AN ABSTRACT OF THE THESIS OF

MIN-CHIH HUANG for the degree of Master of Ocean Engineering
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Title: STABILITY OF PIPELINES UNDER SHOALING FINITE AMPLITUDE WAVES

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The horizontal stability of a bottom-laid pipeline under finite-amplitude surface gravity waves which are shoaling on a mildly sloping beach is considered. Incipient sliding motion conditions are assumed in which the wave-induced hydrodynamic pressure force is statically balanced by the Coulomb bottom frictional force. The Morison equation, which includes both drag and inertia force components, is employed to determine the wave-induced hydrodynamic pressure force exerted on the pipeline. Water particle kinematics computed both analytically from linear wave theory and numerically from the finite-amplitude Dean (1965) stream function wave theory are used to evaluate the wave-induced hydrodynamic pressure force exerted on the pipeline. Both the analytical and experimental methods of determining the drag and inertia force coefficients are reviewed. Experimental values for the Coulomb bottom frictional force coefficients are summarized and values are
recommended for design applications on various benthic materials.

The static balance between the wave-induced hydrodynamic pressure force and the Coulomb bottom frictional force results in a stability criterion that may be expressed in terms of a dimensionless wave force acting on the pipeline, $P_B^*$, a dimensionless stability parameter, $P_s^*$; and a dimensionless force ratio, $W^*$, which includes all of the explicit variables in the pipeline horizontal stability problem. Both the experimental data from Wright (1976) and Yamamoto, et al. (1973) are analyzed to construct a set of design curves for the dimensionless force ratio, $W^*$.

Using water particle kinematics from linear wave theory, the relative magnitude between the inertia and the drag force components is evaluated. Limiting values for this relative ratio are used to delineate three regions of relative importance on a dimensionless relative displacement, $H/D$, relative water depth, $d/L_0$, dissection plane which may be used for design. The horizontal stability criterion of a bottom-laid pipeline is further simplified in each of these three regions of relative importance for design.

Using water particle kinematics computed numerically from the finite-amplitude Dean (1965) stream function wave theory, a set of dimensionless curves for the dimensionless wave force acting on the pipeline, $P_B^*$, under various design
conditions specified by the dimensionless force ratio, $W'$, the depth parameter, $d/T^2$, and the wave height parameter, $H/T^2$, is constructed for design application.

A design procedure for the application of the dimensionless design curves ($W'$ and $P'_B$) is enumerated for determining the limiting depth for the horizontal stabilization of a bottom-laid pipeline.
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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.</td>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>1.1</td>
<td>Review of Previous Studies</td>
<td>1</td>
</tr>
<tr>
<td>1.2</td>
<td>Scope of the Present Study</td>
<td>3</td>
</tr>
<tr>
<td>II.</td>
<td>PROBLEM FORMULATION</td>
<td>5</td>
</tr>
<tr>
<td>2.1</td>
<td>Governing Equation for Horizontal Pipeline Stability</td>
<td>5</td>
</tr>
<tr>
<td>2.2</td>
<td>Determination of the Force Coefficients</td>
<td>10</td>
</tr>
<tr>
<td>2.2.1</td>
<td>Analytical Methods</td>
<td>10</td>
</tr>
<tr>
<td>2.2.2</td>
<td>Experimental Methods</td>
<td>16</td>
</tr>
<tr>
<td>2.3</td>
<td>Determination of the Frictional Coefficient</td>
<td>23</td>
</tr>
<tr>
<td>III.</td>
<td>LINEAR WAVE THEORY AND APPLICATIONS</td>
<td>29</td>
</tr>
<tr>
<td>3.1</td>
<td>Formulation and Solution of the Boundary Value Problem</td>
<td>29</td>
</tr>
<tr>
<td>3.2</td>
<td>Relative Importance Between $F_{Dx}$ and $F_{Ix}$</td>
<td>33</td>
</tr>
<tr>
<td>3.3</td>
<td>Applications to the Stability Problem</td>
<td>39</td>
</tr>
<tr>
<td>IV.</td>
<td>STREAM FUNCTION WAVE THEORY AND APPLICATIONS</td>
<td>45</td>
</tr>
<tr>
<td>4.1</td>
<td>The Stream Function Boundary Value Problem</td>
<td>45</td>
</tr>
<tr>
<td>4.2</td>
<td>Mathematical Model and Solution for Non-Linear Waves on a Steady-Uniform Current</td>
<td>55</td>
</tr>
<tr>
<td>4.3</td>
<td>Applications to Pipe Stability Problem</td>
<td>66</td>
</tr>
<tr>
<td>V.</td>
<td>RESULTS AND DISCUSSIONS</td>
<td>79</td>
</tr>
<tr>
<td>5.1</td>
<td>Comparison of the Stream Function Solutions</td>
<td>79</td>
</tr>
<tr>
<td>5.2</td>
<td>Determination of the Dimensionless Force Ratio $W'$</td>
<td>85</td>
</tr>
<tr>
<td>5.3</td>
<td>Design Curves and Application Procedure</td>
<td>88</td>
</tr>
<tr>
<td>5.4</td>
<td>Summary and Recommendations</td>
<td>97</td>
</tr>
<tr>
<td>REFERENCES</td>
<td></td>
<td>101</td>
</tr>
</tbody>
</table>

Appendix A: Explicit Expression for Matrix Elements | 105 |
Appendix B: List of Notations | 109 |
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1.1</td>
<td>Definition sketch for pipeline stability design.</td>
</tr>
<tr>
<td>3.1.1</td>
<td>Definition sketch for linear wave.</td>
</tr>
<tr>
<td>3.2.1</td>
<td>Relative importance regions between the drag force, ( F_{Dx} ), and the inertia force, ( F_{Ix} ).</td>
</tr>
<tr>
<td>3.3.1</td>
<td>Definition sketch for wave refraction.</td>
</tr>
<tr>
<td>4.1.1</td>
<td>Schematic contours of ( \xi ) and ( z ) in ( (\xi+i\zeta) ) plane and the conformally mapped contours of ( \phi ) and ( \psi ) in ( (\phi+i\zeta) ) plane.</td>
</tr>
<tr>
<td>4.1.2</td>
<td>Linear, bilinear and steady-uniform current profiles.</td>
</tr>
<tr>
<td>4.2.1</td>
<td>Definition sketch for a nonlinear wave on a steady-uniform current.</td>
</tr>
<tr>
<td>4.3.1</td>
<td>Contours of dimensionless wave force acting on the pipeline, ( P'_B ), for design condition: ( W'=0 ).</td>
</tr>
<tr>
<td>4.3.2</td>
<td>Contours of dimensionless wave force acting on the pipeline, ( P'_B ), for design condition: ( W'=0.5 ).</td>
</tr>
<tr>
<td>4.3.3</td>
<td>Contours of dimensionless wave force acting on the pipeline, ( P'_B ), for design condition: ( W'=1.0 ).</td>
</tr>
<tr>
<td>4.3.4</td>
<td>Contours of dimensionless wave force acting on the pipeline, ( P'_B ), for design condition: ( W'=2.0 ).</td>
</tr>
<tr>
<td>4.3.5</td>
<td>Contours of dimensionless wave force acting on the pipeline, ( P'_B ), for design condition: ( W'=5.0 ).</td>
</tr>
<tr>
<td>4.3.6</td>
<td>Contours of dimensionless wave force acting on the pipeline, ( P'_B ), for design condition: ( W'=10.0 ).</td>
</tr>
<tr>
<td>4.3.7</td>
<td>Contours of dimensionless maximum horizontal water particle bottom velocity.</td>
</tr>
<tr>
<td>5.2.1</td>
<td>Contours of dimensionless force ratio, ( W' ), for design condition: ( \mu = 0.33 ).</td>
</tr>
<tr>
<td>5.2.2</td>
<td>Contours of dimensionless force ratio, ( W' ), for design condition: ( \mu = 0.55 ).</td>
</tr>
<tr>
<td>5.2.3</td>
<td>Contours of dimensionless force ratio, ( W' ), for design condition: ( \mu = 1.0 ).</td>
</tr>
<tr>
<td>Table</td>
<td>Description</td>
</tr>
<tr>
<td>---------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>2.2.1</td>
<td>Average force coefficients for zero angle-of-incidence waves from Grace et al.</td>
</tr>
<tr>
<td>2.3.1</td>
<td>Summary of frictional coefficients from Potyondy</td>
</tr>
<tr>
<td>2.3.2</td>
<td>Summary of frictional coefficients from Valent</td>
</tr>
<tr>
<td>2.3.3</td>
<td>Summary of frictional coefficients recommended for design</td>
</tr>
<tr>
<td>4.2.1</td>
<td>Summary of fifty stream function wave cases</td>
</tr>
<tr>
<td>4.2.2</td>
<td>Summary of wave characteristics</td>
</tr>
<tr>
<td>4.2.3</td>
<td>Dimensionless maximum horizontal water particle bottom velocity and acceleration</td>
</tr>
<tr>
<td>5.1.1</td>
<td>Order for stream function solutions</td>
</tr>
<tr>
<td>5.2.1</td>
<td>Dimensionless force ratio, W', calculated from Yamamoto, et al., and from Wright</td>
</tr>
</tbody>
</table>
1.1 Review of Previous Studies

The design of submarine pipelines is of concern due to increasing military and commercial applications. Pipelines may be used to transport fluids from the sea to land, such as the transfer of fuel oil for power generation, of seawater for desalination, or of sewage for disposal at sea; and to stabilize oceanographic sensing cables by encasing them in pipe-like protectors.

These pipelines may be subdivided into long distance offshore pipelines (usually more than 10 Km long) and nearshore coastal pipelines (usually less than 10 Km long) as shown in Anand and Agarwal (1980). These pipelines must be designed to resist the hydrodynamic forces imposed on them, especially in and near the surf zone where the refraction and shoaling of deep sea swells culminate in breakers. Usually these pipelines are buried in the sand or sediments under the breakers as protection against hydrodynamic forces; however, this procedure is not always feasible on hard, impermeable, rock bottoms. Such bottom materials necessitate laying the pipelines or cable protectors on the bottom and anchoring them in contact with the bottom.
Beckmann and Thibodeaux (1962) examined the wave forces exerted on pipelines with circular and trapezoidal cross sections in contact with a smooth, hard-surfaced ocean floor. The size of the pipeline structure that is required to prevent sliding along the bottom has been related to the controlling parameters determined by the contour of the pipelines, the material used, and the flow conditions in the field.

Eager (1971) discussed the applications of split pipe protectors used for cable stabilization. Hudspeth (1971) proposed an analytical expression for a split-pipe stability parameter using linear wave theory and determined the maximum water depth required for the application of such split-pipe protectors to achieve stabilization. Only the drag and lift forces were considered by Hudspeth (1971) as contributing to the hydrodynamic forces exerted on the small split-pipe and cable system. Wave kinematics, and shoaling and refraction coefficients were evaluated using linear wave theory. Hudspeth also recommended the use of the finite-amplitude Dean stream function (1965) wave theory for determining the wave kinematics and the limiting water depth for such split-pipe protectors in the real ocean environments.

For pipelines laid in contact with the bottom, various laboratory experiments have been made to determine the magnitudes of the hydrodynamic forces and the force
coefficients experienced by the pipelines. More recent research has been reported by Yamamoto, Nath and Slotta (1973), and by Wright (1976). Field measurements of such force data are much fewer. Grace (1978) suggested methods for the computation of maximum wave forces exerted on pipelines and discussed some design problems of trenched, buried, and ballasted pipelines. The result of a four-year field measurement program of the wave forces exerted on a small-diameter pipe laid on the bottom in the ocean off Honolulu have been made by Grace et al. (1979). The wave force coefficients calculated from their data provide some useful information for the submarine pipelines in real ocean environments. All of these laboratory and field measurements were obtained in intermediate or deep water wave conditions.

1.2 Scope of the Present Study

The horizontal stability of a submarine pipeline, or the cable and split-pipe protector system, laid on the beach depends on the following factors:

1. waves and current at the design path of pipeline, including their magnitudes and angles of incidence to pipeline.

2. size, type of material, and cross sectional shape of the pipe or protector.
3. structural tensile strength of the pipeline or protector.

4. type of benthic material, slope of the beach, and other benthic and bathymetric properties.

Analytical theories which relate breaking wave heights to wave heights under refraction are not well known and the analytical theories which relate deep water swell heights to wave heights under refraction are only valid seaward of the surf zone and then only for mildly sloping beaches.

The purpose of this thesis is to analyze the horizontal stability of pipelines under shoaling, finite-amplitude waves on mildly sloping beaches. A dimensionless wave force on the pipeline is derived which includes both drag and inertia forces. The relative importance of each of these force components to the total horizontal force under various parametric conditions is examined. Water particle kinematics computed both analytically from linear wave theory and numerically from the finite-amplitude Dean (1965) stream function wave theory are used to evaluate the hydrodynamic forces exerted on the pipelines. Steady, uniform currents are also included in the analysis. The effects of various benthic materials and their corresponding frictional coefficients against sliding are discussed. Engineering applications of the dimensionless wave force on the pipeline to engineering design conclude the thesis.
II. PROBLEM FORMULATION

The first section defines the boundary value problem and the assumptions governing the pipeline stability problem for a pipeline laid on a sloping beach. A stability equation is derived which relates the hydrodynamic, the Coulomb frictional, and the inertial body forces. Sections 2.2 and 2.3 summarize the various laboratory and field measurements of the hydrodynamic force coefficients and the Coulomb frictional coefficient between the pipeline and the beach.

2.1 Governing Equation for Horizontal Pipeline Stability

In unsteady oscillatory flow, such as that induced by waves and a steady-uniform current, $U_o$, the hydrodynamic forces exerted on a pipeline laid on the bottom may be expressed in terms of the horizontal and vertical force components. The total horizontal hydrodynamic force is a linear sum of an inertia force, $\hat{F}_I$, and a drag force, $\hat{F}_D$, derived from the Froude-Krylov hypothesis. The flow over the pipeline also induces a vertical upward lift force, $\hat{F}_L$, which acts perpendicular to both the horizontal force and the ocean bottom. By applying order of magnitude estimates, the inertia force is usually neglected in analyses of small diameter pipe under large waves. Figure
2.1.1 is a definition sketch of a pipeline laid on a sloping beach with incident waves and a steady-uniform current, $U_o$, perpendicular to the pipe.

The horizontal hydrodynamic force per unit length of pipe may be expressed by the vector form of Morison equation as

$$\hat{F}_P = \hat{F}_D + \hat{F}_I$$  \hspace{1cm} (2.1.1)

in which the inertia force is defined by

$$\hat{F}_I = C_M \rho_w \frac{\pi D^2}{4} \hat{q}$$  \hspace{1cm} (2.1.2)

and the drag force is defined by

$$\hat{F}_D = \frac{C_D}{2} \rho_w D |\hat{q}| \hat{q}$$  \hspace{1cm} (2.1.3)

The vertical lift force per unit length may be expressed as

$$\hat{F}_L = \frac{C_L}{2} \rho_w D (\hat{q} \cdot \hat{q}) \hat{k}$$  \hspace{1cm} (2.1.4)

in which the velocity vector is given by

$$\hat{q} = (u \cos \theta + U_o) \hat{i} + u \sin \theta \hat{k}$$  \hspace{1cm} (2.1.5)

with the magnitude given by

$$|\hat{q}| = (u^2 + U_o^2 + 2uU_o \cos \theta)^{\frac{1}{2}}$$  \hspace{1cm} (2.1.6)

The formulation of the pipe stability problem follows closely that given by Hudspeth (1971). The equations of
Figure 2.1.1 Definition sketch for pipeline stability design.
force equilibrium between the hydrodynamic forces, \( \mathbf{F}_p \) and \( \mathbf{F}_L \), the structural tensile strength component of the pipe, \( \mathbf{F}_T \), the torsional resisting moment of the pipe, \( \mathbf{T}_o \), the gravity force, \( \mathbf{F}_g \), the Coulomb frictional force, \( \mathbf{F}_F \), and the normal buoyant reaction force, \( \mathbf{F}_N \), require the following:

1. The sum of moments about the contact point between the pipe and the beach must be zero; i.e.,

\[
\Sigma M = T_o + (F_T \cdot \hat{\mathbf{x}}) \frac{D}{2} - (F_p \cdot \hat{\mathbf{x}}) \frac{D}{2} = 0 \quad (2.1.7)
\]

2. The sum of forces in the vertical \( z \)-direction must be zero; i.e.,

\[
\Sigma F_z = (-F_g + F_L \cos \beta) \cdot \hat{z} = 0 \quad (2.1.8)
\]

in which \( \beta \) is the slope of beach.

