comparing the responses to the Function Test and for comparing the definitions of functions provided by the students, using a significance level of 0.10 would not have been appropriate. According to Good (1984), five tests with a significance level of 0.10 would have produced approximately 0.50 chance of making a Type I error. In order to reduce the possibility of making a Type I error, the significance level of 0.02 was used for each test because the analysis of five tests with this significance level retained a 0.10 significance level for the study.

Each of the questions on the pretest and posttest on knowledge of functions were determined to be correct or incorrect. The correct responses were placed in one group and the incorrect responses were placed into a second group. Then, each group of the responses were analyzed using the constant comparative method (LeCompte & Preissle; 1993). The responses for each group were read through once to get a general impression of the responses. The second reading was performed to develop categories of responses. Then, the responses were sorted into the categories developed by the researcher. Patterns and differences in responses were identified.

For the last phase of the analysis of the Function Test, the students' definitions of function were analyzed. On the basis of the research of Vinner and Dreyfus (1989) and Becker (1991), the following six categories were used to categorize the definitions of functions provided by the students: (a) correspondence, (b) dependence rule, (c) rule, (d)

operation, (e) formula, and (f) representation. Correspondence was the Dirichlet-Bourbaki definition where every element in the first set was assigned to exactly one element in the second set. Sample responses to this category would have been "a correspondence between two sets of elements" and "for every element in A there is only one element in B." Dependence relation was described by a function that was a dependence relation between two variables. Examples of a dependence relation were "y depends on x," "one factor depending on the other one," and "a connection between two magnitudes." Rule was similar to dependence relation. The difference was that a rule connects the variables by a regular pattern, whereas the dependence relation could have been arbitrary or not specified. Examples of rule included "something that connects the value of x with the value of y" and "the result of a certain rule applied to a varying number." Operation denoted manipulation in the manner that a person acted on a given value to obtain another value: for example, "transmitting values to other values according to certain conditions" and "an operation done on certain values of x that assigns to every value of x a value of y." Formulas were algebraic expressions or equations. Examples of formulas were "it is an equation expressing a certain relation between two objects" and "a mathematical expression that gives a connection between two factors." Representation was identified by meaningless ways in one of the representations of functions: for example, "a graph that can be described mathematically," "$y = f(x)$," and "a collection of numbers in a certain order in a table."

In order to ensure that the researcher classified the definitions appropriately, a mathematics educator categorized 20% of the responses. Inter-rater agreement was measured with the standard of 80%. Additionally, any responses that the researcher could not place in a category were discussed with this mathematics educator until full agreement about the classification was reached. When it was agreed that the definition does not fit one of the categories, then the definition was classified as "other."

After the definitions were categorized, the results were analyzed to discover if changes occurred between responses on the pretest and posttest. A 2 by 8 matrix was developed with each position representing a category of functions and the pretest or posttest. Finally, the number of definitions classified in each category for the pretest and posttest were analyzed using a chi-square test.

CHAPTER IV
ANALYSIS OF DATA

Introduction

The purpose of this study was to investigate the relationships among students' knowledge of functions, their symbolic manipulation skill level, and their use of graphing calculators in a precalculus course. The data described in this section were collected to address the three research questions:

1. What knowledge of functions do students gain in a College Algebra course requiring graphing calculators?

2. What knowledge of functions do students of different levels of algebraic skills and with different academic majors gain in a College Algebra course?

3. What role do graphing calculators play in the classroom as students develop an understanding of functions?

Throughout the first five weeks of the term, classroom observations were conducted for each class period. The observations focused on the use of graphing calculators by the instructor and students, the content covered in class, and the instructional techniques. The lessons were videotaped and the researcher prepared comprehensive field notes. Twenty-five students in the first course of the precalculus sequence participated in the study by completing the Algebra Skills Test and both versions of the Function Test.

At the beginning of the term, the students took the Algebra Skills Test used to categorize students into six groups based on their symbolic manipulation skill ability (high and low) and their major (mathematics and science, business, and liberal arts). Twelve students who correctly answered eight or more of the questions were categorized in the High Symbolic Manipulation Skill Level [HSM] and eight students who correctly answered five or less of the questions were categorized in the Low Symbolic Manipulation Skill Level [LSM]. The remaining five students who correctly answered six or seven of the questions were not categorized in either of the groups in order to make

the LSM and HSM groups distinct. Second, the students were categorized on the basis of their academic major: mathematics and science, business, and liberal arts. The combination of these two categories produced six groups: LSM-Mathematics and Science [LSM-MS], LSM-Business [LSM-B], LSM-Liberal Arts [LSM-LA], HSM-Mathematics and Science [HSM-MS], HSM-Business [HSM-B], and HSM-Liberal Arts [HSM-LA]. The LSM-MS group included three students, the LSM-B included two students, the LSM-LA included three students, the HSM-MS group included six students, the HSM-B group included two students, and the HSM-LA group included four students.

Also at the beginning of the term, the pretest version of the Function Test was administered to the students participating in the study. During the sixth week of the term, the posttest version of the Function Test, similar to the pretest, was administered. After the observation period, interviews were conducted with two students from each of the six groups. The purposes of the interviews were to collect data on students' use of graphing calculators, to clarify responses on the Function Test and to further investigate students' understanding of functions. Besides discussing the reasoning behind their responses on the Function Test, the twelve interview students were administered two mathematical tasks at the end of the interviews. Additionally, informal interviews were conducted with the instructor to clarify and gather information about the motivations for incidents observed in the classroom.

Because of the qualitative nature of some of the data collected in the study, a sequence for analyzing the data was followed to reduce bias. First, the classroom observations were analyzed to create a profile of the environment in which the students studied functions. Second, the interviews were analyzed. Once the qualitative analysis of the classroom observations and student interviews were performed, the results of the Function Test were analyzed quantitatively.

This chapter is separated into four sections on the basis of the research questions. First, a profile of the course and the class is presented. This description of the environment in which the students learned the function concept provides a context for the results used to answer the research questions. Next, results of the Function Test, with supporting evidence from the interviews and classroom observations, are presented that pertain to the question about the knowledge of functions students gained in the course.

Following the analysis of the knowledge students gained, the data are presented that pertains to the research question about the difference in knowledge gained for the groups of students. Finally, the observations and interview results about the role of the graphing calculator in students' learning are presented.

Description of Course

Instructor

The instructor, Nellie, graduated with a Master's degree in mathematics. She taught at the community college for five years. Additionally, she taught one year at the high school level and spent several years as a teaching assistant at the college level. Nellie has taught the first course of the precalculus sequence four times prior to the study. For each time she taught the course, the Mathematics Department required students to have access to a graphing calculator.

Nellie viewed teaching as a means to present the mathematical material in a way that students can understand and enjoy mathematics. She had the goal to get students to view mathematics in a positive manner, that "mathematics is cool." Also, Nellie wanted students to develop the ability to think on their own rather than rely on her or other people.

Course on Function Rather than Symbolic Manipulation

Throughout the observation period, Nellie made numerous comments indicating that the course was "not an algebra course." The course focused on the concepts of functions rather than the symbolic manipulation of functions. Nellie emphasized that students needed to have developed their symbolic manipulation skills before enrolling in the course.

In order to enroll in the precalculus course, students had to satisfy the prerequisite requirements. Students were required to have either passed the prerequisite course with a

grade of 2.0 or better, obtained a satisfactory score on the placement test, or passed a high school algebra course that covered the same mathematical content as the prerequisite course. The prerequisite requirements were a Mathematics Department policy, and "If you don't have it, you cannot come into this class." Nellie enforced the policy by checking each student's prerequisites. When a student did not meet the requirements, she advised the student to withdraw from the course and enroll in the prerequisite course.

Also, providing evidence that symbolic manipulation skills were a prerequisite rather than the focus of the course, Nellie made numerous statements about the importance of possessing symbolic manipulation skills for the course. During the discussion of the syllabus on the first day of the term, Nellie said, "You're done with your algebra. Now, you're going to use your algebra to get into some higher mathematics....I'm expecting you to have your algebra down already. It's not an algebra class." Additionally, while passing out the Algebra Skills Test, Nellie said, "I've already discussed that we are not going to be an algebra course. I'm expecting you to walk into this course with your algebra okay....And, you can't do well in this class without having your algebra down." On the second day when the Algebra Skills Test was returned to the students, Nellie reinforced the focus of the course, "For this class you are expected to have your algebra down."

A method to ensure that the students had symbolic manipulation skills prior to enrolling in the course was the Algebra Skills Test as a gateway exam for the course. "You will not receive a passing grade without passing the [Algebra Skills] test." Students were required to take different versions of skills test until they scored at least 80% on the test. However, for students who had difficulty completing the Algebra Skills Test, Nellie warned them that their possible success in the course was in jeopardy. When discussing the results of the test, she said, "...if you missed a good chunk of them [questions on the Algebra Skills Test] and you were totally lost in what we just did, that's a big clue that you're in trouble."

Another method demonstrating the idea that learning symbolic manipulation skills was not the focus of the class was the way Nellie solved some of the problems. During the algebra review on rational expressions, Nellie's expectations were demonstrated by the comment "I've been using the word cancel in this class. I know you're algebraically

mature" regarding the symbolic steps involved with canceling terms in a rational expression. While working a homework problem on the board, Nellie only set up the problem and did not perform the symbolic steps to complete the problem because "from there it's just algebra, simplifying it." Another example for the reduced attention paid to symbolic manipulation occurred when working on inverse functions. Nellie performed the steps for finding the inverse but did not complete the algebra steps because "we have our algebraic skills down and we know how to simplify this."

Nellie did not emphasize teaching the symbolic manipulation skills because she expected students to be proficient with algebra before enrolling in the course. During the lesson on inverse functions, she gave a few functions and asked students to find the inverse function. When verifying the students' solutions, Nellie showed the symbolic steps for finding the inverse functions only when students requested the steps.

> T: Okay, let's talk about these. [S1], did you get the first one, *f* inverse of *x* [$f(x) = \sqrt[3]{x+4}$]?
> S1: *y* equals *x* cubed minus four.
> T: Minus four? Is that what other people got? Okay [S2], did you get this one [$g(x) = 5x + 2$]?
> S2: Um, the inverse would be *x* minus two over five.
> T: Over five? Cool. [S3], did you get C [$h(x) = \frac{1}{x-5}$]?
> S3: Yea, one over *x* plus five.
> T: Okay, quantity one over *x*, plus five [$\frac{1}{x} + 5$]. Oh, I forgot to ask, are these right? Everyone agree with those?
> S4: How do you get C?
> T: How do we get C? Well let's look. That's a good question. Let's see. We're supposed to write *h* of *x* is *y*. We're supposed to switch the *x*'s and the *y*'s. *x* equals one over *y* minus five. And solve for *y*. Okay? Again, this is algebra here, so the big problem is getting the *y* out of the denominator. We've got to multiply both sides by *y* minus five. And, when we do that, we get *y* minus five times *x* equals one. Okay? You don't really want to distribute the *x* there because you're trying to get *y* by itself. Distributing the *x* doesn't really accomplish anything. So, let's just go ahead and divide by *x* right away. And then we can add five to both sides. That's how it works. Questions? Okay?

Though Nellie did conduct a review of symbolic manipulation skills at the beginning of the term, her review consisted of two class periods during the first week.

According to Nellie, "The review will be very quick. It really just gives you time to freshen up if it's been a little while.... Again, because this is not an algebra course, we'll be getting into functions right away." With only two class periods devoted to the review, the students had to work through the algebra review sections outside of class. The review consisted of the following sections in the textbook (Bittenger, et al, 1997): (1) Integer Exponents, Scientific Notation, and Order of Operations; (2) Addition, Subtraction, and Multiplication of Polynomials; (3) Factoring; (4) Rational Expressions; (5) Radical Notation and Rational Exponents; (6) Solving Equations; and (7) Solving Inequalities. Additionally, these topics in the review sections were the ones contained in the Algebra Skills Test.

Course Content

The chapters covered in the first half of the term were the sections in Chapter 1, excluding the data analysis section, and the section on inverse functions in Chapter 3 of the textbook. A list of the sections covered during the observation period is provided in Table 3. Additionally, short descriptions of each section are provided in Appendix H.

Homework

At the beginning of each lesson, except when the test was administered, Nellie wrote the homework assignment for that week on the board, indicating the problems that the students were responsible for completing by the following Tuesday. On each Tuesday at the beginning of the lesson, Nellie asked the students to turn in only one of the sections assigned for the week. Assignments that were not turned in at the beginning of the class were considered late by the instructor. Late assignments received half credit for each day they were late. An assignment turned in two days late received only one-fourth of the value of the assignment.

Table 3. Textbook Sections Covered during the Observation Period

Section Number	Title
R.9	Modeling and Applications
1.1	Functions, Graphs, and Graphers
1.2	Functions and Applications
1.3	Linear Functions and Applications
1.5	Distance, Midpoints, and Circles
1.6	Symmetry
1.7	Transformations of Functions
1.8	The Algebra of Functions
3.1	Inverse Functions

Note. The section number, R.9, refers to ninth review section in the textbook.

The total value of the homework was worth 15% of the final grade for the course. Each homework assignment turned in was worth 10 points. Nellie graded the one section of the homework assignment in two stages. First, if the problems from the homework assignment were attempted with work shown, the student received six points. Then, Nellie graded four random problems worth one point each. She used this method to reduce the amount of grading and to motivate the students to attempt all of the problems. Though the other sections were not graded, she expected the students to "keep on top of yourselves to get that done."

The homework problems were categorized with respect to the functional representation in which the problems were presented and worked. Six categories were identified: (a) symbolic, (b) graphical, (c) numerical, (d) written, (e) written-to-symbolic, and (f) symbolic-to-graphical. Symbolic, graphical, numerical, and written types of questions were presented and solved within the respective representation. Written-to-symbolic questions presented data or the situation in written format and the solution was developed in the symbolic representation. Symbolic-to-graphical questions presented the information symbolically and the solutions were obtained graphically.

A detailed description of the categories is given in Appendix I. All of the homework problems assigned by the instructor were separated into one of the categories. For each section in the textbook, the percentages of questions in the categories were calculated. The results are shown in Table 4. The values in the overall column are the percentages of questions in each representation based on all the sections combined.

Table 4. Percentage of Homework Questions Categorized by Representation of Function for Each Section of the Textbook.

Representation	Sections									
	R.9	1.1	1.2	1.3	1.5	1.6	1.7	1.8	3.1	Overall
Symbolic	0.0	32.4	7.0	58.3	72.0	21.4	13.6	60.5	29.5	35.1
Graphical	0.0	21.6	18.6	2.1	0.0	21.4	0.0	13.2	9.1	10.6
Numerical	0.0	8.1	0.0	14.6	0.0	0.0	0.0	0.0	4.5	4.0
Written	0.0	8.1	2.3	2.1	0.0	0.0	4.5	5.3	2.3	3.0
Written-to-Symbolic	100	5.4	18.6	10.4	4.0	0.0	0.0	18.4	2.3	13.9
Symbolic-to-Graphical	0.0	13.5	48.8	6.3	20.0	64.3	59.0	2.6	52.3	28.8
Other	0.0	10.8	2.3	6.3	4.0	0.0	22.7	0.0	0.0	4.6

Note. The questions in the Other category for Section 1.1 were arrow diagrams used to identify functions and for Section 1.7 were questions about transformations of functions that required matching graphs with their appropriate symbolic function.

Assessment

During the observation period, three quizzes and one test were administered to the students. Nellie developed all of the exams. The first and third quizzes were administered in the classroom, and the students had approximately half of the class period

to complete them. The students were allowed to take home the second quiz to complete it. The students were given an entire 50-minute class period to complete the test.

The quizzes were considered by the instructor to be "mainly" summative because they were administered to assess students for a grade and they covered topics in sections that were finished being presented. The quizzes were also considered "partially" formative because Nellie was prepared to "backtrack" if the whole class did not show understanding of a topic. Additionally, the quizzes were given to "let them [students] know where they're at." The test was considered to be purely summative.

The value of the combined scores of the quizzes was 15% of the final grade and the midterm was 15%. The quizzes had a value of 20 points each and the test had a value of 108 points. The point values for each question were labeled on the quizzes and tests, except for the first quiz. Nellie graded the exams and gave partial credit for incomplete work that was correct.

With regard to the questions that could have been solved using an alternative method, Nellie did not take off points unless the directions for the question specifically stated a method was to be used. However, she expected students to use the most efficient method possible when finding a solution. For example, Nellie expected students to use the symbolic approach to calculate the value of $h(3)$ rather than graph the function and obtain the value using a built-in feature of the calculator. She was prepared to meet with students using alternative approaches to "sit them down and say, 'Hey, you can do it quicker'." Though Nellie was prepared to have this discussion with students, she did not conduct a meeting with any of the students during the observation period.

Overall, of the 46 questions from the quizzes and test given during the observational period, the graphing calculator provided an alternative method for 17.4% of the questions and was required for 19.6% additional questions. Thus, the graphing calculator had an impact on little more than one-third of the questions used to assess the students.

With respect to representation of functions for each question, Table 5 provides the percentage of questions categorized in the same manner as the homework questions from the three quizzes and one test and for all of the quiz and test questions combined.

Approximately 60% of all of the quiz and test questions required students to work within the symbolic representation, whether entirely or partially. Similarly, approximately 45% of all of the quiz and test questions required students to work entirely or partially within the graphical representation.

Table 5. Percentage of Test Questions Categorized by Representation of Function for the Quizzes and Test.

Representation	Exams				
	Quiz #1	Quiz #2	Quiz #3	Test	Overall
Symbolic	45.5	16.7	50.0	38.1	39.1
Graphical	36.4	16.7	50.0	29.6	33.6
Numerical	0.0	0.0	0.0	0.0	0.0
Written	0.0	0.0	0.0	14.3	6.5
Written-to-Symbolic	9.1	33.3	0.0	9.5	10.9
Symbolic-to-Graphical	9.1	33.3	0.0	9.5	10.9

Note. The categories used in this table were the same as the categories used in Table 4. In depth description of the categories is given in Appendix I.

In contrast, none of the quiz and test questions required students to work solely within the numerical representation. For one question on Quiz #2 and two questions on the test, students were presented coordinate points and asked to develop equations of lines and circles. Even though the information was provided in the numerical representation, the students were expected to work within the symbolic representation to develop the equations. Unlike the homework questions involving identifying functions and calculating slopes, students were required to translate the three questions to the symbolic representation.

Typical Class Period

The classroom observations provided information about the environment in which students attempted to develop an understanding of functions. Additionally, informal interviews with Nellie were conducted when the researcher needed clarification of some aspect of the lessons and course. From the observations and interviews, a few incidents and topics were deemed significant such as (a) the lecture format used by Nellie to present material, (b) the selection of the representation of functions to present material and solve problems, and (c) the uses of the graphing calculator.

The class period was composed of two segments: a question-and-answer segment and the presentation of new material. At the beginning of the class hour, Nellie typically conducted a question-and-answer segment for approximately 20 minutes. During this time, the students had the opportunity to ask her questions about homework problems. "The beginning of the class will be usually like asking questions on the homework. So, what I ask of you is that you come in with your homework questions already laid out. When you do the homework the night before, it would be a good idea to write down or circle which questions you had questions on."

Nellie made a list of the problems from the homework assignment of which students had questions. Students called out the number of the problem they wanted worked. When students called out more numbers than time allotted, Nellie took votes to find out which problems were most popular. For example, during Lesson 8, Nellie said, "Let me see which ones are most important. Cause, again, we want to be able to do homework that the class as a whole needs to see. Otherwise, you can just come to my office." Additionally, when two or more of the requested homework problems were similar, she worked just one of them.

When students did not vote on a problem or did not call out for a problem to be worked, Nellie assumed that these students had a sufficient understanding of the problem. During Lesson 3 when the list of problems to be worked out was being made, she stated, "So, the rest of the class is giving the information that you did the homework and it's fine." Also, during Lesson 6, Nellie restated her perspective of not asking questions: "So,

the rest of you got both right [problems requested by students], right? That's the statement you're telling me."

The second segment of the class period consisted of the presentation of new material to the students. Nellie predominantly used a traditional lecture format by presenting the material at the board with the students taking notes in their seats. On occasion she conducted other activities and had students complete worksheets.

When presenting new material and answering students' questions, Nellie used a combination of declarative and interrogative lecture format (Friedman & Stomper, 1988) with interrogative lecture being utilized more often than declarative. When using the declarative format, Nellie made statements to students without attempting to get them involved. Definitions were given and examples demonstrated with little input from students. The interrogative lecture format included interaction between instructor and students. Using this format, Nellie solicited input from students to work problems. When working problems, she asked students for their solution strategies and for the results for each step of the solution.

The following excerpt illustrates how Nellie used the two lecture formats. During the lesson on linear equations, she used the declarative format for the first problem and the interrogative format for the second problem.

> T: What I care about is you being able to use lines, find lines, etc. But, I also care about the meaning. That's why I started out with a meaning. I want you to understand what slope means. It's not just a number. It's not just go up and over and draw another point and draw the line. It has a meaning. And, for the same time, we need to be able to find these lines. So, here is a reminder, the point-slope formula. If you have a point, x-one, y-one, and a slope for a line, your equation of the line is y minus y-one equals m times x minus x-one. Okay, let me give you an example to show you how to do this. Find the equation through two, four [the point *(2,4)*] and negative one, seven [*(-1,7)*]. Find the equation of the line, I should say. I'm doing this fast because it should be review. If it's not review, you should come in and get help because it should be review. [To] find the equation of the line, [you need] slope. That's the first thing you always start with. If you're not given it, you need to find it. Seven minus four [is] the change of y. And, you always have to go the same direction. If you start over here, seven minus four, you've got to go negative one minus two. y-two minus y-one over x-two minus x-one. This is three over negative three, or negative one. We've found the slope. [Then] you pick a point. I'll pick one. I'll pick two, four [*(2,4)*]. I've got the point. I've

got the slope. I plug it into the formula. *y* minus *y*-one. I've picked the point. *y* minus four is the slope, negative one times *x* minus *x*-one. If you're not sure what I am doing, I'm doing the point-slope formula. *y* minus *y*-one is *m* times *x* minus *x*-one. *x* minus two. Distribute. *y* minus four is negative *x* plus two. *y* equals negative *x* plus six. Did I do that right? And, I just found the equation of the line through two, four [*(2, 4)*] and negative one, seven [*(-1, 7)*]. Let me do one last example. We're going to run out of time today. Um, one last example. I want to find the equation of the line through two, negative three [*(2, -3)*] and perpendicular to *y* equals four *x* minus seven. Perpendicular and parallel lines, the slopes are related. I'm just reminding you. What do the slopes of parallel lines have to do with each other?
S: They're equal.
T: Yea, slopes are equal. What about [the slopes of]perpendicular [lines]?
S: Opposite signs and reciprocals.
T: Slopes are negative reciprocals of each other. Again, that should be familiar to you. Anyway, I want to find the equation of the line through two, negative three [*(2, -3)*] and perpendicular to *y* equals four *x* minus seven. How do I do it? What's the first thing you always care about when finding the equation of a line, most of the time?
S: Slope.
T: Slope, right. That's the first thing I did there. I found the slope was negative one. What's the slope that I want?
S: Negative one-fourth.
T: Negative one-fourth because the slope of the line that I want to be perpendicular to is four. Right, this slope is four. I don't want the four though. I want my slope to be perpendicular. The reciprocal of four is one-fourth. Put a negative on it and there's the slope. Now, I've got my slope. I've got my point. And, I just do it all again. *y* minus *y*-one, which is *y* plus three equals the slope, negative one-fourth, times *x* minus *x*-one, two. And, quickly when you solve this for *y*, plus one-half, *y* equals negative one-fourth *x* minus. Help, five-halves is that right? Okay.

Along with the interrogative lecture format, Nellie made comments that reduced her role as the "authority" in the classroom to get students more involved in the solution process for problems done in class. During Lesson 2, she told students, "I do make mistakes, so do watch me. If you had me before, you know that. All instructors make mistakes." When working problems, Nellie prompted students for their strategies for solving problems by asking "What should we do next?...We're just playing here. I'm stuck." When working an application problem for developing a function with time as the variable, Nellie prompted students with "I would love to have you lead me through this. How do I start?" Or, she asked students to work through the problem with "I need

someone to help me out with this one. I'm not quite sure how this one goes." Nellie used these types of questions to prompt students when working problems at the board.

Four incidents that did not follow the typical instructional format were the graphing calculator introduction, the introduction of standard graphs, the review for the test, and the presentation of transformations of functions. After completing the algebra review during Lesson 4, Nellie passed out a worksheet on the introduction to the Texas Instrument calculators addressing the features of the calculator for performing calculations. She had the students get themselves into groups of two or three and had them work through the introduction. At this time Nellie walked around the room answering individual questions.

The second incident that did not follow the typical class format occurred during the lesson on transformations. Nellie wanted the students "to be able to graph [standard graphs used in the textbook] at the drop of the hat without our calculators." On the board, she wrote eight functions that were used to illustrate the transformation concepts: "(a) $y = 2x + 1$, (b) $y = x^2$, (c) $y = x^3$, (d) $y = \sqrt{x}$, (e) $y = \sqrt[3]{x}$, (f) $y = \frac{1}{x}$, (g) $y = |x|$, and (h) $y = 4$."

Then, Nellie paired up students and assigned one function to each pair of students. The students were asked to sketch the graph of the function by hand at their desks: "I want to see it on the paper. No head graphs....It doesn't have to be perfect, just a quick sketch." Then, one student from each pair of students who were assigned a particular function drew his or her graph on the board until all eight functions were graphed. The remaining students not at the board copied the graphs in their notes.

The third incident when the class diverged from the typical lesson was the review for the test. During the class period on the day before the test, Nellie gave the students practice problems that were similar, but not identical to, the problems on the test. The students were separated into groups of two or three, usually students seated next to each other. The students were directed to work through the problems. Nellie walked around the room to answer individual questions. Additionally, there were a few other times during the observation period when Nellie directed students to work problems as practice.

However, these additional incidents involved only a couple of problems at a time and were performed individually by the students rather than by groups.

