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Title: Solving Polynomial Equations From 2000 B.C. Through 20th Century

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This paper is divided into two parts. The first part traces (in details providing proofs and examples) the history of the solutions of polynomial equations(of the first, second, third, and fourth degree) by radicals from Babylonian times (2000 B.C.) through 20th century. Also it is shown that there is no solution by radicals for the quintic (fifth degree) and higher degree equations.

The second part of this thesis illustrates both numerical and graphical solutions of the quintic and higher degree polynomial equations using modern technology such as graphics calculators (TI-85, and HP-48G) and software packages (Matlab, Mathematica, and Maple).

Solving Polynomial Equations From 2000 B.C Through 20^{th} Century

by

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Introduction

Polynomial equations have a very long history. As early as 2000 B. C., the Babylonians could solve pairs of simultaneous equations of the form: $x + y = s$, and $xy = p$, which are equivalent to the quadratic equation $x^2 + p = sx$. At that time, there was no sign of an algebraic formulation, so their solution was based on geometrical constructions.

Many centuries later, *algebra* was invented by the Arabic mathematician Al-khawarizmi (ca. 825). The word algebra came from the title of his book *Hisab al-jabr w'al-muqa-bala.* Al-jabr translates to restoration or completion, while al-muqa-bala means reduction or balancing. Thus, according to Al-khwarizmi, algebra is just the art of reducing and solving equations. His method for solving quadratic equations was remarkable. It has been used in the sixteenth century as a reference to many mathematicians such as G. Cardano (1501-1576) and L. Ferrari (1522-1565) who discovered the solution of the cubic and quartic equations.

Discovering the solutions of the cubic and quartic equations by radicals were the most important mathematical achievements of the sixteenth century. These remarkable discoveries encouraged mathematicians to try to solve the quintic (fifth degree) and higher degree equations by radicals. In the seventeenth and eighteenth centuries, solving the quintic equation had been the object of many mathematicians such as Euler (1707-1783) and Lagrange (1736-1813). Two centuries passed and no one could provide a solution by radicals. At this point, some mathematicians tried to analyze the problem from different prospective.

In 1824, the Norwegian mathematician Neils Abel (1802-1829) proved that it is impossible to solve the general equation of the fifth degree in terms of radicals. By his proof, he closed the door of competition in this dilemma.

At the same period of that time, a French mathematician, Evarist Galois (1811-1832), had another approach to the problem of quintic by using fields and groups. In fact, he created the study of groups, and he was the first one to use the word *"group"* in its technical sense. Galois had a shorter life (21 years) than Abel's.

On the other hand, the quintic and higher degree equations must have solutions, and since there is no algebraic way of finding them, numerical methods are very useful and effective to compute the solutions numerically.

This paper is divided into two parts. In part one, I will trace the history of the solution of polynomial equations from Babylonian times (2000 B.C) until the nineteenth century when Abel and Galois proved the impossibility of the solution of the quintic by radicals.

In part two, I will use modern technology such as graphic calculators and some software packages (Matlab, Mathematica, and Maple) to solve quintic and higher degree equations. In each case, I will pick up one or two examples and then solve them numerically and graphically by using the above tools.

PART ONE

Linear Equations:

The ancient Chinese discovered a method for solving linear equations in any number of unknowns during the Han dynasty (206 B.C.- A.D.220). It appears in the famous book *Nine chapters of mathematical art* which was written during this period. Actually, they started to solve the linear equation of the form ax = b. After that, they developed their methods to solve sets of simultaneous equations in the form:

$$\begin{array}{ccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \ldots + & a_{1n}x_n & = b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \ldots + & a_{2n}x_n & = b_2 \\
\vdots & & \vdots & & & \vdots & \vdots \\
a_{n1}x_1 & + & a_{n2}x_2 & + & \ldots + & a_{nn}x_n & = b_n
\end{array}$$

They subtracted a suitable multiple of each equation from the one below it until a triangular system was obtained:

$$\begin{array}{ccccccc}
c_{11}x_1 & + & c_{12}x_2 & + & \ldots + & c_{1n}x_n & = d_1 \\
 & & c_{22}x_2 & + & \ldots + & c_{2n}x_n & = d_2 \\
 & & & \ddots & & \vdots & \vdots \\
 & & & & & c_{nn}x_n & = d_n
\end{array}$$

Then $x_n = d_n/c_{nn}$, and $x_{n-1}, \ldots, x_2, x_1$ were found by back substitution.

This type of calculation was particularly suited to a Chinese device called the counting board. For example, we solve the following system:

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 39 \\ 2x_1 + 3x_2 + x_3 &= 34 \\ x_1 + 2x_2 + 3x_3 &= 26 \end{aligned}$$

This method begins by displaying the coefficients of the unknowns and the absolute terms in the form of a matrix. This is followed by a series of manipulations with two columns at a time, the goal being to obtain a column consisting of only one nonzero coefficient and an absolute term. Finally, the matrix could be such that the zero terms form a triangle. The main stages of the working of this problem on the *counting board* are shown below (Lay-Yong & Kang Shen, Historia-Mathematica, 16, 1989, pp 113-114):

$$\underset{\mathbf{c_3 \quad\; c_2 \quad\; c_1}}{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \\ 26 & 34 & 39 \end{bmatrix}} \quad \underset{\mathbf{c_2=3c_2-2c_1}}{\begin{bmatrix} 1 & 0 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 1 \\ 26 & 24 & 39 \end{bmatrix}} \quad \underset{\mathbf{c_3=3c_3-c_1}}{\begin{bmatrix} 0 & 0 & 3 \\ 4 & 5 & 2 \\ 8 & 1 & 1 \\ 39 & 24 & 39 \end{bmatrix}} \quad \underset{\mathbf{c_3=5c_3-4c_2}}{\begin{bmatrix} 0 & 0 & 3 \\ 0 & 5 & 2 \\ 36 & 1 & 1 \\ 99 & 24 & 39 \end{bmatrix}}$$

After this, we perform the following operations:

(i) $36 \times 24 - 1 \times 99 = 765$,

(ii) $765/5 = 153$,

(iii) $36 \times 39 - 1 \times 99 - 2 \times 153 = 999$,

(iv) $999/3 = 333$.

The answers are given as

$$x_3 = 99/36,$$

$$x_2 = 153/36, \text{ and}$$

$$x_1 = 333/36.$$

So, the method we know today as Gaussian elimination, according to Carl Gauss (1777-1855), was first discovered by the ancient Chinese about 19 centuries earlier (152 B.C.).

Quadratic Equations:

As early as 2000 B.C., the Babylonians could solve pairs of simultaneous equations of the form:

$$x + y = s, \text{ and } xy = p,$$

which are equivalent to the quadratic equation:

$$x^2 - sx + p = 0.$$

The equation was solved by Babylonians using the following formula:

$$\frac{s}{2} + \sqrt{(\frac{s}{2})^2 - p}\,.$$

Until today, nobody knows how this formula was derived, but most of the mathematicians agree that the derivation was not algebraic in the sense we use the term today. Here is a possible method which based on one figure. It is called an "algebraic geometry" (James K. Bidwell, *A Babylonian Geometrical Algebra*, The College Mathematics Journal, 17 (1986), no. 1, pp. 22-31).

Consider the following expressions:

$$x + y = S\ ,\ x - y = D\ ,\ \text{and } xy = P\ .$$

The algorithms using the numbers S, D, and P involve halving, squaring, addition, subtraction, and square roots. So the following figure (figure 1) was created to derive some of the geometrical algebra techniques of the Babylonians.

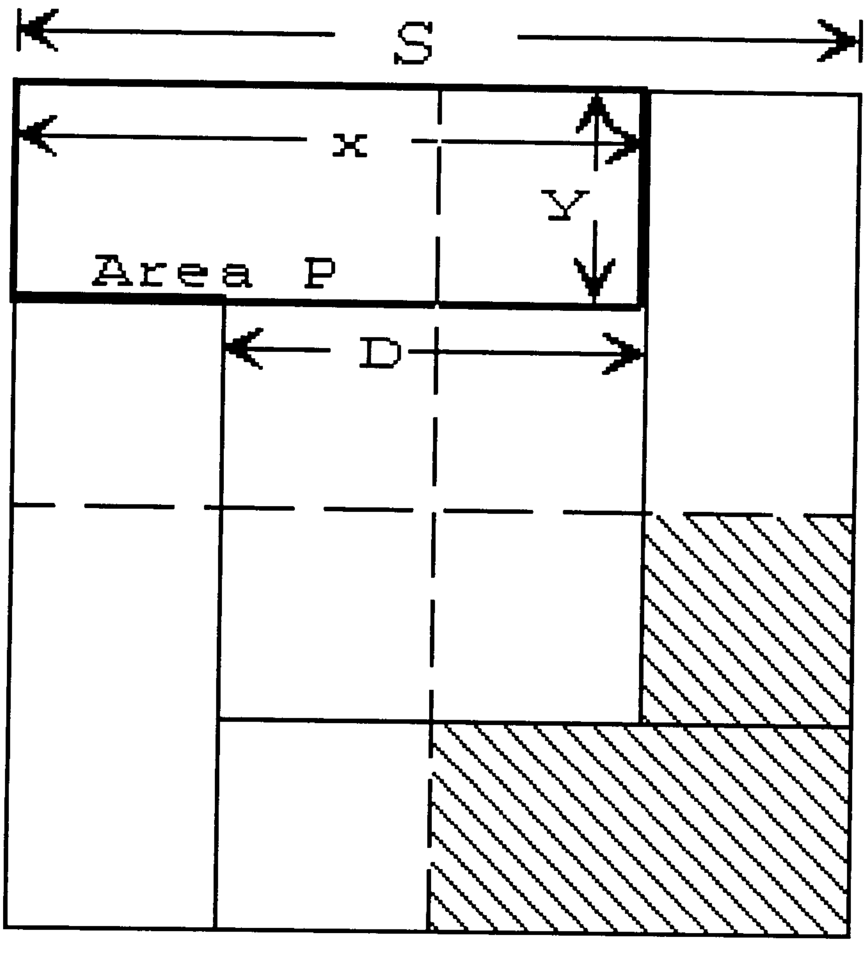


Figure 1 Babylonians' Basic Figure

An examination of figure 1 reveals the segments and areas required in the prescriptions P, S, S/2, $(S/2)^2$, S^2, D, D/2,

$(D/2)^2$, D^2. Also note that the rectangular area P equals the area of the shaded L-shaped region since each constitutes 1/4 of the border.

