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Integral representations provide a useful framework of study and simulation of fractional Browian motion, which has been used in modeling of many natural situations. In this thesis we extend an integral representation of fractional Brownian motion that is supported on a bounded interval of $\mathbb{R}$ to integral representation that is supported on bounded subset of $\mathbb{R}^d$. These in turn can be used to give new series representations of fractional Brownian motion.

Representations of Fractional Brownian Motion

by

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I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Noppadon Wichitsongkram, Author

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REPRESENTATIONS OF FRACTIONAL BROWNIAN MOTION

1. INTRODUCTION

1.1. Preliminary

The fractional Brownian motion was first studied by Kolmogorov in 1940. He called it the Wiener Helix. Then it was further studied by Yaglom in [27]. The name fractional Brownian motion was given by Mandelbrot and Van Ness, who showed a Wiener integral representation of this process in terms of a standard Brownian motion in [20]. The aim of this thesis is to search for new representations of fractional Brownian motion and study their properties.

In this section, the history and some well-known theorems that motivate the idea of this thesis will be given. We define a standard fractional Brownian motion on $\mathbb{R}$ with index $H \in (0,1)$ to be a mean zero real-valued Gaussian process with

$$Cov(B_H(s), B_H(t)) = 1/2(|s|^{2H} + |t|^{2H} - |s-t|^{2H}).$$

More generally, a fractional Brownian motion has

$$Cov(B_H(t,)B_H(s)) = C_H(|s|^{2H} + |t|^{2H} - |s-t|^{2H}).$$

where C_H is a constant.

Fractional Brownian motion can be written in the term of integral representations and series representations. We will see that the representations of fractional Brownian motion are not unique. This fact allows us to be able to find new representations that might be suitable for some particular mathematical framework. In this thesis, we will study the methods to extend representations of a fractional Brownian motion in $\mathbb{R}$ to representation of fractional Brownian motion in $\mathbb{R}^d$. Here are some examples of known representations of Gaussian processes that motivate the idea.

Example 1. Let $\{B(t), 0 \leq t \leq 1\}$ be a Brownian motion. This is the case that $H = 1/2$. Then we have the covariance

$$R(s,t) = \mathbb{E}(B(s)B(t)) = min(s,t), s,t \geq 0.$$

This corresponds to fractional Brownian motion with index $H = 1/2$ on $T = \{t : 0 \leq t \leq 1\}$. The eigenfunctions X of the covariance operator given by

$$\int_0^1 min(s,t)x(t)dt = \lambda x(s)$$

satisfy

$$x''(s) + \lambda x(s) = 0, x(0) = x'(1) = 0$$

It follows that the eigenvalues are given by

$$\lambda_n = \frac{1}{(n+\frac{1}{2})^2\pi^2}$$

with corresponding eigenfunctions

$$e_n = \sqrt{2} sin((n+\frac{1}{2})\pi t)$$

Then we have

$$B(t) = \sqrt{2}\sum_{n=0}^{\infty} \frac{sin((n+\frac{1}{2})\pi t)}{(n+\frac{1}{2})\pi} Z_n$$

This representation is called the Karhunen-Loéve expansion of Brownian motion.

An integral representation is as follows

$$B(t) = \int 1_{[0,t]}(s)dB(s)$$

where B is a standard Brownian motion.

The next example will be another series representation of Brownian motion.

Example 2. From Noda(1987), we have the following theorem.

Theorem 1.1.0.1. *Let X be a Brownian motion with parameter space (S^n, d_G), where S^n is the unit sphere in $\mathbb{R}^n$, d_G is the geodesic distance. Then X can be written as*

$$X(x) = \sum_{i\in I} \lambda_i \xi_i f_i(x).$$

Here ξ_i is an i.i.d sequence of standard Brownian motion, λ_i is a constant and $\{f_i\}$ is an orthonormal basis for S^n.

This representation actually is the Karhunen-Loéve expansion. We have seen from the previous examples that $W(t)$ can be represented by both a series and an integral. This is also true for the case of fractional Brownian motion which we are going to see in the following examples. Some well-known integral representations shall be presented in the next examples. The representations are for $d = 1$. The first two are known as Molchan-Golosov and Mandelbrot-Van Ness representations of fractional Brownian motions, respectively. See Jost [15] for discussion.

Example 3. (a). Molchan-Golosov representation of fractional Brownian motion. For $H \in (0, 1)$, there exists ordinary Brownian motion $(B_t)_{t\in\mathbb{R}}$ such that for all $t \in [0, \infty)$.

$$B^H(t) = C(H) \int_0^t (t-s)^{H-1/2} F(1/2, H-1/2, H+1/2, \frac{s-t}{s}) dB(s)$$

where $C(H) = \frac{2H}{\Gamma(H+1/2)}$ and $F(a, b, c, z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, a, b, c, z \in \mathbb{R}$ with $(a)_k = a(a+1)...(a+k-1), k \in \mathbb{N}$ and $c \in \mathbb{R}/\{..., -2, -1, 0\}$.

(b). Mandelbrot-Van Ness representation.

$$B^H(t) = C(H) \int_{\mathbb{R}} ((t-s)^{H-1/2} 1_{(-\infty,t)}(s) - (-s)^{H-1/2} 1_{(-\infty,0)}(s)) d\widetilde{B}(s), a.s.,$$

Here $\widetilde{B}_s$ represents ordinary Brownian motion.

(c). Another example of an integral representation is given by Lindstrom in [19]. It is an extension of Andreas Stoll's Representation Theorem of Lévy Brownian motion. For $d > 1$ and $p \in (0, 2)$

$$B_p(x) = k_{p,d} \int_{\mathbb{R}^d} (\|x-y\|^{(p-d)/2} - \|y\|^{(p-d)/2}) dB(y)$$

Here B represents the Wiener process in $\mathbb{R}^d$.

Integral representations of fractional Brownian motion can be used to develop series representations of Gaussian process. From (a), we have that a fractional Brownian motion for $\mathbb{R}$ can be written as $B_H(t) = \int_0^t K(t,s)dB(s)$. Then this integral representation can be used to develop other series expansion as follows.

Let $f_n(s)$ be an orthonormal basis for $L^2([0,1])$. For fixed $t \in (0,1)$,

$$K(t,s) = \sum_{n\geq 0} c_n(t) f_n(s),$$

where $c_n(t) =< K(t,\cdot), f_n(\cdot) >$.

Then

$$\begin{aligned} B_H(t) &= \int_0^1 \sum_{n\geq 0} c_n(t) f_n(s) dB(s) \\ &= \sum_{n\geq 0} c_n(t) \int_0^1 f_n(s) dB(s) \\ &= \sum_{n\geq 0} c_n(t) Z_n \end{aligned}$$

where $Z_n = \int_0^1 f_n(s)dB(s)$ is an i.i.d sequence of Gaussian random variables.

We can notice that the covariance

$$Cov(Z_n, Z_m) = \int_0^1 f_n(S) f_m(s) ds = \delta_{n,m}.$$

1.2. Statement of problem and result

We have seen some well known examples of representations of fractional Brownian motion. Notice that the Molchan-Golosov representation requires integration only on a finite interval. This will be useful for extending from a Gaussian process in $\mathbb{R}$ to $\mathbb{R}^d$. In this thesis, we are interested in this kind of integral representations given by integration over compactly supported kernel functions. Roughly speaking, the extension can be achieved from $\mathbb{R}$ to $\mathbb{R}^d$ by an idea of rotating vectors in $\mathbb{R}^d$. Indeed, let $B_H(t) = {}_0^t K(t,u)dB(u)$ be a fractional Brownian motion in $\mathbb{R}$. Then we will prove the following result.

Theorem 1.2.0.2. *For $t \in [-1,1]$, let $B_H(t) = {}_0^\infty K(t,u)dB(u)$ be a fractional Brownian motion. Then for $t \in \mathbb{R}^d$ with $\|t\| \leq 1$*

$$B_H(t) = {}_{\mathbb{R}^d} K(t \cdot \frac{u}{\|u\|}, \|u\|) \frac{1}{\|u\|^{(d-1)/2}} dB(u)$$

is a fractional Brownian motion with index H on the unit disk in $\mathbb{R}^d$.

Now consider a function $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and define

$$B_H(t) = {}_{\mathbb{R}^d} K(f(t,u), \|u\|) \frac{1}{\|u\|^{(d-1)/2}} dB(u)$$

Later on, we shall study the conditions on f that allow this extension to work. That is we shall prove the following theorem.

Theorem 1.2.0.3. *Let $B_H(t) = {}_0^\infty K(t,u)dB(u)$ be a fractional Brownian motion with index H on the interval $[-1,1]$. Let $\|\cdot\|$ be a norm in $\mathbb{R}^d$ induced by an inner product. If the function f defined as above satisfies the following conditions*

(1) For each t such that $\|t\| \leq 1$, $f(t,u) \in [-1,1]$ for all $u \in \mathbb{R}^d$.

(2) For all $t \in \mathbb{R}^d$, $u \in \mathbb{R}^d/\{0\}$, $f(t,u) = f(t, \frac{u}{\|u\|})$.

(3) $f(t,u) + f(s,u) = f(t+s,u)$.

(4) For $c \in [-1,1]$, $f(ct,u) = cf(t,u)$ for all $u \in \mathbb{R}^d$.

(5) $\int_{S^{d-1}} |f(t,u)|^{2H} d\sigma(u) = C_H$ *for all* $\|t\| = 1$, *where* C_H *is a constant.*

Then

$$B_H(t) = \int_{\mathbb{R}^d} K(f(t,u), \|u\|) \frac{1}{\|u\|^{(d-1)/2}} dB(u)$$

is a fractional Brownian motion with index H *on the unit disk in* $\mathbb{R}^d$.

We shall see some example of functions f that satisfy all 5 conditions. The method can be applied to Gaussian processes with stationary increments as stated below.

Theorem 1.2.0.4. *For each* $t \in \mathbb{R}^d$, *let* $Z(t) = \int_0^\infty K(t,u)dB(u), t \geq 0$ *be a one dimensional Gaussian process with stationary increments and assume*

$$g(t) = \mathbb{E}(Z^2(t))$$

is bounded on $[-1, 1]$. *Then we have for* $t \in \mathbb{R}^d$,

$$Z_d(t) = \int_{\mathbb{R}^d} \frac{1}{\|u\|^{(d-1)/2}} K(t \cdot u, \|u\|) dB(u)$$

is a Gaussian process in $\mathbb{R}^d$ *with isotropic property and the covariance function of the form*

$$Cov(Z_d(t)Z_d(s)) = c_d(G_d(\|t\|) + G_d(\|s\|) - G_d(\|t-s\|))$$

where $G_d(u) = \int_{-1}^{1} (1-v^2)^{\frac{d-3}{2}} g(uv)dv$, *and* c_d *is a constant depending only on* d.

This can be used to develop series representation as follows.

Theorem 1.2.0.5. *Let* $B_H(t) = \int_0^1 K(t,u)dB(u)$ *be a fractional Brownian motion in* $[-R, R]$, $R > 0$ *and* g_i *be an orthonormal basis for* $L^2((0,1), dx)$ *and* $\varphi_{n,k}$ *be an orthomormal basis for* $L^2(S^{d-1})$. *Then we have*

$$\begin{aligned} B_H(t) &= \int_{B^d(0,1)} K(t \cdot \frac{u}{\|u\|}, \|u\|) \frac{1}{\|u\|^{(d-1)/2}} dB(u) \\ &= \sum_{n,k,i} c_{n,k,i}(t) \int_{B_d(0,1)} \frac{g_i(\|u\|)}{\|u\|^{(d-1)/2}} \varphi_{n,k}(\frac{u}{\|u\|}) dB(u) \end{aligned}$$

where $c_{n,k,i}(t) = \sigma_{d-1}(\int_{-1}^{1} \int_0^1 K(\|t\|u, r) g_i(r) dr P_n^d(u)(1-u^2)^{\nu} du) \varphi_{n,k}(\frac{t}{\|t\|})$, $\vartheta = (d-3)/2$ *and* P_n^d *is the Legendre polynomial of dimension* d *and degree* n.

1.3. Organization of this Thesis

The mathematical background that is used in this thesis is given in Chapter 2. It is divided into 9 sections. The first two sections discuss inner product spaces and the spherical measure and coordinates in $\mathbb{R}^d$ and provide mathematical tools that are needed in this thesis. Section 2.3 describes the probability background. The definition of random variables, expectation, variance, covariance are all given in this section. Four basic definitions of types of convergence for sequences of random variables are given in Section 2.4. In section 2.5 the definition of stochastic processes and Kolmogorov's Existence theorem are given. Section 2.6 gives the definition of Gaussian random variables, multivariate Gaussian and Gaussian processes. Brownian motion and Brownian sheet definitions are stated in the section 2.7. The last two sections are concerned about the Wiener integral in $\mathbb{R}$ and $\mathbb{R}^d$. We first define the integral of a step function. Then we extend the definition to a larger class of functions by approximation.

Chapter 3 gives a background on fractional Brownian motion. Formal definition of fractional Brownian motion is given in this chapter. Some results and properties are also stated and proved.

Chapter 4 describes series expansion of Gaussian process. The Mercer's theorem and Karhunen-Loéve expansion are stated in the section 4.1. Examples of the Karhunen-Loéve expansion are given in the section 4.2. These are the Karhunen-Loéve expansion of standard Brownian motion and Brownian bridge.

Chapter 5 contains the main results of this thesis. There are four sections in this chapter. The first two sections investigate a method to extend a fractional Brownian motion in $\mathbb{R}$ to a fractional Brownian motion in $\mathbb{R}^d$. The conditions that allows the extension to work are given and proved. Section 5.3 generalizes results of the first two sections. In particular it is shown that this method can also apply to the case of Gaussian processes in $\mathbb{R}$ with stationary increments. Then we obtain a Gaussian process in $\mathbb{R}^d$ with an isotropic property. The last section investigates series expansion of fractional Brownian motion. It is stated and proved in this section that a fractional Brownian motion in $\mathbb{R}^d$ can be written as an infinite summation of terms using the othonormal basis of the unit ball in $\mathbb{R}^d$

The conclusion and future work are described in Chapter 6, the last chapter of this thesis. Through out the thesis, we always consider Gaussian processes with mean zero.

2. MATHEMATICAL BACKGROUND

The following section will give some basic mathematical background, including analytic concepts and fact that will be used in later chapters. Most of this material can be found in standard textbooks on real analysis or functional analysis and it may not be necessary to repeat here the pertinent proofs. However, a few facts of a more special character and not generally known will be mentioned as lemmas and proved.