The final incident involved the use of guided-discovery teaching methods for introducing transformations of functions. For each of the transformations, Nellie gave students a parent function along with one or two similar functions that were transformations of the parent function. Then, Nellie asked students to identify the effects of each transformation by comparing the graphs of the parent and transformed functions:

> T: Okay. We've got two more things to look at. Um, let's consider negative *f* of *x* and *f* of negative *x*. Again, putting a negative on the outside and putting a negative on the inside of a function. What is that going to do? Well, let's play with a different function now. In fact, let's go back to *y* equals the square root of *x*. Negative $f(x)$ is negative square root of *x*? What do you think that will do? [S1] what do you think?
> S1: It's going to reflect it on the *y*.
> T: Reflect it over the *y*-axis? Okay. So, let's try this. [Nellie entered the functions into the calculator.]
> S2: Over the *x*-axis.
> T: The *x*-axis? I hear that it's going to be reflected over the *y*-axis or the *x*-axis. So, first of all, the square root looks like this, just the basic shape. [Nellie drew the graph of $f(x)$ by hand on the board.] Write it up here. And the question is, is it going to be reflected over the *y*-axis? [Nellie drew the graph of $f(-x)$.] Right, are you going to flip it over the *y*-axis or is it going to flip over the *x*-axis? [Nellie drew the graphs of $f(x)$ and $-f(x)$.] Or, maybe something different. Any ideas, one will do? Why is one better? [Nellie is comparing $f(x)$ and $-f(x)$] Okay. How many people think it's going to do this? [Nellie pointed to graphs of $f(x)$ and $f(-x)$.] A few [students with raised hands]. How many people think it's going to do this? [Nellie pointed to graph of $f(x)$ and $-f(x)$.] A few more [students with raised hands]. Anyone want to explain why besides that you've graphed it already and you looked?
> S3: Since $f(x)$ represents y. By going negative $f(x)$, you're going the opposite of the *y*-value. If all of your *y*-values are negative in the first graph, then all of the *y*'s would be positive in the second.
> T: Okay, yea, I would say that's a really good explanation. In essence, you're saying when we say *y* equals radical *x*, we're saying *y* equals $f(x)$, correct? And, we've put a negative in front of *f*, we're changing the *y*-value to a negative $f(x)$. So, if the *y*-value is positive here, it's going to be negative down here. And, let's graph this. Here's the square root of *x*. [Nellie showed graphs on an overhead calculator device.] And, there's negative square root of *x*. Again, it's outside the function, it has to do with *y*. Just like this outside the function [vertical shift] has to do with *y* and that outside of the function [vertical dilation] has to do with *y*. Okay, what about *g* of *x* is the square root of negative *x*?

> S4: It would be the y-axis.
> T: Is that going to flip it over the y-axis? Okay, if the pattern works, that's probably a good guess. When it's inside the function, it has to do with the x-values. All of the x-values change sign. Let's check. The square root of negative x. [Nellie adjusted function in calculator and graphed it] Yea, it did. It flipped it over to the other side. So, this [$-f(x)$] reflects over the x-axis. And this one [$f(-x)$] reflects over the y-axis.

Nellie did not utilize the guided-discovery teaching method as much as she liked because she said that the "pace of the course" did not allow her the time to use that teaching method.

For the discussion about Nellie's use of multiple representations in the solution process demonstrated during class, the stages of the solution process as defined by Brenner et al. (1997) are used. Within the first stage, *problem-representation* phase, Nellie familiarized herself with the problem by listing information and terms, drawing graphs and diagrams, or writing formulas. The next stage was the *solution-planning* phase in which Nellie developed a strategy for determining the solution. The third stage was the *solution-execution* phase in which the strategies that were developed in the solution-planning phase were implemented. In the final stage, the *solution-monitoring* phase, answers were determined to be appropriate for the application situation.

When solving problems, Nellie used multiple representations within the problem-representation and solution-planning phases as well as the solution-monitoring phase. She considered the use of the representations other than the symbolic representation to be important and useful. To solve an application problem, Nellie commented that using "pictures can't go wrong." When setting up a problem, she advised students to draw diagrams or pictures in order to develop a solution strategy. For a question about the graphical combination of functions, she recommended the following approach: "The key issue on the problem is being able to see the relationship between solving the equation and looking at the graph....This is something we're going to be using a lot this quarter." During Lesson 3, while solving a distance-rate problem, she commented, "If there is a picture to be drawn, it never hurts to draw. In fact, it's almost always helpful....You don't have to make a table here, but sometimes tables are nice especially with these kinds of problems [distance/rate/time problems]....That's most of the work right there, setting up a picture and defining our variables." During Lesson 6 Nellie recommended, "I'm

going to draw a picture. It's always good to draw a picture....Notice, I haven't tried attempting the problem. But, my picture is there and I put all of the information in picture form....And, you never get stuck with an empty paper. You get stuck with a picture if at all possible."

Not only were multiple representations used in the problem-representation and solution-planning phases of the solution process, but also they were used to check answers during the solution-monitoring phase. Nellie demonstrated that the numerical representation, as well as the graphical representation, could be used to check answers. While working a problem about transformations of functions, she demonstrated a method for checking whether the symbolic interpretation was correct. "Well, I think I want to move it horizontally. Tell you what, let's plug in a few points and find out. You can always plot points. As long as you've got some sort of a formula....I'm going to make a table here." Additionally, the following statement illustrates Nellie's perspective on the different representations of functions. "There's lots of different ways to represent a function. But my main point is that it's not necessarily just a graph, or it's not necessarily just an equation. It's a general idea." Nellie demonstrated the use of multiple representations throughout all stages of the solution process.

Graphing Calculator Use in the Classroom

The department recommended the TI-83 graphing calculator for the course. The instructor used the TI-83 connected to an overhead projection device. However, students in the class used a variety of calculators. Since some students had purchased and used other types of calculators in previous courses at different schools or before the TI-83 was the calculator recommended by the department, Nellie did not want to place a financial burden on students to purchase a second calculator. Thus, she allowed the various types of graphing calculators. Table 6 presents the distribution of graphing calculators for the students participating in the study.

Table 6. Distribution of Types of Graphing Calculators Used by Students.

	TI-80	TI-81	TI-82	TI-83	TI-85	TI-86
Students	1	1	7	9	4	3

Even though Nellie allowed the various calculators in the classroom, she did not welcome the use of multiple calculators because of the different capabilities of the various models of calculators and the lack of support that she was able to provide students with some calculators:

> I'm going to be using the TI-83...I will be teaching you how to use them [TI-82 and TI-83]. So, if you're not familiar with them, don't worry we'll be working on that...The TI-81, the blue one, it's a very old one. You will be at some disadvantage for the course if you have that, because I can't give you some stuff [calculator programs]. There are some programs that I want to be able to give you and I can't do that on the TI-81...So, if you have a blue one, consider buying a more recent one. TI-85 and 86 are the black ones. And, those are a little bit different. I will not be using them in class. I don't know how to run them, so I will be able to help you outside of class if you have those. However, if you are considering buying one, I would choose the 83 over the 86....The TI-83 is the one we're going to be using. The TI-92 is not allowed in this course...There's other calculators, the HP, Casio. If you choose to use those, you're welcome to. However, I don't know how to run them well. So, you're not going to get a lot of help from me.

As seen by her comments, Nellie did not like having all types of calculators in the class. Her concern was that students using a TI-80 or TI-81 were at a disadvantage. Nellie had programs, such as the quadratic formula, for the TI-82 and TI-83 but not for the TI-80, TI-81 or other types of graphing calculators. Therefore, students with the older versions of graphing calculators did not have the same capabilities of the newer versions.

Another concern for Nellie about students' use of multiple calculators was that she could not provide technical assistance during class for the TI-85, TI-86, and other types of graphing calculators because she was not sufficiently familiar with those calculators. In order to receive comprehensive assistance, students had to meet Nellie

during her office hours. During some lectures, Nellie deferred assistance, "I'll deal with the 85, 86 outside of class."

A further concern for Nellie was the class time squandered by checking students' graphs obtained on different calculators. When working problems that required the use of the graphing calculator, she had to take time to discern whether students could find features in calculators different from the model that she used. Even though the overhead projection device was used, the menus for calculators other than the TI-83 were different enough to cause confusion for students locating the feature accessed by Nellie. It was not uncommon to hear her say "85, 86, did you find it [feature of the calculator]?"

Nellie's allowance of multiple types of calculators in the classroom created three difficulties. First, some students had an advantage over other students because their calculator was more powerful, possessing more built-in features than the older models of calculators. Second, she could not equally assist students with accessing features of some of the types of calculators. Third, time was used up in class checking whether students could follow examples.

Since the graphing calculator was recommended for the precalculus courses but not the prerequisite courses, Nellie determined that an introduction to graphing calculators was necessary for the students. To assist students, Nellie provided an introduction to graphing calculators during the first week of the term and provided continual assistance throughout the observation period.

On Thursday and part of Friday of the first week, an introduction to the graphing calculator was provided to the students. During the class period, students worked in pairs to answer problems on the worksheet, Practicing the Basics of the TI (Appendix J), while Nellie walked around the room answering individual questions. The worksheet covered order of operations and input procedures such as overwrite, insert, store, and evaluate. Another focus on the worksheet was the use of parentheses. Many of the questions required the use of parentheses to evaluate the expressions correctly. Nellie considered student proficiency with the use of parentheses "extremely important. That's [incorrect or lack of use of parentheses] one of the major problems people have with the

calculators. If you do something wrong, about half the time, no matter what the problem is, a parenthesis is missing."

During the second class period for the introduction of the graphing calculator, Nellie guided the students through a couple of examples from a second worksheet, Graphing Practice on the TI (Appendix J). This worksheet addressed calculator skills for obtaining graphs of functions using the standard window. Additionally, a second group of functions provided experience with determining windows, other than the standard window, to view a complete graph. A complete graph was defined as a graph showing the important information such as intercepts, extremas, and end behavior (Demana & Waits, 1990). Guiding students through the worksheet, Nellie was at the front of the room using the graphing calculator overhead projection system.

The introduction of the graphing calculator was not exhaustive, because Nellie did not introduce all of the features that were used by students for the content covered during the observation period. Rather than introduce all of the features of the graphing calculator in the first week, she presented a brief introduction of calculation and graphing skills. Nellie did not want to "overwhelm" the students by introducing all of the features of the graphing calculator at the beginning of the term. Instead, she introduced relevant features of the graphing calculator when needed for a lesson. When finding zeros of a fourth degree polynomial in Lesson 10, Nellie said, "That's where you use your graphing calculator. So, let's get that out, and I'll show you some more functions your graphing calculator has, which makes it really nice." Then, she demonstrated the ROOT feature of the graphing calculator used to obtain zeros of functions. She also demonstrated the feature to find extrema of a function with a graphing calculator. In Lesson 14, she introduced the ZOOMSQUARE feature that changed the value of the axes in order to show a circle as a circle and not as an oval.

With regard to using the physical components of the graphing calculator projection system, Nellie expanded the system's capabilities. Rather than project the image on a screen, Nellie projected the image onto the whiteboard of the classroom. Thus, she was able to mark up the graphs to highlight the concept covered in the lesson. When she introduced a technique of zooming-in to obtain a complete graph of a function, she drew a box around the area of the graph that was to be enlarged. The students were

able to visualize the desired area of the graph that needed to be enlarged. Also, with the image of the graph on the whiteboard, she highlighted intercepts and zeros of functions. When determining the zeros of the function $g(x) = 4x^4 - 3x^2 + 2x - 7$, she obtained a complete graph of the function and highlighted the zeros.

To introduce extrema of functions Nellie used an application problem for finding the maximum area of a rectangle with a known perimeter. After developing the area function based on the width, x, $A(x) = (20 - x)x$, she projected a complete graph onto the whiteboard. Then, she used the graph to explain the formal definition of a maximum:

> So, let me define some things using this graph. A local or relative maximum, let me say, there is a local/relative maximum at $x = c$, if there exists an interval around c, an interval, I, around c such that $f(x)$ is less than or equal to $f(c)$ for all x in the interval. Now, this is a mathematical way of writing something down. You probably don't see this in your everyday life. Let me show you what it means here. What we are saying is that some interval around c [marked c on the x-axis], right? An interval from this value to this value [marked the interval on the x-axis]....If this is x, this is $f(x)$ [marked x and $f(x)$ on the axes]. This is c. This is $f(c)$ [marked $f(c)$]. $f(c)$ is bigger than $f(x)$ no matter which one I choose in here. However, all that means is that we've got the top of a bump on the graph.

In a similar way, the definition of an increasing function was introduced.

Furthermore, when making graphs by hand, Nellie used the overhead device to project axes onto the board. A problem asking to make a graph with a domain of $[-3,-1] \cup [1,5]$ and a range of $\{1,2,3,4\}$ illustrates this use: "So, let's graph this. In fact, let me use my calculator. You don't have to do this yourself, 'cause I just want to use the calculator to make a pretty graph. I'm just going to use the calculator for an axes coordinate system."

Also, making graphs more readable for application problems, Nellie labeled the axes and tickmarks of graphs that were projected onto the whiteboard. For an application problem about developing a graph of postage charges based on weight, she labeled the horizontal axis as the weight and the vertical axis as the charge. These labels helped students to read and interpret the graph.

Another use for a graphing calculator in the class was to evaluate mathematical expressions. While working a problem at the board during Lesson 3, Nellie called out for

assistance for "multiplying this [37^2] out." When attempting to calculate the value for $\frac{475}{3}$ for a step in the solution of an equation presented in Lesson 6, she requested assistance by saying "Help, someone with a calculator." For more involved calculations, Nellie liked using the calculator because "the calculator knows the order of operations" when the expressions were entered with appropriate use of parentheses.

However, a misconception was presented to students with regard to obtaining exact answers with calculators. Exact answers are answers that can be written symbolically in rational, exponential, or radical form. The decimal version of a value is not exact due to the rounding performed by the calculator. The expression $\sqrt{3\pi}$ is an exact value, but *3.06998012384* is a numerical approximation. Since many types of graphing calculators round to approximately 12 decimal places, solutions obtained with a graphing calculator are usually numerical approximations, especially for irrational numbers.

The misconception presented in the class was that the ANSWER feature of the calculator produced exact values. The ANSWER feature used the previous numerical result in the current calculation. However, Nellie taught that using this feature produced exact values. During Lesson 6 she commented "One nice thing on your calculator is that you can keep all of your decimals in there by using your answer button continuously." For intricate calculations within the quadratic formula, Nellie recommended students "keep the number in the calculator; it's exact there." Thus, students were informed that using the ANSWER feature of the graphing calculator was a method for obtaining exact values even though a graphing calculator produced numerical approximations.

With regard to viewing windows, Nellie considered the ability to obtain a complete graph of a function by adjusting the viewing window to be an important skill. "The thing with the calculator is that it doesn't know where you want the graph to be. You've got to tell the calculator where you want the graph to be." Also, she told students, "It's important, in some of the concepts we will be dealing with in this class, to be able to change the window and think about what the window should be." To motivate students to develop the skill for finding appropriate viewing windows, Nellie informed them that a calculator question was on the quiz: "For the calculator, make sure you can

graph with your calculator and find good windows. I'm not going to give you an extremely difficult one, but I promise you that I won't give you a graph that is in the standard viewing window."

The methods Nellie recommended to find a good viewing window required flexibility between representations of functions. Interpreting information from the symbolic expression to determine the window was recommended. She directed students to extract information from the expression in order to determine an appropriate window: "What you've got to be able to do is look at an equation and make some good 'guesstimates' on where it is. When x is zero, your y is 1324 [the function was $y = 50x^2 - 27x + 1324$]. So, if you are going to graph this by hand, I know that it crosses the y-axis at 1324."

Nellie also recommended using the "domain and range [to] help us set up our window." When working the application problem used for the introduction of the extrema of functions, she used the domain restriction of the application problem to find the window:

> And, I want to talk about what window is good. I want to talk about it in terms of the domain and range. Now let's look at the window or the range....It [domain of application] has to be positive, so it's going to start a little past zero. It can't be zero. Anything bigger than zero is fine. And, it can't be 20 either, because then we don't have any length....Okay, let's make our x that. We want to go from 0 to 20.

Besides interpreting the expressions and considering the domain restrictions of an application situation, Nellie provided students with another method for finding the window that did not require as much familiarity with the symbolic notation. During Lesson 12, she demonstrated the technique for finding the vertical dimensions for the window by entering the function into the y-min and y-max locations with the x-min and x-max values. "You don't actually have to put in a number. All I just did was plug 10 into x."

Throughout the observation period Nellie used and recommended using the graphing calculator as an alternative method for solving problems and as a means to check answers obtained symbolically. First, Nellie used the graphing calculator to find a

solution when a symbolic method was not possible. She used the graphing calculator as an alternative method for finding zeros of a function:

> If you have to find the zeros of a function, in essence, with an algebraic expression here, you're apt to set it equal to zero and solve. Now, algebraically we cannot do this. This is a fourth-degree polynomial. I haven't tried to factor it but my guess is, since I made it up off the top of my head, it doesn't factor. So, we're stuck. And there's where you use your graphing calculator.

The use of the graphing calculator was a secondary method when a symbolic method could not be implemented.

However, the alternative method did have its limitations with respect to what types of expressions could be graphed. After symbolically determining the symmetry of $x = 5 + y - y^2$, Nellie hesitated in her recommendation for the use of graphing calculators: "Well, that [$x = 5 + y - y^2$] will actually be hard to graph on this calculator as it sits." Later in the lesson she said, "This one we can't graph" when referring to the equation, $x = y^4 - 4y^2 - 4$. Even though the equation could be graphed by hand, Nellie implied that it could not be graphed because "For it to be a function, you've got to be able to solve for y." Nellie did not recommend the graphing calculator because most graphing calculators could only graph functions with respect to x.

The second approach for using graphing calculators was checking graphs made by hand and checking symbolic work. During Lessons 15 and 16, Nellie used a graphing calculator to check the results of whether a function was odd, even, or neither. "If you can graph it, give me a quick check. That's actually the easiest way to check [whether a function is odd, even, or neither]. The question is to test algebraically whether each function is even, odd, or neither even or odd. Then, check your work graphically using your graph here."

Throughout the observation period, Nellie emphasized using the calculator to check answers more than to obtain solutions. For many of the symmetry and transformation concepts she directed students to solve problems by hand, even when the solution was a graph. She did not want students to become reliant on their calculators: "Graphing is important to be able to do by hand even though you've got the graphing calculator. And, I want you to graph all of the graphs that I asked you to do by hand.

Don't just graph on the calculator and go on." This attitude persisted throughout the observation period. Nellie expected students to develop a familiarity with the standard functions used in the course. When presenting the lesson on transformations she informed students, "I want us to be able to graph [basic functions] at the drop of the hat without our calculators....So, I'm going to break you up and have each graph something by hand and then check it with your calculator....So, those are our functions that we want to be able to know what they look like just off the top of our heads." The graphing calculator was used as a means to check answers that were obtained by hand. Students were expected to have an idea of what the graph looked like before graphing it on the calculator. She reinforced this idea during the next lesson by commenting:

> Again, what our purpose for this [activity] is that we want to be able to get good at drawing a quick sketch of a graph, as well as, approximating or guessing what the equation of the graph might be. So, let's look at $f(x) = -\sqrt{x+4}$. Now, let's do it [sketching transformations] without our calculator first. Then, we'll check it with your calculators. Once you've done them, check it with your calculator to see if you indeed graphed them correctly. I want us to be able to get to the point so that we can graph these in the amount of time that I gave you. Be able to look at the graph and just see an approximation. These are not beautiful graphs. We are not talking about graphing nicely. I'm just talking about getting an idea.

Students were "supposed to see the graph in [their] head" before graphing a function on the graphing calculator.

A reason for Nellie's concern for students having a picture of the graph before they use a graphing calculator was how the calculator graphed discontinuous functions. During Lesson 8 on domain and range, she warned students about the vertical line that graphing calculators make when graphing rational functions in CONNECT mode:

> This line is not supposed to be there. The calculator is not as smart as we are. It plots points by plotting some points and connecting the dots. And what happens, the calculator was plotting all these points and it got to the point way down here. And, all of a sudden the next point it plotted was way up here. So it connected the dots. It's not supposed to connect the dots because there's not supposed to be a value at x equals 2.

Her concern about the vertical line that was drawn by the calculator was addressed again in Lesson 22. "The calculator is not perfectly graphing [the rational function]. That's called an asymptote. You get closer and closer but you never do cross it. That's

something that the calculator doesn't necessarily tell you. In your head you have to realize that this [the graph] actually comes down this way [approaches infinite along the asymptote]."

For graphs of functions that were not continuous everywhere, Nellie informed the students that the calculator was graphing the functions incorrectly by connecting points that should not have been connected. She emphasized that the students should have an idea of the graph before using the calculator. During Lesson 16, she demonstrated the technique for graphing the greatest integer function in DOT mode, the feature that does not connect the pixels:

> Remember how the calculator connects the dots. It's not suppose to. It's supposed to jump. I want to change my window so that it can get a little better look at this...you can turn it on to DOT. And, then it doesn't connect the dots. You get a better-looking picture. However, at the same time, you should know what it looks like without having to put it in DOT mode. But if you're in CONNECT mode, when you graph that, you already know that it connects the dots. You already know that it shouldn't be a connection. In your head you should see what it should do.

Different from the problem of connecting discontinuous functions, graphing calculators produced the opposite effect of not connecting points of a circle. At the end of Lesson 13, Nellie introduced the equation of a circle. The equation for the circle with radius of 1 and center at (-2, 3) was made. Then, she demonstrated graphing the circle on the overhead projection device. In order to graph the circle using the equation, two functions were derived from the equation, $(x-2)^2+(y+3)^2=1$. When the graph was made, Nellie pointed out that the graphs of the two parts of the circle did not connect: "Okay, there's the top one [half of the circle]. There's the bottom one. Now, it [graphing calculator] doesn't connect the dots because they're from two totally different functions. It doesn't know they're related. So in your mind, you've got to connect the dots." When, a student asked "Yeah, but shouldn't it start and end at the same point," Nellie described how the calculator graphs and finished the explanation with "Your calculator isn't that good."

Since the circle looked "kind of squished," Nellie demonstrated the ZOOMSQUARE feature of the graphing calculator. This feature adjusted the window so that the horizontal and vertical parts of the rectangular screen were properly proportioned

to show circles as circles and perpendicular lines as perpendicular. However, this feature had its limitations: "You still need to connect the dots, but it looks more circular." The graphing calculator did not connect the top and bottom halves of the circle.

During the next lesson, Nellie asked students to develop the equation of the circle with a diameter that had endpoints of (5, 3) and (-3, 4). After the equation, $(x-1)^2+\left(y-\frac{7}{2}\right)^2=\frac{65}{4}$, was developed, she directed the students to graph it on their calculator. Then, she demonstrated the ZOOMDECIMAL feature. This feature changed the window so that the horizontal tickmarks had values that terminated at the first decimal. Nellie used this feature to connect the top and bottom halves of the circle. However, for this particular example, the halves of the circle were not connected. After a few minutes of fiddling with the calculator, Nellie was not able to get the graph that she wanted: "And, you can see that I'm kind of stuck and I haven't played with it as much as I should have. And, it's not that important anyway. So, let's back up and go somewhere else."

The reason that the ZOOMDECIMAL feature did not work for the previous problem was that the horizontal diameter did not have endpoints with x-values terminating at one-decimal place. The horizontal endpoints were irrational numbers due to the square root in the function. For the ZOOMDECIMAL feature to produce a connected graph of a circle, the endpoints of the horizontal diameter had to be rational functions terminating at the first decimal place.

During Lesson 2 after introducing circles, Nellie made another attempt at demonstrating features of the graphing calculator to obtain a connected graph of a circle. This time she demonstrated the ZOOMDECIMAL feature using a different function, $x^2+y^2=4$.

> Okay, let's graph this. I figured out why this calculator wasn't working last week. Why don't you go ahead and get that [the equation of circle] on a graph. Make sure that you can do that on your own. I want you to graph this using the square [feature] on the calculator, and then we will do that decimal [feature] thing again....I'm going to graph it first on my own window $[-3,3]_x$ by $[-3,3]_y$. Then, I'm going to square it. Then, we are going to do that decimal thing again....Again, it looks oval. If you want to make it look square, you can ZOOMSQUARE it. And, that will square it

> off so that it looks a lot nicer. And now, let's do that ZOOMDECIMAL thing again. ZOOMDECIMAL. And, it will make it look pretty. But, what we want is the decimal to work with [indicates the endpoints of the horizontal diameter]. These guys [the endpoints] need to be nice numbers [rational numbers terminating at the first decimal].

A final incident arising from the limitation of the calculator to draw complete graphs occurred while students attempted a problem about inverse functions. Nellie gave students the problem of sketching the inverse of $f(x) = 2^x$, even though the students had not been exposed to exponential functions. She expected students to graph the function on their graphing calculator and, then, sketch the graph of the inverse by hand. A misconception was developed by many of the students because the graphing calculator produced a truncated graph when the standard window was used. The students mistook the truncated graph as the complete graph and sketched the inverse based on the incomplete graph. Nellie responded to these graphs:

> I saw a lot of graphs of what that [$f(x) = 2^x$] looks like [instructor drew the truncated graph] and it looks something like this [instructor drew complete graph of $f(x)$]. You've gotta realize that your calculator's not perfect. Even though it looked like it stopped, it didn't. Think about plugging some points in. So, again, don't trust your calculators. You're better than the calculator.

Without prior knowledge of the graph, the students developed misconceptions about the graph and sketched an incorrect inverse function.

In summary, the graphing calculator was used as an auxiliary tool during the course. Primarily, Nellie used the graphing calculator as a tool to develop solution strategies and to check answers. She used the calculator as an alternative method when symbolic approaches were unavailable. Finally, she cautioned the students to have an idea of the graph prior to using the calculator in order to avoid developing misconceptions that arise from the limitations of graphing calculators.

Students' Knowledge of Functions as a Group

Addressing the first research question, "What knowledge of functions do students gain in a College Algebra course requiring graphing calculators," the pretest and posttest of the Function Test (Appendix C) was administered to the students at the beginning and end of the observation period. For each question except the definition of function and example of function questions, the percentages of correct responses were calculated for the questions from both versions of the Function Test, as shown in Table 7.

Since the Function Test consisted of four sections of questions grouped by representation of functions, the use of statistical tests such as *t*-tests and ANCOVA were not applicable. Analyzing the results of the Function Test by comparing the means of the pretest and posttest would have masked differences in gains for questions in different representations of functions. Furthermore, a comparison of the means on the pretest and posttest for questions grouped on the basis of representation of function would have required numerous statistical tests, resulting in a higher probability of producing a Type I error. Thus, the percentages of correct responses for the pretest and posttest were analyzed using a chi-square test that corresponded naturally to the dichotomous grading procedure.