We now turn to the geometrical algebra within the text prescription by manipulating parts of figure 1. Here is the Babylonians' proof of the quadratic equation with its associated geometrical steps.

Problem : (a) $x + y = S$, $xy = P$

Solution: (b) $S/2$, P (halving)

(c)-(d) $(S/2)^2$, P (squaring)

(e) $(S/2)^2 - P = (D/2)^2$, $S/2$ (subtraction)

(f) $\sqrt{((S/2)^2 - P)} = D/2$, $S/2$ (square root)

(g) $x = (S/2) + (D/2)$, $y = (S/2) - (D/2)$

Figure 2 shows the associated steps (a) to (g) that demonstrate the validity of the algorithm using the basic figure and its parts.

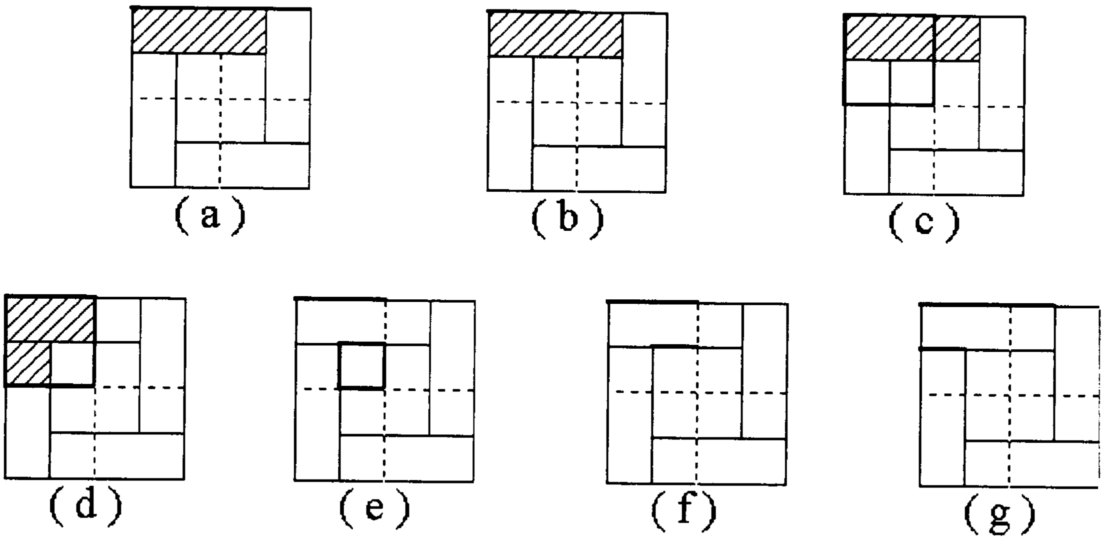


Figure 2 Possible Babylonians' Solution

From step (c) to (d), we have the necessary step from area P to its L-shaped piece within $(S/2)^2$. The keys to this algorithm are the equivalence of area P and the L-shaped area, and the values in step (g). Also from the basic figure, it is clear that $(S/2)^2$ must be greater than P.

Many centuries later (in ca.628), a Hindu mathematician called Brahmagupta gave an explicit solution to the quadratic equation $\mathbf{ax^2 + bx = c}$ by the following words:

> 'To the absolute number multiplied by four times the (coefficient of the) square, add the square of the (coefficient of the) middle term; the square root of the same, less the (coefficient of the) middle term, being divided by twice the (coefficient of the) square is the value'. (Stillwell, 1989, p. 51).

In our modern notation, this solution is:

$$x = \frac{\sqrt{-4ac + b^2} - b}{2a}.$$

In fact, Brahmagupta considered the equation: $\mathbf{x^2 - 10x = -9}$, and found that

$$x = \frac{\sqrt{-4(9) + (-10)^2} - (-10)}{2} = 9$$

Note that the negative numbers were introduced by the Hindus to represent debts, and used in math operations first by Brahmagputa. After 148 years, Brahmagputa's works were brought to Baghdad during the era of the Caliph Al-Mansur (ca. 766) and translated into Arabic.

At that time, many Arabic mathematicians appeared. The most important one was Mohammed Al-Khawarizmi (ca. 825), the

Euclid of the East. He is the inventor of algebra. Also, the origin of the word 'algebra' came from the title of his book "*Hisab al-jabr w'al-muqa-balah*". Al-jabr translates to 'restoration' or 'completion', while al-muqa-balah means "reduction" or "balancing". Thus according to Al-Khawarizmi, algebra is just the art of reducing and solving equations. In addition to that, he introduced a new word into the vocabulary of mathematics. This word is "al-gorithm" which means the art of calculating in any particular way.

Now let's see his contribution in the quadratic equation. A typical problem considered by Al-khawarizmi reads as follows:

"A square and ten of its roots equal to nine and thirty dirhem, that is you add ten roots to one square, the sum is equal to nine and thirty."

The problem leads to an equation of the type:

$$\mathbf{x^2 + px = q}, \text{ with } p = 10, \text{ and } q = 39.$$

To solve this equation, represent x^2 by a square of side x, and 10x by two 5x rectangles (figure 3). Then complete the squre to get the extra square of area 25. Therefore, the square of side x + 5 has the area 25 + 39, since 39 is the given value of $x^2 + 10x$. Thus the big square has area 64, hence its side x + 5 = 8. This gives the solution x = 3.

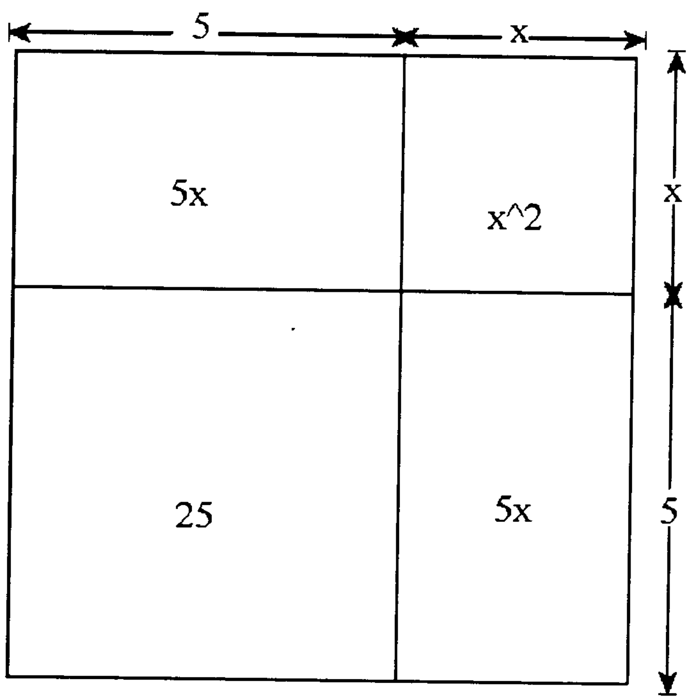


Figure 3 Al-Khwarizmi's Solution

This method of Al-khwararizmi is called the method of completing the square. Also, like Euclid, Al-khawarizmi did not admit negative lengths, so the solution $x = -13$ to the equation $x^2 + 10x = 39$, does not appear.

Let's demonstrate another example by using the same method. Let the square ABCD equal 8x plus 20. In algebraic notation, this will be $x^2 = 8x + 20$ (figure 4). Now suppose that AE is the coefficient of x, namely 8. Then the rectangle AEHD is 8x, and the remainder (the rectangle EBCH) will be exactly 20. Let AE be divided equally at F, and construct the squares of FB and FE, which are FBKP and FELM.

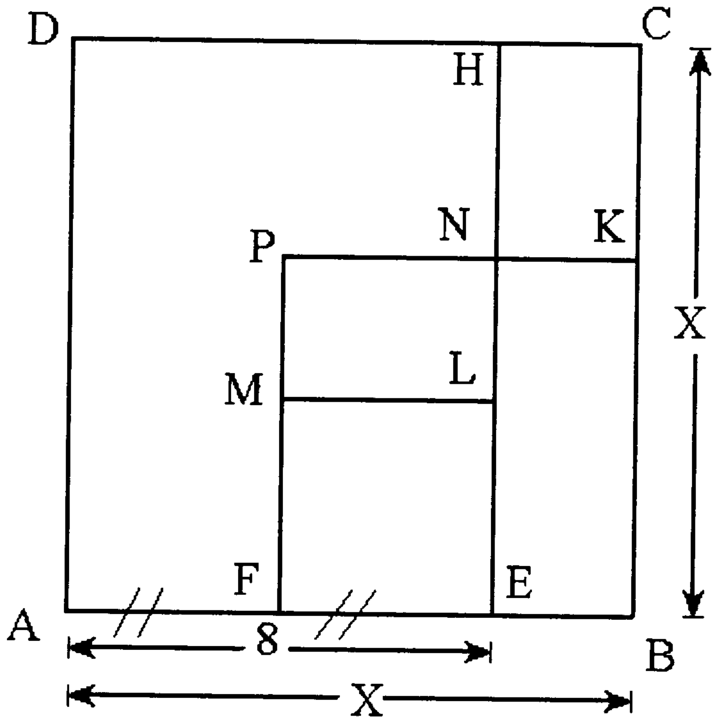


Figure 4 Al-khwarizmi's Method

Since BC = BA and BK = BF, then KC = FA = FE = NP. Also, since EL = EF and EN = BF, then NL = EB = NK. Hence, NK + NH = NP + NL, and the angles at A, E, and N are right angles. Therefore, the rectangles NKCH and NLMP are equal. But the rectangles NKCH + NKBE = 20, and therefore the rectangles NLMP + NKBE = 20, and by adding the square of EFML which is 16 (since FE = 4), the square FBKP will be 36. Hence, the side FB will be 6. Therefore, AB = FB + FA = the value of x = 10 (since FA = 4).

