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William J. Firey

Consider the following problem of random collision: Let P, P' be convex polygons in $\mathbb{S}^2$, the unit sphere in Euclidean space centered at the origin. If we paint Borel subsets σ, σ' of the respective boundaries of P, P', then what is the probability of a paint-to-paint collision?

Let $\mathfrak{C}_0(P, P')$ be the set of all g in the motion group $\mathfrak{G}$ of $\mathbb{S}^2$ for which gP' touches P, i.e. P and gP' meet, but can be separated by a great circle. Suppose σ, σ' are Borel subsets of the boundaries of P, P' and let $\mathfrak{C}_0(P, P'; \sigma, \sigma')$ be the set of all g in $\mathfrak{C}_0(P, P')$ for which $\sigma \cap g\sigma' \neq \emptyset$, i.e. the paint-to-paint collisions. We solve our problem by exhibiting a finite, natural measure $\overline{\mu}(P, P'; \sigma, \sigma')$ of the set $\mathfrak{C}_0(P, P'; \sigma, \sigma')$. That is, for all $\varepsilon > 0$ sufficiently small, sets $\mathfrak{C}_\varepsilon(P, P'; \sigma, \sigma')$ are used to approximate $\mathfrak{C}_0(P, P'; \sigma, \sigma')$. We prove that $\overline{\mu}(P, P'; \sigma, \sigma') := \lim_{\varepsilon \to 0^+} \left[\mu\left(\mathfrak{C}_\varepsilon(P, P'; \sigma, \sigma')\right) \big/ \varepsilon\right]$ exists, where μ is the invariant (Haar) measure on $\mathfrak{G}$. The value of $\overline{\mu}(P, P'; \sigma, \sigma')$ is expressed in terms of geometric quantities (curvature measures). If $\overline{\mu}(P, P')$ is the value obtained when the entire boundaries of P, P' are painted, then the probability of a paint-to-paint collision is taken to be $\overline{\mu}(P, P'; \sigma, \sigma') \big/ \overline{\mu}(P, P')$.

Collision Probabilities of Convex Polygons in Spherical Two-Space

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1. INTRODUCTION

Consider the following problem of random collision: Let K and K' be (nonempty) compact, convex sets. Paint the subsets σ and σ' of the respective boundaries of K and K'. Keep K fixed and let K' move randomly so as to collide with K. What is the probability of a paint-to-paint collision?

This problem was first posed and answered by Firey [**6**] for K, K' in Euclidean space $\mathbb{R}^n$ $(n \geq 2)$. In a related problem, McMullen [**9**] found that if P, P' are two n-polytopes in $\mathbb{R}^n$, then the only collisions with positive probability are those where a m-face of P meets a m'-face of P' with $m + m' = n - 1$. As part of his solution, Firey used approximation by polytopes and allowed only those boundary sets which are inverse images, under the spherical image map, of Borel subsets of the unit sphere. That is, if ω is a Borel subset of the unit sphere, then an allowed boundary set of K consists of all boundary points of K at which there exists an outer unit normal vector that belongs to ω. Consequently, one is not allowed to paint a nonempty, proper subset of a face of K or K'.

Schneider [**16**] gave a solution to the collision problem in $\mathbb{R}^n$ by using approximation with smooth, compact, convex sets. He used the curvature measures of Federer [**3**], which permitted the use of any Borel subsets of the boundaries of K and K'.

We consider the following problem of random collision: Let P, P' be convex polygons in $\mathbb{S}^2$, the unit sphere in $\mathbb{R}^3$ centered at the origin. If we paint Borel subsets σ, σ' of the respective boundaries of P, P', then what is the probability of a paint-to-paint collision?

In more detail, let $\mathfrak{C}_0(P, P')$ be the set of all g in the motion group $\mathfrak{G}$ of $\mathbb{S}^2$ for which gP' touches P, i.e. P and gP' meet, but can be separated by a great circle. Suppose σ, σ' represent Borel subsets of the boundaries of P, P' and let $\mathfrak{C}_0(P, P'; \sigma, \sigma')$ be the set of all g in $\mathfrak{C}_0(P, P')$ for which σ and $g\sigma'$ have nonempty intersection, i.e. the paint-to-paint collisions. We will have a solution to our problem if we exhibit a "natural measure" $\overline{\mu}(P, P'; \sigma, \sigma')$ of the set $\mathfrak{C}_0(P, P'; \sigma, \sigma')$ that is finite for all choices of σ, σ', and if we express $\overline{\mu}(P, P'; \sigma, \sigma')$ in terms of geometric quantities which depend only upon P, P' and σ, σ'. Thus, if $\overline{\mu}(P, P')$ is the (positive) value obtained when the entire boundaries of P, P' are painted, then the probability of a paint-to-paint collision is taken to be

$$\frac{\overline{\mu}(P, P'; \sigma, \sigma')}{\overline{\mu}(P, P')}.$$

The geometric quantities we speak of are the curvature measures defined in Chapter **3**. The motion group of $\mathbb{S}^2$ and its invariant (Haar) measure μ are described in Chapter 4. In Chapter 5 a "thickened" family of μ–measurable sets $\mathfrak{C}_\varepsilon(P, P'; \sigma, \sigma')$ is defined for $0 < \varepsilon < \pi/2$ in order to approximate the set $\mathfrak{C}_0(P, P'; \sigma, \sigma')$. In Chapters 6 and 7 we give upper and lower estimates, respectively, for $\mu\Big(\mathfrak{C}_\varepsilon(P, P'; \sigma, \sigma')\Big)$. If these estimates are divided by ε and a limit taken as ε approaches zero, the same value is obtained. By using a "pinching" argument in Chapter 8 we compute

$$\overline{\mu}(P,P';\sigma,\sigma') = \lim_{\varepsilon\to 0^+} \frac{\mu\Big(\mathfrak{C}_\varepsilon(P,P';\sigma,\sigma')\Big)}{\varepsilon}$$

in terms of curvature measures and also give some examples. Concluding remarks are made in Chapter 9 and prerequisite material is presented in Chapter 2.

2. GEOMETRIC PRELIMINARIES

On the vector space $\mathbb{R}^3$, let $\langle\cdot,\cdot\rangle$ be the Euclidean inner product and $|\cdot| = \sqrt{\langle\cdot,\cdot\rangle}$ be the induced norm. The unit sphere $\mathbb{S}^2$ is the set $\{w \in \mathbb{R}^3\colon |w| = 1\}$. For $m = 0,1,2$ a *subsphere of dimension m*, or *m-subsphere*, of $\mathbb{S}^2$ is the intersection of $\mathbb{S}^2$ with some $(m+1)$-dimensional subspace of $\mathbb{R}^3$. A 0-subsphere is the set of antipodes $\{p, -p\}$, for some $p \in \mathbb{S}^2$, and a 1-subsphere is a great circle. The dimension of $Q \subset \mathbb{S}^2$ is the smallest dimension of a subsphere that contains Q.

We induce a topology on $\mathbb{S}^2$ with the metric $d(p,q) := \operatorname{Arccos}\big(\langle p,q\rangle\big)$, which measures the angle between the vectors $p, q \in \mathbb{R}^3$. Therefore $0 \le d(p,q) \le \pi$. The metric d is equivalent to the metric on $\mathbb{S}^2$ induced by the norm $|\cdot|$ on $\mathbb{R}^3$. If $\{p_n\}$ is a sequence of points in $\mathbb{S}^2$, then the notation $p_n \to p$ means $\lim_{n\to\infty} d(p_n, p) = 0$. The σ-algebra of Borel subsets of $\mathbb{S}^2$ is denoted by $\mathfrak{B}(\mathbb{S}^2)$.

The interior, boundary and closure of Q relative to $\mathbb{S}^2$ are written as int Q, bd Q and cl Q. By relint Q we mean the interior of Q relative to the subsphere of smallest dimension that contains Q. The number of points in Q is denoted by card Q.

If $0 < \lambda < \pi$, then the (closed) disc with center p and radius λ is the set $D(p,\lambda) := \{q \in \mathbb{S}^2\colon d(p,q) \le \lambda\}$. The boundary of $D(p,\lambda)$ is the circle $C(p,\lambda) := \{q \in \mathbb{S}^2\colon d(p,q) = \lambda\}$. The (closed) hemisphere with *pole z* is the set $H(z) := \{q \in \mathbb{S}^2\colon d(z,q) \ge \pi/2\} = \{q \in \mathbb{S}^2\colon \langle z,q\rangle \le 0\}$. The subspace of $\mathbb{R}^3$ orthogonal to z has dimension 2, so $L := \operatorname{bd} H(z)$ is a 1-subsphere, i.e. a great circle. Conversely, for each great circle L there exists a point z such that L is the common boundary of the two hemispheres $H(z)$ and $H(-z)$; we call z and $-z$ the *poles* of L.

Suppose $p, q \in \mathbb{S}^2$ are distinct and not antipodal, so that $p \neq \pm q$. Then p, q are linearly independent vectors in $\mathbb{R}^3$ and the intersection of $\mathbb{S}^2$ with the subspace of $\mathbb{R}^3$ spanned by $\{p, q\}$ is a great circle L. Apply Gram-Schmidt to obtain $p \vdash q := (q - \langle p, q\rangle p)/\mid q - \langle p, q\rangle p \mid$; then $\{p, p \vdash q\}$ is orthonormal. The set $\{(\cos s)p + (\sin s)(p \vdash q)\colon -\pi < s \leq \pi\}$ describes L with the orientation "from p towards q." The (closed) arc with endpoints p, q is the set $arc[pq] := \{(\cos s)p + (\sin s)(p \vdash q)\colon\ 0 \leq s \leq d(p,q)\}$. When $s = d(p,q)/2$ we obtain the midpoint $(p+q)/\mid p+q \mid$ of $arc[pq]$. Each point in $arc[pq]$ also has the form $\lambda_1 p + \lambda_2 q$ where $\lambda_i \geq 0$ for $i = 1, 2$.

Let $\mathcal{H}^m$ denote the m-dimensional Hausdorff measure in $\mathbb{R}^3$. As defined in [4], $\mathcal{H}^1$ and $\mathcal{H}^2$ agree with the classical notions of length and area on $\mathbb{S}^2$, respectively, and $\mathcal{H}^0$ is the counting measure. If $\sigma \in \mathcal{B}(\mathbb{S}^2)$, then we write Length($\sigma$) and Area($\sigma$) for $\mathcal{H}^1(\sigma)$ and $\mathcal{H}^2(\sigma)$, respectively.

Convex Bodies and Convex Polygons

A subset Q of $\mathbb{S}^2$ is *(spherically) convex* if Q is contained in the interior of a hemisphere and $arc[pq] \subset Q$ whenever p, q are distinct points in Q. For example, the disc $D(p, \lambda)$ is convex when $0 \leq \lambda < \pi/2$ and not convex when $\lambda \geq \pi/2$. Also, note that the interior of a hemisphere H is convex, but H itself is not. However, if $p \neq \pm q$ are points in H, then $arc[pq] \subset H$.

We say Q is a *convex body* if Q is convex, closed and int $Q \neq \varnothing$. Any convex body is contained in a disc of radius less than $\pi/2$. If the intersection of a finite number of hemispheres is a convex body P, then we call P a *polygonal* *convex body*, or simply a *convex polygon*. Unless noted otherwise, P will denote a convex polygon.

A great circle $L = \mathrm{bd}\, H(z)$ *supports* P *at* x if $x \in P \cap L$ and either $P \subset H(z)$ or $P \subset H(-z)$. If $P \subset H(z)$, then we say that $H(z)$ supports P at x. Necessarily $x \in \mathrm{bd}\, P$. A great circle L *supports* P if L supports P at some point x. If the great circle L supports P, then $f := L \cap P$ is called a *face* of P. If the dimension of f is m, then f is called an m-face of P and the collection of m-faces of P is denoted by $\mathfrak{F}^m(P)$. The 0-faces and 1-faces of P are its vertices and edges. We note that $\mathrm{bd}\, P$ is the disjoint union of its vertices and the relative interiors of its edges. If $\{x_1, \cdots, x_k\}$ are the vertices of P, then each point of P can be written as $\lambda_1 x_1 + \cdots + \lambda_k x_k$ where $\lambda_i \geq 0$ for each i.

The great circle $L = \mathrm{bd}\, H(z)$ *separates* P and P' if $P \subset H(z)$ and $P' \subset H(-z)$, or vice versa.

Spherical Trigonometry

A spherical *triangle* is the intersection $\bigcap_{i=1}^{3} H(z_i)$, where $\{z_1, z_2, z_3\}$ is linearly independent. It follows that any triangle is a convex polygon.

Suppose T is a spherical right triangle with vertices A, B, C, interior angles $\alpha := d(A \vdash B, A \vdash C)$, $\beta := d(B \vdash A, B \vdash C)$, $\gamma := d(C \vdash A, C \vdash B) = \pi/2$, and edge-lengths $a := d(B, C)$, $b := d(A, C)$, $c := d(A, B)$. Then

$$\sin b = \tan a \cot \alpha \qquad \text{and} \qquad \cos c = \cos a \cos b,$$

see [**2**; p. 198].

Outer Parallels

If Q is a nonempty closed subset of $\mathbb{S}^2$ and $\lambda \geq 0$, then the *outer λ-parallel of Q* is defined to be the set $Q_\lambda = \{p \in \mathbb{S}^2 \colon d(p,q) \leq \lambda \text{ for some } q \in Q\}$.

If P is a convex polygon and $0 < \varepsilon < \pi/2$, then bd P_ε is a simple closed curve consisting of portions of circles of radius ε and $\pi/2 + \varepsilon$; see Figure 2.1.

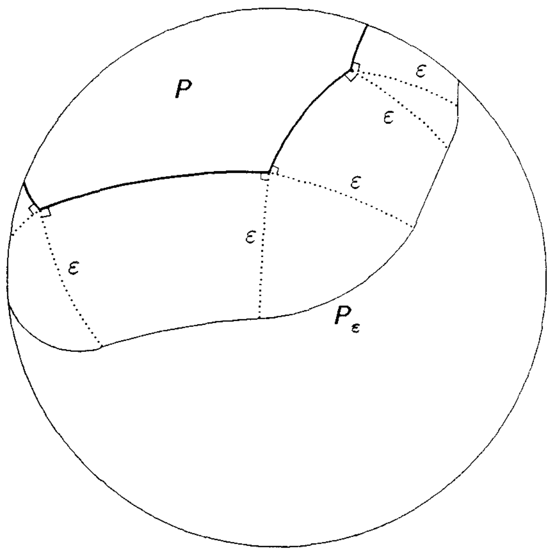

Figure 2.1. P and its outer parallel P_ε.

If $0 < \varepsilon < \pi/2$, then the boundary of P_ε is C^1 and piecewise C^∞. Note that P_ε is not convex; this contrasts sharply with the situation in $\mathbb{R}^n$ where the outer parallel of a polygonal convex body is itself a convex body.

An important set to consider is $P_{\pi/2}$. Note that bd $P_{\pi/2}$ consists of the poles z of supporting hemispheres $H(z)$ of P. For example, suppose $\mathbf{e}_1, \mathbf{e}_2$ are edges of P with common vertex x and $H(z_1), H(z_2)$ are supporting hemispheres such that $\mathbf{e}_i \subset \text{bd}\, H(z_i)$ for $i = 1, 2$. Let α be the interior angle of P at x. Then z_i is a pole of the great circle that contains $\mathbf{e}_i$ and $arc[z_1 z_2]$ consists of the poles

z of hemispheres $H(z)$ that support P at x. If we set $L := \text{bd}\, H(z)$, then the points $z \in arc[z_1 z_2]$ are in 1-1 correspondence with great circles that support P at x. Moreover, the length of $arc[z_1 z_2]$ is $\pi - \alpha$. See Figure 2.2. If we continue this process for each vertex x of P, then observe that $\text{bd}\, P_{\pi/2}$ is a simple closed curve consisting of geodesic arcs.

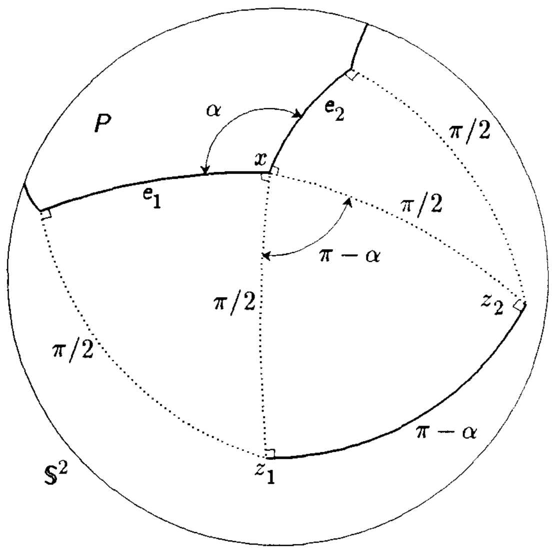

Figure 2.2. $arc[z_1 z_2] \subset \text{bd}\, P_{\pi/2}$.

Spherical Distance and the Hausdorff Metric

Let Q, Q' be nonempty, closed subsets of $\mathbb{S}^2$. The *(spherical) distance* between Q and Q' is $\rho(Q, Q') := \min\{d(q, q')\colon\ q \in Q,\ q' \in Q'\}$. If $p \in \mathbb{S}^2$, then we define the distance between p and Q to be $\rho(p, Q) := \rho(\{p\}, Q)$. Note that $Q_\lambda = \{p \in \mathbb{S}^2\colon\ \rho(p, Q) \leq \lambda\}$, and this set is nonempty and closed. We have $\rho(Q, Q') = 0$ iff $Q \cap Q' \neq \varnothing$, and $\rho(Q, Q') = 0$ need not imply $Q = Q'$. Therefore ρ is not a metric, but a natural metric does exist.

Define $d_H(Q, Q') := \min\{\lambda \geq 0\colon\ Q \subset (Q')_\lambda \text{ and } Q' \subset Q_\lambda\}$. Then d_H is a metric on the collection of nonempty, closed subsets of $\mathbb{S}^2$ and is called the Hausdorff metric. If $\{Q_n\}$ is a sequence of nonempty, closed sets, then $Q_n \to Q$ means that $\lim_{n\to\infty} d_H(Q_n, Q) = 0$.

Spherical distance ρ has the following continuity properties. If $p_n \to p$, then $\rho(p_n, Q) \to \rho(p, Q)$. Similarly, if $Q_n \to Q$, then $\rho(p, Q_n) \to \rho(p, Q)$ and $\rho(Q_n, Q') \to \rho(Q, Q')$.

Polarity

Suppose Q is a nonempty closed convex set or a hemisphere. The *polar set* to Q is $Q^* := \{p \in \mathbb{S}^2\colon\ \rho(p, Q) \geq \pi/2\} = \{p \in \mathbb{S}^2\colon\ \langle p, q\rangle \leq 0 \ \ \forall q \in Q\}$. The polar set to a hemisphere $H(z)$ is $\{z\}$ and the polar set to $\{z\}$ is $H(z)$. Thus, $\left[H(z)^*\right]^* = H(z)$. If P is a convex polygon, then we observe the following.

1. $P^* = \mathbb{S}^2 \setminus \operatorname{int} P_{\pi/2}$ and $\operatorname{int} P^* = \mathbb{S}^2 \setminus P_{\pi/2}$. Therefore $\operatorname{bd} P^* = \operatorname{bd} P_{\pi/2}$, P^* is contained in an open hemisphere and $\operatorname{int} P^* \neq \varnothing$. If $q_1, q_2 \in P^*$ and $\lambda_1 q_1 + \lambda_2 q_2 \in \mathit{arc}[q_1\, q_2]$ with $\lambda_i \geq 0$, then $\langle p, \lambda_1 q_1 + \lambda_2 q_2\rangle = \lambda_1\langle p, q_1\rangle + \lambda_2\langle p, q_2\rangle \leq 0$ for all $p \in P$. Thus, P^* is a convex body. If $\{x_1, \cdots, x_k\}$ are the vertices of P, then $P^* = \bigcap_{i=1}^{k} H(x_i)$; hence P^* is a convex polygon. Moreover, $(P^*)^* = P$.

2. There is a 1-1 correspondence between the m-faces of P and the $(1-m)$-faces of P^*. If α is the measure of the interior angle at a vertex x of P, then $\pi - \alpha$ is the length of the edge of P^* that corresponds to x; see Figure 2.2. If s is the length of an edge $\boldsymbol{e}$ of P, then $\pi - s$ is the measure of the interior angle at the vertex of P^* that corresponds to $\boldsymbol{e}$.

3. There is a 1-1 correspondence between points z in $\mathrm{bd}\, P^*$ and great circles L that support P. If $L = \mathrm{bd}\, H(z)$ supports P at a vertex x, then z lies in the edge of P^* that corresponds to x. If $L = \mathrm{bd}\, H(z)$ and $e = L \cap P$ is an edge of P, then z is the vertex of P^* that corresponds to e.

Nearest-Point Map

If P is a nonempty, closed, convex set or a hemisphere and $0 < \rho(P,q) < \pi/2$, then there is a unique point p_0 in P that is nearest to q, i.e. $d(p_0,q) = \rho(P,q)$. This fact, whose proof is essentially the same as that for Euclidean space (see [**10**; pp. 30-31]), allows us to define the nearest-point map $N(P,\cdot)\colon \mathbb{S}^2 \setminus P^* \to P$ by

$$N(P,q) = \begin{cases} q & \text{if } q \in P \\ p_0 & \text{if } 0 < \rho(P,q) < \pi/2. \end{cases}$$

The nearest point map $N(P,\cdot)$ is continuous on $\mathbb{S}^2 \setminus P^*$; the proof given in [8; p. 119] for Euclidean space also works for $\mathbb{S}^2$.

Suppose P, P' are convex polygons such that $0 < \rho(P,P') = d(x,x') < \pi/2$ for some points $x \in \mathrm{bd}\, P$, $x' \in \mathrm{bd}\, P'$. From the definition of the nearest-point map it follows immediately that $x = N(P,x')$ and $x' = N(P',x)$.

The nearest-point map has several other properties. The proofs are similar to the Euclidean case (see [8], [**10**] and [14]). For each of the following assume that P is a either a convex polygon or a hemisphere, and that $0 < \rho(P,\overline{x}) < \pi/2$, $x := N(P,\overline{x})$, $z := x \vdash \overline{x}$.

1. If $q \in \mathrm{relint}\,(\mathit{arc}[xz])$, then $N(P,q) = x$.

2. $H(z)$ supports P at x. The great circle $L := \text{bd}\, H(z)$ through x and perpendicular to $arc[x\overline{x}]$ supports P at x.

3. Suppose P is a convex polygon. If e is an edge of P and $x \in \text{relint}\, e$, then $arc[x\overline{x}]$ is perpendicular to e.

4. Suppose $P = H$ is a hemisphere. Then $arc[x\overline{x}]$ is perpendicular to $L := \text{bd}\, H$.

Finally, we consider two useful theorems with no Euclidean analogs.

Theorem 2.1. *Suppose P, P' are convex polygons. If there are points $x \in \text{bd}\, P$, $x' \in \text{bd}\, P'$ with $0 < \rho(P, P') = d(x, x') < \pi/2$, then x is a vertex of P or x' is a vertex of P' (or both).*

Proof. Here $x = N(P, x')$ and $x' = N(P', x)$. We assume that neither x, x' are vertices and show that a contradiction results.