Through out this thesis we let $\mathbb{R}^d$ denote the Euclidian d dimensional space. If x is a point of $\mathbb{R}^d$ the coordinates of x will be denoted by x_i, that is $x = (x_1, x_2, ..., x_d)$. If $u, v \in \mathbb{R}^d$ we let $u \cdot v$ denote the inner product, and $\|u\|$ the Euclidian norm in $\mathbb{R}^d$. For points in $\mathbb{R}$, $|\cdot|$ is the ordinary absolute value. The Lebesgue measure of a subset S of $\mathbb{R}^d$ will usually be called the volume of S and denoted by $|S^{d-1}|$. We shall begin this chapter by giving some backgrounds on the inner product space of $\mathbb{R}^d$.

2.1. Inner product space

For the background that is used in this section, we refer to [8].

Definition 2.1.0.1 (Friedberg[8]). *An inner product on a real vector space V is a function from $V \times V \to \mathbb{R}$ that for any two vectors $u, v \in V$, there is a real number $< u, v >$, satisfying the following properties.:*

(1) Linearity : $< au + bv, w >= a < u, w > +b < v, w >$, where $a, b \in \mathbb{R}$.

(2) Symmetric : $< u, v >=< v, u >$.

(3) Positive definite : For any $u \in V$, $< u, u >\geq 0$ and $< u, u >= 0$ if and only if $u = 0$.

The vector space V with an inner product is called an inner product space.

For each vector $u \in V$, the norm of u is defined as the number

$$\|u\| = \sqrt{< u, u >}.$$

If $\|u\| = 1$ we call u a unit vector and u is said to be normalized. For any vector $v \neq 0 \in V$, we have the unit vector

$$\bar{v} = \frac{v}{\|v\|}.$$

This process is called normalizing v.

Let $\mathfrak{B} = \{u_1, u_2, ..., u_n\}$ be a basis of an n dimensional inner product space V. For vector $u, v \in V$, we can write

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$$

and

$$v = \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n.$$

The linearity implies that

$$\begin{aligned} < u, v > &= < \sum_{i=1}^{n} \alpha_i u_i, \sum_{j=1}^{n} \beta_i u_i > \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \beta_j < u_i, u_j > \end{aligned}$$

we call the $n \times n$ matrix

$$A = \begin{pmatrix} < u_1, u_1 > & < u_1, u_2 > & \cdots & < u_1, u_n > \\ < u_2, u_1 > & < u_2, u_2 > & \cdots & < u_2, u_n > \\ \vdots & \vdots & \ddots & \vdots \\ < u_n, u_1 > & < u_n, u_2 > & \cdots & < u_n, u_n > \end{pmatrix}$$

the matrix of the inner $< \cdot, \cdot >$ product relative to the basis $\mathfrak{B}$. Thus by using coordinate vectors

$$[u]_{\mathfrak{B}} = [\alpha_1, \alpha_2, ..., \alpha_n]^T, \; [v]_{\mathfrak{B}} = [\beta_1, \beta_2, ..., \beta_n]^T,$$

we have

$$< u, v >= [u]_{\mathfrak{B}}^T A[v]_{\mathfrak{B}}.$$

A subset $S = \{u_1, u_2, ..., u_k\}$ of non zero vectors of V is called an orthogonal set if every pair of vectors are orthogonal, that is,

$$< u_i, u_j >= 0,\ 1 \leq i < j \leq k.$$

An orthogonal set $S = \{u_1, u_2, ..., u_k\}$ is called an orthonormal set if we further have

$$\|u_i\| = 1,\ 1 \leq i \leq k.$$

An orthonormal basis of V is a basis which is also an orthonormal set. For the next theorem, we shall give some background on diagonalizing symmetric matrix.

Theorem 2.1.0.6 (Friedberg[8])**.** *Any real symmetric matrix A can be diagonalized by an orthogonal matrix. That is there exists an orthonormal basis $\mathfrak{B} = \{u_1, u_2, ...u_n\}$ of $\mathbb{R}^n$ such that*

$$Au_1 = \lambda_i u_i,\ 1 \leq i \leq n,$$

$$Q^{-1}AQ = Q^T AQ = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where $Q = [u_1, u_2, ..., u_n]$ and spectral decomposition

$$A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \cdots + \lambda_n u_n u_n^T.$$

An $n \times n$ matrix is called positive definite if, for any non zero vector $u \in \mathbb{R}^n$,

$$< u, Au >= u^T Au > 0.$$

Theorem 2.1.0.7 (Friedberg[8]). *Let A be real symmetric matrix. Let $< \cdot, \cdot >$ be defined by*

$$< u, v >= u^T A v, \;\; u, v \in \mathbb{R}^n.$$

The $< \cdot, \cdot >$ is an inner product in $\mathbb{R}^n$ if and only if A is positive definite.

Theorem 2.1.0.8 (Friedberg[8]). *Let A be the matrix of n dimensional inner product space V relative to a basis $\mathfrak{B}$. Then for $u, v \in V$,*

$$< u, v >= [u]_{\mathfrak{B}}^T A [v]_{\mathfrak{B}}.$$

Moreover, A is positive definite.

The following section shall give a basic knowledge on some computations, change of variable and measure of spherical coordinate in $\mathbb{R}^d$.

2.2. Spherical measure and Spherical Coordinate in $\mathbb{R}^d$

For the background material in this section, we refer to [9] and [14].

Through out this section, we write $B^d(p, r)$ for the closed ball in $\mathbb{R}^d$ of radius r centered at p, and $B^d = B^d(0, 1)$ for the closed unit ball in $\mathbb{R}^d$ centered at 0. Furthermore, we let S^{d-1} denote the boundary of B^d, that is the unit sphere in $\mathbb{R}^d$. The spherical Lebesgue measure on S^{d-1} will be denoted by σ, the volume of B^d by $|B^d|$, and the surface area of B^d by $|S^{d-1}|$.

We let $L(S^{d-1})$ denote the class of integrable function on S^{d-1} and $L^2(S^{d-1})$ the class of square integrable function on S^{d-1}. Thus $L^2(S^{d-1})$ consists of all real-valued Lebesgue measurable functions F on S^{d-1} with the property that

$$_{S^{d-1}} F^2(u) d\sigma(u) < \infty,$$

where the underlying measure space is always $(S^{d-1}, \mathcal{M}, \sigma)$ with $\mathcal{M}$ denoting the class of subsets of S^{d-1} that are measurable with respect to the spherical Lebesgue measure σ. If $F, G \in L^2(S^{d-1})$ the inner product $< F, G >$ is defined by

$$< F, G >= \int_{S^{d-1}} F(u)G(u)d\sigma(u).$$

Now we will recall the background on change of variable to spherical coordinate which we will benefit from in the main result section. But first, we shall give the definition of the Jacobian matrix and then we state the theorem giving the general change of variable formula.

Let π_i be the projection of the i component of $\mathbb{R}^m$ on to $\mathbb{R}$, that is $\pi_i(x) = x_i$ for $x = (x_1, x_2, ..., x_m) \in \mathbb{R}^m$. For a mapping T of an open set Ω in $\mathbb{R}^n$ into $\mathbb{R}^m$, let $(g_1, g_2, ..., g_m)$ be its component functions, that is g_i is a real-valued function on Ω defined by $g_i = \pi_i \circ T$ on Ω. If all of the partial derivatives $\frac{\partial g_i}{\partial x_j}$ for $i = 1, ..., m$ and $j = 1, ..., n$ exist at $p \in \Omega$, then we call the $m \times n$ matrix

$$J_T(p) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \end{pmatrix}$$

the Jacobian matrix of the mapping T at p.

Now we are going to state the theorem giving change of variable formula.

Theorem 2.2.0.9 (McDonald[14]). *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, $(\Lambda, \mathcal{S})$ a measurable space, and T a measurable transformation from $(\Omega, \mathcal{A})$ to $(\Lambda, \mathcal{S})$. Then, for any $\mathcal{S}$ measurable function f on Λ,*

$$\int_\Omega f \circ T(x)d\mu(x) = \int_\Lambda f(y)d\mu \circ T^{-1}(y),$$

in the sense that if one of the integrals exists, then so does the other, and they are equal.

As an immediate consequence of the above Theorem, we have the following corollary.

Corollary 2.2.0.1 (McDonald[14]). *Let Ω be an open set in $\mathbb{R}^d$ and let T be a one to one mapping of Ω of class of $C^1(\Omega)$ into $\mathbb{R}^d$. Then for any d dimensional measurable function f on $T(\Omega)$, we have*

$$\int_{T(\Omega)} f(y)d\lambda^d(y) = \int_{\Omega} f \circ T(x)|\det J_T|d\lambda^d(x),$$

in the sense that the existence of one side implies that of the other and the equality of the two. In particular, the integral holds for every nonnegative real-valued d dimensional measurable function f on $T(\Omega)$.

Now consider $\Omega = (0,\infty) \times (-\pi,\pi)$ in $\mathbb{R}^2$ and a mapping $T = (g_1, g_2)$ of the open set Ω in $\mathbb{R}^2$ into $\mathbb{R}^2$ defined for $(r,\theta) \in (0,\infty) \times (-\pi,\pi)$ by

$$\begin{aligned} x_1 &= g_1(r,\theta) = r\cos\theta, \\ x_2 &= g_2(r,\theta) = r\sin\theta. \end{aligned}$$

T maps Ω one to one onto $\mathbb{R}^2 \backslash A$, where $A = (-\infty, 0] \times \{0\} \subset \mathbb{R}^2$. The Jacobian matrix of T is given by

$$J_T(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

Thus T is of class $C^1(\Omega)$ with

$$\det J_T(r,\theta) = r \text{ for } (r,\theta) \in (0,\infty) \times (-\pi,\pi).$$

If f is a real-valued measurable function on $\mathbb{R}^2$, then since $\lambda^2(A) = 0$, we have

$$\begin{aligned} \int_{\mathbb{R}^2} f(x_1,x_2)d\lambda^2(x_1,x_2) &= \int_{\mathbb{R}^2 \backslash A} f(x_1,x_2)d\lambda^2(x_1,x_2) \\ &= \int_{(0,\infty)\times(-\pi,\pi)} f(r\cos\theta, r\sin\theta) r dr d\theta \end{aligned}$$

For general $d \geq 2$, let $(r, \theta_1, \theta_2, ...\theta_{d-1}) \in (0,\infty) \times (0,\pi) \times \cdots \times (0,\pi) \times (-\pi,\pi) = \Omega_d$ and define T_d from Ω_d to $\mathbb{R}^d$ by

$$T_d(r, \theta_1, ...\theta_{d-1}) = (x_1, x_2, ..., x_d),$$

where

$$\begin{aligned}
x_1 &= r\cos\theta_1 \\
x_2 &= r\sin\theta_1\cos\theta_2 \\
x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3 \\
&\vdots \\
x_{d-1} &= r\sin\theta_1\cdots\sin\theta_{d-2}\cos\theta_{d-1} \\
x_d &= r\sin\theta_1\cdots\sin\theta_{d-2}\sin\theta_{d-1}
\end{aligned}$$

Notice that $x_1^2 + x_2^2 + \cdots + x_d^2 = r^2$ and the Jacobian matrix of T_d is given by

$$J(r, \theta_1, ..., \theta_{d-1}) = r^{d-1}\prod_{i=1}^{d-2} sin^{d-i-1}\theta_i,$$

where $\prod_{i=1}^{0}\sin\theta_i = 1$.

Hence by the change of variable formula, we have for real-valued function $F \in L^2(S^{d-1})$,

$$\int_{S^{d-1}} F(u)d\sigma(u) = \int_{(0,\pi)\times\cdots\times(0,\pi)\times(-\pi,\pi)} F(u_1, ..., u_d)\prod_{i=1}^{d-2}\sin^{d-i-1}(\theta_i)d\theta_1\cdots d\theta_{d-1},$$

where

$$\begin{aligned}
u_1 &= \cos\theta_1 \\
u_2 &= \sin\theta_1\cos\theta_2 \\
u_3 &= \sin\theta_1\sin\theta_2\cos\theta_3 \\
&\vdots \\
u_{d-1} &= \sin\theta_1\cdots\sin\theta_{d-2}\cos\theta_{d-1} \\
u_d &= \sin\theta_1\cdots\sin\theta_{d-2}\sin\theta_{d-1}
\end{aligned}$$

The next lemma relates integration over S^{d-1} to a particular integration over $[-1, 1]$. This lemma is a special case of the Funk-Hecke theorem, see [7].

Lemma 2.2.0.1 (Groemer[9])**.** *If Φ is a bounded Lebesgue integrable function on $[-1,1]$, and if p is a given point on S^{d-1}, $d \geq 2$, then $\Phi(u \cdot p)$ considered as a function of u on S^{d-1} is σ integrable and*

$$\int_{S^{d-1}} \Phi(u \cdot p) d\sigma(u) = |S^{d-1}| \int_{-1}^{1} \Phi(\zeta)(1-\zeta^2)^{\frac{d-3}{2}} d\zeta.$$

The next corollary will be a consequence of the previous lemma.

Corollary 2.2.0.2. *Let $X_1 = (\mathbb{R}^d, \cdot)$ and $X_2 = (\mathbb{R}^d, < \cdot, \cdot >)$ be respectively dot product and inner product space in $\mathbb{R}^d$, for $d \geq 2$. Let $\{e_1, ..., e_d\}$ be the usual orthonormal basis of X_1, that is all of the components of e_i are zeroes but ith is 1, and $\{t, f_2, ..., f_d\}$ be an orthonomal basis of X_2. Define a function $f : X_1 \to X_2$ by*

$$f(u) = v = v(u),$$

where $u = \alpha_1 e_1 + \cdots + \alpha_d e_d$ and $v = \alpha_1 t + \alpha_2 f_2 + \cdots + \alpha_d f_d$.
So f is isometric isomorphism from X_1 to X_2.
Then for t in $\mathbb{R}^d$ with $< t, t >= 1$,

$$\int_{S^{d-1}} | < v(u)/\|v(u)\|, t > | d\sigma(u) = |S^{d-1}| \int_{-1}^{1} |\zeta|(1-\zeta^2)^{(d-3)/2} d\zeta$$

is a constant.

Proof. By the previous Lemma,

$$\begin{aligned} \int_{S^{d-1}} | < v, t > | d\sigma(u) &= \int_{S^{d-1}} | < v(u), t > | d\sigma(u) \\ &= \int_{S^{d-1}} |u \cdot e| d\sigma(u) \\ &= |S^{d-1}| \int_{-1}^{1} |\zeta|(1-\zeta^2)^{(d-3)/2} d\zeta. \end{aligned}$$

is clearly a constant. □

The following sections contain the mathematical backgrounds of probability and random process.