3. The sum of forces in the horizontal \( x \)-direction must be zero; i.e.,

\[
\Sigma F_x = (F_p - F_T - F_L) \cdot \hat{x} = 0 \quad (2.1.9)
\]

in which the frictional force per unit length of pipe, \( F_F \), may be determined by the product of a frictional coefficient \( \mu \) times the normal force, \( F_N \), reduced by the lift force, \( F_L \); i.e.,

\[
F_F = \mu(F_N - F_L) \cos \beta \hat{z} \quad (2.1.10)
\]

in which
\[ \mathbf{F}_N = (\rho_s - \rho_w) \frac{\pi D^2}{4} g \hat{k} \]  
(2.1.11)

and \( \mathbf{F}_N = \mathbf{\hat{F}}_N \cdot \hat{k} \).

The horizontal stability of the pipeline requires that the horizontal hydrodynamic force, \( \mathbf{\hat{F}}_P \), be less than the sum of the two resisting forces, \( \mathbf{\hat{F}}_T \) and \( \mathbf{\hat{F}}_F \). This stable equilibrium condition may be expressed as

\[ \frac{\mathbf{\hat{F}}_P \cdot \hat{i}}{\mathbf{\hat{F}}_F (1 + \frac{|\mathbf{\hat{F}}_T|}{|\mathbf{\hat{F}}_F|}) \cdot \hat{i}} < 1 \]  
(2.1.12)

Neglecting the ratio \( |\mathbf{\hat{F}}_T|/|\mathbf{\hat{F}}_F| \) results in a conservative stability criterion. Substituting for the force vectors in Eq. (2.1.12) yields

\[ \frac{F_{Px}}{F_F} = \frac{F_{Dx} + F_{Ix}}{\mu (F_N - F_L) \cos \beta} < 1 \]  
(2.1.13)

in which the x-component of the horizontal drag and inertia forces on the pipe per unit length are

\[ F_{Dx} = \frac{C_D}{2} \rho_w D (u^2 + U_o^2 + 2 u U_o \cos \theta) \left( u \cos \theta + U_o \right) \]  
(2.1.14.a)

\[ F_{Ix} = C_M \rho_w \frac{\pi D^2}{4} \hat{u} \cos \theta \]  
(2.1.14.b)

Eq. (2.1.13) represents a horizontal stability equation for the pipeline. In order to evaluate Eq. (2.1.13), it is necessary to express the hydrodynamic forces in terms of
the water particle kinematics. Chapters 3 and 4 will discuss linear wave theory and finite-amplitude Dean (1965) stream function wave theory, respectively.

2.2 Determination of the Force Coefficients

In order to gain some appreciation for the mechanics of wave forces on horizontal pipes, we will first briefly discuss the analytical methods available for determining the inertia and lift force coefficients using results from potential flow theory.

2.2.1 Analytical Methods

2.2.1.a Inertia force coefficient $C_M$

The inertia force coefficient, $C_M$, may be calculated analytically from the potential flow theory for the following flow conditions surrounding a horizontal pipe:

A. Nonoscillatory flow

1. Fixed cylinder
   
   a. infinite accelerating flow
   
   b. infinite accelerating flow with a rigid boundary

2. Accelerating cylinder

   a. infinite fluid
   
   b. infinite fluid with a rigid boundary
B. Oscillatory flow

1. Infinitely deep water waves
2. Finite depth water waves

2.2.1.b Lift force coefficient $C_L$

There are a few analytical solutions presently available for the lift force coefficient which may be determined from potential flow theory. The few that are available may be summarized by the following flow conditions:

A. Fixed cylinder submerged in a steady, unidirectional flow
   1. Infinite boundaries
   2. Close rigid boundaries

B. Cylinder submerged under finite-depth water waves

2.2.1.c Analytical results

From potential flow theory, an added mass coefficient value of $C_m = 1.0$ may be computed analytically for a cylinder fixed in an accelerating stream of inviscid fluid away from any boundary (Lamb, 1932, p. 93). The inertia coefficient, $C_M$, is defined as $C_M = 1.0 + C_m$ and accordingly has the value of 2.0 for this case.

For a cylinder near to or in contact with a rigid impermeable boundary in an otherwise infinite mass of accelerating fluid, the added mass force is increased due to the presence of this rigid boundary. Taylor (1930)
calculated the added mass coefficients, \( C_m \), for a family of lenses consisting of two circular arcs. The family passes from a flat plate through a single circle to a pair of tangent circles. This last limiting case may be regarded as a circle tangent to a ground plane and an added mass coefficient value of \( C_m = 2.29 \), therefore, an inertia coefficient value of \( C_M = 3.29 \) was calculated (Goodman, 1972). Garrison (1972) also calculated an inertia coefficient value of \( C_M = 3.29 \) for a circular cylinder in contact with a plane boundary using the Milne-Thomson circle theorem (Milne-Thomson, 1968, pp. 157-163).

For a cylinder accelerated in an infinite mass of inviscid fluid, an inertia coefficient value of \( C_M = 2.0 \) was obtained by Milne-Thomson (1968, p. 246). This value was obtained by considering the increase in total kinetic energy of the mass and the fluid due to an unbalanced force on the cylinder.

Yamamoto, Nath and Slotta (1973) have derived an analytical expression for the added mass coefficient, \( C_m' \), for a cylinder accelerated toward a rigid boundary from an infinite distance. A horizontal cylinder was represented in the potential flow by a doublet with the axis parallel to the direction of motion. Applying the method of images, the cylinder motion toward the boundary may be approximated by two doublets moving against each other with their axes in the direction of motion. The value of the inertia
coefficient, $C_M$, was shown to increase from 2.0 to 3.29 as the cylinder approached the boundary from an infinite distance.

Both Dean (1948) and Ursell (1950) have considered the interaction of small sinusoidal surface waves with a submerged cylinder in an infinitely deep water domain using potential flow theory. The velocity potential was obtained in terms of pulsating sources and vortices placed at the center of the cylinder. Ogilvie (1963) has obtained the solution for the first-order oscillatory forces due to the acceleration of water particle using the velocity potential derived by Dean (1948) and Ursell (1950). The numerical results were calculated for the following three cases: 1) fixed cylinder, 2) oscillating cylinder, and 3) neutrally-buoyant cylinder allowed to respond to the first-order oscillatory forces. These numerical results were presented in terms of the amplitudes of the oscillatory forces as a function of the depth of submergence of the cylinder below the free surface. Inertia coefficient values may thus be calculated from these results. These results were not obtained for cylinders located near a fixed boundary and should not be used for bottom-laid pipelines.

The interaction of finite-depth, small-amplitude surface waves with a submerged object near the bottom boundary introduces more difficulties to the potential flow problem. Wehausen and Laitone (1960) derived a linearized potential
flow solution in which the velocity-squared terms in the Bernoulli equation were neglected. The velocity potential was expressed by pulsating sources distributed continuously over the surface of the object. Garrison and Rao (1971) used this method to calculate the wave forces exerted on a submerged hemisphere resting on the bottom by numerically determining the strength of these pulsating sources. The results of the calculated inertia coefficient, $C_M$, were presented as a function of the ratio of the hemisphere radius to wavelength, with the ratio of the water depth to hemisphere radius as a varying parameter. No numerical results were calculated for a submerged cylinder resting on the bottom.

For a horizontal cylinder immersed in an infinite mass of steady fluid flow, there exists no vertical lift force from potential flow theory (Lamb, 1932, p. 93). The effect of a rigid boundary in an otherwise infinite steady flow is to induce a lift force on the cylinder which acts perpendicularly to the direction of steady flow. Yamamoto, Nath and Slotta (1973) have also used potential flow theory and obtained an analytical expression for the lift coefficient, $C_L$, for a horizontal cylinder near a boundary. They demonstrated that the value of the lift coefficient, $C_L$, was a function of the gap between the cylinder and a rigid boundary. For a cylinder in contact with a rigid boundary, the lift coefficient, $C_L$, was computed analytically to be
equal to 4.49. When the gap between the cylinder and the boundary was less than two cylinder diameters, the lift coefficient, $C_L$, was of considerable magnitude and always negative (i.e., the lift force acted downward toward the boundary). Analytical expressions for the cylinder-boundary gap problem may also be found in Milne-Thomson (1968, Chapter 8).

Ogilvie (1963) has also obtained a solution for the second-order steady force (i.e., lift force) exerted on a cylinder submerged in an infinitely deep water with small sinusoidal surface waves at the free surface. The numerical results were also calculated for the following three cases: 1) fixed cylinder, 2) oscillating cylinder and 3) neutrally-buoyant cylinder. These results were presented in terms of the amplitudes of the steady forces as a function of the depth of submergence of the cylinder. Lift coefficient values may also be calculated from these results. These results were not obtained for cylinders located near a fixed boundary and should not be used for bottom-laid pipelines.

In unsteady oscillatory flow, such as that induced by waves, considerable complexity is added to the flow phenomena around a horizontal cylinder near a plane boundary. The force coefficients may be determined by the following experimental methods.
2.2.2 Experimental Methods

2.2.2.a Experimental methods for Morison equation force coefficients

The horizontal wave-induced drag and inertia force coefficients, $C_D$ and $C_M$, respectively, in the Morison equation may be determined by any of the following methods:

A. Methods of analyses

1. Deterministic wave fitting methods
   a. Fourier analysis
   b. Least-squares fitting using a
      i. numerical wave theory
      ii. linear filter
      iii. measured kinematics
   c. Maximum force fitting

2. Nondeterministic wave fitting methods
   a. Drag/inertia dominate frequency fitting
   b. Method of moments
   c. Spectral least-squares fitting

Basically, there are two types of experiments which may be conducted to obtain wave forces and their corresponding force coefficients. These two types of experiments are summarized by the following:

B. Types of experiments

1. Lab tests
   a. Standing waves
   b. Progressive waves
c. U-tube test
d. Oscillating cylinder

2. Oscillatory ocean waves

Next, a brief description will be given of the three deterministic wave fitting methods that are usually employed to determine the force coefficients $C_D$ and $C_M$.

C. Description of the deterministic wave fitting methods

Fourier analysis

The force coefficients $C_D$ and $C_M$ may be determined according to the following Fourier methods (Keulegan and Carpenter, 1958; Sarpkaya, 1975):

\[
F_m(t) = \frac{C_D}{2} \rho_w D U_m \cos \frac{2\pi t}{T} |U_m \cos \frac{2\pi t}{T}|
- C_m \rho_w \frac{\pi D^2}{4} \cdot \frac{2\pi}{T} \cdot U_m \sin \frac{2\pi t}{T}
\]

(2.2.1)

\[
C_D = \frac{3\pi}{2T} \int_0^T \frac{F_m(t)}{\rho_w D U_m^2} \cos \frac{2\pi t}{T} \, dt
\]

(2.2.2)

\[
C_M = -4U_m \frac{\pi}{D} \int_0^T \frac{F_m(t)}{\rho_w D U_m^2} \sin \frac{2\pi t}{T} \, dt
\]

(2.2.3)

in which $F_m(t)$ = the measured time dependent horizontal force per unit length on the cylinder, $U_m$ = the amplitude of the horizontal water particle velocity, $T$ = the oscillation or wave period.
Least-squares fitting

The method of least-squares fitting consists of the minimizing the error between the measured and calculated forces. The force coefficients $C_D$ and $C_M$ may be determined by the following equations (Sarpkaya, 1977):

$$C_D = \frac{16}{3T} \int_0^T \frac{F_m(t)}{\rho_w D U_m^2} \left| \cos \frac{2\pi t}{T} \right| \cos \frac{2\pi t}{T} \, dt$$

(2.2.4)

in which $F_m(t) =$ the measured horizontal force given by Eq. (2.2.1). The inertia coefficient, $C_M'$, is also given by Eq. (2.2.3); i.e., the inertia coefficient, $C_M'$, calculated from least-squares fitting gives the same value as that calculated from Fourier analysis.

Maximum force fitting

For a measured horizontal force, $F_m(t)$, given by Eq. (2.2.1), the force coefficients $C_D$ and $C_M$ are given by the following equations (Grace, 1979):

$$C_D = \frac{F_m(0)}{\rho_w D U_m^2/2}$$

(2.2.5)

$$C_M = \frac{F_m(-T/4)}{\rho_w \frac{\pi D^2}{4} \left( \frac{2\pi}{T} U_m \right)}$$

(2.2.6)

in which $F_m(0)$ and $F_m(-T/4)$ are the measured horizontal force values per unit length of cylinder at the instant when $t = 0$ and when $-T/4$, respectively.
2.2.2.b Experimental method for the lift force coefficient

For a small bottom-laid pipe, the hydrodynamic vertical force exerted on the pipe is due to the lift force only. This lift force may be obtained by using the same experiments summarized in Section 2.2.2.a. The vertical lift force coefficient, $C_L$, may be determined by any of the following methods (Sarpkaya, 1977): 1) maximum lift force method, 2) semi peak-to-peak value, and 3) normalized root-mean-square value.

The most often used is the maximum lift method in which the lift force, $P_m(t)$, and the lift coefficient, $C_L$, are given by:

$$P_m(t) = \frac{C_L}{2} \rho_w D |U_m \cos 2\pi t| U_m \cos \frac{2\pi t}{T}$$

(2.2.7)

$$C_L = \frac{P_m(0)}{\rho_w D U_m^2/2}$$

(2.2.8)

in which $P_m(t)$ and $P_m(0)$ are the measured vertical force values per unit length of cylinder as a function of time and at the instant when $t = 0$, respectively.

Next, the published results obtained from various experiments will be discussed.
2.2.2.c Experimental results

Standing waves

By using the Fourier analysis method, Keulegan and Carpenter (1958) found that for oscillatory flow under simple harmonic waves, the drag coefficient, $C_D$, had no clear dependency on the Reynolds number. It was shown that both force coefficients, $C_D$ and $C_M$, were dependent upon a dimensionless period parameter (or Keulegan-Carpenter number), $N_{KC} = U_m T/D$, in which $U_m$ = the amplitude of the water particle horizontal velocity, and $T$ = the wave period. The force coefficients $C_D$ and $C_M$ achieved maximum and minimum values, respectively, for a period parameter of around 15. It was also shown that the force coefficients $C_D$ and $C_M$ varied within one cycle of the oscillatory flow in which $C_M$, the inertia coefficient, varied with a cyclical period twice that of wave period, $T$.

Progressive waves

By using the maximum force method, Wright (1976) conducted a series of wave tank experiments at the Oregon State University - Wave Research Facility for a horizontal cylinder placed on the bottom. For these test conditions, the dimensionless relative water depth, $d/L_o$, varied from 0.027 to 0.5 and the dimensionless relative displacement, $H/D$, varied from 0.202 to 2.776. The resulting force coefficients indicated that the average inertia coefficient, $C_M$, varied between the two maximum values of 3.47 and 2.96.
during a wave period, the average value of the drag coefficient was found to be a constant of $C_D = 0.93$. The average value of the lift coefficient was found to be a constant of $C_L = 1.90$.

**U-tube test**

Using the Fourier analysis method, Sarpkaya (1975) conducted experiments by oscillating flow past fixed cylinders in a U-shaped vertical water tunnel. The resulting curves of the drag coefficient, $C_D$, versus the period parameter, $N_{kc}$, from his tests demonstrated that the maximum value of the drag coefficient, $C_D = 2.3$ for a period parameter of $N_{kc} = 12$ compared very well with the values obtained by Keulegan and Carpenter (1958). Sarpkaya obtained values of the inertia coefficient, $C_M$, which decreased from a maximum value of $C_M = 2.2$ for a period parameter of $N_{kc} = 2$ to a minimum value of $C_M = 0.7$ for a period parameter of $N_{kc} = 12$. The agreement of the inertia coefficient, $C_M$, values between Sarpkaya (1975), Keulegan and Carpenter (1958) for period parameter, $N_{kc}$, greater than 10 is not as good as the drag coefficient, $C_D$, values. By using the maximum lift force method, the values of the lift coefficient, $C_L$, were found to reach as high as 3.0 and exhibited several maxima.