The result of the chi-square test ($\chi^2(18, N = 25) = 186.2, p < 0.001$) indicated that the percentages of correct responses were not equivalent for the pretest and posttest. The percentages of correct responses on the pretest and posttest were found to be significantly different. Although the results of the chi-square test were significant, blanket statements about the gain in percentage of correct responses for each question can not be made. Additional methods for distinguishing practical significant gains for individual questions were applied. The scores for the individual questions were examined using two criteria: (a) the percentage of correct responses on the posttest must exceed the expected value obtained from the application of the chi-square test and (b) the percentages of correct responses on the posttest must be above 70%. As a result of the chi-square test, a score on the posttest that was higher than the expected value was proportionally better than the score on the pretest. Due to the research question regarding differences in students' symbolic manipulation skills, a score of 70% on the posttest was selected as a minimum

Table 7. Percentage of Correct Responses for the Class on the Pretest and Posttest

Question	Pretest		Posttest	
	% Correct	Expected Value	% Correct	Expected Value
Identification				
1*	16	(41.4)	92	(66.6)
2	16	(27.6)	56	(44.4)
3*	16	(38.3)	84	(61.7)
4	8	(21.4)	48	(34.6)
Numerical				
7	100	(76.6)	100	(123.4)
8	100	(76.6)	100	(123.4)
9*	16	(35.2)	76	(56.8)
10	76	(61.3)	84	(98.7)
11	68	(50.5)	64	(81.5)
Graphical				
12	92	(73.5)	100	(118.5)
13	80	(68.9)	100	(111.1)
14*	20	(41.4)	88	(66.6)
15	76	(58.2)	76	(93.8)
16	28	(18.4)	20	(29.6)
Symbolic				
17	20	(19.9)	32	(32.1)
18	16	(16.8)	28	(27.2)
19	4	(7.7)	16	(12.3)
20	8	(18.4)	40	(29.6)
21	12	(19.9)	40	(32.1)

Note. $\chi^2(18, N = 25) = 186.2, p < 0.001$. The numbers in the %-columns are the percentages of students with correct responses for the corresponding question on the pretest and posttest of the Functions Test. The numbers in the expected columns are the expected values obtained from the chi-square test. Test questions marked with an asterisk had gains that were determined to have practical significance.

value for practical significance. For example, the gain for Question #21 (Table 7) was not deemed practically significant because the posttest score was 40%, even though this score was greater than the expected value. Because this score was below 50%, it could have been attributed solely to the students with high symbolic skills rather than the group as a whole. Seventy percent was chosen so that a combination of students from the high and low symbolic groups was required.

Identification of Functions on the Pretest and Posttest

With a gain of 76 percent (Table 7), Question #1, assessing students' abilities to identify functions graphically, showed the largest gain in the percentage of correct responses for any of the questions on the Function Test. The explanations for the responses by many students demonstrated an improved understanding for identifying functions graphically in their explanations for their responses. A few students determined that the scatterplot on the posttest was not a function "because when doing the vertical line test you pass through more than one point" or because "it [the scatterplot] did not pass the vertical line test."

Rather than use the vertical line test within their explanation for identifying the graphical relationship, approximately 65% of the students who answered correctly, gave reasons related to the formal definition of functions. "A function cannot have two range elements for the same domain element." And, "If you evaluate this problem by doing the vertical [line] test, you find that there are times when there are two y-values for an x-value." As an indication of the gain in students' understanding, a student who left the answer blank on the pretest responded on the posttest: "This [the scatterplot] is not a function because several x values share y values. In order for it to be a function a given x value must correspond exactly W/one [with one] y value."

With regard to the incorrect responses on the pretest for Question #1, explanations for why the scatterplot was not a function included that the data points were not continuous, the scatterplot was a set of points, or the points appeared "too random." Two students reasoned that "a function would have a continuous [*sic*] line or arc." Another

student said, "Since it is presented as a set of points, the vertical line test cannot be applied." Two other students reasoned incorrectly that "The points on the graph are random, scattered. They do not represent a function."

Another example of an incorrect explanation was a student considering the scatterplot to be a function because "an equation could be plugged in to give a graph looking like this." This justification was given after the student connected the points making the plot appear to be a graph of a polynomial function.

During the interviews students indicated that they had a preference for the graphical representation for identifying functions on the Function Test. Interview Student #1 from the LSM-MS group stated, "I'd just rather see it on a graph. Saying look, it has to pass the vertical line test. That [the written identification question] gets confusing, because to me there are too many variables in there. You can look at it too many different ways and flip it."

The results of the first mathematical task administered in the interviews also indicated that students had a preference for the graphical representation to identify functions. For the first task, students were given the function in symbolic, graphical, and numerical representations. Eight of the twelve students interviewed used the graph to identify the function. Unsuccessful with using the symbolic representation, one student switched to the graphical representation to finally identify the function.

With a gain of 68%, the percentage of correct responses for Question #3 on the pretest and posttest were found to be significantly different. Students used three approaches to identify the relationship given in the numerical representation. First, some students incorporated their definition of functions within their explanations. These students identified the relationship as a function "because every input has exactly one output." Another explanation was that the relationship was a function because "the coordinates are one-to-one."

Another method for identifying the numerical relationship in Question #3 on the Function Test was a graphical approach. Two students graphed the points and applied the vertical line test.

The most common approach used for Question #3 involved an algorithm. Students identified the number of times that each domain element was listed. This

approach was considered the numerical equivalence of applying the vertical line test. Students inspected the domain values and determined if a domain value was repeatedly listed. When a domain value repeated, students determined that the relationship was not a function. Students considered a function to have no repeated domain values. The following exchange with Interview Student #7 from the LSM-MS group demonstrated this method:

> R: What I want to ask you, first off, is this relationship a function?
> S: Um, yes.
> R: Okay, how did you determine that?
> S: Well, first I looked at each x-value. There's [*sic*] not two x-values.
> R: So, I don't have another -2 there? [The researcher examined the list of data to determine whether -2 was listed twice.]
> S: Right, and then here [the graph of the function] it passes the vertical line test.
> R: Okay, so first you went to the table looking for the x's to see if the x's were doubled up.
> S: Right.
> R: And, then you looked at the graph and did the vertical line test?
> S: Right.

This algorithmic approach for identifying functions used for relationships given in the numerical representation was not presented by Nellie in class and not found in the textbook. Nellie used arrow diagrams to determine whether relationships given numerically were functions. The textbook also used a correspondence method. For the coordinate pairs, {(9, -5), (9, 5), (2, 4)}, the textbook presented the following explanation. "The relation is not a function because the order pairs (9, -5) and (9, 5) have the same first coordinate and different second coordinates. The domain is the set of all the first coordinates: {9, 2}. The range is the set of all second coordinates: {-5, 5, 4}." (Bittenger et al., 1997) Thus, this algorithmic method that students used to identify the relationship given numerically was either developed by the students or obtained from outside sources.

As seen in Table 7, the gains for the two remaining questions within the identification section of the Function Test were not found to be significant. The percentage of correct responses on the posttest for Question #2 and Question #4 were 56% and 48%, respectively. Though the values for these two questions did not meet the

criteria for significant gains, information regarding students' lack of success warrants discussion.

Question #2 assessed whether students were able to identify functions given as a written statement. Some of the incorrect justifications that students provided for this question involved using irrelevant information, not distinguishing between a relation and a function, and requiring the situation to be one-to-one. A few of the students incorporated irrelevant information into the decision process. One student incorporated the delivery person into the justification: "Depending on what type of pizza you ordered and how long the drive is for the delivery person, the price might change." Another student introduced time into the justification: "The prices of shoes is [*sic*] different depending on the situation. Are they new, old, on sale, etc. and those may also depend on the # of shoes (6 old shoes could be less expensive than 2 new brand & design shoes.)" Also, one student responded, "without the price no one would take initiative enough for the pizza to be made."

Another incorrect justification within the responses for this question was that students did not distinguish between relations and functions. Four students did not consider the restriction on the correspondence between domain and range. The type of relationship was not addressed: "There is a relationship between price and pizza." A student wrote, "one relates to the other" and another wrote "because they are both related." Another student wrote, "The relationship between the type of pizza ordered has a direct (cause-and-effect) relationship on its price. Therefore, it would constitute a function."

Alternatively, a few students required functions to be one-to-one. On the pretest, one student wrote that the shoe situation was a function "because each pair of shoes has a different price." Another student wrote that the shoe situation was not a function because "Two shoes can have the same price. Again, not a one to one ratio [relationship], so not a function." A third student justified the shoe situation to be a function because "there are different kinds of shoes that have varying prices." Thus, one justification was that functions are one-to-one relationships.

Question #4 assessed whether students could identify a symbolic expression as a function. A piecewise function was used for the Function Test. One reason for the low

scores for this question on the pretest and posttest was that many of the incorrect student responses were actually blank. Eight students did not respond to the question on the posttest. Interview Student #4 from the LSM-B group remarked, "I still don't understand them." Additionally, though unable to identify the function three students stated that the expression looked familiar: "I think that it is [a function] because I saw something familiar in Math 90." Another student decided the expression was a function because it had "exactly the format of a function."

The small gain shown for this question may have been attributed to students refreshing their memories. The student who expressed being familiar with the form of the function responded on the posttest: "For each of the x (domain) from this piece-wise function, there is only one output." When asked about the change from a blank response on the pretest to a correct one on the posttest, Interview Student #5 from the HSM-LA group said: "I kind of forgot. I mean I learned it, but just forgot about how it works."

Results of the chi-square test supported by the explanations provided by students on the Function Test and by the interviews indicated that students were able to identify functions within the graphical and numerical representations better than within the written and symbolic representations. However, the difference for student performance with respect to representation of functions had an algorithmic aspect to it. The common approaches used to identify functions within the graphical and numerical representations were algorithms. The poor performance by students for identifying functions in written and symbolic representations may have been attributed to students inability to develop an algorithm within these representations. Additionally, familiarity with the representation was also a factor. Students expressed a lack of familiarity for the symbolic question and other students left it blank. Thus, two factors to students' success for identifying functions were access to an algorithm and familiarity to the problem.

Domain and Range

Only two questions from the application sections of the Function Test were found to have a significant gain for correct responses. Both questions assessed whether students

could determine the domain and range of functions. Question #9 and Question #14 asked for the domain and range for a function given in the numerical representation and the graphical representation, respectively. However, students did not demonstrate a gain in ability to determine the domain and range of the function given in the symbolic representation, Question #19.

One factor for the gains for Questions #9 and #14 was that students had learned interval notation. Interview Student #2 from the HSM-MS group commented about learning the notation, "This is just the proper notation. I guess I picked that up." Interview Student #9 from the HSM-B group did not respond to Question #14 on the pretest because "I couldn't think of how to write it [the domain and range]." However, both of these students were able to respond on the posttest using the interval notation appropriately.

Another factor for the gains in correct responses to the domain and range questions was that students learned to distinguish which set of values was the domain and which set of values was the range. Interview Student #6 from the HSM-MS group had difficulty with the numerical representation on the pretest due to an inability to determine whether the numbers in the first column consisted of the domain values or the range values. When asked about not responding on the pretest, the student said, "I knew what they [domain and range] were. I just couldn't remember which one was which, which was the x-value and which was the y-value." The exchange between the researcher and Interview Student #7 from the LSM-MS group further illustrated the benefit of the graphical representation:

> R: Okay, so one of the things that I'm seeing is that you picked up this idea of domain and range. How did you pick that up?
> S: I'm just learning it better. I've learned that the x-value is the domain and range is the y-value. I can visualize that. How far over and how far up the graph.

Interview Student #10 from the LSM-B group, also preferred the graphical representation for determining domain and range of a function. The student preferred identifying functions graphically because "domain is X and range is Y." The consistent format for domain and range within the graphical representation may have been the key for this student's success. For Question #9, the numerical identification question on the

Function Test, this student provided an incorrect response because functions presented in the numerical representation did not have a standard format for presenting the data since data could be displayed in columns or rows. The student had difficulties determining the order of the domain and range because of the various forms that the values were presented in the numerical representation. Thus, the consistent placement of the domain on the horizontal axis and range on the vertical axis provided a cognitive clue for students to determine which variable was the domain and which was the range.

The results of the first mathematical task given during the student interviews also demonstrated students' preference for the graphical representation. Four of the nine interview students who correctly determined the domain of the function given in the first task used the graphical representation. Initially, Interview Student #11 from the LSM-LA group had difficulty determining the domain of the function when using the numerical representation. However, after switching to the graphical representation of the rational function the student was able to determine the domain.

With respect to the symbolic representation, the lack of success for students determining the domain of the function was evident by the low percentage of correct responses, 16%, for Question #19, the domain and range question in the symbolic representation. Almost half of the students, 44%, did not provide a response.

One factor for the low score on this question may have been the context of the application problem because the interviewed students had better success for identifying the domain of a rational function not given in an application situation. Seven of the eleven interview students correctly determined the domain and range of the rational function in the second task by identifying the restriction within the denominator. Since these students demonstrated an ability to determine the domain of a rational function in symbolic form, the context of the problem on the Function Test may have been a factor for the low success rate for the domain and range application problem in the symbolic representation.

Although a few students were able to apply information from the application situation to determine the domain for Question #19, six students did not apply the information to the range. These students were able to determine the domain with the

restriction that money invested could not be negative values. But, they did not apply the restriction to the range as well and determined the range to be unbounded.

The results of the interview supported the results of the Function Test. Of the students interviewed, only Interview Student #1 from the LSM-MS group commented about determining the domain and range of the symbolic problem with information ascertained from the application situation:

> S: I realized what was going on inside the function. And I had to have this output, reasonable output, of positive dollars. Otherwise, it was invalid. Input had to obviously be positive dollars. I had to spend some money.
> R: Since it seemed to be a real life situation, you had a better idea of what to do?
> S: I guess I didn't really know how to put it. But, like on this one, the domain obviously had to be. The function wouldn't be realistic if you have negative 10 dollars or no dollars. So, I knew that. And, your range, I guess, I mean like the function could be like. Realistically, I thought it had to be a positive number too, because you know. I guess, theoretically you could spend so little that it would do no good. But I thought it would still give zero. So, I got that with logic and not with anything mathematically to come up with the domain and range.
> R: So, is domain and range for an application problem different than just an algebra, abstract problem?
> S: I definitely think so. It's just like the problem with throwing the rock off the top. You had a huge domain and range for the actual function because it would go on forever. But, it wasn't realistic, because all you needed was the ground and the building. Unless you dug a hole, you'd get a few extra feet. But, that's it. There is a difference in all. I think that, that is the hard part about math and taking the numbers that you need. And, not the numbers you don't [need]. And, understanding how to do that.

Thus, domain and range were concepts that students demonstrated a gain in performance at the end of the observation period for functions presented in the numerical or graphical representation. For the first mathematical task when students were presented the function symbolically, graphically, and numerically, students demonstrated a preference for the graphical representation. With regard to the numerical representation, students expressed initial confusion about the order of the domain and range due to the variety of displays for tables of data. Finally, students did not show much proficiency for determining domain and range when given a function symbolically. Students did not

necessarily use information given with the symbolic function for determining the domain and range. Thus, the representation of functions was a factor in students' success for determining domain and range.

Evaluation of Functions

Evaluation questions involved determining the range value for a given domain value and vice versa. Each of the application sections of the Function Test had two questions assessing students' ability to evaluate functions. The first question in each application section assessed whether students were able to determine the corresponding range value for a given domain value. The second question assessed students' ability to determine the domain value for a given range value.

Although the questions about the evaluation of functions did not meet the criteria for significant gains, issues regarding the percentages warranted discussion. First, the percentages of correct responses for Questions #7 and #8, the evaluation questions in the numerical representation, were 100% on the pretest and posttest. Thus, no gain was possible. Additionally, on the pretest the percentages for Questions #12 and #13, the evaluation questions within the graphical representation, were 92% and 80%, respectively. On the posttest the percentage was 100% for both questions. The results of the pretest and posttest indicated that students possessed the skill to evaluate functions in the numerical and graphical representations prior to the course.

However, the results of Questions #17 and #18 indicated that students had difficulty evaluating a function in the symbolic representation for an application situation. Evident in the written work on the pretest and posttest, many students did not convert the units of given values to the form required for the function. The symbolic function was given in terms of hundreds of dollars, where the questions were presented in terms of dollars. Twelve of eighteen students with incorrect responses did not convert $2000 into the appropriate input of 20 as illustrated by comments of Interview Student #1 from the LSM-MS group: "I can probably sit down and figure it out now, but I know that I need to put in 20 rather than 2000." Additionally, three students did not convert the units back to

dollars. These students gave the response, $210. Thus, 15 of the 18 students who responded incorrectly did not convert the units of the dollar amounts at the beginning or end of the solution process.

Even though students did not perform well on the symbolic evaluation questions, differences between the percentages of correct responses for the evaluation questions within the symbolic representation and the other representations could not be attributed to the course. The results of the Function Test and interview responses indicated students were initially proficient in evaluating functions given graphically or numerically. Furthermore, hindering the comparison of the results for the different representations of functions was the carelessness of the students for keeping track of the constraints in the application situation used for the questions presented symbolically.

Interpretation Difficulties

With regard to the interpretation questions, only Questions #10 and #15 had percentages above 70% on the posttest (Table 7). Both of the questions were about determining intervals where the function was increasing or decreasing. Question #10 and #15 presented the information numerically and graphically, respectively. However, the practical significant scores could not be attributed to a gain in understanding. The scores on the pretest indicated that students were already familiar with the concept of increasing and decreasing functions within the numerical and graphical representations. With regard to the symbolic representations, the scores on the pretest indicated that students were not successful at the beginning of the term. Yet, the posttest scores indicated only a slight gain. Therefore, students were familiar with the concept within the numerical and graphical representation and unable to develop a proficiency within the symbolic representation.

With regard to the other interpretation questions, difficulties for the students were evident. The scores on the posttest for Question #11 and #16 were lower than the scores on the pretest. Question #11 asked students to determine the day that the stock market was closed from the table of data. The students who responded correctly as well as the

other students who responded incorrectly were able to focus on the rows of the table that had the same data. Students were able to determine that the stock market was closed because "stock remained the same #'s." However, most of the students who did not respond correctly, selected the incorrect day of the two days with the same value. The most common mistake was listing the both the day when the market was closed and the day before when the market was closed.

For Question #16, the graphical interpretation question, many students demonstrated a lack of understanding of the physics for hitting a softball by not knowing that the most powerful hit would occur when the bat was at its highest speed. Many of the students thought that the time for the most powerful hit was when the hands and bat were going at the same speed. One student from the HSM-MS group wrote on the pretest, "the power of the swing would be greatest at the point when both the hands and the bat are traveling at the same speed." Another student from the HSM-MS group selected the intersection as the most powerful hit "because that is the time that the ball and bat should be to get the maximum power hit. That is when the hands and bat come together to hit the ball most powerfully." A student from the LSM-B group selected the intersection because "It is when the hands and bat are at the same speed causing the most force."

Another interpretation error for the graphical interpretation question was assuming that the most powerful hit would occur when the hands were at a maximum velocity. One student from the LSM-B group wrote, "Your hands are at their max velocity creating the max power. Your hands and bat are the most stable they can get (most force)." A student from the HSM-B group justified the selection of the maximum of the hands because "That's when the hands [are] at [their] highest." Even though the these interpretations were incorrect, many of the students read the graphs correctly for their false assumptions regarding the physics aspect of the problem.

However, some students not only had incorrect interpretations of the physics but also read the graph incorrectly. Selecting the intersection of the two graphs, a student from the LSM-MS group justified the selection because "At the time period of .14 seconds, both the hands and the bat reach their top speed. This would enable you to hit the ball at your best speed possible." A student from the HSM-MS wrote, "When bat

velocity and hand velocity intersect, both are peaked creating greater total force transferred to ball." Another incorrect graphical analysis was thinking that the sum of the graphs was greatest at the intersection. A student from the LSM-MS group wrote, "You want the ball to hit the bat when the bat & hands combined are at their greatest velocity." This student gave the time when the graphs intersected. These examples of two incorrect interpretations incorporated concepts from the course such as maximums and intersection of graphs, and combination of functions. Regardless of the faulty assumptions of the physics, these students incorrectly analyzed the graph.

With regard to the interpretation questions given in the symbolic representation, a correlation appeared between students successful on Question #20 and #21. All of the students, who responded correctly on Question #20, responded correctly on Question #21. As seen in Table 7, the discrepancy between the percentages for these two questions on the pretest was attributed to one student who made a simple arithmetic error in the calculation of the difference quotient on Question #20. Then for Question #21, the student correctly interpreted the negative value as a decrease in profit.

The most common mistake made by students on Question #20 was not recognizing the function notation. Most students were able to substitute the values for a and b. However, many students considered P to be a coefficient rather than the function's name. And, when simplifying the difference quotient, these students combined the values of a and b and reduced the fraction to get P. A glimpse of this approach was seen by the following explanation for Question #21: "No, it looks as if the amount of profit does not increase when the amount of money spent on advertising changes from 2000 to 5000. From problem 20 you can see that the two cancel each other out [the 30's in the numerator and denominator]."

Then, in Question #21, obtaining the result of P created some difficulty for students to interpret. Either they did not provide an explanation or they interpreted the result as the profit being constant. Three students similarly wrote "you are left with the same profit." A fourth student interpreted the result of P as the profit being constant but wrote the justification in the context for identifying inverse functions. The student referred to the function's name, P, as if it were the input variable x: "No [not increasing], because you end up with what you started with." Three other students did not provide an

explanation when getting P for Question #20. Additionally, for students who partially reduced the fractions and retained a symbolic expression such as $\frac{P(10)}{10}$, $\frac{P(40)-P(30)}{10}$, or $\frac{P(50)}{30}-\frac{P(20)}{30}$, 11 of 13 students did not respond to Question #21.

However, for the students correctly responding to Question #20 on the pretest and posttest, all of these students interpreted the result correctly for Question #21. Furthermore, some of these students were able to recognize that the difference quotient was the slope of the function and to include this concept into their explanations. A student from the HSM-LA group wrote, "No, the result in [Question] #20 indicates that there is a decline from 20 to 50 by a slope of -1. Since the slope is negative, we know that it does decrease." A student from the HSM-MS group wrote, "It decreases, as it shows a negative relationship in [Question] #20....The actual profit dropped from 210 to 180." A student from the HSM-MS group was able to explain the result in terms of the situation, "No, the given formula indicates that the more money that is spent on advertising, the smaller the profit."

Thus, no significant gain was found for the scores on the interpretation questions on the Function Test in the numerical, graphical, and symbolic representations of functions. However, the concept of increasing and decreasing functions within the numerical and graphical representations of functions was found to be a concept that students were familiar prior to enrolling in the course. Students not familiar with the application situation had difficulties with interpreting results even when they were able to extract the appropriate information. Finally, lack of symbolic skills hindered the interpretation ability for some students.

Definition of Function

The definitions of function provided by all of the students on the Function Test, were categorized similar to the methods used by Vinner and Dreyfus (1989) and Becker (1991). The categories included (a) correspondence, (b) dependence rule, (c) rule, (d) operation, (e) formula, (f) representation, (g) blank/I don't know, and (h) other.

Descriptions of the categories were covered in Chapter 3. In order to ensure proper categorization of the definitions, a mathematics educator familiar with the course was given the descriptions of the categories and sorted the students' definitions. The agreement rate between the researcher and the mathematics educator was 0.88. After discussions with the mathematics educator, the remaining three definitions were categorized in a manner agreeable to the researcher and the mathematics educator.

For each of the categories, the percentage of definitions assigned to each category was calculated. These percentages were used to make a 2×8 matrix with respect to categories of functions and with respect to the pretest and posttest (Table 8). A chi-square test was applied to the matrix to determine whether the categories were independent with respect to pretest and posttest. The results of the chi-square test indicated that the categories were independent ($\chi^2(7, N = 25) = 77.0,\ p < 0.001$). Thus, the percentages of definitions for each category on the posttest were statistically different than the percentages on the pretest, indicating that many students changed their definition.

Table 8. Categorization of the Students' Definitions of Function on the Pretest and Posttest

Category of function	Pretest (%)	Expected Value	Posttest (%)	Expected Value
Correspondence	4	(24.2)	44	(23.8)
Dependence rule	4	(2.0)	0	(2.0)
Rule	4	(4.0)	4	(4.0)
Operation	8	(8.1)	8	(7.9)
Formula	8	(10.1)	12	(9.9)
Representation	12	(16.2)	20	(15.8)
Blank/I don't know	48	(29.3)	12	(28.7)
Other	12	(6.1)	0	(5.9)

Note. $\chi^2(7, N = 25) = 77.0,\ p < 0.001$. The values in parentheses are the expected values for each category of function derived from the chi-square test.

Further analysis of the difference in definitions between the pretest and posttest for individual students indicated that 68% of the students changed their definition towards the formal definition: a function is a correspondence between a first set, called the domain, and a second set, called the range, such that each member of the domain corresponds to exactly one member of the range (Bittenger et al., 1997, 85). Even though most of this change was due to students who did not respond on the pretest, only three of the ten students providing a definition on the pretest did not change their definition towards the formal definition. Two of these three students responded on the pretest and posttest that a function must pass the vertical line test. For instance, a function is "something that, when graphed, passes the vertical line test."

Differences in Knowledge of Functions between the Six Student Groups

For the second research question, "What knowledge of functions do students of different levels of symbolic manipulation skills and with different academic majors gain in a College Algebra course requiring calculators," differences were found for the performance of students on a few questions within the Function Test. The student interviews corroborated the results of the Function Test regarding differences in performance between the students of the six groups: LSM-MS, LSM-B, LSM-LA, HSM-MS, HSM-B, and HSM-LA.

The percentages of correct responses for the questions on the Function Test were tabulated for each of the six groups. The percentages of correct responses on the pretest, on the posttest, and for the gain in scores were arranged into three tables. Each of the three tables can be found in Appendix K.

Since the ANCOVA on 19 questions from the Function Test would have produced a high probability for making a Type-I error, the three tables of scores were analyzed using the chi-square test. Additionally, the chi-square test corresponded well with the dichotomous grading procedure of right or wrong. The results of each of the three chi-square tests indicated that the table of scores for the pretest, posttest, and gain were independent with respect to group and question ($p < 0.001$). Since the results of a

chi-square test do not indicate which questions had significant difference, further analysis of the scores was required to identify relationships between the groups of students and the questions on the Function Test. Significance was determined by the method applied to the scores of the whole class on the Function Test based: (a) for practical significance the posttest score for the groups needed to be greater than 70% and (b) the gain needed to be greater than the expected value.

Differences in Responses of the Six Student Groups for the Identification Questions on the Function Test

The first four questions on the Function Test assessed students' ability to identify relationships as functions when given in one of the four representations of functions. In Question #1, the relationship was presented in the form of a scatterplot. Shown in Table 9, the gains in scores for the groups were similar to their expected scores except for the

Table 9. Percentage of Correct Responses on Question #1 for the Six Student Groups.

Test	Low Symbolic Skill Level (LSM)			High Symbolic Skill Level (HSM)		
	Math & Science	Business	Liberal Arts	Math & Science	Business	Liberal Arts
Pretest	25 (17.7)	0 (23.6)	0 (23.6)	14 (24.5)	100 (26.6)	0 (22.9)
Posttest	75 (73.2)	100 (82.5)	100 (93.8)	100 (106.0)	100 (127.6)	100 (91.9)
Gain	50 (57.3)	100* (60.6)	100* (60.6)	86* (78.8)	0 (114.6)	100* (64.0)

Note. The values in parentheses are the expected values obtained from the chi-square test applied separately to scores on the pretest, posttest, and gain. The gains in scores for the groups marked with an (*) were determined to have practical significance.