2.3. General concepts of the probability measure

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a measure space, where Ω is a non-empty set and $\mathcal{F}$ is a σ-field of subsets of Ω. Then we call $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space if $\mathbb{P}$ is a measure on $\mathcal{F}$ satisfying

1) $0 \leq \mathbb{P}(A) \leq 1$, for $A \in \mathcal{F}$,

2) $\mathbb{P}(\varnothing) = 0$ and $\mathbb{P}(\Omega) = 1$,

3) If $A_1, A_2, \ldots$ is a disjoint sequence in $\mathcal{F}$ and $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$, then

$$\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

where $\varnothing$ denotes the empty set.

We define the Euclidean norm of $x \in \mathbb{R}^d$, $d \geq 1$ by

$$\|x\| = (x_1^2 + \cdots + x_d^2)^{1/2}.$$

Define a half open rectangle in $\mathbb{R}^d$ to be the set of the form

$$I = (a_1, b_1] \times \cdots \times (a_d, b_d],$$

and denote $\mathcal{B}^d$ as the Borel σ algebra generated by the half open intervals in $\mathbb{R}^d$.

Now let X be a measurable function from $(\Omega, \mathcal{F})$ into $(\mathbb{R}^d, \mathcal{B}^d)$. That is $X = (X_1, \ldots, X_d)$, is a vector of length d, where X_i is a real-valued random variable. Then for every Borel set $B \in \mathcal{B}^d$, the set

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\}$$

is an element of $\mathcal{F}$. Then X is called an $\mathbb{R}^d$-valued random variable. When $d = 1$, X is said to be real-valued.

We can also define a probability measure F_X or simply F on $\mathcal{B}^d$ from the probability measure $\mathbb{P}$ by

$$F_X(B) = \mathbb{P}\{X^{-1}(B)\}$$

for every $B \in \mathcal{B}^d$. We define a distribution function F of the random variable X by

$$\begin{aligned} F(x) &= F_X((-\infty, x_1] \times \cdots \times (-\infty, x_d]) \\ &= \mathbb{P}[X_1 \leq x_1, \ldots, X_d \leq x_d]. \end{aligned}$$

Denote λ_d, or simply λ, to be Lebesgue measure on $\mathbb{R}^d$. Then for every absolutely continuous function F, there exists a non-negative Borel function $f_X : \mathbb{R}^d \to \mathbb{R}$ such that

$$F_X(B) = \int_B f_X(x) d\lambda(x)$$

for every $B \in \mathcal{B}^d$. The function f_X is called the probability density function of the random variable X.

If X is a $\mathbb{R}^d$-valued random variable then we define the expectation of X by

$$\mathbb{E}(X) = \int_\Omega X(\omega) d\mathbb{P}(\omega) = \int_{\mathbb{R}^d} x dF_X(x)$$

and we denote $\mu_X = \mathbb{E}(X)$ to be the mean of X.
If $g : \mathbb{R}^d \to \mathbb{R}^k$ is a measurable function, then the expectation of $g(X)$ is defined as the Lebesgue- Stieltjes integral

$$\mathbb{E}(g(X)) = \int_\Omega g(X) d\mathbb{P} = \int_{\mathbb{R}^d} g(x) dF_X(x)$$

provided the integral exists. Since $g(X) = (g_1(X), \ldots, g_k(X))$ is $\mathbb{R}^k$ valued, so is $\mathbb{E}(g(X))$. Note that $\mathbb{E}(g(X))$ exists if, for $i = 1, \ldots, m$, $\mathbb{E}(g_i(X)) < \infty$.

If X is a real-valued random variable and $\mathbb{E}(|X|) < \infty$, X is said to be integrable. If $\mathbb{E}(|X|^2) < \infty$, we call X square integrable.

We define absolute moments of order k of X by

$$\mathbb{E}(\|X\|^k) = \int_{\mathbb{R}^d} \|x\|^k dF_X(x).$$

Since $j \leq k$ implies $\|x\|^j \leq 1 + \|x\|^k$, if X has finite absolute moment of order k, then it has all finite absolute moments of order $1 \leq j \leq k$.

The kth moment of a real-valued random variable X is defined by

$$\mathbb{E}(X^k) = \ {}_{\mathbb{R}} x^k dF_X(x).$$

We call $Var(X) = \mathbb{E}\{|X|^2\} - \mu_X^2$ the variance of X and $Cov(X,Y) = \mathbb{E}\{(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))\}$ the covariance of X and Y.

2.4. Stochastic convergence

There are four basic definitions of convergence for sequences X_n of random variables.

Definition 2.4.0.2 (Billingsley[4]). *1) X_n is said to converge to X with probability one, or almost surely, if there exists a set $N \subset \Omega$ such that $\mathbb{P}(N) = 0$ and, for every $\omega \in N^c$,*

$$\lim_{n\to\infty} \|X_n(\omega) - X(\omega)\| = 0.$$

2) X_n is said to converge to X in vth mean $(v > 0)$ if

$$\mathbb{E}(\|X_n - X\|^v) \to 0 \ \textit{as } n \to \infty.$$

3) X_n is said to converge to X in probability if, for every $\epsilon > 0$,

$$\mathbb{P}(\|X_n - X\| > \epsilon) \to 0 \ \textit{as } n \to \infty.$$

4) X_n is said to converge to X weakly, or in distribution, if

$$F_{X_n}(x) \to F_X(x) \ \textit{as } n \to \infty$$

at every continuity point x of $F_X(x)$.

In the following lemma we will summarize a number of results relating to the types of convergence and giving sufficient conditions for convergence. The proofs can be found in [4]. First we shall give a definition of the characteristic function.

Let $X = (X_1, \ldots, X_d)$ be a random variable with its distribution F_X in $\mathbb{R}^d$. Let $t \cdot x = \sum_{i=1}^{d} t_i x_i$ denote inner product. We define the characteristic function of X over $\mathbb{R}^d$ by

$$\begin{aligned} \varphi_X(t) &= \mathbb{E}(e^{it \cdot X}) \\ &= \int_{\mathbb{R}^d} e^{i \sum_{i=1}^{d} t_i x_i} dF_X(x) \end{aligned}$$

The characteristic function in nonprobability context is called the Fourier transform.

The characteristic function has two important properties that are used in this thesis.

1) The distribution of a random variable is completely determined by its characteristic function, and vice versa.

2) From the pointwise convergence of characteristic functions follows the weak convergence of the corresponding distributions.

Now we are going to state the Lemma mentioned above.

Lemma 2.4.0.2 (Billingsley[4]). *Let X_n be a sequence of random variables.*

1) If X_n converges to X almost surely then X_n converges to X in probability.

2) If X_n converges to X in vth mean then X_n converges to X in probability.

3) If X_n converges to X in probability then there is a sequence X_{n_k} of X_n such that X_{n_k} converges to X almost surely.

4) X_n converges to X in distribution if and only if $\varphi_{X_n}(t) \to \varphi_X(t)$ for every t.

2.5. Stochastic process

We start this section by recall the definition of stochastic process.

Definition 2.5.0.3 (Billingsley[4]). *A stochastic process is a collection $[X_t : t \in T]$ of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{R})$.*

In many standard cases, T is the set of integers and time is discrete, or else T is an interval and time is continuous. However, for general theory of this section we will regard T as an arbitrary index set. Now for each k-tuple $(t_1, \ldots, t_d)$ of distinct elements of T, we define a probability measure $\mu_{t_1,\ldots,t_d}$ on $\mathbb{R}^d$ by

$$\mu_{t_1,\ldots,t_d}(H) = \mathbb{P}[(X_{t_1}, \ldots, X_{t_d}) \in H], H \in \mathcal{B}^d.$$

Then we call $\mu_{t_1,\ldots,t_d}$ the finite dimensional distributions of the stochastic process $[X_t : t \in T]$. A natural question to ask is whether or not one can always find a stochastic process that has these $\mu_{t_1,\ldots,t_d}$ as its finite dimensional distribution. A result of Kolmogorov is that a necessary and sufficient condition for the existence of such a stochastic process is that the given family of measures satisfies the following two consistency conditions:

1) Suppose $H = H_1 \times \cdots \times H_d$ $(H_i \in \mathcal{B})$, and consider a permutation π of $(1, 2, \ldots, d)$. Since $[(X_{t_1}, \ldots, X_{t_d}) \in (H_1 \times \cdots \times H_d)]$ and $[(X_{\pi 1}, \ldots, X_{\pi d}) \in (H_{\pi 1} \times \cdots \times H_{\pi d})]$ are the same event. Then

$$\mu_{t_1,\ldots,t_d}(H_1 \times \cdots \times H_d) = \mu_{t_{\pi 1},\ldots,t_{\pi d}}(H_{\pi 1} \times \cdots \times H_{\pi d}).$$

2)

$$\mu_{t_1,\ldots,t_{d-1}}(H_1 \times \cdots \times H_{d-1}) = \mu_{t_1,\ldots,t_{d-1},t_d}(H_1 \times \cdots \times H_{d-1} \times \mathbb{R}).$$

The Kolmogorov's Existence Theorem can be stated as the following

Theorem 2.5.0.10 (Billingsley[4]). *If $\mu_{t_1,\ldots,t_d}$ are a system of distributions satisfying the consistency conditions 1) and 2), then there exists on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a stochastic process $[X_t : t \in T]$ having the $\mu_{t_1,\ldots,t_d}$ as its finite dimensional distributions.*

2.6. Gaussian process

An important class of stochastic processes are the class of Gaussian processes. We shall start with the definition of a Gaussian random variable.

Definition 2.6.0.4 (Billingsley[4])**.** *A real-valued random variable X is said to be Gaussian or normally distributed if it has finite mean $\mu = \mathbb{E}(X)$ and variance $\sigma^2 = \mathbb{E}(|X - \mu|^2) > 0$, and its distribution function is given by*

$$F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} (2\pi\sigma^2)^{-1/2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx,$$

That is, X has the density function

$$f_X(x) = (2\pi\sigma^2)^{-1/2} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

and characteristic function

$$\varphi_X(t) = e^{it\mu - \frac{1}{2}t^2\sigma^2}.$$

We abbreviate this by writing $X \sim N(\mu, \sigma^2)$. For the case $\mu = 0$ and $\sigma = 1$, we say that X has a standard normal distribution. So the distribution function of the standard normal random variable is

$$\varphi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-\frac{1}{2}x^2} dx.$$

Now consider $X = (X_1, \ldots, X_d)$ with independent components each having the standard normal distribution. Since each X_i has density $e^{-x^2/2}/\sqrt{2\pi}$, X has density

$$f(x) = \frac{1}{(2\pi)^{d/2}} e^{-\|x\|^2/2}.$$

Its characteristic function is

$$\mathbb{E}(\prod_{i=1}^{d} e^{it_iX_i}) = \prod_{i=1}^{d} e^{-t_i^2/2} = e^{-\|t\|^2/2}.$$

Let $A = [a_{ij}]$ be a $d \times d$ matrix, and let $Y = AX^T$, where X^T is the matrix transpose of X. Since $\mathbb{E}(X_iX_j) = \delta_{ij}$, the matrix $V = [v_{ij}]$ of the covariance of Y has entries $v_{ij} = \mathbb{E}(Y_iY_j) = \sum_{k=1}^{d} a_{ik}a_{jk}$. Thus $V = AA^T$. The matrix V is symmetric and positive definite, that is $\sum_{ij} v_{ij}x_ix_j = \|A^Tx^T\|^2 \geq 0$. Thus the characteristic function of

AX^T is

$$\begin{aligned}\mathbb{E}(e^{it(AX^T)}) &= \mathbb{E}(e^{i(A^Tt^T)^TX^T}) \\ &= e^{-\|A^Tt^T\|^2/2} \\ &= e^{-tVt^T/2}.\end{aligned}$$

Define a centered Gaussian distribution as any probability measure whose characteristic function has this form for some symmetric positive definite V.

If V is symmetric and positive definite, then for an orthogonal matrix U, $U^TVU^{=}D$ is diagonal matrix whose diagonal elements are the eigenvalues of V. So they are non-negative. If $\sqrt{D}$ is the diagonal matrix whose elements are the square roots of those of D, and if $A = U\sqrt{D}$, then $V = AA^T$. Thus for every positive definite V there exists a centered Gaussian distribution(namely the distribution of AX) with covariance matrix V and characteristic function $e^{-\frac{1}{2}tVt^T}$.

If V is nonsingular then A is as well. Since X has the density $f(x) = \frac{1}{(2\pi)^{d/2}}e^{-\|x\|^2/2}$, $Y = AX^T$ has density $f(A^{-1}x^T)|det(A^{-1})|$. Since $V = AA^T$, $|det(A^{-1})| = (det(V))^{-1/2}$. Moreover, $V^{-1} = (A^T)^{-1}A^{-1}$, so that $\|A^{-1}x^T\|^2 = xV^{-1}x^T$. Thus the normal distribution has density $(2\pi)^{d/2}(det(V))^{-1/2}e^{-\frac{1}{2}xV^{-1}x^T}$ if V is nonsingular.

Next we shall give a definition of multivariate Gaussian random variable.

Definition 2.6.0.5 (Billingsley[4])**.** *An $\mathbb{R}^d$-valued random variable X is said to be multivariate Gaussian if for every d-tuple of real numbers $(\alpha_1, \ldots, \alpha_d)$ the real-valued variable $Y = \quad {}_{i=1}^{d} \alpha_i X_i$ is Gaussian.*

In this case, the probability density of the d-dimensional vector X is given by

$$f_X(x) = (2\pi)^{-d/2}(det(V))^{-1/2}e^{-\frac{1}{2}(x-\mu)V^{-1}(x-\mu)^T}$$

where μ is the d vector with element $\mu_j = \mathbb{E}(X_j)$ and V is the positive definite $d \times d$ covariance matrix with elements

$$v_{ij} = Cov(X_i, X_j) = \mathbb{E}[(X_i - \mu_i)(X_j - \mu_j)].$$

We can check that the corresponding characteristic function of X is

$$\varphi_X(t) = e^{it\mu^T - \frac{1}{2}tVt^T}.$$

By the uniqueness property of characteristic function, a Gaussian distribution is completely determined by its covariance matrix.

We can now define a Gaussian process.