Sarpkaya (1975) also concluded that there was no correlation between the force coefficients and the Reynolds number. However, Miller (1975) replotted Sarpkaya's values
(1975) for the force coefficients $C_D$ and $C_M$ on the dimensionless Reynolds number -- period parameter dissection plane and demonstrated a clear Reynolds number dependency. Based on these results from Miller (1975), Sarpkaya (1977) later used both the Fourier analysis and the least-square fitting methods and replotted the data from Keulegan and Carpenter (1958) and from Sarpkaya (1975). These replotted results demonstrated that the force coefficients do indeed depend on both the Reynolds number and the period parameter (or the ratio of these two parameters).

**Oscillatory ocean waves**

Grace, Castiel, Shak and Zee (1979) conducted a field test in the ocean offshore from Honolulu using a pipe faired into bottom. During these tests, wave periods varied between 12 to 17 seconds and wave heights up to 17 feet (5.12 m) were measured. The test site depth was 37 feet (11.28 m) and the steel pipe used was 16 inches (40.64 cm) in diameter. Although this work focused primarily on the peak forces, instantaneous force coefficients for the Morison equation for only the zero angle-of-incidence waves were calculated by using the maximum force method and are shown in Table 2.2.1.

Davis and Ciani (1976) summarized various studies for the hydrodynamic forces on submarine pipelines laid on the bottom. The force coefficients suggested for design purposes were separated into the following three regions:
Table 2.2.1. Average force coefficients for zero angle-of-incidence waves from Grace et al. (1979).

<table>
<thead>
<tr>
<th>Test sequence</th>
<th>Roughness</th>
<th>Surface finish</th>
<th>Clearance</th>
<th>Average values</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>εr/D</td>
<td>(2)</td>
<td>e/D</td>
<td>CM (5)</td>
</tr>
<tr>
<td>(3)</td>
<td></td>
<td>(4)</td>
<td></td>
<td>CD (6)</td>
</tr>
<tr>
<td>(7)</td>
<td></td>
<td></td>
<td></td>
<td>CL (7)</td>
</tr>
<tr>
<td>1A</td>
<td>0.0001</td>
<td>natural</td>
<td>0.188</td>
<td>2.12</td>
</tr>
<tr>
<td>1B</td>
<td>0.0025</td>
<td>corroded</td>
<td>0.031</td>
<td>2.42 0.86</td>
</tr>
<tr>
<td>1C</td>
<td>0.016</td>
<td>rib</td>
<td>0.031</td>
<td>2.70 0.77 0.58</td>
</tr>
</tbody>
</table>

1) $C_M = 0$, $C_D = 1.0 - 2.0$, and $C_L = 4.5$ for the region in which the drag force dominated over the inertia force;
2) $C_M = 3.3 - 4.5$, $C_D = 0$, and $C_L = 4.5$ for the region in which the inertia force dominated over the drag force; and
3) between these two regions, the values of force coefficients were $C_M = 3.3 - 4.5$, $C_D = 1.0 - 2.0$ and $C_L = 4.5$.

2.3 Determination of the Frictional Coefficient

For a horizontal pipeline laid on the bottom, the frictional resistance between the pipeline and the beach depends on the following factors (Anand and Agarwal, 1980): 1) surface coating of the pipe; 2) beach material and its properties, such as angle of internal friction, cohesion, and degree of saturation; 3) contact area between the pipeline and the beach; and 4) direction of pipe movement, i.e., longitudinal or horizontal movements.

Potyondy (1961) estimated values for the frictional coefficients, $\mu$, between various soils and pipe materials
by conducting direct shear tests; i.e., the contact surface between the pipe materials and soils was flat. Potyondy's results indicated that the values for frictional coefficient, $\mu$, for rough concrete specimens on sand or dense cohesionless silt were approximately 10% greater than the corresponding values for smooth specimens. The values for rough steel specimens on sand or dense cohesionless silt were approximately 10-25% greater than those values for the smooth specimens. These test results from Potyondy (1961) are summarized in Table 2.3.1.

Table 2.3.1. Summary of frictional coefficients from Potyondy (1961).

<table>
<thead>
<tr>
<th>Pipeline material (1)</th>
<th>Soil material (2)</th>
<th>$\mu$ (3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Smooth steel</td>
<td>Sand</td>
<td>0.64</td>
</tr>
<tr>
<td></td>
<td>Dense cohesionless silt</td>
<td>0.68</td>
</tr>
<tr>
<td></td>
<td>Clay</td>
<td>0.50</td>
</tr>
<tr>
<td>Rough steel</td>
<td>Sand</td>
<td>0.80</td>
</tr>
<tr>
<td></td>
<td>Dense cohesionless silt</td>
<td>0.75</td>
</tr>
<tr>
<td></td>
<td>Clay</td>
<td>0.80</td>
</tr>
<tr>
<td>Smooth concrete</td>
<td>Sand</td>
<td>0.80</td>
</tr>
<tr>
<td>(made in iron forms)</td>
<td>Dense cohesionless silt</td>
<td>0.87</td>
</tr>
<tr>
<td></td>
<td>Clay</td>
<td>1.00</td>
</tr>
<tr>
<td>Grained concrete</td>
<td>Sand</td>
<td>0.88</td>
</tr>
<tr>
<td>(made in wooden forms)</td>
<td>Dense cohesionless silt</td>
<td>0.96</td>
</tr>
<tr>
<td></td>
<td>Clay</td>
<td>1.00</td>
</tr>
<tr>
<td>Rough concrete</td>
<td>Sand</td>
<td>0.90</td>
</tr>
<tr>
<td>(made on adjusted ground)</td>
<td>Dense cohesionless silt</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td>Clay</td>
<td>1.00</td>
</tr>
</tbody>
</table>
Lyons (1973) conducted both small and large scale tests for soil resistance to the lateral sliding of submarine pipelines. The tests included 1 inch (2.54 cm), 9 inch (22.86 cm) and 16 inch (40.64 cm) diameter concrete-coated pipes resting on sand with loading in the horizontal direction only. The coefficients of friction, $\mu$, for large scale tests on a sandy beach ranged from 0.65 to 0.71. For the model scale tests, the frictional coefficient, $\mu$, was greater than unity due to the formation of a sand wedge that formed in front of the pipe following application of the lateral force. The horizontal force would, therefore, have to overcome not only the friction but also the soil resistance of the sand wedge piled up before the cylinder.

Karal (1977) proposed an analytical model for predicting the frictional coefficient, $\mu$, based on an upper bound approximation from the theory of perfect plasticity. The calculated values of the frictional coefficient, $\mu$, varied between 0.6 to 1.2 for pipes on sand. The values for the frictional coefficient, $\mu$, varied between 0.18 to 0.4 for pipes on clay.

Valent (1979) also experimentally determined the frictional coefficients, $\mu$, between three sea floor sand materials and two building materials for pipes (both steel and concrete) by conducting direct shear tests. Samples of the calcareous sand used for these tests included coralline sand obtained from an atoll beach, foraminiferal sand-silt
from a deep ocean site, and oolitic sand from a shallow ocean site. For both steel and concrete specimens, rough and smooth surface conditions were used in the tests. In general, the frictional coefficient reached its peak value, $\mu_{\text{peak}}$, shortly after the test started and subsided to a constant residual value, $\mu_{\text{residual}}$, for the rest of the direct shear test. The effect of different sand materials on the frictional coefficient was not significant. The effect of surface roughness was shown to be very little for the concrete specimen while the frictional coefficient for smooth steel was approximately one-third of that found for rough steel specimen. These test results from Valent (1979) are summarized in Table 2.3.2.

Table 2.3.2. Summary of frictional coefficients from Valent (1979).

<table>
<thead>
<tr>
<th>Pipeline Material (1)</th>
<th>Soil Material (2)</th>
<th>$\mu_{\text{peak}}$ (3)</th>
<th>$\mu_{\text{residual}}$ (4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Smooth steel</td>
<td>Coralline sand</td>
<td>0.20-0.21</td>
<td>0.17-0.18</td>
</tr>
<tr>
<td></td>
<td>Oolitic sand</td>
<td>0.32</td>
<td>0.31</td>
</tr>
<tr>
<td></td>
<td>Foram sand-silt</td>
<td>0.40</td>
<td>0.37</td>
</tr>
<tr>
<td>Rough steel</td>
<td>Coralline sand</td>
<td>0.63</td>
<td>0.55</td>
</tr>
<tr>
<td></td>
<td>Oolitic sand</td>
<td>0.54-0.58</td>
<td>0.50-0.51</td>
</tr>
<tr>
<td></td>
<td>Foram sand-silt</td>
<td>--</td>
<td>0.66</td>
</tr>
<tr>
<td>Smooth concrete</td>
<td>Coralline sand</td>
<td>0.63</td>
<td>0.56</td>
</tr>
<tr>
<td></td>
<td>Oolitic sand</td>
<td>0.58-0.59</td>
<td>0.52-0.54</td>
</tr>
<tr>
<td></td>
<td>Foram sand-silt</td>
<td>--</td>
<td>0.67</td>
</tr>
<tr>
<td>Rough concrete</td>
<td>Coralline sand</td>
<td>0.66</td>
<td>0.59</td>
</tr>
<tr>
<td></td>
<td>Oolitic sand</td>
<td>0.74</td>
<td>0.57</td>
</tr>
</tbody>
</table>
Anand and Agarwal (1980) conducted frictional resistance tests using different diameter concrete-coated pipes for model and prototype studies. In these tests, pipes were pulled in the horizontal direction on both sandy and silty soils. For horizontal pulls in which rolling occurred, the range of the frictional coefficient, $\mu$, from model tests was found to vary between 0.145-0.185 for fine sands, between 0.08-0.14 for coarse sands, and between 0.10-0.25 for silty soils. It was considered that for a long pipe under field conditions, the horizontal force acting on the pipe would slide the pipe rather than roll it along its longitudinal axis. Accordingly, an anti-roll device was provided in the prototype study. The range of frictional coefficients, $\mu$, from their prototype study was found to vary between 0.12-0.24 for fine sands, between 0.07-0.17 for coarse sands, and between 0.33-0.67 for silty soils. Anand and Agarwal (1980) concluded that for a concrete-coated pipe on sandy or silty soils, the frictional coefficient, $\mu$, may vary between 0.5-0.6.

There exist no laboratory or field measurements of the frictional coefficients, $\mu$, between pipeline materials and a rock beach. For design purposes (if no accurate site survey data are available), a value of 0.3 for the frictional coefficient, $\mu$, between stone masonry and wet undisturbed ground (Eshbach, 1974, p. 480) is recommended.
For design purposes, the frictional coefficients, $\mu$, between various beach and pipeline materials are suggested in Table 2.3.3.

Table 2.3.3. Summary of frictional coefficients recommended for design.

<table>
<thead>
<tr>
<th>Beach material (1)</th>
<th>Pipeline material (2)</th>
<th>$\mu$ (3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rock</td>
<td>Smooth steel</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td>Rough steel, smooth</td>
<td>0.30</td>
</tr>
<tr>
<td></td>
<td>and rough concrete</td>
<td></td>
</tr>
<tr>
<td>Sand</td>
<td>Smooth steel</td>
<td>0.18</td>
</tr>
<tr>
<td></td>
<td>Rough steel, smooth</td>
<td>0.55</td>
</tr>
<tr>
<td></td>
<td>and rough concrete</td>
<td></td>
</tr>
<tr>
<td>Silty-soil, clay</td>
<td>Smooth steel</td>
<td>0.33</td>
</tr>
<tr>
<td></td>
<td>Rough steel, smooth</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>and rough concrete</td>
<td></td>
</tr>
</tbody>
</table>
III. LINEAR WAVE THEORY AND APPLICATIONS

The boundary value problem and solution for linear wave theory is defined in Section 3.1. Section 3.2 discusses the relative importance between the contributions of the drag and inertia components to the horizontal hydrodynamic force and defines three design regions for the pipeline stability problem on a dimensionless relative displacement (H/D) - relative water depth (d/L_o) dissection plane. The general forms of the stability equations for each of these three regions are derived. Section 3.3 discusses applications of linear wave theory kinematics to the pipeline stability problem in the various design regions identified on the dimensionless dissection plane.

3.1 Formulation and Solution of the Boundary Value Problem

The water-wave phenomenon may be idealized as a two-dimensional boundary value problem for ideal flow shown in Figure 3.1.1 Ideal flow assumes that the motion is irrotational and that the fluid is inviscid and incompressible without capillarity. For monochromatic waves with frequency \( \omega = 2\pi/T \), the boundary value problem may be expressed in terms of a scalar velocity potential (Mei, 1978) given by:

\[
\phi(\xi, z, t) = \text{Re}[\phi(\xi, z)\exp - i(\omega t)]
\]  

(3.1.1)
\[ \eta(\xi, t) = \frac{H}{2} \cos(\kappa \xi - \omega t) \]

Horizontal, impermeable bottom, \( z = -d \)

Figure 3.1.1. Definition sketch for linear wave.
from which the velocity components may be determined according to the following directional derivatives:

\[ u(\xi, z, t) = \text{Re}\left[ \frac{\partial \phi}{\partial \xi} \exp \left( -i\omega t \right) \right] \]  (3.1.2)  
\[ w(\xi, z, t) = \text{Re}\left[ \frac{\partial \phi}{\partial z} \exp \left( -i\omega t \right) \right] \]  (3.1.3)  

in which \( \text{Re}[\cdot] \) denotes the real part of a complex-valued function.

The governing field equation is the Laplace equation.

\[ \nabla^2 \phi = 0; \quad |\xi| < \infty; \quad -d \leq z \leq 0 \]  (3.1.4)  

with boundary conditions given by

\[ \frac{\partial \phi}{\partial z} = 0; \quad |\xi| < \infty; \quad z = -d \]  (3.1.5)  

\[ \frac{\partial \phi}{\partial z} - \frac{\omega^2}{g} \phi = 0; \quad |\xi| < \infty; \quad z = 0 \]  (3.1.6)  

The free surface profile, \( \eta(\xi, t) \), may be determined from either the kinematic free surface boundary condition,

\[ \text{KFSBC} \quad \frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} ; \quad z = 0 \]  (3.1.7)  

or the dynamic free surface boundary condition,

\[ \text{DFSBC} \quad \eta = \frac{1}{g} \frac{\partial \phi}{\partial t} ; \quad z = 0 \]  (3.1.8)  

Eq. (3.1.6) combines both the KFSBC and DFSBC by eliminating the unknown free surface profile, \( \eta(\xi, t) \). The velocity potential for a plane incident wave travelling in
the positive $\xi$-direction is given by

$$\phi(\xi,z,t) = \frac{H}{2} \frac{\omega}{\kappa} f(z) \sin(\kappa \xi - \omega t) \quad (3.1.9)$$

in which $H$ = the wave height and

$$f(z) = \cosh \frac{\kappa (d+z)}{\sinh \kappa d} \quad (3.1.10)$$

in which $\kappa = 2\pi/L$ and $L$ = the wavelength, $d$ = the water depth, provided that

$$\omega^2 = g \kappa \tanh \kappa d \quad (3.1.11)$$

The free surface profile $\eta(\xi,t)$ may now be obtained from the solution for $\phi(\xi,z,t)$, according to either Eqs. (3.1.7) or (3.1.8), as

$$\eta(\xi,t) = \frac{H}{2} \cos(\kappa \xi - \omega t) \quad (3.1.12)$$

The horizontal water particle velocity, $u(\xi,z,t)$, and acceleration, $\dot{u}(\xi,z,t)$, may be determined from

$$u(\xi,z,t) = u_{\text{max}} \cdot f(z) \cdot \cos(\kappa \xi - \omega t) \quad (3.1.13)$$

$$\dot{u}(\xi,z,t) = \dot{u}_{\text{max}} \cdot f(z) \cdot \sin(\kappa \xi - \omega t) \quad (3.1.14)$$

in which the over-dot denotes temporal derivatives and

$$u_{\text{max}} = \frac{H \pi}{T} \quad (3.1.15)$$

$$\dot{u}_{\text{max}} = \frac{2H \pi^2}{T^2} = \left( \frac{2\pi}{T} \right) u_{\text{max}} \quad (3.1.16)$$
Other forms of solution to the linear wave theory boundary value problem may be found in Ippen (1966, Chapter 1), Horikawa (1978, Chapter 1), Wehausen and Laitone (1960) inter alios.