HSM-B and LSM-MS groups. Due to the high score on the pretest, no gain was expected for the HSM-B. However, the LSM-MS group did not have the expected gain in score as compared to the other groups for Question #1.

For Question #2, the identification question given in the written representation, gains in scores by the groups were mixed (Table 10). The HSM-B and LSM-MS groups had gains of 100% and 75%, respectively. The other groups had gains less than their expected values obtained from the chi-square test.

A difficulty for some students on Question #2 was linked to the concept of domain and range. Rather than thinking of the actual price that a person paid for a pizza or a pair of shoes, some students considered that all possible prices corresponded to a pizza or pair of shoes. For example a student from the HSM-LA group wrote, "Each pizza ordered can have different sizes and shapes with any number of toppings, thus, creating a different price for each of the accommodations." This student's misconception was comparable to stating that $f(x) = 3x + 6$ was not a function because the range contained more than one value even though the univalence requirement was preserved.

Table 10. Percentage of Correct Responses on Question #2 for the Six Student Groups.

Test	Low Symbolic Skill Level (LSM)			High Symbolic Skill Level (HSM)		
	Math & Science	Business	Liberal Arts	Math & Science	Business	Liberal Arts
Pretest	0 (13.3)	0 (17.7)	50 (17.7)	29 (18.3)	0 (19.9)	25 (17.1)
Posttest	75 (42.2)	0 (47.7)	50 (54.2)	57 (61.2)	100 (73.7)	50 (53.1)
Gain	75* (30.0)	0 (31.7)	0 (31.7)	28 (41.2)	100* (59.9)	25 (33.5)

Note. The values in parentheses are the expected values obtained from the chi-square test applied separately to scores on the pretest, posttest, and gain. The gains in scores for the groups marked with an (*) were determined to have practical significance.

Another difficulty students had on Question #2 of the Function Test was distinguishing domain and range. Some students transposed the domain and range for the situation. A student from the LSM-LA group determined the relationship between shoes and price not to be a function "because different pairs of shoes could cost the same amount. Then, you would have more than one output for an input." A student from the LSM-B group did not identify the function because "two different shoes could be the same price."

A third difficulty for students regarding the identification of the function given in the written representation was connected to the one-to-one concept. One student from the HSM-MS group determined the pizza situation to be a function because "no two numbers [prices] are shared." And, this student determined the shoe situation to be a function because "each pair of shoes has a different price." This student had the limited view that functions were required to be one-to-one. Another student had the misconception that a one-to-one relation was not a function: "Two shoes can have the same price. Again, [it is not] a one-to-one ratio [relation], so not a function."

Thus, students had difficulties with identifying the function in the written representation because they could not distinguish aspects of the domain and range. Six students had difficulty with the direction of the univalence requirement of functions. Two students associated the whole set of range values to each domain value. And, three students limited the relationship to be one-to-one.

As seen from Table 11 providing the gains in scores on Question #3, the numerical identification question, two of the LSM groups did not perform as well the other groups. Of the LSM groups, only the LSM-LA group had a gain in scores greater than its expected value. The LSM-MS had a gain below its expected value and the LSM-B group had a decrease in scores. Regarding the gains for the HSM groups, the gains for the HSM-MS and HSM-B groups were not above their expected values because the scores on the posttest were 100%. The upper bound from the use of percentages produced a gain that was less than the expected value because the chi-square test does not take into account of the upper bound of 100.

Table 11. Percentage of Correct Responses on Question #3 for the Six Student Groups.

Test	Low Symbolic Skill Level (LSM)			High Symbolic Skill Level (HSM)		
	Math & Science	Business	Liberal Arts	Math & Science	Business	Liberal Arts
Pretest	0 (16.5)	100 (21.9)	0 (21.9)	29 (22.7)	0 (24.7)	0 (21.3)
Posttest	50 (63.6)	50 (71.8)	100 (81.6)	100 (92.2)	100 (110.9)	100 (79.9)
Gain	50 (55.3)	-50 (58.6)	100* (58.6)	71* (76.1)	100* (110.6)	100* (61.8)

Note. The values in parentheses are the expected values obtained from the chi-square test applied separately to scores on the pretest, posttest, and gain. The gains in scores for the groups marked with an (*) were determined to have practical significance.

Although the identification of functions in the numerical representation was not a skill that was assessed in the tests and quizzes, the instructor did consider it a valid concept for students to understand. Due to the amount of material in the course, the instructor was unable to assess every concept covered in the course. Thus, analysis of the methods used by students to identify functions in the numerical representation was warranted.

For Question #3, the numerical identification question on the Function Test, students typically used one of four approaches: (a) applying the definition of functions, (b) determining whether the relationship was one-to-one, (c) graphing the coordinates and applying the vertical line test, and (d) identifying whether domain values repeated. The two students who used a graphical approach to correctly identify the numerical relationship as a function were both from the HSM-LA group. Regarding the other approaches for Question #3, no clear pattern was evident between the approach and group, LSM and HSM. However, the LSM and HSM groups differed by the successful application of the chosen method. For example, Interview Student #4 from the LSM-B group who did not properly identify a function confused the direction of the univalence requirement in the definition of function:

> Looking at the x-values, the f of whatever to $f(-1)$ is 0. Seeing that there are no similar y-values between them. Seeing that $f(-2)$ is -4. $f(2)$ is -3. Seeing that they are not the same y-values. If $f(-2)$ and $f(2)$, both, well actually that's $f(-1)$ and $f(-2)$ both have the same as $f(-4)$, it wouldn't work. It would be two y-values for that or there would be one y-value for two x-values.

For Question #4, the symbolic identification problem, the gain in scores for the LSM-LA group was the only group to exceed its expected value (Table 12). The explanations given on the Function Test by the students in the LSM-LA group indicated an understanding not evident in the explanations by the students in the other LSM groups. One student in the LSM-LA group examined the individual parts of the piecewise function to identify the function: "Because each of the three formulas [within the piecewise function] is a function. So, no matter what number x is, it [the piecewise function] will always make a function." Another student in the LSM-LA group responded in terms of the definition of functions: "For each input you get one output, answer." Except for one student in the LSM-MS group, all of the other students within a LSM group did not provide responses for this question.

Table 12. Percentage of Correct Responses on Question #4 for the Six Student Groups.

	Low Symbolic Skill Level (LSM)			High Symbolic Skill Level (HSM)		
Test	Math & Science	Business	Liberal Arts	Math & Science	Business	Liberal Arts
Pretest	0 (3.7)	0 (4.9)	0 (4.9)	29 (5.1)	0 (5.5)	0 (4.8)
Posttest	25 (34.5)	0 (38.9)	100 (44.2)	71 (50.0)	50 (60.1)	25 (43.3)
Gain	25 (31.8)	0 (33.7)	100* (33.7)	42 (43.8)	50 (63.6)	25 (35.5)

Note. The values in parentheses are the expected values obtained from the chi-square test applied separately to scores on the pretest, posttest, and gain. The gains in scores for the groups marked with an (*) were determined to have practical significance.

With respect to gains in scores, there was no definite support to indicate that gains in performance differed for the LSM and HSM groups on the identification questions on the Function Test. There was no recognizable pattern for comparing the gains of the LSM groups and the HSM groups. However, the LSM-LA group had significant gains for three of the four identification questions. The HSM-MS group was nearly comparable to the LSM-LA group with significant gains for two of the problems and near significant gain for a third identification question on the Function Test. The gains on Question #4 for the HSM-MS group was not considered significant because the gain was lower than the expected value by 1.8 percentage points.

Differences in the groups' performance was most pronounced on Question #3 where the LSM-LA group was the only LSM group to show significant gain. Yet, all three of the HSM groups showed significant gain in performance for the identification question in the numerical representation. For the identification questions in the graphical, written, and symbolic representations, no recognizable pattern was found with respect to comparing LSM groups to the HSM groups.

Relationship between the Two Business Groups and the Types of Application Situations

Within the application section of the Function Test, two of the three sets of questions were situations related to business. The application questions given in the numerical representation, Questions #7 through #11, consisted of a table listing the values of three stocks. The application questions given in the symbolic representation,

Questions #17 through #21, involved profit for a business. As seen in Table 13, the LSM and HSM business groups performed well on the posttest for Questions #7 through #11, except Question #9. The HSM-B group scored 100% on all of the application questions in the numerical representation. The LSM-B group scored 100% on all of the questions except for Question #9, the domain and range question presented numerically. The students in the LSM-B group either did not respond or confused range with domain.

Table 13. Percentage of Correct Responses for the Business Groups on the Numerical Application Questions within the Pretest and Posttest

Business Group	Question Number				
	7	8	9	10	11
Pretest					
LSM	100	100	0	50	100
	(102.1)	(102.1)	(10.9)	(74.2)	(88.6)
HSM	100	100	50	100	100
	(114.8)	(114.8)	(12.2)	(83.4)	(71.8)
Posttest					
LSM	100	100	0	100	100
	(86.1)	(86.1)	(61.0)	(73.4)	(71.2)
HSM	100	100	100	100	100
	(133.1)	(133.1)	(94.3)	(113.4)	(110.0)

Note. The values in parentheses are the expected values obtained from the chi-square test applied separately to scores on the pretest, posttest, and gain.

For Questions #20 and 21, the symbolic interpretation questions, both of the business groups scored 100% on the posttest (Table 14). Even though the students in the LSM-B group were not able to answer the first three questions in the symbolic application section of the Function Test, they were able to evaluate the difference quotient correctly. The common mistake made by the students in the LSM-B group on Questions #17 and #18 was not converting the units of the domain and range values in terms of $100. However, since Questions #20 and #21 did not require students to convert the units, the students from the two business groups demonstrated proficiency with the symbolic representation. They were able to perform the symbolic manipulations in Question #20 and were able to interpret slope correctly in Question #21. A student from the LSM-B group, who missed the evaluation problems by not converting the values to hundreds of dollars, interpreted correctly the negative slope: "No, they lose $100 more money in the problem."

Table 14. Percentage of Correct Responses for the Business Groups on the Symbolic Application Questions within the Pretest and Posttest

Business Group	Question Number				
	17	18	19	20	21
Pretest					
LSM	0 (7.3)	0 (7.3)	0 (2.4)	0 (6.6)	0 (6.6)
HSM	0 (8.2)	0 (8.2)	0 (2.7)	0 (7.5)	0 (7.5)
Posttest					
LSM	0 (27.7)	0 (24.1)	0 (16.4)	100 (40.5)	100 (40.5)
HSM	100 (42.8)	100 (37.3)	50 (25.3)	100 (63.6)	100 (63.6)

Note. The values in parentheses are the expected values obtained from the chi-square test applied separately to scores on the pretest, posttest, and gain.

However, the business groups' performance on the interpretation questions in the graphical representation, Questions #15 and #16 (Table 15), was not as remarkable as their performance on the interpretation questions given in the numerical and symbolic representations. Questions #15 and #16 required the students to interpret a graph representing the speed of a person's hands and bat when hitting a softball. The students from the LSM-B and HSM-B groups had difficulty interpreting the physics of the example. The most common misconception exhibited by their explanations was thinking that the maximum force applied by the bat to the softball was at the time when the hands were at a maximum speed. A student from the LSM-B group, who responded correctly on the pretest, changed the response on the posttest to a time at the end of the swing. "This [the time at 0.22 seconds] is when the bat is three-quarters the way through with its swing" and the bat will transfer its energy to the ball "taking it farther."

Table 15. Percentage of Correct Responses for the Business Groups on the Graphical Application Questions within the Pretest and Posttest

Business Group	Question Number				
	12	13	14	15	16
Pretest					
LSM	100 (99.7)	100 (86.2)	0 (15.1)	100 (86.2)	50 (36.4)
HSM	100 (112.1)	100 (97.0)	0 (17.0)	100 (97.0)	50 (41.0)
Posttest					
LSM	100 (86.1)	100 (86.1)	100 (71.8)	50 (64.6)	0 (23.5)
HSM	100 (133.1)	100 (97.0)	100 (110.9)	50 (99.8)	50 (41.0)

Note. The values in parentheses are the expected values obtained from the chi-square test applied separately to scores on the pretest, posttest, and gain.

Students from the LSM-B and HSM-B groups scored consistently well on the interpretation questions that were business-related and had gains on the symbolic interpretation questions that were greater than the expected gains calculated by the chi-square test. However, on the graphical interpretation questions that were not business-related, the two groups did not have the same success. Thus, for the business groups, a relationship appeared to exist between the types of application situations and students' success.

Flexibility between Representations of Functions

Students demonstrated flexibility with regard to the representations of functions through proficiency for finding solutions within a given representation or by working between different representations. A student who could identify a function given in the

written representation by translating to the graphical representation but who could also identify the function within the written representation demonstrated flexibility with respect to representations of functions. However, a predominant reliance on a single algorithmic approach within a particular representation did not indicate flexibility between representations of functions. For instance, rigid use of the vertical line test for identifying functions even when functions were given in representations other than the graphical representation did not demonstrate flexibility.

Initially, many of the results of the Function Test and the student interviews indicated that students from both groups demonstrated flexibility between representations of functions. However, with regard to the symbolic representation, further study of the results indicated that the students from the LSM groups demonstrated less flexibility than students from the HSM groups. The results indicated that students from the LSM groups demonstrated a lack of preference for the symbolic representation. In comparison, the students from the HSM groups demonstrated flexibility through their willingness to work within the symbolic representation as well as the other representations.

For the examples and nonexamples of functions given by students on the pretest of the Function Test, differences in preferences towards certain representations of functions were found between the student groups. On the pretest, half of the LSM students gave graphical examples and nonexamples (Table 16). Only one of the LSM students gave an example of functions within the symbolic representation. In comparison, all but two of the HSM students provided examples given in the symbolic representation.

Although seven students provided nonexamples in the symbolic representation, all of the nonexamples were incorrect for the six students from the HSM groups and the one student from a LSM group. The nonexample function given by these students involved the square root. The students did not include the plus-minus symbol to make the expression non-representative of a function.

On the posttest, students from both groups provided examples and nonexamples that were similar to the written situations presented by the instructor during lessons. A student from the HSM-LA group provided the following example of a function: "Students to desks, there can only be one student in a desk." And, the nonexample was

Table 16. Categories of Examples and Nonexamples of Functions

Representation Group	Pretest		Posttest	
	Example	Nonexample	Example	Nonexample
Symbolic				
HSM	8	7	6	6
LSM	1	1	1	1
Graphical				
HSM	0	0	0	1
LSM	3	3	2	2
Written				
HSM	0	0	6	3
LSM	2	2	4	4
Arrow				
HSM	2	2	1	2
LSM	0	0	1	1

Note. Two students from LSM groups and three students from HSM groups did not provide examples on the pretest.

"Students to color of shirts, two students could be wearing the same color shirt." In comparison to responses given on the pretest, six students from both groups changed their examples and five of the six students changed their nonexamples to the written representation from other representations used on the pretest (Table 16).

Even though the number of examples and nonexamples given the written representation increased more than the other representations, the symbolic representation was still used predominantly by HSM students. Only one LSM student used the symbolic representation.

In isolation, the examples and nonexamples of functions provided by the HSM and LSM groups did not demonstrate conclusively that the groups differed with respect to flexibility. Yet, the examples and nonexamples did provide a piece of evidence that fits with results from other sources. The student interviews provided support about

differences between the student groups in terms of flexibility between representations of functions.

Students in the HSM groups demonstrated flexibility between representations of function when determining whether a function was continuous. During the first task, students were presented the function in the symbolic, graphical, and numerical representations. During the second task, students were presented the function in only the symbolic representation. For each task, the number of students working within each of the representations of functions was tabulated. As seen in Table 17, during the first task, five of six of the interview students from the LSM groups used the graphical representation to determine whether the function was continuous. Whereas, only half of the interview students from the HSM groups used the graphical representation.

However, when the function was given in the symbolic representation for the second task, five out of six students from the HSM groups were able to determine the continuity of the function by using the symbolic representation (Table 17). The students' ability to adjust to the given representation demonstrated flexibility. On the first task, at least one student from the HSM group chose one of the different representations of function. When the function was given symbolically, most of the students from the HSM groups were able to adjust to the given representation.

Table 17. Representation Used by Students to Determine Continuity during the First and Second Interview Tasks

Representation Group	Interview Task	
	First	Second
Symbolic		
LSM	1	3
HSM	1	5
Graphical		
LSM	5	3
HSM	3	1
Numerical		
LSM	0	0
HSM	2	0

On the contrary, the LSM groups demonstrated little flexibility because only two students adjusted to the representation in which the function was presented. Though the function was given symbolically, interview students from the LSM groups continued to use the graphical representation to determine continuity. Whereas, five of the six HSM students were able to work within the symbolic representation, half of the LSM students needed to translate the function into the graphical representation in order to determine continuity. Interview Student #10 from the LSM-B group expressed frustration for working within the symbolic representation: "I have no idea of how to interpret it [the function given symbolically]. Is this a trick question?"

The students in the HSM groups indicated a preference for the symbolic representation. Interview Student #2 from the HSM-MS group stated, "I do like algebra. I've always enjoyed that. Algebra is my favorite. I really like it." Also, Interview Student #9 from the HSM-B stated, "I particularly liked the algebra. I just use that [algebra] more." However, students from the HSM groups also commented about their facility to work within the graphical representation. Interview Student #12 from the HSM-B group stated, "I guess it's [symbolic representation] funner [*sic*]. I liked playing with the numbers more than looking at a graph. But, I don't mind looking at graphs." Furthermore, Interview Student #6 from the HSM-MS group, the only student from a HSM group to use the graphical representation to determine the continuity of the function for the second task, commented that "Graphs are easier to work with, for me anyway." Although the HSM students had a preference for the symbolic representation, they were willing to work within other representations of functions.

On the other hand, students from the LSM groups preferred working within the graphical and numerical representations and not within the symbolic representation. Both Interview Student #4 from the LSM-B group and Interview Student #7 from the LSM-LA group preferred the graphical representation because it was more "visual." Interview Student #11 from the LSM-LA group expressed a preference for the numerical representation because tables were more exact, as seen in the following exchange:

> R: Is there any particular one of those [representations of function] that you had a preference for?

S: The first one [numerical representation] because it was good. It was all exact answers. Like on the second one [graphical representation], you had to kind of look and trace your way up to about where it was on the graph.
R: Okay, so the more precision, the better, the more comfortable you felt.

Flexibility for working in representations of function was affected by students' confidence for working within the symbolic representation. Interview Student #2 from the LSM-MS group expressed difficulty working within the symbolic representation: "I guess it's harder for me to look at an algebraic thing and decide whether it's [a function]. I'm not confident whether you gave me a bunch of $g(x)$'s or $f(x)$'s. I'm not confident if I understand enough to just look at the algebra." Furthermore, even though Interview Student #10 from the LSM-B group did not express a preference for a particular representation, the student used only the graphical representation to answer all of the questions for the two tasks.

As a final indication of a lack of flexibility, half of the interview students from the LSM groups interviewed expressed that the three representations of functions were separate entities. At the beginning of the first task, interview students were presented the function given in the symbolic, graphical, and numerical representations. The researcher informed the interview students that the expression, graph, and table represented the same relationship: "What I've got is a relationship in the different representations. It's the same relationship, but just in symbolic, numerical, and graphical form. So, my first question is, is this relationship a function?" Yet, when presented the graphical, numerical, and symbolic representation of the function, three of the six LSM students interviewed had difficulties recognizing that the three representations were of the same function. Interview Student #1 from the LSM-MS group was unsure whether the representations were the same function, as illustrated by the following exchange:

R: Is this relationship a function?
S: Between all three of them?
R: All three of them are the same relationship. Just different.
S: Representations. Are they a function?

Then, when asked whether there were other options besides the vertical line test to identify the function, this student continued to speak as if the representations were

different entities rather than the same function, "To figure out, if I'm given these two, are these a function?"

When presented the first task, Interview Student #13 from the LSM-LA group asked, "They're all the same?" Interview Student #4 from the LSM-B group had the same difficulty of viewing the representations as separate entities, as shown in the following exchange:

R: Is this relationship a function?
S: These three relationships together?
R: Yea, they're all the same relationship. Just different ways of presenting it. So, is the relationship a function?
S: So, these two, undefined at 2. Yeah.
R: How, in what way did you determine that it was a function?
S: I'm not so sure about this one.

In summary, the students in the HSM groups exhibited flexibility for working within and between representations of functions. They were less dependent than the LSM students on the graphical representation and were able to adjust to the given representation. The comments of the HSM students indicated that they had a preference for the symbolic representation but were willing to work within the other representations of functions. During the planning stage of students' solution process, the HSM students demonstrated the ability to distinguish between representations for an optimum approach to the solution, whereas, the LSM students limited their approaches to the graphical and numerical representations. Additionally, the LSM students viewed the different representations of the same function as separate entities.

The Role of the Graphing Calculator

Mathematics educators have recommended the use of graphing calculators to aid students in their development of an understanding of functions. However, information concerning how students use graphing calculators is needed in order to utilize them as effectively as possible. Interview Student #2 from the HSM-MS group stated that the graphing calculator was "definitely a tool that can help you. Yet, you need to know the basics [about using graphing calculators], so you know when it is wrong." This section

addresses the research question, "What role do graphing calculators play in the classroom as students develop an understanding of functions?"

The graphing calculator as a tool helped students in three main ways: (a) an alternative method for obtaining solutions to mathematical problems, (b) a guide for students during the planning stage of the solution process, and (c) a reference or resource for students to check their solutions. Similar to the benefits obtained from any tool, the graphing calculator helped save time for the instructor and students to perform calculations and develop graphs of functions. However, for the proper use of any tool, training was an important issue that was observed in class and raised by students during the interviews. Otherwise, without a good understanding of the limitations of the tool, the graphing calculator was a source for students' misconception.

One consideration for the use of graphing calculators in the precalculus course was whether students were sufficiently prepared to use it. As discussed earlier, the instructor considered student proficiency with graphing calculators an important issue. So, two class periods during the first week of the term was devoted to introducing the features of the graphing calculators. Also, throughout the observation period, Nellie provided much assistance during class periods to students on the use of graphing calculators. Rather than assume the students were proficient with the advanced features of graphing calculators, Nellie demonstrated features of graphing calculators when the corresponding mathematical topic was covered in class. With regard to the timing of the demonstrations, Nellie connected the features of the graphing calculator with the mathematical topics. Additionally, Nellie continuously invited students to visit her during office hours for further help on learning the features of graphing calculators. Thus, student proficiency with graphing calculators was important to the instructor as evident by the time spent assisting students learn the use of their calculators in and outside the classroom.

Background information gained from the cover sheet of the pretest indicated that training on the use of graphing calculators was necessary because few students perceived themselves as proficient with their calculators. In response to the question, "How proficient are you with using your calculator," only three students claimed to be "very" proficient with their calculators. Two students claimed to be "not at all" proficient. The

remaining students expressed themselves to be "somewhat" proficient with their calculators.

Supporting the classroom observations and the background information, students during the interviews expressed concern about their proficiency with their calculators. Interview Student #13 from the LSM-LA group expressed concern about the time that it took to become proficient with the graphing calculator: "I mean, it took me half the quarter just to learn how to use them [graphing calculators]." The time that it took this student to become proficient with the calculator hindered the student's grade because the first quiz assessed students' abilities with graphing calculators. Interview Student #1 from the LSM-MS group also was concerned about how the calculator impacted success on tests:

> There's a lot to learn on the calculator....The graphs come quickly now, but not before the first test and, actually, the first two tests. For the first one especially, I had a problem with my calculator. And, I didn't do well on the test because it threw me off. Everything, it can be very frustrating using it. It's just a matter of getting your window right and everything like that, which now seems much easier. But, I do notice that every once in awhile I'll get stuck on time. And I just can't think clearly to get it. But, you know, if your window gets wrong, it's hard to get a good reading, a good answer.

This student's comments about the importance of obtaining appropriate viewing windows for graphs of functions supported the class observations. As stated earlier, Nellie considered the ability to find appropriate viewing windows to be an important skill. She demonstrated various methods for finding appropriate viewing windows. One method was to interpret the symbolic form of the function for information, such as the y-intercept, in order to find a complete graph of the function. Also, she demonstrated a method for entering the function into y-min and y-max with the corresponding x-min and x-max values. Thus, the skill for finding viewing windows of complete graphs was a skill that was important to the instructor and the students.

Since proficiency with graphing calculators was an issue addressed by the instructor and students, effective methods for training students were needed. Interview Student #6 from the HSM-MS group favored the method that Nellie used to teach the

features of the calculators. Rather than teach all of the features of the graphing calculator at the beginning of the term, the student recommended introducing features when needed:

> R: So this was the first term using the calculator. Did you have any difficulty learning the calculator at the same time as the material?
> S: This is all straightforward. We didn't get into any part of the calculator that wasn't relevant. We didn't go off just playing with it. Maybe if we've done that, it would have been too overwhelming. But, just as every topic that we went through, [Nellie] would go back and show us, here's where you could use the calculator. And, then doing the problems and using the calculator to double-check. It kind of runs it through your mind at the same time. You learn all of it at the same time. I know how to do the math and I know how to double check, because we've done it over and over again.
> R: Okay, I've got two ways of doing the calculator, introducing it. I can take two days in the beginning, and just show all of the techniques and all of the steps of the calculator or I can spread that out and put it in when I need it.
> S: Yeah, I would do it section by section. Yeah, if I'm covering the basic functions, I would start with the algebra behind the basic functions. Show what they are. And, then, when you start working the problem, and have the students working the problems and then show them this is how to double check it on the calculator. Then you spend however long it takes to show them that particular example. And they work both at the same time. They learn both how to do the algebra and to use the computer. They're doing the algebra and going back to double check on the calculator. They can learn the process for both of them.

After becoming proficient with the calculator, students commented that its use had reduced the time for doing their mathematical work. For instance, Interview Student #9 from the HSM-B group commented, "It [graphing calculator] helps a lot without actually drawing out the graphs by hand. That's time consuming. Maybe for the first couple of times so that you know how to do it [graph by hand] and, so you can use it [graphing calculator]. It's just faster and allows the course to proceed at a faster rate." The student's comments about graphing by hand for the "first couple of times" corresponded to the class observations. Nellie wanted the students to be familiar with basic functions such as linear, quadratic, cubic and absolute value functions. During one class period, Nellie had different students draw the graphs of basic functions on the board without the use of graphing calculators. Nellie wanted the students to "see the graph in [your] head" before accessing graphing calculators. Thus, with regards to graphing basic

functions, students used graphing calculators because they produced graphs quickly rather than because they could not produce them by hand.