Definition 2.6.0.6 (Billingsley[4]). *A Gaussian process is a stochastic process* $[X(t) : t \in T]$ *possesing the finite dimensional distributions all of which* $\mu_{t_1,\ldots,t_k}$ *are multivariate Gaussian.*

From the definition and above notice, we can see that all the finite dimensional distributions of a real-valued Gaussian process are completely determined by mean and covariance functions:

$$\mu(t) = \mathbb{E}(X(t))$$

$$Cov(X(s), X(t)) = R(s,t) = \mathbb{E}((X(s) - \mu(s))^T (X(t) - \mu(t))).$$

2.7. Brownian motion and Brownian sheet

We start this section with the definition of increments of a stochastic process.

Definition 2.7.0.7. *Let* $\{X(t), t \in T\}$ *be a stochastic process with an index set* T*. Then the increment between any two points* s *and* $t \in T$ *is defined as a random variable*

$$X(s) - X(t).$$

Definition 2.7.0.8. *A stochastic process* $\{X(t), t \in T\}$ *with an index set* T *has stationary increments if for any* $s, t \in T$,

$$\{X(t) - X(s)\} =^d \{X(t-s) - X(0)\}.$$

Definition 2.7.0.9. *A stochastic process* $\{X(t), t \in \mathbb{R}\}$ *has independent increments if for any* $t_0 \leq t_1 \cdots \leq t_k$,

$$\mathbb{P}(X(t_i) - X(t_{i-1}) \in H_i, i \leq k) = \prod_{i \leq k} \mathbb{P}(X(t_i) - X(t_{i-1}) \in H_i),$$

where $H_i \in \mathcal{B}$ *for all* $0 \leq i \leq k$.

Note that if X is a Gaussian process then the Definition 2.7.0.9 is equivalent to

$$\mathbb{E}((X(t) - X(s))(X(s) - X(0))) = 0, \text{ for } 0 < s < t.$$

Definition 2.7.0.10 (Billingsley[4])**.** *A Brownian motion or Wiener process is a stochastic process* $[B(t), t \in \mathbb{R}]$*, on some probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, with these three properties.*

1) $\mathbb{P}(B(0)) = 1$.

2) The increments are independent.

3) B *is a Gaussian process with the covariance function of the form*

$$\mathbb{E}(B(s)B(t)) = min\{s, t\}.$$

Definition 2.7.0.11. *A Brownian sheet* B *in* $\mathbb{R}^d$, $d \geq 2$, *is a Gaussian process with stationary increments and the covariance function of the form*

$$\mathbb{E}(B(s)B(t)) = \prod_{i=1}^{d} min\{|s_i|, |t_i|\},$$

where $s = (s_1, \ldots, s_d)$, $t = (t_1, \ldots, t_d)$ *are in the same quadrant in* $\mathbb{R}^d$.

2.8. Wiener integral in $\mathbb{R}$

In ordinary calculus, we define the Riemann integral as following. First, we define the integral of a step function in a way that the integral represents the area beneath the graph. Then we extend the definition to a larger class of functions by approximation. They are called the Riemann integrable functions. We define the integral for general function f to be the limit of the integrals of step functions that converge to f. So we will do the same for the Wiener integral.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space also let $\{B(t) : t \in \mathbb{R}\}$ be a Brownian motion on $\mathbb{R}$ and let $\{f(t) : t \in \mathbb{R}\}$ be a well-behaved function whose specific qualifications shall be given later. Then we shall define the Wiener integral with respect to Brownian motion as a random variable.

$$I(f) = \int_{-\infty}^{\infty} f(t)dB(t).$$

This integration may be interpreted in the same way as classical Riemann-Stieltes integral. However, such an interpretation is not well-defined because almost all paths of Brownian motion are not of bounded variation. So the definition of the Wiener integral needs some more subtle development to make it well-defined.

We first let $\mathcal{B}$ denote the smallest σ-field that contains all of the half open subsets of the form $(a, b]$; that is, $\mathcal{B}$ is the set of Borel subsets of $\mathbb{R}$. Now we consider the class $L^2(\mathbb{R}, \mathcal{B}, dt)$ of all measurable functions f that are square integrable in the sense that

$$\int_{-\infty}^{\infty} f^2(t)dt < \infty.$$

For the simplicity, we will use the notation $L^2(dt)$ instead of $L^2(\mathbb{R}, \mathcal{B}, dt)$.

Next if we take f to be the indicator of the interval $(a, b] \subset \mathbb{R}$, then clearly f is an element of $L^2(dt)$. Then we define the Wiener integral by

$$I(f) = \int_{a}^{b} dB(t) = B(b) - B(a).$$

Also we want the Wiener integral to be linear so that it will determine how $I(f)$ must be defined for a large class of integrands. So we let $\mathcal{S}$ denote the subset of $L^2(dt)$ that consists of all functions that can be written as a finite sum of the form

$$f(t) = \sum_{i=0}^{n-1} a_i 1_{(t_i < t \leq t_{i+1})},$$

where a_i is a constant for all i and $a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$. Clearly, $f \in L^2(dt)$. We call $\mathcal{S}$ the subspace of all step functions of $L^2(dt)$. Then for all functions in $\mathcal{S}$ we simply define $I(f)$ by

$$I(f) = \sum_{i=0}^{n-1} a_i(B(t_{i+1}) - B(t_i)).$$

Next step we need to extend the domain of I from $\mathcal{S}$ to all of $L^2(dt)$ and the key is to show that $I : \mathcal{S} \to L^2(d\mathbb{P})$ is a continuous mapping. So we have the following lemma.

Lemma 2.8.0.3. *For $f \in \mathcal{S}$ we have*

$$\|I(f)\|_{L^2(d\mathbb{P})} = \|f\|_{L^2(dt)}.$$

Proof. Since the Brownian motion has independent increments,

$$\begin{aligned} \|I(f)\|^2_{L^2(d\mathbb{P})} &= \int_\Omega \Big(\sum_{i=0}^{n-1} a_i(B(t_{i+1}) - B(t_i))\Big)^2 d\mathbb{P} \\ &= \sum_{i=0}^{n-1} a_i^2 |t_{i+1} - t_i| \\ &= \sum_{i=0}^{n-1} a_i^2 \lambda((t_i, t_{i+1}]) \\ &= \|f\|^2_{L^2(dt)}, \end{aligned}$$

where λ is the Lebesgue measure on $\mathbb{R}$. $\square$

By the linearity of I, it implies that I takes equal distant points in $\mathcal{S}$ to equal distant points in $L^2(d\mathbb{P})$. Thus I maps a Cauchy sequence in $\mathcal{S}$ into a Cauchy sequence in $L^2(d\mathbb{P})$. The importance of this leads to the next lemma which shows that any $f \in L^2(dt)$ can be approximated arbitrarily by elements in $\mathcal{S}$.

Lemma 2.8.0.4. *For any $f \in L^2(dt)$, there exists a sequence $\{f_n\}$ with $f_n \in \mathcal{S}$ such that*

$$\|f - f_n\|_{L^2(dt)} \to 0 \text{ as } n \to \infty.$$

Now for any $f \in L^2(dt)$, this approximation lemma tells us that there is a sequence $\{f_n\} \in \mathcal{S}$ such that f_n converges to f in $L^2(dt)$. Before we give a formal definition it is necessary to check that the random variable $I(f)$ does not depend on the specific choice of a sequence f_n.

Lemma 2.8.0.5. *Let $f \in L^2(dt)$. Assume f_n and g_n are sequences in S such that $f_n \to f$ and $g_n \to f$ then*

$$\lim_{n\to\infty} I(f_n) = \lim_{n\to\infty} I(g_n) = I(f) \text{ a.s.}$$

Proof. Since f_n and g_n are Cauchy sequences in S and I is linear, $I(f_n)$ and $I(g_n)$ are also Cauchy sequences in $L^2(d\mathbb{P})$. Since $L^2(d\mathbb{P})$ is complete, $I(f_n)$ converges to F and $I(g_n)$ converges to G in $L^2(d\mathbb{P})$. Hence

$$\begin{aligned}
\|F - G\|_{L^2(d\mathbb{P})} &\leq \|I(f_n) - F\|_{L^2(d\mathbb{P})} + \|I(g_n) - G\|_{L^2(d\mathbb{P})} + \|I(f_n) - I(g_n)\|_{L^2(d\mathbb{P})} \\
&\leq \|I(f_n) - F\|_{L^2(d\mathbb{P})} + \|I(g_n) - G\|_{L^2(d\mathbb{P})} \\
&\quad + \|I(f_n) - I(f)\|_{L^2(d\mathbb{P})} + \|I(g_n) - I(f)\|_{L^2(d\mathbb{P})} \to 0 \text{ as } n \to \infty
\end{aligned}$$

□

Now we are ready to give a formal definition of the Wiener integral that is used in this thesis.

Definition 2.8.0.12. *Let $f \in L^2(dt)$ then there exists a sequence f_n in $\mathcal{S}$ such that f_n converges to f. Then we define the Wiener integral of f by*

$$I(f) = \lim_{n\to\infty} I(f_n),$$

where the interpretation is that the random variable $I(f)$ is the a.s. unique element of $L^2(d\mathbb{P})$ such that $\|I(f) - I(f_n)\|_{L^2(d\mathbb{P})} \to 0$ as $n \to \infty$.

This completes the definition of $I(f)$. Next we will prove some basic properties of $I(f)$, that is a proposition stating that limits of Gaussian random variables are Gaussian.

Proposition 2.8.0.1. *If f_n is a sequence of Gaussian random variables with mean zero and variance $\|f_n\|^2_{L^2(dt)}$ and f is a random variable such that*

$$\lim_{n\to\infty} f_n = f \text{ in } L^2(d\mathbb{P}),$$

then $I(f)$ is Gaussian with mean zero and variance $\|f\|^2_{L^2(dt)}$.

Proof. Since f_n is Gaussian with mean zero and variance $\|f_n\|^2_{L^2(dt)}$, by the dominated convergence theorem we get

$$\begin{aligned}
\mathbb{E}(e^{itI(f)}) &= \lim_{n\to\infty} \mathbb{E}(e^{itI(f_n)}) \\
&= \lim_{n\to\infty} e^{-\frac{1}{2}t^2\|f_n\|^2_{L^2(dt)}} \\
&= e^{-\frac{1}{2}t^2\|f\|^2_{L^2(dt)}}
\end{aligned}$$

Hence $I(f)$ is Gaussian with mean zero and variance $\|f\|^2_{L^2(dt)}$. □

For the next section, we will extend the definition of Wiener integral from $\mathbb{R}$ to $\mathbb{R}^d$.

2.9. Wiener integral in $\mathbb{R}^d$

We can define the Wiener integral with respect to Brownian motion in $\mathbb{R}^d$ in the similar way we did for the real line. First let J be a collection of subsets of $\mathbb{R}^d$ of the form $K = \{t \in \mathbb{R}^d : a_1 < t_1 \le b_1, \ldots, a_d < t_d \le b_d\}$. Also let $\{B(t) : t \in \mathbb{R}^d\}$ be a Brownian motion on $\mathbb{R}^d$. Let $\mathcal{B}$ be the smallest σ-field containing all of the half open subsets of J. It is similar to the case $d = 1$. We consider the class $L^2(K, \mathcal{B}, dt)$ of all measurable functions f that are square integrable. That is

$${}_{\mathbb{R}^d} f^2(t)dt < \infty.$$

For the simplicity, we shall use the notation $L^2(dt)$ instead of $L^2(\mathbb{R}^d, \mathcal{B}, dt)$. Now consider an indicator function f of $A = (a_1, b_1] \times \cdots \times (a_d, b_d] \in J$. That is $f(t) = 1_A(t)$. We similarly define the Wiener integral for f by

$$\begin{aligned} I(f) &= \mathop{}\limits_{A} dB(t) \\ &= B(A) \\ &= B(b_1,\ldots,b_d) - \mathop{}\limits_{i=1}^{d} B(b_1,\ldots,b_{i-1},a_i,b_{i+1},\ldots,b_d) \\ &\quad + \mathop{}\limits_{1\le i<j\le d} B(b_1,\ldots,b_{i-1},a_i,b_{i+1},\ldots,b_{j-1},a_j,b_{j+1},\ldots,b_d) - \cdots (-1)^d B(a_1,\ldots,a_d) \\ &=^d B(b_1-a_1,\ldots,b_d-a_d). \end{aligned}$$

Notice that

$$\begin{aligned} \mathop{}\limits_{\Omega} I^2(f)d\mathbb{P} &= \mathop{}\limits_{\Omega} B^2(b_1-a_1,\ldots,b_d-a_d)d\mathbb{P} \\ &= (b_1-a_1)\cdots(b_d-a_d) \\ &= \lambda_d(A) \\ &= \|f\|^2_{L^2(dt)}, \end{aligned}$$

where λ_d is the Lebesgue measure in $\mathbb{R}^d$.

Now let $A_i = (a_1(i), b_1(i)] \times \cdots \times (a_d(i), b_d(i)] \subset \mathbb{R}^d$ with A_i is mutually disjoint and we consider step function f of the form

$$f(t) = \mathop{}\limits_{i=0}^{n-1} a_i 1_{A_i}.$$

We denote $\mathcal{S}$ as a subspace of all step functions f of $L^2(dt)$. Then we define $I(f)$ by

$$I(f) = \mathop{}\limits_{i=0}^{n-1} a_i B(A_i),$$

So we have the following lemmas. Since some proofs are similar to the case $d = 1$, some of the proof shall be skipped.

Lemma 2.9.0.6. *For $f \in \mathcal{S}$ we have*

$$\|I(f)\|_{L^2(d\mathbb{P})} = \|f\|_{L^2(dt)}.$$

Proof. By the property of Brownian sheet in $\mathbb{R}^d$ and since A_i is mutually disjoint, we have

$$\begin{aligned}
\int_\Omega I^2(f)d\mathbb{P} &= \int_\Omega \Big(\sum_{i=0}^{n-1} a_i B(A_i)\Big)^2 d\mathbb{P} \\
&= \sum_{i=0}^{n-1} a_i^2 \int_\Omega B^2(A_i)d\mathbb{P} \\
&= \sum_{i=0}^{n-1} a_i^2 \lambda_d(A_i) \\
&= \|f\|^2_{L^2(dt)}.
\end{aligned}$$

$\square$

Lemma 2.9.0.7. *For any $f \in L^2(dt)$, there exists a sequence $\{f_n\}$ with $f_n \in \mathcal{S}$ such that*

$$\|f - f_n\|_{L^2(dt)}, \to 0 \ \text{a.s.}$$

Lemma 2.9.0.8. *Let $f \in L^2(dt)$. Assume f_n and g_n are sequences in S such that $f_n \to f$ and $g_n \to f$ then*

$$\lim_{n\to\infty} I(f_n) = \lim_{n\to\infty} I(g_n) = I(f) \ \text{a.s.}$$

Similarly, we define the Wiener integral for any f in $L^2(dt)$ as the following.