3.2 Relative Importance Between $F_{Dx}$ and $F_{Ix}$

The inertia force component in Eq. (2.1.2) is proportional to the acceleration of a water particle and, therefore, is of out phase with the drag and lift forces which are in phase with the water particle velocity. Various analyses have been made to determine the relative importance between the drag and inertia force components, $F_{Dx}$ and $F_{Ix}$, by using linear wave theory kinematics. Some of these analyses will be reviewed separately below.

Beckmann and Thibodeau (1962) established a stability criterion for the case when the inertia force component dominates the total force on a bottom-laid horizontal pipe. Using linear wave theory kinematics, the maximum value of the inertia force was equated to the maximum force resulting from the sum of the drag and the Coulomb frictional force, i.e.,

$$C_M \rho_w \frac{\pi D^2}{4} \dot{u}_{\text{max}} f(-d) = \frac{1}{2}(C_D + \mu C_L) \rho_w D u_{\text{max}}^2 f^2(-d)$$

(3.2.1)

which may be reduced using Eq. (3.1.16) to
\[ N'_k = \frac{u_{\text{max}} \cdot f(-d) \cdot T}{D} = \pi^2 \left[ \frac{C_M/C_D}{1 + \mu(C_L/C_D)} \right] \]  

(3.2.2)

in which \( N'_k \) is a form of the Keulegan-Carpenter number which is obtained when an oscillatory linear wave theory velocity is used for \( u_{\text{max}} \). A wave steepness coefficient may be defined by

\[ B_t = \frac{u_{\text{max}} f(-d) T}{4s} \]  

(3.2.3)

in which \( s \) = the path length travelled by a wave particle in one-half cycle. For the case in which the inertia force component dominates, the ratio of the pipe diameter, \( D \), to path length, \( s \), was determined to be given by

\[ \frac{D}{s} = \frac{4}{\pi^2} B_t \left[ \frac{1 + \mu(C_L/C_D)}{(C_M/C_D)} \right] \]  

(3.2.4)

Beckmann and Thibodeaux determined experimentally that the value of \( B_t \) was in the interval between \( \pi/4 < B_t < 1 \).

Yamamoto, Nath and Slotta (1973) and Davis and Ciani (1976) determined that the maximum dimensionless horizontal hydrodynamic force exerted on a bottom-laid pipe could be divided into three distinct regions on a dimensionless relative displacement, \( H/D \) -- relative water depth, \( d/L_o \), dissection plane. These three regions were determined from the relative magnitudes of the ratio of the drag force to inertia force components in the Morison equation. The ratio of these two force components may be expressed using linear wave theory by
\[ \alpha = \left( \frac{2}{\pi D} \right) \left( \frac{C_D}{C_M} \right) \left( \frac{u_{\text{max}}^2}{u_{\text{max}}^2} \right) f(z) \]  

(3.2.5)

in which \( f(z) \) is defined by Eq. (3.1.10). Substituting Eqs. (3.1.15) and (3.1.16) for \( u_{\text{max}} \) and \( \dot{u}_{\text{max}} \) yields

\[ \alpha = \left( \frac{1}{\pi} \right) \left( \frac{C_D}{C_M} \right) \left( \frac{H}{D} \right) f(z) \]  

(3.2.6)

Assuming that the pipe diameter is small compared to both the water depth, \( d \), and the wavelength, \( L \); the small argument approximation for the hyperbolic cosine may be invoked \( [\text{i.e.}, f(-d + D/2) = \cosh \kappa (D/2)/\sinh \kappa d \sim \sinh^{-1} \kappa d] \). Yamamoto, Nath and Slotta (1973) used values of the drag coefficient, \( C_D = 1.0 \) and the inertia coefficient \( C_M = 3.29 \) to determine these three regions. Davis and Ciani (1976) used values of the drag coefficient \( C_D = 1.5 \) and the inertia coefficient \( C_M = 3.9 \) to also determine these three regions.

For design purposes, two constant-value \( \alpha \) contours may be drawn on a dimensionless dissection plane which combines the principal parameters usually specified in design (i.e., relative displacement, \( H/D \) versus relative water depth, \( d/L_0 \)). These two constant-value \( \alpha \) contours divide the dimensionless dissection plane into three separate regions for design. The values chosen were \( \alpha = 20 \) (drag force component dominates) and \( \alpha = 0.05 \) (inertia force component dominates).
Region A in the upper-left corner of Figure 3.2.1 applies to design conditions in which the drag force dominates over the inertia force. In this region the inertia force amounts to less than 5% of the drag force (i.e., $a > 20$). In general, high waves in shallow water create the conditions encountered in this region.

In the lower-right corner, Region C, the situation is reversed; the drag force amounts to less than 5% of the inertia force (i.e., $a < 0.05$). In general, lower relative waves and waves in deep water create the conditions encountered in this region.

In Region B which lies between Regions A and C, both the drag force and inertia force components should be considered in the design problem. Figure 3.2.1 provides some guidance for design purposes; however, it should be used with caution since the use of linear wave theory in the surf zone does not represent near-breaking waves very well.

The stability equation, Eq. (2.1.13), when applied to design conditions in which both the drag and inertia forces are included (such as the wave conditions in Region B, Fig. 3.2.1), may be reduced to

$$C_D [1 + \mu \left( \frac{C_L}{C_D} \right) \cos \beta] (u^2 + U_o^2 + 2uU_o \cos \theta) \frac{1}{2} (u \cos \theta + U_o)$$

$$+ C_M \frac{\pi D}{2} u \cos \theta < \mu (\gamma_s - 1) \frac{\pi D}{2} g \cos \beta$$

in which the specific gravity of pipe, $\gamma_s = \rho_s / \rho_w$. 
Figure 3.2.1. Relative importance regions between the drag force, $F_{Dx'}$, and the inertia force, $F_{Ix'}$. 

Region A
$C_M/C_D = 3.29, \alpha = 20$
$\alpha > 20$

Region B
$0.05 < \alpha < 20$

Region C
$C_M/C_D = 3.29, \alpha = 0.05$
$C_M/C_D = 2.6, \alpha = 0.05$

$\alpha < 0.05$
As a measure of the relative contributions between the drag and lift forces, a dimensionless force coefficient ratio, $W$, may be defined as

$$W = 1 + \mu \left( \frac{C_L}{C_D} \cos \beta \right)$$ \hspace{1cm} (3.2.8)

By using a dimensionless form of the horizontal water particle velocity and acceleration, $u'$ and $\ddot{u}'$, and a dimensionless steady-uniform horizontal current, $U_o'$, which are defined by

$$u' = \frac{u}{H_T}, \quad \ddot{u}' = \frac{\ddot{u}}{H_T^2}, \quad U_o' = \frac{U_o}{H_T}$$ \hspace{1cm} (3.2.9)

Eq. (3.2.7) may be further reduced to the following dimensionless form:

1. **Region B**

$$\frac{H(y)}{L_o} \left( \frac{H(y)}{L_o} \right) C_D W(u')^2 \left[ 1 + \left( \frac{U_o'}{u'} \right)^2 + 2 \left( \frac{U_o'}{u'} \right) \cos \theta(y) \right]^{\frac{1}{2}}$$

$$\cdot \left[ \frac{U_o'}{u'} + \cos \theta(y) \right] + C_M \frac{D}{L_o} \ddot{u}' \cos \theta(y)$$

$$< \mu (\gamma_s - 1) \pi^2 \left( \frac{D}{L_o} \right)^2 \cos \beta$$ \hspace{1cm} (3.2.10)

Eq. (3.2.10) may be further simplified for Regions A and C to,
2. Region A

\[
\left[ \frac{H(y)}{L_O} \right]^2 \cdot (u')^2 \left[ 1 + \left( \frac{U_C'}{u'} \right)^2 + 2 \left( \frac{U_C'}{u'} \right) \cos \theta(y) \right]^{\frac{3}{2}} \left( \frac{U_C'}{u'} \right) \\
+ \cos \theta(y) \right] < \mu (\gamma_s - 1) \pi^2 \left( \frac{D}{L_O} \right) (C_D W)^{-1} \cos \beta 
\]

(3.2.11)

3. Region C

\[
\frac{H(y)}{L_O} \cdot u' \cos \theta(y) < 2 \mu (\gamma_s - 1) \pi C_M^{-1} \cos \beta 
\]

(3.2.12)

Eqs. (3.2.10-3.2.12) may be evaluated using any wave theory. To satisfy this stability requirement in which the right-hand-side of Eqs. (3.2.10-3.2.12) is computed in terms of the specified design parameters, maximum values of the terms in the left-hand-side of these equations should be employed. Next, Eqs. (3.2.10-3.2.12) will be evaluated explicitly using linear wave theory.

3.3 Applications to the Stability Problem

The wave height, \( H(y) \), in Eqs. (3.2.10-3.2.12) may be expressed in terms of the deep water swell height, \( H_O \), a refraction coefficient, \( K_R(y) \), and a shoaling coefficient, \( K_S(y) \), by the following (Ippen, 1966, p. 166):

\[
H(y) = K_R(y)K_S(y)H_O 
\]

(3.3.1)

\[
K_R(y) = \left[ \frac{b_o}{b(y)} \right]^{\frac{3}{2}} 
\]

(3.3.2)

\[
K_S(y) = \left[ 2n(y)L(y)/L_O \right]^{-\frac{3}{2}} 
\]

(3.3.3)
in which

\[ n(y) = \frac{1}{3}[1+2\frac{d(y)}{\sinh 2k d(y)}] \quad (3.3.4) \]

and \( L_0 = \frac{gT^2}{2\pi} \) = the wavelength in deep water, \( b_0 \) and \( b(y) \) are the horizontal distances between wave orthogonals in deep water and at a specific pipeline location, \( y \), respectively, shown in Figure 3.3.1.

Substituting Eqs. (3.1.13), (3.1.14) and (3.3.1) for \( u, \dot{u} \) and \( H(y) \) from linear wave theory and invoking the small argument approximation as before, the general equation given by Eq. (3.2.10) may now be expressed by:

\[
K_r(y)K_s(y)\left(\frac{H_0}{L_0}\right)^2\left[K_r(y)K_s(y)\left(\frac{H_0}{L_0}\right)C_D W \sinh^{-2} \kappa d \cos^2 \omega t - 0.5\left(1+\frac{1}{U_0'}\right)\sinh \kappa d \cos \omega t \cos \theta(y)\right] \frac{1}{L_0} \\
\cdot \left[U_0' \pi \kappa d \cos \omega t + \cos \theta(y)\right] - C_M \pi \left(\frac{D}{L_0}\right) \sinh^{-1} \kappa d \sin \omega t
\]

\[
\cos \theta(y) < \frac{g}{L_0} (\gamma - 1)(D - \frac{D}{L_0}) \cos \beta \quad (3.3.5)
\]

Eq. (3.3.5) may be further simplified for the following cases:

1. **Region B without current**

Substituting \( U_0' = 0 \) into Eq. (3.3.5) yields

\[
E < \frac{g}{L_0} (\gamma - 1) \cos \beta \quad (3.3.6)
\]

in which
Figure 3.3.1. Definition sketch for wave refraction.
The maximum value of \( E \) may be obtained using an approach similar to that given by Horikawa (1978, p. 106); i.e., the maximum occurs when

\[
\frac{\partial E}{\partial (\omega t)} = 0 \quad @ \quad \omega t = (\omega t)_{\text{max}}
\]  

(3.3.10)

or

\[
[G - 2F \sin(\omega t)_{\text{max}}] \cos(\omega t)_{\text{max}} = 0
\]  

(3.3.11)

There are two possible solutions to Eq. (3.3.11). The solution for \( \cos(\omega t)_{\text{max}} = 0 \) applies to cases in which the inertia force dominates and the Morison equation may not be valid. The second solution is given by

\[
\sin(\omega t)_{\text{max}} = \frac{G}{2F} = -\frac{C_M \pi D \sinh \kappa d}{2K_r(y)K_s(y)H_0 C_D W}
\]

(3.3.12)

\[
\cos(\omega t)_{\text{max}} = \left[1 - \left(\frac{G}{2F}\right)^2\right]^\frac{1}{2} = \left[1 - \left(\frac{C_M \pi D \sinh \kappa d}{2K_r(y)K_s(y)H_0 C_D W}\right)\right]^\frac{1}{2}
\]

(3.3.13)

Substituting Eqs. (3.3.12) and (3.3.13) into Eq. (3.3.7) gives the maximum value of \( E \) as
$$E_{\text{max}} = K_r(y)K_s(y) \frac{H_o}{L_o} \sinh^{-1} \kappa d \cos \theta(y) \cdot F \cdot [1 + \left(\frac{G}{2F}\right)^2]$$

(3.3.14)

The stability equation may thus be reduced to

$$K_r^2(y)K_s^2(y) \sinh^{-2} \kappa d \left[1 + \left(\frac{\kappa d}{2K_r(y)K_s(y)H_oC_D W}\right)^2 \right] \cos \theta(y)$$

$$< \mu (\gamma_s - 1) \left( \frac{L_o}{H_o} \right) \left( \frac{D}{H_o} \right) (C_D W)^{-1} \cos \beta$$

(3.3.15)

2. **Region A**

Substituting an inertia coefficient of $C_M = 0$ into Eq. (3.3.5) yields

$$K_r^2(y)K_s^2(y) \frac{H_o}{L_o} C_D W \sinh^{-2} \kappa d \cos^2 \omega t \cdot$$

$$\left[1 + (U_o' \pi ^{-1} \sinh \kappa d \cos^{-1} \omega t)^2 + 2U_o' \pi \sinh^{-1} \kappa d \cos \omega t \cos \theta(y)\right]^{1/2}$$

$$\cdot [U_o' \pi \sinh^{-1} \kappa d \cos \omega t + \cos \theta(y)] < \mu (\gamma_s - 1) \left( \frac{D}{L_o} \right) \cos \beta$$

(3.3.16)

The maximum value of the left-hand-side of Eq. (3.3.16) occurs when $\omega t = 0$. The stability equation may thus be reduced to

$$K_r^2(y)K_s^2(y) \sinh^{-2} \kappa d \left[1 + (U_o' \pi ^{-1} \sinh \kappa d)^2 + 2U_o' \pi \sinh^{-1} \kappa d \cos \theta(y)\right]^{1/2}$$

$$\cdot [U_o' \pi \sinh^{-1} \kappa d + \cos \theta(y)] < \mu (\gamma_s - 1) \left( \frac{L_o}{H_o} \right) \left( \frac{D}{H_o} \right)^2 (C_D W)^{-1} \cos \beta$$

(3.3.17)
3. **Region C**

Substituting a drag coefficient of $C_D = 0$ into Eq. (3.3.5) yields

$$K_r(y)K_s(y)\sinh^{-1}kd\cos\theta(y) < \mu(\gamma - 1)(\pi C_M)^{-1}\frac{L_o}{H_o}\cos\beta$$

(3.3.18)
IV. STREAM FUNCTION WAVE THEORY
AND APPLICATIONS

Section 4.1 describes the general features of the stream function wave theory and various published solution techniques. Section 4.2 formulates the equations for solving the stream function boundary value problem for nonlinear waves on a steady-uniform current. Section 4.3 describes the applications of the stream function wave kinematics to the horizontal pipe stability problem.

4.1 The Stream Function Boundary Value Problem

Waves in the ocean environment never have the symmetric water surface profile assumed in the linear wave theory solution. A finite amplitude wave theory is required to provide a better description of the free surface profile for real ocean waves. Among the nonlinear wave theories presently available are the Cnoidal, Solitary, Stoke's higher-order wave theories, and Stream Function wave theory (Dean, 1965).

All of the finite amplitude water wave theories represent solutions to boundary value problems. The formulation of the finite amplitude wave boundary value problem is basically the same as that of the small amplitude linear
wave boundary value problem except now the higher order nonlinear terms are retained in the analysis.

It is assumed that the wave profile propagates with constant speed and without change of form in an ideal inviscid fluid. It is then possible to choose a coordinate system moving with the speed of and in the same direction as the wave profile. In this moving reference frame, the wave profile does not change shape and the motion is steady relative to this moving coordinate system. The water particle velocity components, \( u \) and \( w \), may be defined in terms of a scalar stream function, \( \psi \), by

\[
u - C = -\frac{\partial \psi}{\partial z} \tag{4.1.1}
\]

\[
w = \frac{\partial \psi}{\partial \xi} \tag{4.1.2}
\]

in which \( C \) is the constant wave celerity.