Cubic Equations:

The solution of the cubic equation was the most important mathematical achievement in the sixteenth century. This solution was discovered by the Italian mathematician Girolamo Cardano (1501-1576). He presents his algebra in the geometric style of Al-Khwarizmi (whom he describes as the originator of algebra).

Cardano found a rule to solve the depressed cubic equation of the form: $\mathbf{y^3 + py = q}$. The rule is:

> "Cube one-third the coefficient of y; add to it the square of one-half the constant of the equation; and take the square root of the whole. You will duplicate (repeat) this, and to one of the two you add one-half the number you already squared and from the other you subtract one-half the same. You will then have a *binomium* and its *apotome*. Then, subtracting the cube root of the *apotome* from the cube root of the *binomium*, the remainder [or that which is left] is the value of y"(G.Cardano, The Great Art or The Rules of Algebra, 1968,pp. 98-99).

Here is how Cardano derived the above solution. He imagined a large cube (figure 5), having side AC, whose length is t. Side AC is divided at B into segment BC of length u and segment AB of length t-u. Here t and u are auxiliary variables whose values we must find.

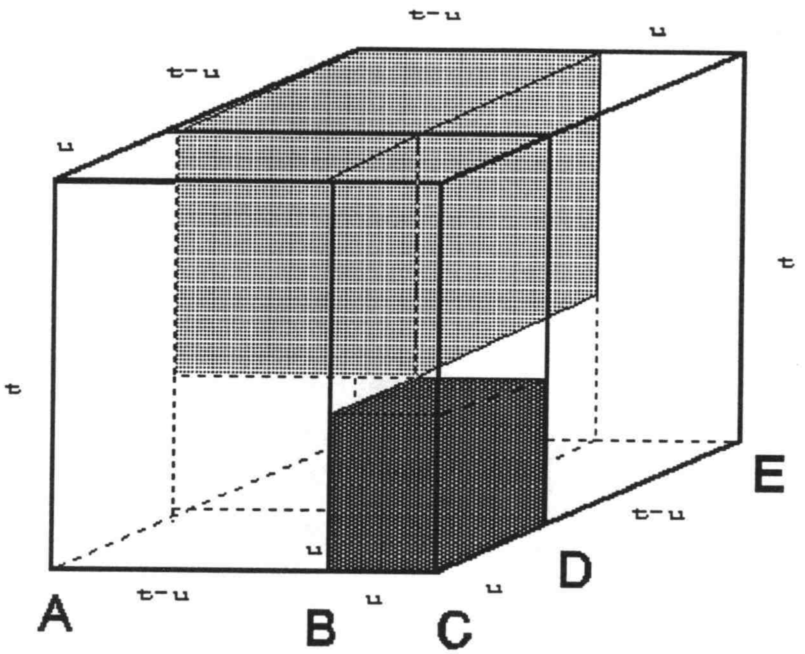


Figure 5 Cardano's Solution

From this diagram, the large cube can be divided into six pieces such that:

1) a small cube in the lower front corner with volume u^3
2) a large cube in the upper back corner with volume $(t-u)^3$
3) two upright slabs, one facing front along AB and the other is facing the right side DE with volume $tu(t-u)$
4) a tall block in the upper front corner, standing upon the small cube, with volume $u^2(t-u)$
5) a flat block in the lower back corner, beneath the large cube, with volume $u(t-u)^2$. (Dunham, Journey Through Genius, 1990, pp.142-145).

In fact, the large cube's volume, t^3, equals the sum of these six component volumes. That is,

$$\mathbf{t^3 = u^3 + (t - u)^3 + 2tu(t - u) + u^2(t - u) + u(t - u)^2}$$

or

$$(t - u)^3 + [2tu(t - u) + u^2(t - u) + u(t - u)^2] = t^3 - u^3$$

Then by factoring the common $(t - u)$, we get

$$(t - u)^3 + (t - u)[2tu + u^2 + u(t - u)] = t^3 - u^3$$

Now let's simplify these terms such that

$$\mathbf{(t-u)^3 + 3tu(t-u) = t^3-u^3.}$$

In the last equation, if we put $y = t - u$, $3tu = p$, and $t^3 - u^3 = q$, we will get our original depressed cubic:

$$\mathbf{y^3 + py = q.}$$

So, if we can determine the values of t and u in terms of p and q from the depressed cubic, then $y = t - u$ will be the solution we seek.

To get the solution, let's consider the following two conditions such that

$$\mathbf{3tu = p} \text{ and } \mathbf{t^3-u^3 = q}$$

Eliminating u gives a quadratic in t^3,

$$\mathbf{t^3 - p^3/27t^3 = q}$$

Multiply both sides by t^3 to get the quadratic equation:

$$\mathbf{t^6-qt^3-p^3/27 = 0}, \text{ or}$$

$$\mathbf{(t^3)^2-q(t^3)-p^3/27 = 0}$$

with roots

$$t^3 = \frac{q}{2} + \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}\ .$$

By using only the positive square root, we have

$$t = \sqrt[3]{\frac{q}{2} + \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}}\ .$$

But we know that $u^3 = t^3 - q$, therefore, we have

$$u^3 = \frac{q}{2} + \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3} - q$$

or

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}}\ .$$

Finally, we have the Cardano's rule for solving the depressed cubic equation $\mathbf{y^3 + py = q}$:

$\mathbf{y = t - u}$

or

$$y = \sqrt[3]{\frac{q}{2} + \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}} - \sqrt[3]{-\frac{q}{2} + \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}}\ .$$

This expression is called a "solution by radicals" or an "algebraic solution" for the depressed cubic.

In fact, Cardano used his rule to solve the following example: $\mathbf{x^3 + 6x = 20}$, (Cardano, The Great Art, p. 99). The solution is:

> "Cube 2, one-third of 6, making 8; square 10, one-half the constant; 100 results. Add 100 and 8 making 108, the square root of which is $\sqrt{(108)}$. Thus you will obtain the *binomium* $\sqrt{(108)} + 10$ and its *apotome* $\sqrt{(108)} - 10$. Take the cube roots of these. Subtract [the cube root of the] *apotome*

from that of the *binomium* and you will have the value of x:

$$x = \sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10} = 2$$

Note that, having found one solution to the cubic, we are now in a position to find any others. Since $y = 2$ in the above example, we know that $y - 2$ is one factor of $y^3 + 6y = 20$, and long division will generate the other, second degree factor. This means that: $y^3 + 6y - 20 = (y - 2)[y^2 + 2y + 10]$. The solutions to the original cubic thus comes from solving the linear and quadratic equations: $y - 2 = 0$ and $y^2 + 2y + 10 = 0$.

In his book, the Great Art, Cardano solved three kinds of equations: $y^3 + py = q$, $y^3 + q = py$, and $y^3 = py + q$ (where p and q are positive numbers). In the sixteenth century, negative numbers were rejected. Today, if we allow p and q to be negative, then the above three equations will the same and will be solved by using the above formula.

But, solving the general third-degree equation of the form

$$\mathbf{ax^3 + bx^2 + cx + d = 0}$$

was his great discovery. To do that, he introduced a new variable y by substituting

$\mathbf{x = y - b/3a}$, which yields

$$a(y - b/3a)^3 + b(y - b/3a)^2 + c(y - b/3a) + d = 0.$$

By expanding this equation we get

$$[ay^3 - by^2 + (b^2/3a)y - (b^3/27a^2)] +$$
$$[by^2 - (2b^2/3a)y + (b^3/9a^2)] + [cy - (cb/3a)] + d = 0, \text{ or}$$
$$ay^3 + [c - (b^2/3a)]y + [d - (cb/3a) + (2b^3/27a^2)] = 0$$

Note that the second-degree term, y^2, was cancelled out. So, the result is the depressed cubic equation of the form:

$$\mathbf{y^3 + py = q}\text{ , where}$$
$$p = (3ac - b^2)/3a^2, \quad \text{and}$$
$$q = (27a^2d - 9abc + 2b^3)/27a^3.$$

Now use Cardano's formula to calculate the value of y. After that, it is easy to compute the value of x from the relation of x and y (above).

To see this process in action, consider the following example:

$$2x^3 - 30x^2 + 162x - 350 = 0$$

with the substitution

$$\begin{aligned} x &= y - b/3a \\ &= y - (-30/6) \\ &= y + 5, \quad \text{we get} \end{aligned}$$

$$2(y + 5)^3 - 30(y + 5)^2 + 162(y + 5) - 350 = 0$$

which is simplified to the following form:

$$2y^3 + 12y - 40 = 0, \text{ or}$$
$$y^3 + 6y - 20 = 0.$$

Note that, the second-degree term was cancelled out as desired. This is the same depressed cubic equation we solved earlier, and we found that $y = 2$. Hence $x = y + 5 = 7$.