Assume that e, e' are edges of P, P' such that $x \in \text{relint}\, e$, $x' \in \text{relint}\, e'$. Then $arc[xx']$ is perpendicular to both e and e'. Let L, L' be the great circles that contain e, e' and let z be the pole of L in the hemisphere that does contain x'; see Figure 2.3a. Suppose $q' \in L'$ satisfies $0 < d(q', x') < \pi/2$ and set $q := z \vdash q'$, so that $0 < d(q, x) < \pi/2$ and $d(q, q') + d(q', z) = \pi/2$.

Consider the spherical right triangle with vertices z, x', q' as shown in Figure 2.3b. Apply spherical trigonometry to obtain

$$\sin[\pi/2 - d(x, x')] = \tan d(q', x') \ \cot d(q, x)$$

which gives

$$\cos d(x, x') = \tan d(q', x') \ \cot d(q, x)$$

and therefore $d(q,x) = \text{Arctan}\left[\sec d(x,x') \ \tan d(q',x')\right]$. This shows that

$$q' \to x' \quad \Rightarrow \quad q \to x. \tag{2.1}$$

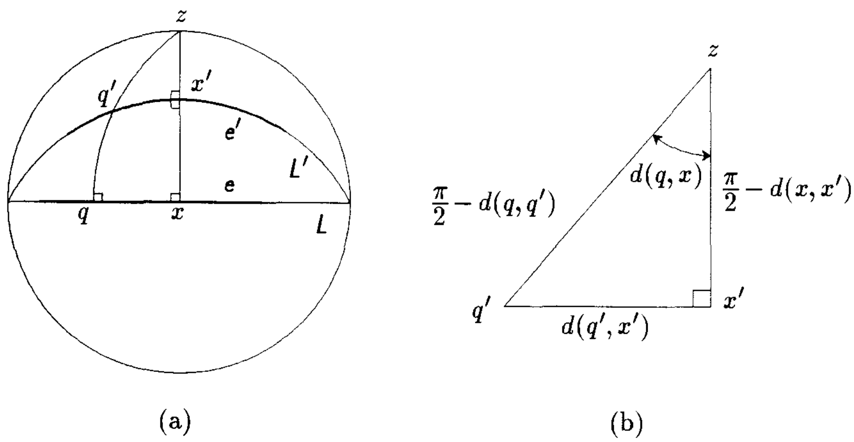

Figure 2.3. (a) $q' \in \mathbf{e}'$ and $q \in \mathbf{e}$, (b) the triangle z, x', q'.

Since $x \in \text{relint}\ \mathbf{e}$, $x' \in \text{relint}\ \mathbf{e}'$ and (2.1) holds, we know that for all $q' \in L'$ sufficiently close to x' we have $q' \in \mathbf{e}'$ and $q \in \mathbf{e}$. For all such q' we again consider the triangle with vertices z, x', q'. Use spherical trigonometry to obtain

$$\cos\left[\pi/2 - d(q,q')\right] = \cos\left[\pi/2 - d(x,x')\right] \cos d(q',x'). \tag{2.2}$$

Now $0 < d(q',x') < \pi/2$ implies $0 < \cos d(q',x') < 1$, and when this is used in (2.2), we get

$$\cos\left[\pi/2 - d(q,q')\right] < \cos\left[\pi/2 - d(x,x')\right].$$

Therefore

$$\frac{\pi}{2} - d(q,q') > \frac{\pi}{2} - d(x,x')$$

which gives $d(q,q') < d(x,x')$. This is a contradiction because $d(x,x')$ is minimal. □

Let H be a hemisphere with boundary L. If P, e are replaced by H, L in the proof of Theorem 2.1, then we have a proof of the following theorem.

Theorem 2.2. *Suppose P' is a convex polygon and H is a hemisphere. If there are points $x \in L := \text{bd}\, H$, $x' \in \text{bd}\, P'$ with $0 < \rho(H, P') = d(x,x') < \pi/2$, then x' is a vertex of P'.*

3. CURVATURE MEASURES

Brush Sets

If $\sigma \in \mathfrak{B}(\mathbb{S}^2)$ and $0 < \varepsilon < \pi/2$, then

$$B_\varepsilon(P,\sigma) := \{q \in P_\varepsilon \setminus P\colon\ N(P,q) \in \sigma\}$$

is called a ***brush set***. Note that $B_\varepsilon(P,\mathbb{S}^2) = P_\varepsilon \setminus P$.

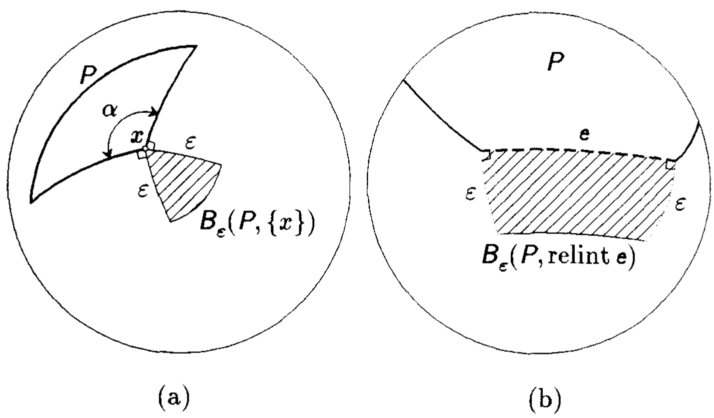

Figure 3.1. Brush sets for (a) vertex x and (b) edge e.

For example, suppose x is a vertex of P and α is the measure of the interior angle of P at x. The brush set $B_\varepsilon(P,\{x\})$ is shown in Figure 3.1a; its closure is a sector with central angle $\pi - \alpha$ of the disc $D(x,\varepsilon)$. The area of $B_\varepsilon(P,\{x\})$ is $(1-\cos\varepsilon)(\pi-\alpha)$. Note that $\pi - \alpha$ is the length of the edge of P^* polar to x and, if x^* denotes this edge, we have

$$\text{Area}\Big(B_\varepsilon(P,\{x\})\Big) = (1-\cos\varepsilon)\cdot\text{Length}(x^*). \tag{3.1}$$

Suppose e is an edge of P. Figure 3.1b shows the brush set $B_\varepsilon(P, \text{relint } e)$. In this case we have

$$\text{Area}\Big(B_\varepsilon(P, \text{relint } e)\Big) = (\sin\varepsilon)\cdot \text{Length}(e). \tag{3.2}$$

Note that

$$P_\varepsilon \setminus P = \Big[\bigcup_{x\in \mathcal{F}^0(P)} B_\varepsilon(P,\{x\})\Big] \cup \Big[\bigcup_{e\in \mathcal{F}^1(P)} B_\varepsilon(P, \text{relint } e)\Big] \tag{3.3}$$

and this union is disjoint. When (3.3) is combined with (3.1) and (3.2) we obtain

$$\begin{aligned}\text{Area}\Big(B_\varepsilon(P,\mathbb{S}^2)\Big) &= \text{Area}(P_\varepsilon \setminus P)\\ &= (1-\cos\varepsilon)\sum_{x\in\mathcal{F}^0(P)}\text{Length}(x^*) + (\sin\varepsilon)\sum_{e\in\mathcal{F}^1(P)}\text{Length}(e)\\ &= (1-\cos\varepsilon)\cdot\text{Length}(\text{bd } P^*) + (\sin\varepsilon)\cdot\text{Length}(\text{bd } P). \end{aligned}\tag{3.4}$$

Equation (3.4) is related to a noneuclidean version of Steiner's formula, see [**13**; p. 322].

Curvature Measures

In this section we give a local version of (3.4) which, for any $\sigma \in \mathcal{B}(\mathbb{S}^2)$, expresses the area of $B_\varepsilon(P,\sigma)$ in terms of certain finite measures on $\mathcal{B}(\mathbb{S}^2)$.

A study of (3.4) leads to the following observation.

$$\text{Area}\Big(B_\varepsilon(P,\sigma)\Big) = (1-\cos\varepsilon)\sum_{x\in\mathcal{F}^0(P)\cap\sigma}\text{Length}(x^*) + (\sin\varepsilon)\sum_{e\in\mathcal{F}^1(P)}\text{Length}(e\cap\sigma)$$

$$= (1-\cos\varepsilon) \sum_{x \in \mathcal{F}^0(P)} \mathcal{H}^0(\{x\} \cap \sigma)\, \mathcal{H}^1(x^*) + (\sin\varepsilon) \sum_{e \in \mathcal{F}^1(P)} \mathcal{H}^1(e \cap \sigma)\, \mathcal{H}^0(e^*). \tag{3.5}$$

Here $\mathcal{H}^0$ is the counting measure and e^* is the set consisting of the vertex of P^* polar to e. The sum with coefficient $(1-\cos\varepsilon)$ is the contribution from the vertices of P and the sum with coefficient $(\sin\varepsilon)$ is the contribution from the edges of P. Equation (3.5) is a local version of (3.4). We are now in a position to make the next definition.

Definition 3.1. Let $\sigma \in \mathcal{B}(\mathbb{S}^2)$ and P be a polygonal convex body. For $m = 0, 1$, if f is an m-face of P, let f^* be the $(1-m)$-face of P^* polar to f. Define

$$\Phi_m(P,\sigma) := \sum_{f \in \mathcal{F}^m(P)} \mathcal{H}^m(f \cap \sigma)\, \mathcal{H}^{1-m}(f^*).$$

Each $\Phi_m(P, \cdot\,)$ is a finite measure on $\mathcal{B}(\mathbb{S}^2)$. We call these measures the *curvature measures* of P.

With this definition (3.5) becomes

$$\text{Area}\Big(B_\varepsilon(P,\sigma)\Big) = (1-\cos\varepsilon)\, \Phi_0(P,\sigma) + (\sin\varepsilon)\, \Phi_1(P,\sigma). \tag{3.6}$$

We now give some motivation for the term "curvature measure." Suppose $\mathbb{S}^2$ is oriented with outward normals and bd P_ε is oriented with P_ε "on the left." The (signed) geodesic curvature κ on smooth portions of bd P_ε that correspond to vertices and sides of P is then $\cot\varepsilon$ and $\cot(\pi/2+\varepsilon) = -\tan\varepsilon$, respectively. If $ds(\varepsilon)$ is the corresponding element of arc-length on smooth portions of bd P_ε, then it can be shown that

$$\Phi_0(P,\sigma) = \lim_{\varepsilon \to 0^+} \int_{\{q \in \mathrm{bd}\, P_\varepsilon :\ N(P,q) \in \sigma\}} \kappa \, ds(\varepsilon). \tag{3.7}$$

Curvature measures of convex bodies in Euclidean n-space are studied in **[15]**, where a definition similar to Definition **3**.1 is given for polytopes. Also, in **[13**; pp. **302-303]** there is material related to both the integral of κ and the limit in (3.7).

4. MOTIONS AND KINEMATIC MEASURE

Frames and Motions

The set $\{x, y, z\}$ is a called a *frame* if it is a right-handed orthonormal ordered basis for $\mathbb{R}^3$. We use the notation $[\![x, y, z]\!]$ to denote both the matrix with first column x, second column y, third column z, and also the frame $\{x, y, z\}$. The phrase "right-handed" means the determinant of $[\![x, y, z]\!]$ is $+1$. The spherical triangle with vertices x, y, z has $\pi/2$ for each of its interior angles and its edge lengths. The standard basis $[\![\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3]\!]$ is the *reference frame* and also the 3×3 identity matrix e.

A *motion* in $\mathbb{S}^2$ is defined to be the restriction to $\mathbb{S}^2$ of a linear transformation g in $\mathbb{R}^3$ whose matrix relative to the reference frame is an element of the special orthogonal group SO(3). The collection of all motions is denoted by $\mathfrak{G}$.

We identify the linear transformation g with the 3×3 real matrix $[\![g^i_j]\!]$ such that

$$i\text{th coordinate of } gp = (gp)^i = \sum_{j=1}^{3} g^i_j\, p^j \qquad \text{for all } 1 \leq i \leq 3.$$

Under this identification, the composition $g \circ h$ agrees with the matrix multiplication gh. If each point p in $\mathbb{R}^3$ is identified with a 3×1 column matrix, then the matrix multiplication gp gives the image of p under g. In this way we identify $\mathfrak{G}$ with SO(3). Each matrix in SO(3) has determinant $+1$ and its columns form a frame. We can therefore identify $\mathfrak{G}$ with the collection of frames in $\mathbb{R}^3$.

The topology on $\mathfrak{G}$ is that induced by the metric

$$d_{\mathfrak{G}}(g, g') := \max\{d(gp, g'p) \colon p \in \mathbb{S}^2\}.$$

This metric is equivalent to the metric d_O induced by the linear operator norm, i.e. $d_O(g, g') := \| g - g' \| = \max\{ | gp - g'p | : p \in \mathbb{S}^2\}$. If $\mathfrak{G}$ is regarded as a subset of $\mathbb{R}^9$, then $d_{\mathfrak{G}}$ is also equivalent to the Euclidean metric on $\mathbb{R}^9$. The σ-algebra of Borel subsets of $\mathfrak{G}$ is denoted by $\mathfrak{B}(\mathfrak{G})$.

Representation of Motions; Translations and Rotations

How can a general motion $g = [\![x, y, z]\!]$ in $\mathfrak{G}$ be described? Consider the possible choices for x, z, y in that order. First, x can be any point in $\mathbb{S}^2$. Once x is fixed, then z can be any point in the unit circle contained in the orthogonal complement of the line span$\{x\}$. If both x and z are fixed, then y must be the unique point in the orthogonal complement of the plane span$\{x, z\}$ such that $\det [\![x, y, z]\!] = 1$. Similar comments hold if we use the order z, x, y. With these considerations, we parametrize $\mathfrak{G}$ as follows.

Define

$$\mathrm{Rot}_{12}(t) := \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathrm{Rot}_{13}(t) := \begin{bmatrix} \cos t & 0 & -\sin t \\ 0 & 1 & 0 \\ \sin t & 0 & \cos t \end{bmatrix},$$

and

$$\mathrm{Rot}_{23}(t) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}.$$

Note that $\mathrm{Rot}_{ij}(t)$ is a rotation in the plane span $\{e_i, e_j\}$ with t equal to the angle measured positively from e_i towards e_j.

For $-\pi/2 \leq u \leq \pi/2$ and $-\pi < v \leq \pi$ define $g_{\mathrm{t}} := \mathrm{Rot}_{12}(v) \cdot \mathrm{Rot}_{13}(u)$. Then

$$g_{\mathrm{t}} = \begin{bmatrix} \cos u \, \cos v & -\sin v & -\sin u \, \cos v \\ \cos u \, \sin v & \cos v & -\sin u \, \sin v \\ \sin u & 0 & \cos u \end{bmatrix}.$$

Note that the first column of g_{t}, which is $g_{\mathrm{t}} e_1$, describes any point x in $\mathbb{S}^2$ with latitude u and longitude v, see [11; p. 134].

Suppose $g = [\![x, y, z]\!]$. Then there exist u, v such that $x = g_{\mathrm{t}} e_1$. Since g_{t} is a motion, its columns form the frame $[\![x, g_{\mathrm{t}} e_2, g_{\mathrm{t}} e_3]\!]$. But $[\![x, y, z]\!]$ is also a frame. This implies that there is a unique θ with $-\pi < \theta \leq \pi$ such that $y = (\cos\theta)\, g_{\mathrm{t}} e_2 + (\sin\theta)\, g_{\mathrm{t}} e_3$ and $z = (-\sin\theta)\, g_{\mathrm{t}} e_2 + (\cos\theta)\, g_{\mathrm{t}} e_3$. See Figure 4.1.

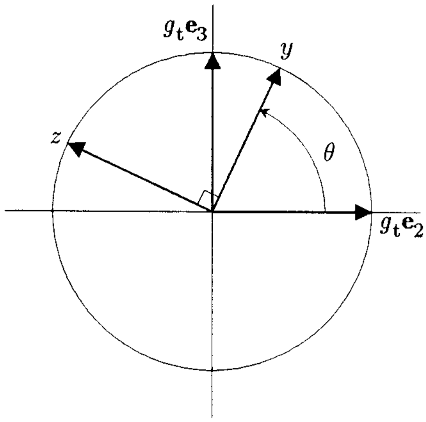

Figure 4.1. The rotation g_{r} about x.

Said differently, there is a unique θ that determines a motion g_{r} whose matrix is $\mathrm{Rot}_{23}(\theta)$ relative to the frame $[\![x, g_{\mathrm{t}} e_2, g_{\mathrm{t}} e_3]\!]$. Here $g_{\mathrm{r}} x = x$ and so g_{r} is a rotation about x. Note that the collection of all rotations about x is isomorphic to SO(2). With respect to the reference frame we have $g_r = g_t\big[\mathrm{Rot}_{23}(\theta)\big]g_t^{-1}$.

To summarize, for each $g = [\![x, y, z]\!]$ in $\mathfrak{G}$ there exist parameters u, v, θ with $-\pi/2 \le u \le \pi/2$, $-\pi < v \le \pi$, $-\pi < \theta \le \pi$ and the corresponding motions $g_t = g_t(u, v)$, $g_r = g_r(u, v, \theta)$ such that $g = g_r g_t$. This representation of g is unique when $u \neq \pm\,\pi/2$, i.e. when x is neither the north nor south pole of $\mathbb{S}^2$. With respect to the reference frame we have

$$g = [\![x, y, z]\!] = g_r g_t = \mathrm{Rot}_{12}(v) \cdot \mathrm{Rot}_{13}(u) \cdot \mathrm{Rot}_{23}(\theta)$$

$$= \begin{bmatrix} \cos u\, \cos v & -\sin v\, \cos\theta - \sin u\, \cos v\, \sin\theta & \sin v\, \sin\theta - \sin u\, \cos v\, \cos\theta \\ \cos u\, \sin v & \cos v\, \cos\theta - \sin u\, \sin v\, \sin\theta & -\cos v\, \sin\theta - \sin u\, \sin v\, \cos\theta \\ \sin u & \cos u\, \sin\theta & \cos u\, \cos\theta \end{bmatrix}.$$

The decomposition of g into $g_r g_t$ is analogous to the situation in $\mathbb{R}^2$, where a proper rigid motion can be represented as a translation followed by a rotation. For this reason we call g_t a *translation* in $\mathbb{S}^2$ and g_r a *rotation* in $\mathbb{S}^2$.

Kinematic Density

In the previous section we described $\mathfrak{G}$ as a matrix group with parameters u, v, θ. The group SO(3) is a compact, smooth manifold of dimension three, see [17]. Let $U = \{(u, v, \theta)\colon -\pi/2 < u < \pi/2,\ -\pi < v < \pi,\ -\pi < \theta < \pi\}$. If we consider $\mathfrak{G}$ as subset of $\mathbb{R}^9$, then the mapping $\phi\colon U \to \mathfrak{G}$ given by

$$g = \phi(u, v, \theta) = \mathrm{Rot}_{12}(v) \cdot \mathrm{Rot}_{13}(u) \cdot \mathrm{Rot}_{23}(\theta)$$

is one-to-one and regular on U and is therefore a coordinate system. Note that

$\mathfrak{G}$ is the image of the closure of U under ϕ. Also, $g_t = \phi(u, v, 0)$ and $\phi(0,0,0)$ is the identity matrix.

According to [**13**; p. 153-154], the kinematic density is the volume element of the matrix group $\mathfrak{G}$ given by

$$dK := \left| \omega_2^1 \wedge \omega_3^1 \wedge \omega_3^2 \right|,$$

where ω_j^i is the differential 1-form located in the ith row and jth column of the matrix $g^{-1}dg$, and $\wedge$ denotes exterior multiplication.

A calculation shows that

$$\omega_2^1 = -\sin\theta\ du - \cos u\ \cos\theta\ dv, \quad \omega_3^1 = -\cos\theta\ du + \cos u\ \sin\theta\ dv,$$

and

$$\omega_3^2 = -\sin u\ dv - d\theta.$$

Then $\omega_2^1 \wedge \omega_3^1 = -\cos u\ du \wedge dv$ and so $dK = |\cos u\ du \wedge dv \wedge d\theta|$. Note that the area element dA of $\mathbb{S}^2$ at $x = g_t \mathbf{e}_1$ is $\cos u\ du \wedge dv$.

Kinematic Measure

Kinematic measure μ is the invariant (Haar) measure obtained by integration of the kinematic density dK, see [**13**; p. 157]. (We discuss the invariance of μ in the next section.) If $\mathfrak{M}$ is a Borel subset of $\mathfrak{G}$, then

$$\mu(\mathfrak{M}) = \int_{\mathfrak{M}} dK = \int_{\phi^{-1}(\mathfrak{M})} d\theta\ \cos u\ du\ dv.$$

Let $x = g_t \mathbf{e}_1$. For calculation purposes we have

$$\mu(\mathfrak{M}) = \int_{\{(u,v,\theta) \in U:\ \phi(u,v,\theta) \in \mathfrak{M}\}} d\theta\ \cos u\ du\ dv$$

$$= \int_{\{(u,v,0)\in U:\ \phi(u,v,0)\in \mathfrak{G}\}} \left[\int_{\{\theta:\ \phi(u,v,\theta)\in \mathfrak{M}\}} d\theta \right] \cos u \ du\ dv$$

$$= \int_{\{(u,v,0)\in U:\ g_{\mathrm{t}}\mathbf{e}_1\in \mathbb{S}^2\}} \left[\int_{\{g_{\mathrm{r}}:\ g_{\mathrm{r}}g_{\mathrm{t}}\in \mathfrak{M}\}} d\theta \right] \cos u \ du\ dv$$

$$= \int_{\{x\in \mathbb{S}^2\}} \int_{\mathfrak{R}(x)} d\theta \ dA$$

where $\mathfrak{R}(x) := \{\text{rotations } g_{\mathrm{r}} \text{ about } x\colon\ g_{\mathrm{r}}g_{\mathrm{t}} \in \mathfrak{M}\}$.

We complete the measure space $(\mathfrak{G}, \mathfrak{B}(\mathfrak{G}), \mu)$ and denote the completed measure also by μ. Elements of the completion of $\mathfrak{B}(\mathfrak{G})$ with respect to μ will be called the *μ–measurable* sets.

Properties of Kinematic Measure

Since the group $\mathfrak{G}$ is compact, the kinematic measure μ is unique up to a multiplicative constant, see [12; p. 380]. Moreover, kinematic measure is left invariant, right invariant, and invariant under inversion, refer to [**13**].

Left invariance of μ means $\mu(g_0\mathfrak{M}) = \mu(\mathfrak{M})$ for any fixed g_0 in $\mathfrak{G}$. For example, suppose P, P' are two polygons and consider the set $\mathfrak{M} = \{g \in \mathfrak{G}\colon\ P \cap gP' \neq \varnothing\}$. Then $g_0\mathfrak{M} = \{g \in \mathfrak{G}\colon\ P \cap g_0gP' \neq \varnothing\} = \{g \in \mathfrak{G}\colon\ g_0^{-1}P \cap gP' \neq \varnothing\}$. The left invariance of μ implies that we could use the polygon $g_0^{-1}P$ rather than P and still obtain the same measure. Thus, the initial position of P is irrelevant.

Right invariance of μ means $\mu(\mathfrak{M}g_0) = \mu(\mathfrak{M})$ for any fixed g_0 in $\mathfrak{G}$. To continue with our example, consider $\mathfrak{M}g_0 = \{g \in \mathfrak{G}\colon\ P \cap gg_0P' \neq \varnothing\}$. Now the right invariance of μ implies that the polygon g_0P' could be used instead of P'. Therefore the initial position of P' is irrelevant.

Invariance of μ under inversion means $\mu(\mathfrak{M}^{-1}) = \mu(\mathfrak{M})$. In our example, $\mathfrak{M}^{-1} = \{g \in \mathfrak{G}\colon\ P \cap g^{-1}P' \neq \varnothing\} = \{g \in \mathfrak{G}\colon\ gP \cap P' \neq \varnothing\}$. Here invariance of μ

under inversion implies that it is irrelevant which polygon moves and which polygon is stationary.