Definition 2.9.0.13. *Let $f \in L^2(dt)$ then there exists a sequence f_n in $\mathcal{S}$ such that f_n converges to f. Then we define the Wiener integral of f by*

$$I(f) = \lim_{n\to\infty} I(f_n),$$

where the interpretation is that the random variable $I(f)$ is the a.s. unique element of $L^2(d\mathbb{P})$ such that $\|I(f) - I(f_n)\|_{L^2(d\mathcal{P})} \to 0$ as $n \to \infty$.

Also the basic properties of $I(f)$ can be proved similarly.

Proposition 2.9.0.2. *If f_n is a sequence of Gaussian random variables with mean zero and variance $\|f_n\|^2_{L^2(dt)}$ and f is a random variable such that*

$$\lim_{n\to\infty} f_n = f \ \text{in } L^2(d\mathbb{P}),$$

then $I(f)$ is Gaussian with mean zero and variance $\|f\|^2_{L^2(dt)}$.

Corollary 2.9.0.3. *For f and g in $L^2(dt)$,*

$$Cov(I(f), I(g)) = \mathbb{E}(I(f)I(g)) = \int_{\mathbb{R}^d} fgdt.$$

In this thesis, we are interested in integral representations of the form

$$Z(t) = \int_T K(t,u)dB(u)$$

where K is a function defined on $\mathrm{T} \times T$, T is a compactly supported subset of $\mathbb{R}^d$.

3. FRACTIONAL BROWNIAN MOTION

In this chapter, we review the main properties that make fractional Brownian motion interesting for many applications.

3.1. Fractional Brownian motion and its definition

The fractional Brownian motion was first introduced in a Hilbert space framework by Kolmogorov in 1940. It was then studied by Yaglom. In 1968, the name *fractional Brownian motion* was used by Mandelbrot and Van Ness to describe the process. It was Mandelbrot who named the parameter H of B_H after the name of the hydrologist Hurst, who made a statistical study of water run-offs of the Nile river for many years.

Next we give a formal definition of the fractional Brownian motion.

Definition 3.1.0.14 (Embrechts[6]). *Let $0 < H \leq 1$. A real-valued Gaussian process $\{B_H(t), t \in \mathbb{R}^d\}$ is called fractional Brownian motion if $\mathbb{E}[B_H(t)] = 0$ and*

$$\mathbb{E}[B_H(t)B_H(s)] = \frac{1}{2}[\|t\|^{2H} + \|s\|^{2H} - \|t-s\|^{2H}].$$

For the case $H = 1/2$, $B_{1/2}$ is the standard Brownian motion in $\mathbb{R}^d$.

Recall that the distribution of a Gaussian process is determined by its mean and covariance structure. That is the distribution of a process is determined by all joint distributions and the density of a multidimensional Gaussian distribution is explicitly given through its mean and covariance matrix. Thus, the condition in the above Definition determine a unique Gaussian process.

Next we shall discuss the well known properties of fractional Brownian motion. Only important proofs shall be given. For more details, we refer to [6]. We start with the selfsimilar property.

Definition 3.1.0.15 (Embrechts[6]). *An $\mathbb{R}^d$-valued stochastic process $\{X(t), t \in \mathbb{R}^d\}$ is said to be selfsimilar if for any $a > 0$, there exists $b > 0$ such that*

$$\{X(at)\} =^d \{bX(t)\}.$$

Theorem 3.1.0.11 (Embrechts[6]). *If $\{X(t)\}, t \in \mathbb{R}\}$ is nontrivial, probabilistic continuous at $t = 0$ and selfsimilar, then there exists a unique $H \geq 0$ such that*

$$\{X(at)\} =^d \{a^H X(t)\}.$$

We call H the exponent of selfsimilarity of the process $\{X(t), t \in \mathbb{R}\}$. We refer to such a process as H-selfsimilar(or H-ss, for short).

Proposition 3.1.0.3 (Embrechts[6]). *If $\{X(t), t \in \mathbb{R}\}$ is H-ss and $H > 0$, then $X(0) = 0$ almost surely.*

Proof. By the previous Theorem,

$$X(0) =^d a^H X(0).$$

Then we let $a \to 0$. □

Theorem 3.1.0.12 (Embrechts[6]). *Let X be H-ss with stationary increments in $\mathbb{R}^d$ and $\mathbb{E}(X^2(1)) < \infty$. Then*

$$\mathbb{E}(X(s)X(t)) = \frac{\mathbb{E}(X^2(1))}{2}(\|t\|^{2H} + \|s\|^{2H} - \|t-s\|^{2H}).$$

Next, some properties of fractional Brownian motion shall be stated and proved here.

Theorem 3.1.0.13 (Embrechts[6]). *A fractional Brownian motion $\{B_H(t), t \in \mathbb{R}^d\}$ is H-ss.*

Proof. We have that

$$\begin{aligned}
\mathbb{E}[B_H(at)B_H(as)] &= \frac{1}{2}[\|at\|^{2H} + \|as\|^{2H} - \|a(t-s)\|^{2H}] \\
&= a^{2H}\mathbb{E}[B_H(t)B_H(s)] \\
&= \mathbb{E}[(a^H B_H(t))(a^H B_H(s))].
\end{aligned}$$

Since all processes are mean zero Gaussian, $\{B_H(at)\} =^d \{a^H B_H(t)\}$. □

Note that from this theorem, we can conclude that the only Gaussian process with stationary increments and H-ss is fractional Brownian motion with index H.

Theorem 3.1.0.14 (Embrechts[6])**.** *A fractional Brownian motion $\{B_H(t), t \in \mathbb{R}^d\}$ has stationary increments.*

Proof. Again, it suffices to consider only covariances. We have

$$\begin{aligned}
\mathbb{E}[(B_H(t+h) - B_H(h))(B_H(s+h) - B_H(h))] &= \mathbb{E}[B_H(t+h)B_H(s+h)] - \mathbb{E}[B_H(t+h)B_H(h)] \\
&\quad -\mathbb{E}[B_H(s+h)B_H(h)] + \mathbb{E}[B_H(t)^2] \\
&= \frac{1}{2}[\|t+h\|^{2H} + \|s+h\|^{2H} - \|t-s\|^{2H} \\
&\quad -\|t+h\|^{2H} + \|h\|^{2H} - \|t\|^{2H} \\
&\quad -\|s+h\|^{2H} + \|h\|^{2H} - \|s\|^{2H} + 2\|h\|^{2H}] \\
&= \frac{1}{2}[\|t\|^{2H} + \|s\|^{2H} - \|t-s\|^{2H}] \\
&= \mathbb{E}[B_H(t)B_H(s)],
\end{aligned}$$

We can conclude that

$$\{B_H(t+h) - B_H(h)\} =^d \{B_H(t)\}.$$

Therefore $B_H(t)$ has stationary increments. □

Theorem 3.1.0.15 (Embrechts[6])**.** *For $0 < H < 1$, a fractional Brownian motion $\{B_H(t), t \in \mathbb{R}\}$ has a Wiener integral representation*

$$C_H \quad {}_{\mathbb{R}} (t-u)^{H-1/2} 1_{(-\infty,t)} - (-u)^{H-1/2} 1_{(-\infty,0)} dB(u),$$

where

$$C_H = [\int_{-\infty}^{0} ((1-u)^{H-1/2} - (-u)^{H-1/2})^2 du + \frac{1}{2H}]^{-1/2}.$$

Proof. First we prove for the case $t \geq 0$.

Let

$$X_H(t) = C_H \int_{\mathbb{R}} (t-u)^{H-1/2} 1_{(-\infty,t)} - (-u)^{H-1/2} 1_{(-\infty,0)} dB(u).$$

We can see that

$$\begin{aligned}
\int_{\mathbb{R}} [(t-u)^{H-1/2} 1_{(-\infty,t)} - (-u)^{H-1/2} 1_{(-\infty,0)}]^2 du &= \int_{-\infty}^{0} ((t-u)^{H-1/2} - (-u)^{H-1/2})^2 du \\
&\quad + \int_{0}^{t} (t-u)^{2H-1} du \\
&= t^{2H} \int_{0}^{\infty} ((1+u)^{H-1/2} - u^{H-1/2})^2 du + \frac{t^{2H}}{2H} \\
&\leq t^{2H} (\int_{0}^{1} max((1+u)^{2H-1}, u^{2H-1}) du \\
&\quad + \int_{1}^{\infty} (H-1/2)^2 (\int_{u}^{u+1} v^{H-3/2} dv)^2 du) + \frac{t^{2H}}{2H} \\
&\leq \frac{t^{2H}}{2H} (max(2^{2H}-1, 1) + 1) \\
&\quad + t^{2H} (H-1/2)^2 \int_{1}^{\infty} u^{2H-3} du \\
&= \frac{t^{2H}}{2H} (max(2^{2H}-1, 1) + 1) \\
&\quad + t^{2H} \frac{(H-1/2)^2}{2-2H} \\
&\leq \infty. \qquad (3.1)
\end{aligned}$$

So X_H is well defined. Then by change of variable ($v = \frac{u}{t}$), we have

$$\begin{aligned}
\mathbb{E}[X_H(t)^2] &= C_H^2 \{ [\int_{-\infty}^{0} (t-u)^{H-1/2} - (-u)^{H-1/2} dB(u)]^2 + [\int_{0}^{t} (t-u)^{H-1/2} dB(u)] \}^2 \\
&= C_H^2 [\int_{-\infty}^{0} [(t-u)^{H-1/2} - (-u)^{H-1/2}]^2 du + \int_{0}^{t} (t-u)^{2H-1} du] \\
&= t^{2H}
\end{aligned}$$

Moreover,

$$
\begin{aligned}
\mathbb{E}[(X_H(t+h)-X_H(t))^2] &= C_H^2\mathbb{E}[(\int_{-\infty}^{h}((t+h-u)^{H-1/2}-(h-u)^{H-1/2})dB(u) \\
&\quad + \int_{h}^{h+t}(t+h-u)^{h-1/2}dB(u))^2] \\
&= C_H^2\{\int_{-\infty}^{h}((t+h-u)^{H-1/2}-(h-u)^{H-1/2})^2du \\
&\quad + \int_{h}^{h+t}(t+h-u)^{2H-1}du\} \\
&= C_H^2\{\int_{-\infty}^{0}((t-u)^{H-1/2}-(-u)^{H-1/2})^2du+\int_{0}^{t}(t-u)^{2H-1}du\} \\
&= t^{2H}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathbb{E}[X_H(t)X_H(s)] &= \frac{1}{2}\{\mathbb{E}[X_H(t)^2]+\mathbb{E}[X_H(s)^2]-\mathbb{E}[(X_H(t)-X_H(s))^2] \\
&= \frac{1}{2}\{t^{2H}+s^{2H}-|t-s|^{2H}\}
\end{aligned}
$$

For the case $t<0$, we can use the change of variable by letting $v=t+u$. Then the proof is similar to the case $t \geq 0$. Therefore, $X_H(t)$ for $0<H<1$ is fractional Brownian motion. □

Note that for the case $H=1$, we have

$$\mathbb{E}[B_1(t)-B_1(s)]^2 = ts.$$

Then

$$
\begin{aligned}
\mathbb{E}[(B_1(t)-t)^2] &= \frac{1}{2}[\mathbb{E}[B_1(t)^2]-2t\mathbb{E}[B_1(t)]+t^2 \\
&= t^2-2t^2+t^2 \\
&= 0,
\end{aligned}
$$

so that $B_1(t)=t$ almost surely.

Theorem 3.1.0.16 (Embrechts[6]). *Fractional Brownian motion is unique in the sense that the class of all fractional Brownian motions on* $\mathbb{R}^d$ *coincides with that of all Gaussian processes on* $\mathbb{R}^d$ *with stationary and selfsimilar increments.*

Proof. Assume $X_H(t)$ is $H-ss$ and has stationary increments. Then

$$\begin{aligned}
\mathbb{E}[X_H(t)X_H(s)] &= \frac{1}{2}\{\mathbb{E}[X_H(t)^2]+\mathbb{E}[X_H(s)^2]-\mathbb{E}[(X_H(t)-X_H(s))^2]\} \\
&= \frac{1}{2}\{\mathbb{E}[X_H(t)^2]+\mathbb{E}[X_H(s)^2]-\mathbb{E}[X_H(\|t-s\|)^2]\} \\
&= \frac{1}{2}\{\|t\|^{2H}+\|s\|^{2H}-\|t-s\|^{2H}\}
\end{aligned}$$

So X_H has the same covariance structure as the fractional Brownian motion.

Since $\{X_H(t)\}$ is mean zero Gaussian, it is the same as $B_H(t)$ in distribution. □

Theorem 3.1.0.17 (Embrechts[6]). $\{B_H(t), t \geq 0\}$ *has independent increments if and only if* $H = 1/2$.

Proof. For $0 < s < t$,

$$\begin{aligned}
\mathbb{E}[B_H(s)(B_H(t)-B_H(s))] &= \frac{1}{2}\{t^{2H}+s^{2H}-(t-s)^{2H}-2s^{2H}\} \\
&= \frac{1}{2}\{t^{2H}-s^{2H}-(t-s)^{2H}\}.
\end{aligned}$$

It is easy to see that the equality equals to 0 if and only if $H = \frac{1}{2}$.

Therefore B_H has independent increments when $H = \frac{1}{2}$. □

The Mandelbrot Van Ness representation of fractional Brownian motion is popular, but there is another useful representation as a Wiener integral over a finite interval. This attributed to Molchan-Golosov, see [21].

Theorem 3.1.0.18 (Embrechts[6]). *When* $0 < H < 1$ *and* $t \geq 0$,

$$B_H(t) =^d C_H \int_0^t K(t,u)\,dB(u),$$

where

$$K(t,u) = \{(\frac{t}{u})^{H-1/2}(t-u)^{H-1/2} - (H-\frac{1}{2})u^{1/2-H} \int_u^t x^{H-3/2}(x-u)^{H-1/2}dx\}$$

and C_H is a normalizing constant.

For the details about the proof, we refer to [21].

4. SERIES EXPANSIONS OF GAUSSIAN PROCESS

In this chapter, we shall discuss series representation of fractional Brownian motion. Then we will give a well known example called the Karhunen-Loéve representation and some well-known results.