The boundary value problem for the irrotational motion may now be formulated by the following (Dean, 1965):

**DE:** \( \nabla^2 \psi = 0; \quad |\xi| < \infty; \quad -d \leq z \leq \eta(\xi) \tag{4.1.3} \)

**BBC:** \( \frac{\partial \psi}{\partial \zeta} = 0; \quad |\xi| < \infty; \quad z = -d \tag{4.1.4} \)

**KFSBC:** \( \frac{\partial \eta}{\partial \zeta} = -\frac{\partial \psi}{\partial \xi}/(\frac{\partial \psi}{\partial z}); \quad |\xi| < \infty; \quad z = \eta(\xi) \tag{4.1.5} \)

**DFSBC:** \( \eta + \frac{1}{2g}[\left(\frac{\partial \psi}{\partial z}\right)^2 + \left(\frac{\partial \psi}{\partial \xi}\right)^2] = \frac{C^2}{2g} = Q; \quad |\xi| < \infty; \quad z = \eta(\xi) \tag{4.1.6} \)
Periodicity: $\psi(\xi+L,z) = \psi(\xi,z)$ \hspace{1cm} (4.1.7)

in which $Q$ is the Bernoulli constant and $L$ is the wavelength.

Dean (1965) assumed the following stream function solution:

$$\psi(\xi,z) = \frac{Lz}{T} + \sum_{N=1}^{NN} X(n) \sinh\left[\frac{2\pi n}{L} (d+z)\right] \cos\left(\frac{2\pi n}{L} \xi\right) \hspace{1cm} (4.1.8)$$

in which $T$ is the wave period and $d$ is the water depth.

Eq. (4.1.8) satisfies the governing differential equation, Eq. (4.1.3), the bottom boundary condition, Eq. (4.1.4), the kinematic free surface boundary condition, Eq. (4.1.5), and the periodicity requirement, Eq. (4.1.7), exactly for arbitrary finite values of the wavelength, $L$, of the stream function value at the free surface, $\psi(\xi,\eta)$, and of the $NN$ stream function coefficients, $X(n)$. The numerical procedures for determining these unknown values for $X(n)$, $L$, and $\psi(\xi,\eta)$ are based on the concepts of minimizing the errors associated with the dynamic free surface boundary condition, Eq. (4.1.6).

Dean (1968) compared the analytical validity of eight different wave theories and concluded that the stream function wave theory of fifth order (i.e., $NN = 5$) provided the best fit over a wide range of wave conditions. For very
shallow water waves, the linear wave theory and the first order Cnoidal wave theory compared best with the stream function solution. Dean (1968) concluded that by increasing the order of stream function solution, it would provide the best fit even for most shallow water waves. Dean and Le Mehaute (1970) also compared experimentally these same wave theories and found that, on an overall basis, the stream function theory provided a significantly better fit to the measured water particle velocities than did the other theories.

Next we will review other forms of boundary value problem formulations and solution techniques.

**Von Schwind and Reid (1972)**

Von Schwind and Reid (1972) developed a stream function wave theory with basic similarities to the theory developed by Dean (1965). The principal difference between the two theories is that Von Schwind and Reid used a conformal transformation of the coordinates for the boundary value problem from the complex $(\xi + iz)$ plane to the complex $(\phi + i\psi)$ plane. The schematic contours for such a transformation are shown in Figure 4.1.1. After the equations which result from the transformation are given in a dimensionless form, the Fourier coefficients of the solutions to the boundary value problem are determined through a numerical iterative process similar to that used by Dean (1965).
Figure 4.1.1. Schematic contours of $\xi$ and $z$ in $(\xi+i\eta)$ plane and the conformally mapped contours of $\phi$ and $\psi$ in $(\phi+i\psi)$ plane.

Figure 4.1.2. Linear, bilinear and steady-uniform current profiles.
Benzi, Salusti and Sutera (1979)

Benzi, Salusti and Sutera (1979) have formulated the stream function wave theory by means of a variational principle. For a periodic, finite amplitude wave in a coordinate frame moving with wave celerity, $C$, the boundary value problem was formulated as:

**DE:** $\nabla^2 \psi = F(\psi); \quad |\xi| < \infty, \quad -d \leq z \leq \eta(\xi)$ (4.1.9)

**BBC:** $\psi(\xi, -d) = \psi_{-d} = \text{constant}; \quad |\xi| < \infty, \quad z = -d$ (4.1.10)

**KFSBC:** $\psi(\xi, \eta) = \psi_{\eta} = \text{constant}; \quad |\xi| < \infty, \quad z = \eta(\xi)$ (4.1.11)

**DFSBC:** $\frac{1}{2}(\nabla \psi)^2 + g\eta + Q = \text{constant}; \quad |\xi| < \infty, \quad z = \eta(\xi)$ (4.1.12)

The variational principle states that if a functional of the free surface, $\eta$, and the stream function, $\psi; I(\eta, \psi)$ is defined by:

$$I(\eta, \psi) = \int_{L}^{\eta} \int_{0}^{-d} \left[ \frac{1}{2} |\nabla \psi|^2 + g\eta + G(\psi) \right] \, d\xi \, dz$$ (4.1.13)

in which $F(\psi) = \frac{dG(\psi)}{d\psi}$ is related to the pressure; then the boundary value problem, Eqs. (4.1.9-4.1.12) may be proved to be equivalent to the equations:

$$\frac{\delta I}{\delta \psi} = 0; \quad \frac{\delta I}{\delta \eta} = 0$$ (4.1.14)
with the boundary conditions specified by Eqs. (4.1.10-11) and the periodicity requirements specified by:

\[ \eta(\xi + L) = \eta(\xi) \]  
(4.1.15)

\[ \psi(\xi + L, z) = \psi(\xi, z) \]  
(4.1.16)

Chaplin (1980)

Chaplin (1980) reformulated the problem solution technique by choosing as unknowns the wavelength, \( L \), and the surface elevations of the wave profile at a discrete number of points, \( \eta_j \), which were equally distributed between the crest and the trough. For a wave condition specified by the wave height, \( H \), the wave period, \( T \), and the water depth, \( d \); the unknowns are defined in the dimensionless forms by the following equations:

\[ S_j = \frac{d + \eta_j}{L} \quad , \quad j = 1, \ldots, J \]  
(4.1.17)

\[ R = \frac{d}{L} \]  
(4.1.18)

Two of these unknowns are eliminated immediately by the constraints that the numerical iterative process should converge both to the specified wave height, \( H \), and to the still water level, \( d \). These two constraints may be expressed dimensionlessly as

\[ S_J = S_1 - \frac{H}{d} R \]  
(4.1.19)
in which \( w_j \) are the weights used in the numerical integration; e.g., Simpson's rule.

The dimensionless stream function is of the form of Eq. (4.1.8), given by:

\[
\psi = \sum_{n=1}^{J-1} A_n \sinh(2\pi n S) \cos(2\pi n \bar{X}) + A_j S \quad (4.1.21)
\]

in which the dimensionless variables are defined according to

\[
\bar{\psi} = \frac{\psi}{\psi(\zeta, \eta)} ; \quad \bar{X} = \frac{\xi}{L} ; \quad A_n = \frac{X(n)}{\psi(\zeta, \eta)} ; \\
S = \frac{d+z}{L} \quad (4.1.22)
\]

and the dimensionless coefficient \( A_1, A_2, \ldots, A_J \) must satisfy the free surface requirements specified by

\[
1 = \sum_{n=1}^{J-1} A_n \sinh 2\pi n S_j \cos 2\pi n \bar{X}_j + A_j S_j, \\
j = 1, \ldots, J \quad (4.1.23)
\]

For a particular free surface profiles, \( S, \) and wavelength, \( L, \) Eq. (4.1.23) is considered to be given by a sum of linearly independent functions of \( \bar{X}_j. \) A set of orthonormal functions may be constructed from this set by the Gram-Schmidt process. These may then be treated by generalized Fourier analysis to compute the stream function coefficients, \( A_n. \) A detailed and reasoned development is
given by Hamming (1962).

The dynamic free surface boundary condition, Eq. (4.1.6), may be expressed in dimensionless form by dividing by the water depth, \( d \), i.e.,

\[
Q_j = \frac{d}{R^2} \frac{d}{A_j^2} \left[ \left( \frac{\partial \psi}{\partial x} \right)^2 + \left( \frac{\partial \psi}{\partial S} \right)^2 \right] + \frac{S_j}{R},
\]

\( j = 1, \ldots, J \) \hspace{1cm} (4.1.24)

in which the derivatives are to be evaluated at each discrete point, \((\bar{x}_j, S_j)\). The numerical iterative process is then carried out through minimizing the errors associated with Eq. (4.1.24)

Dalrymple (1974)

Dalrymple (1974) presented a stream function model for the nonlinear water waves on shear current in which the shear current velocity was modeled either by a linear or a bilinear profile. Figure 4.1.2 is a definition sketch for the linear, bilinear and steady-uniform current profiles.

For a linear shear current, the velocity profile may be expressed by

\[
U(z) = U_b + \frac{(U_s - U_b)(d+z)}{d},
\]

\( in which \( U_b \) and \( U_s \) are the magnitudes of the bottom and surface current. The governing differential equation may be expressed in the form of the Poisson equation,
Figure 4.2.1. Definition sketch for a nonlinear wave on a steady-uniform current.
\[ \nabla^2 \psi = \frac{U_s - U_b}{d} = -\Omega_o \quad (4.1.26) \]

in which \(-\Omega_o\) represents the uniform vorticity of the fluid.

For the case of a steady-uniform current, a special case of the linear shear current model; i.e.,

\[ U(z) = U_b = U_s = U_o \quad (4.1.27) \]

the governing differential equation becomes that of the Laplace equation, Eq. (4.1.3).

Next, the solution techniques for this model will be discussed in detail in Section 4.2.

4.2. Mathematical Model and Solution for Non-linear Waves on a Steady-Uniform Current

Figure 4.2.1 is a definition sketch of the boundary value problem for the nonlinear waves on a steady-uniform current. The water particle velocity components, \(u\) and \(w\), may be defined in terms of a scalar stream function, \(\psi\), by

\[ u + U_o - C = -\frac{\partial \psi}{\partial z} \quad (4.2.1) \]

\[ w = \frac{\partial \psi}{\partial \zeta} \quad (4.2.2) \]

The boundary value problem may then be specified by Eqs. (4.1.3-4.1.7). A stream function perturbation series is assumed to be given by:
\[ \psi(\xi, z) = \left[ \frac{X(3)}{T} - U_o \right] z - \sum_{n=4}^{N+2} X(n) \sinh \left[ \frac{2\pi(n-3)(z+d)}{X(3)} \right] \cos \left[ \frac{2\pi(n-3)\xi}{X(3)} \right] \] (4.2.3)

in which \( X(3) \) is the undetermined wavelength, \( L \). Eq. (4.2.3) satisfies all the equations of the boundary value problem except for the DFSBC, Eq. (4.1.6).

Evaluating the stream function, \( \psi \), on the free surface, \( z = \eta(\xi) \), yields the following transcendental equation for the free surface, \( \eta(\xi) \), as:

\[ \eta(\xi) = \frac{X(N+3)}{(X(3)/T - U_o)} + \sum_{n=4}^{N+2} \frac{X(n)}{(X(3)/T - U_o)} \sinh \left[ \frac{2\pi(n-3)(\eta+d)}{X(3)} \right] \cos \left[ \frac{2\pi(n-3)\xi}{X(3)} \right] \] (4.2.4)

in which the stream function at the free surface, \( \psi(\xi, \eta) \), is identified by \( X(N+3) \). Once the \( N-1 \) stream function coefficients, \( X(n) \), wavelength, \( X(3) \), and the stream function at the free surface have been computed, the free surface elevation may be determined by solving Eq. (4.2.4) iteratively, or by a Newton–Raphson iterative method given by Hudspeth and Slotta (1978).

By specifying the wave height, \( H \), wave period, \( T \), water depth, \( d \), and the steady-uniform current, \( U_o \); the \( N+1 \) unknowns, \( (X(n), n=3, \ldots, N+3) \), may be solved iteratively. At each step of the iterative process, small corrections to the previous estimates of the unknown coefficients are made to improve the solution.
The N+1 unknowns may be determined so that the dynamic free surface boundary condition (DFSBC) is satisfied in a best least-squares sense. The error in the DFSBC may be expressed as:

\[ \varepsilon_1 = \frac{1}{\pi} \int_0^\pi [R_1(\theta)]^2 d\theta = \frac{1}{\pi} [Q(\theta) - \bar{Q}]^2 d\theta \] (4.2.5)

in which

\[ \bar{Q} = \frac{1}{\pi} \int_0^\pi Q(\theta) d\theta, \quad \theta = \frac{2\pi \xi}{X(3)} \] (4.2.6)

Dalrymple (1974) introduced two Lagrangian multipliers, \( X(1) \) and \( X(2) \), into the iterative process so that the wave profile may converge to the mean water level and wave height. The total error associated with the iteration is thus defined by:

\[ \varepsilon_T = \varepsilon_1 + X(1) \varepsilon_2 + X(2) \varepsilon_3 \]

\[ = \frac{1}{\pi} \int_0^\pi [R_1(\theta)]^2 d\theta + \frac{X(1)}{\pi} \int_0^\pi \eta(\theta) d\theta \]

\[ + X(2) [-H + \eta(0) - \eta(\pi)] \] (4.2.7)

in which

\[ \varepsilon_2 = \frac{1}{\pi} \int_0^\pi \eta(\theta) d\theta \] (4.2.8)

\[ \varepsilon_3 = (-H + \eta(0) - \eta(\pi)) \] (4.2.9)

where the two Lagrangian constraints have been introduced.
The numerical technique used to minimize the total error is based on an algorithm given by Marquardt (1963), in which the total error, $\varepsilon_T$, is expanded in a Taylor series for the $N+3$ unknowns, $(X(n), n=3, \ldots, N+3)$, according to:

$$
\varepsilon_T' = \frac{1}{\pi} \int_0^\pi \left[ R_1(\theta) + \sum_{n=3}^{N+3} \frac{\partial R_1(\theta)}{\partial X(n)} \delta X(n) \right]^2 d\theta 

+ \left[ X(1) + \delta X(1) \right] \int_0^\pi \left[ \eta(\theta) + \sum_{n=3}^{N+3} \frac{\partial \eta(\theta)}{\partial X(n)} \delta X(n) \right] d\theta 

+ \left[ X(2) + \delta X(2) \right] \left[ -H + \eta(0) + \sum_{n=3}^{N+3} \frac{\partial \eta(0)}{\partial X(n)} \delta X(n) - \eta(\pi) 

- \sum_{n=3}^{N+3} \frac{\partial \eta(\pi)}{\partial X(n)} \delta X(n) \right] 

(4.2.10)

The corrections to these unknowns may be computed by solving the following equations:

$$
\frac{\partial \varepsilon_T'}{\partial \delta X(1)} = 0 

(4.2.11)

\frac{\partial \varepsilon_T'}{\partial \delta X(2)} = 0 

(4.2.12)

\frac{\partial \varepsilon_T'}{\partial \delta X(m)} = 0, \; m = 3, \ldots, N+3 

(4.2.13)

These $N+3$ equations may be expressed as:

$$
\frac{1}{\pi} \int_0^\pi \left[ \eta(\theta) + \sum_{n=3}^{N+3} \frac{\partial \eta(\theta)}{\partial X(n)} \delta X(n) \right] d\theta = 0 

(4.2.14)
\[ H - \eta(0) - \sum_{n=3}^{N+3} \frac{\partial \eta(0)}{\partial x(n)} \delta x(n) + \eta(\pi) \]

\[ + \sum_{n=3}^{N+3} \frac{\partial \eta(\pi)}{\partial x(n)} \delta x(n) = 0 \]  \hspace{1cm} (4.2.15)

\[ \frac{1}{\pi} \int_{0}^{\pi} 2[R_1(\Theta) + \sum_{n=3}^{N+3} \frac{\partial R_1(\Theta)}{\partial x(n)} \delta x(n)] \frac{\partial R_1(\Theta)}{\partial x(m)} d\Theta \]

\[ + \frac{[x(1) + \delta x(1)]}{\pi} \int_{0}^{\pi} \frac{\partial \eta(\Theta)}{\partial x(m)} d\Theta + [x(2) + \delta x(2)] \frac{\partial \eta(0)}{\partial x(m)} \]

\[ - \frac{\partial \eta(\pi)}{\partial x(m)} = 0, \quad m = 3, \ldots, N+3 \]  \hspace{1cm} (4.2.16)

or, in matrix form as:

\[ [A] \cdot \{\delta x(n)\} = \{B\}, \quad n = 1, \ldots, N+3 \]  \hspace{1cm} (4.2.17)

in which \([A] = \]

\[
\begin{bmatrix}
1 & 0 & 0 & \ldots & \frac{1}{\pi} \int_{0}^{\pi} \frac{\partial \eta(\Theta)}{\partial x(3)} d\Theta & \ldots & \frac{1}{\pi} \int_{0}^{\pi} \frac{\partial \eta(\Theta)}{\partial x(N+3)} d\Theta \\
0 & 1 & 0 & \ldots & \frac{\partial \eta(0)}{\partial x(3)} - \frac{\partial \eta(\pi)}{\partial x(3)} & \ldots & \frac{\partial \eta(0)}{\partial x(N+3)} - \frac{\partial \eta(\pi)}{\partial x(N+3)} \\
\frac{1}{\pi} \int_{0}^{\pi} \frac{\partial \eta(\Theta)}{\partial x(3)} d\Theta & \frac{\partial \eta(0)}{\partial x(3)} - \frac{\partial \eta(\pi)}{\partial x(3)} & \ldots & \frac{2}{\pi} \int_{0}^{\pi} \frac{\partial R_1(\Theta)}{\partial x(c)} \cdot \frac{\partial R_1(\Theta)}{\partial x(r)} d\Theta \\
. & . & . & \ldots & \ldots & . & . \\
. & . & . & \ldots & \ldots & . & . \\
. & . & . & \ldots & \ldots & . & . \\
. & . & . & \ldots & \ldots & . & . \\
\frac{1}{\pi} \int_{0}^{\pi} \frac{\partial \eta(\Theta)}{\partial x(N+3)} d\Theta & \frac{\partial \eta(0)}{\partial x(N+3)} - \frac{\partial \eta(\pi)}{\partial x(N+3)} & \ldots & \ldots & \ldots & . & . \\
\end{bmatrix}
\]
Explicit expressions for the derivatives with respect to the unknowns are described in detail in Appendix A.