5. MEASURE OF NEAR-COLLISIONS

Paint-to-Paint Collisions

Definition 5.1. The set of motions that cause P' to *collide* with P is $\mathfrak{C}_0(P,P') := \{g \in \mathfrak{G}\colon\ P \cap gP' \neq \varnothing,\ P$ and gP' can be separated by a great circle$\}$. For $\sigma, \sigma' \in \mathfrak{B}(\mathbb{S}^2)$, define $\mathfrak{C}_0(P,P';\sigma,\sigma') := \{g \in \mathfrak{C}_0(P,P')\colon\ (\sigma \cap \operatorname{bd} P) \cap g(\sigma' \cap \operatorname{bd} P') \neq \varnothing\}$. If we regard $\sigma \cap \operatorname{bd} P$ and $\sigma' \cap \operatorname{bd} P'$ as "painted" sets, then $\mathfrak{C}_0(P,P';\sigma,\sigma')$ is the set of motions that cause P' to collide with P *paint-to-paint.*

Observe that $\mathfrak{C}_0(P,P';\mathbb{S}^2,\mathbb{S}^2) = \mathfrak{C}_0(P,P') = \mathfrak{C}_0(P',P)^{-1}$ and $\mathfrak{C}_0(P,P';\sigma,\sigma') = \mathfrak{C}_0(P',P;\sigma',\sigma)^{-1}$. Let μ be the completion of kinematic measure as given in Chapter 4. Then μ is invariant under inversion and this implies $\mu\big(\mathfrak{C}_0(P,P';\sigma,\sigma')\big) = \mu\Big(\mathfrak{C}_0(P',P;\sigma',\sigma)^{-1}\Big)$, provided these sets are μ–measurable. This symmetry means that P, rather than P', can be considered as the moving polygon.

To compute the probability of a paint-to-paint collision it is natural to consider the quotient $\mu\big(\mathfrak{C}_0(P,P';\sigma,\sigma')\big) / \mu\big(\mathfrak{C}_0(P,P')\big)$. A difficulty arises in that the denominator, and hence also the numerator, is zero. Before proving this in Theorem 5.5, it is convenient to describe paint-to-paint collisions by using great circles L and L' that support P and P', respectively.

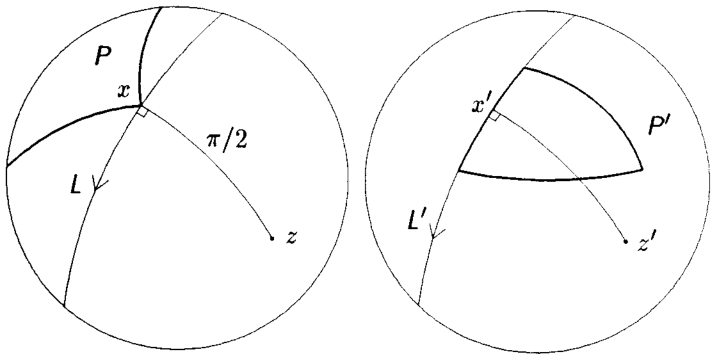

Figure 5.1. (a) The pair x, L (b) the pair x', L'.

Definition 5.2. The motion $g(x, L; x', L')$ is defined as follows.

1. Suppose $x \in \mathrm{bd}\, P$ and $x' \in \mathrm{bd}\, P'$.

2. Suppose L is any great circle that supports P at x. Let z be the pole of L in the hemisphere that does not contain P. Choose $y \in L$ such that $[\![x, y, z]\!]$ is a frame. See Figure 5.1a.

3. Suppose L' is any great circle that supports P' at x'. Let z' be the pole of L' in the hemisphere that does contain P'. Choose $y' \in L'$ such that $[\![x', y', z']\!]$ is a frame. See Figure 5.1b.

4. Define $g(x, L; x', L')$ by the matrix $[\![x, y, z]\!][\![x', y', z']\!]^{-1}$ so that $x' \mapsto x$, $y' \mapsto y$, $z' \mapsto z$.

Remark 5.3. The points $x, y \in L$ determine the orientation of L "from x towards y" via the parametrization $(\cos s)x + (\sin s)y$, $-\pi < s \leq \pi$. In figures, an arrow on L will indicate this orientation. Similar comments hold for L'. Consequently

$g(x, L; x', L')$ takes x' to x and L' to L, consistent with the orientations of L and L'.

The set of paint-to-paint collisions can be characterized in terms of Definition 5.2.

Theorem 5.4. $\mathfrak{C}_0(P, P'; \sigma, \sigma') = \{g(x, L; x', L')\colon\ x \in \sigma$ and L supports P at x, $x' \in \sigma'$ and L' supports P' at $x'\}$.

Proof. Let $g \in \mathfrak{C}_0(P, P'; \sigma, \sigma')$. Then there is a great circle L that separates P and gP', and there exists $x \in \sigma \cap \operatorname{bd} P$ and $x' \in \sigma' \cap \operatorname{bd} P'$ such that $x = gx'$. Let z be the pole of L in the hemisphere that does not contain P. Then $d(p, z) \geq \pi/2$ for all $p \in P$ and $d(gp', z) \leq \pi/2$ for all $p' \in P'$. The previous inequalities imply $d(x, z) = d(gx', z) = \pi/2$. Thus L supports P at x and L also supports gP' at gx'. Choose $y \in L$ so that $[\![x, y, z]\!]$ is a frame.

Define $L' := g^{-1}L$, $z' := g^{-1}z$, and $y' := g^{-1}y$. Then $[\![x', y', z']\!]$ is a frame, $y' \in L'$, L' supports P' at $x' \in \sigma'$, and z' is the pole of L' in the hemisphere that does contain P'. Therefore $g = [\![x, y, z]\!][\![x', y', z']\!]^{-1} = g(x, L; x', L')$.

Suppose $x \in \sigma, x' \in \sigma'$ and let $g := g(x, L; x', L')$ be as in Definition 5.2. Then $x = gx'$ implies $(\sigma \cap \operatorname{bd} P) \cap g(\sigma' \cap \operatorname{bd} P') \neq \varnothing$. It now suffices to show that $L = gL'$ separates P and gP'. This is proved below:

$$L \text{ supports } P \text{ at } x \quad \Rightarrow \quad d(p, z) \geq \pi/2 \quad \forall p \in P$$

and

$$\begin{aligned} L' \text{ supports } P' \text{ at } x' \quad &\Rightarrow \quad gL' \text{ supports } gP' \text{ at } gx' \\ &\Rightarrow \quad d(gp', gz') \leq \pi/2 \quad \forall p' \in P' \\ &\Rightarrow \quad d(gp', z) \leq \pi/2 \quad \forall p' \in P'. \end{aligned}$$

□

Theorem 5.5. $\mu\Big(\mathfrak{C}_0(P, P')\Big) = 0.$

Proof. Note that bd P' is the finite disjoint union of the vertices of P' and the relative interiors of the edges of P'. Therefore

$$\mathfrak{C}_0(P,P') = \bigcup_{x' \in \mathfrak{F}^0(P')} \Big\{g(x, L; x', L') \colon L \text{ supports } P \text{ at } x,\ L' \text{ supports } P' \text{ at } x'\Big\}$$

$$\cup \bigcup_{e' \in \mathfrak{F}^1(P')} \Big\{g(x, L; x', L') \colon L \text{ supports } P \text{ at } x,\ x' \in \operatorname{relint} e' \subset L'\Big\}$$

$$=: \bigcup_{x' \in \mathfrak{F}^0(P')} \mathfrak{S}(x') \quad \cup \quad \bigcup_{e' \in \mathfrak{F}^1(P')} \mathfrak{S}(e').$$

Consider any one of the sets $\mathfrak{S}(x')$. If $g = g_r g_t \in \mathfrak{S}(x')$, then $g_t x' = x \in \operatorname{bd} P$ and x' is a vertex of P'. By the right invariance of μ, we may assume $x' = \mathbf{e}_1$. Then

$$\mu\Big(\mathfrak{S}(x')\Big) \le \int_{\operatorname{bd} P} \int_{-\pi}^{\pi} d\theta\, dA = 2\pi \cdot 0 = 0.$$

Since we have a finite union of such sets the measure of the first union is zero.

For any $e' \in \mathfrak{F}^1(P')$ we have relint $e' \subset L'$, thus there is a unique pole z' of L' in the hemisphere that does contain P'. Without loss of generality $z' = \mathrm{e}_1$. For $g \in \mathfrak{S}(e')$ we must have $L = gL'$, where L supports P. This implies $gz' = g_t z' = z \in \operatorname{bd} P^*$. Then

$$\mu\Big(\mathfrak{S}(e')\Big) \le \int_{\operatorname{bd} P^*} \int_{-\pi}^{\pi} d\theta\, dA = 2\pi \cdot 0 = 0.$$

Again, we have a finite union of such sets and so the measure of the second union is zero. The conclusion now follows. □

Near Paint-to-Paint Collisions

Because of Theorem 5.5, another approach is needed. In this section we approximate $\mathfrak{C}_0(P, P'; \sigma, \sigma')$ by constructing a family of μ-measurable sets $\mathfrak{C}_\varepsilon(P, P'; \sigma, \sigma')$ for $0 < \varepsilon < \pi/2$.

Definition 5.6. Suppose $P \cap P' = \varnothing$. Define $R(P, P')$ to be the set of all points (x, x') in $\text{bd}\, P \times \text{bd}\, P'$ that realize the distance between P and P', i.e. $d(x, x') = \rho(P, P')$.

Definition 5.7. The set of motions that cause a *near* paint-to-paint collision of P' with P is $\mathfrak{C}_\varepsilon(P, P'; \sigma, \sigma') := \{g \in \mathfrak{G} \colon 0 < \rho(P, gP') \leq \varepsilon,\ R(P, gP') \cap \sigma \times g\sigma' \neq \varnothing\}$. The set of motions that cause a *near-collision* of P' with P is $\mathfrak{C}_\varepsilon(P, P') := \mathfrak{C}_\varepsilon(P, P'; \mathbb{S}^2, \mathbb{S}^2)$.

We can ignore any motion g in $\mathfrak{C}_\varepsilon(P, P'; \sigma, \sigma')$ which causes $R(P, gP')$ to have more than a single element. This will be implied from Theorem 5.10. First, two lemmas are necessary.

Lemma 5.8. *Suppose* $C(x, \xi)$ *and* $C(q, \eta)$ *are circles in* $\mathbb{S}^2$ *with* $0 < \xi < \pi$ *and* $0 < \eta < \pi/2$. *If* $x \neq \pm p$, *then* $\text{card}[C(x, \xi) \cap C(p, \eta)] \leq 2$.

Proof. The circle $C(x, \xi)$ lies in the plane $\Pi(x, \xi)$ in $\mathbb{R}^3$ through $(\cos \xi)x$ with unit normal x. Similarly, $C(p, \eta)$ lies in the plane $\Pi(p, \eta)$ in $\mathbb{R}^3$ through $(\cos \eta)p$ with unit normal p. Now for each ξ and η

$$x \neq \pm p \quad \Rightarrow \quad x \text{ and } p \text{ are linearly independent}$$

$\Rightarrow$ the planes $\Pi(x,\xi)$ and $\Pi(p,\eta)$ are not parallel
$\Rightarrow$ the planes $\Pi(x,\xi)$ and $\Pi(p,\eta)$ intersect in a line Λ.

Then

$$C(x,\xi) \subset \Pi(x,\xi) \text{ and } C(p,\eta) \subset \Pi(p,\eta) \Rightarrow C(x,\xi) \cap C(p,\eta) \subset \Pi(x,\xi) \cap \Pi(p,\eta) = \Lambda$$
$$\Rightarrow C(x,\xi) \cap C(p,\eta) \subset \Lambda \cap C(p,\eta).$$

Note that $C(p,\eta)$ is a circle of radius $\sin\eta$ in the plane $\Pi(p,\eta)$ and this plane also contains the line Λ. In the plane $\Pi(p,\eta)$, a line and a circle cannot intersect more than twice. Therefore $\text{card}[\Lambda \cap C(p,\eta)] \leq 2$, and hence $\text{card}[C(x,\xi) \cap C(p,\eta)] \leq 2$. □

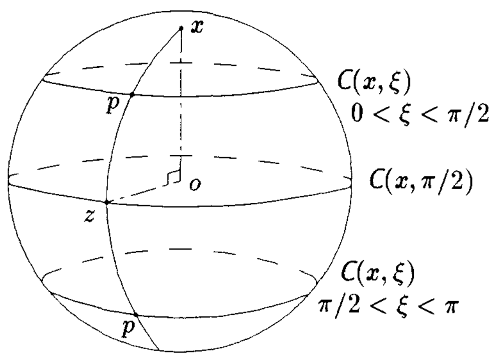

Figure 5.2. $\{$rotations about x of angle $\theta\} \cong C(x,\pi/2) \cong C(x,\xi)$.

Lemma 5.9. *Suppose* $0 < \xi < \pi$ *and let* $x \in \mathbb{S}^2$. *There is a 1-1 correspondence between points on the circle* $C(x,\xi)$ *and rotations about* x *of angle* θ, $-\pi < \theta \leq \pi$.

Proof. Note that there is a 1-1 correspondence between the rotations about x of angle θ and points z in the great circle $C(x,\pi/2)$ with pole x. For any ξ with $0 < \xi < \pi$, the points p in $C(x,\xi)$ are in 1-1 correspondence with the points z in

$C(x, \pi/2)$. This last correspondence is given by $p = (\cos\xi)x + (\sin\xi)z$. See Figure 5.2. □

Consider any motion g for which the points in bd P, bd P' that realize the closest distance are not unique. This means that there are points $(x, x') \neq (p, p')$ in bd $P \times$ bd P' such that $0 < \rho(P, gP') = d(x, gx') = d(p, gp') < \pi/2$. Since P and P' are polygons, by Theorem 2.1 we know that

$$x \text{ is a vertex of } P \text{ or } x' \text{ is a vertex of } P'$$

and

$$p \text{ is a vertex of } P \text{ or } p' \text{ is a vertex of } P'.$$

(It is possible that both x, x' are vertices and the same holds for p, p'.) These observations will be used in the proof of the next theorem.

As in [16], Theorem 5.10 is an essential tool in the proofs of Theorem 5.12 and Theorem 5.13.

Theorem 5.10. *The set* $\mathfrak{N}(P, P') := \{g \in \mathfrak{G} \colon 0 < \rho(P, gP') < \pi/2,$ $\operatorname{card} R(P, gP') > 1\}$ *has* μ*–measure zero.*

Proof. If $g \in \mathfrak{N}(P, P')$, then $\rho(P, gP') = \rho(g^{-1}P, P')$ and $R(P, gP') = R(g^{-1}P, P')$. Therefore $\mathfrak{N}(P, P') = \mathfrak{N}(P', P)^{-1}$.

For $x' \in \operatorname{bd} P'$ define $\mathfrak{N}(x') := \left\{g \in \mathfrak{N}(P, P') \colon gx' \text{ is the second coordinate of a point in } R(P, gP')\right\}$. Similarly, for $x \in \operatorname{bd} P$ define $\mathfrak{N}(x) := \left\{g \in \mathfrak{N}(P', P) \colon gx \text{ is the second coordinate of a point in } R(P', gP)\right\}$. Then $\bigcup_{x' \in \operatorname{bd} P'} \mathfrak{N}(x') = \mathfrak{N}(P, P') = \mathfrak{N}(P', P)^{-1} = \bigcup_{x \in \operatorname{bd} P} \mathfrak{N}(x)^{-1}$. Note that

$$\begin{aligned}
\mathfrak{N}(x)^{-1} &= \left\{g^{-1} \colon\ g \in \mathfrak{N}(P', P) \text{ and } gx \text{ is the second coord of a point in } R(P', gP)\right\} \\
&= \left\{g \in \mathfrak{N}(P, P') \colon\ g^{-1}x \text{ is the second coord of a point in } R(P', g^{-1}P)\right\} \\
&= \left\{g \in \mathfrak{N}(P, P') \colon\ x \text{ is the second coord of a point in } R(gP', P)\right\} \\
&= \left\{g \in \mathfrak{N}(P, P') \colon\ x \text{ is the first coord of a point in } R(P, gP')\right\}.
\end{aligned}$$

If $g \in \mathfrak{N}(P, P')$, then there exist points $x' \in \operatorname{bd} P'$ and $x \in \operatorname{bd} P$ such that $(x, gx') \in R(P, gP')$. The preceding calculation then shows that $g \in \mathfrak{N}(x') \cap \mathfrak{N}(x)^{-1}$. Moreover x or x' must be a vertex of P or P', respectively. This allows us to write

$$\mathfrak{N}(P, P') = \bigcup_{x' \in \mathfrak{F}^0(P')} \mathfrak{N}(x') \ \cup \bigcup_{x \in \mathfrak{F}^0(P)} \mathfrak{N}(x)^{-1}.$$

It now suffices to prove $\mu\big(\mathfrak{N}(x')\big) = 0 = \mu\Big(\mathfrak{N}(x)^{-1}\Big)$ for any points x', x in $\operatorname{bd} P', \operatorname{bd} P$ respectively. We consider $\mathfrak{N}(x')$ first.

Assume $x' \in \operatorname{bd} P'$ is fixed. Without loss of generality $x' = e_1$. We use the decomposition $g = g_r g_t$, so that g_r is rotation about $g_t x'$. Suppose $g \in \mathfrak{N}(x')$. Then $\exists p' \neq x'$ in $\operatorname{bd} P'$ and $\exists x \neq p$ in $\operatorname{bd} P$ such that $0 < \rho(P, gP') = d(x, gx') = d(p, gp') < \pi/2$.

Define $\xi := d(g_t x', g_t p') = d(x', p')$ and $\eta := d(x, g_t x') = d(x, gx')$. Then $0 < \xi < \pi$ and $0 < \eta < \pi/2$. Consider the circle $C(g_t x', \xi)$ that contains $g_t p'$ and let $C(p, \eta)$ be the circle with center p and radius η. Since $g \in \mathfrak{N}(x')$ we know $d(p, gp') = \eta$ and so $gp' \in C(p, \eta)$. Also $d(gx', gp') = d(g_t x', g_t p') = \xi$ whereby $gp' \in C(g_t x', \xi)$. See Figure 5.3. Therefore $gp' = g_r(g_t p') \in C(g_t x', \xi) \cap C(p, \eta)$. Note that $g_t x' \notin P$ and so $g_t x' \neq p$. By Lemma 5.8 we know

$$g_t x' \neq \pm p \quad \Rightarrow \quad \operatorname{card}[C(g_t x', \xi) \cap C(p, \eta)] \leq 2.$$

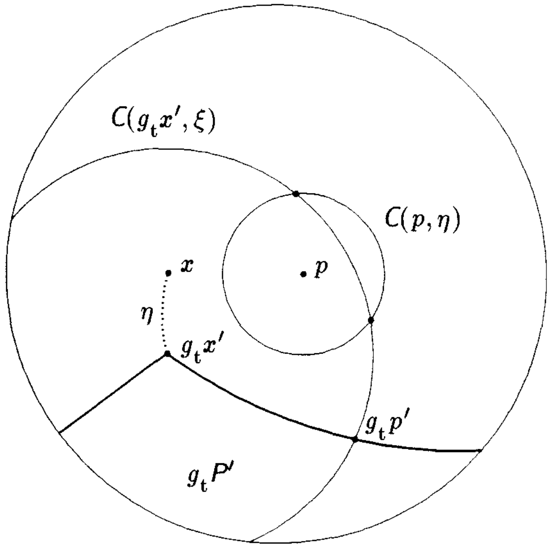

Figure 5.3. $g_t x' \neq \pm p \Rightarrow \mathrm{card}[C(g_t x', \xi) \cap C(p, \eta)] \leq 2.$

Suppose $g_t x' \neq -p$. Now use Lemma 5.9, which shows the correspondence between the rotations about $g_t x'$ and the points of the circle $C(g_t x', \xi)$, and Lemma 5.8 to conclude that there are at most two possibilities for g_r, i.e. $\mathrm{card}\big(\{g_r\colon g_r g_t \in \mathfrak{N}(x')\}\big) \leq 2$. This is true no matter what $p' \neq x'$ in $\mathrm{bd}\, P'$ and $p \neq x$ in $\mathrm{bd}\, P$ happen to be.

Let $-\mathrm{bd}\, P = \{-p\colon\ p \in \mathrm{bd}\, P\}$ and consider the set $\{g \in \mathfrak{N}(x')\colon g_t x' \in -\mathrm{bd}\, P\}$. This set has μ–measure zero because the area of $-\mathrm{bd}\, P$ is zero:

$$\mu\Big(\{g \in \mathfrak{N}(x')\colon\ g_t x' \in -\mathrm{bd}\, P\}\Big) \leq \int_{-\mathrm{bd}\, P} \int_{-\pi}^{\pi} d\theta\, dA = 2\pi \cdot \mathrm{Area}(-\mathrm{bd}\, P) = 0.$$

If we use this and the fact that $\mathrm{card}\big(\{g_r\colon\ g_r g_t \in \mathfrak{N}(x')\}\big) \leq 2$, then we get

$$\begin{aligned}
\mu\big(\mathfrak{N}(x')\big) &= \mu\Big(\{g \in \mathfrak{N}(x')\colon\ g_t x' \notin -\mathrm{bd}\, P\}\Big) \\
&= \int_{\{g_t x' \in \mathbb{S}^2 \setminus (\pm\, \mathrm{bd}\, P)\}} \left[\int_{\{g_r\colon\ g_r g_t \in \mathfrak{N}(x')\}} d\theta\right] dA \\
&= \int_{\mathbb{S}^2 \setminus (\pm\, \mathrm{bd}\, P)} 0\, dA = 0.
\end{aligned}$$

A similar argument with the roles of P and P' reversed shows that $x \in \operatorname{bd} P \Rightarrow \mu\big(\mathfrak{N}(x)\big) = 0$. By the invariance of μ under inversion we have $\mu\Big(\mathfrak{N}(x)^{-1}\Big) = \mu\big(\mathfrak{N}(x)\big) = 0$ as well. □

Measurability of Near-Collisions

In this section we prove that $\mathfrak{C}_\varepsilon(P, P'; \sigma, \sigma')$ is μ–measurable when $0 < \varepsilon < \pi/2$ and σ, σ' are Borel subsets of $\mathbb{S}^2$. The statements and proofs given here are essentially the same as that found in [**16**; pp. 7–8].

Lemma 5.11. *Let φ be a map from $\mathfrak{B}(\mathbb{S}^2)$ to the collection of all subsets of $\mathfrak{G}$ which has the following properties.*

(a) $\sigma_1, \sigma_2 \in \mathfrak{B}(\mathbb{S}^2)$ *disjoint* $\Rightarrow$ $\varphi(\sigma_1) \cap \varphi(\sigma_2)$ *has μ–measure zero.*

(b) $\sigma_i \in \mathfrak{B}(\mathbb{S}^2)$ *for* $i = 1, 2, 3, \ldots$ $\Rightarrow$ $\varphi\left(\bigcup_{i=1}^{\infty} \sigma_i\right) = \bigcup_{i=1}^{\infty} \varphi(\sigma_i)$.

(c) $\sigma \subset \mathbb{S}^2$ *closed* $\Rightarrow$ $\varphi(\sigma)$ *is μ–measurable.*

Then for each $\sigma \in \mathfrak{B}(\mathbb{S}^2)$, the set $\varphi(\sigma)$ is μ–measurable.