Let $\{Z_i, i \geq 1\}$ be an i.i.d sequence of Gaussian processes. Now let C be a compact subset in $\mathbb{R}^d$. Then we define for $t \in C$

$$Z(t) = \sum_{k\geq 1} \sqrt{\lambda_k} f_k(t) Z_k. \tag{4.1}$$

Then we have the following theorem.

Theorem 4.0.0.19. *Let $\{Z_i, i \geq 1\}$ be an i.i.d sequence of standard Gaussian processes and define*

$$Z(t) = \sum_{k\geq 1} \lambda_k f_k(t) Z_k$$

if $\sum_{k\geq 1} \lambda_k^2 f_k^2(t) < \infty$ then $Z(t)$ converges and is a Gaussian process in a compact subset C of $\mathbb{R}^d$.

Proof. Assume $\sum_{k\geq 1} \lambda_k f_k^2 f(t) < \infty$. Clearly $Z(t)$ has mean zero. So it suffices to show $Z(t)$ has finite variance.

$$\begin{aligned}
\mathbb{E}(Z^2(t)) &= \int_\Omega Z(t)^2 d\mathbb{P} \\
&= \int_\Omega \Big(\sum_{k\geq 1} \lambda_k f_k(t) Z_k(t)\Big)^2 d\mathbb{P} \\
&\leq \sum_{k\geq 1} \int_C \lambda_k^2 f_k(t)^2 \\
&< \infty
\end{aligned}$$

Therefore $Z(t)$ converges and is a Gaussian process. $\square$

Let C be a compact subset in $\mathbb{R}^d$ and K be a function defined on $C \times C$. Now define an operator T on $L^2(C)$ by

$$T(x)(s) = \int_C K(t,s)x(t)dt.$$

We say λ is an eigenvalue of T if there is a nonzero vector $x \in L^2(C)$ such that $T(x)(s) = \lambda x(s)$. We call the corresponding x an eigenfunction. This means that λ is an eigenvalue of T if and only if $T - \lambda I$ is not 1-1. If f_k is orthonormal in (4.1) above then we have the following corollary.

Corollary 4.0.0.4 (Hernandez[10])**.** *If $Z(t) = \sum_{k\geq 1} \lambda_k f_k(t) Z_k$ with f_k orthonormal then each f_k is an eigenfunction of the operator $T(x)(s) = \int_C \mathbb{E}(Z(s)Z(t))x(t)dt$.*

Proof. Since Z_k is i.i.d,

$$\begin{aligned}
\mathbb{E}(Z(s)Z(t)) &= \int_\Omega \Big(\sum_{k\geq 1} \lambda_k f_k(s) Z_k\Big)\Big(\sum_{l\geq 1} \lambda_l f_l(t) Z_l\Big) d\mathbb{P} \\
&= \sum_{k\geq 1} \lambda_k^2 f_k(s) f_k(t)
\end{aligned}$$

So

$$\begin{aligned}
\int_C \mathbb{E}(Z(t)Z(s)) f_k(s) ds &= \int_C \sum_{l\geq 1} \lambda_l^2 f_l(s) f_l(t) f_k(s) ds \\
&= \lambda_k^2 f_k(t)
\end{aligned}$$

Hence f_k is eigenfunction of the operator T. □

In the next section, we will discuss a well known series representation namely the Karhunen-Loéve expansion.

4.1. Karhunen-Loéve expansion

We start this section by the following lemma.

Lemma 4.1.0.9. *Let T be a self adjoint compact operator on a Hilbert space H. Then H has an orthonormal basis consisting of eigenfunctions of T.*

Now assume K is positive definite, symmetric and continuous. Then T is self adjoint and compact operator on the infinite dimensional Hilbert space $L^2(C)$. By the Lemma, we have $L^2(C)$ has an orthonormal basis consisting of eigenfunction of T, say $e_n, n \geq 1$. Since K is positive definite, all eigenvalues of T are positive, say $\lambda_n, n \geq 1$. Without loss of generality, we assume $\lambda_1 \geq \lambda_2 \geq \cdots > 0$. Next we shall state the Mercer's Theorem.

Theorem 4.1.0.20 (Hernandez[10]). *If K is a continuous, symmetric and positive definite function, then*

$$\underset{k\geq 1}{} \lambda_k < \infty$$

and

$$K(s,t) = \underset{k\geq 1}{} \lambda_k e_k(s) e_k(t)$$

is absolutely and uniformly convergent in both variables.

Since all covariance functions are symmetric and positive definite, we have the following corollary.

Corollary 4.1.0.5 (Hernandez[10]). *If K is a continuous covariance function of a stochastic process X, then*

$$\underset{k\geq 1}{} \lambda_k < \infty$$

and

$$K(s,t) = \underset{k\geq 1}{} \lambda_k e_k(s) e_k(t)$$

is absolutely and uniformly convergent in both variables.

Definition 4.1.0.16 (Hernandez[10]). *A stochastic process X over $S \subset \mathbb{R}^d$ is mean square(m.s.) continuous at $x_0 \in S$ if $\mathbb{E}|X(x) - X(x_0)|^2 \to 0$ as $x \to x_0$. X is said to be mean square (m.s.) continuous if it is continuous at every point in S.*

For example, fractional Brownian motion with index H is mean square continuous. Now we are going to state the Kahunen Loeve expansion theorem.

Theorem 4.1.0.21 (Hernandez[10]). *[the Karhunen-Loéve Representation]A stochastic process X with mean zero and the continuous covariance function K on a compact set $C \subset \mathbb{R}^d$ is m.s continuous if and only if*

$$X(x) = \sum_{k=1}^{\infty} \lambda_k Z_k e_k(x),$$

converges uniformly in $L^2(C)$, where

$$Z_k = \frac{1}{\lambda_k} \int_C X(t) e_k(t) dt$$

and the $\{Z_k, k \geq 1\}$ are orthogonal random variables with $\mathbb{E}(Z_n) = 0$, $\mathbb{E}(Z_n^2) = 1$.

Note that in the case where $\{X(t), t \in C\}$ is a Gaussian process, the $\{Z_k\}$ are independent standard Gaussian. The next section, we give an example of the Kahunen Loeve expansion for a Gaussian process, namely standard Brownian motion.

4.2. Karhunen-Loéve expansion of Brownian motion

Let $\{B(t), t \geq 0\}$ be a standard Brownian motion. Then we have $K(s,t) = \mathbb{E}(B(s)B(t)) = min(s,t), s \geq 0, t \geq 0$ Now let $C = [0,1]$. Consider

$$\int_0^1 min(s,t)x(t)dt = \lambda x(s) \Leftrightarrow \lambda x''(s) + x(s) = 0, x(0) = 0 = x'(1).$$

So

$$x(s) = C_1 \cos(\frac{1}{\sqrt{\lambda}} s) + C_2 \sin(\frac{1}{\sqrt{\lambda}} s),$$

where C_1 and C_2 are constant. By the boundary conditions, we get

$$\lambda_n = \frac{1}{(n-1/2)^2 \pi^2}, n \geq 1$$

and

$$e_n(s) = \sqrt{2}\sin((n - \frac{1}{2})\pi t).$$

Therefore the corresponding Karhunen-Loeve expansion is

$$B(t) = \sqrt{2}\sum_{k\geq 1}\frac{\sin((n-\frac{1}{2})\pi t)}{(n-\frac{1}{2})\pi}Z_n$$

Similarly, let $\{W(t), 0 \leq t \leq 1\}$ be a Brownian bridge. Then we have

$$K(s,t) = \mathbb{E}(W(s)W(t)) = min(s,t) - st, 0 \leq s, t \leq 1.$$

So the corresponding equation is

$$\int_0^1 (min(s,t) - st)x(s)ds = \lambda x(t) \Leftrightarrow \lambda x''(t) + x(t) = 0, x(0) = 0 = x(1).$$

Hence the eigenvalues and eigenfunctions are given by

$$\lambda_k = \frac{1}{k^2\pi}, e_k(t) = \sqrt{2}\sin(k\pi t).$$

The corresponding Karhunen-Loeve expansion is

$$W(t) = \sqrt{2}\sum_{k\geq 1}\frac{\sin(k\pi t)}{k\pi}Z_k,$$

where $\{Z_k\}$ are independent and standard Gaussian.

5. MAIN RESULTS

5.1. An integral representation of fractional Brownian motion in $\mathbb{R}^2$

In this chapter, we wish to create methods for extending integral representations of fractional Brownian motion in $\mathbb{R}$ to a representation of fractional Brownian motion in higher dimension of Euclidian spaces. Then we will obtain series representations of fractional Brownian motion from the integral representation.

First, we shall study the case of $\mathbb{R}^2$. Later we shall consider more complicated cases. Now suppose we have an integral representation on an interval $[-1,1]$ of the form

$$B_H(t) = \int_0^\infty K(t,u)dB(u).$$

Then we want to extend this integral representation to an integral representation in $\mathbb{R}^2$. It makes sense that we try to extend this by rotating vectors u in $\mathbb{R}^2$. So we consider the dot product in $\mathbb{R}^2$ and define $K'(t,u) = K(t \cdot \frac{u}{\|u\|}, \|u\|)$, where $\|u\|^2 = u \cdot u$. Since $-1 \leq t \cdot \frac{u}{\|u\|} \leq 1$, K' is well defined for all $u \in \mathbb{R}^2$. Then we have the following theorem.

Theorem 5.1.0.22. *For $t \in \mathbb{R}^2$ and $\|t\| \leq 1$ we define*

$$B_H(t) = \int_{\mathbb{R}^2} K'(t,u)\frac{1}{\|u\|^{1/2}}dB(u),$$

where $K'(t,u) = K(t \cdot \frac{u}{\|u\|}, \|u\|)$. Then B_H is a fractional Brownian motion with index H on the unit disk in $\mathbb{R}^2$.

Proof. We need to check that B_H is fractional Brownian motion with index H. It suffices to show

$$E(B_H(t)B_H(s)) = C_H(\|t\|^{2H} + \|s\|^{2H} - \|t-s\|^{2H}).$$

Now we will need to change the integral to polar coordinate. That is for $u = (u_1, u_2) \in \mathbb{R}^2$ let $u_1 = r\cos(\theta)$ and $u_2 = r\sin(\theta)$. Notice that $t \cdot \frac{u}{\|u\|} = \|t\|(\frac{t}{\|t\|} \cdot \frac{u}{\|u\|}) = \|t\|\cos(\theta)$, where

θ is the angle between vectors t and u.

So

$$\begin{aligned}
\mathbb{E}(B_H(t)B_H(s)) &= \mathbb{E}(\int_{\mathbb{R}^2} K(t \cdot \frac{u}{\|u\|}, \|u\|)\frac{1}{\|u\|^{1/2}}dB(u) \int_{\mathbb{R}^2} K(s \cdot \frac{v}{\|v\|}, \|v\|)\frac{1}{\|v\|^{1/2}}dB(v)) \\
&= \int_{\mathbb{R}^2} K(t \cdot \frac{u}{\|u\|}, \|u\|)K(s \cdot \frac{u}{\|u\|}, \|u\|)\frac{1}{\|u\|}du \\
&= \int_0^{2\pi}\int_0^{\infty} K(\|t\| \cdot \cos(\theta), r)K(\|s\| \cdot \cos(\theta), r)drd\theta \\
&= \frac{1}{2}\int_0^{2\pi} \|t\|^{2H}|\cos(\theta)|^{2H} + \|s\|^{2H}|\cos(\theta)|^{2H} - \|(t-s)\|^{2H}|\cos(\theta)|^{2H}d\theta \\
&= C_H(\|t\|^{2H} + \|s\|^{2H} - \|t-s\|^{2H})
\end{aligned}$$

where $C_H = 1/2 \int_0^{2\pi} |\cos(\theta)|^{2H}d\theta$ is a constant.

Therefore B_H is a fractional Brownian motion with index H on the unit disk. □

In general, if we have $B_H(t) = \int_0^{\infty} K(t,u)dB(u)$ is a fractional Brownian motion on interval $[a,b]$, are we able to obtain a fractional Brownian motion in $\mathbb{R}^2$ with the same method we did as above? We are going to answer the question now.

Let $B_H(t) = \int_0^{\infty} K(t,u)dB(u)$ be a fractional Brownian motion in the interval $[0,1]$. We shall use the same method as above to try to extend this fractional Brownian motion to one in $\mathbb{R}^2$. Now define for each $\|t\| \leq 1$,

$$B_H(t) = \int_{\mathbb{R}^2} K(t \cdot \frac{u}{\|u\|}, \|u\|)\frac{1}{\|u\|^{1/2}}dB(u)$$

Notice that since $t \cdot \frac{u}{\|u\|}$ needs to be a value in $[0,1]$ for all $u \in \mathbb{R}^2$, $K(t \cdot \frac{u}{\|u\|}, \|u\|)$ is not defined for all $u \in \mathbb{R}^2$. Now consider β be the angle between t and x-axis. Thus

$$\begin{aligned}
\mathbb{E}(B_H^2(t)) &= \int_{[t\cdot\frac{u}{\|u\|}\geq 0]} K^2(t\cdot\frac{u}{\|u\|}, \|u\|)\frac{1}{\|u\|}du \\
&= \int_{\beta-\pi/2}^{\beta+\pi/2}\int_0^{(cos\theta,sin\theta)\cdot t} K^2((cos\theta, sin\theta)\cdot t, r)drd\theta \\
&= \int_{\beta-\pi/2}^{\beta+\pi/2} |(cos\theta, sin\theta)\cdot t|^{2H}d\theta \\
&= \int_{\beta-\pi/2}^{\beta+\pi/2} \|t\|^{2H}|cos\beta cos\theta + sin\beta sin\theta|^{2H}d\theta \\
&= \|t\|^{2H}\int_{\beta-\pi/2}^{\beta+\pi/2} cos^{2H}(\theta-\beta)d\theta \\
&= C_H\|t\|^{2H}
\end{aligned}$$

C_H does not depend on β since $\frac{d}{d\beta}\int_{\beta-\pi/2}^{\beta+\pi/2} cos^H(\theta-\beta)d\theta = 0$. Thus C_H is a constant depending only on H.