More Discussion:

From Cardano's formula, let

$$A = \frac{q}{2} + \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3},$$

$$B = -\frac{q}{2} + \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3},$$

and then set $t^3 = A$, and $u^3 = B$, such that the possible values of t will be:

$$t = A^{1/3}, \quad t = \omega A^{1/3}, \quad t = \omega^2 A^{1/3}$$

and the associated values of u will be:

$$u = B^{1/3}, \quad u = \omega^2 B^{1/3}, \quad u = \omega B^{1/3}$$

where ω has the following value:

$$\omega = \frac{-1 + i\sqrt{3}}{2}.$$

Hence, the equation $\mathbf{y^3 + py = q}$, will have the following roots:

$$y_1 = \sqrt[3]{A} - \sqrt[3]{B}$$

$$y_2 = \omega\sqrt[3]{A} - \omega^2\sqrt[3]{B}$$

$$y_3 = \omega^2\sqrt[3]{A} - \omega\sqrt[3]{B}.$$

Let $\mathbf{\Delta = q^2/4 + p^3/27}$, which is called the *discriminant*. Now suppose p and q are real numbers. Then, it is clear that Δ could be positive, zero, or negative.

Case 1:

Suppose $\Delta > 0$. Then, the square root of Δ will be real and also the values of $A = t^3$, and $B = u^3$ will be real and satisfy the relation $tu = p/3$, or

$$\sqrt[3]{A}\,\sqrt[3]{B} = \frac{p}{3}.$$

Therefore, the first root $y_1 = t - u$, is real. In this case, the other two values of y are:

$$y_2 = \omega t - \omega^2 u, \text{ and } \; y_3 = \omega^2 t - \omega u, \text{ or}$$

$$y_2 = \frac{-\sqrt[3]{A} + \sqrt[3]{B}}{2} + i\sqrt{3}\,\frac{\sqrt[3]{A} + \sqrt[3]{B}}{2}$$

$$y_3 = \frac{-\sqrt[3]{A} + \sqrt[3]{B}}{2} - i\sqrt{3}\,\frac{\sqrt[3]{A} + \sqrt[3]{B}}{2}.$$

So, the conclusion is: if $\Delta > 0$, two roots must be complex, and the third one must be real.

Case 2:

Suppose $\Delta = 0$. In this case,

$$A = t^3 = q/2, \quad \text{and} \quad B = u^3 = -\,q/2.$$

Then, the roots of the equation $y^3 + py = q$, are

$$y_1 = \sqrt[3]{\frac{q}{2}} - \sqrt[3]{-\frac{q}{2}} = 2\sqrt[3]{\frac{q}{2}}$$

$$y_2 = \frac{-\sqrt[3]{\frac{q}{2}} + \sqrt[3]{-\frac{q}{2}}}{2} + i\sqrt{3}\,\frac{\sqrt[3]{\frac{q}{2}} + \sqrt[3]{-\frac{q}{2}}}{2}$$

$$= -2\,\frac{\sqrt[3]{\frac{q}{2}}}{2}$$

$$= -\sqrt[3]{\frac{q}{2}}$$

$$= y_3.$$

Thus, $y_2 = y_3$ is a double root unless $q = 0$, which implies $p = 0$ when all three roots are equal to 0, and the equation $y^3 = 0$ is trivial. In conclusion, if $\Delta = 0$, we have two equal roots.

Case 3:

Suppose $\Delta < 0$. Since $\Delta = q^2/4 + p^3/27$, then

$$i\sqrt{-\Delta} = \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

is purely imaginary. Therefore, we have

$$A = t^3 = \frac{q}{2} + i\sqrt{-\Delta}$$

$$B = u^3 = -\frac{q}{2} + i\sqrt{-\Delta}.$$

In other words, we can write:

t^3 = a + ib, and **u^3 = -a + ib**, where $a = q / 2$, and $b = \sqrt{-\Delta}$. Using Demoivre's theorem, we have

$a = r \cos \theta$, and **$b = r \sin \theta$**, where r and θ are real.

Then, by Cardano's formula, we have

$$y = \sqrt[3]{A} - \sqrt[3]{B} = t - u$$

$$y = \sqrt[3]{a + ib} - \sqrt[3]{- a + ib}$$

$$y = r^{\frac{1}{3}}[\cos\theta + i\sin\theta]^{\frac{1}{3}} - r^{\frac{1}{3}}[- \cos\theta + i\sin\theta]^{\frac{1}{3}}$$

$$= r^{\frac{1}{3}}[\cos\frac{\theta + 2k\pi}{3} + i\sin\frac{\theta + 2k\pi}{3} + \cos\frac{\theta + 2k\pi}{3} - i\sin\frac{\theta + 2k\pi}{3}]$$

$$= r^{\frac{1}{3}}(\cos\frac{\theta + 2\pi k}{3} + \cos\frac{\theta + 2\pi k}{3})$$

$$= 2r^{\frac{1}{3}} \cos\frac{\theta + 2\pi k}{3}$$

where k is any integer. By taking k = 0, 1, 2 we obtain three distinct values of $\cos[(\theta + 2\pi k)/3]$, and therefore find three distinct values of y, each of which satisfies the cubic equation. If you take further values of k, you will repeat the same values of y, and all these values are real. So, our conclusion is: if $\Delta < 0$, all three roots are real.

After completing his work on the cubic, Cardano started to think about the equations of higher degree such as the fourth degree (quartic) and fifth degree (quintic) equations. Finally, his mind led him to the following pattern: to solve the general quadratic equation of the form

$$ax^2 + bx + c = 0,$$

he used the linear transformation

$$x = z - b/2a,$$

and got the right solution. Also, he used the linear transformation

$$x = z - b/3a,$$

to depress the general cubic equation and got the right answer. Finally, he concluded that it is possible to continue in this pattern to solve the quartic (4th degree) and quintic (5th degree) equations. For this reason, he asked his student Lodovico Ferrari to try to solve the quartic equation.

The Quartic Equations:

In response to Cardano's request, Ferrari (1522-1565) solved the general quartic equation:

$$ax^4 + bx^3 + cx^2 + dx + e = 0.$$

His method can be summarized as follows. He reduced the above quartic equation by the linear transformation

$$x = y - b/4a$$

and then divided through by a to generate a depressed quartic in y:

$$\mathbf{y^4 + py^2 + qy + r = 0.}$$

It can be further written as

$$y^4 + 2py^2 + p^2 = py^2 - qy - r + p^2$$

or

$$(y^2 + p)^2 = py^2 - qy + p^2 - r.$$

Then for any arbitrary z, we have

$$\begin{aligned}(y^2 + p + z)^2 &= [(y^2 + p) + z]^2 \\ &= (y^2 + p)^2 + 2z(y^2 + p) + z^2 \\ &= (py^2 - qy + p^2 - r) + 2z(y^2 + p) + z^2 \\ &= (p + 2z)y^2 - qy + (p^2 - r + 2pz + z^2) \\ &= Ay^2 + By + C.\end{aligned}$$

So, the quadratic equation on the right-hand side

$$Ay^2 + By + C$$

will be square only if

$$B^2 - 4AC = 0, \text{ or}$$

$$q^2 - 4(p + 2z)(p^2 - r + 2pz + z^2) = 0$$

which is a cubic equation for z, and may be solved by Cardano's rule. Note that such a value of z reduces the original problem to nothing but the extraction of square roots.

Now let's use Ferrari's method to solve the following equation:

$$y^4 + y^2 + 4y - 3 = 0.$$

Note that $p = 1$, $q = 4$, and $r = -3$. We know that

$$(y^2 + p + z)^2 = (p + 2z)y^2 - qy + (p^2 - r + 2pz + z^2).$$

By substitution with the values of p and q, we have

$$(y^2 + 1 + z)^2 = (1 + 2z)y^2 - 4y + (4 + 2z + z^2)$$

$$= Ay^2 + By + C, \text{ where}$$

$$A = 1 + 2z,$$

$$B = -4, \quad \text{and}$$

$$C = 4 + 2z + z^2.$$

Therefore, the quadratic equation

$$Ay^2 + By + C$$

will be square only if the discriminant

$$B^2 - 4AC = 0, \text{ or}$$

$$16 - 4(1 + 2z)(4 + 2z + z^2) = 0, \text{ or}$$

$$(1 + 2z)(4 + 2z + z^2) = 4, \text{ which is a cubic in z.}$$

Thus $\quad 2z^3 + 5z^2 + 10z = 0$, or

$$z(2z^2 + 5z + 10) = 0.$$

Then, either $z = 0$, or

$$2z^2 + 5z + 10 = 0.$$

Therefore, it is sufficient to choose only one of the three roots (say $z = 0$) which satisfies the above condition (the discriminant equals zero). Then, substitute (for $z = 0$) in the above equation such that

$$(y^2 + 1 + 0)^2 = (1 + 2(0))y^2 - 4y + (4 + 2(0) + (0)^2), \text{ or}$$

$$(y^2 + 1)^2 = y^2 - 4y + 4.$$

Note that the right-hand side of this equation is square since its discriminant is zero. Therefore

$$(y^2 + 1)^2 = (y - 2)^2, \text{ or}$$

$$y^2 + 1 = y - 2, \text{ or}$$

$$y^2 - y + 3 = 0$$

Now to get the other quadratic equation, let's divide the original polynomial

$$y^4 + y^2 + 4y - 3, \text{ by } \; y^2 - y + 3,$$

and get

$$y^2 + y - 1.$$

Therefore, our original problem (the quartic) became a product of two quadratic equations such that

$$y^4 + y^2 + 4y - 3 = (y^2 + y -1)(y^2 - y + 3).$$

In other words, we reduced our original problem to nothing but the extraction of square roots. So the four roots of the quadratic factors are

$$y_1 = \frac{-1 + \sqrt{5}}{2}$$

$$y_2 = \frac{-1 - \sqrt{5}}{2}$$

$$y_3 = \frac{1 + i\sqrt{11}}{2}$$

$$y_4 = \frac{1 - i\sqrt{11}}{2}.$$

Ferrari's results made Cardano very optimistic to go ahead and try to solve equations of higher degree such as the quintic (fifth degree) equation. Cardano believed that a solution by radicals is possible and he recommended that the

the same pattern be continued. In other words, to solve the quintic

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0,$$

he introduced the linear transformation $x = y - b/5a$ to depress it to the following form:

$$y^5 + my^3 + ny^2 + py + q = 0$$

and then searched for some auxiliary variables to reduce this to a quartic equation, which is known to be solvable by radicals.