Proof. We want to show $\mathcal{A} := \{\sigma \in \mathfrak{B}(\mathbb{S}^2);\ \varphi(\sigma) \text{ is } \mu\text{–measurable}\} = \mathfrak{B}(\mathbb{S}^2)$. It suffices to show $\mathcal{A}$ is a σ-algebra that contains the closed sets. Note that property (c) implies $\mathcal{A}$ contains the closed subsets of $\mathbb{S}^2$, including $\mathbb{S}^2$ itself. From property (b) we know $\mathcal{A}$ is closed under countable unions. Therefore we need only to prove $\mathcal{A}$ is closed under complementation, i.e. if $\sigma \in \mathfrak{B}(\mathbb{S}^2)$ and $\varphi(\sigma)$ is μ–measurable, then $\varphi(\mathbb{S}^2 \setminus \sigma)$ is μ–measurable.

Assume $\sigma \in \mathcal{A}$ and define $\sigma^{c} := \mathbb{S}^2 \setminus \sigma$, $\varphi(\sigma)^{c} := \mathfrak{G} \setminus \varphi(\sigma)$. By property (a) we know $\varphi(\sigma^{c}) \cap \varphi(\sigma)$ is μ–measurable. Since $\mathbb{S}^2 = \sigma^{c} \cup \sigma$, property (b) implies $\varphi(\mathbb{S}^2) = \varphi(\sigma^{c}) \cup \varphi(\sigma)$. Note that

$$\varphi(\mathbb{S}^2) \cap \varphi(\sigma)^{c} = [\varphi(\sigma) \cup \varphi(\sigma^{c})] \cap \varphi(\sigma)^{c} = \varphi(\sigma^{c}) \cap \varphi(\sigma)^{c}$$

and so

$$\varphi(\sigma^{c}) = [\varphi(\sigma^{c}) \cap \varphi(\sigma)] \cap [\varphi(\sigma^{c}) \cap \varphi(\sigma)^{c}] = [\varphi(\sigma^{c}) \cap \varphi(\sigma)] \cup [\varphi(\mathbb{S}^2) \cap \varphi(\sigma)^{c}].$$

The far right-hand side of the last equation is the union of μ–measurable sets. □

Theorem 5.12. *If $\sigma, \sigma' \in \mathfrak{B}(\mathbb{S}^2)$, then $\mathfrak{C}_{\varepsilon}(P, P'; \sigma, \sigma')$ is μ–measurable.*

Proof. Suppose $\sigma, \sigma' \subset \mathbb{S}^2$ are closed, $0 < \varepsilon < \pi/2$, and write
$\mathfrak{C} := \mathfrak{C}_{\varepsilon}(P, P'; \sigma, \sigma') \cup \mathfrak{C}_{0}(P, P'; \sigma, \sigma')$.

We show $\mathfrak{C}$ is a closed subset of $\mathfrak{G}$. Let $\{g_j\}$ be a sequence in $\mathfrak{C}$ that converges to $g \in \mathfrak{G}$. Then $\{g_j P'\}$ converges to gP' in the Hausdorff metric and therefore $\rho(P, g_j P') \to \rho(P, gP')$. Consequently, $\rho(P, g_j P') \le \varepsilon \ \forall j$ implies

$$\rho(P, gP') \le \varepsilon, \tag{5.1}$$

i.e. the distance from P to gP' is at most ε.

Also $\operatorname{int} P \cap g_j P' = \varnothing \ \forall j$ implies $\operatorname{int} P \cap gP' = \varnothing$. To prove this, first note that

$$\operatorname{int} P \cap gP' \neq \varnothing \quad \Rightarrow \quad \exists p \in \operatorname{int} P,\ \exists p' \in P' \text{ such that } p = gp'.$$

Choose $\lambda > 0$ so that $D(p, \lambda) \subset \operatorname{int} P$. Since $g_j p' \to gp'$, $\exists J \in \mathbb{Z}^{+}$ such that $g_j p' \in D(p, \lambda) \ \forall j \ge J$. But this implies $g_j P' \cap \operatorname{int} P \neq \varnothing \ \forall j \ge J$, hence $g_j \notin \mathfrak{C}$

$\forall j \geq J$. With this contradiction we have proved

$$\text{int}\, P \cap gP' = \varnothing. \tag{5.2}$$

For each $j \in \mathbb{Z}^+$, $g_j \in \mathfrak{C}$ implies $\exists x_j \in \text{bd}\, P \cap \sigma$, $\exists x'_j \in \text{bd}\, P' \cap \sigma' \ni d(x_j, g_j x'_j) = \rho(P, g_j P')$. Since $\text{bd}\, P \cap \sigma$ and $\text{bd}\, P' \cap \sigma'$ are compact, $(\text{bd}\, P \cap \sigma) \times (\text{bd}\, P' \cap \sigma')$ is compact. Thus there is a subsequence $\{j_k\}$ and a point $(x, x') \in (\text{bd}\, P \cap \sigma) \times (\text{bd}\, P' \cap \sigma')$ such that $x_{j_k} \to x$ and $x'_{j_k} \to x'$ as $k \to \infty$.

Note that d and ρ are continuous and $g_{j_k} x'_{j_k} \to gx'$ as $k \to \infty$. Thus, if $k \to \infty$ in $d(x_{j_k}, g_{j_k} x'_{j_k}) = \rho(P, g_{j_k} P')$, we obtain

$$d(x, gx') = \rho(P, gP'). \tag{5.3}$$

If $\rho(P, gP') = 0$, then (5.2) implies $g \in \mathfrak{C}_0(P, P')$. If $\rho(P, gP') > 0$, then (5.1) and (5.3) imply $g \in \mathfrak{C}_\varepsilon(P, P'; \sigma, \sigma')$. Therefore $g \in \mathfrak{C}$ and $\mathfrak{C}$ is closed.

Let $\sigma' \subset \mathbb{S}^2$ be a fixed, closed set. Define

$$\varphi(\sigma) := \mathfrak{C}_\varepsilon(P, P'; \sigma, \sigma') \quad \text{for } \sigma \in \mathfrak{B}(\mathbb{S}^2).$$

Suppose σ_1, σ_2 are disjoint Borel sets. If $g \in \varphi(\sigma_1) \cap \varphi(\sigma_2)$, then $R(P, gP') \cap \sigma'_i \times g\sigma' \neq \varnothing$ for $i = 1, 2$. Since $\sigma_1 \cap \sigma_2 = \varnothing$, we know $\text{card}\, R(P, gP') > 1$. Apply Theorem 5.10 to conclude that $\varphi(\sigma_1) \cap \varphi(\sigma_2)$ has μ–measure zero, which establishes property (a) of Lemma 5.11.

If σ is closed, then $\mathfrak{C}_\varepsilon(P, P'; \sigma, \sigma') \cup \mathfrak{C}_0(P, P'; \sigma, \sigma')$ is μ–measurable because it is closed, as shown above. The second set in the union, $\mathfrak{C}_0(P, P'; \sigma, \sigma')$, has μ–measure zero, being a subset of $\mathfrak{C}_0(P, P')$. Therefore the set $\varphi(\sigma) = \mathfrak{C}_\varepsilon(P, P'; \sigma, \sigma')$ is μ–measurable, which is property (c) of Lemma 5.11.

Property (b) of Lemma 5.11 is also satisfied, i.e.

$$\varphi\Big(\bigcup_{i=1}^{\infty}\sigma_i\Big)=\mathfrak{C}_\varepsilon\Big(P,P';\bigcup_{i=1}^{\infty}\sigma_i,\sigma'\Big)=\bigcup_{i=1}^{\infty}\mathfrak{C}_\varepsilon(P,P';\sigma_i,\sigma')=\bigcup_{i=1}^{\infty}\varphi(\sigma_i).$$

Therefore, by Lemma 5.11, the set $\varphi(\sigma)=\mathfrak{C}_\varepsilon(P,P';\sigma,\sigma')$ is μ–measurable for each $\sigma\in\mathfrak{B}(\mathbb{S}^2)$.

Fix $\sigma\in\mathfrak{B}(\mathbb{S}^2)$. Define

$$\psi(\sigma'):=\mathfrak{C}_\varepsilon(P,P';\sigma,\sigma')\quad\text{for } \sigma'\in\mathfrak{B}(\mathbb{S}^2).$$

If σ' is closed, then $\psi(\sigma')$ is μ–measurable. Therefore properties (a)-(c) of Lemma 5.11 hold and so $\psi(\sigma')$ is μ–measurable for each $\sigma'\in\mathfrak{B}(\mathbb{S}^2)$. □

Measure of Near-Collisions

In this section we prove that if σ,σ' are Borel subsets of $\mathbb{S}^2$, then $\mu\big(\mathfrak{C}_\varepsilon(P,P';\cdot,\sigma')\big)$ and $\mu\big(\mathfrak{C}_\varepsilon(P,P';\sigma,\cdot)\big)$ are both finite measures on the Borel subsets of $\mathbb{S}^2$. Theorem 5.13 and its proof are basically the same as that of [**16**; Lemma 7].

Theorem 5.13. *For each $\sigma'\in\mathfrak{B}(\mathbb{S}^2)$ the function $\mu\big(\mathfrak{C}_\varepsilon(P,P';\cdot,\sigma')\big)$ is a finite measure on $\mathfrak{B}(\mathbb{S}^2)$.*

Proof. Since μ is a measure, $\mu\big(\mathfrak{C}_\varepsilon(P,P';\cdot,\sigma')\big)$ is a nonnegative, extended real-valued function on $\mathfrak{B}(\mathbb{S}^2)$ ([1; Def. 1.2.3]). Therefore only σ-additivity must be established.

Assume $\sigma'\in\mathfrak{B}(\mathbb{S}^2)$ is fixed. Suppose $\sigma_1,\sigma_2\in\mathfrak{B}(\mathbb{S}^2)$ with $\sigma_1\cap\sigma_2=\varnothing$. If $g\in\mathfrak{C}_\varepsilon(P,P';\sigma_1,\sigma')\cap\mathfrak{C}_\varepsilon(P,P';\sigma_2,\sigma')$, then $\operatorname{card}R(P,gP')>1$. By Theorem

5.10,

$$\mu\Big(\mathfrak{C}_\varepsilon(P,P';\sigma_1,\sigma')\cap\mathfrak{C}_\varepsilon(P,P';\sigma_2,\sigma')\Big)=0. \tag{5.4}$$

Since μ is a measure and (5.4) holds, we have ([1; Theorem 1.2.5(b)])

$$\begin{aligned}\mu\Big(\mathfrak{C}_\varepsilon(P,P';\sigma_1,\sigma')\Big)&+\mu\Big(\mathfrak{C}_\varepsilon(P,P';\sigma_2,\sigma')\Big)\\ &=\mu\Big(\mathfrak{C}_\varepsilon(P,P';\sigma_1,\sigma')\cup\mathfrak{C}_\varepsilon(P,P';\sigma_2,\sigma')\Big)\\ &\quad+\mu\Big(\mathfrak{C}_\varepsilon(P,P';\sigma_1,\sigma')\cap\mathfrak{C}_\varepsilon(P,P';\sigma_2,\sigma')\Big)\\ &=\mu\Big(\mathfrak{C}_\varepsilon(P,P';\sigma_1,\sigma')\cup\mathfrak{C}_\varepsilon(P,P';\sigma_2,\sigma')\Big).\end{aligned}$$

This shows that $\mu\Big(\mathfrak{C}_\varepsilon(P,P';\,\cdot\,,\sigma')\Big)$ is finitely additive ([1; p. 6]). In order to prove the σ-additivity, consider the following.

Let $\{\sigma_i\}$ be a sequence in $\mathfrak{B}(\mathbb{S}^2)$ such that $\sigma_i\downarrow\varnothing$, i.e. $\sigma_1\supset\sigma_2\supset\cdots$ and $\bigcap_{i=1}^\infty\sigma_i=\varnothing$. Here $\sigma_i\supset\sigma_{i+1}$ $\forall i$ implies $\mathfrak{C}_\varepsilon(P,P';\sigma_1,\sigma')\supset\mathfrak{C}_\varepsilon(P,P';\sigma_2,\sigma')\supset\cdots$. Set $\mathfrak{C}:=\bigcap_{i=1}^\infty\mathfrak{C}_\varepsilon(P,P';\sigma_i,\sigma')$. Then $\mathfrak{C}_\varepsilon(P,P';\sigma_i,\sigma')\downarrow\mathfrak{C}$ which, along with the σ-additivity of the finite measure μ, implies

$$\lim_{i\to\infty}\mu\Big(\mathfrak{C}_\varepsilon(P,P';\sigma_i,\sigma')\Big)=\mu(\mathfrak{C}),$$

see [1; Theorem 1.2.7(b)].

If $\mu(\mathfrak{C})=0$, then we are done because in this case the measure $\mu\Big(\mathfrak{C}_\varepsilon(P,P';\,\cdot\,,\sigma')\Big)$ is continuous from above at $\varnothing$ and therefore σ-additive ([1; Theorem 1.2.8(b)]).

Assume $\mathfrak{C}\neq\varnothing$. Then

$$\begin{aligned}g\in\mathfrak{C}\quad&\Leftrightarrow\quad g\in\mathfrak{C}_\varepsilon(P,P';\sigma_i,\sigma')\quad\forall i\\ &\Rightarrow\quad\forall i\;\exists(x_i,gx_i')\in R(P,gP')\cap(\operatorname{bd}P\cap\sigma_i)\times g(\operatorname{bd}P'\cap\sigma').\end{aligned}$$

Fix i and consider $x_i \in \operatorname{bd} P \cap \sigma_i$ as just described. Then

$$\bigcap_{j=1}^{\infty} \sigma_j = \varnothing \quad \Rightarrow \quad \bigcap_{j=1}^{\infty} (\operatorname{bd} P \cap \sigma_j) = \varnothing \quad \Rightarrow \quad \exists k \text{ such that } x_i \notin (\operatorname{bd} P \cap \sigma_k).$$

Now $g \in \mathfrak{C} \Rightarrow \exists (x_k, gx'_k) \in R(P, gP') \cap (\operatorname{bd} P \cap \sigma_k) \times g(\operatorname{bd} P' \cap \sigma')$. Notice that

$$\begin{aligned} x_i \notin (\operatorname{bd} P \cap \sigma_k) \quad &\Rightarrow \quad x_i \neq x_k \\ &\Rightarrow \quad (x_k, gx'_k) \neq (x_i, gx_i) \\ &\Rightarrow \quad \operatorname{card} R(P, gP') > 1. \end{aligned}$$

This implies $\mathfrak{C} \subset \{g\colon 0 < \rho(P, gP') < \pi/2,\ \operatorname{card} R(P, gP') > 1\}$, from which we have $\mu(\mathfrak{C}) = 0$ by applying Theorem 5.10. □

Remark 5.14. From the description of $\mathfrak{C}_\varepsilon(P, P'; \sigma, \sigma')$ in Definition 5.7, we know that

$$\mathfrak{C}_\varepsilon(P, P'; \sigma, \sigma') = \mathfrak{C}_\varepsilon(P', P; \sigma', \sigma)^{-1}.$$

Since μ is invariant under inversion, there is

$$\mu\Big(\mathfrak{C}_\varepsilon(P, P'; \sigma, \sigma')\Big) = \mu\Big(\mathfrak{C}_\varepsilon(P', P; \sigma', \sigma)\Big).$$

Now application of Theorem 5.13 shows that $\mu\Big(\mathfrak{C}_\varepsilon(P, P'; \sigma, \cdot\,)\Big)$ is a measure on $\mathfrak{B}(\mathbb{S}^2)$ for each $\sigma \in \mathfrak{B}(\mathbb{S}^2)$.

6. AN UPPER ESTIMATE FOR THE MEASURE OF NEAR-COLLISIONS

Once it is known that $\mu\Big(\mathfrak{C}_\varepsilon(P,P';\,\cdot\,,\sigma')\Big)$ and $\mu\Big(\mathfrak{C}_\varepsilon(P,P';\sigma,\,\cdot\,)\Big)$ are finite measures on $\mathfrak{B}(\mathbb{S}^2)$, there remains the task of computing the limit

$$\lim_{\varepsilon\to 0^+} \frac{\mu\Big(\mathfrak{C}_\varepsilon(P,P';\sigma,\sigma')\Big)}{\varepsilon}$$

in terms of geometric quantities which depend only on P, P' and σ, σ'. In this chapter we prove that $\mu\Big(\mathfrak{C}_\varepsilon(P,P';\sigma,\sigma')\Big)\Big/\varepsilon$ is bounded above by an expression whose limit exists as ε approaches zero.

A Set Containing the Near-Collisions

In this section we define a set $\mathfrak{U}_\varepsilon(P,P';\sigma,\sigma')$ of motions which has the near-collisions $\mathfrak{C}_\varepsilon(P,P';\sigma,\sigma')$ as a subset.

A minor modification of parts 2 and 4 of Definition 5.2 can produce motions that often cause a near paint-to-paint collision of P' with P. Parts 1-3 in the following definition are exactly like the corresponding parts of Definition 5.2.

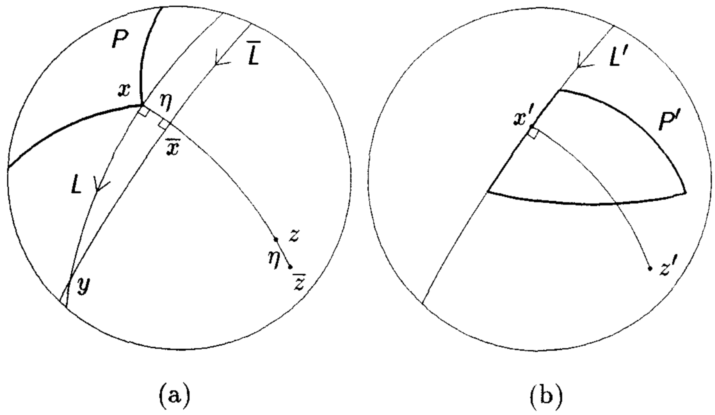

Figure 6.1. (a) The pairs x, L and $\overline{x}, \overline{L}$ (b) the pair x', L'.

Definition 6.1. Suppose $0 < \varepsilon < \pi/2$ and $0 < \eta \leq \varepsilon$. The motion $g(x, L, \eta; x', L')$ is defined as follows.

1. Suppose $x \in \operatorname{bd} P$ and $x' \in \operatorname{bd} P'$.

2. Suppose L is any great circle that supports P at x. Let z be the pole of L in the hemisphere that does not contain P. Choose $y \in L$ such that $[\![x, y, z]\!]$ is a frame. See Figure 6.1a.

3. Suppose L' is any great circle that supports P' at x'. Let z' be the pole of L' in the hemisphere that does contain P'. Choose $y' \in L'$ such that $[\![x', y', z']\!]$ is a frame. See Figure 6.1b.

4. Let $\overline{x}$ be the unique point in $arc[xz]$ such that $d(x, \overline{x}) = \eta$. Let $\overline{L}$ be the unique great circle through $\overline{x}$ that is perpendicular to $arc[x\overline{x}]$. Let $\overline{z}$ be the pole of $\overline{L}$ in the hemisphere that does not contain x. See Figure 6.1a.

5. Define $g(x, L, \eta; x', L')$ by the matrix $[\![\overline{x}, y, \overline{z}]\!][\![x', y', z']\!]^{-1}$ so that $x' \mapsto \overline{x}$, $y' \mapsto y$, $z' \mapsto \overline{z}$.

The motion $g(x', L', \eta; x, L)$ is defined by reversing the roles of P and P' in the following way. If parts 1-3 are unchanged, then replace parts 4 and 5 by:

4'. Let $\overline{x}$ be the unique point in $arc[x'(-z')]$ such that $d(x', \overline{x}) = \eta$. Let $\overline{L}$ be the unique great circle through $\overline{x}$ that is perpendicular to $arc[x'\overline{x}]$. Let $\overline{z}$ be the pole of $\overline{L}$ in the hemisphere that does not contain x'.

5'. Define $g(x', L', \eta; x, L)$ by the matrix $[\![\overline{x}, y', -\overline{z}]\!][\![x, y, z]\!]^{-1}$ so that $x \mapsto \overline{x}$, $y \mapsto y'$, $z \mapsto -\overline{z}$.

Remark 6.2. In part 4 of Definition 6.1 the points $\overline{x}, \overline{z}$ are obtained from x, z by a rotation about y of angle η. Therefore $\eta = d(x, \overline{x}) = d(z, \overline{z})$. Also, $\overline{L}$ is the image of L under this rotation. From part 5, the motion $g(x, L, \eta; x', L')$ takes x' to $\overline{x}$ and L' to $\overline{L}$, consistent with the orientations of $\overline{L}$ and L'. Note also that the points $\overline{x}$ and $\overline{z}$ determine each other, i.e. $\overline{x}$ is the unique point on $arc[xz]$ with $d(x, \overline{x}) = \eta$ iff $\overline{z}$ is the unique point on $arc[(-x)z]$ with $d(z, \overline{z}) = \eta$. This means that in part 4 we could have chosen $\overline{z}$ before $\overline{x}$. Similar comments can be made regarding the $\overline{x}, \overline{z}$ that occur in parts 4' and 5'.

Remark 6.3. If L and z are as part 2 of Definition 6.1, then z must lie in bd $P_{\pi/2}$ = bd P^*. However, the point $\overline{z}$ defined in part 4 may or may not lie in P^*. If $\overline{z}$ lies outside $\pm P^*$, then $\overline{L}$ intersects the interior of P. Figure 6.2 illustrates how this can happen when x is a vertex and y belongs to the relative interior of an adjacent edge. If $\overline{z}$ lies in the boundary of P^*, then $\overline{L}$ supports P at some point other than x. If $\overline{z}$ lies in the interior of P^*, then $\overline{L}$ does not intersect P and so P must be contained in one of the two open hemispheres determined by $\overline{L}$.

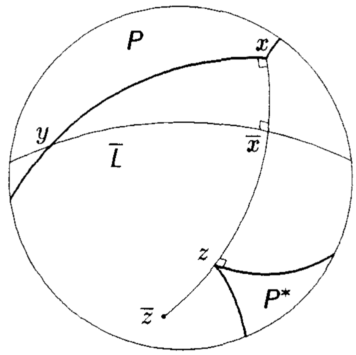

Figure 6.2. $\overline{z} \notin \pm P^*$ and $\overline{L} \cap \operatorname{int} P \neq \varnothing$.

Definition 6.4. For each ε with $0 < \varepsilon < \pi/2$ define $\mathfrak{U}_\varepsilon(P, P'; \sigma, \sigma') :=$ $\{g(x, L, \eta; x', L') \colon x \in \sigma,\ x' \in \sigma',\ 0 < \eta \le \varepsilon\} = \{g(x, L, \eta; x', L') \colon x \in \sigma$ and L supports P at x, $x' \in \sigma'$ and L' supports P' at x', $0 < \eta \le \varepsilon\}$. Define $\mathfrak{U}_\varepsilon(P, P') := \mathfrak{U}_\varepsilon(P, P'; \mathbb{S}^2, \mathbb{S}^2)$.

Evidently $\mathfrak{U}_\varepsilon(P, P'; \sigma_1 \cup \sigma_2, \sigma') = \mathfrak{U}_\varepsilon(P, P'; \sigma_1, \sigma') \cup \mathfrak{U}_\varepsilon(P, P'; \sigma_2, \sigma')$ and $\mathfrak{U}_\varepsilon(P, P'; \sigma, \sigma'_1 \cup \sigma'_2) = \mathfrak{U}_\varepsilon(P, P'; \sigma, \sigma'_1) \cup \mathfrak{U}_\varepsilon(P, P'; \sigma, \sigma'_2)$. Also, it is clear that $\mathfrak{U}_\varepsilon(P, P'; \sigma \cap \operatorname{bd} P, \sigma' \cap \operatorname{bd} P') = \mathfrak{U}_\varepsilon(P, P'; \sigma, \sigma')$.