Now we consider the covariance function $E(B_H(t)B_H(s))$ of B_H.

$$\mathbb{E}(B_H(t)B_H(s)) = \int_{[0\leq\frac{u}{\|u\|}\cdot t]\bigcap[0\leq\frac{u}{\|u\|}\cdot s]} K(t\cdot\frac{u}{\|u\|}, \|u\|)K(s\cdot\frac{u}{\|u\|}, \|u\|)\frac{1}{\|u\|}du$$

So we let $s = -t$. Then

$$\begin{aligned}
\mathbb{E}(B_H(t)B_H(-t)) &= \int_{[0\leq\frac{u}{\|u\|}\cdot t]\bigcap[0\leq\frac{u}{\|u\|}\cdot(-t)]} K(t\cdot\frac{u}{\|u\|}, \|u\|)K(-t\cdot\frac{u}{\|u\|}, \|u\|)\frac{1}{\|u\|}du \\
&= 0 \\
&= C_H(2\|t\|^{2H} - \|2t\|^{2H})
\end{aligned}$$

Hence this process gives a different covariance function from fractional Brownian motion.

We have seen that the process we defined above was not fractional Brownian motion. This is because the area of the integration does not overlap whenever vector s and t are parallel with $s\cdot t < 0$.

However, we can modify the covariance function K to move the fractional Brownian motion from the interval $[0,1]$ to $[-1/2,1/2]$ by

$$\begin{aligned}\tilde{B}_H(t) &= B_H(t+1/2) - B_H(1/2)\\ &= \int_0^\infty (K(t+1/2,u) - K(1/2,u))dB(u)\\ &= \int_0^\infty \tilde{K}(t,u)dB(u)\end{aligned}$$

where $\tilde{K}(t,u) = K(t+1/2,u) - K(1/2,u)$. So we have the following lemma

Lemma 5.1.0.10. *$\tilde{B}_H$ defined as above is a one dimensional fractional Brownian motion on $[-1/2,1/2]$.*

Proof. Clearly, $\tilde{B}(t) = B_H(t+1/2) - B_H(1/2)$ is a Gaussian process. So we need only to check its covariance function.

$$\begin{aligned}\mathbb{E}(\tilde{B}_H(t)\tilde{B}_H(s)) &= \mathbb{E}(\int_{\mathbb{R}} \tilde{K}(t,u)dB(u) \int_{\mathbb{R}} \tilde{K}(s,v)dB(v))\\ &= \int_0^\infty \tilde{K}(t,u)\tilde{K}(s,u)du\\ &= (\int_0^\infty K(t+1/2,u)K(s+1/2,u)du - \int_0^\infty K(t+1/2,u)K(1/2,u)du\\ &\quad - \int_0^\infty K(s+1/2,u)K(1/2,u)du + \int_0^\infty K^2(1/2,u)du)\\ &= (|t+1/2|^{2H} + |s+1/2|^{2H} - |t-s|^{2H} - |t+1/2|^{2H}\\ &\quad -(1/2)^{2H} + |t|^{2H} - |s+1/2|^{2H} - (1/2)^{2H} + |s|^{2H} + 2(1/2)^{2H})\\ &= (|t|^{2H} + |s|^{2H} - |s-t|^{2H}).\end{aligned}$$

Therefore $\tilde{B}_H$ is a one dimensional fractional Brownian motion on the interval $[-1/2,1/2]$ with the index H. □

In general, if we have $B_H(t) = \int_0^\infty K(t,u)dB(u)$ defines a fractional Brownian motion in an interval $[a,b]$. We could define $\tilde{B}_H(t) = B_H(t+\frac{a+b}{2}) - B(\frac{a+b}{2})$ as a fractional Brownian motion in the interval $[\frac{a-b}{2}, \frac{b-a}{2}]$. So this idea can be extended to any disk of radius $R = (b-a)/2$ by using the kernel

$$K_R(t,u) = K(t + \frac{a+b}{2}, u) - K(\frac{a+b}{2}, u),$$

where $K(t,u)$ is a covariance function of a fractional Brownian motion in the interval $[a,b]$.

Corollary 5.1.0.6. *For $t \in \mathbb{R}^2$ and $\|t\| \leq R$ we define*

$$B_H(t) = \int_{\mathbb{R}^2} \frac{1}{\|u\|^{1/2}} K_R(t \cdot \frac{u}{\|u\|}, \|u\|) dB(u)$$

then B_H is a fractional Brownian motion with index H on the disk of radius R.

Proof. The proof is similar to that of the Theorem 5.1.0.22, that is we need to check that B_H is fractional Brownian motion with index H. Thus it suffices to show

$$E(B_H(t)B_H(s)) = C'_H(\|t\|^{2H} + \|s\|^{2H} - \|t-s\|^{2H}).$$

$$\begin{aligned}
\mathbb{E}(B_H(t)B_H(s)) &= \mathbb{E}(\int_{\mathbb{R}^2} K_R(t \cdot \frac{u}{\|u\|}, \|u\|) \frac{1}{\|u\|^{1/2}} dB(u) \int_{\mathbb{R}^2} K_R(s \cdot \frac{v}{\|v\|}, \|v\|) \frac{1}{\|v\|^{1/2}} dB(v)) \\
&= \int_{\mathbb{R}^2} K_R(t \cdot \frac{u}{\|u\|}, \|u\|) K_R(s \cdot \frac{u}{\|u\|}, \|u\|) \frac{1}{\|u\|} du \\
&= \int_0^{2\pi} \int_0^{\infty} K_R(\|t\|\cos(\theta), r) K_R(\|s\|\cos(\theta), r) dr d\theta \\
&= \frac{1}{2} \int_0^{2\pi} |\|t\|\cos(\theta)|^{2H} + |\|s\|\cos(\theta)|^{2H} - |\|(t-s)\|\cos(\theta)|^{2H} d\theta \\
&= C_H(\|t\|^{2H} + \|s\|^{2H} - \|t-s\|^{2H})
\end{aligned}$$

where $C_H = 1/2 \int_0^{2\pi} |\cos(\theta)|^{2H} d\theta$ is a constant. □

Example 1. *From [3], we can represent the fractional Brownian motion over a finite interval by*

$$B_H(t) = C_H \int_0^t K(t,u) dB(u)$$

where $0 < H < 1/2$ and $t \geq 0$

$$K_H(t,u) = \{(\frac{t}{u})^{H-1/2}(t-u)^{H-1/2} - (H - \frac{1}{2})u^{1/2-H} \int_u^t x^{H-3/2}(x-u)^{H-1/2} dx\}$$

and for $H > 1/2$ and $t \geq 0$.

$$K_H(t,u) = u^{1/2-H} \int_u^t |x-u|^{H-3/2} x^{H-1/2} dx.$$

For the proof, we refer to [3].

We can also apply the method above to this fractional Brownian motion on $[0,\infty]$. We have seen some examples that we could be able to extend and could not. In the next step we will study what properties that allow us to be able to extend a fractional Brownian motion from an interval subset of $\mathbb{R}$ to one of a subset of $\mathbb{R}^2$.

First let $B_H(t) = \int_0^\infty K(t,u)dB(u)$ be a fractional Brownian motion in the interval $[-1,1]$. Then we define

$$B_H(t) = \int_{\mathbb{R}^2} K(f(t,u), g(t,u))h(t,u)dB(u),$$

where f, g and h are functions from $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$.

Now we want to study what conditions on f, g and h that allow B_H to be defined as a fractional Brownian motion in $\mathbb{R}^2$. We have seen an example above that if $f(t,u) = t \cdot \frac{u}{\|u\|}$, $g(t,u) = \|u\|$ and $h(t,u) = \frac{1}{\|u\|^{1/2}}$ then B_H is a fractional Brownian motion in a subset of $\mathbb{R}^2$. In this thesis, we consider only the case $g(t,u) = \|u\|$, where $\|\cdot\|$ is a norm induced by an inner product in $\mathbb{R}^2$.

Let $<\cdot,\cdot>$ be an inner product in $\mathbb{R}^2$ then we have

$$A = \begin{pmatrix} <e_1,e_1> & <e_1,e_2> \\ <e_2,e_1> & <e_2,e_2> \end{pmatrix}$$

as the matrix representation of the inner product $<\cdot,\cdot>$, that is $<u,v>= u^T Av$, where $\{e_1, e_2\}$ is a basis of $\mathbb{R}^2$.

We can see that A is symmetric and

$$\begin{aligned} u^T Au &= <u,u> \\ &\geq 0 \end{aligned}$$

So A is positive definite. Thus A is diagonalizable. That is there is a 2×2 unitary matrix U such that $A = U^T DU$, where D is a diagonal matrix. Suppose

$$D = \begin{pmatrix} \alpha_1^2 & 0 \\ 0 & \alpha_2^2 \end{pmatrix} \text{ and } U = \begin{pmatrix} u_1 & u_2 \\ u_2 & u_3 \end{pmatrix}$$

Let $(x, y) \in \mathbb{R}^2$ and define

$$r\cos\theta = \alpha_1(u_1 x + u_2 y)$$

and

$$r\sin\theta = \alpha_2(u_2 x + u_3 y).$$

Then we can solve for x and y in the term of variable r and θ. That is

$$x = \frac{r}{\alpha_1\alpha_2 \det(U)}(\alpha_2 u_3 \cos\theta - \alpha_1 u_2 \sin\theta)$$

and

$$y = \frac{r}{\alpha_1\alpha_2 \det(U)}(\alpha_1 u_1 \sin\theta - \alpha_2 u_2 \cos\theta).$$

It is easy to see that

$$|J(r,\theta)| = |\frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial y}{\partial r}\frac{\partial x}{\partial \theta}| = \frac{r}{\alpha_1\alpha_2}.$$

Then we have the following Theorem.

Theorem 5.1.0.23. *Let $B_H(t) = {}_0^\infty K(t,u)dB(u)$ be a fractional Brownian motion with index H on the interval $[-1,1]$. If the function f defined as above satisfies the following conditions*

(1) For each t such that $\|t\| \le 1$, $f(t,u) \in [-1,1]$ for all $u \in \mathbb{R}^2$,

(2) For all $t \in \mathbb{R}^2$, $u \in \mathbb{R}^2/\{0\}$, $f(t,u) = f(t, \frac{u}{\|u\|})$,

(3) $f(t,u) + f(s,u) = f(t+s,u)$,

(4) For $c \in [-1,1]$, $f(ct,u) = cf(t,u)$ for all $u \in \mathbb{R}^2$.

(5) ${}_{S^1} |f(t,u)|^{2H} d\sigma(u) = C_H$ for all $\|t\| = 1$, where C_H is a constant.

then we have

$$B_H(t) = \int_{\mathbb{R}^2} K(f(t,u), \|u\|) \frac{1}{\|u\|^{1/2}} dB(u)$$

defines a fractional Brownian motion with index H in a unit disk in $\mathbb{R}^2$.

Proof. The first condition makes $K(f(t,u), \|u\|)$ well defined for all $\|t\| \leq 1$ and $u \in \mathbb{R}^2$. Now we need to check the covariance function of the defined process. WLOG, assume $\alpha_1 = \alpha_2 = 1$. So

$$\begin{aligned}
\mathbb{E}(B_H(s)B_H(t)) &= \int_{\Omega} (\int_{\mathbb{R}^2} K(f(s,u), \|u\|) \frac{1}{\|u\|^{1/2}} dB(u) \int_{\mathbb{R}^2} K(f(t,u), \|u\|) \frac{1}{\|u\|^{1/2}} dB(u)) d\mathbb{P} \\
&= \int_{\mathbb{R}^2} K(f(s,u), \|u\|) K(f(t,u), \|u\|) \frac{1}{\|u\|} dB(u) \\
&= \int_0^{2\pi} \int_0^{\infty} (K(f(s,(r,\theta)), r) K(f(t,(r,\theta)), r) dr d\theta \\
&= \int_0^{2\pi} \int_0^{\infty} (K(f(s,(1,\theta)), r) K(f(t,(1,\theta)), r) dr d\theta \\
&= \int_0^{2\pi} |f(s,(1,\theta))|^{2H} + |f(t,(1,\theta))|^{2H} - |f(s,(1,\theta)) - f(t,(1,\theta))|^{2H} d\theta \\
&= \int_0^{2\pi} |f(s,(1,\theta))|^{2H} + |f(t,(1,\theta))|^{2H} - |f(s-t,(1,\theta))|^{2H} d\theta \\
&= \int_0^{2\pi} \|s\|^{2H} |f(\frac{s}{\|s\|}, (1,\theta))|^{2H} + \|t\|^{2H} |f(\frac{t}{\|t\|}, (1,\theta))|^{2H} \\
&\quad - \|s-t\|^{2H} |f(\frac{(s-t)}{\|s-t\|}, (1,\theta))|^{2H} d\theta \\
&= \int_0^{2\pi} |f(\frac{s}{\|s\|}, (1,\theta))|^{2H} d\theta (\|s\|^{2H} + \|t\|^{2H} - \|s-t\|^{2H}) \\
&= C_H(\|s\|^{2H} + \|t\|^{2H} - \|s-t\|^{2H})
\end{aligned}$$

Therefore $B_H(t) = \int_{\mathbb{R}^2} K(f(t,u), \|u\|) \frac{1}{\|u\|^{1/2}} dB(u)$ is a fractional Brownian motion with index H in $\mathbb{R}^2$. □

The next corollary is an example of a kind of function that satisfies the previous theorem.

Corollary 5.1.0.7. *Let $< \cdot, \cdot >$ be an inner product on $\mathbb{R}^2$. Let $B_H(t) = \int_0^{\infty} K(t,u) dB(u)$ is a fractional Brownian motion on the interval $[-1, 1]$ also define $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ as*

$f(t,u) =< t, \frac{u}{\|u\|} >$ then

$$B_H(t) = \int_{\mathbb{R}^2} K(f(t,u), \|u\|)\sqrt{\frac{\alpha_1\alpha_2}{\|u\|}}dB(u)$$

is a fractional Brownian motion on the unit disk in $\mathbb{R}^2$, where α_1^2 and α_2^2 are the eigenvalues of the matrix A.

Proof. Notice that f obviously satisfies first 4 conditions and the condition 5 is true by the Corollary 2.2.0.2.

Hence by change of variable, we can show that

$$\mathbb{E}(B_H(t)B_H(s)) = C_H(\|t\|^{2H} + \|s\|^{2H} - \|s-t\|^{2H}).$$

Therefore $B_H(t) = \int_{\mathbb{R}^2} K(< t, \frac{u}{\|u\|} >, \|u\|)\sqrt{\frac{\alpha_1\alpha_2}{\|u\|}}dB(u)$ is a fractional Brownian motion whose support is the unit disk in $\mathbb{R}^2$. $\square$

The next section we will extend this method to higher dimensional Euclidian space.