Besides graphing the basic functions quickly, graphing calculators were a quick resource for graphs of complicated functions different from the basic functions. The graphing calculator was a tool to obtain graphs in a manner more quickly and accurate than ones developed by hand. For instance, Nellie used the graphing calculator to produce a graph of a function representing the total area enclosed by a fence in order to find the dimensions that produced the maximum area. The graphing calculator allowed Nellie to keep the students focused on the topic of extrema of functions and not get distracted by the steps for graphing the function by hand.

During the interviews, students indicated that graphing calculators were helpful because of the quick and accurate graphs that calculators produced. When asked how the calculator was helpful, Interview Student #5 from the HSM-LA group responded: "I can put any equation in. See what it looks like on the graph. For some real complicated functions, we can see patterns that I wouldn't see just by doing it by hand." This student used the calculator because it provided graphs that were beneficial for discovering concepts that would have been missed if graphed by hand. Interview Student #11 from the LSM-LA group liked the security of the accurate graphs produced by calculators: "If I didn't have my calculator, it would force me to graph each thing by hand. Then, I might graph things incorrectly." Students used the calculator because it provided them with accurate information more quickly and reliably than if they graphed them by hand.

In addition to graphing functions, students used the calculator to evaluate mathematical expressions, especially the quadratic formula. Occasionally, Nellie requested students to perform calculations that arose within the solution process of examples that she demonstrated. To calculate the solutions of quadratic equations, students accessed programs or features in graphing calculators. Additionally, the calculator helped reduce anxiety for obtaining complex solutions of quadratic equations. Interview Student #13 from the LSM-LA group used the calculator to avoid finding complex solutions because "I'm scared of them...complex numbers." Thus, graphing calculators reduced time spent on developing graphs and performing calculations. Also, it reduced anxiety for students.

Besides using the calculator to produce graphs or to evaluate expressions, the calculator was used by students in the classroom in three significant ways: alternative method, check, and guide. As discussed earlier, Nellie used graphing calculators to find solutions to problems that were not solvable with the symbolic skills covered in the course. Due to the goals of the course, finding the zeroes of polynomial functions with degrees larger than two and determining the extrema of functions were not solved using symbolic approaches. The necessary symbolic skills using derivatives were introduced in a later course. One method that Nellie used graphing calculators was as an alternative method to the symbolic approach.

Additionally, Nellie advocated that students use the graphing calculator to check answers. For instance, she requested students to use their graphing calculator to check solutions when determining whether a function was even or odd. Also, after determining the symmetry of a function used as an example for the introduction of the symmetry topic, she requested that students check their results with graphs produced by the calculator.

The results of the student interviews corresponded to these two methods and introduced a third method, the graphing calculator as guide or compass. As discussed earlier, Nellie advocated that students draw a picture of the problem situation during the initial stages of the solution process. During the interviews, students indicated that they used graphing calculators in order to get a "ballpark" estimate of the solution. Then, students worked towards the solution using a symbolic approach.

Of these three ways, the alternative method was the most controversial among students for much the same reasons that have been debated within the mathematics community. The classroom observations indicated that Nellie expected students to be proficient solving problems within the symbolic representation. But, there were examples presented in class that were solvable only through graphical means because the symbolic approach was not an objective for the course.

Whether students should use graphical methods rather than symbolic methods for finding solutions was an issue for the students. On one end of the spectrum of the issue, Interview Student #1 from the LSM-MS group favored the use of the graphing calculator as a means for obtaining solutions without knowing the symbolic approach. While

enrolled in the precalculus course, this student monitored the prerequisite course, for which the Mathematics Department did not recommend graphing calculators. Talking about the use of the graphing calculator to obtain answers to symbolic questions, the student said:

> S: Now, I know it when I'm sitting in math 90 [the prerequisite course], I figure out answers twice as quick as kids who don't use 'em [graphing calculators] in the class, when the teacher puts them [mathematical problems] up on the board.
> R: What kind of questions are those that you're getting quicker?
> S: Mostly algebra, just being able to figure out like the algebra and, uh, also the functions that [the instructor of math 90] puts on the board. They [students in math 90] would figure it out algebraically. And, I would just quickly graph them.
> R: You mean answers. Uh, equations you're finding solutions to?
> S: Given functions, like [Nellie] did the function of throwing [an object] from the building. And, I can just graph it up while everyone else is just trying to figure it out. I was just going along the graph and finding the point [Nellie] was looking for. The calculator seems to offer a lot of things with graphing it, if you have a function.

At the other end of the spectrum, Interview Student #2 from the HSM-MS group, did not appreciate the use of graphing calculators to obtain solutions:

> I think in math 90, you're learning the basics [symbolic manipulation skills]. This [graphing calculator] would be too much of a crutch. I mean, you wouldn't want it. You'll use the calculator for only easier ways of doing things. You need to know by yourself first. You need to know the process. And, this [graphing calculator] just helps you get through the quick things. It just makes it quicker.

Interview Student #6 from the HSM-MS group agreed with Interview Student #2 about the importance of the symbolic approach: "If you don't know how to go back and do it algebraically, then you're stuck in the water."

This exchange with Interview Student #10 from the LSM-B group further illustrated a common theme about symbolic versus alternative method.

> R: So, I'm going to give you a hypothetical situation. Let's say that you're the administrator at school. And, it's been brought up, they want to make a policy about using graphing calculators. And, you've got to decide whether calculators will be or will not be allowed in math 131. What would you decide and what are the reasons?

> S: I would say yes. Because, speaking now as a student, I wouldn't say that it allowed me to cheat, because it doesn't. Being able to use it and seeing the graph helps most students.
> R: So, kind of looking at this cheating idea, even though you're not saying it is, you still have this idea of cheating?
> S: Because you can come up to a point that it is. Because if you graph it and it shows you what it is.
> R: For some particular problems?
> S: Yeah, for this [the first interview task]. I can graph that [the symbolic form of the function] and it [graphing calculator] will show me what it is.
> R: And, so, you would get those two horizontal lines for this particular function: x minus 2 over x minus 2. So, what you're saying is, that there is another method to determine whether this is a function or not?
> S: Yeah.
> R: You can do it algebraically in a sense.
> S: Yeah.
> R: So, rather than saying you could do it algebraically, you could do it graphically on the graphing calculator?
> S: Faster. It just speeds up the process.
> R: And, so, because you have this alternative, you still feel like that you should be able to do it algebraically?
> S: Before, I will do it on my calculator. Like, I'll make sure I do the homework and know how to do it algebraically in case [Nellie] says no graphing calculator on the test. Just to see if we would understand it algebraically, cause all this is is a shortcut [graphical approach].

As seen in the end of the exchange above, a motivation for the student to learn the symbolic approach was in case Nellie did not allow the graphical approach on tests. With this same motivation, five of the twelve interview students advocated that there should be a balance between symbolic manipulation skills and graphical skills for finding solutions to mathematical problems. Interview Student #2 from the HSM-MS group talked about keeping a balance because the instructor expected them to show their work on test questions. "We're supposed to show our work. So, obviously it [graphing calculator] didn't get in the way [of learning symbolic manipulation skills], cause we still have to put down [symbolic work]." Concern about Nellie limiting access to the graphing calculator on tests affected how Interview Student #10 from the LSM-B group approached homework:

> Before I will do it on my calculator, I'll make sure I do the homework and know how to do it algebraically in case [Nellie] says no graphing calculator on the test just to see if we would understand it algebraically. Cause all of this is just shortcuts [using graphing calculators].

Interview Student #6 from the HSM-MS group also was concerned about Nellie forcing students to show work on tests:

> If [Nellie] didn't make us show work, it would be real easy to jump to the calculator and graph real quick, like finding zeros. Just graph it real quick and knock out the answers and go to the next one. And, I guess, in real-life applications, that's what you would do anyway. You wouldn't sit there doing algebra if you could do it faster on the calculator. But, not really because you have to show your work and you don't know, sometimes, it's harder to graph a function than do it algebraically. On certain cases that I ran into on some of the tests it was like that. If you don't know how to go back and do it algebraically, then you're stuck in the water. But, mainly it's to show the work part. You got to be able to know what you're doing first. Use your graph later to make sure you're right or get you going towards the right answer.

Not only concerned about Nellie requiring the symbolic approach, Interview Student #13 from the LSM-LA group advocated a balanced approach as a "safety guard" in case the instructor for the next mathematics course did not allow graphing calculators. The student wanted to make "sure that the instructor does it both ways so that we know how to do it by hand without the calculator."

Another motivation for students wanting a balanced presentation of symbolic and graphical approaches was the prevalence of technology in the work place. Interview Student #12 from the HSM-B group, commented on the "real-life application" motivation for keeping a balance. Besides being prepared for the next test, this student wanted to be prepared to use computers in a higher course or in the workplace while possessing the necessary symbolic skills:

> R: What would you decide [whether to allow the use graphing calculators] and how would you decide it? What are your reasons?
> S: I would probably say, I'd say yes they could use them. Because in precalculus, there are people going on to be science majors. Which means that they are going to get into bigger problems in the work place using calculators. They'll be using computers to do a lot of stuff. So, it would be nice to get a foundation in doing something with the calculator. But, keep it balanced. So they know how to do them.
> R: What do you mean by balanced?
> S: Um, not doing everything on the calculator. Doing enough so that they know when they do it, then know how and why.
> R: In terms of how to do it?

> S: The different [symbolic] steps, rather than just plugging in the numbers. So, you know how to do it, you also know how to use the calculator.

Besides an alternative approach, graphing calculators were observed in the classroom to be a device to check work obtained symbolically. As discussed earlier, Nellie recommended students to use graphing calculators to check answers for topics such as transformations and symmetry of functions. When introducing transformations of functions with a guided-discovery teaching method, she used the graphing calculator to verify students' responses. For determining whether a function was odd or even, Nellie recommended students to determine it symbolically and, then, check graphically.

Supporting the classroom observations, a common use of graphing calculators for students from the HSM groups was to check their answers obtained symbolically. As Interview Student #2 from the HSM-MS group stated, "I can do it the equation way and check my answers by graphing." Also, Interview Student #5 from the HSM-LA group stated, "I do it algebraically and then check it on my calculator." The strongest statement for the use of the calculator as a checking device was made by Interview Student #3 from the HSM-LA group: "I use the calculator like I use the back of the book."

The calculator was used to confirm answers on test questions in a manner similar to looking at the answers in the back of the book for homework questions. Interview Student #6 from the HSM-MS group described a testing situation where the use of the calculator assisted in finding an error in the calculation of the extrema of a quadratic function:

> It caught me a couple of times on tests. I'd have a whole section that I was doing wrong. And, because I didn't memorize what the formula was supposed to be correctly [did not include the negative sign for the first *b*-value], I did everyone wrong. And, everyone of the graphs were completely opposite of what the answer I was coming up with. I went back and figured out what went wrong. Boom, fixed every question on there, so that I could get the answer.

By being able to check answers on tests and homework, this student felt more confident taking tests with access to a graphing calculator: "I can sit down and do a test. Probably pass it okay without it. But, my confidence level is way lower because I'm not sure cause I could have had one sign backwards. Flipped the entire graph over and I didn't even realize it cause I couldn't check it."

Besides being a tool for an alternative approach or for checking answers, students used graphing calculators as a guide or compass to direct themselves towards finding the solution to a problem. This method was similar to the use of the graphing calculator to check answers but was different with respect to the timing within the solution process for accessing the calculator. When used as a checking device, students accessed the graphing calculator at the end of the solution process. Yet, when used as a compass, students accessed the graphing calculator at the beginning of the solution process to develop an strategy for solving the problem.

As discussed earlier, Nellie recommended that "pictures can't go wrong" for setting up a problem. Similar to how Nellie set up problems with pictures and diagrams, students indicated that they used graphing calculators to develop a strategy. Interview Student #1 from the LSM-MS group used a graphing calculator to get started or to figure out where to go during a problem: "I like to be able to do the algebra myself. But, a lot of times, if I think that I can, I will. But a lot of times, I just plug in the calculator first to see if I'm getting answers that I like. And, then I usually try to figure it out algebraically. I use it as a directional tool. Just to make sure that I'm going in the right direction." This student's use of the graphical representation with the calculator was demonstrated during the first task for determining the domain of the function. Using information obtained from the graph, the student determined the solution in the symbolic representation. While analyzing the graph, the student became aware of the undefined value at $x = 2$. This awareness gained in the graphical representation helped the student to determine the solution in the symbolic representation:

> I can see the graphical domain, but seeing that it [the graph of the function] goes in a seemingly continuous line. I assume it goes on for eternity, except it's missing one [point]. It's missing (2,0). Which is undefined, right here. So, I guess that's what couldn't be in your domain because x minus 2 is 2 minus 2 is 0. That's an undefined equation. It's an undefined expression cause it's a fraction, because you're going to get something over 0. So, that is your one limitation on your input.

Interview Student #6 from the HSM-MS group also used the graphing calculator as a compass. During the interview, the student was asked, "If you were to approach a problem algebraically and couldn't come up with a solution, then would you also go to the graph?" The student responded:

> Absolutely, yeah. Just so I would know what ball park I was supposed to be in. For me sometimes, just looking at the graph and knowing where I'm supposed to be, and looking at my original problem is, I can get an idea of where I was supposed to go. Or, what kind of answer I'm looking for, to start with. Then I can go back and try to figure out which algebraic function or method I was supposed to use to come up with that. That helps a lot. Before I found discriminants or anything like that, I would graph it first and know what my answer was supposed to be. Then, I would go back through the algebra and make sure that I came up with that answer. If I didn't, I could know, look and know where the differences were. Then, try to figure out where I'm making my mistake.

To support students' comments about using the graphing calculator in the class, students were observed for their use of the graphing calculator during the second mathematical task given in the interview. While attempting the second interview task, one student from a HSM group and one student from a LSM group used graphing calculators at the beginning of the problem. Three students from a LSM group used the graphing calculator after unsuccessful attempts in the symbolic representation. Surprisingly, these three LSM students also demonstrated a reluctance to use the graphing calculator. They made comments that they could solve it graphically, but did not pursue solving it graphically. Only after being prompted by the researcher did these students use their graphing calculators. Even when the calculators were positioned on the desk next to them, they were hesitant to reach for it and use it. Part of the reluctance could have been attributed to not being given specific directions to use the graphing calculator. When asked for topics or occasions where the calculator was beneficial, Interview Student #10 from the LSM-B group responded, "The only time that it was really used [was] when [Nellie] would say graph this on a calculator."

Common within the three uses of graphing calculators discussed above was that the graphing calculator was a resource of information for students to develop solution strategies, solve problems, and check answers. For students who used a graphical approach to solve problems, they were dependent on graphing calculators to provide reliable information. However, two students expressed concern about the difficulty in finding their errors when using graphing calculators. Since little record of students' work existed when using graphing calculators, errors were difficult to find. Interview Student #3 from the HSM-LA group stated: "If you're doing something like that [a problem

filling up all three chalk boards] in your head or in the calculator, and when you screwed up, you had to go all the way back through. When you're going 10 steps, you lose track. So, you're not going to remember exactly what you did. You can play with numbers on paper a lot easier than you can with a calculator." Interview Student #5 from the HSM-LA group also was concerned about lack of records for using graphing calculators: "If they [students who rely predominantly on the graphical approach] put something wrong in the calculator...with a calculator you don't know if you messed up somewhere."

Besides the difficulty with input errors, Nellie was concerned that limitations of graphing calculators would assist students in developing misconceptions. As discussed earlier, Nellie spent time during three class periods informing students of methods to graph circles that appeared circular and were continuous. She warned students of the graphing calculator's limitation for graphing rational functions with vertical asymptotes. Nellie's concerns about students developing misconceptions based on limitations of graphing calculators came to fruition for a problem requiring students to draw the inverse of the exponential function, $f(x) = 2^x$. Students developed the misconception that the graph of the exponential had a finite domain because the calculator showed a truncated graph. For large negative x-values, the graph of the exponential function could not be distinguished from the horizontal axis. The graph obtained from the calculator appeared to have a limited domain. Students used this truncated graph to draw a truncated graph of the inverse function.

To gather additional data about the impact of the limitations of the graphing calculator on students' understanding of functions, the second interview task involved the limitation of calculators for graphing rational functions with vertical asymptotes. When graphing rational functions in viewing windows not obtained with a special feature of the calculators, graphing calculators drew a continuous graph with a vertical line on the graph at the position of the vertical asymptote. This vertical line was drawn because when the calculator was in CONNECT mode, it connected two points on opposite sides of the undefined value. In order to explore the effect of this limitation of the graphing calculator, students who did not use their graphing calculator for the second task were asked to graph the function to see their reaction to the vertical line in the graph.

Four students from a HSM group and one student from a LSM group were not distracted by the vertical line. These students recalled Nellie's comments about the limitations of graphing calculators, as illustrated by comments of Interview Student #2 from the HSM-MS group: "I remember us having one like that. I just remembered what the instructor said. It was that the calculator just wasn't really as smart as it should be. It didn't realize that we shouldn't have had a point there. It just drew a straight line, because it needs to connect the dots." Interview Student #6 from the HSM-MS group also recalled Nellie's explanation for the calculator showing continuous graphs of discontinuous functions: "That's the way [Nellie] explained, and it makes sense to me too. That would be because the calculator has to put some points as it's going across [the screen], filling it in. It jumps up to the next one, so it naturally keeps continuing on. Although realistically, that's the asymptote of the two different curves."

However, the vertical line drawn by the graphing calculator for discontinuous rational functions created confusion for the remaining two students from the HSM groups and five students from the LSM groups. Initially, Interview Student #13 from the LSM-LA group was unable to identify whether the graph was of a function. Yet, when the window was changed to ZOOMDECIMAL to avoid the vertical line, the student quickly identified the graph as a function.

Not only did the vertical line cause hesitation for students who initially used the graphing calculator to identify the function, but also some students changed their correct responses after seeing the graph presented with the vertical line. Using the symbolic representation, Interview Student #9 from the HSM-B group correctly determined that the relationship was a function and that it was not continuous. However, when asked to graph the function on the calculator and determine whether the function was continuous, the student changed responses: "Uh, yeah, it looks like it is [continuous]. Actually, it doesn't look like a function though." The student had difficulty determining whether the vertical line presented by the calculator was part of the graph.

After identifying the function and getting the domain correct using the symbolic representation, Interview Student #7 from the LSM-MS group attempted to identify the function using the graphing calculator. When seeing the graph given by the calculator, the student changed responses:

S: So, it's not [a function].
R: How come?
S: Because right here you have many y-values, many y-values for the x-value.
R: Which x-value would you have many y-values?
S: Negative three.

The vertical line created difficulties for the student. When asked to draw the graph on a piece of paper, the student included the vertical line in the graph, indicating that the vertical line was part of the graph of the function. Using this graph, the student determined that the relationship was not a function and it was continuous. However, when WINDOW DECIMAL was used to draw the graph, the student changed responses and asked, "How does it do that?"

Even though the vertical line created difficulties for some students, other students used the vertical line produced by the calculator to locate the undefined value and determine the domain. Interview Student #4 from the LSM-B group used the vertical line to an advantage: "That vertical line is not supposed to technically be there. It's pointing out that, it's an asymptote. There will be no y-values for that x-value. At negative three, it won't be possible, so technically it's not supposed to be there. But it's there. The calculator put it in." Interview Student #10 from the LSM-B group also used the vertical line to help determine the domain:

S: At the point negative three, there's the line.
R: The vertical line?
S: It's undefined at that point.
R: So, whenever you see that on your calculator, you know that it's just undefined?
S: Yeah.

Interview Student #6 from the HSM-MS group also interpreted the vertical line as an asymptote: "If this line is here, that's just telling me that it's an asymptote of these two curves."

However, not all of the students used the vertical line without difficulties. Even knowing that the vertical line was not part of the graph, Interview Student #11 from the LSM-LA group had some confusion as seen in this exchange:

S: I think that it would be a function. But, it's going to be undefined.
R: Where?

S: At negative three.
R: And, how are you determining that it's undefined at negative three.
S: The asymptote.
R: The vertical line that you see on the calculator?
S: I'm not sure if the asymptote is kind of considered. It's kind of like a thing with the calculator. Sometimes it has that. So, the line isn't really there. It probably just connects it on its own. So, yeah it is [a function].
R: Okay, because you've seen the graph and it passes the vertical line test?
S: Except for that one line that isn't part of it.
R: All right. What's the domain of this function?
S: Negative infinity to infinity.
R: Okay, so using everybody. Then, is this continuous?
S: Uh, no. Neither one of the curves, or whatever, are ever going to touch negative three. It's a hole.
R: So, it's a hole at negative three. But, what's your domain?
S: Yeah, that's true. I guess it would be negative infinity to 3 and 3 to infinity.

In summary, the graphing calculator was a time-saving device used by the instructor and students for in-class work, homework, and tests. The graphing calculator provided quick and mostly reliable graphs and calculations. However, student proficiency with graphing calculators was a concern of both the instructor and students. The instructor considered proficiency with graphing calculators to be important by spending two days during the first week of the term to introduce features of graphing calculators. Students indicated concern that becoming proficient with the calculators distracted them from learning the mathematics covered during the early stages of the class. The time used for learning the features of the graphing calculator reduced the amount of time devoted for homework and studies, especially when quizzes and tests included questions that assessed students' abilities to use graphing calculators. Thus, the instructor's approach for introducing basic features at the beginning of the term and specialized features throughout the term lessened the anxiety for students unfamiliar with graphing calculators.

As seen from the classroom observations and student interviews, students used the calculator in three main ways. First, the calculator provided alternative methods to present topics and to obtain solutions to mathematical problems. The instructor was able

to present the higher level mathematical concepts such as extrema without spending time graphing functions by hand.

However, the use of graphing calculators for alternative approaches was as controversial for the students as it has been for the mathematics community. Students from the HSM groups called for a balance of symbolic and graphical knowledge because they perceived mathematics as a symbolic endeavor. Additionally, both groups of students were concerned with meeting the instructor's expectations for answering test questions using symbolic approaches.

Second, students used the calculator to check their work during the solution-monitoring phase of the solution process. As one student said, "I use the calculator like I use the back of the book." This use of the calculator to confirm answers provided security for students with little confidence. For problems on the homework or the test, students could check their answers and proceed to the next problem with confidence. Also, other students used it to adjust their solution strategies when incorrect answers were obtained.

Third, students used graphing calculators as a guide. Though this method was similar to the second method described above, the two methods for using graphing calculators differed in the timing and motivation. When used as a guide, students accessed graphing calculators during the problem-representation and planning-solution phases of the solution process. During the initial stages of the solution process, students used calculators as a compass as a "directional tool" to determine a strategy and to obtain answers that provided direction for determining the symbolic approach. For instance, a student who got lost in the algebra used the calculator to check his bearings to proceed with the symbolic approach.

Finally, the use of graphing calculators must be cautioned. For each of the above methods for using graphing calculators, students relied on them to provide accurate and reliable information. Even though the instructor repeatedly warned students about the limitations of the graphing calculator, inaccurate graphs still created confusion or doubts for students. Students readily accepted graphs that the calculator provided as if it was an authority similar to the teacher. The misconception about exponential functions having a bounded domain was an example of the students' naiveté The instructor had to use class

time to address the misconception that exponential functions had bounded domain. Without prior knowledge of the properties of the graphs of exponential functions, students accepted without question the graph provided by the calculator. Additionally, even with the warnings about the vertical line that graphing calculators draw to connect points across an undefined value, the vertical line was a cognitive obstacle for many of the students from the LSM groups.

CHAPTER V

DISCUSSIONS AND IMPLICATIONS

This study investigated undergraduate students' learning of the function concept and the role of the graphing calculator in the first course of a two-course precalculus sequence. Much of the earlier research conducted on the impact of graphing calculators compared classes with access to graphing calculators to classes without access. Yet, confounding factors were introduced into the studies because graphing calculators were usually accompanied by changes in course goals or instructional practices. The design of many studies did not allow for an appropriate discussion of the impact of graphing calculators on students' learning of functions (Hiebert, 1999).

To address the first research question, "What knowledge of functions do students gain in a College Algebra course requiring graphing calculators," a pretest and posttest was administered to students. Interviews of twelve students were conducted to verify students' responses on the Function Test. Additionally, classroom observations were performed to ascertain the manner in which students encountered the function concept.

Much of the reasoning behind the recommendations for incorporating graphing calculators into precalculus courses was that students with low symbolic manipulation skills would be able to use graphing calculators to overcome the algebraic barrier (American Mathematical Association of Two-Year Colleges [AMATYC], 1995). The expectation was that mathematical concepts would become more accessible to students as they explore the concepts in the various representations of functions rather than solely within the symbolic representation. However, few researchers explored whether students with low symbolic manipulation skills learned the mathematical concepts in a manner similar to students with high symbolic manipulation skills. So, this study investigated the question of "What knowledge of functions do students of different levels of algebraic skills and with different academic majors gain in a College Algebra course?"

On the basis of students' academic major and of the results of an algebra skills test given at the beginning of the term, 21 of the 25 students participating in the study were placed into one of six groups: High Symbolic Manipulation Level-math and science [HSM-MS], High Symbolic Manipulation Level-business [HSM-B], High Symbolic

Manipulation Level-liberal arts [HSM-LA], Low Symbolic Manipulation Level-math and science [LSM-MS], Low Symbolic Manipulation Level-business [LSM-B], and Low Symbolic Manipulation Level-liberal arts [LSM-LA]. The remaining four students did not have scores that placed them in either the Low or High Symbolic Manipulation Levels. The percentages of correct responses on the pretest and posttest for each of the groups were tabulated and analyzed. To verify the responses of the students on the Function Test, two students from each of the groups were interviewed.

Finally, many of the earlier research studies did not include classroom observations. Without conducting observations, researchers could not verify whether graphing calculators were used appropriately or uniformly for studies with numerous classes. Information was not gathered about the role of graphing calculators on students' learning. For this study, classroom observations were conducted to gather information about the manner in which the instructor and students used graphing calculators. Additionally, student interviews were conducted to verify classroom observations. This information helped address the third research question: "What role do graphing calculators play in the classroom as students develop an understanding of functions?"

Discussion of the Results

With respect to the first research question concerning what knowledge of functions do students gain, students' personal definition progressed towards the formal definition of functions. Yet, students had difficulties with the univalence requirement in three areas: (a) order of domain and range, (b) preference for simple algorithms, and (c) the restriction that functions were one-to-one. Regarding the second research question comparing the six student groups for the knowledge of functions gained, differences in flexibility were observed. Compared to students with low symbolic manipulation skills, students with high symbolic manipulation skills were more flexible working between representations of functions. Half of the interviewed students with low symbolic manipulation skills perceived a single function given in numerical, graphical, and symbolic representations as separate entities.