If we set $\mathfrak{U}_0(P, P'; \sigma, \sigma') := \{g(x, L; x', L') \colon x \in \sigma$ and L supports P at x, $x' \in \sigma'$ and L' supports P' at $x'\}$, then Theorem 5.4 states that $\mathfrak{C}_0(P, P'; \sigma, \sigma') = \mathfrak{U}_0(P, P'; \sigma, \sigma')$. However the analogous result with 0 replaced by ε is not true, as Theorem 6.5 and Example 6.6 will demonstrate.

Theorem 6.5. $\mathfrak{C}_\varepsilon(P, P'; \sigma, \sigma') \subset \mathfrak{U}_\varepsilon(P, P'; \sigma, \sigma')$.

Proof. Assume $g \in \mathfrak{C}_\varepsilon(P, P'; \sigma, \sigma')$ and set $\eta := \rho(P, gP')$, so that $0 < \eta \le \varepsilon$. Suppose $(x, gx') \in R(P, gP') \cap \sigma \times g\sigma'$, so that $\eta = d(x, gx')$. Define

$\overline{x} := gx'$,

$L :=$ the great circle through x that is perpendicular to $arc[x\overline{x}]$,

$z :=$ the pole of L in the hemisphere that does contain $\overline{x}$,

$\overline{L} :=$ the great circle through $\overline{x}$ that is perpendicular to $arc[x\overline{x}]$,

$\overline{z} :=$ the pole of $\overline{L}$ in the hemisphere that does not contain x,

$L' := g^{-1}\overline{L}$,

$z' := g^{-1}\overline{z}$.

Choose $y \in L \cap \overline{L}$, $y' \in L'$ so that $1 = \det[\![x, y, z]\!] = \det[\![\overline{x}, y, \overline{z}]\!] = \det[\![x', y', z']\!]$. To prove $g \in \mathfrak{U}_\varepsilon(P, P'; \sigma, \sigma')$, it suffices to show

(*i*) $d(x, \overline{x}) = \eta$,

(*ii*) $gy' = y$,

(*iii*) L supports P at x and L' supports P' at x',

(*iv*) z is the pole of L in the hemisphere that does not contain P and z' is the pole of L' in the hemisphere that does contain P'.

Note that (*i*) follows from the definition of $\overline{x}$. Here $gx' = \overline{x}$, $gz' = \overline{z}$, and $[\![x', y', z']\!], [\![\overline{x}, y, \overline{z}]\!]$ are frames in $\mathbb{R}^3$; hence $gy' = y$, which is (*ii*).

To prove (*iii*), note that $x = N(P, \overline{x})$, $z = x \vdash \overline{x}$, and L is perpendicular to $arc[x\overline{x}]$. Therefore, by a property of the nearest-point map, L supports P at x. Similarly, $\overline{x} = gx' = N(gP', x)$, $-\overline{z} = -gz' = \overline{x} \vdash x$, and $gL' = \overline{L}$ is perpendicular to $arc[x\overline{x}]$. Hence gL' supports gP' at gx', and therefore L' supports P' at x'.

For the first part of (*iv*), note that $\overline{x} \in arc[xz]$ since z is the pole of L in the hemisphere that does contain $\overline{x}$. If $r \in \text{relint}(arc[xz])$, then $x = N(P, r)$ by one of the properties of the nearest-point map. This implies $d(r, p) \geq d(r, x)$ for all $p \in P$. If $r \to z$, then $d(r, x) \to d(z, x) = \pi/2$ and $d(r, p) \to d(z, p)$ for each $p \in P$. Thus $d(z, p) \geq \pi/2$ for all $p \in P$ and so z is the pole of L in the hemisphere that does not contain P. For the second part of (iv), note that $-\overline{z}$ is the pole of $\overline{L}$ in the hemisphere that does contain x. Apply the same sort of argument to conclude that $d(-\overline{z}, gp') \geq \pi/2$ for all $p' \in P'$. This implies $d(z', p') = d(gz', gp') = d(\overline{z}, gp') \leq \pi/2$ for all $p' \in P'$ and so z' is the pole of L' in the hemisphere that does contain P'. □

The set containment in Theorem 6.5 may be proper. This is illustrated by the following example.

Example 6.6. We show that $\mathfrak{U}_\varepsilon(P, P')$ may contain motions that are not in $\mathfrak{C}_\varepsilon(P, P')$. Let $[\![x, y, z]\!]$ be a frame and suppose P is a convex polygon as in Remark 6.3 and Figure 6.2, so that x is a vertex of P and y belongs to the relative interior of an adjacent edge of P. Furthermore, suppose P' has an edge e' whose length is greater than $\pi/2$. Let L and L' be the great circles that contain $arc[xy]$ and e', respectively. Let $g_0 \in \mathfrak{C}_0(P, P')$ be a motion such that $g_0 e'$ properly contains $arc[xy]$. See Figure 6.3a. Let $x' = g_0^{-1} x \in e'$.

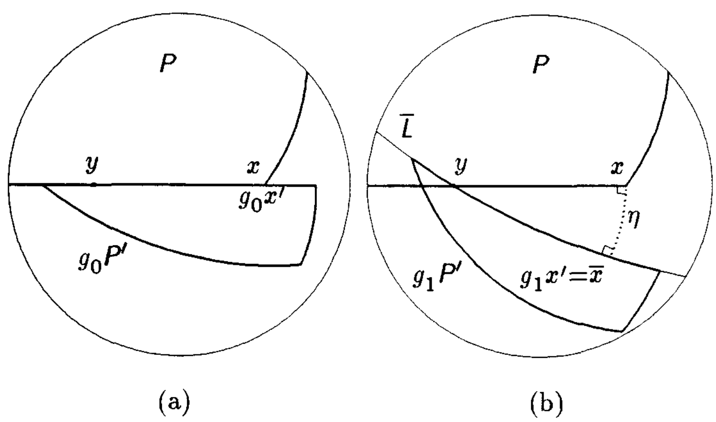

Figure 6.3. (a) $g_0 \in \mathfrak{C}_0(P, P')$ (b) $g_1 \in \mathfrak{U}_\varepsilon(P, P')$ but $g_1 \notin \mathfrak{C}_\varepsilon(P, P')$.

Assume $0 < \varepsilon < \pi/2$ and $0 < \eta \leq \varepsilon$. Let g_1 be the composition of g_0 with a rotation of angle η about y as shown in Figure 6.6b. Then $g_1 \in \mathfrak{U}_\varepsilon(P, P')$. Consider the points $\overline{x}, \overline{z}$ which are the images of x, z under this rotation. By choosing ε smaller if necessary, we can assume the point $\overline{z}$ does not belong to $\pm P^*$. This implies that $\overline{L}$ meets the interior of P; compare Figure 6.2 with Figure 6.3b. Therefore $g_1 P$ meets the interior of P, and so $g_1 \notin \mathfrak{C}_\varepsilon(P, P')$.

An Upper Estimate for the Measure of Near-Collisions

We now give a very important result of this chapter, an upper estimate for the measure of $\mathfrak{U}_\varepsilon(P, P'; \sigma, \sigma')$, by considering different cases for σ and σ'. Once this estimate is known, we will then have an upper estimate for the measure of the near-collisions $\mathfrak{C}_\varepsilon(P, P'; \sigma, \sigma')$.

Theorem 6.7. *If* $0 < \varepsilon < \pi/2$ *and* $\sigma, \sigma' \in \mathfrak{B}(\mathbb{S}^2)$, *then*

$$\mu\Big(\mathfrak{U}_\varepsilon(P, P'; \sigma, \sigma')\Big) \leq (1 - \cos\varepsilon)\Big[\Phi_0(P, \sigma)\ \Phi_0(P', \sigma') + \Phi_1(P, \sigma)\ \Phi_1(P', \sigma')\Big] + (\sin\varepsilon)\Big[\Phi_0(P, \sigma)\ \Phi_1(P', \sigma') + \Phi_1(P, \sigma)\ \Phi_0(P', \sigma')\Big].$$

Proof.

Case 1 Vertex to vertex

Assume x, x' are fixed vertices of P, P' and suppose $\sigma = \{x\},\ \sigma' = \{x'\}$. Let α, α' be the interior angles of P, P' at x, x'.

There are the 1-1 correspondences

L supports P at x	$\longleftrightarrow$	z in the edge of P^* polar to $x =: Z$ (which is a subset of the great circle with pole x)

$$\longleftrightarrow \qquad \left\{\overline{x} \in arc[xz] \colon 0 < d(x,\overline{x}) \leq \varepsilon\right\}.$$

This illustrated in Figure 6.4. Since x is fixed, each pair x, L corresponds to a choice of L and, for a given ε, this corresponds to a portion of $arc[xz]$. The collection of all possible pairs x, L is then described by the set

$$\overline{X} := \bigcup_{z \in Z} \left\{\overline{x} \in arc[xz] \colon 0 < d(x,\overline{x}) \leq \varepsilon\right\} = B_\varepsilon(P, \{x\}).$$

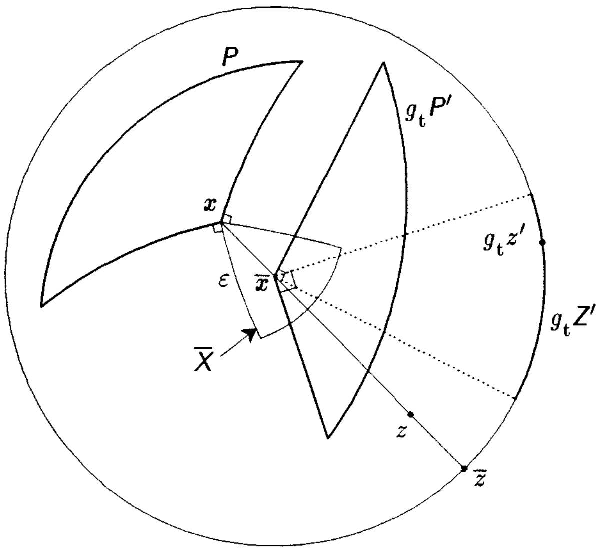

Figure 6.4. Case 1, vertex to vertex.

We also have the 1-1 correspondence

$$\begin{array}{c} L' \\ \text{supports } P' \\ \text{at } x' \end{array} \qquad \longleftrightarrow \qquad \begin{array}{c} z' \\ \text{in } \{-p' \colon p' \in \text{the edge of } (P')^* \text{ polar to } x'\} =: Z'. \end{array}$$

Thus, x' fixed implies each pair x', L' corresponds to a choice of L', and therefore to a choice of z'. The set of all possible z' consists of those points whose antipodes lie in the edge of $(P')^*$ polar to x', which is a subset of the great circle with pole x'.

Now, in order to compute the measure of $\mathfrak{U}_\varepsilon(P, P'; \{x\}, \{x'\})$, we set $\eta = d(x, \overline{x})$ and consider the motions $g(x, L, \eta; x', L')$ such that

$$\begin{array}{rll} & x' \mapsto \overline{x} & \text{for each } \overline{x} \in \overline{X}, \\ & y' \mapsto y, & \\ \text{for each } z' \in Z' & z' \mapsto \overline{z}. & \end{array} \tag{6.1}$$

Assume $x' = \mathbf{e}_1$. For each $\overline{x}$ suppose $g_t x' = \overline{x}$ and define $\Re(\overline{x}) := \big\{g_r \colon g_r g_t \in \mathfrak{U}_\varepsilon(P, P'; \{x\}, \{x'\})\big\} = \{\text{rotations } g_r \text{ about } \overline{x} \colon g_r(g_t y') = y \text{ and } g_r(g_t z') = \overline{z}\}$. Since any rotation about $\overline{x}$ which takes $g_t z'$ to $\overline{z}$ also takes $g_t y'$ to y, we have $\Re(\overline{x}) = \{\text{rotations } g_r \text{ about } \overline{x} \colon g_r(g_t z') = \overline{z}\}$. Note that each $g_t z'$ lies in the set $g_t Z'$, which is part of the great circle with pole $\overline{x}$ and (see Figure 6.4)

$$\text{Length}(g_t Z') = \int_{g_t Z'} ds = \int_{Z'} ds = \pi - \alpha'.$$

Then

$$\begin{aligned} \mu\big(\mathfrak{U}_\varepsilon(P, P'; \{x\}, \{x'\})\big) &= \int_{\overline{X}} \int_{\Re(\overline{x})} d\theta \, dA \\ &= \int_{\overline{X}} (\pi - \alpha') \, dA = (\pi - \alpha') \cdot \text{Area}(\overline{X}) \\ &= (1 - \cos\varepsilon)\big[(\pi - \alpha)(\pi - \alpha')\big] \\ &= (1 - \cos\varepsilon) \, \Phi_0(P, \{x\}) \, \Phi_0(P', \{x'\}). \end{aligned}$$

Case 2 Vertex to edge

Assume σ is a fixed Borel subset of the relative interior of an edge of P. Suppose x' is a fixed vertex of P' with corresponding interior angle α'. There is a unique great circle L that supports P at each point x in σ. In fact, the vertex

z of P^* polar to the edge of P that contains σ is a pole of L. We have the 1-1 correspondence

$$\underset{\text{in } \sigma}{x} \quad \longleftrightarrow \quad \left\{\overline{x} \in arc[xz]\colon\ 0 < d(x, \overline{x}) \leq \varepsilon\right\}.$$

In this way each pair x, L corresponds to a choice of x in σ. The collection of all possible pairs x, L is then described by the set (see Figure 6.5)

$$\overline{X} := \bigcup_{x \in \sigma} \left\{\overline{x} \in arc[xz]\colon\ 0 < d(x, \overline{x}) \leq \varepsilon\right\} = B_\varepsilon(P, \sigma).$$

The pairs x', L' are as in case 1.

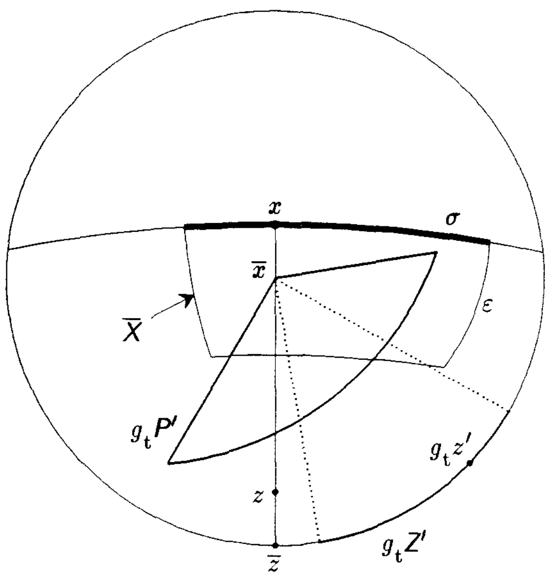

Figure 6.5. Case 2, vertex to edge.

To compute the measure of $\mathfrak{U}_\varepsilon(P, P'; \sigma, \{x'\})$ we set $\eta = d(x, \overline{x})$ and consider the motions $g(x, L, \eta; x', L')$ that satisfy (6.1). Assume $x' = \mathbf{e}_1$ and let $g_t, \Re(\overline{x})$ be as in case 1. Then

$$\mu\Big(\mathfrak{U}_\varepsilon(P,P';\sigma,\{x'\})\Big) = \int_{\overline{X}} \int_{\Re(\overline{x})} d\theta \, dA$$

$$= \int_{\overline{X}} (\pi-\alpha')\, dA = (\pi-\alpha')\cdot \text{Area}(\overline{X})$$

$$= (\sin\varepsilon)\big[\text{Length}(\sigma)\cdot(\pi-\alpha')\big]$$

$$= (\sin\varepsilon)\; \Phi_1(P,\sigma)\; \Phi_0(P',\{x'\}).$$

Case **3** Edge to vertex

Assume x is a fixed vertex of P with corresponding interior angle α. Suppose σ' is a fixed Borel subset of the relative interior of an edge of P'. There are the 1-1 correspondences

L supports P at x	$\longleftrightarrow$	z in the edge of P^* polar to $x = Z$
	$\longleftrightarrow$	$\Big\{\overline{z}\in arc[(-x)z]\colon\ 0 < d(z,\overline{z}) \le \varepsilon\Big\}.$

The last correspondence follows from the observation made in Remark 6.2 that the points $\overline{x}$ and $\overline{z}$ determine each other. For a given ε, each pair x, L corresponds to a choice of a portion of $arc[(-x)z]$. The collection of all possible pairs x, L is then described by the set

$$\overline{Z} := \bigcup_{z\in Z} \Big\{\overline{z}\in arc[(-x)z]\colon\ 0 < d(z,\overline{z}) \le \varepsilon\Big\} = B_\varepsilon(H(-x), Z).$$

There is a unique great circle L' which supports P' at each point x' in σ', and so there is only one z' for all x'. Therefore each pair x', L' corresponds to a choice of x' in σ'. Note that σ' is a subset of a great circle with pole z'. See Figure 6.6.

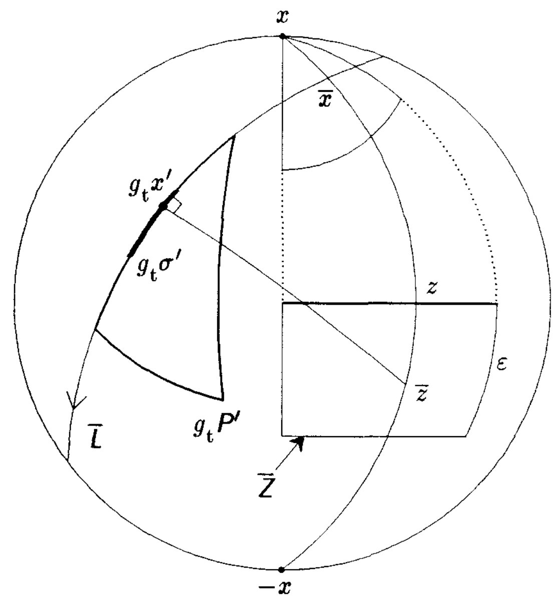

Figure 6.6. Case 3, edge to vertex.

To compute the measure of $\mathfrak{U}_\varepsilon(P, P'; \{x\}, \sigma')$ we set $\eta = d(z, \overline{z}) = d(x, \overline{x})$ and consider the motions $g(x, L, \eta; x', L')$ such that

$$\begin{aligned} &\text{for each } x' \in \sigma', \qquad && x' \mapsto \overline{x} \\ & && y' \mapsto y \\ & && z' \mapsto \overline{z}, \qquad \text{for each } \overline{z} \in \overline{Z}. \end{aligned} \tag{6.2}$$

Assume $z' = \mathbf{e}_1$. For each $\overline{z}$ suppose $g_t z' = \overline{z}$ and define $\Re(\overline{z}) := \{g_r \colon g_r g_t \in \mathfrak{U}_\varepsilon(P, P'; \{x\}, \sigma')\} = \{\text{rotations } g_r \text{ about } \overline{z} \colon g_r(g_t x') = \overline{x} \text{ and } g_r(g_t y') = y\}$. Since any rotation about $\overline{z}$ which takes $g_t x'$ to $\overline{x}$ also takes $g_t y'$ to y, we have $\Re(\overline{z}) = \{\text{rotations } g_r \text{ about } \overline{z} \colon g_r(g_t x') = \overline{x}\}$. Note that each $g_t x'$ lies in $g_t \sigma'$, which is part of the great circle with pole $\overline{z}$ and (see Figure 6.6)

$$\text{Length}(g_t \sigma') = \int_{g_t \sigma'} ds = \int_{\sigma'} ds = \text{Length}(\sigma').$$

Then

$$\mu\Big(\mathfrak{U}_\varepsilon(P,P';\{x\},\sigma')\Big) = \int_{\overline{Z}} \int_{\mathfrak{R}(\overline{z})} d\theta\, dA$$

$$= \int_{\overline{Z}} \text{Length}(\sigma')\, dA = \text{Length}(\sigma')\cdot \text{Area}(\overline{Z})$$

$$= (\sin\varepsilon)\big[(\pi-\alpha)\cdot \text{Length}(\sigma')\big]$$

$$= (\sin\varepsilon)\ \Phi_0(P,\{x\})\ \Phi_1(P',\sigma').$$

Case 4 Edge to edge

Assume σ, σ' are fixed Borel subsets of the relative interiors of edges of P, P'. There is only one great circle L that supports P at each x in σ. Thus z is the same for all x. In fact, z is the vertex of P^* that is polar to the edge that contains σ. Now we have the correspondence

$$\underset{\text{in } \sigma}{x} \qquad \longleftrightarrow \qquad \Big\{\overline{z} \in arc[(-x)z]\colon\ 0 < d(z,\overline{z}) \le \varepsilon\Big\}.$$

Note that σ is a subset of the great circle with pole z. The collection of all possible pairs x, L is described then by the set

$$\overline{Z} := \bigcup_{x\,\in\,\sigma} \Big\{\overline{z} \in arc[(-x)z]\colon\ 0 < d(z,\overline{z}) \le \varepsilon\Big\}.$$

The pairs x', L' are as in case 3.

To compute the measure of $\mathfrak{U}_\varepsilon(P,P';\sigma,\sigma')$ we set $\eta = d(z,\overline{z}) = d(x,\overline{x})$ and consider the motions $g(x,L,\eta;x',L')$ that satisfy (6.2). Assume $z' = \mathrm{e}_1$, and let g_{t} and $\mathfrak{R}(\overline{z})$ be in case 3. Each point $g_{\mathrm{t}}x'$ lies in $g_{\mathrm{t}}\sigma'$, which is part of the great circle with pole $\overline{z}$ and (see Figure 6.7)

$$\text{Length}(g_{\mathrm{t}}\sigma') = \int_{g_{\mathrm{t}}\sigma'} ds = \int_{\sigma'} ds = \text{Length}(\sigma').$$

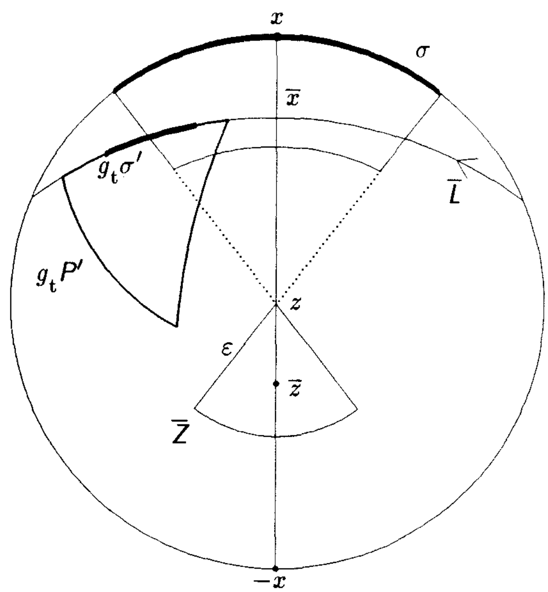

Figure 6.7. Case 4, edge to edge.