Small corrections to the unknown coefficients, \((\delta X(n), n=3, \ldots, N+3)\), found from the matrix inversion of Eq. (4.2.17) are then added to the previous iterative estimates for the unknowns and the iterations are terminated when the total error is acceptably small. For near-breaking wave conditions and for the extreme shallow water wave conditions, the iterative corrections oscillate rapidly and only a small fraction of the differential corrections obtained are added to the estimates from the previous iterations to achieve a stable solution.

The starting values are found from the linear wave theory approximation. The wavelength, \(X(3)\), is found by solving the following equation using a Newton-Raphson iterative method,
\[ \frac{d}{L_0} = \frac{d}{X(3)} \tanh \left[ \frac{2\pi d}{X(3)} \right] \]  

(4.2.20)

in which \( L_0 \) is the deep water wavelength from linear wave theory \( (=gT^2/2\pi) \). The starting coefficients are all initialized to zero except \( X(4) \), which is estimated from linear wave theory to be

\[ X(4) = \frac{H}{2} \frac{X(3)}{T} \sinh^{-1} \frac{2\pi d}{X(3)} \]  

(4.2.21)

Dean (1974) has tabulated the principal wave field variables from forty wave cases that extend over three decades of dimensionless relative depth from shallow water wave conditions \((d/X(3)<1/25)\) to deep wave conditions \((d/X(3) > \frac{1}{2})\). The dimensionless relative depth, \( d/L_0 \), ranged from 0.002 to 2.0. The wave cases are also characterized by the dimensionless relative wave steepness, \( H/L_0 \), which corresponds to the following ratios of wave height to the breaking wave height, \( H/H_B \): 0.25, 0.5, 0.75, 1.0. Table 4.2.1 summarizes the magnitudes of the dimensionless relative wave steepness, \( H/L_0 \); dimensionless relative water depth, \( d/L_0 \); wave height parameter, \( H/T^2 \); and depth parameter, \( d/T^2 \) for the 40 wave cases tabulated by Dean.

Table 4.2.2 summarizes the wave characteristics calculated in this study for the forty wave cases when no current is present. Column 2 tabulates the ratio of wavelength to deep water wavelength, \( X(3)/L_0 \); column 3 tabulates the ratio of the wave crest elevation to wave height,
Table 4.2.1. Summary of fifty stream function wave cases [tabulated by Dean (1974) and by Chaplin (1980)].

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<th>H/T²</th>
<th>d/L₀</th>
<th>d/T²</th>
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</table>
Table 4.2.2 (continued)

<table>
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<tr>
<th>Case</th>
<th>X(3)/L₀</th>
<th>η₀/H</th>
<th>U₀/C</th>
<th>RMSQ (Dean)</th>
<th>εₚ</th>
<th>No. of iterations</th>
</tr>
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<td>9A</td>
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<td>0.533760</td>
<td>0.149212</td>
<td>1.3E-6</td>
<td>2.9E-5</td>
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</tr>
<tr>
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<td>0.568995</td>
<td>0.319677</td>
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<td>2.2E-5</td>
<td>5.4E-3</td>
</tr>
<tr>
<td>9C</td>
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<td>0.609476</td>
<td>0.512297</td>
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<td>6.8E-4</td>
<td>2.3E-3</td>
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<td>9D'</td>
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<td>7.7E-3</td>
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<td>0.149207</td>
<td>1.2E-6</td>
<td>9.7E-5</td>
<td>7.8E-3</td>
</tr>
<tr>
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<td>8.5E-3</td>
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<tr>
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<td>0.792835</td>
<td>4.9E-6</td>
<td>5.4E-3</td>
<td>4.1E-3</td>
</tr>
</tbody>
</table>

*(Note: Convergence to 0.01 ft (0.003 m) not satisfied.)*
column 4 tabulates the ratio of the horizontal water particle velocity at the wave crest to wave celerity, \( \frac{u_c}{C} \); columns 5 and 6 tabulate the root mean square errors of the DFSBC for the forty wave cases from this study and those tabulated by Dean (1974). The root mean square error of DFSBC is defined by the following equation as:

\[
RMSQ = (\varepsilon_1)^{\frac{1}{2}}
\]

in which the DFSBC error, \( \varepsilon_1 \), is defined in Eq. (4.2.5). Columns 7 and 8 tabulate the total error, \( \varepsilon_T \), defined in Eq. (4.2.7) and the number of iterations to achieve stable solutions with respect to a minimum specified error.

Table 4.2.3 summarizes the dimensionless horizontal water particle bottom velocity and acceleration from this study and those tabulated by Dean (1974). The dimensionless horizontal water particle bottom velocity and acceleration, \( \frac{u_b}{H} \) and \( \frac{\dot{u}_b}{T^2} \), are defined according to Eq. (3.2.9) by

\[
\frac{u_b}{H} = \left( \frac{u_b}{H} \right); \quad \frac{\dot{u}_b}{T^2} = \left( \frac{\dot{u}_b}{T^2} \right)
\]

4.3 Applications to Pipe Stability Problem

For a small pipeline laid on the bottom, the horizontal water particle bottom velocity and acceleration, \( \frac{u_b}{H} \) and \( \frac{\dot{u}_b}{T^2} \), are used to represent the water kinematics for the
Table 4.2.3. Dimensionless maximum horizontal water particle bottom velocity and acceleration.

<table>
<thead>
<tr>
<th>Case (1)</th>
<th>( u'_b ) This study (2)</th>
<th>( u'_b ) Dean (1974) (3)</th>
<th>( \dot{u}'_b ) This study (4)</th>
<th>( \dot{u}'_b ) Dean (1974) (5)</th>
</tr>
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<td>1A</td>
<td>34.752</td>
<td>44.197</td>
<td>715.854</td>
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<tr>
<td>1B</td>
<td>33.710</td>
<td>40.078</td>
<td>797.490</td>
<td>768.243</td>
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<tr>
<td>1C</td>
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<td>36.286</td>
<td>761.434</td>
<td>779.499</td>
</tr>
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<td>1D'</td>
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<td>774.679</td>
<td>--</td>
</tr>
<tr>
<td>1D</td>
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<td>33.271</td>
<td>778.012</td>
<td>760.443</td>
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<tr>
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<td>26.223</td>
<td>323.568</td>
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<td>323.411</td>
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<td>21.208</td>
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<tr>
<td>3D</td>
<td>11.258</td>
<td>12.320</td>
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<tr>
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</tr>
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<td>1.580</td>
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Table 4.2.3 (continued)

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<th>( u_b' ) Dean (1974) (3)</th>
<th>( u_b' ) This study (4)</th>
<th>( u_b' ) Dean (1974) (5)</th>
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</tr>
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</tr>
<tr>
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</tr>
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<td>0.025</td>
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<td>0.000</td>
<td>0.000</td>
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<td>0.000</td>
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</tr>
<tr>
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</tr>
<tr>
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<td>0.000</td>
<td>0.001</td>
<td>0.001</td>
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</tbody>
</table>
whole depth of pipe by invoking the small argument approximations.

The general stability equation, Eq. (3.2.10) may now be expressed in terms of the dimensionless horizontal water particle bottom velocity and acceleration, \( u'_b \) and \( \dot{u}'_b \), by:

\[
P'_b < P'_s \tag{4.3.1}
\]

in which the dimensionless pipeline stability parameter, \( P'_s \), and a dimensionless force acting on the pipeline, \( P'_b \), are defined as:

\[
P'_b = \frac{H(y)}{L_0} \left( \frac{H(y)}{L_0} \right) C_D W (u'_b)^2 \left[ 1 + \left( \frac{U'}{u'_b} \right)^2 + \left( \frac{U''}{u'_b} \right) \cos\theta(y) \right]^{\frac{1}{2}}.
\]

\[
\left[ \frac{U'}{u'_b} + \cos\theta(y) \right] + C_M \frac{\pi D}{2L_0} \dot{u}'_b \cos\theta(y) \right]\tag{4.3.2}
\]

\[
P'_s = \mu (\gamma_s - 1) \pi^2 \left( \frac{D}{L_0} \right) \cos\beta \tag{4.3.3}
\]

For the special case in which no current is present, the above equation may be further reduced to:

\[
\tilde{P}'_b < P'_s \tag{4.3.4}
\]

in which

\[
\tilde{P}'_b = \left( \frac{H(y)}{L_0} \right)^2 C_D W \left( \frac{P'_s}{\tilde{P}_B} \right) \cos\theta(y) \tag{4.3.5}
\]

is the dimensionless wave force acting on the pipeline due to the finite amplitude shoaling wave, and
\[ P'_B = u'_b^2 + \frac{\pi}{2} \frac{C_M}{C_D} \frac{D}{H(y)} \cdot u'_D \]  

(4.3.6)

is the dimensionless hydrodynamic bottom force.

Eq. (4.3.4) represents the horizontal stability requirement for a pipeline under shoaling finite-amplitude wave. As stated in Sec. 3.2, the maximum value of the dimensionless wave force acting on the pipeline, \( \bar{P}_b' \), in Eq. (4.3.5) should be employed to satisfy this stability requirement.

The values of the wave height, \( H(y) \), and wave direction, \( \theta(y) \), at any specific location may be estimated iteratively following the procedures proposed by Dean (1974) or measured explicitly from a detail site survey.

Defining a dimensionless force ratio, \( W' \), by the following:

\[ W' = \frac{C_M}{C_D} \frac{D}{H(y)} = \frac{C_M}{C_L(C_D/C_D) \cos \beta} \frac{D}{H(y)} \]  

(4.3.7)

the maximum values of the dimensionless force, \( P'_B \), may then be calculated for the forty wave cases tabulated by Dean (1974) for a set of the dimensionless force ratio, \( W' \), values which varies between 0 to 10. The results are presented as contours of \( P'_B \) on a wave height parameter, \( H/T^2 \), versus depth parameter, \( d/T^2 \), dissection plane. Figures 4.3.1 to 4.3.6 demonstrate the dimensionless force, \( P'_B \), for the dimensionless force ratio values, \( W' \), equal to 0, 0.5, 1.0, 2.0, 5.0 and 10.0, respectively.
Figure 4.3.1. Contours of dimensionless wave force acting on the pipeline, $P'_B$, for design condition: $W'=0$. 
Figure 4.3.2. Contours of dimensionless wave force acting on the pipeline, $P'_B$, for design condition: $W'=0.5$. 
Figure 4.3.3. Contours of dimensionless wave force acting on the pipeline, $P_B'$, for design condition: $W'=1.0$. 
Figure 4.3.4. Contours of dimensionless wave force acting on the pipeline, $P_B'$, for design condition: $W'=2.0$. 
Figure 4.3.5. Contours of dimensionless wave force acting on the pipeline, $P'_B$, for design condition: $W'=5.0$.\[\text{Figure 4.3.5. Contours of dimensionless wave force acting on the pipeline, } P'_B, \text{ for design condition: } W'=5.0.\]
Figure 4.3.6. Contours of dimensionless wave force acting on the pipeline, $F_B'$, for design condition: $W'=10.0$. 
The dimensionless force ratio, \( W' = 0 \), represents the design condition in which the inertia forces may be neglected. The stability equation, Eq. (4.3.4), may be further reduced for this case to the following:

\[
    u'_b < \left[ \frac{\mu (\gamma_S - 1) \pi^2 (D/L) \cos \beta}{[H(y)/L_0]^2 C_D W \cos \theta(y)} \right]^\frac{1}{2}
\]  

(4.3.8)

The maximum dimensionless horizontal bottom velocity, \( u'_b \), in Eq. (4.3.8) is calculated for the forty wave cases. These maximum values are used to construct the contours of \( u'_b \) on a wave height parameter, \( H/T^2 \), versus depth parameter \( d/T^2 \), dissection plane. Figure 4.3.7 demonstrates these contours for \( u'_b \).
Figure 4.3.7. Contours of dimensionless maximum horizontal water particle bottom velocity.
V. RESULTS AND DISCUSSIONS

In the first section, numerical results from the stream function solutions calculated from this study for the forty wave cases defined by Dean (1974) and for the ten wave cases defined by Chaplin (1980) are compared with the numerical results computed by Dean (1974) and by Chaplin (1980). Section 5.2 discusses the determination of the dimensionless force ratio, $W'$, and the sensitivity of $W'$ to the various design parameters. Section 5.3 summarizes the results of the dimensionless design curves and the procedure for engineering applications to the horizontal pipeline stability problem.

5.1 Comparison of the Stream Function Solutions

The stream function solutions obtained for the forty wave cases tabulated by Dean (1974) from this study are expressed in terms of the dimensionless characteristics summarized in Tables 4.2.2 and 4.2.3. Chaplin (1980) did not obtain numerical values for the near-breaking waves; i.e., Case-D waves, in his study. Instead, he defined a set of wave cases with wave heights equal to ninety percent of a near-breaking wave (Case-D) height. Also tabulated in Table 4.2.1 are the dimensionless relative water depth, $d/L_0$, the dimensionless relative wave steepness, $H/L_0$, the
depth parameter, \(d/T^2\), and the wave height parameter, \(H/T^2\), for these ten wave cases (specified as Case-D'waves).

The stream function solutions for these ten cases are also calculated in this study and summarized in Tables 4.2.2 and 4.2.3.

The stream function solutions for these fifty wave cases are obtained iteratively from the Marquardt method (1963). Iterations were terminated when the combined Bernoulli (i.e., the dynamic free surface boundary condition), wave height, and still water level errors were less than some arbitrary small value; 0.01 ft (0.003 m) in this study except for the wave cases 1C, 1D' and 1D. For wave cases 1C, 1D' and 1D, the Marquardt method was terminated after sixty iterations even though the 0.01 ft (0.003 m) convergence error was not satisfied. It is clear that for these three extremely shallow and near-breaking wave cases more iterations are needed if the reduction of the total error to less than 0.01 ft (0.03 m) is to be realized.

For all of the shallow water waves (i.e., Cases 1-, 2- and 3-) and for all of the wave cases near-breaking (i.e., Cases-D' and -D), the damping coefficient required by the Marquardt iterative process was equal to 0.3 and was increased to 0.8 after ten iterations. The only exceptions were the three extremely shallow water wave cases, 1C, 1D' and 1D, where stable solutions were obtained only by using a very small damping coefficient equal to 0.1 for all
iterations. A damping coefficient equal to 0.8 was used for all of the other wave cases.