The Quintic Equations:

After the solutions of the third degree (cubic) equation and the fourth degree (quartic) equation had been found by Cardano and Ferrari, the problem of solving the fifth degree (quintic) equation had been the object of many mathematicians, particularly in the 17th and 18th centuries. For example, in 1750, Leonhard Euler (1707-1783) tried to reduce the solution of the general quintic equation to that of an associated quartic equation, but he failed. Thirty years later, Joseph Lagrange (1736-1813) tried the same method, but he failed too.

Two centuries passed and no one could provide a "solution by radicals" for the quintic. After that, some mathematicians tried to solve this problem from different direction. For example, an Italian physician, Paola Ruffini

(1765-1822) proved that a general fifth, or higher, degree equation cannot be expressed by means of radicals in terms of the coefficients of the equation. However, his proof was incomplete.

This remarkable fact was established later in 1824 by the young Norwegian mathematician Niels Henrik Abel (1802-1829) at the age of twenty-two. His proof can be found in David Eugene Smith's *Source Book in Mathematics*, 1929, pp.261-266. His conclusion was: *It is impossible to solve the general equation of the fifth degree in terms of radicals.* This proof shocked most of the mathematicians around the world.

Abel lived a very short life (26 years), but his contribution is great. By his proof, he closed the door of 200 year story of race on the solution of the quintic. He showed that there is no algebraic way of finding the solution the fifth degree, or higher degree, equations. In other words, there is no formula into which we may substitute the coefficients of the general equation of the given degree (higher than four) to obtain the desired roots.

At the same period of that time, a French mathematician, Evarist Galois (1811-1832), had another approach to the problem of quintic by using fields and groups. In fact, he created the study of groups, and he was the first one to use the word *"group"* in its technical sense. Galois had a shorter life (21 years) than Abel's.

Now, we will try to examine Galois' contribution on proving the impossibility of a solution of the quintic. For example,

$\mathbf{x^5 - 6x + 3}$ over Q is not soluble by radicals.

Before we prove this theory, we introduce the following preliminaries.

Fundamental Theorem of Algebra:

In 1799, Carl Gauss proved the fundamental theory of algebra and got his doctorate degree for that from the University of Helmstadt. We can state this theorem as follows:

"Every algebraic equation with arbitrary given complex coefficients has always at least one real or imaginary root." (Uspensky 1948, p. 55).

Eisenstein's irreducibility criterion:

"Let $f(x) = a_0 + a_1x + \ldots + a_nx^n$

be a polynomial over $\mathbf{Z}$. Suppose that there is a prime q such that:

(1) $q \nmid a_n$

(2) $q \mid a_i$ $(i = 0, 1, \ldots, n - 1)$

(3) $q^2 \nmid a_0$.

Then f is irreducible over $\mathbf{Q}$". (Stewart, Ian (1989) p. 20).

Theorems:

(1) "The symmetric group $\mathbf{S}_n$ of degree n is not soluble if $n \geq 5$.

(2) "Let f be a polynomial over a field K of characteristic zero. If f is soluble by radicals, then the Galois group of f over K is a soluble group.

(3) Let p be a prime, and f an irreducible polynomial of degree p over Q. Suppose that f has precisely two non-real roots in C. Then, the Galois group of f over Q is the symmetric group $\mathbf{S}_p$" (Stewart, 1989).

(4) "If an equation with real coefficients has an imaginary root a + bi of multiplicity α, it has also the conjugate root a - bi of the same multiplicity, or, imaginary roots occur in conjugate parts" (Uspensky, 1948, p. 59).

Now let's come back and show that:

$x^5 - 6x + 3$ over $\mathbf{Q}$ is not soluble by radicals.

By Eisenstein's criterion, f is irreducible over Q.
To apply theorem 3 above, we have to show that $x^5 - 6x + 3$ has precisely two non-real roots. The following graph (figure 6) shows f(x) with three real zeros (as theorem 3 cited above), but we must be very careful.

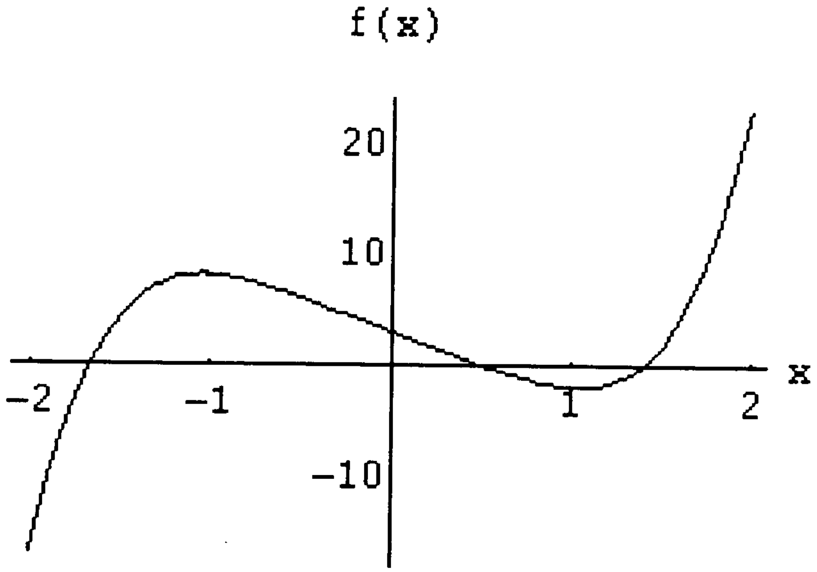


Figure 6 A quintic with three real zeros.

Since the derivative of f(x), $f'(x) = 5x^4 - 6$, is positive for $x \leq -(6/5)^{1/4}$ and $x \geq (6/5)^{1/4}$, the graph of $x^5 - 6x + 3$ intersects the x-axis in exactly three places. Thus the other two roots must be complex according to the fundamental theorem of algebra, and these two roots must satisfy the condition of theorem 4 above.

Finally, since f is irreducible over Q and since 5 is a prime, then by theorem 3 above, the Galois group of f over Q is $\mathbf{S}_5$. But by theorem 1 above $\mathbf{S}_5$ is not soluble. Finally, by theorem 2 above, f is not soluble by radicals.

PART TWO

Introduction:

Mathematicians know that quintic and higher degree equations must have solutions, but there is no algebraic way of finding them using radicals, as Abel proved. Today, numerical methods can be used to find the zeros (real or complex) to any required degree of accuracy. In fact, they are the only useful and effective methods. In this part, I will try to approach the problem by using these methods. Modern technology such as graphics calculators and some software packages (e.g. Matlab, Mathematica, Maple) will be my tools.

Graphics Calculators:

Graphics calculators are powerful tools. The wise use of them will help students to improve their skills in algebra and other topics in mathematics. In other words, we can say that weakness in algebraic skills need no longer prevent students from exploring more advanced topics in mathematics.

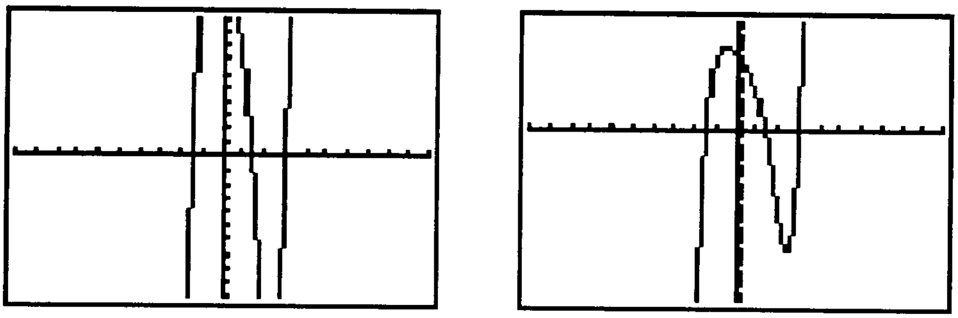
One of these advanced topics is finding the roots of a polynomial equation of the fifth degree or higher. Suppose we have a function f(x). Before calculating the roots of the function f(x), it is preferable to study its graphics behavior. Looking at the graph will help us estimate the roots' locations and know their types (real or complex). However, we must interpret results carefully. Also it is very important to understand the issue of scaling and changing in the small viewing screen of the calculators.

There are different brands of graphics calculators. Each one has its own features. The difference between them is like the difference between IBM and Macintosh. But the most important thing that they are available everywhere at reasonable prices so that most of the students can afford to buy one. Next, I will demonstrate two different examples by using two different graphics calculators: TI-85, and HP-48G.

Let's start with the following example by using the calculator TI-85. To solve the polynomial

$$x^5 - 3x^4 - x^2 - 4x + 14 = 0,$$

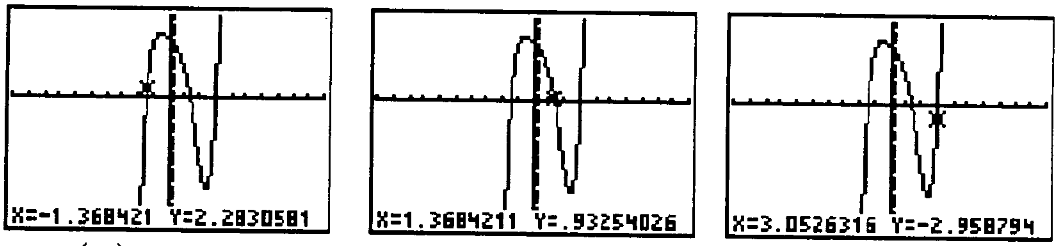
let's first graph it.