Then

$$\mu\Big(\mathfrak{U}_\varepsilon(P,P';\sigma,\sigma')\Big) = \int_{\overline{Z}} \int_{\Re(\overline{z})} d\theta\, dA$$
$$= \int_{\overline{Z}} \text{Length}(\sigma')\, dA = \text{Length}(\sigma')\cdot \text{Area}(\overline{Z})$$
$$= (1-\cos\varepsilon)\big[\text{Length}(\sigma)\cdot \text{Length}(\sigma')\big]$$
$$= (1-\cos\varepsilon)\ \Phi_1(P,\sigma)\ \Phi_1(P',\sigma').$$

With the calculations in each case completed, we consider the situation when σ, σ' are general Borel subsets of of $\mathbb{S}^2$. That is, if $\sigma, \sigma' \in \mathfrak{B}(\mathbb{S}^2)$, then consider their intersections with the vertices and the relative interiors of the edges of P, P'. The subsets of bd P and bd P' that result are the kinds considered in cases 1-4. In fact,

$$\sigma \cap \text{bd}\, P = \sigma\ \cap\ \Big[\mathfrak{F}^0(P)\ \cup\ \bigcup_{e\,\in\,\mathfrak{F}^1(P)} \text{relint}\ e\Big]$$

$$= \bigcup_{x \in \mathfrak{F}^0(P)} [\sigma \cap \{x\}] \quad \cup \quad \bigcup_{e \in \mathfrak{F}^1(P)} [\sigma \cap \operatorname{relint} e]$$

and the union is disjoint. A similar expression holds for $\sigma' \cap \operatorname{bd} P'$.

Now

$$\mathfrak{U}_\varepsilon(P, P'; \sigma, \sigma') = \mathfrak{U}_\varepsilon(P, P'; \sigma \cap \operatorname{bd} P, \sigma' \cap \operatorname{bd} P')$$

$$= \bigcup_{\substack{x \in \mathfrak{F}^0(P) \\ x' \in \mathfrak{F}^0(P')}} \mathfrak{U}_\varepsilon\Big(P, P'; \sigma \cap \{x\}, \sigma' \cap \{x'\}\Big) \quad \cup$$

$$\bigcup_{\substack{e \in \mathfrak{F}^1(P) \\ x' \in \mathfrak{F}^0(P')}} \mathfrak{U}_\varepsilon\Big(P, P'; \sigma \cap \operatorname{relint} e, \sigma' \cap \{x'\}\Big) \quad \cup$$

$$\bigcup_{\substack{x \in \mathfrak{F}^0(P) \\ e' \in \mathfrak{F}^1(P')}} \mathfrak{U}_\varepsilon\Big(P, P'; \sigma \cap \{x\}, \sigma' \cap \operatorname{relint} e'\Big) \quad \cup$$

$$\bigcup_{\substack{e \in \mathfrak{F}^1(P) \\ e' \in \mathfrak{F}^1(P')}} \mathfrak{U}_\varepsilon(P, P'; \sigma \cap \operatorname{relint} e, \sigma' \cap \operatorname{relint} e').$$

This union is not disjoint, and so

$$\mu\Big(\mathfrak{U}_\varepsilon(P, P'; \sigma, \sigma')\Big) \leq \sum_{\substack{x \in \mathfrak{F}^0(P) \\ x' \in \mathfrak{F}^0(P')}} \mu\Big(\mathfrak{U}_\varepsilon(P, P'; \sigma \cap \{x\}, \sigma' \cap \{x'\})\Big) \quad +$$

$$\sum_{\substack{e \in \mathfrak{F}^1(P) \\ x' \in \mathfrak{F}^0(P')}} \mu\Big(\mathfrak{U}_\varepsilon(P, P'; \sigma \cap \operatorname{relint} e, \sigma' \cap \{x'\})\Big) \quad +$$

$$\sum_{\substack{x \in \mathfrak{F}^0(P) \\ e' \in \mathfrak{F}^1(P')}} \mu\Big(\mathfrak{U}_\varepsilon(P, P'; \sigma \cap \{x\}, \sigma' \cap \operatorname{relint} e')\Big) \quad +$$

$$\sum_{\substack{e \in \mathfrak{F}^1(P) \\ e' \in \mathfrak{F}^1(P')}} \mu\Big(\mathfrak{U}_\varepsilon(P, P'; \sigma \cap \operatorname{relint} e, \sigma' \cap \operatorname{relint} e')\Big).$$

Apply the results of cases 1-4 to the four sums in the previous inequality to obtain

$$\mu\Big(\mathfrak{U}_\varepsilon(P,P';\sigma,\sigma')\Big) \le \sum_{\substack{x\,\in\,\mathcal{F}^0(P)\\ x'\,\in\,\mathcal{F}^0(P')}} (1-\cos\varepsilon)\ \Phi_0(P,\sigma\cap\{x\})\ \Phi_0(P',\sigma'\cap\{x'\}) \quad +$$

$$\sum_{\substack{e\,\in\,\mathcal{F}^1(P)\\ x'\,\in\,\mathcal{F}^0(P')}} (\sin\varepsilon)\ \Phi_1(P,\sigma\cap \operatorname{relint} e)\ \Phi_0(P',\{x'\}) \quad +$$

$$\sum_{\substack{x\,\in\,\mathcal{F}^0(P)\\ e'\,\in\,\mathcal{F}^1(P')}} (\sin\varepsilon)\ \Phi_0(P,\sigma\cap\{x\})\ \Phi_1(P',\sigma'\cap \operatorname{relint} e') \quad +$$

$$\sum_{\substack{e\,\in\,\mathcal{F}^1(P)\\ e'\,\in\,\mathcal{F}^1(P')}} (1-\cos\varepsilon)\ \Phi_1(P,\sigma\cap \operatorname{relint} e)\ \Phi_1(P',\sigma'\cap \operatorname{relint} e'). \tag{6.3}$$

For each $x \in \mathcal{F}^0(P)$ the definition of the curvature measure $\Phi_0(P,\,\cdot\,)$ implies that

$$\Phi_0(P,\sigma\cap\{x\}) = \mathcal{H}^0(\{x\}\cap\sigma)\ \mathcal{H}^1(x^*)$$

and so

$$\sum_{x\,\in\,\mathcal{F}^0(P)} \Phi_0(P,\sigma\cap\{x\}) = \Phi_0(P,\sigma). \tag{6.4}$$

Also, for each $e \in \mathcal{F}^1(P)$ we have

$$\Phi_1(P,\sigma\cap \operatorname{relint} e) = \mathcal{H}^1(e\cap\sigma)\ \mathcal{H}^0(e^*)$$

whereby

$$\sum_{e\,\in\,\mathcal{F}^1(P)} \Phi_1(P,\sigma\cap \operatorname{relint} e) = \Phi_1(P,\sigma). \tag{6.5}$$

Similar statements to (6.4) and (6.5) describe $\Phi_0(P',\sigma')$ and $\Phi_1(P',\sigma')$, respectively.

Therefore, when (6.4), (6.5) and their counterparts for P' are used in (6.3) we find that

$$\mu\Big(\mathfrak{U}_\varepsilon(P,P';\sigma,\sigma')\Big) \leq (1-\cos\varepsilon)\Big[\Phi_0(P,\sigma)\,\Phi_0(P',\sigma') + \Phi_1(P,\sigma)\,\Phi_1(P',\sigma')\Big] + (\sin\varepsilon)\Big[\Phi_0(P,\sigma)\,\Phi_1(P',\sigma') + \Phi_1(P,\sigma)\,\Phi_0(P',\sigma')\Big]. \tag{6.6}$$

□

Remark 6.8. Although the results of case 3 were proved directly, they are a direct consequence of case 2. This is because if the roles of P and P' and reversed in case 2, so that P' is fixed and P is mobile, then the motions that result are the inverses of those considered in case 3. Since the measure μ is invariant under inversion the results of case 3 follow immediately from case 2.

However, case 3 is considered in detail for two reasons. It leads nicely into case 4 and, more importantly, it will be modified in the next chapter to define a set of motions that will be used to obtain a lower estimate for the measure of near-collisions.

If the right-hand side of (6.6) is divided by ε and then a limit taken as ε approaches zero, we obtain the value

$$\Phi_0(P,\sigma)\,\Phi_1(P',\sigma') + \Phi_1(P,\sigma)\,\Phi_0(P',\sigma').$$

7. A LOWER ESTIMATE FOR THE MEASURE OF NEAR-COLLISIONS

In Theorem 6.7 we allowed all possible supporting great circles at a vertex of P or P'. In this section we will use certain subcollections of supporting great circles at vertices of P and P' to define a subset $\mathfrak{L}_\varepsilon(P,P';\sigma,\sigma')$ of $\mathfrak{U}_\varepsilon(P,P';\sigma,\sigma')$ so that, for all ε sufficiently small,

$$g \in \mathfrak{L}_\varepsilon(P,P';\sigma,\sigma') \;\Rightarrow\; 0 < \rho(P,gP') \le \varepsilon \quad \text{and} \quad \operatorname{card} R(P,gP') = 1 \tag{7.1}$$

and

$$\lim_{\varepsilon \to 0^+} \frac{\mu\Big(\mathfrak{L}_\varepsilon(P,P';\sigma,\sigma')\Big)}{\varepsilon} = \Phi_0(P,\sigma)\,\Phi_1(P',\sigma') + \Phi_1(P,\sigma)\,\Phi_0(P',\sigma') \tag{7.2}$$

Condition (7.1) will be used to prove $\mathfrak{L}_\varepsilon(P,P';\sigma,\sigma') \subset \mathfrak{C}_\varepsilon(P,P';\sigma,\sigma')$; the cardinality condition is motivated by Theorem 5.10. The right-hand side of (7.2) is the same as that noted at the end of Chapter 6. Furthermore, we want to define $\mathfrak{L}_\varepsilon(P,P';\sigma,\sigma')$ so that calculation of its measure is straightforward with the techniques used in the proof of Theorem 6.7.

A Subset of the Near-Collisions

Suppose x is a fixed, but arbitrary, vertex of P. Let $\overline{z}$ be as in case 3 of the proof of Theorem 6.7. We wish to restrict the choice of $\overline{z}$ so that x is the unique point in P closest to $\overline{x}$. The choice of $\overline{z}$ is somewhat involved because we want both (7.1) and (7.2) to hold. We begin by letting α be the measure of the interior angle of P at x. For $i = 1, 2$ define

$x_i :=$ a vertex of P adjacent to x,

$s_i := d(x, x_i) = \text{Length}(arc[x\,x_i])$,

$z_i :=$ the vertex of P^* polar to the edge $arc[x\,x_i]$ of P.

$p_i :=$ the midpoint of $arc[(-x)x_i]$, i.e. $(x_i - x)/\,|\,x_i - x\,|$, whereby $d(x, p_i) > \pi/2$,

$\delta_i = \delta_i(x, \varepsilon) :=$ Arcsin$[\tan\varepsilon\,\tan(s_i/2)]$ for $0 < \varepsilon < \pi/2$.

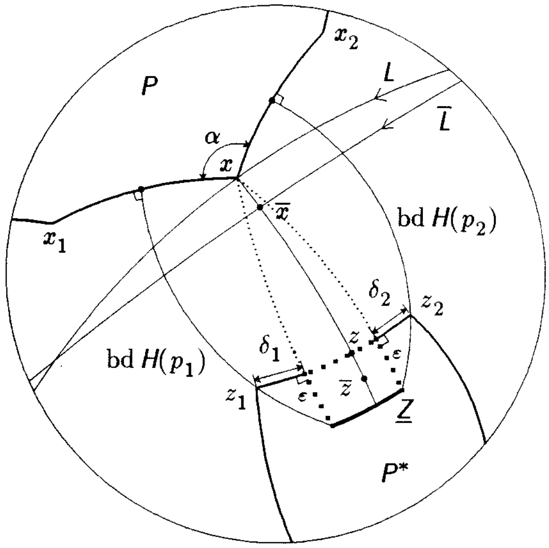

Figure 7.1. The setting for the definition of δ_i and $\underline{Z}$.

Note that $arc[z_1 z_2]$ is the edge of P^* polar to x, $\pi - \alpha$ is the length of $arc[z_1 z_2]$, $\pi - s_i$ is the measure of the interior angle of P^* at z_i. The great circle bd $H(p_i)$ contains both the midpoint of $arc[x\,x_i]$ and the point z_i; therefore it bisects the interior angle of P^* at z_i (note that $arc[xz_i]$ is perpendicular to $arc[z_1 z_2]$, see Figure 7.4). Refer to Figure 7.1 and to Definition 6.1.

Consider the Taylor series expansions Arcsin$(\xi) = \xi + \frac{1}{6}\xi^3 + \cdots$ and $\tan\varepsilon = \varepsilon + \frac{1}{3}\varepsilon^3 + \cdots$. By properties of power series (see [7; p. 559], for example), substitution yields

$$\delta_i = (\varepsilon + \tfrac{1}{3}\varepsilon^3 + \cdots)\tan\frac{s_i}{2} + \frac{1}{6}\Big[(\varepsilon + \tfrac{1}{3}\varepsilon^3 + \cdots)\tan\frac{s_i}{2}\Big]^3 + \cdots$$

$$= \left(\tan\frac{s_i}{2}\right)\varepsilon + \left[\frac{1}{3}\tan\frac{s_i}{2} + \frac{1}{6}\left(\tan\frac{s_i}{2}\right)^3\right]\varepsilon^3 + \cdots$$

$$= O(\varepsilon).$$

Note that $\delta_i \to 0$ as $\varepsilon \to 0$.

The definition of δ_i comes from consideration of the spherical right triangle with $(\pi - s_i)/2$ as one angle, ε the opposite leg, and δ_i the other leg, see Figure 7.2a.

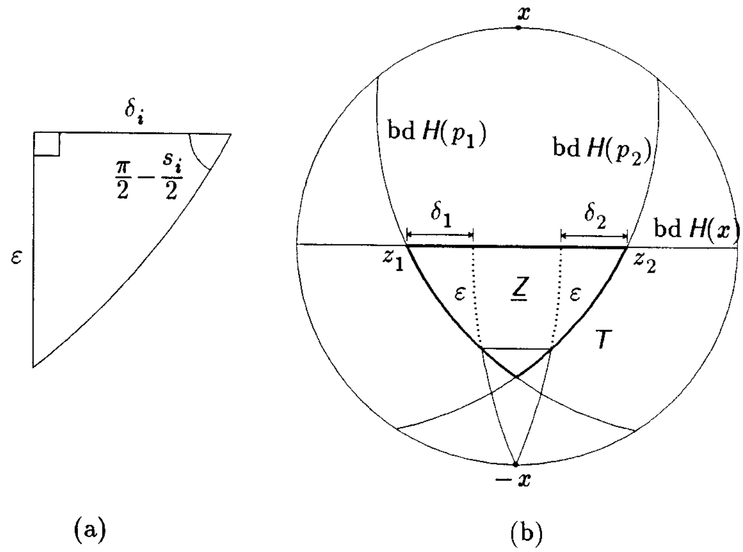

Figure 7.2. (a) The quantity δ_i, (b) the set $\underline{Z}$ and the triangle T.

If spherical trigonometry is applied to this triangle, then

$$\sin\delta_i = \tan\varepsilon\,\cot\left(\frac{\pi}{2} - \frac{s_i}{2}\right) = \tan\varepsilon\,\tan\frac{s_i}{2}. \tag{7.3}$$

Rather than consider any supporting great circle L to P at x, as in cases 1 and **3** of Theorem 6.7, we now require that L supports P at x and its pole z satisfies $d(z, z_i) > \delta_i$ for $i = 1, 2$. By making this change to the definition of $\overline{Z}$ given in case **3** of Theorem 6.7, we define

$$\underline{Z} := \bigcup_{z \in *} \left\{ \overline{z} \in arc[(-x)z] \colon 0 < d(z, \overline{z}) \le \varepsilon \right\}, \tag{7.4}$$

where

$$* = \{p \in arc[z_1 z_2] \colon d(p, z_i) > \delta_i \text{ for } i = 1, 2\}.$$

See Figures 7.1, 7.2b and 7.3. Note that $\underline{Z}$ depends on both x and ε and is a subset of $\overline{Z}$.

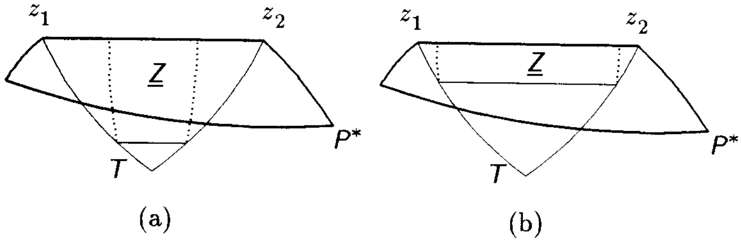

Figure 7.3. (a) $\underline{Z} \not\subset P^*$ (b) $\underline{Z} \subset \operatorname{int} P^*$.

Remark 7.1. Because of the definition of δ_i, each point $\overline{z}$ in $\underline{Z}$ will not lie on the angle bisector at z_i; therefore the great circle $\overline{L}$ cannot be equidistant from two adjacent vertices of P. Later we show that if ε is "sufficiently small," then for each such $\overline{L}$ the set $R\left(P, \overline{L}\right)$ consists a single element. This is desirable because we wish to construct a subset of the near-collisions so that the distance is realized by a unique point-pair, refer to (7.1).

Observe that $\{x, x_1, x_2\}$ is linearly independent in $\mathbb{R}^3$ and so $\{x, p_1, p_2\}$ is as well. Therefore $T := H(x) \cap H(p_1) \cap H(p_2)$ is a triangle with $arc[z_1 z_2]$ as one of its edges, see Figures 7.2b and 7.3. Moreover, T is contained in the triangle $H(x) \cap H(x_1) \cap H(x_2)$. Each point in the interior of T is closer to the extended edge of P^* that contains $arc[z_1 z_2]$ than to the extended adjacent edges of P^*, i.e.

$$q \in \operatorname{int} T \;\Rightarrow\; \rho\Big(q, \operatorname{bd} H(x)\Big) < \rho\Big(q, \operatorname{bd} H(x_i)\Big) \quad \text{for } i = 1, 2. \tag{7.5}$$

We summarize properties of $\underline{Z}$ and begin consideration of what it means for ε to be "sufficiently small" in the next theorem.

Theorem 7.2. *Suppose* $\underline{Z}$ *is given by (7.4). There exists* $\varepsilon(x)$ *with* $0 < \varepsilon(x) < \pi/2$ *such that if* $0 < \varepsilon < \varepsilon(x)$, *then the following hold.*

(a) $0 < \delta_i < (\pi - \alpha)/2$ *for* $i = 1, 2$.

(b) $\underline{Z} \subset \operatorname{int} P^*$.

(c) *If* $\overline{z} \in \underline{Z}$, *then* $\rho\big(\overline{z}, \operatorname{bd} H(x)\big) < \rho\big(\overline{z}, \operatorname{bd} H(\widehat{x})\big)$ *for any vertex* $\widehat{x} \neq x$ *of* P.

(d) *If* $\overline{Z} = \bigcup_{z \in arc[z_1 z_2]} \big\{\overline{z} \in arc[(-x)z] \colon 0 < d(z, \overline{z}) \leq \varepsilon\big\}$, then $\operatorname{Area}(\underline{Z}) = \operatorname{Area}(\overline{Z}) - O(\varepsilon^2)$.

Proof. (a) In (7.3) replace δ_i by $(\pi - \alpha)/2$. This motivates the definition

$$\varepsilon_1(x) := \min\Big\{\operatorname{Arctan}\Big(\sin\tfrac{\pi - \alpha}{2} \cot\tfrac{s_i}{2}\Big) \colon i = 1, 2\Big\}.$$

Then $0 < \varepsilon_1(x) < \pi/2$ and if $0 < \varepsilon < \varepsilon_1(x)$, then $0 < \delta_i < (\pi - \alpha)/2 < \pi/2$ for $i = 1, 2$.

(b) There exists $\varepsilon_2(x)$ with $0 < \varepsilon_2(x) < \varepsilon_1(x)$ such that $0 < \varepsilon < \varepsilon_2(x)$ implies that z_1, z_2 are the only vertices of P^* contained in the disc $D(x, \pi/2 + \varepsilon)$. Indeed, for these ε we have

$$\begin{aligned} D(x, \pi/2 + \varepsilon) \cap \operatorname{int} P^* &= D(x, \pi/2 + \varepsilon) \cap \operatorname{int}[H(x) \cap H(x_1) \cap H(x_2)] \\ &\supset D(x, \pi/2 + \varepsilon) \cap \operatorname{int} T \\ &\supset \underline{Z}. \end{aligned}$$

(c) The dependence of $\underline{Z}$ upon ε is indicated by writing $\underline{Z}(\varepsilon)$. We assume that no such $\varepsilon(x)$ exits. This is equivalent to the following:

Suppose for each positive integer $J \geq J_0$ we have $0 < 1/J < \varepsilon_2(x)$ and that there exist $j > J$ and $q_j \in \underline{Z}(1/j)$ such that $\rho\left(q_j, \operatorname{bd} H(x)\right) \geq \rho\left(q_j, \operatorname{bd} H(p)\right)$ for some vertex $p \neq x$ of P.

Part (b) implies $\underline{Z}(1/j) \subset \operatorname{int} P^*$ for each $j > J$. Since P^* is compact there is a subsequence $\left\{q_{j_k}\right\}$ and a point $q \in P^*$ such that $q_{j_k} \to q$ as $k \to \infty$. Note that

$$0 \leq \rho\left(q_{j_k}, \operatorname{bd} H(p)\right) \leq \rho\left(q_{j_k}, \operatorname{bd} H(x)\right) \leq 1/j_k \to 0 \quad \text{as} \quad k \to \infty.$$

Therefore $\rho\left(q, \operatorname{bd} H(x)\right) = 0 = \rho\left(q, \operatorname{bd} H(p)\right)$ by the continuity of ρ. Thus q belongs to both of the edges $P^* \cap \operatorname{bd} H(x) = arc[z_1 z_2]$ and $P^* \cap \operatorname{bd} H(p)$ of P^*. This can only happen if $P^* \cap \operatorname{bd} H(p)$ is an edge of P^* adjacent to $arc[z_1 z_2]$, i.e. $p = x_1$ or $p = x_2$.

If $p = x_1$, then our assumption implies $\rho\left(q_{j_k}, \operatorname{bd} H(x)\right) \geq \rho\left(q_{j_k}, \operatorname{bd} H(x_1)\right)$ for each k. This contradicts (7.5), which implies $\rho\left(q_{j_k}, \operatorname{bd} H(x)\right) < \rho\left(q_{j_k}, \operatorname{bd} H(x_i)\right)$ for $i = 1, 2$ and each k. The argument is the same if $p = x_2$. Because of these contradictions we conclude that there exists $\varepsilon(x)$ with $0 < \varepsilon(x) < \varepsilon_2(x)$ such that part (c) of the theorem holds.

(d) $\operatorname{Area}(\underline{Z}) = (\sin\varepsilon)[(\pi - \alpha) - (\delta_1 + \delta_2)] = (\sin\varepsilon)(\pi - \alpha) - (\sin\varepsilon)(\delta_1 + \delta_2) = \operatorname{Area}(\overline{Z}) - (\sin\varepsilon)\, O(\varepsilon) = \operatorname{Area}(\overline{Z}) - O(\varepsilon^2)$. □

Corollary 7.3. *Suppose* $0 < \varepsilon < \varepsilon(x)$ *and* $\underline{Z}$ *is given by (7.4). Then* $\overline{z} \in \underline{Z}$ *implies* $0 < \rho\left(P, \overline{L}\right) \leq \varepsilon$ *and* $R\left(P, \overline{L}\right) = \left\{(x, \overline{x})\right\}$.