Addressing the third research question about the role of the graphing calculator in students' learning of functions, it played a role in all phases of the solution process. During the initial phases, students used calculators to develop a symbolic approach. The prime motivation for using graphing calculators during the solution-execution phase was to avoid careless errors. The most common use of graphing calculators was to check answers during the solution-monitoring phase. However, graphing calculators did create difficulties for students who viewed them as an authority and accepted graphs at face value.

Function Concept

The definition of functions historically has become more abstract in order to encompass a wide range of mathematical relationships (Malik, 1980). The definition has been refined over time to embrace the new developments in mathematics. The results of this study suggest that students demonstrated a similar growth. Students modified their personal definitions more closely towards the formal definition of functions. This growth was also seen in students' understanding of the univalence requirement for functions.

Concerning the definition of functions written on the pretest and posttest, students demonstrated a progression in their understanding towards the formal definition. Students who provided a graph or formula as their definition on the pretest provided definitions that were categorized as correspondences or relationships. Those students who did not provide a definition on the pretest gave definitions on the posttest in terms of a representation, formula or operation. Thus, in terms of levels of abstraction for their personal definition of functions, a progression was found towards developing the formal definition. While students began the study with less abstract definitions, at the end of the observation period, they demonstrated a more abstract definition relative to their initial definition.

Skemp (1971) claimed that the mathematics to which students are exposed affects the development of their understanding of functions. The classroom observations in the study provided information concerning the manner in which students encountered

functions. At times, students were confronted with functions presented in the various perspectives of functions, action, process and object. During the observation period, students were presented an action view of functions through the techniques used to evaluate functions and to verify the composition of functions. The numerical method for verifying transformations involved an action perspective of functions. In the sections on the algebra of functions and transformations of functions, students manipulated functions as objects. Also, a significant portion of the discussion of rational functions involved the analysis of the graphical features of the asymptotes, which treated graphs as objects. Finally, students encountered functions as abstract relations between two variables or concepts from an application situation. The instructor introduced the definition of functions in the process perspective by using abstract relationships between people's names and people's family relationships. The methods used by the instructor to present function concepts were similar to the recommendations of Adams (1997). The instructor complemented the formal definition with other definitions and examples to aid students developing their understanding of functions. Thus, students were presented and worked with functions within the various perspectives and representations of functions.

The definitions that students initially provided indicated that they viewed functions as an action for obtaining output values or as an object such as a graph. To encompass the various perspectives of functions that they encountered, their personal definition became more abstract. As seen in the progression of students' definition towards the more abstract formal definitions, students modified their personal definitions to resemble more closely the formal definitions.

The conclusion that students made a progression towards the formal definition corresponded to the results of Vinner and Dreyfus (1989) and of Slavit (1994). Vinner and Dreyfus found that the definitions provided by college students progressed towards the formal definition. Examination of Slavit's study provided support that students' personal definitions did not automatically correspond to the formal definition.

During Slavit's study (1994), a test was given to high school students at the end of each of the three sections of the pre-calculus course. Each test included a question asking students to provide a definition of functions. Slavit found differences in students' definitions given on the tests. On the test given at the end of the first section in which the

formal definition was presented, students' definitions were more formal than the definitions provided on the second and third tests. Slavit attributed this reverse trend to students merely memorizing the formal definition for the first test.

According to Mansfield (1985), Slavit's (1994) observations indicated that the students did not integrate the formal definition into their knowledge structure. The formal definition coexisted with the students' personal definition. Because the formal definition was not assimilated into students' knowledge structures, the formal definition was readily forgotten as demonstrated by the less formal definitions written on the second and third tests. The responses on the second and third test were more representative of the students' personal definitions than the formal definitions memorized for the first test.

For this study, students began the class with definitions at various levels of abstraction. At the end of the observation period, most of the students had refined their definitions closer to the formal definition. Thus, with regard to the definition of function, the results of this study support of other studies indicating that students develop their definitions of functions progressively rather than leaping to the formal definition.

Besides the level of abstraction, another aspect of the function concept that became apparent in the analysis of the results was the univalence requirement, that for each element in the first set there corresponds a unique element of the second set. Students' difficulty with the univalence requirement was observed in three areas: (a) in their techniques for determining domain and range, (b) in their preferences towards simple algorithms to identify functions, and (c) in their restriction that functions were one-to-one.

First, when determining the domain and range for functions given in the written and numerical representations of functions, students had difficulties with the order of the mathematical relationship. On the Function Test and during the interviews, students indicated that they were familiar with the concept of domain and range but not with the order of domain and range. Students preferred the graphical representation to identify domain and range because domain was always the horizontal axis and range was always the vertical axis. For the numerical identification question on the Function Test, students did not have difficulties with the order of domain and range because the relationship was presented as list of coordinate pairs rather than a table of data. The standard form of

coordinate pairs reduced students' difficulty for distinguishing the order of domain and range.

However, when the mathematical relationship possessed ambiguity in form or context, students had difficulty with domain and range. Within the numerical representation, tables of data did not have a standard form like coordinate pairs. Tables were constructed either horizontally or vertically. The initial difficulties with the domain and range for the numerical application on the Function Test were due to the lack of a standardized method for presenting tables of data.

The context of the application situations on the Function Test also produced difficulties for students within symbolic and written representations. The domain and range question given in the symbolic application on the Function Test was difficult for students because they did not incorporate the application aspect to the domain. The written identification question on the Function Test created difficulties for students because they were unable to determine whether shoe or price was the domain. When no standard form was displayed, students had difficulties with determining domain and range. Students' difficulties with certain representations of functions corresponded with the results of Army (1991) where students had mixed results for determining the domain of trigonometric functions given in different representations.

Second, students preferred simple algorithms for identifying functions. Within the graphical representation, the vertical line test was applied by students without an essential understanding for the rationale of the test. For the numerical identification, students developed a simple algorithm to identify functions. Within the symbolic representations, some students, mainly from the high symbolic groups, identified the rational function as a function because there was no plus-minus symbol. The identification of a relationship given in the written representation was difficult for students because they were unable to use or develop a simple algorithm.

Third, similar to other studies (Becker, 1991; Sfard, 1992; Slavit, 1994) students in this study restricted functions to be one-to-one correspondences. This restriction was especially evident within students' explanations for the identification of functions. Students incorrectly reasoned that the pizza-price relationship was not a function because different pizzas had the same price. For the numerical identification question on the

Function Test, students determined the relationship not to be a function because it was not one-to-one.

Although developing efficient methods for solving problems is a goal within mathematics education, the search for simple algorithms and their use can guide students in developing a common misconception of functions. Students initially viewed functions as one-to-one not to designate a particular type of function, but because one-to-one functions were the easiest types of functions to identify. Requiring functions to be one-to-one greatly simplified the process for identifying functions, especially since the order of the domain and range was difficult for students. Students did not have to distinguish whether the domain was restricted or the range was restricted. With regard to the univalence requirement of functions, restricting functions to be one-to-one reduced the cognitive demands for identifying functions.

Students' preference for simple algorithms to identify functions supported the view that students' requiring functions to be one-to-one was an immature understanding of functions rather than an entrenched misconception (Gardner, 1991). Students had limited mathematical experiences that would have created cognitive dissonance leading to changes in mental structures. The algorithms for identifying functions did not necessarily force students to reevaluate their limited view of functions because the test was applied to each isolated x-value in order to determine that there was only one y-value. Other points on the graph were ignored when individual x-values were checked. Additionally, for the function given in the numerical representation on the Function Test, students' algorithm did not require them to examine the range values. They focused solely on the domain values in order to check the number of times each domain value was listed.

The algorithms used by students did not force them to look at functions with regard to the order of the univalence requirement. Within the algorithms, the key issue was to guarantee that a particular value was found only once. Applying the vertical line test, students checked only an element of the range. For the numerical representation, students checked only an element of the domain. By looking through the domain for determining whether a value repeated, students could have generalized that there was

only one domain value for each range value. Thus, combining these two methods could have guided a student to the misconception that functions were one-to-one.

The algorithmic procedures for identifying functions in the numerical representation attributed to the view that functions were one-to-one. This restrictive view of functions reduced the effort for understanding the direction of the univalence requirement. Thus, this misconception probably was developed in the initial stage for students' development of their concept image of functions rather than transferring the one-to-one concept onto all types of functions.

Flexibility between Representations of Function

Janvier (1987) recommended using multiple representations to provide students with the opportunity to make connections between properties of functions. Additionally, the AMATYC (1995) and NCTM (2000) proposed that students need to possess the flexibility to work within and between the various representations of functions. One of the expected outcomes from using graphing calculators was that students would become flexible with regard to representations of functions by developing the skills to work between representations and within representations (Williams, 1993). However, Goldenberg (1988) was concerned about which type of student would benefit from the use of technology. He thought that students successful in the traditional instructional setting would be the ones successful in the reform classes. A prime motivation for the recommendation of graphing calculators has been that graphing calculators can be an equalizer for students without sufficient symbolic manipulation skills. This study investigated the role of the graphing calculator by comparing students with different levels of symbolic manipulation skills.

The results of this study indicated that students in the HSM groups exhibited more flexibility compared to students in the LSM groups. When given a function symbolically, students from the HSM groups were able to identify the function within the symbolical representation rather than translating it to another representation. Even though students from the LSM groups were able to translate functions into the graphical

representation to apply the vertical line test, this technique did not necessarily imply that the students were flexible. Similar to the results of the study by Army (1991), this algorithm for identifying functions was used when students were not familiar with the representation in which the relationship was presented. The students in the LSM groups used the graphical representation as a last resort when they were not familiar with the symbolic approach or to avoid the symbolic representation altogether.

Another justification for the AMATYC's (1995) recommendation for the use of technology and multiple representations was the expectation that students would be able to make connections between the representations and develop a rich understanding of the function concept. Further evidence of the lack of flexibility for the LSM students was that half of the LSM students interviewed considered each of the representations of the given function as separate entities. In a sense, the LSM students compartmentalized the representations. An example of this compartmentalization was when students with low symbolic manipulation skills were not successful in identifying a piecewise function given symbolically. Many of students from the LSM groups were unable to make connections between the symbolic and graphical representations of piecewise functions. Similar to the results of Tall and Bakar (1992), this lack of success could be attributed to students' lack of familiarity of the symbolic form of piecewise functions. Even though the students were exposed to many graphs of piecewise functions in homework problems and test questions, the symbolic forms of the functions were not presented simultaneously with the graphs. Thus, for the LSM students, the representations of functions were distinct entities and they were unable to make the mathematical connections between the different representations of functions.

Role of the Graphing Calculator

The graphing calculator played many roles in students' learning of functions. First, students welcomed the use of graphing calculators but recommended a balanced approach with regard to its use and the use of symbolic approaches. Next, the graphing calculator was a tool used by students within the solution solving process. Students were

observed to use calculators as a "directional tool," as a resource for verifying solutions, and as a security blanket. Additionally, graphing calculators provided students access to the concept of domain and range in a form familiar to them. Yet, graphing calculators were a distraction for students as a source for misconceptions and as cognitive hurdle to overcome while learning the mathematical concepts.

Much of the debate over the use of graphing calculators centered on the reduced emphasis on symbolic manipulation skills. However, in this study, students were not concerned with the instructor's perspective that symbolic manipulation skills were still important even though graphing calculators were readily available. During the interviews, students from the LSM and HSM groups recommended that the course retain a balance between the use of graphing calculators and symbolic approaches. Students, especially the HSM students, appeared to accept the manner in which the instructor used graphing calculators in a course that regularly worked within the symbolic representation. Yet, they also indicated that they were motivated to learn the symbolic approaches in order to perform well on tests and quizzes in this class as well as future classes.

One role of the graphing calculator was as a tool used by students during the cognitive phases of the problem solving process (Brenner, et al., 1997). Within the first phase, problem-representation phase, students used calculators to graph functions in order to familiarize themselves with the problem. During the interview, a few students from the LSM groups graphed the rational function not necessarily to solve the problem but to get an image of the function. The graphing calculator provided information for students to get them started with a problem.

The next phase was the solution-planning phase in which students developed a strategy for determining the solution. Within this phase, students used graphing calculators like a compass as a means to get pointed in the right direction for solving the problem. The use of graphing calculators assisted students in their problem solving by helping them develop and confirm their solution strategies. Using the solutions, students were able to develop a symbolic approach.

Students using graphing calculators during the solution-planning phase corresponded to the results of a questionnaire given by Quesada and Maxwell (1994) which also indicated that students used the graphing calculator in the solution-planning

phase. Ruthven (1990) labeled this stage as the refinement stage in which students used the calculator to adjust and confirm their solution strategies for translating functions between the graphical and symbolic representations.

The third phase was the solution-execution phase in which the strategies that were developed in the solution-planning phase were implemented. During this phase, students did not exhibit much reliance on their calculators. Much of the strategies that students used were worked within the symbolic representation. Also, students demonstrated a reluctance to use their graphing calculators for the mathematical tasks given during the interviews. Many of the students used the graphing calculator only after being prompted to use them.

Hesitation by students to use graphing calculators was also seen in the case study conducted by Lauten, Graham, and Ferrini-Mundy (1994). Unlike the student in the case study who used the calculator only for a short time before the interview, the students in this study used the calculator from the beginning of the course. A number of students had prior experience with graphing calculators. Thus, rather than not being familiar and comfortable with graphing calculators as indicated by Lauten et al., students were reluctant to use a graphing calculator to replace their symbolic approaches because they were concerned about developing the appropriate symbolic manipulation skills required by the instructor. The students viewed the graphing calculator as a tool to support symbolic manipulation skills rather than as a tool to replace symbolic skills.

Another reason some students were hesitant to use graphing calculators was that it did not provide a record of their solution process. Students expressed concern that a small error made while entering numbers, functions, and data would lead to an incorrect solution. Usiskan (1998) addressed this lack of a visible record in a discussion about the benefits of written algorithms. Without a visible record, students expressed concern that they could not backtrack to locate and correct a mistake. After becoming aware of an incorrect solution, a student would have to repeat the whole process and hope not to make a similar mistake. Also, students without an expectation of the correct answer would blindly accept the answer provided by the calculator. Thus, since graphing calculators did not provide a record of an approach, students expressed caution for relying on the sole use of graphing calculators.

In the final phase, the solution-monitoring phase, answers were determined to be appropriate for the application situation. Within this phase, students used the graphing calculator as a device to check their answers obtained from approaches other than graphical. As one student stated, "I use my calculator like I use the back of the book." The calculator was used as a tool for self-assessment. When the instructor assigned problems that did not have solutions given in the textbook, students used the graphing calculator to check their answers. Students were able to identify calculation errors or mistakes in problem solutions. Quesada and Maxwell (1994) also found that students used their calculators in this manner to check their answers.

A benefit that the graphing calculator provided students in the solution-monitoring phase was increased confidence and reduced anxiety during tests. The graphing calculator was like a security blanket for students. Students expressed satisfaction for the use of graphing calculators to avoid small mistakes on tests and homework problems. Similar to Paschal (1997), students expressed little frustration about minor arithmetic and algebraic errors. Similar to the results of other studies (Berger, 1998; Quesada & Maxwell, 1994; Smith & Shotsberger, 1997), students indicated that they had more confidence when using the calculator. The calculator was a support mechanism for students' learning. With the calculator, students were able to focus more on the mathematical topics being assessed because they were less worried about cumbersome calculations. The confidence gained from using graphing calculators was not because students thought the calculator was going to obtain the solution for them, but because the use of the calculator would reduce their careless errors in calculating expressions or making graphs.

Besides being a tool in problem solving, the graphing calculator played a role in students' learning of the domain and range concept. As discussed earlier, the graphical representation was one of the most successful representation of functions for students determining domain and range of functions because the order of the domain and range had a standard form. The horizontal axis was always the domain and the vertical range was always the range. This consistent pattern allowed students to keep track of the order. Looking at the graphs provided by the calculator, students easily distinguished the order

of the domain and range. The graphing calculator matched up well with students' difficulties for distinguishing the order of domain and range.

Yet, the graphing calculator did not always play a positive role in students' learning of functions. As cautioned by Goldenberg (1988), the graphing calculator assisted students in developing a misconception. The students were given a problem to draw the inverse of the exponential function, $y = 2^x$. However, at the time of the assignment, students had not yet encountered exponential functions. Many of the students obtained the graph of the exponential from the graphing calculator. When graphed in the STANDARD screen, the graphing calculator produced a truncated graph because the graph was imperceptible from the x-axis. Students were unable to distinguish the horizontal asymptote of the graph from the horizontal axis. Rather than explore the graph to see whether the graph of the exponential function continued in the negative direction, students accepted the truncated graph without question. Since students develop their understanding based on the examples of functions that they encounter, the limitation of graphing calculators was a possible source for assisting students developing misconception that the graph of the exponential function had a bounded domain.

In the study by Slavit (1994), students considered an equation to be a function only when the equation could be solved for y. Slavit concluded that the limitation of graphing calculators assisted students in developing this misconception. However, none of the students in this study provided a definition on the posttest that was related to this misconception. This result that no student developed this misconception was unexpected because classroom observations identified three potential sources to aid students in developing this misconception. First, the calculators used by the students were designed to graph equations that could be only solved for y. Second, all but two of the equations used in class were ones that could be solved for y. Third, the instructor made references about using only equations that were solvable for y. Even though there were numerous sources to assist students in developing this misconception, there was no evidence to support that students limited functions to be solvable for y.

However, the difference in the lengths in the time for high school and college courses may have had an impact. The high school students in Slavit's study had a longer

period of time to develop the misconception that functions are limited to expressions that can be solved for y. Additionally, the curriculum for the high school mathematics course may have differed to the college course used in this study.

Limitations of the Study

The obvious limitation of this study was the number of classes used in the study. Generalizing the results obtained from only one class is tenuous at best. The students' recommendation for a balanced approach between use of graphing calculators and use of solving strategies involving the symbolic representation may have been attributed to the influence of the instructor.

The small number of discovery-based activities administered in class may have reduced the role of graphing calculators in student learning. One justification for the recommendation of introducing graphing calculators into precalculus courses was that students could use it as a tool in discovery-based activities. Due to time limitations of the course, the instructor administered few discovery-based activities. Students were not exposed to a particular use of graphing calculators and this lack of exposure may have been unique to this class. On the other hand, the few number of discovery-based activities also may have prevented students from developing misconceptions as they did during the inverse function activity.

Another aspect of the class that made generalizing difficult was that the students were not randomly selected. The class was selected on the basis of obtaining at least two students in each of the groups formed on the basis of students' academic majors and symbolic manipulation skill levels. Without the random selection, the distribution of students with respect to academic major and symbolic skill level has a chance for being not representative of the typical College Algebra class, especially when other sections of the course were not chosen for the study because they did not have the desired distribution.

Another aspect that may have made the class unique was the allowance of multiple types of graphing calculators. This allowance may have hindered the

generalizations of the results because the instructor spent class time verifying whether students with different calculators had accessed similar graphs or same feature as the instructor. The time used to confirm students' graphs could have been used for additional examples or deeper discussions of the mathematical material. Interactions with students during these times focused on technical aspects rather than on mathematical concepts.

Regarding the Function Test, two aspects of the test limited the results of the study. First, the conversion required for the solutions to the symbolic application problems may have masked students' performances. Possible trends were not discovered because the conversion error was a common mistake. Second, the application sections may have been biased towards specific fields. The application section about hitting a softball was biased towards students with a physics background. The other two application sections were business related. Because two of the three application problems were focused on business applications, the Function Test may have been biased towards the students with a business major. Thus, multiple applications were needed to reduce the bias against students who did not understand or have experience with a particular context.

The results of the Function Test and interviews provided support that students develop their personal definitions in a progressive manner. However, periodic interviews would have obtained data providing information about the manner in which students refined their personal definitions of functions. Periodic interviews would have helped identify possible stages in the progression of personal definitions towards a formal definition.

Implications for Teaching Functions with Graphing Calculators

Since the students in this study were more successful with the numerical and graphical problems on the Function Test, these representations may be a cognitive root for students learning functions. Tall (1992) described cognitive roots as concepts that were familiar to students and provided a basis for learning mathematics. The students in the study demonstrated proficiency for determining the domain and range within the

numerical and graphical representations and exhibited more familiarity with these two representations of functions for the application sections on the Function Test. Because the students preferred the graphical and numerical representations, college instructors should introduce function concepts within these representations. When students have gained an initial understanding of the function concept, then connections to the concept within the symbolic representation should be made.

Much work is needed to assist students to work within and between the representations of functions. Presenting function concepts from the different representations was not enough. Students from the LSM groups viewed functions in the representations as different entities. Independently, students were unable to make the connections between the symbolic and graphical representations with respect to piecewise functions. Instructors should not expect students to make the connections between the representations without some guidance.

One-to-one functions were another concept that students needed assistance with. Even though students' personal definition may be as simple as "for each input you get one output," students had difficulty with internalizing the word "each." Gardner (1991) discussed the difficulties students have learning concepts when their intuition gets in the way. When looking at the word "each," students could easily construe it as being synonymous with "one."

Gardner (1991) recommended that students needed to confront examples to force students to abandon their intuitions in favor of the academic viewpoint. The use of simple algorithms to identify functions did not challenge students' limited view that functions were one-to-one. The search for clear-cut examples of functions within each of the representations of functions is needed to overcome the rigid application of simple algorithms. The use of the graphical representation did not force students to expand their understanding because of frequent use of the vertical line test. The algorithm used in the numerical representation also did not challenge students' understanding of functions. In fact, the algorithm reinforced the one-to-one limitation of functions through the search for repeated listings of domain values. Through generalizing the method for searching a table for only one domain value, students could easily interchange the word 'one' for 'each' in the personal definition above.

One way to circumvent the algorithm applied in the numerical representation is to list data pairs twice in order to force students to look beyond the domain values. However, tables with repeated data pairs are contrived and would not correspond with the AMATYC's (1995) recommendation for using real-life situations. Although real-life situations given in written form satisfy the recommendation, students had difficulties with determining domain and range for the situations. Thus, arrow diagrams are recommended to highlight relationships that are not one-to-one because they can arise from real-life situations without students becoming distracted by interpreting the information given in the written representation.

Also, assessment questions need to move beyond merely identifying functions. Since students used algorithms without having a complete understanding of the function concept, test questions need go beyond merely assessing their skill for identifying functions. Test questions need to be developed that assess students' conceptual understanding of functions rather than their skills.

With regard to preparing students to use graphing calculators, students recommended that instructors introduce features of the calculator when needed. The objectives of training sessions conducted at the beginning of the term should include only a basic overview of graphing calculators. The timing for introducing the different sophisticated features of graphing calculators should correspond to when the mathematical topic is covered in class. For instance, the ROOT feature of graphing calculators should be introduced during the section of the course on zeros of functions. Students need time to develop their skills on the basic features before they can master the more advanced skills. Introducing features of graphing calculator when needed may reduce the anxiety of students who are concerned about falling behind at the beginning of the term due to learning to use the calculator while learning the mathematics.

One misconception where teachers need awareness is that the calculator does not always provide exact values. For most calculations, the calculator presents values rounded off to a certain number of decimal places. Since rounding off does not produce a significant error for many problems, students and teachers develop the misconception that exact values are obtained by using the answer key to carry a value through a series of computations. The impact of this misconception may be more noticeable when students

encounter rounding methods specific to their fields. Discussions of methods for rounding off or different levels of significant digits may be confusing to students with the misconception that calculators provide exact answers. Thus, teachers need to inform students of the limitations of calculators for obtaining exact answers.

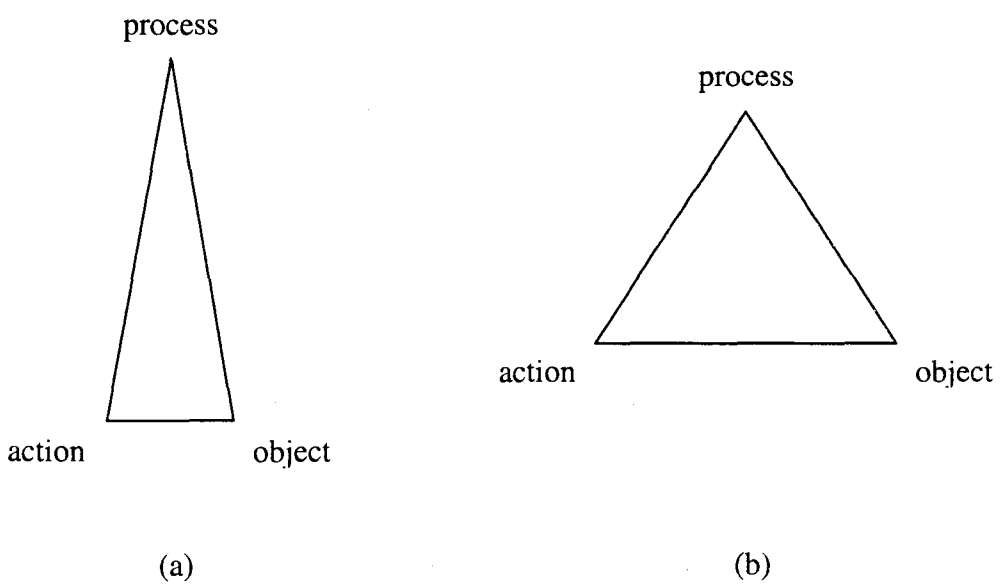
Time issues regarding the quantity of material and the pace of the course were heightened with the use of graphing calculators. Learning to use the graphing calculator played a role in creating frustration within some students about time issues. Students without initial proficiency with graphing calculators expressed concern that they had to learn not only the mathematical material but also the features of graphing calculators. For a course that students considered to be overly packed with information and to be fast paced (Carlson, 1998), becoming proficient with the graphing calculator created an additional source of apprehension for students without prior experiences with the calculator. Adjustments to the curriculum should be made to allow the time for students to develop their skills.

On the basis that the all aspects of curriculum should be connected, lesson plans should be developed in two ways: include opportunities for students without graphing calculator experience to develop their calculator skills and present features of the graphing calculator when needed. Having students learn features of the calculator that are unrelated to the current mathematical topics creates apprehension for students regarding the time they have to learn the calculator and the topics. Lesson plans need to schedule time at the beginning of the term to provide opportunities for students to become proficient with their calculator. Also, not all of the features of the calculator should be taught at the beginning of the course. Students expressed satisfaction that the instructor taught calculator features when they were needed rather than all at one time. Students may be able to develop a rich understanding of the mathematical concepts when they can make cognitive connections between the topics and features of the calculator.

Recommendations for Future Research

Sfard (1992) suggested that the action, process, object perspectives of functions were not developed in a linear fashion. The researcher suggests that the three perspectives can be viewed as an isosceles triangle with each perspective at the vertices. The action and object are placed at the vertices on the base of the triangle (Figure 1a). The triangle is not viewed as an equilateral triangle because the students did not demonstrate an initial possession of the process perspective. Students appeared to be comfortable with the action and object perspectives. As students experience the different perspective of functions, their understanding of functions begin to assimilate the process perspective and their understanding of the action and object perspectives begin to accommodate. The sides of the triangle begin to shrink and the isosceles triangle becomes an equilateral triangle demonstrating a richer understanding of functions (Figure 1b).