It is not clear why Chaplin (1980) could not achieve the stream function solutions for the Dean Case-D waves. The convergence criteria used for Chaplin's iterative process were 1) the wavelength, 2) the surface water particle velocity at the wave crest, and 3) the wave height at the crest. These three convergence criterion were also calculated in this study and are summarized in columns 2, 3 and 4 in Table 4.2.2. Chaplin's numerical results were obtained when these three criterion first remained unchanged to three or five significant figures, respectively. For each case, the higher order nonlinear solutions had to be started from the previous lower order solution. If the convergence criteria could not be satisfied for a given solution order, the order was increased and the solution was restarted using the previous lower order solution as the starting estimates. This stacking procedure was somewhat difficult to determine from his study.

In general, the order of the solutions by Chaplin (1980) were higher than those used by either Dean (1974) or this study. Table 5.1.1 summarizes the order of the solutions used in each of these three studies. The order of the solutions used in this study were the same as those used by Chaplin (1980) with the exception of the extremely shallow water wave cases in which lower order solutions were used in this study. For all of the fifty wave cases
Table 5.1.1. Order for stream function solutions. (Note: $N_3$ and $N_5$ are the orders in which $X(3)/L_0$, $\eta_C/H$, and $u_C/C$ first remained unchanged to 3 and 5 significant figures, respectively.)

<table>
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<th>Case (1)</th>
<th>This study (2)</th>
<th>Chaplin (1980) N$_3$ (3)</th>
<th>N$_5$ (4)</th>
<th>Dean (1974) (5)</th>
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</tr>
<tr>
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</tr>
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<td>33</td>
<td>51</td>
<td>19</td>
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analyzed in this study, the three convergence criteria defined by Chaplin (1980) were always satisfied even though much lower order solutions were employed.

For all the forty wave cases tabulated by Dean (1974), columns 5 and 6 in Table 4.2.2 indicated that the root mean square errors in the DFSBC from this study are somewhat smaller.

For the forty wave cases tabulated by Dean (1974), the three dimensionless parameters summarized in columns 2, 3 and 4 in Table 4.2.2 (i.e., X(3)/L0, ηc/H and uc/C) from this study agree very well with those calculated from the Dean (1974) stream function tables except for the extremely shallow water waves (i.e., Cases 1- and 2-). For these two extremely shallow water wave cases, the Dean tables overestimate the wavelength, X(3), by about 5% to 10% and overestimate the wave crest height, ηc, by about 10% to 26% compared to the results obtained by this study. The differences between the surface particle velocity ratio at the wave crest, uc/C, between this study and the Dean tables varied between -24% to 15%.

The values of the dimensionless maximum horizontal water particle bottom velocity, u_b', and acceleration, û_b', for the fifty wave cases analyzed are summarized in Table 4.2.3. These values agree with those from the Dean tables very well with the exception of the extremely shallow water wave cases (i.e., Cases 1- and 2-) and the near-breaking
wave cases (i.e., Cases-D). For Cases 1- and 2-, the Dean tables for the dimensionless maximum horizontal water particle bottom velocities, \( u_b' \), were larger than those calculated by this study. The values of the dimensionless maximum horizontal water particle bottom accelerations, \( \dot{u}_b' \), obtained from the Dean tables were, in general, smaller than those calculated from this study. For the near-breaking waves (i.e., Cases-D), the values of the dimensionless maximum horizontal water particle bottom velocity and acceleration, \( u_b' \) and \( \dot{u}_b' \), were, in general, smaller than those calculated from this study.

5.2 Determination of the Dimensionless Force Ratio \( W' \)

From the definition of the dimensionless force ratio, \( W' \), given by Eq. (4.3.7), it may be shown that the value of this ratio depends on the following dimensionless parameters: 1) the force coefficient ratios, \( C_M/C_D \) and \( C_L/C_D \); 2) the slope of the beach, \( \beta \), the frictional coefficient, \( \mu \), and 3) the relative displacement, \( H/D \).

For a beach with mild slope (i.e., \( \beta < 1/5 \)), the effect of slope, \( \beta \), on the dimensionless force ratio, \( W' \), is very small. For example, for a slope of \( \beta = 1/5 \), \( \cos \beta = 0.98 \approx 1.0 \); therefore, the slope \( \beta \) may be assumed as constant for most design cases. The value of the frictional coefficient, \( \mu \), may be estimated from a site survey
or by using one of the suggested values listed in Table 2.3.3.

To determine the dependency of the dimensionless force ratio, $W'$, on the force coefficient ratios, $C_M/C_D$ and $C_L/C_D'$, and on the dimensionless relative displacement $H/D$, the experimental data from Yamamoto, Nath and Slotta (1973) and from Wright (1976) have been analyzed in order to estimate the values of the dimensionless force ratio, $W'$. The force coefficients were calculated by using the maximum force method (Section 2.2.2).

From the experiments of Yamamoto, Nath and Slotta (1973), the values of the dimensionless relative water depth, $d/L_0$, varied between 1.16 to 1.95 and the values of the dimensionless relative displacement, $H/D$, varied between 0.217 to 0.517. The drag forces were negligible in these experiments; therefore, a constant value for the drag coefficient $C_D = 0$ was used. The values of the inertia coefficient, $C_M$, varied between 2.20 to 3.17. For all of the experiments, the pipe was placed at an elevation of $e$ above the bottom of the wave tank given a gap ratio of $e/D= 0.021$. A constant value of the lift coefficient $C_L = 1.9$ was used for all the calculations of the dimensionless force ratio, $W'$.

From the experiments of Wright (1976), the values of the dimensionless relative water depth, $d/L_0$, varied between 0.027 to 0.5 and the values of the dimensionless
relative displacement, H/D, varied between 0.202 to 2.776.
For the wave conditions in which the dimensionless relative water depth, \( \frac{d}{L_o} > 0.082 \), the measured horizontal force data were very small and none of the previously discussed methods for determining force coefficient values of \( C_D \) (Section 2.2.2) would be reliable for engineering design. Therefore, a value of the drag coefficient \( C_D = 0.446 \) was computed by using Eq. (2.2.5) from the data for wave condition in which the dimensionless relative water depth \( \frac{d}{L_o} = 0.082 \) and was assumed to be constant for wave conditions in which \( \frac{d}{L_o} > 0.082 \). For the wave conditions in which \( \frac{d}{L_o} \leq 0.056 \), the values of the drag coefficient, \( C_D \), varied between 0.686 to 1.27 while the values of the inertia coefficient, \( C_M \), varied between 2.32 to 3.75. For all of the experiments, an average value of the lift coefficient \( C_L = 1.9 \) was calculated by Wright (1976). This constant value of the lift coefficient, \( C_L \), was used for all of the calculations of the dimensionless force ratio, \( W' \), in this study.

For the experiments conducted both by Yamamoto, Nath and Slotta (1973) and by Wright (1976), the value of the inertia coefficient, \( C_M \), was found to have no clear dependency on either the dimensionless relative water depth, \( \frac{d}{L_o} \), or the dimensionless relative displacement, \( H/D \). In contrast, the value of the drag coefficient, \( C_D \), was found to increase as the value of the dimensionless relative water
depth, \( \frac{d}{L_0} \), is decreased. The effect of the lift coefficient, \( C_L \), on the value of the dimensionless force ratio, \( W' \), was included in the calculations by specifying a constant value of the lift coefficient, \( C_L \).

The values of the dimensionless force ratio, \( W' \), were calculated for the following three design cases: 1) \( \mu = 0.33 \) (i.e., a hard-rock beach), 2) \( \mu = 0.55 \) (i.e., a sandy beach), and 3) \( \mu = 1.0 \) (i.e., a silty-soil beach), with zero-slope (i.e., \( \cos \beta = 1.0 \)). These results are tabulated in Table 5.2.1 and are also presented as contours of the dimensionless force ratio, \( W' \), on a dimensionless relative displacement, \( H/D \)--dimensionless relative water depth, \( \frac{d}{L_0} \), dissection plane; which are shown in Figures 5.2.1-5.2.3.

These results provide an approximate range for values of the dimensionless force ratio, \( W' \), which may be used in the horizontal pipeline stability design problem as design aids. From Figures 5.2.1-5.2.3, it may be seen that the values of the dimensionless force ratio, \( W' \), increases from the upper-left corner (drag force component dominates) to the lower-right corner (inertia force component dominates).

5.3 Design Curves and Application Procedure

Although a steady-uniform current, \( U_0 \), is included in the general stability equation, Eq. (4.3.2), and the stream
Table 5.2.1. Dimensionless force ratio, \( W' \), calculated from Yamamoto, et al. (1973) and Wright (1976). (Note: \( \mu_1 = 0.33, \mu_2 = 0.55, \mu_3 = 1.0 \))

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<td>1.97</td>
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Table 5.2.1 (continued)

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<th>C_M (3)</th>
<th>C_D (4)</th>
<th>C_L (5)</th>
<th>W'</th>
<th>μ₁ (6)</th>
<th>μ₂ (7)</th>
<th>μ₃ (8)</th>
<th>Reference</th>
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<td>2.56</td>
<td>0.915</td>
<td>1.9</td>
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<td>1.71</td>
<td>1.19</td>
<td></td>
<td>Wright (1976)</td>
</tr>
<tr>
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<td>0.027</td>
<td>2.54</td>
<td>0.988</td>
<td>1.9</td>
<td>1.38</td>
<td>1.10</td>
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Figure 5.2.1. Contours of dimensionless force ratio, \( W' \), for design condition: \( \mu = 0.33 \).
Figure 5.2.2. Contours of dimensionless force ratio, $W'$, for design condition: $\mu = 0.55$. 
Figure 5.2.3. Contours of dimensionless force ratio, $W'$, for design condition: $u = 1.0$. 
function solution for a finite-amplitude wave on current may be solved by using the techniques discussed in Sec. 4.2. The tabulation of the dimensionless force acting on a bottom-laid pipeline, $P'_B$, for the infinite combinations which are possible is beyond the scope of this study. Therefore, Eq. (4.3.4) may be used to represent the horizontal stability requirement for a bottom-laid pipeline under a shoaling, finite-amplitude wave without a current. The dimensionless wave force acting on the pipeline, $P'_B$, in Eq. (4.3.4) is proportional to a dimensionless force $P'_B$; therefore, the maximum values of the dimensionless force $P'_B$ are calculated for the fifty wave cases and have been used to construct a set of design curves on a wave height parameter, $H/T^2$, depth parameter, $d/T^2$, dissection plane.

From the definition of the dimensionless force, $P'_B$, given by Eq. (4.3.6), it may be seen that the values of $P'_B$ depend on the horizontal drag and inertia forces and on the vertical lift forces. There is no contribution from the vertical inertia force since the vertical water particle acceleration is zero at the bottom.

The general features of the dimensionless force $P'_B$ curves, shown in Figures 4.3.1-4.3.6, and the general stability equation for pipelines under a shoaling finite-amplitude wave, Eq. (4.3.4), may be summarized as follows:

1. The value of the dimensionless force, $P'_B$, decreases slightly as the wave height parameter, $H/T^2$,
increases. The contours of $P_B'$, therefore, are curved slightly toward the shallow water wave region in the wave height parameter, $H/T^2$—depth parameter, $d/T^2$, dissection plane. The value of the dimensionless wave force acting on the pipeline, $P_B'$, increases in proportion to the value of the wave height parameter, $H/T^2$, according to Eq. (4.3.5). The value of the dimensionless pipeline stability parameter, $P_s'$, remains unchanged as the value of the wave height parameter, $H/T^2$, increases according to Eq. (4.3.3). Therefore, from Eq. (4.3.4), the pipeline becomes less stable under higher waves.

2. The value of the dimensionless force, $P_B'$, decreases as the depth parameter, $d/T^2$, increases. The value of the dimensionless pipeline stability parameter, $P_s'$, remains unchanged as the depth parameter, $d/T^2$, is increased. Therefore, the pipeline becomes more stable if moved seaward.

3. From Figs. 4.3.1-4.3.6, it may be seen that the value of the dimensionless force, $P_B'$ (for any wave condition specified by the dimensionless design parameters, $d/T^2$ and $H/T^2$) increases as the dimensionless force ratio, $W'$, increases.

In an ocean environment where pipelines are placed on the beach and are then subjected to shoaling finite-amplitude waves, the limiting depth of the pipeline without horizontal protection should be determined for design.
Between the shoreline and this limiting depth, it is necessary to either bury the pipelines in trenches or to use other means of stabilization in the case of a hard, impermeable beach. This limiting depth may be determined from the dimensionless design curves of $P'_B$ in the following procedure similar to that proposed by Hudspeth (1971):

1. Predict the maximum deep water design wave height, $H_o$, wave period, $T$, and the direction of wave propagation along the proposed pipeline path.

2. Construct refraction and shoaling diagrams for the predicted wave climate. The procedure is described in detail in Ippen (1966, p. 257-263) or Dean (1974).

3. From the refraction and shoaling diagrams, determine the wave height and wave direction as a function of depth along the proposed path. Steps 2 and 3 may be omitted if a site survey of wave height and wave direction is available.

4. Beginning at the shoreward contour of the breaker zone determined by Steps 2 and 3, compute the wave height parameter, $H/T^2$, depth parameter, $d/T^2$, and the dimensionless force ratio, $W'$, either by using Eq. (4.3.7) or by estimating from Figs. 5.2.1 to 5.2.3.

5. Estimate the value of the dimensionless force, $P'_B$, from Figs. 4.3.1 to 4.3.6, by entering a vertical line through the depth parameter, $d/T^2$, and a horizontal line through the wave height parameter, $H/T^2$. The value of the
dimensionless force, \( P'_B \), may then be estimated from the point of intersection of the two lines and the values of the adjacent contours of constant \( P'_B \). If the value of the dimensionless force ratio, \( W' \), is not the same as those specified by Figs. 4.3.1 to 4.3.6, the value of the dimensionless force, \( P'_B \), may be estimated by numerical interpolation.

6. Compute the dimensionless wave force acting on the pipeline, \( \tilde{P}'_b \), from Eq. (4.3.5). Compute the dimensionless pipeline stability parameter, \( P'_s \), from Eq. (4.3.3).

7. If the value of the dimensionless force acting on the pipeline, \( \tilde{P}'_b \), is greater than the value of the dimensionless pipeline stability parameter, \( P'_s \), the pipeline is unstable, move seaward along the proposed pipeline path to the next adjacent wave height shown on the refraction and shoaling diagram.

8. Repeat steps 4 to 7 by moving seaward of the breaker zone along the proposed path until the dimensionless wave force acting on the pipeline, \( \tilde{P}'_b \), is less than the dimensionless pipeline stability parameter, \( P'_s \). The limiting depth for pipeline stabilization is determined under such conditions.

5.4 Summary and Recommendations

From the numerical stream function solutions calculated in this study, the following conclusions may be
1. The stability of bottom laid pipelines under shoaling, finite-amplitude waves may be expressed in terms of the dimensionless wave force acting on the pipeline, $P'_b$, and the dimensionless stability parameter, $P'_s$, as shown in Eq. (4.3.4). The pipeline is stable if the dimensionless wave force acting on the pipeline, $P'_b$, is less than the dimensionless stability parameter, $P'_s$.

2. The value of the dimensionless wave force acting on the pipeline, $P'_b$, depends on the following parameters: 1) wave height, $H$, wave period, $T$, wave direction, $\theta$, water depth, $d$, pipe diameter, $D$; 2) the force coefficients, $C_D$, $C_M$ and $C_L$; 3) the slope of beach, $\beta$, and the frictional coefficient, $\mu$.

3. The values of 0.33, 0.55 and 1.0 are suggested for the frictional coefficients, $\mu$, for the rock, sand and silty-soil seafloors, respectively.

4. The value of the drag coefficient, $C_D$, varies between 0.686 to 1.27 with increasing $C_D$ value as the value of the dimensionless relative displacement ratio, $H/D$ is increased at the range of $0.027 < d/L_o < 0.082$. The drag coefficient, $C_D$, may be assumed to be a constant value of 0.446 for the range of $0.082 < d/L_o < 0.5$. The effect of the drag force may be neglected for the range of $d/L_o > 0.5$ and, therefore, $C_D = 0$. 
5. The value of the inertia coefficient, $C_M$, varies between 2.20 to 3.75 for the range of the dimensionless relative water depth, $d/L_o > 0.027$ and the dimensionless relative displacement, $H/D < 2.776$ and may be assumed to increase as $d/L_o$ is increased and $H/D$ is decreased.