(a) (b)

Figure 7 Graph of $f(x) = x^5 - 3x^4 - x^2 - 4x + 14$

Note that the graph of figure 7(a) is incomplete because of using standard range. In figure 7(b), we changed the range of y-axis so that we can see a good picture of f(x). Also, by using the derivative, we can say that the polynomial f(x) has exactly three real roots. These three real roots can be approximated if we activate the Trace feature (figure 8) on the intersections of f(x) with the x-axis.



(a) (b) (c)

Figure 8 The same graph shown in figure 7(b) but with the Trace feature activated

When the Trace feature is activated, you will be able to see the value of x and its corresponding value of y at the bottom of the small viewing screen of the calculator. These values of x are good approximations of the desired real roots. The other two roots (the complex ones) can not be found by using the graphic technique.

The TI-85 has other features for finding the exact solutions of the polynomial f(x). In other words, by these features you will be able to compute all the roots including the complex ones. Also, it is very easy to use it. Just press *2nd* button and then press *poly* button. Next the calculator will ask you to input the order of your polynomial. After that, you only need to input the coefficients of your polynomial in descending order, and, finally, by pressing *solve* button, you will see the desired solutions on the small viewing screen. If your polynomial is of order seven and higher, you may not be able to see all the roots at once on the small viewing screen. In this case, you may use the *Arrows* features to scroll the screen up and down , or left and right.

By following the above procedures, I found that the polynomial

$$f(x) = x^5 - 3x^4 - x^2 - 4x + 14,$$

has the following solutions:

$$x_1 = 3.08674533988$$

$$x_2 = -1.41421356237$$

$$x_3 = 1.41421356237$$

$$x_4 = -0.043372669941 + 1.50528388856\ i$$

$$x_5 = -0.043372669941 - 1.50528388856\ i.$$

Comparing these values of the real roots to those which are found by using the graphic techniques, the difference is very small. More precisely, the values of x_1, x_2, and x_3 which are found in the previous method (the graphic method) are underestimated the exact values by 0.03411, -0.04597, and 0.04579 respectively.

Now let's demonstrate another example by using the calculator HP-48G. To find all the zeros of the polynomial

$$24x^5 + 143x^4 - 136x^3 + 281x^2 + 36x - 140,$$

let's first graph it on the standard range of HP-48 viewing screens. This standard range is between -6.5 to 6.5 horizontally and between -3.1 to 3.2 vertically. The following graph (figure 9) is created by using this standard range.

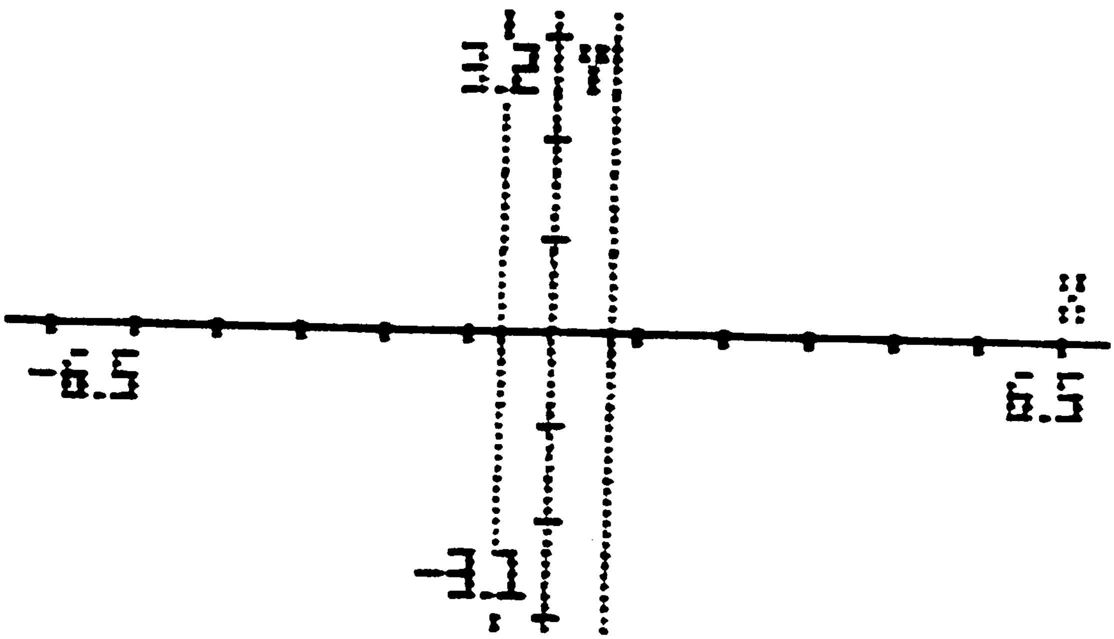


Figure 9 Graph of $f(x) = 24x^5 + 143x^4 - 136x^3 + 281x^2 + 36x - 140$ on the standard range of HP-48 calculators' viewing screen.

Note that this graph is not clear enough to estimate the number of the real roots since you see only two parallel lines. To get a clearer graph, you have to increase the range of the viewing screen horizontally and vertically (figure 10).

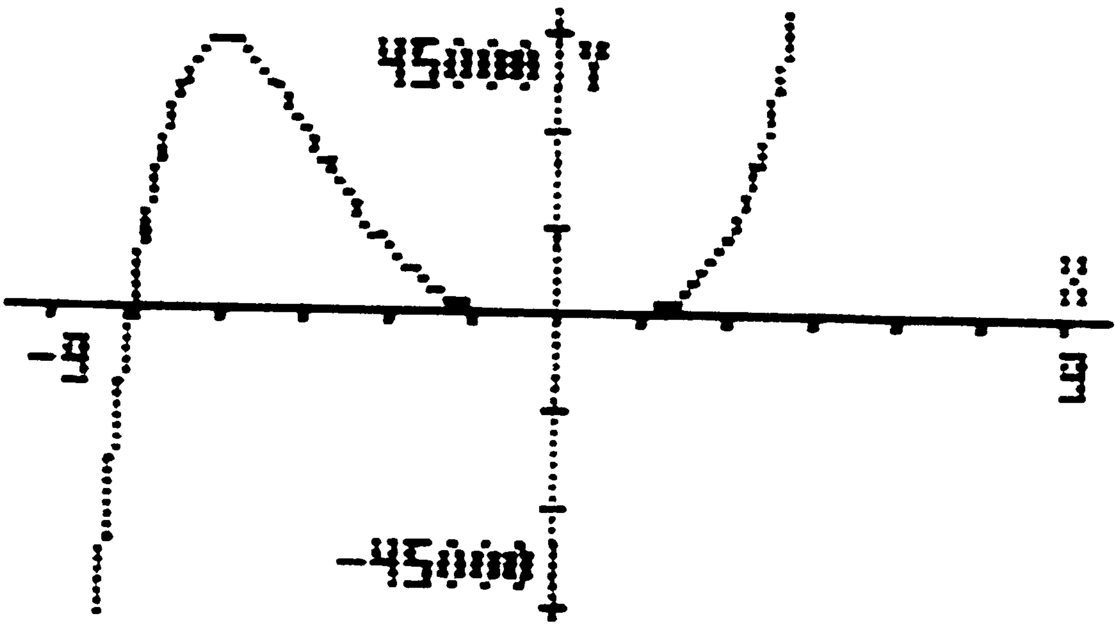


Figure 10 Graph of the same polynomial shown in figure 9, but with different range.

From figure 10, it appears that our polynomial has three real zeros. We can determine the values of these zeros one by one by pointing the cursor close to the root's location and then by using *menu keys* and then pressing the *Root* feature which is available inside *FCN* (function) folder of the *Plot* menu. The value of the desired root will appear below the graph on the left bottom of the viewing screen (figure 11).

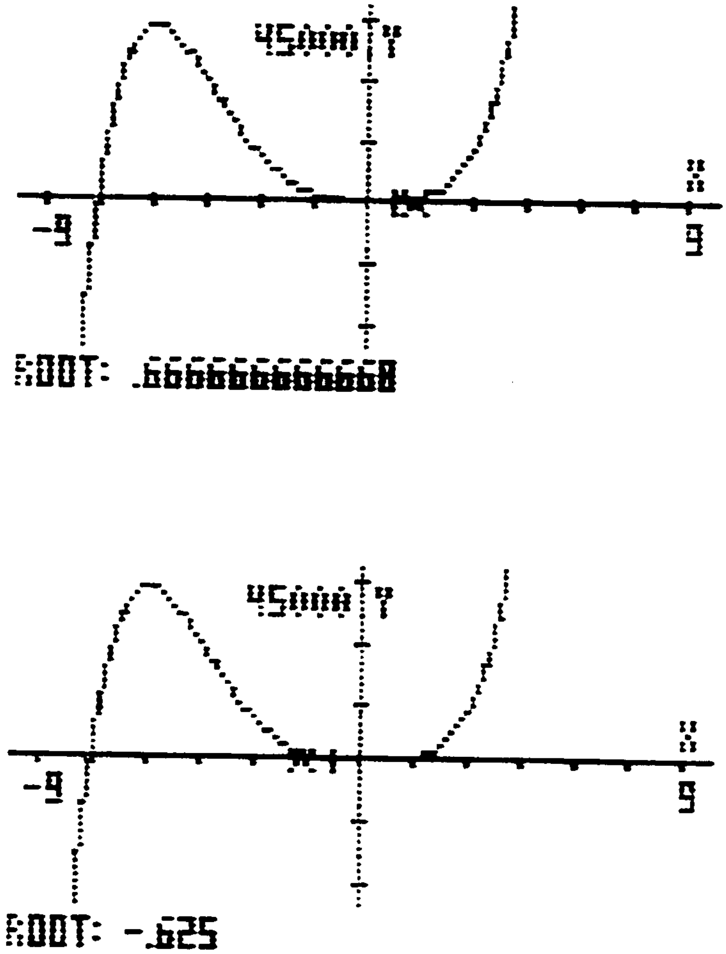


Figure 11 The same graph shown in figure 10 but with the Root feature activated.