Figure 1. Visualization of the Growth in Understanding of the Three Perspectives of Functions



Although this study found that students' personal definitions became more similar to the formal definition, further research is needed to investigate students' development of their understanding of the perspectives of functions. A study including student interviews on a regular basis throughout the observational period is recommended. The interviews provide information of the students' perspective as the material is covered in class. Hopefully, the interviews are constructed to provide information regarding changes in students' perspective of functions.

The source of the restriction that functions are only one-to-one was not conclusively identified during this study. Observations suggest that students developed this restriction based on the examples they encountered rather than them transferring the one-to-one requirement for inverse functions onto functions in general. The one-to-one concept was not covered until the later part of the observation period during the section on inverse functions. Students' had little exposure to the formal presentation of the one-to-one concept for that concept to have such an impact. Their understanding was developed from other aspects of the class.

An alternative source for students to place the one-to-one restriction on functions may have been the ease for which learning the definition of function was reduced. The restriction for one-to-one reduces the cognitive difficulties for determining whether the domain elements or the range elements was restricted. By restricting both sets of elements, a cognitive obstacle was avoided. Further studies need to be conducted to verify whether students naturally place the one-to-one restriction on functions or generalize the requirement for inverse functions to functions in general.

Similar to recommendations of Berger (1998), context may be helpful for students. The business groups performed better on the symbolic questions of the Function Test than the other groups. The groups' success may have been due to the context of the questions. The use of real-life examples was recommended to help students make abstract ideas more concrete (AMATYC, 1995; Mortensen, 1992; NCTM, 1989). However, the results of the study suggested that the context of the examples make a difference to students. The examples to be used need to be ones with which students are familiar. The success for the business students on the business-oriented questions lends support to this idea. For the graphical interpretation question on the Function Test,

the business students had less success because they were unfamiliar with the physics of body motions and collisions. Thus, much consideration is needed for the choice of real-life examples used to make mathematics less abstract. Physics examples may not be effective for students who do not understand the physics concepts used to illustrate the mathematical concepts. In a course already containing many mathematical topics, little room exists to provide background of the real-life examples. Research is needed to study the types of real-life problems that can be used to assist students in developing a rich understanding of functions, especially in courses that are connected to other scientific fields.

A concern about the use of graphing calculators was an issue of authority. It was thought that access to graphing calculators would reduce the 'authority' role of the instructor and change the role to consultant. The change in teacher's roles needs to be further examined. The instructor demonstrated multiple roles in the classroom. At different times, she took on roles of a facilitator, an authority, and presenter of knowledge. The instructor used multiple roles even during times when graphing calculators were not used. The impact of graphing calculators on the instructor's roles could not be determined in this study; the instructor probably used these techniques in a class without access to graphing calculators. The manner in which graphing calculators effect the role of the instructor needs to be studied.

The students in this study developed misconceptions through the use of graphing calculators in a discovery-based homework activity. Because students did not explore the graph provided by the calculator, students took the truncated graph as the complete graph. Without much mathematical experience, students were not aware when further explorations were needed for graphs provided by the calculator. Students needed the expertise of the instructor in order to avoid misinterpreting the graph. Thus, guided-discovery activities may be more beneficial to students. Further study is needed to compare the use of discovery-based activities versus guided-discovery activities in order to determine whether students could avoid developing misconceptions that arise from limited graphs by calculators.

Smith and Shotsberger (1997) cautioned that there were topics for which graphing calculators were not beneficial. The equation of a circle was a topic that did not benefit

from access to the graphing calculator. Due to the technical difficulties that graphing calculators have for drawing a complete graph of a circle, graphs of circles were obtained using a combination of two functions. Graphing circles that were continuous and circular, not oval, took up much time in class. Additionally, whether the difficulties for producing graphs of circles using two functions had a positive or negative impact on students' understanding of functions needs to be examined.

Another topic that may or may not benefit from graphing calculators is transformations of functions. The types of functions that assist students in developing an understanding of transformations such as horizontal and vertical dilations need to be investigated. The instructor used the top half of a circle to illustrate dilations. Possibly the half circle assists students more than the graph of a quadratic function. With the half circle, a vertical dilation will produce an effect that only occurs in the vertical direction. Whereas, a vertical dilation on a parabola can easily be interpreted as a horizontal dilation because the parabola simultaneously appears to be thinner as well as taller. Thus, with regard to transformations, future study needs to be conducted to determine which types of functions best illustrate transformations.

In addition to the time taken up with one type of calculator in the class (Oster, 1995), the use of multiple calculators in the course took up class time, when the instructor addressed the needs of students with different types of calculators. Whether valuable class time that could have been used to provide more examples or explore concepts was lost needs to be further examined, especially since students already find the pace of the course to be fast (Carlson, 1998). Research is needed about the impact of multiple types of calculators in the classroom in order to provide departments with information to make recommendations on graphing calculator.

Finally, due to the use of only one class and the small number of students in each of the groups, the ability to generalize the results was reduced. The results of this study indicated that all students did not benefit equally from the use of graphing calculators. Since a main motivation for the use of graphing calculators was to create an equal playing field for students with and without strong symbolic manipulation skills, additional studies to be conducted to confirm this result study. Studies need to have

similar designs to this study but with larger sample sizes so that the ability to generalize can be raised.

In summary, the recommendations for the use of graphing calculators have been motivated in part by the expectation that students with little symbolic manipulation skill could be successful in developing a rich understanding of functions. Further research is needed to examine students' development of their personal definition of functions, their understanding of the three perspectives of functions, and their understanding of one-to-one functions. Whether students' restriction that one-to-one functions was a misconception or a stepping stone towards a proper understanding of functions needs to be examined.

The results of this study indicated that not all of the students benefited from the use of graphing calculators to the same degree and in the same manner. Graphing calculators are tools that some students utilize more effectively than other students. Research needs to be conducted in order to determine whether certain concepts of functions are more readily accessible and, then, to develop means so that all students can use the tool effectively. Also, comparisons between the effectiveness of discovery-based activities or guided-discovery activities are needed to develop instructional material that does not promote misconceptions. Additionally, whether the use of multiple types of calculators detracts from learning needs to be explored to provide school administrators with reliable information on which to make recommendations. Finally, the use of one class limited the ability to generalize. Similar studies with larger sample sizes need to be conducted to support the results of this study.

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APPENDICES

APPENDIX A

SKILLS TEST

The following is the Skills Test that was developed by one instructor and was administered to students in four sections during the first or second day of the term. The instructors of the sections used it as a means to assess the students' symbolic manipulation skills. The test consisted of 10 questions on topics covered in the prerequisite course for the precalculus course.

Algebra Pre-Test A

Name_____

Do the following operations, simplifying your answer.

1. $(2x^3 - 5y)^2$

2. $\frac{5}{x-3} - \frac{2x}{x+1}$

3. $(3a^3)^3(2ab^2)$

4. $\sqrt{6x} \cdot \sqrt{10x^3}$

5. Solve the following system:

$$2x + 4y = -2$$
$$x = y + 8$$

Solve the following equations:

6. $3x - 11 = 6(x+1)$

7. $2x^2 - 7x = 4$

8. $\frac{3}{2x-4} = \frac{4}{x-3}$

9. $\frac{4ab}{x} = \frac{3}{mz}$ (solve for x)

10. Simplify $\frac{4x^{-2}y}{16x^5y^3}$, writing your answer with positive exponents only.

APPENDIX B

VALIDATION OF SKILLS TEST

According to Borg and Gall (1989) content validity is found in order to ensure that the questions of the test represent the content covered in a course. The Skills Test was given by the educators to assess the students' symbolic manipulation ability on topics that were covered in the prerequisite course. Due to the expertise required to determine content validity of the prerequisite skills for College Algebra, a committee of five instructors in the mathematics department was made. Each member was chosen because of his/her experience with College Algebra and the prerequisite course. Each member recently taught both courses or was a member of the curriculum committee for the courses. These instructors were deemed appropriate to determine whether the content of the Skills Test was valid for the students completing the prerequisite course. The following cover letter was given to the validation committee along with the test, outcomes for the prerequisite course, and a table that was completed by each member. In order to insure their cooperation, the members were given a small gift in appreciation for their time and consideration.

Validation of Skills Test

Graphing calculators have been required for many lower level college mathematics courses. It has been recommended by national mathematics organizations that the graphing calculator should be used as a tool to enhance the presentation of mathematical topics in multiple representations: symbolic, numerical, graphical, and verbal. The graphing calculator has been recommended in order to provide a variety of mathematical experiences to a diverse student population who enroll in precalculus courses. Yet, few studies have been conducted to investigate how this tool has been used in courses by students with various symbolic manipulation ability. Whether the use of the graphing calculator promotes better understanding of the function concept for students of different symbolic manipulation abilities has not been determined.

The study will investigate the impact of the graphing calculator on student understanding of functions with regard to their symbolic manipulation ability. The following test will be used to collect data on students' symbolic manipulation skills at the beginning of the course. This data will be used to categorize the students on the basis of symbolic manipulation skills. I am requesting that you look over this test to determine whether the content for each question is appropriate for students entering the first course of the precalculus sequence. In order to assist you in your decision, I have included the list of outcomes for the prerequisite course, Math 90. Please identify which outcome(s) in the list correspond to each of the questions. Place the number(s) of the outcome in the chart. For example, if you determine that outcomes 3 and 8 correspond to question five, then write a 3 and 8 in the chart on line five in the second column.

For your time and consideration, I have included a token of my appreciation.

Outcomes:

Students who successfully complete Math 90 should:

1. be able to perform basic operations on polynomials, rational expressions, radicals, and expressions with rational exponents;
2. be able to solve basic linear, rational, quadratic, and radical equations;
3. understand the concept of a linear function;
4. be able to graph linear functions, and find equations to lines;
5. be able to solve systems of linear equations in two variables;
6. understand basic terminology associated with the above concepts, and be able to use related mathematical notation correctly;
7. be able to construct equations which model situations described in words;
8. be able to use the above abilities to solve word problems, and be able to express solutions clearly.

Question	Outcome(s)
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

APPENDIX C

PRETEST AND POSTTEST OF THE FUNCTION TEST

In order to gather information about the change in students' understanding of functions, two versions of the Function Test were administered to the students. The pretest was administered on Friday of the first week of the term and the posttest was administered the first day after the unit on functions was completed. The pretest and posttest had different cover pages. To adhere to the guidelines of human subjects, consent of the students was asked on each of the cover pages. When students consented to allow the researcher to have access to their tests, the students provided general background information. When students did not consent, the students did not provide the background information. The researcher did not have access to students who did not consent to participate.

For students participating in the survey students provided the last digits of their social security number. The researcher did not have access to tests for students who did not consent by providing their social security number.

The posttest was a parallel form of the pretest. Only the numbers and graphs were changed. The wording and format of the questions were the same.

Pretest Cover Sheet

This questionnaire is being given in order to assess college students' understanding of the concept of function. I am collecting this information as partial fulfillment of the requirements for my degree. All responses will be kept strictly confidential. Your name will be removed from the questionnaire in order to retain your privacy. Allowing access to this questionnaire has no affect on your grade in the class.

Thank you for your assistance.

_____No, I do not want to participate in the study by allowing the researcher access to my questionnaire.

_____Yes, I will participate in the study by allowing the researcher access to my questionnaire.

If you are participating in the study, please fill provide the following information.

Background Information:

Social Security Number: (last 6 digits) ***-_____-_____

Major/Program:_____

Grade level (circle): Freshmen Sophomore Junior Senior

Last mathematics course taken:_____ Year taken:_____

Mathematics Courses you will be taking in the future that are required by your major (circle)

None Math 132 Math 151 Math 152 Math 150 Math 240

Which graphing calculator are you using for the course?

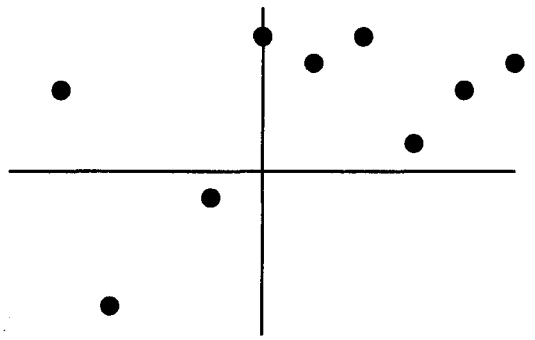
How proficient are you with using your calculator? not at all somewhat very

Pretest

Directions: Please read each of the following questions carefully and check your responses. In all cases, explain the reason you made that particular choice. The written responses are very important, so please give a complete explanation.

1. Determine whether the following graph represents a function:

_____ a) Yes
_____ b) No
_____ a) I do not know
Explanation:



2. Determine whether the following statement represents a function:
The relationship between a pizza ordered at a restaurant and the price of the pizza.
_____ a) Yes
_____ b) No
_____ a) I do not know
Explanation:

3. Determine whether the following represents a function:
$f(-1) = 5, f(1) = 2, f(0) = -4$, and $f(2) = 5$.
_____ a) Yes
_____ b) No
_____ a) I do not know
Explanation:

4. Determine whether the following represents a function:

$$f(x) = \begin{cases} x^2 & \text{for } x < 2 \\ 1 & \text{for } x = 2 \\ x - 3 & \text{for } x > 2 \end{cases} \quad \text{for any real number } x?$$

_____ a) Yes
_____ b) No
_____ a) I do not know
Explanation:

5. In your opinion, what is a function?

6. Give one example of a function and one example that is not a function.
Function Example:
Not a Function Example:

Use the following table of data to answer questions 7 - 11. The table of data represents the price of stock at the end of the day.

Stock Market			
Day	Dow Jones Industrial Average	PA Industries	The AERA Group
1	3829.16	11.72	12.01
2	3834.44	11.77	12.09
3	3834.44	11.77	12.09
4	3828.48	11.63	12.46
5	3857.65	11.99	12.12
6	3858.92	12.57	11.91
7	3867.41	13.02	11.36

7. What was the Dow Jones Industrial Average on Day 6?

8. For what day was the stock price of The AERA Group at $12.46?

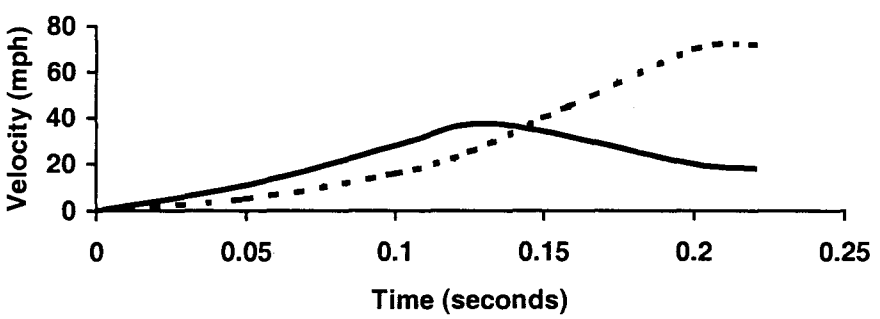
9. Identify the domain and range.

Domain:_____
Range: _____

10. What was the amount of decrease in the stock price of The AERA Group between Day 4 and Day 7?

11. For what day(s) do you think the stock market was closed?
Explanation:

Use the following graph for questions 12 - 16. The graph shows how the velocity of the hands and the softball bat vary with the time of the swing.



12. What is the speed of the hands at 0.15 seconds?

13. At what time is the bat speed 20 mph?

14. Identify the domain and range of the velocity of the softball bat.
    Domain:
    Range:

15. By how fast does the speed of the bat increase between 0.05 and 0.15 seconds?

16. In order to get the most powerful hit possible, at what time do you want the bat to hit the ball? Explanation:

The following function represents the amount of profit (P) based on the amount of money spent on advertising (x): $P(x) = 2(115 - 0.5x)$. Both x and $P(x)$ are in terms of hundreds of dollars.

17. What is the amount of profit if 3,000 dollars are spent on advertising?

18. How much was spent on advertising in order to get a profit of 15,000 dollars?

19. Identify the domain and range
    Domain:
    Range:

20. Let $a = 30$ and $b = 40$, evaluate $\dfrac{P(b) - P(a)}{b - a}$.

21. On the basis on the result in (20), does profit increase when the amount of money spent on advertising changes from 3000 to 4000 dollars? Explain why/why not.

Posttest Cover Sheet

This questionnaire is being given in order to assess college students' understanding of the concept of function. I am collecting this information as partial fulfillment of the requirements for my degree. All responses will be kept strictly confidential. Your name will be removed from the questionnaire in order to retain your privacy. Allowing access to this questionnaire has no affect on your grade in the class.

Thank you for your assistance.

_____No, I do not want to participate in the study by allowing the researcher access to my questionnaire.

_____Yes, I will participate in the study by allowing the researcher access to my questionnaire.

If you are participating in the study, please fill provide the following information.

Background Information:

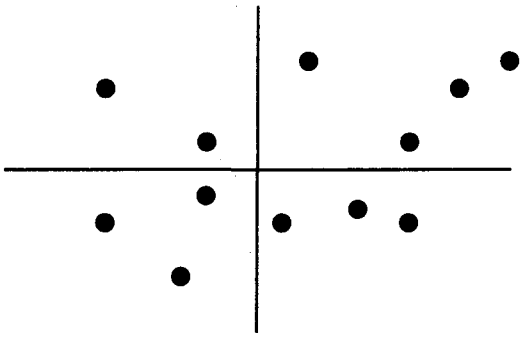
Social Security Number: (last 6 digits) ***-_____-_____

Posttest

Directions: Please read each of the following questions carefully and check your responses. In all cases, explain the reason you made that particular choice. The written responses are very important, so please give a complete explanation.

1. Determine whether the following graph represents a function:

_____ a) Yes
_____ b) No
_____ a) I do not know
Explanation:



2. Determine whether the following statement represents a function:
The relationship between shoes purchased at a store and the price of the shoes.
_____ a) Yes
_____ b) No
_____ a) I do not know
Explanation:

3. Determine whether the following represents a function:
$f(-2) = -4, f(-1) = -2, f(0) = 1$, and $f(2) = -3$.
_____ a) Yes
_____ b) No
_____ a) I do not know
Explanation:

4. Determine whether the following represents a function:

$$f(x) = \begin{cases} x - 25 & \text{for } x < 5 \\ 100 & \text{for } x = 5 \\ x^3 & \text{for } x > 5 \end{cases} \quad \text{for any real number } x?$$

_____ a) Yes
_____ b) No
_____ a) I do not know
Explanation:

5. In your opinion, what is a function?

6. Give one example of a function and one example that is not a function.
Function Example:
Not a Function Example:

Use the following table of data to answer questions 7 - 11. The table of data represents the price of stock at the end of the day.

Stock Market			
Day	Dow Jones Industrial Average	PA Industries	The AERA Group
1	3829.16	11.72	22.01
2	3834.44	11.77	22.09
3	3828.48	11.83	22.46
4	3828.48	11.63	22.46
5	3857.65	11.99	22.12
6	3858.92	12.57	21.91
7	3867.41	13.02	21.36

7. What was the Dow Jones Industrial Average on Day 4?

8. For what day was the stock price of The AERA Group at $21.91?

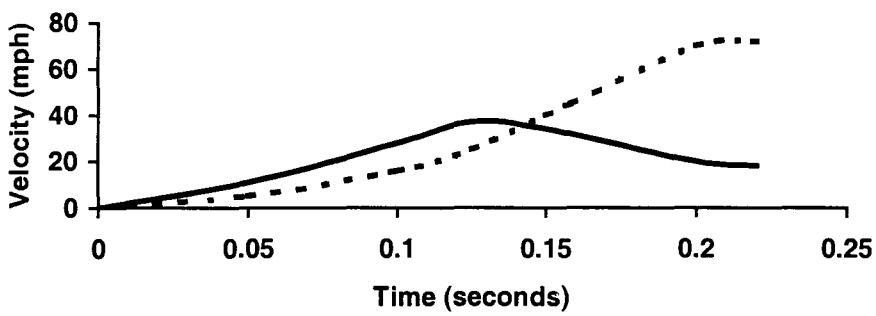
9. Identify the domain and range.

Domain:_____
Range: _____

10. What was the amount of decrease in the stock price of The AERA Group between Day 4 and Day 7?

11. For what day(s) do you think the stock market was closed?
Explanation:

Use the following graph for questions 12 - 16. The graph shows how the velocity of the hands and the softball bat vary with the time of the swing.



12. What is the speed of the hands at 0.2 seconds?

13. At what time is the bat speed 40 mph?

14. Identify the domain and range for the velocity of the hands.
    Domain:
    Range:

15. By how fast does the speed of the bat increase between .05 and .15 seconds?

16. In order to get the most powerful hit possible, at what time do you want the bat to hit the ball? Explanation:

The following function represents the amount of profit (P) based on the amount of money spent on advertising (x): $P(x) = 2(115 - 0.5x)$. Both x and $P(x)$ are in terms of hundreds of dollars.

17. What is the amount of profit if 2,000 dollars are spent on advertising?

18. How much was spent on advertising in order to get a profit of 20,000 dollars?

19. Identify the domain and range
    Domain:
    Range:

20. Let $a = 20$ and $b = 50$, evaluate $\dfrac{P(b) - P(a)}{b - a}$.

21. On the basis on the result in (20), does profit increase when the amount of money spent on advertising changes from 2000 to 5000 dollars? Explain why/why not.

APPENDIX D

VALIDATION OF THE FUNCTION TEST

According to Borg and Gall (1989) content validity is found in order to ensure that the questions of the test represent the content covered in the course. Due to the expertise required to determine content validity, a committee of five educators in the mathematics department was made. Each member was chosen because of his/her experience with college algebra. Each member recently taught the course or was a member of the curriculum committee for the course. These educators were deemed appropriate to determine whether the content of the Function Test was valid for the students in the course used in the study. The following is the cover letter was given to the validation committee along with the test, test objectives, and the table of specifications. In order to insure their cooperation, the members were given a $5 gift certificate to a local coffee shop in appreciation for their time and consideration.

Validation of Function Test

Graphing calculators have been required for many lower level college mathematics courses. It has been recommended by national mathematics organizations that the graphing calculator should be used as a tool to enhance the presentation of mathematical topics in multiple representations: symbolic, numerical, graphical, and verbal. The graphing calculator has been recommended in order to provide a variety of mathematical experiences to a diverse student population who enroll in precalculus courses. Yet, few studies have been conducted to investigate how this tool has been used in these courses. Whether the use of the graphing calculator promotes better understanding of the function concept has not been determined conclusively.

The study will investigate the impact of the graphing calculator on student understanding of functions. The following test will be used to collect data on student understanding of functions. I am requesting that you look over this test to determine whether the content for each question is appropriate for students in the first course of the precalculus sequence. In order to assist you in your decision, I have included the list of objectives that were used to develop the test. Please identify which objective(s) in the list correspond to each of the questions. Place the number(s) of the objective in the chart. For example, if you determine that objectives 3 and 8 correspond to question five, then write a 3 and 8 in the chart on line five in the second column. Also, in the third column indicate with yes or no whether the question can be solved without the use of the graphing calculator.

For your time and consideration, I have included a token of my appreciation.

Objectives:

Students will be able to:
1. identify whether a relationship is a function when represented either graphically, numerically, verbally, or symbolically.
2. provide examples and nonexamples of functions.
3. define functions.
4. determine domain and range of functions.
5. analyze functions for increasing/decreasing.
6. determine maximum/minimum of functions from graphs.
7. use functions to mode real-world relationships.
8. evaluate functions given symbolically.
9. read and interpret graphs of real data.
10. read and interpret charts of real data.

Question	Objective(s)	Can be solved <u>without</u> the graphing calculator.
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		
13		
14		
15		
16		
17		
18		
19		
20		
21		

Results of Validation Test

Listing of objectives from each member of validation committee.

Question	#1	#2	#3	#4	#5	80% Agreement
1	1	1,9	1	1,3	1,3	1
2	1	1,3,7	1	1,3	1,3	1
3	1	1,10	1	1,2,3	1,3	1
4	1	1,8	1	1,3,8	1,3	1
5	3	3	3	1,2,3	3	3
6	2	2,3	2	1,2,3	2	2
7	7,10	7,10	10	7,10	7,10	7,10
8	7,10	7,10	10	7,10	7,10	7,10
9	4,7,10	4,7,10	4,7,10	1,4,7	4,7,10	4,7,10
10	5,7,10	5,7,10	5,10	5,7,10	5,7,10	5,7,10
11	7,10	5,7,10	5,10	7,10	5,7,10	7,10
12	7,9	7,9	9	7,9	7,9	7,9
13	7,9	7,9	9	7,9	7,9	7,9
14	4,7,9	4,7,9	4,7,9	4,7,9	4,9	4,7,9
15	5,7,9	5,7,9	5,9	5,7,9	5,7,9	5,7,9
16	6,7,9	6,7,9	6,9	6,7,9	6,7,9	6,7,9
17	7,8	7,8	7,8	7,8	7,8	7,8
18	7,8	7,8	7,8	7,8	7,8	7,8
19	4,7,8	4,7,8	4,7,8	4,7	4	4,7
20	7,8	7,8	7,8	7,8	8	7,8
21	5,7	5,7,8	5,7,8	5,7,8	5,7,8	5,7,8

APPENDIX E

STUDENT INTERVIEW PROTOCOL

This section contains the student interview protocol that was followed during the interviews with two students from each of the following groups: (a) High Symbolic Manipulation-math and science, High Symbolic Manipulation-business, High Symbolic Manipulation-liberal arts, Low Symbolic Manipulation-math and science, Low Symbolic Manipulation-business, and Low Symbolic Manipulation-liberal arts. The protocol was piloted with one student from the High Symbolic Manipulation-math and science group. The interviews lasted no more than an hour and no less than one-half hour. The students were allowed to use paper-and-pencil and their graphing calculator at any time during the interview. The interviews were videotaped and all papers were collected.

Student Interview Protocol

1. Introduction

a) Put the student [S] at ease with questions:

How's the term going so far?
What classes are you taking?
How are you doing in your other classes?

b) Explain the procedures to S.

There will be three parts to the interview. First, we'll discuss the test that you took in the class. Next, we'll talk about the instruction of the class. Finally, I'll give you two math problems to do. I am interested in having you talk as you think. You may use paper and pencil or your calculator at any time during the interview.

c) Explain confidentiality.

I want to reassure you that the tape recordings of this interview will be destroyed at the end of the study. Your identity will be kept confidential and your name will never be used. Also, comments made in the interview will not be shared with your teacher.

Turn on the recorder.

2. Function Test

Present S the two tests.

How did you feel about taking the tests?