6. The value of the lift coefficient, $C_L$, may be assumed to be a constant value of 1.9 for the range of $d/L > 0.027$ and $H/D < 2.776$.

7. The value of the dimensionless force ratio, $W'$, may be determined by using Eq. (4.3.7) or estimated graphically from the $H/D$ versus $d/L_o$ dissection plane in Figs. 5.2.1 to 5.2.3.

8. The value of the dimensionless force $P'_B$ may be graphically determined on the wave height parameter, $H/T^2$, versus depth parameter, $d/T^2$, dissection plane in Figs. 4.3.1 to 4.3.6.

**Recommendations for Further Study**

1. More study is required in order to determine and to quantify the effects of the dimensionless relative displacement, $H/D$, and the dimensionless relative water depth, $d/L_o$, on the value of the dimensionless force, $W'$; especially for the wave conditions in which $d/L_o < 0.027$.

2. For a more accurate analysis of the pipeline stability problem, the vertical inertia force should be included in determining the frictional force between pipeline and seafloor.
3. The effect of current on the pipeline stability problem should be included in the numerical Stream Function solution in further studies.
REFERENCES


APPENDIX A. Explicit Expression for Matrix Elements (Chapter 4).

The purpose of this appendix is to give explicit expressions for the matrix elements in Eq. (4.2.17) which are used in the numerical stream function solutions.

Using the identity \( \Theta = \frac{2\pi x}{X(3)} \), the water particle velocity components at the free surface \( \eta(\Theta) \) are derived from Eqs. (4.2.1-4.2.2) and expressed by:

\[
\begin{align*}
\text{u}(\Theta) & = \sum_{n=4}^{N+2} X(n) \cdot \frac{2\pi(n-3)}{X(3)} \cosh \left[ \frac{2\pi(n-3)(d+\eta(\Theta))}{X(3)} \right] \cos(n-3) \Theta \\
\text{w}(\Theta) & = \sum_{n=4}^{N+2} X(n) \cdot \frac{2\pi(n-3)}{X(3)} \sinh \left[ \frac{2\pi(n-3)(d+\eta(\Theta))}{X(3)} \right] \sin(n-3) \Theta
\end{align*}
\]

(A.1)  

(A.2)

in which the free surface \( \eta(\Theta) \) may be expressed by:

\[
\eta(\Theta) = \frac{X(N+3)}{X(3)} - \frac{X(n)}{T} - U_o \sum_{n=4}^{N+2} \frac{X(n)}{X(3)} \sinh \left[ \frac{2\pi(n-3)(d+\eta(\Theta))}{X(3)} \right] \cos(n-3) \Theta
\]

(A.3)

The derivatives of \( \eta(\Theta) \) with respect to the unknowns may be expressed by:

\[
\frac{\partial \eta(\Theta)}{\partial X(3)} = \frac{C \cdot \eta(\Theta) + \text{u}(\Theta)(d+\eta(\Theta))}{(\text{u}(\Theta) + U_o-C) \cdot X(3)} \]

(A.4)

\[
\frac{\partial \eta(\Theta)}{\partial X(m)} = \frac{-\sinh \left[ \frac{2\pi(m-3)(d+\eta(\Theta))}{X(3)} \right]}{\text{u}(\Theta) + U_o-C} \cdot \cos(m-3) \Theta,
\]

\( m=4, \ldots, N+2 \)  

(A.5)
\[ \frac{\partial \eta(\theta)}{\partial X(N+3)} = \frac{-1}{u(\theta)+U_o-C} \]  
(A.6)

in which

\[ C = X(3)/T \]  
(A.7)

The derivatives of the dynamic free surface boundary condition (DFSBC) error with respect to the unknowns may be expressed by:

\[ \frac{\partial Q(\theta)}{\partial X(m)} = \frac{\partial Q(\theta)}{\partial \eta(\theta)} \frac{\partial \eta(\theta)}{\partial X(m)} + \frac{\partial Q(\theta)}{\partial u(\theta)} \left[ \frac{\partial u(\theta)}{\partial X(m)} + \frac{\partial u(\theta)}{\partial \eta(\theta)} \frac{\partial \eta(\theta)}{\partial X(m)} \right] \]

\[ + \frac{\partial Q(\theta)}{\partial w(\theta)} \left[ \frac{\partial w(\theta)}{\partial X(m)} + \frac{\partial w(\theta)}{\partial \eta(\theta)} \frac{\partial \eta(\theta)}{\partial X(m)} \right] \]  
(A.8)

in which

\[ \frac{\partial Q(\theta)}{\partial \eta(\theta)} = 1 \]  
(A.9)

\[ \frac{\partial Q(\theta)}{\partial u(\theta)} = \frac{u(\theta)+U_o-C}{g} \]  
(A.10)

\[ \frac{\partial Q(\theta)}{\partial w(\theta)} = \frac{w(\theta)}{g} \]  
(A.11)

\[ \frac{\partial u(\theta)}{\partial X(3)} = \sum_{n=4}^{N+2} -X(n) \frac{2\pi(n-3)}{X(3)} \cosh \left[ \frac{2\pi(n-3)(d+\eta(\theta))}{X(3)} \right]^2 \] 

\[ \cos(n-3)\theta \]

\[ + \sum_{n=4}^{N+2} X(n) \frac{2\pi(n-3)}{X(3)} \cosh \left[ \frac{2\pi(n-3)(d+\eta(\theta))}{X(3)} \right]^2 \] 

\[ \left[ -\frac{2\pi(n-3)(d+\eta(\theta))}{X(3)} \right] \sinh \left[ \frac{2\pi(n-3)(d+\eta(\theta))}{X(3)} \right] \]

\[ \cos(n-3)\theta \]  
(A.12)
\[
\frac{\partial u(\Theta)}{\partial X(m)} = \frac{2\pi(m-3)}{X(3)} \cosh \left[ \frac{2\pi(m-3)(d+n(\Theta))}{X(3)} \right] \cos(m-3) \Theta, \quad m=4, \ldots, N+2 \tag{A.13}
\]

\[
\frac{\partial u(\Theta)}{\partial X(N+3)} = 0 \tag{A.14}
\]

\[
\frac{\partial w(\Theta)}{\partial X(3)} = \sum_{n=4}^{N+2} -X(n) \frac{2\pi(n-3)}{(X(3))^2} \sinh \left[ \frac{2\pi(n-3)(d+n(\Theta))}{X(3)} \right] \sin(n-3) \Theta + \sum_{n=4}^{N+2} X(n) \frac{2\pi(n-3)}{X(3)}
\]

\[
\left[ -\frac{2\pi(n-3)(d+n(\Theta))}{(X(3))^2} \right] \cosh \left[ \frac{2\pi(n-3)(d+n(\Theta))}{X(3)} \right] \sin(n-3) \Theta \tag{A.15}
\]

\[
\frac{\partial w(\Theta)}{\partial X(m)} = \frac{2\pi(m-3)}{X(3)} \sinh \left[ \frac{2\pi(m-3)(d+n(\Theta))}{X(3)} \right] \sin(m-3) \Theta, \quad m=4, \ldots, N+2 \tag{A.16}
\]

\[
\frac{\partial w(\Theta)}{\partial X(N+3)} = 0 \tag{A.17}
\]

\[
\frac{\partial u(\Theta)}{\partial n(\Theta)} = \sum_{n=4}^{N+2} X(n) \left[ \frac{2\pi(n-3)}{X(3)} \right]^2 \sinh \left[ \frac{2\pi(n-3)(d+n(\Theta))}{X(3)} \right] \cos(n-3) \Theta \tag{A.18}
\]

\[
\frac{\partial w(\Theta)}{\partial n(\Theta)} = \sum_{n=4}^{N+2} X(n) \left[ \frac{2\pi(n-3)}{X(3)} \right]^2 \cosh \left[ \frac{2\pi(n-3)(d+n(\Theta))}{X(3)} \right] \sin(n-3) \Theta \tag{A.19}
\]
The derivatives of $R_1(\Theta)$ with respect to the unknowns may be expressed by:

$$\frac{\partial R_1(\Theta)}{\partial X(c)} = \frac{\partial Q(\Theta)}{\partial X(c)} - \frac{\partial \bar{Q}(\Theta)}{\partial X(c)}, \quad c=3, \ldots, N+3 \quad (A.20)$$

By using Eqs. (A.4-A.6, A.20), the matrix equation, Eq. (4.2.17) may be formulated and solved by the matrix inversion for every iteration step.
APPENDIX B. List of Notations

[A] = Matrix used in Eq. (4.2.17)

$A_n$ = Dimensionless stream function coefficient

{B} = Vector used in Eq. (4.2.17)

$B_t$ = Wave steepness coefficient

$b, b(y)$ = Distance between wave orthogonals

$b_o$ = Distance between wave orthogonals in deep water

$C$ = Wave celerity

$C_D$ = Drag force coefficient

$C_L$ = Lift force coefficient

$C_M$ = Inertia force coefficient

$C_m$ = Added-mass force coefficient

$D$ = Pipe diameter

d, d(y) = Water depth, still water level

$E, F, G$ = Dimensionless forces used in Section 3.3.

e = Gap between the pipe and the bottom

$F(\psi), G(\psi)$ = Pressure functions in the Bernoulli equation

$F_D$ = Horizontal drag force on pipe per unit length

$F_{DX}$ = x-component of horizontal drag force on pipe per unit length $F_D$

$F_F$ = Coulomb frictional force on pipe per unit length

$F_g$ = Gravity force per unit length

$F_I$ = Horizontal inertia force on pipe per unit length

$F_L$ = Vertical lift force on pipe per unit length
\( F_m(t) = \) Measured horizontal force on the pipe per unit length

\( F_N = \) Normal buoyant reaction force per unit length

\( F_P = \) Horizontal hydrodynamic force on pipe per unit length

\( F_{px} = \) x-component of horizontal hydrodynamic force on pipe per unit length, \( F_P \)

\( F_T = \) Structural tensile strength component of pipe per unit length

\( F_x = \) x-component of total force on pipe per unit length

\( F_z = \) z-component of total force on pipe per unit length

\( f(z) = \cosh k(d+z)/\sinh kd \)

\( g = \) Gravitational constant

\( H, H(y) = \) Wave height

\( H_0 = \) Deep water wave height

\( I(\eta, \psi) = \) Functional of the free surface and the stream function

\( K_r(y) = \) Refraction coefficient

\( K_s(y) = \) Shoaling coefficient

\( L, L(y) = \) Wavelength

\( L_0 = g\pi^2/2T = \) deep water wavelength

\( M = \) Moment about the contact point between the pipe and the beach

\( N_{KC} = U_m T/D = \) Keulegan-Carpenter number

\( N'_{K} = \) A form of the Keulegan-Carpenter number

\( n(y) = \frac{1}{2} [1+2\kappa d(y) /\sinh 2\kappa d(y)] = \) ratio of wave group velocity to wave celerity based on linear wave theory
\( P'_B = \) Dimensionless force
\( P'_b = \) Dimensionless force acting on the pipe
\( \bar{P}'_b = \) Dimensionless wave force acting on the pipe
\( P_m(t) = \) Measured vertical force on the pipe per unit length
\( P'_s = \) Dimensionless pipeline stability parameter

\( Q, Q_j, Q(\theta) = \) Bernoulli constant
\( \bar{Q} = \) Mean value of Bernoulli constant
\( q = \) Horizontal water particle velocity due to both wave and current
\( \dot{q} = \) Horizontal water particle acceleration due to both wave and current

\( R = \) Dimensionless water depth

\( \text{RMSQ} = \) Root-mean-square error of the dynamic free surface boundary condition

\( R_1(\theta) = \) Dynamic free surface boundary condition error

\( S, S_j = \) Dimensionless free surface elevation from bottom

\( s = \) Path length traveled by a wave particle in one-half cycle

\( T = \) Wave period

\( T_0 = \) Torsional resisting moment of pipe

\( t = \) time variable

\( U_b = \) Shear current at the bottom

\( U_m = \) Amplitude of horizontal water particle velocity

\( U_o = \) Steady-uniform current

\( U'_o = \) Dimensionless steady-uniform current

\( U_s = \) Shear current at the free surface
U(z) = Horizontal shear current at vertical elevation, z

\( u, u(\xi,z,t) \) = Instantaneous horizontal water particle velocity due only to wave

\( \dot{u} \) = Instantaneous horizontal water particle acceleration due only to wave

\( u', \dot{u}' \) = Dimensionless instantaneous horizontal particle velocity, \( u \), and acceleration, \( \dot{u} \), respectively.

\( u_b \) = Instantaneous horizontal water particle bottom velocity due only to wave

\( \dot{u}_b \) = Instantaneous horizontal water particle bottom acceleration due only to wave

\( u'_b, \dot{u}'_b \) = Dimensionless instantaneous horizontal water particle bottom velocity, \( u_b \), and acceleration, \( \dot{u}_b \), respectively

\( u_c \) = Maximum horizontal wave water particle velocity measured at the wave crest

\( u_{\text{max}} \) = \( H\pi/T \) = maximum horizontal wave water particle velocity measured at the free surface from linear wave theory

\( \dot{u}_{\text{max}} \) = \( 2H\pi^2/T^2 \) = maximum horizontal wave water particle acceleration measured at the free surface from linear wave theory

\( W \) = Dimensionless force coefficient ratio

\( W' \) = Dimensionless force ratio

\( w, w(\xi,z,t) \) = Instantaneous vertical water particle velocity due only to wave

\( w_j \) = Weighting function for numerical integration

\( X_1 \) = Lagrangian multiplier for mean water level

\( X_2 \) = Lagrangian multiplier for wave height

\( X(n) \) = Stream function coefficient

\( \bar{x}, \bar{x}_j \) = Dimensionless x-coordinate of the free surface
\( \delta X(n) = \) Small correction to stream function coefficient

\( x = \) Horizontal coordinate direction perpendicular to the longitudinal axis of pipe

\( y = \) Horizontal coordinate direction parallel to the longitudinal axis of pipe

\( z = \) Vertical coordinate direction measured positive upwards from still water level

\( \kappa = 2\pi/L = \) wave number

\( \zeta = \) Horizontal coordinate direction parallel to wave direction

\( \rho_s = \) Density of pipe material

\( \rho_w = \) Density of water

\( \gamma_s = \rho_s/\rho_w = \) specific gravity of the pipe

\( \omega = 2\pi/T = \) wave angular frequency

\( \varepsilon_r = \) Roughness of the pipe

\( \Omega_o = \) Uniform vorticity of the fluid

\( \varepsilon_1 = \) Dynamic free surface boundary condition error

\( \varepsilon_2 = \) Wave height error

\( \varepsilon_3 = \) Mean water depth error

\( \varepsilon_T = \) Total error

\( \varepsilon'_T = \) Taylor series expansion of total error

\( \theta = 2\pi x/X(3) = \) dimensionless phase angle

\( \psi, \psi(\xi,z,t) = \) Scalar stream function

\( \psi_d, \psi_\eta = \) Stream function value at the bottom and free surface, respectively

\( \bar{\psi} = \) Dimensionless stream function

\( \phi(\xi,z), \phi(\xi,z,t) = \) Scalar velocity potential

\( \eta, \eta(\Theta), \eta_j, \eta(\xi,t) = \) Free surface elevation
\( \eta_c \) = Free surface elevation at the wave crest

\( \alpha \) = Ratio of maximum drag force component to inertia force component

\( \beta \) = Beach slope

\( \Theta, \Theta(y) \) = Angle between wave crest and the longitudinal axis of pipe

\( \mu \) = Frictional coefficient

\( \mu_{\text{peak}} \) = Peak value of frictional coefficient

\( \mu_{\text{residual}} \) = Residual value of frictional coefficient

\( \cdot \) = Vector notation

\( \hat{i} \) = Unit vector in x-direction

\( \hat{k} \) = Unit vector in z-direction

\( \Sigma \) = Summation notation

\( \nabla^2 \) = Laplace operator

\( \text{Re}[\cdot] \) = Real part of [\cdot]

\( ' \) = Dimensionless quantities

Parameters and Dimensionless Ratios:

\( e_r/D \) = Dimensionless roughness ratio of the pipe

\( e/D \) = Clearance (gap) ratio of the pipe

\( H/D \) = Dimensionless relative displacement

\( d/L_0 \) = Dimensionless relative water depth

\( d/T^2 \) = Depth parameter

\( H/L_0 \) = Dimensionless wave steepness

\( H/T^2 \) = Wave height parameter

Subscripts

\( \text{max} \) = Maximum value