Like the TI-85, HP-48G has other features to calculate all the zeros of the polynomial including the complex ones. To do that just press the *right-shift* key and then press the *solve* button. Next, a new small window with a list of five options will appear. By using the down arrow, choose *solve polynomial*. Then, move the highlight to the *coefficients* field, if necessary. After that, enter the coefficients of the polynomial in descending order as a row vector. Finally, press *solve* from the *Menu Labels*, so that the desired solutions will be displayed as a row vector. To be able to see all the roots in this row vector, you have to press *ON* key. This means that your solution will be sent to the first level of the *Stack*. Then, by pressing the *VIEW* key, you will able to scroll your solution left and right by using the left and the right arrows.

By following the above procedures, I found that the polynomial

$$24x^5 + 143x^4 - 136x^3 + 281x^2 + 36x - 140,$$

has the following roots:

$$x_1 = 0.666666666667$$

$$x_2 = -0.625$$

$$x_3 = -7$$

$$x_4 = 0.5 + 1.32287565553i$$

$$x_5 = 0.5 - 1.32287565553i.$$

Note that there is no difference between the values of the real roots in the two methods.

Fifteen years ago, arithmetic fell to the power of inexpensive hand calculators; ten years ago, scientific calculators offered at the touch of a button more sophisticated numerical mathematics than most students knew anything about. Today's calculators can do a large fraction of all techniques taught in the first two years of college mathematics. Tomorrow's calculators will do what computers do today (National Research Council, "Teaching," *Everybody Counts*, p. 63, National Academy Press, 1989).

Computers:

Computers are the magic of this century. They have caused a big revolution in many aspects of our lives. As a result, they became an important factor in our education, particularly, in mathematics. As a result, many software packages such as *Matlab, Mathematica,* and *Maple* were introduced and developed to help students and researchers.

Next, I will use these three software packages to solve the polynomial equations of degree five and more. To do that, I will choose arbitrary examples and analyze them numerically and graphically.

MATLAB:

MATLAB is a software package for numerical computation and graphics developed by The Math Works, Inc. It stands for mathematics laboratory. In MATLAB, polynomials are represented as row vectors containing the coefficients ordered by descending powers. For example,

p = [1 0 0 0 -6 3]

is the MATLAB representation of the polynomial

$$X^5 - 6X + 3.$$

The zeros of this equation can be obtained by using the following command:

r = roots(p).

So the roots are

$$x_1 = -1.6709$$

$$x_2 = -0.1181 + 1.5874i$$

$$x_3 = -0.1181 - 1.5874i$$

$$x_4 = 1.4016$$

$$x_5 = 0.5055.$$

Note that since the coefficient of x^4 equals to zero, then

$$x_1 + x_2 + x_3 + x_4 + x_5 = 0,$$

which provides a check of our solution. Also, we can use MATLAB to graph this polynomial (figure 12). From this graph, we can obtain valuable information such as how many real zeros there are. Now, to graph the above function, just type the following simple commands at the prompt of

MATLAB (»):

```
x = -2 : .02 : 2;
y = x.^5 - 6*x + 3;
plot(x,y)
```

The results of these commands is the graph on figure 12.

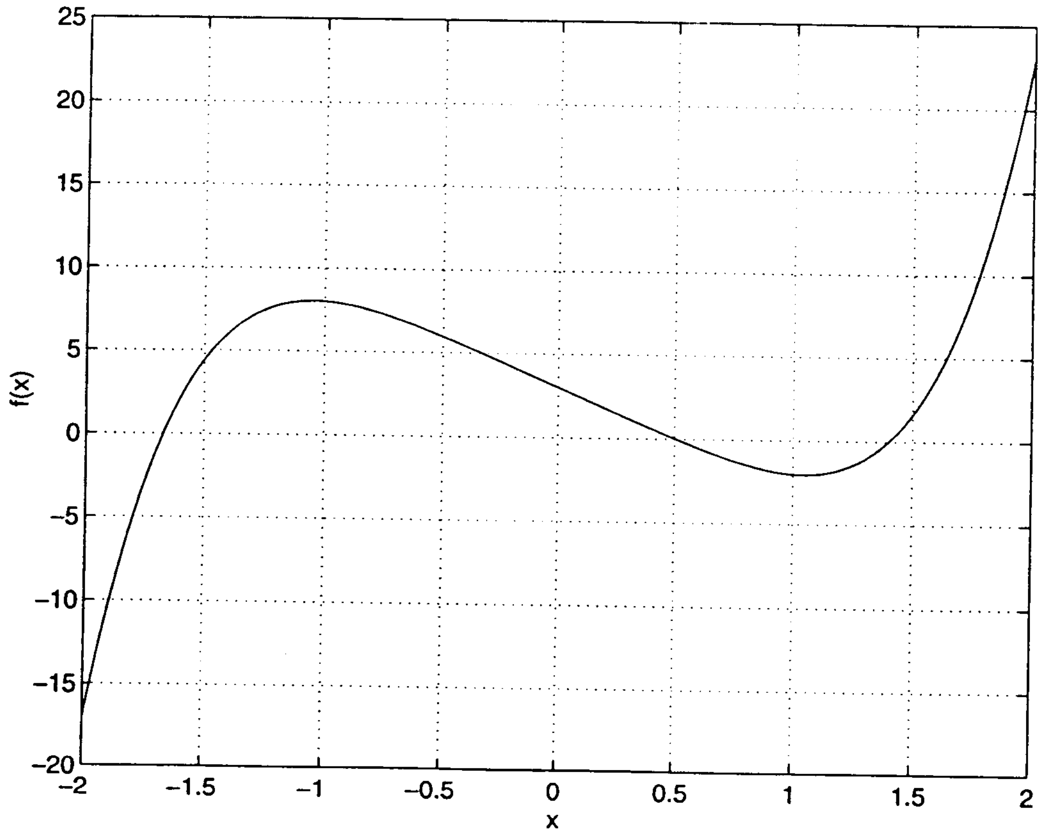


Figure 12 Graph of $f(x) = x^5 - 6x + 3$ created by Matlab.

Let's try another example. Consider the polynomial

$$6x^8 + 5x^6 + 12x^4 + 2x^2 + 1 = 0.$$

The graph of this function (see figure 13) doesn't go through the x-axis. This means that this function has no real solutions, thus all of its eight roots are complex.

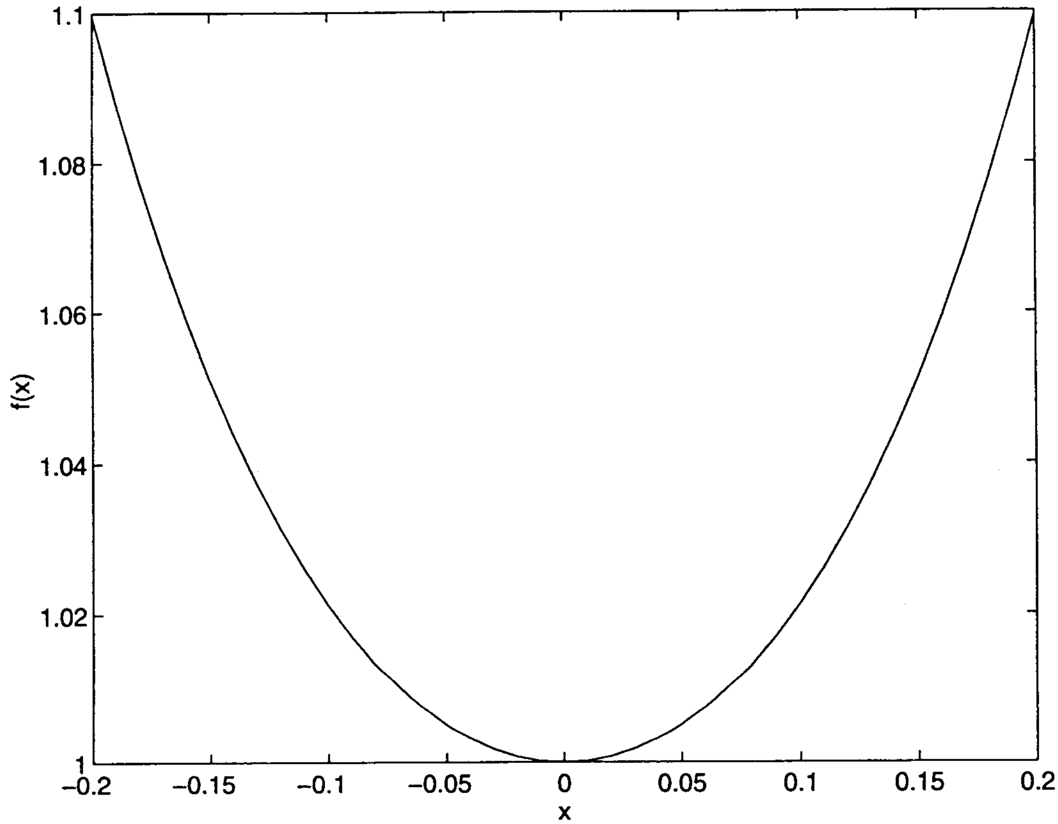


Figure 13 Graph of $f(x) = 6x^8 + 5x^6 + 12x^4 + 2x^2 + 1$

Since there is no radical formula to compute these eight roots, MATLAB will be the perfect tool to do this job too. Here, the MATLAB representation of this equation is

```
p = [6 0 5 0 12 0 2 0 1]
```

and the roots of this equation are returned in a column vector by roots:

```
r = roots(p)
```

Therefore, the eight complex roots are:

$$x_1 = -0.7077 + 0.9180i$$
$$x_2 = -0.7077 - 0.9180i$$
$$x_3 = 0.7077 + 0.9180i$$
$$x_4 = 0.7077 - 0.9180i$$
$$x_5 = -0.3384 + 0.4351i$$
$$x_6 = -0.3384 - 0.4351i$$
$$x_7 = 0.3384 + 0.4351i$$
$$x_8 = 0.3384 - 0.4351i$$

Note that each complex number is occured with its conjugate which satisfies the fundamental theorem of algebra.