A) Clarify responses on test.

a) Follow any train of thought that S takes as long as the discussion is on the tests, the class, or the instructor in relation to the test.

b) If S does not address unclear or omitted responses, then direct S to clarify written or omitted explanations.

B) Investigate changes in responses.

I've noticed that you answer for question ## has changed, can you give me some information for the reason for the change?

This line of questioning will proceed until all of the changes have been investigated.

C) Tendencies for a representation of functions.

Was there any part of the test that you found particularly easy? difficult?
Was there a group of questions that you were the most comfortable with? How come?

Was there a group of questions that you were the least comfortable with? How come?

3) Graphing calculators.

Suppose you are an administrator here at the school. You are the one that has to make the decision of whether to allow or not allow graphing calculators in Math 131. What would you decide and why?

If given a positive response,
For what part of the class did you find graphing calculators to be beneficial?
If given a negative response,
For what part of the class did you find graphing calculators distracting?

Did you use your graphing calculator on your homework?
If so, how did you use it?
If not, why did you not use it?

Follow any line of thought regarding the use of graphing calculators in the course.

4) Mathematical Tasks.

A) First Task.

Present S with the algebraic, symbolic, and numerical representation of the function $f(x) = \frac{|x-2|}{x-2}$.

a) Determine whether a function.

Is this relationship a function?
How did you determine whether this is a function?

b) Determine domain.

What is the domain of this function?
How did you determine the domain of this function?

c) Determine whether continuous.

What does continuous mean to a function?
Is this function continuous?

B) Second Task.

Present S the function $g(x) = \frac{4}{x+3}$ in only the symbolic representation.

a) Determine whether a function.

Is this relationship a function?
How did you determine whether this is a function?

b) Determine domain.

What is the domain of this function?
How did you determine the domain of this function?

c) Determine whether continuous.
Is this function continuous?
How did you determine whether this function is continuous?

5. Explain the answers to the problems if the student does not answer. Answer any questions. Thank them for helping.

APPENDIX F

VALIDATION OF STUDENT INTERVIEW PROTOCOL

In order to determine whether the questions for the student interviews were valid and unbiased, a committee of five mathematics educators was identified. The following is the cover letter that was given to the validation committee along with the interview protocol and interview goals. In order to insure their cooperation, the members were given a small gift in appreciation for their time and consideration.

Validation of Student Interview

Graphing calculators have been required for many lower level college mathematics courses. It has been recommended by national mathematics organizations that the graphing calculator should be used as a tool to enhance the presentation of mathematical topics in multiple representations: symbolic, numerical, graphical, and verbal. The graphing calculator has been recommended in order to provide a variety of mathematical experiences to a diverse student population who enroll in precalculus courses. Yet, few studies have been conducted to investigate how this tool has been used in these courses. Whether the use of the graphing calculator promotes better understanding of the function concept has not been determined conclusively.

The study will investigate the impact of the graphing calculator on student understanding of functions. The following interview will be used to collect data on student understanding of functions. I am requesting that you look over the interview protocol to ensure that the questions match the goals of the interview and that the questions are not biased. In order to assist you in your decision, I have included the list of goals that were used to develop the interview.

For your time and consideration, I have included a token of my appreciation.

Goals:

To gather information about

1. students' responses on the Function Test;
2. students' preference towards a representation of functions on the Function Test;
3. reasons for changes in responses on the pretest and posttest;
4. students' use of the graphing calculator throughout the course;
5. students' motivation to use the graphing calculator;
6. whether students can apply their understanding of functions to a new mathematical task;
7. the impact of graphing calculators on students' approaches to the mathematical task;

Do you think the interview matches the goals of the interview? Yes No

If not, which part of the interview or which goal is not matched?

_____

_____

_____

Do you think the interview questions are unbiased? Yes No

If not, which questions are considered biased?

_____

_____

_____

APPENDIX G

OBSERVATION AND DOCUMENT SUMMARY FORMS

After each classroom observation, the researcher completed the following observation summary (Miles & Huberman; 1994). These summaries were short forms guided the researcher to plan for interviews, to suggest new codes, to reorient the researcher to conduct write-ups, and to help with data analysis. Also, summaries were made for any handouts given by the instructors. These summaries helped the researcher sort through volumes of data.

Observation Summary

Time:_____

Date:_____

Lesson: _____

Topics:_____

_____

_____

Graphing Calculator Use by Instructor: Demonstration/Alternative Method/Discovery

Graphing Calculator Use by Student: _____

_____

_____

_____

Anything about the lesson and calculator use that was salient, interesting, illuminating, or important?

_____

_____

_____

_____

_____

Anything about the lesson and calculator use that needs to be followed up in the interviews?

_____

_____

_____

_____

_____

Document Summary

Time:_____

Date:_____

Type: Test/Quiz/Activity/Other:_____

Topics: _____

_____

_____

Graphing Calculator: Allowed/Not allowed/Restricted

Representation: Symbolic/Graphical/Numerical/Word

Brief summary of contents: _____

_____

_____

_____

_____

_____

_____

Anything about the document that was salient, interesting, illuminating, or important?

_____

_____

_____

_____

_____

_____

_____

APPENDIX H

DESCRIPTION OF TEXTBOOK SECTIONS

Review section R.9, Modeling and Applications, demonstrated a five-step problem solving method. The steps recommended students should (1) familiarize themselves with the problem situation, (2) translate the problem situation to mathematical language or symbolism, (3) perform some type of mathematical manipulation, (4) check the solution with the problem situation, and (5) state the answer clearly. This approach was used to solve problem situations such as "An investment is made at 8%, compounded annually. It grows to $702 at the end of 1 yr. How much was original invested?" (Bittenger, et al., 1997, p. 78)

Section 1.1, Functions, Graphs, and Graphers, was the section that introduced the function concept. The definition of function was given: "A function is a correspondence between a first set, called the domain, and a second set, called the range, such that each member of the domain corresponds to exactly one member of the range" (Bittenger, et al., 1997, p. 85). Domain and range were introduced in this section as well. Techniques for identifying functions were discussed in this section. Additionally, notation for functions was introduced, and evaluation of functions was demonstrated.

Three concepts of functions and a type of function were covered in Section 1.2, Functions and Applications. Zeros of functions were defined. Also, the algebraic and graphical skills for obtaining zeros of functions were demonstrated. The second concept of functions involved increasing, decreasing, and constant functions. With the definition, graphical skills were demonstrated for determining intervals of increasing, decreasing, and constant functions. The third concept of functions was extrema. The graphical approach for locating extrema was demonstrated. Finally, piecewise functions were introduced in this section. The application problems used in this section were situations modeled with piecewise functions.

Section 1.3, Linear Functions and Applications, covered developing linear equations and applying them for problem situations. The horizontal and vertical lines and slope were defined. Linear equations were developed using the slope-intercept equation,

the point-slope equation, and the two-point equation. Also, parallel and perpendicular lines were explored. Problem situations involved the application of linear functions.

Section 1.5, Distance, Midpoints, and Circles, incorporated less function notation than the other sections in the textbook. The distance formula, midpoint formula, and the equation of a circle were introduced along with the development of the distance formula. Many of the examples dealt with using the formulas or making the equation of a circle given points or a graphical situation.

Symmetry of functions was introduced in Section 1.6. The graphical and symbolic methods for determining the symmetry of a function were demonstrated. Also, even and odd functions were defined in this section.

Section 1.7 covered the concept of transformations of functions. Twelve functions were reviewed to develop a foundation for the exploration of transformations, such as the quadratic, cubic, square root, absolute value, and rational functions. The transformations covered were horizontal and vertical shifts, horizontal and vertical dilations, and reflections. Graphical and symbolic approaches were presented.

Section 1.8, The Algebra of Functions, covered the methods for combining functions. The symbolic and graphical skills for manipulating the sums, differences, products, and quotients of functions were demonstrated. The composition of functions was also defined in this section. Also, the techniques for evaluating the combination of functions were demonstrated. Finally, skills for decomposing a function as a composition of two or more functions were addressed. Application problems were presented, such as developing the profit function by combining the revenue and cost functions. Problem situations required the composition of functions.

The last section covered in the observation period was Section 3.1, Inverse Functions, which was the last section not on a specific family of functions. In this section, one-to-one function was defined and the horizontal-line test was demonstrated for identifying one-to-one functions. Additionally, a technique for obtaining the expressions for the inverse. Composition of functions was revisited to demonstrate the special relationship of inverse functions. Additionally, techniques for determining the domain of functions were reviewed to prepare students for the domain restriction that

functions require for an inverse. Inverse functions were required to obtain solutions to the application problems.

APPENDIX I

DESCRIPTION OF CATEGORIES OF HOMEWORK AND EXAM PROBLEMS

The homework and exam problems were categorized with respect to the functional representation in which the problems were presented and worked. Homework problems are used to illustrate the categories.

Problems that were categorized as symbolic presented or required work with the symbolic representation. Question #23 from Section 1.1, "Given that $g(x) = 3x^2 - 2x + 1$, find $g(0)$ and $\frac{g(a+h) - g(a)}{h}$," was categorized as a symbolic question. Question #35 from Section 1.3 required the students to write a slope-intercept equation for a line when given the slope, negative three-fifths, and the point, $(-4, -1)$. Question #27 from Section 1.8 asked students to find $(f \circ g)(x)$ for the given functions $f(x) = x + 3$ and $g(x) = x - 3$. Symbolic questions were presented in the symbolic representation and the symbolic representation was used to obtain the answer.

Graphical homework problems were presented in the graphical representation, and the answers were obtained from the graphs. Question #21 from Section 1.1 presented a graph of a fifth degree polynomial with three points labeled. The question required an evaluation of the function, given the x-values of the three points. Question #1 from Section 1.6 asked students to determine visually whether the graph was symmetric with respect to the x-axis, the y-axis, or the origin. Question #13 from Section 3.1 asked students to use the horizontal-line test to determine whether the function shown graphically was one-to-one.

Numerical questions were presented in numerical form such as a table of data or in paired data using the ordered-pair format. Question #15 from Section 1.1 presented the domain and range in the form, {(2, 10), (3, 15), (4, 20)} and asked students to determine whether each relation was a function and to identify the domain and range. Question #1 from Section 1.8 instructed students to find the inverse for a given set of data points. Question #1 from Section 1.3 presented the data in tabular form and asked students to determine whether there was a change in the inputs x, whether there was a change in the outputs y, and whether the data could be represented by a linear function.

Written questions presented the information in words, and responses were written in words rather than symbols. Questions requiring an explanation were typical of this type of question. Question #9 from Section 1.1 presented the domain as "a set of cars in a parking lot", the correspondence as "each car's license number" and the range as "a set of numbers." The students were asked to decide whether the correspondence was a function. Question #81 from Section 1.7 asked students to explain why the graph of $y = f(-x)$ was a reflection of the graph of $y = f(x)$ across the y-axis. Question #75 form Section 3.1 was "Suppose that you have graphed a function using a grapher and you see that it is one-to-one. How could you then use the TRACE feature to make a hand-drawn graph of the inverse?"

Written-to-symbolic questions presented the data or situation in written format and the question was worked in the symbolic representation. Question #51 from Section 1.2 presented information about an enclosed rectangular garden and asked students to express the garden's area as a function of the unknown length. The information was given in the written representation and the result was obtained using the symbolic representation. Question #53 from Section 1.8 described ripples made by a stone thrown into a pond and asked students to develop functions of the radius and the area of the ripples. When the written part of a question was merely the directions, then the question was not considered a written-to-symbolic question.

Symbolic-to-graphical questions were presented symbolically and the solutions were obtained graphically. Question #29 from Section 1.2 asked students to find where the function, $f(x) = \frac{8x}{x^2+1}$, was increasing or decreasing. The function was given symbolically, rather than graphically, and the students were asked to find the intervals using their graphing calculators. Question #9 from Section 1.6 asked students to graph the equation, $5y = 4x + 5$, and determine whether it was symmetric with respect to the x-axis, the y-axis, or the origin by graphing it and checking it visually. The distinguishing feature between graphical and symbolic-to-graphical types of questions was whether the graph was or was not initially given. Questions not providing a graph were considered symbolic-to-graphical questions. When the graph was given, then the question was considered a graphical question.

APPENDIX J

INTRODUCTION TO GRAPHING CALCULATOR WORKSHEETS

The two worksheets in this appendix are the worksheets that were given to the students. Educators within the Mathematics Department developed both worksheets. The first worksheet, Practicing the Basics of the TI, was given to the students on Thursday of the first week and the second one, Graphing Practice on the TI, was given to the students the next day. The instructor allowed the students to work in small groups on the first worksheet. For the graphing worksheet, the instructor guided the students, while demonstrating the keystrokes on the graphing calculator connected to the overhead projection device.

PRACTICING THE BASICS OF THE TI

The **2nd** key allows you to do the operations printed above the regular keys.
Use this key to evaluate the following, then *round to the nearest hundredth*:

1) 3π _____

2) $\sqrt{5}$ _____

3) $4 \cdot 1^3$ _____
(the ^ key indicates you want an exponent)

4) $3\sqrt{6} - 2\sqrt{3}$ _____

Parenthesis are EXTREMELY important when using this calculator. The TI knows the order of operations, so if you input expressions correctly it will evaluate them correctly. Use your TI to evaluate the following.
(*Round to the nearest hundredth when appropriate.*)

5) $3^2 - 4(-2)(3)$
You must use the negation key (-) to indicate negative numbers!
The subtraction key is an operation.

6) $3\{4 + 3[2 - 5(1+6)]\}$
(You'll want to enter 3(4+3(2-5(1+6))) into the TI)

7) $\dfrac{\pi + 34}{3 - \pi}$
You need to use parenthesis.

8) $(-2 + 3(-6))^5$ _____

9) $(5.4 \times 10^{-13})(1.2 \times 10^7)$ _____
The scientific notation key is the EE key.

10) $\dfrac{-5}{7} - \dfrac{3}{11}$ _____
Find the decimal representation.

11) $4^{-2} - 2^{-5}$ _____

12) $\sqrt{\sqrt{6 - 2.2^2}}$ _____

13) $-5^4 + (-2)^2$ _____

Follow the directions below to learn how to **edit** and do other fun things!
(*Round to the nearest hundredth.*)

14) Enter "$5^2 - 2\pi$" and evaluate: _____

15) Press the **2nd** key and **Entry**, then use your **arrow keys** to overwrite the "5" and create the new entry "$3^2 - 2\pi$", then evaluate: _____

16) As in #15, bring your **Entry** back and change it to "$31^2 - 2\pi$" by positioning the square cursor over the "^{2}", press the **2nd** key, and then **INS** (insert). Your cursor should change to a *blinking line*. Insert a "**1**" and press **Enter**.
The result is: _____ (INSert places new characters in front of existing ones)

17) As in #15, bring your **Entry** back and change it to " $31^2 - \pi$ " by positioning the cursor over the "2" and pressing **DEL**. Press **Enter**: _____

18) Evaluate " $(31^2 - \pi)/27$ " by pressing the **2nd** key, **ANS**, then "**/27**".
(You should see "Ans/27" on the screen) The result is: _____
(The **ANS** key takes the previously-computed result and represents it on the screen with "Ans".)
(Every time you compute a new value by pressing ENTER, the result is stored in"**ANS**".)

19) Find the reciprocal of 3.2 by entering "3.2" and then using the x^{-1} key.
(*This is a faster way than using the ^ key.*) Result: _____
(Unrounded)

20) Find 45^2 quickly by entering "45" then using the x^2 key: _____

21) Store 4.5123 into X by entering "4.5123", pressing the *STO* ▷ key and indicating you want it to be stored in X. (You should see "4.5123 → X" on your screen.) Be sure to press **Enter** to tell the calculator to do it. Now evaluate " $3x^2 - 3s + 2$ " (with X being equal to 4.5123) by entering this expression into your calculator (use the "X" key). Then press **Enter**:
Result: _____ (Notice how you don't need to use the multiplication symbol between "3" and "X".)

Evaluate the following expressions with the given values for the variables:
[DO NOT round off your values for these two problems.]

22) *STOre*: $x = 2.314$, $y = 12.34$
Then evaluate: $3xy + x^2 - 6y^3$
(you need to store 2 #'s)
Result: _____

23) *STOre*: $A = 148.84$
Then evaluate: $2A - A^2 + \sqrt{A}$
Result: _____

Answers:

1)	9.42	7)	-262.31	13)	-621	19)	.3125
2)	2.24	8)	-3200000	14)	18.72	20)	2025
3)	68.92	9)	6.48×10^{-6}	15)	2.72	21)	49.55
4)	3.88	10)	-.99	16)	954.72	22)	-11183.46655
5)	33	11)	.03125	17)	957.86	23)	-21843.4656
6)	-285	12)	1.04	18)	35.48		

Graphing Practice on the TI

Graph the following on your calculator **using the standard viewing window**, checking the answers on the back. Be careful about parentheses!

1. $y = 0.03x^3 - 4$

2. $y = \dfrac{3}{x-5}$

3. $y = 3^{0.2x}$

4. $y = \dfrac{3x^2 - 8}{x^2 - 4}$

5. $y = 5\sqrt{4x - x^2}$

6. $y = 10^{\frac{2x-1}{x+4}}$

Graph the following on your calculator, **determining a good window** for that graph. Again, check your answers on the back (the window may be different). Consider the type of equation and the numbers involved when determining the best window.

7. $y = x^2 + 25$

8. $y = 0.001x$

9. $y = x^3 - 20x^2 + 69x + 90$

10. $y = \sqrt{x + 200}$

APPENDIX K

COMPARISON OF GROUPS' SCORES ON FUNCTION TEST

The percentage of correct responses for each of the six groups on the pretest and posttest were tabulated. Also, the gains for each group were tabulated. The three tables of scores were analyzed using a chi-square test. The chi-square test was used because analysis of covariance would have been inappropriate for two reasons. First, the overall mean would have used masked information. Second, the use of the means of the groups for each question would have resulted in a high probability of performing a Type I error.

The results of the chi-square test indicated that the tables of scores were independent ($p < 0.001$) with respect to the percentage of correct responses for each group and question on the Function Test. Thus, the groups responded differently on some of the questions and further analysis was required.

Table A1. Percentage of Correct Responses on the Pretest for the Groups

Question	Low Symbolic Skill Level			High Symbolic Skill Level		
	Math & Science	Business	Liberal Arts	Math & Science	Business	Liberal Arts
Identification						
1	25 (17.7)	0 (23.6)	0 (23.6)	14 (24.5)	100 (26.6)	0 (22.9)
2	0 (13.3)	0 (17.7)	50 (17.7)	29 (18.3)	0 (19.9)	25 (17.1)
3	0 (16.5)	100 (21.9)	0 (21.9)	29 (22.7)	0 (24.7)	0 (21.3)
4	0 (3.7)	0 (4.9)	0 (4.9)	29 (5.1)	0 (5.5)	0 (4.8)
Numerical						
7	100 (76.5)	100 (102.1)	100 (102.1)	100 (105.6)	100 (114.8)	100 (98.9)
8	100 (76.5)	100 (102.1)	100 (102.1)	100 (105.6)	100 (114.8)	100 (98.9)
9	0 (8.2)	0 (10.9)	0 (10.9)	14 (11.3)	50 (12.2)	0 (10.5)
10	0 (55.6)	50 (74.2)	100 (74.2)	86 (76.8)	100 (83.4)	100 (71.8)
11	75 (66.5)	100 (88.6)	100 (88.6)	71 (91.7)	100 (99.7)	75 (85.9)

(table continues)

Table A1 *(continued)*

Question	Low Symbolic Skill Level			High Symbolic Skill Level		
	Math & Science	Business	Liberal Arts	Math & Science	Business	Liberal Arts
Graphical						
12	100 (74.8)	100 (99.7)	100 (99.7)	86 (103.2)	100 (112.1)	100 (96.6)
13	100 (64.7)	100 (86.2)	50 (86.2)	57 (89.3)	100 (97.0)	100 (83.5)
14	25 (11.4)	0 (15.1)	50 (15.1)	14 (15.7)	0 (17.0)	0 (14.7)
15	50 (64.7)	100 (86.2)	100 (86.2)	57 (89.3)	100 (97.0)	100 (83.5)
16	25 (27.3)	50 (36.4)	50 (36.4)	14 (37.7)	50 (41.0)	25 (35.3)
Symbolic						
17	0 (5.5)	0 (7.3)	0 (7.3)	43 (7.6)	0 (8.2)	0 (7.1)
18	0 (5.5)	0 (7.3)	0 (7.3)	43 (7.6)	0 (8.2)	0 (7.1)
19	0 (1.8)	0 (2.4)	0 (2.4)	14 (2.5)	0 (2.7)	0 (2.3)
20	0 (5.0)	0 (6.6)	0 (6.6)	14 (6.9)	0 (7.5)	25 (6.4)
21	0 (5.0)	0 (6.6)	0 (6.6)	14 (6.9)	0 (7.5)	25 (6.4)

Note: $\chi^2(90, N = 20) = 2077,\ p < 0.001$. The values in the parentheses are the expected values obtained from the chi-square test.

Table A2. Percentage of Correct Responses on the Posttest for the Groups

Question	Low Symbolic Skill Level			High Symbolic Skill Level		
	Math & Science	Business	Liberal Arts	Math & Science	Business	Liberal Arts
Identification						
1	75 (73.2)	100 (82.5)	100 (93.8)	100 (106.0)	100 (127.6)	100 (91.9)
2	75 (42.2)	0 (47.7)	50 (54.2)	57 (61.2)	100 (73.7)	50 (53.1)
3	50 (63.6)	50 (71.8)	100 (81.6)	100 (92.2)	100 (110.9)	100 (79.9)
4	25 (34.5)	0 (38.9)	100 (44.2)	71 (50.0)	50 (60.1)	25 (43.3)
Numerical						
7	100 (76.3)	100 (86.1)	100 (97.9)	100 (110.6)	100 (133.1)	100 (95.9)
8	100 (76.3)	100 (86.1)	100 (97.9)	100 (110.6)	100 (133.1)	100 (95.9)
9	75 (54.1)	0 (61.0)	100 (69.3)	100 (78.4)	100 (94.3)	50 (67.9)
10	50 (65.0)	100 (73.4)	100 (83.4)	86 (94.2)	100 (113.4)	75 (81.7)
11	50 (63.1)	100 (71.2)	100 (80.9)	71 (91.5)	100 (110.0)	75 (79.3)

(table continues)

Table A2 (*continued*)

Question	Low Symbolic Skill Level			High Symbolic Skill Level		
	Math & Science	Business	Liberal Arts	Math & Science	Business	Liberal Arts
Graphical						
12	100 (76.3)	100 (86.1)	100 (97.9)	100 (110.6)	100 (133.1)	100 (95.9)
13	100 (76.3)	100 (86.1)	100 (97.9)	100 (110.6)	100 (97.0)	100 (95.9)
14	75 (63.6)	100 (71.8)	50 (81.6)	100 (92.2)	100 (110.9)	75 (79.9)
15	50 (57.3)	50 (64.6)	100 (73.4)	100 (83.0)	50 (99.8)	100 (71.9)
16	0 (20.9)	0 (23.5)	50 (26.8)	14 (30.2)	50 (41.0)	25 (26.2)
Symbolic						
17	25 (24.6)	0 (27.7)	0 (31.5)	43 (35.6)	100 (42.8)	25 (30.9)
18	0 (21.4)	0 (24.1)	0 (27.4)	43 (31.0)	100 (37.3)	25 (26.9
19	25 (14.5)	0 (16.4)	0 (18.6)	14 (21.0)	50 (25.3)	25 (18.2)
20	0 (35.9)	0 (40.5)	0 (46.0)	57 (52.0)	100 (63.6)	25 (45.1)
21	0 (35.9)	0 (40.5)	0 (46.0)	57 (52.0)	100 (63.6)	25 (45.1)

Note: $\chi^2(90, N = 20) = 1435,\ p < 0.001$. The values in the parentheses are the expected values obtained from the chi-square test.

Table A3. Gain in Percentage of Correct Responses on the Function Test for the Groups

Question	Low Symbolic Skill Level			High Symbolic Skill Level		
	Math & Science	Business	Liberal Arts	Math & Science	Business	Liberal Arts
Identification						
1	50 (57.3)	100 (60.6)	100 (60.6)	86 (78.8)	0 (114.6)	100 (64.0)
2	75 (30.0)	0 (31.7)	0 (31.7)	28 (41.2)	100 (59.9)	25 (33.5)
3	50 (55.3)	-50 (58.6)	100 (58.6)	71 (76.1)	100 (110.6)	100 (61.8)
4	25 (31.8)	0 (33.7)	100 (33.7)	42 (43.8)	50 (63.6)	25 (35.5)
Numerical						
7	0	0	0	0	0	0
8	0	0	0	0	0	0
9	75 (47.4)	0 (50.2)	100 (50.2)	86 (65.3)	50 (94.9)	50 (53.0)
10	50 (13.1)	50 (13.9)	0 (13.9)	0 (18.1)	0 (26.3)	-25 (14.7)
11	0	-50	0	0	0	0

(table continues)

Table A3 (*continued*)

Question	Low Symbolic Skill Level			High Symbolic Skill Level		
	Math & Science	Business	Liberal Arts	Math & Science	Business	Liberal Arts
Graphical						
12	0 (1.8)	0 (1.9)	0 (1.9)	14 (2.5)	0 (3.7)	0 (2.1)
13	0 (12.2)	0 (12.9)	50 (12.9)	43 (16.8)	0 (24.4)	0 (13.7)
14	50 (54.0)	100 (57.2)	0 (57.2)	86 (74.3)	100 (108.0)	75 (60.3)
15	0 (5.6)	-50 (6.0)	0 (6.0)	43 (7.8)	-50 (11.3)	0 (6.3)
16	-25 (3.3)	-50 (3.5)	0 (3.5)	0 (4.5)	0 (6.6)	25 (3.7)
Symbolic						
17	25 (79.7)	0 (20.9)	0 (20.9)	0 (27.1)	100 (39.4)	25 (22.0)
18	0 (16.4)	0 (17.4)	0 (17.4)	0 (22.6)	100 (32.8)	25 (18.4)
19	25 (13.1)	0 (13.9)	0 (13.9)	0 (18.1)	50 (26.3)	25 (14.7)
20	0 (31.9)	100 (33.8)	0 (33.8)	43 (43.9)	100 (63.8)	0 (35.7)
21	0 (31.9)	100 (33.8)	0 (33.8)	43 (43.9)	100 (63.8)	0 (35.7)

Note: $\chi^2(90, N = 20) = 2725,\ p < 0.001$. The values in the parentheses are the expected values obtained from the chi-square test. Questions #7, #8, and #11 do not have expected values because these rows were omitted in the chi-square test because the row of values were zero (Freund & Simon, 1997). Negative values were entered as zeros in the matrix when the chi-square test was performed.