Proof. Let $\overline{z} \in \underline{Z}$ and suppose $\overline{L}$ has pole $\overline{z}$. By the proof of part (b) of Theorem 7.2 we know that $\underline{Z} \subset D(x, \pi/2 + \varepsilon) \cap \operatorname{int} P^*$. Since $\overline{z} \in \operatorname{int} P^*$, the distance from $\overline{z}$ to P is greater than $\pi/2$. Therefore the distance from $\overline{L}$ to P is positive, i.e. $0 < \rho\left(P, \overline{L}\right)$. Note that this implies $P \cap \overline{L} = \varnothing$, hence $R\left(P, \overline{L}\right) \neq \varnothing$. Since $d(x, \overline{x}) = d(z, \overline{z}) \leq \varepsilon$, we have $\rho\left(P, \overline{L}\right) \leq \varepsilon$.

If $(p, q) \in R\left(P, \overline{L}\right)$, then p must be a vertex of P. Therefore

$$\begin{aligned}
\rho\left(P, \overline{L}\right) &= \min\left\{d(p, q)\colon\ p \in \operatorname{bd} P,\ q \in \overline{L}\right\} \\
&= \min\left\{d(\widehat{x}, q)\colon\ \widehat{x} \text{ is a vertex of } P,\ q \in \overline{L}\right\} \\
&= \min_{\widehat{x} \in \mathcal{F}^0(P)} \left[\min\left\{d(\widehat{x}, q)\colon\ q \in \overline{L}\right\}\right] \\
&= \min_{\widehat{x} \in \mathcal{F}^0(P)} \rho\left(\widehat{x}, \overline{L}\right).
\end{aligned}$$

Note that $\rho\left(\overline{z}, \operatorname{bd} H(\widehat{x})\right) = \rho\left(\widehat{x}, \overline{L}\right)$. If part (c) of Theorem 7.2 is used, then $\rho\left(\overline{z}, \operatorname{bd} H(\widehat{x})\right) > \rho\left(\overline{z}, \operatorname{bd} H(x)\right)$ if $\widehat{x} \neq x$. Therefore $\rho\left(\widehat{x}, \overline{L}\right) > \rho\left(x, \overline{L}\right)$ if $\widehat{x} \neq x$ and so x is the only vertex for which $\rho\left(P, \overline{L}\right) = \rho\left(x, \overline{L}\right)$. Moreover, $\overline{x}$ is the only point in $\overline{L}$ for which $\rho\left(x, \overline{L}\right) = d(x, \overline{x})$, i.e. $\overline{x} = N\left(\overline{L}, x\right)$. This proves that $R\left(P, \overline{L}\right) = \left\{(x, \overline{x})\right\}$. □

Consider the same procedure applied to a vertex x' of P'. If $\alpha', x_i', s_i', z_i'$, $p_i', \delta_i' = \delta_i'(x', \varepsilon)$, T' are defined analogously to $\alpha, x_i, s_i, z_i, p_i, \delta_i$, T, respectively, then we require that L' supports P' at x' and its pole z' satisfies $d(-z', z_i') > \delta_i'$ for $i = 1, 2$. Define

$$\underline{Z}' := \bigcup_{z' \in *'} \left\{\overline{z} \in \mathit{arc}[(-x')(-z')]\colon\ 0 < d(-z', \overline{z}) \leq \varepsilon\right\}, \tag{7.6}$$

where

$$*' := \{-p'\colon\ p' \in \mathit{arc}[z_1' z_2'],\ d(p', z_i') > \delta_i' \text{ for } i = 1, 2\}.$$

This set $\underline{Z}'$ depends upon both x' and ε. The following analogs to Theorem 7.2 and Corollary 7.3 are immediate.

Theorem 7.4. *Suppose $\underline{Z}'$ is given by (7.6). There exists $\varepsilon(x')$ with $0 < \varepsilon(x') < \pi/2$ such that if $0 < \varepsilon < \varepsilon(x')$, then the following hold.*

(a) $0 < \delta_i' < (\pi - \alpha')/2$ *for* $i = 1, 2$.

(b) $\underline{Z}' \subset \operatorname{int}(P')^*$.

(c) *If* $\overline{z} \in \underline{Z}'$, *then* $\rho\left(\overline{z}, \operatorname{bd} H(x')\right) < \rho\left(\overline{z}, \operatorname{bd} H(\widehat{x})\right)$ *for any vertex* $\widehat{x} \neq x'$ *of* P'.

(d) *If* $\overline{Z}' = \bigcup_{-z' \in \mathit{arc}[z_1' z_2']} \left\{\overline{z} \in \mathit{arc}[(-x')(-z')]:\ 0 < d(-z', \overline{z}) \leq \varepsilon\right\}$, *then* $\operatorname{Area}(\underline{Z}') = \operatorname{Area}(\overline{Z}') - O(\varepsilon^2)$.

Corollary 7.5. *Suppose $0 < \varepsilon < \varepsilon(x')$ and $\underline{Z}'$ is given by (7.6). Then $\overline{z} \in \underline{Z}'$ implies $0 < \rho\left(P', \overline{L}\right) \leq \varepsilon$ and $R\left(P', \overline{L}\right) = \left\{(x', \overline{x})\right\}$.*

Since the vertices x, x' were arbitrary, we can use the preceding theorems to define $\varepsilon(P) := \min\{\varepsilon(x):\ x$ is a vertex of $P\}$ and $\varepsilon(P') := \min\{\varepsilon(x'):\ x'$ is a vertex of $P'\}$. From now on the phrase "ε sufficiently small" will mean $0 < \varepsilon < \min\{\varepsilon(P), \varepsilon(P')\}$.

Definition 7.6. The set of motions $\mathfrak{L}_\varepsilon(P, P'; \sigma, \sigma')$ is the subset of $\mathfrak{U}_\varepsilon(P, P'; \sigma, \sigma')$ defined by the following steps.

1. If x is a vertex of P, then define $\mathfrak{L}_\varepsilon\left(P, P'; \{x\}, \sigma'\right) := \left\{g(x, L, \eta; x', L'):\right.$ $x' \in \sigma',\ d(z, z_i) > \delta_i$ for all $i,\ 0 < \eta \leq \varepsilon\left.\right\}$.

2. Define $\mathfrak{L}_\varepsilon\big(P, P'; \sigma \cap \mathfrak{F}^0(P), \sigma'\big) := \bigcup_{x \in \sigma \cap \mathfrak{F}^0(P)} \mathfrak{L}_\varepsilon\big(P, P'; \{x\}, \sigma'\big)$.

3. If x' is a vertex of P', then define $\mathfrak{L}_\varepsilon\big(P', P; \{x'\}, \sigma\big) := \big\{g(x', L', \eta; x, L)$: $x \in \sigma,\ d(-z', z_i') > \delta_i'$ for all i, $0 < \eta \leq \varepsilon\big\}$.

4. Define $\mathfrak{L}_\varepsilon\big(P', P; \sigma' \cap \mathfrak{F}^0(P'), \sigma\big) := \bigcup_{x' \in \sigma' \cap \mathfrak{F}^0(P')} \mathfrak{L}_\varepsilon\big(P', P; \{x'\}, \sigma\big)$.

5. Define $\mathfrak{L}_\varepsilon(P, P'; \sigma, \sigma') := \mathfrak{L}_\varepsilon\big(P, P'; \sigma \cap \mathfrak{F}^0(P), \sigma'\big) \cup \mathfrak{L}_\varepsilon\big(P', P; \sigma' \cap \mathfrak{F}^0(P'), \sigma \setminus \mathfrak{F}^0(P)\big)^{-1}$.

The main results of this section are the next theorem and its corollary. The theorem shows that $\mathfrak{L}_\varepsilon(P, P'; \sigma, \sigma')$ satisfies the properties given in (7.1).

Theorem 7.7. *Assume that ε is sufficiently small and $0 < \eta \leq \varepsilon$. If $g = g(x, L, \eta; x', L')$ or $g = g(x', L', \eta; x, L)^{-1}$ is a motion in $\mathfrak{L}_\varepsilon(P, P'; \sigma, \sigma')$, then $0 < \rho(P, gP') \leq \varepsilon$ and $R(P, gP') = \big\{(x, gx')\big\}$.*

Proof. Assume $g = g(x, L, \eta; x', L') \in \mathfrak{L}_\varepsilon\big(P, P'; \sigma \cap \mathfrak{F}^0(P), \sigma'\big)$. Here $g = [\![\overline{x}, y, \overline{z}]\!]\,[\![x', y', z']\!]^{-1}$ and $gz' = \overline{z}$ lies in the set $\underline{Z}$ of (7.4). By Corollary 7.3 we know that $R(P, gL') = \big\{(x, gx')\big\}$. Define $H := \{p \in \mathbb{S}^2 \colon d(p, gz') \leq \pi/2\}$. Then $gL' = \operatorname{bd} H$ and so $R(P, H) = R(P, gL') = \big\{(x, gx')\big\}$. Hence $gx' \in gP' \subset H$ implies $R(P, gP') = \big\{(x, gx')\big\}$. Also note that $\eta = d(x, gx') = \rho(P, gP')$ and therefore $0 < \rho(P, gP') \leq \varepsilon$.

Assume $g^{-1} = g(x', L', \eta; x, L) \in \mathfrak{L}_\varepsilon\big(P', P; \sigma' \cap \mathfrak{F}^0(P'), \sigma \setminus \mathfrak{F}^0(P)\big)$. Here $g^{-1} = [\![\overline{x}, y', -\overline{z}]\!]\,[\![x, y, z]\!]^{-1}$ and $g^{-1}(-z) = \overline{z}$ lies in the set $\underline{Z}'$ of (7.6). If Corollary 7.5 is used in an argument similar to that in the preceding paragraph, then we get $R(g^{-1}P, P') = \{(g^{-1}x, x')\}$. Hence $R(P, gP') = \{(x, gx')\}$. Also as before, we have $\eta = d(g^{-1}x, x') = d(x, gx')$ which implies $0 < \rho(P, gP') \leq \varepsilon$. □

Corollary 7.8. $\mathfrak{L}_\varepsilon(P, P'; \sigma, \sigma') \subset \mathfrak{C}_\varepsilon(P, P'; \sigma, \sigma')$ *for all* ε *sufficiently small.*

Proof. This follows directly from Theorem 7.7 and the meaning of $\mathfrak{C}_\varepsilon(P, P'; \sigma, \sigma')$ given in Definition 5.7. □

There remains the problem of establishing (7.2). This will be accomplished in the next section, but the following properties of $\mathfrak{L}_\varepsilon(P, P'; \sigma, \sigma')$ will be essential.

Theorem 7.9. *For* $i = 1, 2$, *suppose* $x_i \in \sigma \cap \mathfrak{F}^0(P)$ *and* $x_i' \in \sigma' \cap \mathfrak{F}^0(P')$. *Then the following hold for all* ε *sufficiently small.*

(a) $x_1 \neq x_2 \;\Rightarrow\; \mathfrak{L}_\varepsilon(P, P'; \{x_1\}, \sigma') \cap \mathfrak{L}_\varepsilon(P, P'; \{x_2\}, \sigma') = \varnothing$.

(b) $x_1' \neq x_2' \;\Rightarrow\; \mathfrak{L}_\varepsilon(P', P; \{x_1'\}, \sigma) \cap \mathfrak{L}_\varepsilon(P', P; \{x_2'\}, \sigma) = \varnothing$.

(c) *The sets* $\mathfrak{L}_\varepsilon\big(P, P'; \sigma \cap \mathfrak{F}^0(P), \sigma' \cap \mathfrak{F}^0(P')\big)$, $\mathfrak{L}_\varepsilon\big(P, P'; \sigma \cap \mathfrak{F}^0(P), \sigma' \setminus \mathfrak{F}^0(P')\big)$ *and* $\mathfrak{L}_\varepsilon\big(P', P; \sigma' \cap \mathfrak{F}^0(P'), \sigma \setminus \mathfrak{F}^0(P)\big)^{-1}$ *are pairwise disjoint.*

Proof. (a) By Theorem 7.7 we know that $g_i \in \mathfrak{L}_\varepsilon\big(P, P'; \{x_i\}, \sigma'\big) \Rightarrow R(P, g_iP') = \{(x_i, g_iq_i')\}$ for some $q_i' \in \sigma'$. Now $g_1 = g_2 \Rightarrow R(P, g_1P') = R(P, g_2P') \Rightarrow x_1 = x_2$.

(b) In this case $g_i \in \mathfrak{L}_\varepsilon\big(P', P; \{x_i'\}, \sigma\big) \Rightarrow R(g_iP, P') = \{(g_iq_i, x_i')\}$ for some $q_i \in \sigma$. Then $g_1 = g_2 \Rightarrow R(g_1P, P') = R(g_2P, P') \Rightarrow x_1' = x_2'$.

(c) Suppose

$$g_1 \in \mathfrak{L}_\varepsilon\Big(P, P'; \sigma \cap \mathfrak{F}^0(P), \sigma' \cap \mathfrak{F}^0(P')\Big),$$
$$g_2 \in \mathfrak{L}_\varepsilon\Big(P, P'; \sigma \cap \mathfrak{F}^0(P), \sigma' \setminus \mathfrak{F}^0(P')\Big),$$
$$g_3 \in \mathfrak{L}_\varepsilon\Big(P', P; \sigma' \cap \mathfrak{F}^0(P'), \sigma \setminus \mathfrak{F}^0(P)\Big)^{-1}.$$

Then from Theorem 7.7 we have

$$R(P, g_1 P') = \Big\{(x_1, g_1 x_1')\Big\} \quad \text{for some } x_1 \in \sigma \cap \mathfrak{F}^0(P) \text{ and } x_1' \in \sigma' \cap \mathfrak{F}^0(P'),$$
$$R(P, g_2 P') = \Big\{(x_2, g_2 x_2')\Big\} \quad \text{for some } x_2 \in \sigma \cap \mathfrak{F}^0(P) \text{ and } x_2' \in \sigma' \setminus \mathfrak{F}^0(P'),$$
$$R(P, g_3 P') = \Big\{(x_3, g_3 x_3')\Big\} \quad \text{for some } x_3 \in \sigma \setminus \mathfrak{F}^0(P) \text{ and } x_3' \in \sigma' \cap \mathfrak{F}^0(P').$$

Now $g_1 = g_2 \Rightarrow x_1' = x_2'$, $g_2 = g_3 \Rightarrow x_2' = x_3'$, $g_3 = g_1 \Rightarrow x_3 = x_1$. But $x_1' \neq x_2'$, $x_2' \neq x_3'$, and $x_3 \neq x_1$. Therefore the g_i are distinct and the sets are pairwise disjoint. □

Theorem 7.10. $\mathfrak{L}_\varepsilon(P, P'; \sigma, \sigma') = \mathfrak{L}_\varepsilon\Big(P, P'; \sigma \cap \mathfrak{F}^0(P), \sigma' \cap \mathfrak{F}^0(P')\Big) \cup \mathfrak{L}_\varepsilon\Big(P, P'; \sigma \cap \mathfrak{F}^0(P), \sigma' \setminus \mathfrak{F}^0(P')\Big) \cup \mathfrak{L}_\varepsilon\Big(P', P; \sigma' \cap \mathfrak{F}^0(P'), \sigma \setminus \mathfrak{F}^0(P)\Big)^{-1}$ *and the union is disjoint for each* ε *sufficiently small.*

Proof. The set equality follows from the fact that $\mathfrak{L}_\varepsilon\Big(P, P'; \sigma \cap \mathfrak{F}^0(P), \sigma'\Big)$ is the union

$$\mathfrak{L}_\varepsilon\Big(P, P'; \sigma \cap \mathfrak{F}^0(P), \sigma' \cap \mathfrak{F}^0(P')\Big) \cup \mathfrak{L}_\varepsilon\Big(P, P'; \sigma \cap \mathfrak{F}^0(P), \sigma' \setminus \mathfrak{F}^0(P')\Big)$$

and step 5 of Definition 7.6. From Theorem 7.9(c), the union in the statement of the theorem is disjoint. □

Theorem 7.11. *Suppose x and x' represent vertices in $\sigma \cap \mathfrak{F}^0(P)$ and $\sigma' \cap \mathfrak{F}^0(P')$, respectively. Then for each ε sufficiently small*

(a) $\mathfrak{L}_\varepsilon\big(P, P'; \sigma \cap \mathfrak{F}^0(P), \sigma' \cap \mathfrak{F}^0(P')\big) = \bigcup_x \bigcup_{x'} \mathfrak{L}_\varepsilon\big(P, P'; \{x\}, \{x'\}\big),$

(b) $\mathfrak{L}_\varepsilon\big(P, P'; \sigma \cap \mathfrak{F}^0(P), \sigma' \setminus \mathfrak{F}^0(P')\big) = \bigcup_x \mathfrak{L}_\varepsilon\big(P, P'; \{x\}, \sigma' \setminus \mathfrak{F}^0(P')\big),$

(c) $\mathfrak{L}_\varepsilon\big(P', P; \sigma' \cap \mathfrak{F}^0(P'), \sigma \setminus \mathfrak{F}^0(P)\big) = \bigcup_{x'} \mathfrak{L}_\varepsilon\big(P', P; \{x'\}, \sigma \setminus \mathfrak{F}^0(P)\big),$

and all unions are disjoint.

Proof. Clearly, each set equality is true. From parts (a) and (b) of Theorem 7.9, it follows that all unions are disjoint. □

A Lower Estimate for the Measure of Near-Collisions

We now show that (7.2) holds by applying the techniques of Theorem 6.7 to compute the measure of the set $\mathfrak{L}_\varepsilon(P, P'; \sigma, \sigma')$. By Corollary 7.8, this will give a lower estimate for the measure of near-collisions for all ε sufficiently small.

Theorem 7.12. *For each ε sufficiently small the following hold.*

(a) $\mu\Big(\mathfrak{L}_\varepsilon(P, P'; \sigma \cap \mathfrak{F}^0(P), \sigma' \cap \mathfrak{F}^0(P'))\Big) = (1 - \cos\varepsilon)\big[\Phi_0(P, \sigma) - O(\varepsilon)\big] \cdot \Phi_0(P', \sigma').$

(b) $\mu\Big(\mathfrak{L}_\varepsilon(P, P'; \sigma \cap \mathfrak{F}^0(P), \sigma' \setminus \mathfrak{F}^0(P'))\Big) = (\sin\varepsilon)\big[\Phi_0(P, \sigma) - O(\varepsilon)\big] \cdot \Phi_1(P', \sigma').$

(c) $\mu\Big(\mathfrak{L}_\varepsilon(P', P; \sigma' \cap \mathfrak{F}^0(P'), \sigma \setminus \mathfrak{F}^0(P))\Big) = \Phi_1(P, \sigma) \cdot (\sin\varepsilon)\big[\Phi_0(P', \sigma') - O(\varepsilon)\big].$

Proof. Suppose x, x' are vertices of P', P with interior angles α', α, respectively.

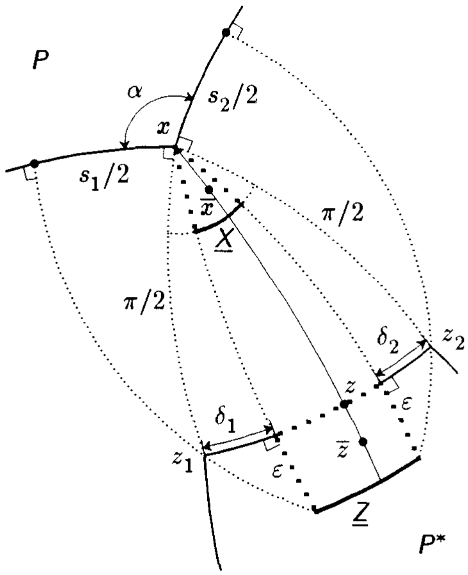

Figure 7.4. Zoom of Figure 7.1.

(a) Let $arc[z_1z_2]$ be the edge of P^* polar to x, see Figure 7.4. For each ε sufficiently small we use the methods of case 1 of Theorem 6.7 to define

$$\underline{X} := \bigcup_{z \in *} \left\{ \bar{x} \in arc[xz] \colon 0 < d(x, \bar{x}) \leq \varepsilon \right\},$$

where

$$* = \{ p \in arc[z_1z_2] \colon d(p, z_i) > \delta_i \text{ for } i = 1, 2 \}.$$

Here $\underline{X}$ is a proper subset of $B_\varepsilon(P, \{x\})$, which was used in case 1 of Theorem 6.7, see Figure 7.5. Also,

$$\begin{aligned} \mu\Big(\mathfrak{L}_\varepsilon(P, P'; \{x\}, \{x'\})\Big) &= (\pi - \alpha') \cdot \text{Area}(\underline{X}) \\ &= (\pi - \alpha') \cdot (1 - \cos\varepsilon) \Big[(\pi - \alpha) - (\delta_1 + \delta_2) \Big]. \end{aligned}$$

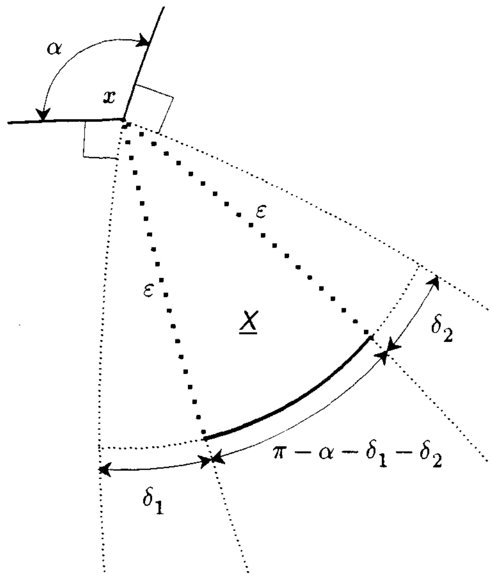

Figure 7.5. Detail of the set $\underline{X}$, c.f. Figure 7.4.

Using the disjoint union in Theorem 7.11(a) we have

$$\begin{aligned}
\mu\Big(\mathfrak{L}_\varepsilon(P,P';\sigma\cap\mathfrak{F}^0(P),\sigma'\cap\mathfrak{F}^0(P'))\Big) &= \sum_x\sum_{x'}\mu\Big(\mathfrak{L}_\varepsilon(P,P';\{x\},\{x'\})\Big)\\
&=(1-\cos\varepsilon)\left(\sum_x[(\pi-\alpha)-(\delta_1+\delta_2)]\right)\left[\sum_{x'}(\pi-\alpha')\right]\\
&=(1-\cos\varepsilon)\left[\sum_x(\pi-\alpha)-\sum_x(\delta_1+\delta_2)\right]\left[\sum_{x'}(\pi-\alpha')\right]\\
&=(1-\cos\varepsilon)\left[\Phi_0(P,\sigma)-O(\varepsilon)\right]\cdot\Phi_0(P',\sigma'),
\end{aligned}$$

where $\sum_x (\delta_1+\delta_2)=O(\varepsilon)$.