Mathematica:

Mathematica is a software package for symbolic computation, numerical computation and graphics. It was developed by Wolframe Research, Inc. In *Mathematica*, the lines labelled *In[n]* are what you would type in, while the lines labelled *Out[n]* are what *Mathematica* would type back.

Mathematica can always find exact solutions to polynomials of degree four and less. For cubic and quartic equations, however, the results can be extremely complicated. (Wolfram P. 395)

In trying to find the exact solutions of polynomial equations with degrees higher than four, Mathematica runs into some fundamental mathematical difficulties. For example, the solutions to an arbitrary polynomial equation of degree five or more cannot necessarily be written as algebraic expressions. More specifically, the solutions cannot as combinations of arithmetic functions and k^{th} roots. On the other hand, factorization is one important way of braking down polynomials into simper parts. If someone factors a polynomial P(x), he must write it as a product $p_1(x)p_2(x)$ ... of polynomials $p_i(x)$.

Here is a polynomial equation, $x^5 - 7x^3 + 9$, for which explicit algebraic solutions cannot be found by applying the function *Solve[poly == 0, x]*. Mathematica returns a symbolic form of the result.

In[1]:= Solve[x^5 - 7 x^3 + 9 == 0, x]

Out[1]:= {ToRules[Roots[-7 x^3 + x^5 == -9, x]]}

In this case, you can get approximate numerical solutions to the polynomial equation by applying the function *N*.

In[2]:= N[%]

Out[2]:= {{x -> -2.72822}, {x -> -0.49031 - 0.933087I}, {x -> -0.49031 + 0.933087I}, {x -> 1.16903}, {x -> 2.53982}} where % stands for the last result. The answer is given in the transformation rules. Moreover, you can get the numerical solutions without first trying to find the exact results just by applying the function *NRoots[poly==0, x]* or *NRoots[poly==0, x, n]* to approximate the solution to n-digit precision.

In[3]:= NRoots[x^5 - 7x^3 + 9 == 0, x, 20]

Out[3]:=

x_1 == -2.72822341844705802477

x_2 == -0.49031005579943292349 - 0.93308697396167678865 I

x_3 == -0.49031005579943292349 + 0.93308697396167678865 I

x_4 == 1.16902716547715074669

x_5 == 2.5398163645687731251

NRoots will always provide you with complete set of roots for any polynomial equation in one variable. Also, this gives the numerical solutions to 20-digit precision. This solution will be more clear if you graph it by using the Mathematica *Plot* function as follows:

In[4]:= Plot[x^5 - 7x^3 + 9, {x, -3, 3}]

Out[4]:= (see figure 14)

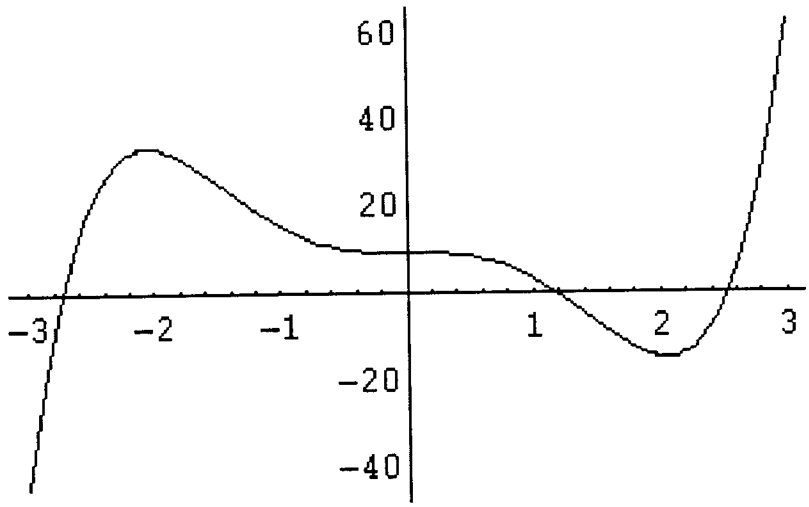


Figure 14 Graph of $f(x) = x^5 - 7x + 9$.

This figure (figure 14) shows that the above polynomial has three real solutions in the range -3 to 3.

Maple:

Maple is a software package that is devoted primarily to symbolic computation. It was developed at the University of Waterloo in Waterloo, Ontario.

Maple statements are input after the > prompt. Also, every Maple statement must end with a semicolon ; (or a colon : if the result is not to be printed). If the (semi)colon is forgotten, no further commands will work until a (semi)colon is typed.

In fact, Maple has many mathematical functions. Here, I would like to discuss one of those functions:

fsolve (<eqns>, <vars>, <options>); where

In fact, Maple has many mathematical functions. Here, I would like to discuss one of those functions:

***fsolve* (*<eqns>*, *<vars>*, *<options>*);** where

<eqns> : an equation or set of equations,

<vars> : an unknown or set of unknowns, and

<options>: parameters controlling solutions.

This function, *fsolve*, is very important to our topic, the solutions of the polynomials.

For a general equation, ***fsolve,*** seeks to compute a single real root. However, for polynomials, it will compute all real (non-complex) roots. To calculate all roots of a polynomial over the field of complex numbers, apply the *complex* option. In addition to the complex option, there are two more options: *maxsols = n*, and *intervals*. By applying the first one, the complex option, you can find one root (or all roots for polynomials) over the complex floating-point numbers. The second one, the maxsols = n option, was created specially for finding all the real roots of the polynomials. Finally, the intervals option, searchs for the roots in the given intervals only. The ranges are closed intervals, i.e. the endpoints are included in the range.

Let's apply the above features and solve the following polynomial:

$$x^7 - 19x^5 + 29x^2 - 13.$$

If you want to find two real solutions of the above polynomial, just type the following statement after the > prompt and the results will be followed as the following:

fsolve(x^7 - $19x^5$ + $29x^2$ -13, x, maxsols = 2);

$$x_1 = -4.396957651$$

$$x_2 = -0.6229502511.$$

On the other hand, if your goal is to determine five real roots of the above polynomial, repeat the same step and replace the value of maxsols by 5 such as:

fsolve(x^7 - $19x^5$ + $29x^2$ -13, x, maxsols = 5);

$$x_1 = -4.396957651$$

$$x_2 = -0.6229502511$$

$$x_3 = 4.318396722.$$

This answer means that there are only three real solutions to the above polynomial.

These three solutions can be found by using the intervals option(specifically a .. b or x = a .. b)such as:

fsolve(x^7 - $19x^5$ + $29x^2$ -13, x, 3 .. 5);

$$x = 4.318396722, \text{ or}$$

fsolve(x^7 - $19x^5$ + $29x^2$ -13, x, -3 .. 3);

$$x = -0.6229502511, \text{ or}$$

fsolve(x^7 - $19x^5$ + $29x^2$ -13, x, -5 .. infinity);

$$x_1 = -4.396957651$$

$$x_2 = -0.6229502511$$

$$x_3 = 4.318396722.$$

Note that the function *fsolve* may fail to find a root in spite of its existing. To avoid a problem like this, you should specify a range in which one of its endpoints point is converged a solution.

Finally, by applying the complex option, you can compute all the seven roots (reals and complex) of the above polynomial as follows:

fsolve(x^7 - $19x^5$ + $29x^2$ -13, x, complex);

$$x_1 = -4.396957651$$

$$x_2 = -0.6229502511$$

$$x_3 = 4.318396722$$

$$x_4 = -0.5149729864 - 1.094504516\ i$$

$$x_5 = -0.5149729864 + 1.094504516\ i$$

$$x_6 = 0.8657285765 - 0.0408647009\ i$$

$$x_7 = 0.8657285765 + 0.0408647009\ i.$$

After all, it is a good idea to clarify your algebraic solutions by graphing the above polynomial as follows:

plot(x^7 - $19x^5$ + $29x^2$ -13, x = -5 .. 5);

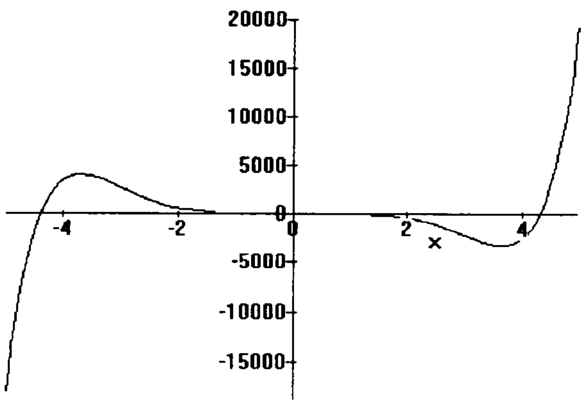


Figure 15   Graph of $f(x) = x^7 - 19x^5 + 29x^2 - 13$

In conclusion, major changes in the mathematics education curriclum are occuring today due to the use of graphics calculators and to some extent computers. In the near future, graphics calculators will likely be required for all teaching, homework, and testing in mathematics. As a result of this, students will improve their skills in algebra and other parts in mathematics. As a powerful mathematics software applications become more accessible, a true revolution in the way we teach and learn mathematics will become a reality.

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