(b) Let $\underline{Z}$ be given by (7.4), see Figure 7.6. By the technique used in case **3** of Theorem 6.7 we have

$$\begin{aligned}
\mu\Big(\mathfrak{L}_\varepsilon(P,P';\{x\},\sigma'\setminus\mathfrak{F}^0(P'))\Big) &= \text{Length}\Big(\sigma'\setminus\mathfrak{F}^0(P')\Big)\cdot\text{Area}(\underline{Z})\\
&=\text{Length}(\sigma')\cdot\text{Area}(\underline{Z})
\end{aligned}$$

$$= \Phi_1(P', \sigma') \cdot (\sin \varepsilon) \left[(\pi - \alpha) - (\delta_1 + \delta_2) \right].$$

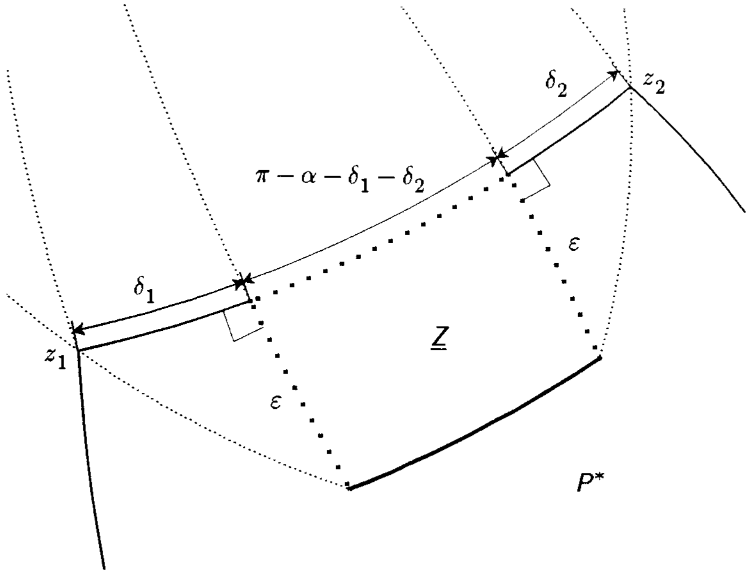

Figure 7.6. Detail of the set $\underline{Z}$, c.f. Figure 7.4.

Use the disjoint union in Theorem 7.11(b) to obtain

$$\mu\Big(\mathfrak{L}_\varepsilon(P, P'; \sigma \cap \mathfrak{F}^0(P), \sigma' \setminus \mathfrak{F}^0(P'))\Big) = \sum_x \mu\Big(\mathfrak{L}_\varepsilon(P, P'; \{x\}, \sigma' \setminus \mathfrak{F}^0(P'))\Big)$$

$$= (\sin \varepsilon) \left[\sum_x (\pi - \alpha) - \sum_x (\delta_1 + \delta_2) \right] \cdot \Phi_1(P', \sigma')$$

$$= (\sin \varepsilon) \left[\Phi_0(P, \sigma) - O(\varepsilon) \right] \cdot \Phi_1(P', \sigma').$$

(c) Suppose $arc[z_1' z_2']$ is the edge of $(P')^*$ polar to x' and assume $\underline{Z}'$ is given by (7.6). Use the method of (b) to get

$$\mu\Big(\mathfrak{L}_\varepsilon(P', P; \{x'\}, \sigma \setminus \mathfrak{F}^0(P))\Big) = \text{Length}\Big(\sigma \setminus \mathfrak{F}^0(P')\Big) \cdot \text{Area}(\underline{Z}')$$

$$= \text{Length}(\sigma) \cdot \text{Area}(\underline{Z}')$$

$$= \Phi_1(P, \sigma) \cdot (\sin \varepsilon) \left[(\pi - \alpha') - (\delta_1' + \delta_2') \right].$$

Then, by the disjoint union in Theorem 7.11(c), we have

$$\begin{aligned}\mu\Big(\mathfrak{L}_\varepsilon(P',P;\sigma'\cap\mathfrak{F}^0(P'),\sigma\setminus\mathfrak{F}^0(P))\Big) &= \sum_{x'}\mu\Big(\mathfrak{L}_\varepsilon(P',P;\{x'\},\sigma\setminus\mathfrak{F}^0(P))\Big)\\ &= \Phi_1(P,\sigma)\cdot(\sin\varepsilon)\Big[\sum_{x'}(\pi-\alpha')-\sum_{x'}(\delta_1'+\delta_2')\Big]\\ &= \Phi_1(P,\sigma)\cdot(\sin\varepsilon)\Big[\Phi_0(P',\sigma')-O(\varepsilon)\Big],\end{aligned}$$

where $\sum_{x'}(\delta_1'+\delta_2')=O(\varepsilon)$. □

Corollary 7.13. *If $\sigma,\sigma'\in\mathfrak{B}(\mathbb{S}^2)$ and ε is sufficiently small, then*

$$\begin{aligned}\mu\Big(\mathfrak{L}_\varepsilon(P,P';\sigma,\sigma')\Big) = &\Big[\Phi_0(P,\sigma)-O(\varepsilon)\Big]\Big[\Phi_0(P',\sigma')(1-\cos\varepsilon)+\Phi_1(P',\sigma')(\sin\varepsilon)\Big]\\ &+\Phi_1(P,\sigma)\cdot(\sin\varepsilon)\Big[\Phi_0(P',\sigma')-O(\varepsilon)\Big].\end{aligned}$$

Proof. From the invariance of μ under inversion we have

$$\mu\Big(\mathfrak{L}_\varepsilon(P',P;\sigma'\cap\mathfrak{F}^0(P'),\sigma\setminus\mathfrak{F}^0(P))\Big)=\mu\Big(\mathfrak{L}_\varepsilon(P',P;\sigma'\cap\mathfrak{F}^0(P'),\sigma\setminus\mathfrak{F}^0(P))^{-1}\Big).$$

Now for all ε sufficiently small, Theorem 7.10 and Theorem 7.12 imply that the measure of $\mathfrak{L}_\varepsilon(P,P';\sigma,\sigma')$ is as claimed. □

As a result of Corollary 7.13, the limit result in (7.2) holds.

8. COLLISION PROBABILITIES

In this chapter we combine the results of Chapters 6 and 7 and use a "pinching" argument to prove our main result.

Theorem 8.1. *The limit*

$$\overline{\mu}(P,P';\sigma,\sigma') := \lim_{\varepsilon\to 0^+} \frac{\mu\Big(\mathfrak{C}_\varepsilon(P,P';\sigma,\sigma')\Big)}{\varepsilon}$$

exists and

$$\overline{\mu}(P,P';\sigma,\sigma') = \Phi_0(P,\sigma)\,\Phi_1(P',\sigma') + \Phi_1(P,\sigma)\,\Phi_0(P',\sigma'). \tag{8.1}$$

Proof. Combine Corollary 7.8 and Theorem 6.5 to get

$$\mu\Big(\mathfrak{L}_\varepsilon(P,P';\sigma,\sigma')\Big) \le \mu\Big(\mathfrak{C}_\varepsilon(P,P';\sigma,\sigma')\Big) \le \mu\Big(\mathfrak{U}_\varepsilon(P,P';\sigma,\sigma')\Big)$$

for all ε sufficiently small. Now use Corollary 7.13 on the left, Theorem 6.7 on the right, divide by ε, and let $\varepsilon \to 0^+$ to obtain the desired result. □

Because of Theorem 8.1, both $\overline{\mu}(P,P';\sigma,\,\cdot\,)$ and $\overline{\mu}(P,P';\,\cdot\,,\sigma')$ are finite measures on $\mathfrak{B}(\mathbb{S}^2)$ for any polygons P, P' and $\sigma,\sigma' \in \mathfrak{B}(\mathbb{S}^2)$. Let $\overline{\mu}(P,P')$ denote $\overline{\mu}(P,P';\mathbb{S}^2,\mathbb{S}^2)$. We use (8.1) to write

$$\frac{\overline{\mu}(P,P';\sigma,\sigma')}{\overline{\mu}(P,P')} = \frac{\Phi_0(P,\sigma)\,\Phi_1(P',\sigma') + \Phi_1(P,\sigma)\,\Phi_0(P',\sigma')}{\Phi_0(P,\mathbb{S}^2)\,\Phi_1(P',\mathbb{S}^2) + \Phi_1(P,\mathbb{S}^2)\,\Phi_0(P',\mathbb{S}^2)} \tag{8.2}$$

as the probability of a paint-to-paint collision. Note that $\overline{\mu}(P,P')$ is positive and the right-hand side of (8.2) is symmetric in P and P'. As an application of

(8.2) we consider examples of colliding triangles, which have all the essential elements found in the more general case of colliding convex polygons.

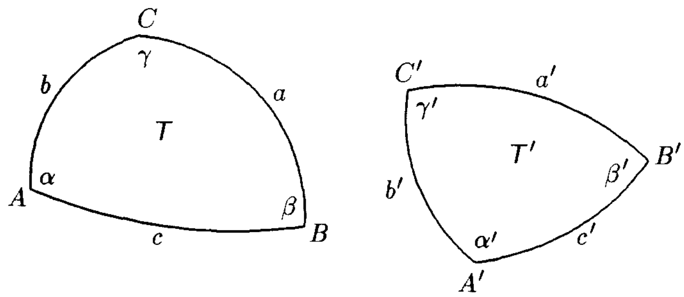

Figure 8.1. The triangles T and T'.

Example 8.2. *Vertex-to-edge collision.* Let T be the geodesic triangle with vertices A, B, C, interior angles α, β, γ and edge-lengths a, b, c. Let T' be the geodesic triangle with vertices A', B', C', interior angles α', β', γ' and edge-lengths a', b', c'. See Figure 8.1. Note that

$$\Phi_0(T, \mathbb{S}^2) = (\pi - \alpha) + (\pi - \beta) + (\pi - \gamma) \quad \text{and} \quad \Phi_1(T', \mathbb{S}^2) = a' + b' + c',$$
$$\Phi_1(T, \mathbb{S}^2) = a + b + c \quad \text{and} \quad \Phi_0(T', \mathbb{S}^2) = (\pi - \alpha') + (\pi - \beta') + (\pi - \gamma').$$

We compute the probability T' collides with T so that the vertex C' touches $\mathit{arc}[BC]$. By this we mean $\sigma = \mathit{arc}[BC]$ and $\sigma' = \{C'\}$ in (8.2). Note that $\mathit{arc}[BC]$ and $\{C'\}$ are in $\mathfrak{B}(\mathbb{S}^2)$ and

$$\Phi_0(T, \mathit{arc}[BC]) = 0 = \Phi_1(T', \{C'\}),$$
$$\Phi_1(T, \mathit{arc}[BC]) = a \quad \text{and} \quad \Phi_0(T', \{C'\}) = \pi - \gamma'.$$

We then obtain the probability

$$\frac{\overline{\mu}\big(T, T'; arc[BC], \{C'\}\big)}{\overline{\mu}(T, T')} = \frac{a(\pi - \gamma')}{(3\pi - \alpha - \beta - \gamma)(a' + b' + c') + (a + b + c)(3\pi - \alpha' - \beta' - \gamma')}.$$

In Example 8.1 we get the same result if $arc[BC]$ is replaced by relint$\big(arc[BC]\big)$. This seems to indicate that the probability T' collides with T so that vertex C' touches the vertex B is zero. That is indeed the case, as the next example shows.

Example 8.3. *Vertex-to-vertex collision.* Let T, T' be as in Example 8.1. The probability that T' collides with T so that vertex C' touches vertex B is zero. If $\sigma = \{B\}$ and $\sigma' = \{C'\}$, then

$$\Phi_0(T, \{B\}) = \pi - \beta \quad \text{and} \quad \Phi_1(T', \{C'\}) = 0,$$
$$\Phi_1(T, \{B\}) = 0 \quad \text{and} \quad \Phi_0(T', \{C'\}) = \pi - \gamma'.$$

Application of (8.2) gives the probability

$$\frac{\overline{\mu}\big(T, T'; \{B\}, \{C'\}\big)}{\overline{\mu}(T, T')} = 0.$$

Intuitively, the result of Example 8.2 is appealing. One would also expect that the probability of an edge-to-edge collision is zero. The next example confirms this.

Example 8.4. *Edge-to-edge collision.* Let T, T' be as in Example 8.1. The probability that T' collides with T so that $arc[A'C']$ touches $arc[BC]$ is zero. Let $\sigma = arc[BC]$ and $\sigma' = arc[A'C']$. Then

$$\Phi_0(T, arc[BC]) = 0 \quad \text{and} \quad \Phi_1(T', arc[A'C']) = b',$$
$$\Phi_1(T, arc[BC]) = a \quad \text{and} \quad \Phi_0(T', arc[A'C']) = 0.$$

Now apply (8.2) to obtain the probability

$$\frac{\overline{\mu}\Big(T, T'; arc[BC], arc[A'C']\Big)}{\overline{\mu}(T, T')} = 0.$$

Remark 8.5. These examples show that the only positive probabilities occur when 1 is the sum of the dimensions of the faces that touch. This is the same kind of result as that proved for n-polytopes in $\mathbb{R}^n$, see [**9**].

9. CONCLUDING REMARKS

Collision Probabilities of Convex Bodies

One of our original goals was the study of collision probabilities of convex bodies in $\mathbb{S}^2$. Since convex polygons can be used to approximate general convex bodies, it was natural to consider polygons first. That has been accomplished. However, with our approach there are several problems in passing to the general case of convex bodies. In this section we briefly state some of those problems and give suggestions for further research.

One of the first difficulties that must be overcome is the definition of the curvature measures for a convex body $Q \subset \mathbb{S}^2$. It appears that the procedure in [**15**] used to define curvature measures of convex bodies in $\mathbb{R}^n$ can be adapted to $\mathbb{S}^2$. If this can be done, then if Q is a convex body in $\mathbb{S}^2$ and $0 < \varepsilon < \pi/2$, there are finite measures $\Phi_m(Q,\,\cdot\,)$ such that $\text{Area}(B_\varepsilon(Q,\sigma)) = (1-\cos\varepsilon)\,\Phi_0(Q,\sigma) + (\sin\varepsilon)\,\Phi_1(Q,\sigma)$ for each $\sigma \in \mathfrak{B}(\mathbb{S}^2)$.

Another useful result was given in Theorem 5.10. For convex bodies Q and Q' its statement becomes: *The set* $\mathfrak{N}(Q,Q') := \{g \in \mathfrak{G}\colon\ 0 < \rho(Q,gQ') < \pi/2,\ \operatorname{card} R(Q,gQ') > 1\}$ *has μ–measure zero.* As yet we have not determined the truth value of this statement, although a similar theorem for convex bodies in $\mathbb{R}^n$ was proved in [**16**]. If Theorem 5.10 holds for convex bodies Q and Q', then $\mu\Big(\mathfrak{C}_\varepsilon(Q,Q';\,\cdot\,,\sigma')\Big)$ and $\mu\Big(\mathfrak{C}_\varepsilon(Q,Q';\sigma,\,\cdot\,)\Big)$ will be finite measures on $\mathfrak{B}(\mathbb{S}^2)$, see Theorem 5.12 and Theorem 5.13.

If $\{P_n\}$ is a sequence of convex polygons with $P_n \to Q$, then for an approximation procedure to work we must have $\mu\Big(\mathfrak{C}_\varepsilon(P_n,Q';\sigma,\sigma')\Big) \to \mu\Big(\mathfrak{C}_\varepsilon(Q,Q';\sigma,\sigma')\Big)$ and $\Phi_m(P_n,\sigma) \to \Phi_m(Q,\sigma)$ for all σ in a reasonably large class of subsets of $\mathbb{S}^2$. In [**16**] the concept of weak convergence of Borel measures was used. From [**1**; p. 196], the sequence of finite Borel measures $\{\nu_n\}$ on $\mathfrak{B}(\mathbb{S}^2)$ converges weakly to the finite Borel measure ν iff

$$(i) \quad \liminf_{n\to\infty} \nu_n(S) \geq \nu(S) \quad \text{for every open set } S \subset \mathbb{S}^2$$

and

$$(ii) \quad \nu_n(\mathbb{S}^2) \to \nu(\mathbb{S}^2).$$

Consider a problem related to (ii), i.e. to show that

$$\mu\Big(\mathfrak{C}_\varepsilon(P_n, Q'; \mathbb{S}^2, \mathbb{S}^2)\Big) \to \mu\Big(\mathfrak{C}_\varepsilon(Q, Q'; \mathbb{S}^2, \mathbb{S}^2)\Big).$$

To prove this it may be helpful to compute $\mu\Big(\mathfrak{C}_\varepsilon(Q, Q'; \mathbb{S}^2, \mathbb{S}^2)\Big)$ for any convex bodies Q, Q'.

How can this measure be calculated? For convex bodies in $\mathbb{R}^n$, Schneider [**16**] and Firey [**6**] used the Fundamental Kinematic Formula, see [**13**]. In particular, they relied on the fact that $\chi(K_\varepsilon \cap gK') = 1$, where χ is the Euler characteristic, K, K' are convex bodies in $\mathbb{R}^n$, and g is a proper rigid motion in $\mathbb{R}^n$ for which $K_\varepsilon \cap gK' \neq \emptyset$. However, since Q_ε need not be convex (as K_ε must be), the value of $\chi(Q_\varepsilon \cap gQ')$ may be any positive integer. See Figure 9.1 for two examples when Q' is a convex polygon. If we take the viewpoint that $\mathbb{S}^2$ is like $\mathbb{R}^2$ in the small, then perhaps as ε approaches zero the measure of those motions for which $\chi(Q_\varepsilon \cap gQ') > 1$ becomes negligible. This deserves more study.

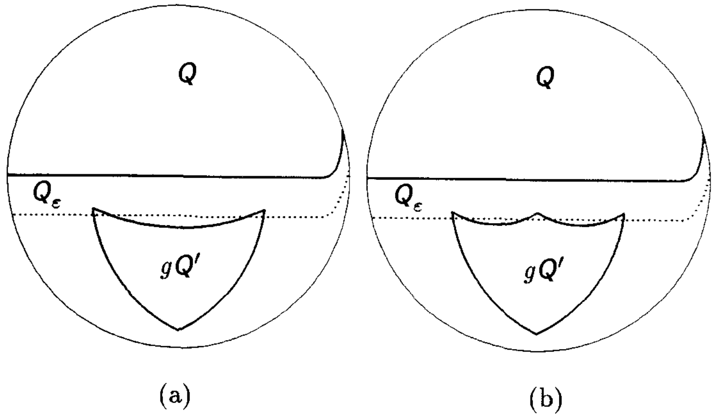

Figure 9.1. (a) $\chi(Q_\varepsilon \cap gQ') = 2$, (b) $\chi(Q_\varepsilon \cap gQ') = 3$.

Integral-Geometric Interpretation of Curvature Measures

An m-subsphere $L_m, m = 0, 1$, is said to *support* P *in* $\sigma \in \mathfrak{B}(\mathbb{S}^2)$ if $\sigma \cap P \cap L_m \neq \varnothing$ and L_m is contained in a great circle that supports P. Using the Euclidean case as a model, see [**5**] and [**15**; § 5], we conjecture that $\Phi_m(P, \sigma)$ is a natural measure of those $(2-m-1)$-subspheres that support P in σ.

Evidence for this conjecture is found in cases 1 and 3 of Theorem 6.7, if we consider each point $\overline{x}$ as a 0-subsphere which "nearly" supports P in σ, and each point $\overline{z}$ as a pole of a 1-subsphere $\overline{L}$ which "nearly" supports P in σ. Perhaps a thickening procedure similar to that of Chapter 5, the estimation techniques given in Chapters 6-7, and the limit process in Chapter 8 can be used to prove the conjecture.

BIBLIOGRAPHY

1. R. B. Ash, *Measure, Integration, and Functional Analysis.* Academic Press, New York, 1972.

2. *CRC Standard Mathematical Tables*, 19th ed. The Chemical Rubber Company, 1971.

3. H. Federer, Curvature measures, *Trans. Amer. Math. Soc.* **93** (1959), 418–491.

4. H. Federer, *Geometric Measure Theory.* Springer-Verlag, New York, 1969.

5. W. J. Firey, An integral-geometric meaning for the lower order area functions of convex bodies, *Mathematika* **19** (1972), 205–212.

6. W. J. Firey, Kinematic measures for sets of support figures, *Mathematika* **21** (1974), 270–281.

7. W. Fulks, *Advanced Calculus*, 3rd ed. Wiley, New York, 1978.

8. P. J. Kelly and M. L. Weiss, *Geometry and Convexity: A Study in Mathematical Methods.* Wiley (Interscience), New York, 1979.

9. P. McMullen, A dice probability problem, *Mathematika* **21** (1974), 193–198.

10. P. McMullen and G. C. Shephard, *Convex Polytopes and the Upper Bound Conjecture.* Cambridge Univ. Press, 1971.

11. B. O'Neill, *Elementary Differential Geometry.* Academic Press, San Diego, CA, 1966.

12. H. L. Royden, *Real Analysis*, 3rd ed. Macmillan, New York, 1988

13. L. A. Santaló, *Integral Geometry and Geometric Probability* (Encyclopedia of Mathematics and Its Applications 1). Addison-Wesley, Reading, MA, 1976.

14. R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory* (Encyclopedia of Mathematics and Its Applications **44**). Cambridge Univ. Press, 1993.

15. R. Schneider, Curvature measures of convex bodies, *Ann. Mat. Pura Appl.* **116** (1978), 101–134.

16. R. Schneider, Kinematic measures for sets of colliding convex bodies, *Mathematika* **25** (1978), 1–12.

17. M. Spivak, *A Comprehensive Introduction to Differential Geometry*, vol. 1, 2nd ed. Publish or Perish, Houston, 1979.

APPENDIX

LIST OF SYMBOLS

Symbol	Meaning	Page
g, $\mathfrak{G}$	a motion, and the motion group of $\mathbb{S}^2$	18
SO(3)	special orthogonal group of $\mathbb{R}^3$	18
$d_{\mathfrak{G}}(g, g')$	$\max\{d(gp, g'p)\colon p \in \mathbb{S}^2\}$; a metric on $\mathfrak{G}$	19
$\mathfrak{B}(\mathfrak{G})$	σ-algebra of Borel subsets of $(\mathfrak{G}, d_{\mathfrak{G}})$	19
g_{t}	translation in $\mathbb{S}^2$	20, 21
g_{r}	rotation in $\mathbb{S}^2$	20, 21
dA	element of area on $\mathbb{S}^2$	22
μ	kinematic measure	22
$\mathfrak{R}(x)$		23
$\mathfrak{M}^{-1}$	$\{g^{-1} \in \mathfrak{G}\colon g \in \mathfrak{M}\}$	23
$\mathfrak{C}_0(P, P')$	motions that cause P' to collide with P	25
$\mathfrak{C}_0(P, P'; \sigma, \sigma')$	paint-to-paint collisions	25
$g(x, L; x', L')$		26
$R(P, P')$	point-pairs that realize the distance between P, P'	29
$\mathfrak{C}_\varepsilon(P, P'; \sigma, \sigma')$	near paint-to-paint collisions	29
$\mathfrak{C}_\varepsilon(P, P')$	near-collisions; $\mathfrak{C}_\varepsilon(P, P'; \mathbb{S}^2, \mathbb{S}^2)$	29
$g(x, L, \eta; x', L')$		41
$\overline{x}$, $\overline{z}$, $\overline{L}$		41, 42
$g(x', L', \eta; x, L)$		42
$-P$	the set of antipodes of points in P; $\{-p\colon p \in P\}$	42
$\mathfrak{U}_\varepsilon(P, P'; \sigma, \sigma')$	set containing the near paint-to-paint collisions	43
Z, Z'		46, 47
$\overline{X}$		47, 49
ds	element of arc-length	48
$\overline{Z}$		50, 52
δ_i		58
$O(\varepsilon)$	order of ε; $\lim_{\varepsilon \to 0} [O(\varepsilon)/\varepsilon]$ is finite	59
$\underline{Z}$		60
$\underline{Z}'$		63
$\mathfrak{L}_\varepsilon(P, P'; \sigma, \sigma')$	subset of $\mathfrak{U}_\varepsilon(P, P'; \sigma, \sigma')$ and $\mathfrak{C}_\varepsilon(P, P'; \sigma, \sigma')$, ε small	64
$\underline{X}$		69
$\overline{\mu}(P, P'; \sigma, \sigma')$		73
$\overline{\mu}(P, P')$		73