5.2. An integral representation of fractional Brownian motion in $\mathbb{R}^d$ where $d \geq 2$

In this section we will extend the result from the previous section to higher dimensional spaces. Recall that $\{B(u) : u \in \mathbb{R}^d\}$ represents the Brownian sheet in $\mathbb{R}^d$. In particular, $B(A) = \int_A dB(u)$ is a Gaussian random measure with

$$Cov(B(A_1), B(A_2)) = \lambda(A_1 \cap A_2)$$

where λ is d-dimensional Lebesgue measure.

The following theorem is analogous to the case of $d = 2$.

Theorem 5.2.0.24. *Let $B_H(t) = \int_0^\infty K(t,u)dB(u)$ be a fractional Brownian motion on the interval $[-1,1]$. Then for $t \in \mathbb{R}^d$ with $\|t\| \leq 1$*

$$B_H(t) = \int_{\mathbb{R}^d} K(t \cdot \frac{u}{\|u\|}, \|u\|) \frac{1}{\|u\|^{(d-1)/2}} dB(u)$$

is a fractional Brownian motion with index H on the unit disk in $\mathbb{R}^d$.

Proof. It suffices to show $\mathbb{E}(B_H(t)B_H(s)) = C'_H(\|t\|^{2H} + \|s\|^{2H} - \|t-s\|^{2H})$.

$$\begin{aligned}
\mathbb{E}(B_H(t)B_H(s)) &= \mathbb{E}(\int_{\mathbb{R}^d} K(t \cdot \frac{u}{\|u\|}, \|u\|)K(s \cdot \frac{u}{\|u\|}, \|u\|)\frac{1}{\|u\|^{d-1}} dB(u)) \\
&= C_H^2 \int_{S^{d-1}} \int_0^\infty K(t \cdot \theta, r)K(s \cdot \theta, r) dr d\sigma(\theta) \\
&= \frac{C_H^2}{2} \int_{S^{d-1}} (|t \cdot \theta|^{2H} + |s \cdot \theta|^{2H} - |(t-s) \cdot \theta|^{2H}) d\sigma(\theta) \\
&= C'_H(\|t\|^{2H} + \|s\|^{2H} - \|t-s\|^{2H})
\end{aligned}$$

where $\theta = \frac{u}{\|u\|} \in S^{d-1}$, $r = \|u\|$ and $C'_H = \int_{S^{d-1}} |\frac{t}{\|t\|} \cdot \theta|^{2H} d\sigma(\theta)$ is a constant for all $t = 0$ and σ is the uniform measure on S^{d-1}. □

Similarly, we can extend the result to a disk of radius R in $\mathbb{R}^d$ by using the kernel

$$K_R(t,u) = K(t+R,u)1_{[0 \leq \|u\| \leq t+R]} - K(R,u)1_{[0 \leq \|u\| \leq R]}.$$

Corollary 5.2.0.8. *For $t \in \mathbb{R}^d$ and $\|t\| \leq R$ we define*

$$B_H(t) = \int_{\mathbb{R}^d} \frac{1}{\|u\|^{(d-1)/2}} K_R(t \cdot \frac{u}{\|u\|}, \|u\|) dB(u)$$

then B_H is a fractional Brownian motion with index H on the disk of radius R.

Proof. It suffices to show

$$\mathbb{E}(B_H(t)B_H(s)) = C'_H(\|t\|^{2H} + \|s\|^{2H} - \|t-s\|^{2H}).$$

$$\begin{aligned}
\mathbb{E}(B_H(t)B_H(s)) &= \mathbb{E}(\int_{\mathbb{R}^d} K_R(t \cdot \frac{u}{\|u\|}, \|u\|)K_R(s \cdot \frac{u}{\|u\|}, \|u\|)\frac{1}{\|u\|^{d-1}} dB(u)) \\
&= C_H^2 \int_{S^{d-1}} \int_0^\infty K_R(t \cdot \theta, r)K_R(s \cdot \theta, r) dr d\sigma(\theta) \\
&= \frac{C_H^2}{2} \int_{S^{d-1}} (|t \cdot \theta|^{2H} + |s \cdot \theta|^{2H} - |(t-s) \cdot \theta|^{2H}) d\sigma(\theta) \\
&= C'_H(\|t\|^{2H} + \|s\|^{2H} - \|t-s\|^{2H})
\end{aligned}$$

where $\theta = \frac{u}{\|u\|} \in S^{d-1}$, $r = \|u\|$, and $C'_H = {}_{S^{d-1}} |\frac{t}{\|t\|} \cdot \theta|^{2H} d\sigma(\theta)$ is a constant for all $t = 0$ and σ is the uniform measure on S^{d-1}. □

Similar to the case $d = 2$, if we have $B_H(t) = {}_0^{\infty} K(t,u)dB(u)$ a fractional Brownian motion in an interval $[a,b]$, we could define $\tilde{B}_H(t) = B_H(t + \frac{a+b}{2}) - B(\frac{a+b}{2})$ as a fractional Brownian motion in the interval $[\frac{a-b}{2}, \frac{b-a}{2}]$. So this idea can be extended to any disk in $\mathbb{R}^d$ of radius $R = (b-a)/2$ by using the kernel

$$K_R(t,u) = K(t + \frac{a+b}{2}, u) - K(\frac{a+b}{2}, u),$$

where $K(t,u)$ is a covariance function of a fractional Brownian motion in the interval $[a,b]$. So we have the following corollary that is analogous to the case $d = 2$.

Corollary 5.2.0.9. *For $t \in \mathbb{R}^2$ and $\|t\| \leq R$ we define*

$$B_H(t) = {}_{\mathbb{R}^d} \frac{1}{\|u\|^{(d-1)/2}} K_R(t \cdot \frac{u}{\|u\|}, \|u\|)dB(u)$$

then B_H is a fractional Brownian motion with index H on the disk in $\mathbb{R}^d$ of radius R.

Proof. The proof is similar to the case $d = 2$. □

Now we are ready to state the Theorem that tells us the conditions for the extension to work. The proof is not only true for Euclidian norm but also true for any norm that is induced by an inner product.

Let $< \cdot, \cdot >$ be an inner product on $\mathbb{R}^d$. Now let A be the matrix representation of an inner product $< \cdot, \cdot >$ in $\mathbb{R}^d$, that is $< u, v >= u^T A v$ for all $u, v \in \mathbb{R}^d$. Consider the matrix representation A of the inner product $< \cdot, \cdot >$. That is,

$$\begin{pmatrix} < e_1, e_1 > & \dots & < e_1, e_d > \\ \vdots & \ddots & \vdots \\ < e_d, e_1 > & \dots & < e_d, e_d > \end{pmatrix}$$

where $\{e_1, \ldots, e_d\}$ is a basis in $\mathbb{R}^d$.

Then A is symmetric and positive definite.

So there exists a self adjoint and unitary matrix

$$Q = \begin{pmatrix} q_1 \\ \vdots \\ q_d \end{pmatrix}$$

and a diagonal matrix

$$D = \begin{pmatrix} \alpha_1^2 & \ldots & 0 \\ 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \alpha_d^2 \end{pmatrix}$$

such that $A = QDQ$, where $q_i, \ldots, q_d$ are orthonomal row vectors of Q.

Now let

$$\begin{aligned}
f_1(r, \theta_1, \ldots, \theta_{d-1}) &= r\cos\theta_1 = \alpha_1 q_1 x \\
f_2(r, \theta_1, \ldots, \theta_{d-1}) &= r\sin\theta_1\cos\theta_2 = \alpha_2 q_2 x \\
&\vdots \\
f_{d-1}(r, \theta_1, \ldots, \theta_{d-1}) &= r\sin\theta_1 \cdots \sin\theta_{d-2}\cos\theta_{d-1} = \alpha_{d-1} q_{d-1} x \\
f_d(r, \theta_1, \ldots, \theta_{d-1}) &= r\sin\theta_1 \cdots \sin\theta_{d-1} = \alpha_d q_d x
\end{aligned}$$

where $x = [x_1, \cdots, x_d]^T$ is a vector in $\mathbb{R}^d$.

Then

$$\begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} = DQx.$$

So

$$QD^{-1}\begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}.$$

Then

$$J_x = \begin{pmatrix} \frac{\partial x_1}{\partial r} & \cdots & \frac{\partial x_1}{\partial \theta_{d-1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_d}{\partial r} & \cdots & \frac{\partial x_d}{\partial \theta_{d-1}} \end{pmatrix} = QD^{-1}\begin{pmatrix} \frac{\partial f_1}{\partial r} & \cdots & \frac{\partial f_1}{\partial \theta_{d-1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_d}{\partial r} & \cdots & \frac{\partial f_d}{\partial \theta_{d-1}} \end{pmatrix}.$$

Hence

$$|\det(J_x)| = |\det(Q)\det(D^{-1})\det\begin{pmatrix} \frac{\partial f_1}{\partial r} & \cdots & \frac{\partial f_1}{\partial \theta_{d-1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_d}{\partial r} & \cdots & \frac{\partial f_d}{\partial \theta_{d-1}} \end{pmatrix}| = \frac{r^{d-1}\ {}_{i=1}^{d-2}\sin\theta_i^{d-i-1}}{\alpha_1\cdots\alpha_d}.$$

Then we have a result analogous to that of the Theorem in the previous section.

Theorem 5.2.0.25. *Let* $B_H(t) = {}_0^{\infty} K(t,u)dB(u)$ *be a fractional Brownian motion with index* H *on the interval* $[-1,1]$*. If the function* f *defined as above satisfies the following conditions*

(1) For each t *such that* $\|t\| \le 1$, $f(t,u) \in [-1,1]$ *for all* $u \in \mathbb{R}^d$,

(2) For all $t \in \mathbb{R}^d$, $u \in \mathbb{R}^d/\{0\}$, $f(t,u) = f(t, \frac{u}{\|u\|})$,

(3) $f(t,u) + f(s,u) = f(t+s,u)$,

(4) For $c \in [-1,1]$, $f(ct,u) = cf(t,u)$ *for all* $u \in \mathbb{R}^d$.

(5) ${}_{S^{d-1}} |f(t,u)|^{2H} d\sigma(u) = C_H$ *for all* $\|t\| = 1$*, where* C_H *is a constant.*

then we have

$$B_H(t) = {}_{\mathbb{R}^d} K(f(t,u), \|u\|)\frac{1}{\|u\|^{(d-1)/2}}dB(u)$$

is a fractional Brownian motion with index H *in a unit disk in* $\mathbb{R}^d$.

Proof. The first condition makes $K(f(t,u),\|u\|)$ well defined for all $\|t\| \le 1$ and $u \in \mathbb{R}^d$. Now we need to check the covariance function of the defined process. Without loss of generality, assume $\alpha_1 = \cdots = \alpha_d = 1$. So

$$\begin{aligned}
\mathbb{E}(B_H(s)B_H(t)) &= \int_\Omega \Big(\int_{\mathbb{R}^d} K(f(s,u),\|u\|)\frac{1}{\|u\|^{(d-1)/2}}dB(u) \int_{\mathbb{R}^d} K(f(t,u),\|u\|)\frac{1}{\|u\|^{(d-1)/2}}dB(u)\Big)d\mathbb{P} \\
&= \int_{\mathbb{R}^2} K(f(s,u),\|u\|)K(f(t,u),\|u\|)\frac{1}{\|u\|^{d-1}}dB(u) \\
&= \int_{S^{d-1}}\int_0^\infty (K(f(s,ru),r)K(f(t,ru),r)drd\sigma(u) \\
&= \int_{S^{d-1}}\int_0^\infty (K(f(s,u),r)K(f(t,u),r)drd\sigma(u) \\
&= \int_{S^{d-1}} |f(s,u)|^{2H} + |f(t,u)|^{2H} - |f(s,u) - f(t,u)|^{2H} d\sigma(u) \\
&= \int_{S^{d-1}} |f(s,u)|^{2H} + |f(t,u)|^{2H} - |f(s-t,u)|^{2H} d\sigma(u) \\
&= \int_{S^{d-1}} \|s\|^{2H}|f(\frac{s}{\|s\|},u)|^{2H} + \|t\|^{2H}|f(\frac{t}{\|t\|},u)|^{2H} \\
&\quad -\|s-t\|^{2H}|f(\frac{(s-t)}{\|s-t\|},u)|^{2H}d\sigma(u) \\
&= \int_{S^{d-1}} |f(\frac{s}{\|s\|},u)|^{2H}(\|s\|^{2H} + \|t\|^{2H} - \|s-t\|^{2H})d\sigma(u) \\
&= C_H(\|s\|^{2H} + \|t\|^{2H} - \|s-t\|^{2H})
\end{aligned}$$

Therefore $B_H(t) = \int_{\mathbb{R}^d} K(f(t,u),\|u\|)\frac{1}{\|u\|^{(d-1)/2}}dB(u)$ is a fractional Brownian motion with index H in $\mathbb{R}^d$. □

Corollary 5.2.0.10. *Let* $<\cdot,\cdot>$ *be an inner product on* $\mathbb{R}^d$. *Let* $B_H(t) = \int_0^\infty K(t,u)dB(u)$ *is a fractional Brownian motion on the interval* $[-1,1]$ *also define* $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ *as* $f(t,u) = < t, \frac{u}{\|u\|} >$ *then*

$$B_H(t) = \int_{\mathbb{R}^d} K(f(t,u),\|u\|)(\frac{\alpha_1\cdots\alpha_d}{\|u\|})^{(d-1)/2}dB(u)$$

is a fractional Brownian motion on the unit disk in $\mathbb{R}^d$, *where* $\alpha_1^2, \ldots, \alpha_d^2$ *are the eigenvalues of the matrix* A.

Corollary 5.2.0.11. *Define for* $u, v \in \mathbb{R}^d$ *and* $j = 1,...,d$, $f(u,v) = u\cdot\frac{v_j'}{\|v\|}$, *where* $v_j' = (v_1,...,-v_j,...,v_d)$. *Then* f *satisfies all conditions of the theorem 5.2.0.25.*

Corollary 5.2.0.12. *Let $< \cdot, \cdot >$ be an inner product on $\mathbb{R}^d$. Define for $u, v \in \mathbb{R}^d$, $f(u,v) = r < u, \frac{v}{\|v\|} >$, where $0 < |r| \leq 1$. Then f satisfies all conditions of the theorem 5.2.0.25.*

5.3. An integral representation of Gaussian processes in $\mathbb{R}^d$

Next we are going to generalize the result of the previous section to the case of Gaussian processes on $\mathbb{R}$. Now a natural question arises that, if we have a stochastic process

$$Z(t) = \int_0^\infty K(t,u)dB(u)$$

in $\mathbb{R}$ with $K(t,u)$ defined on $[-R,R] \times (0,\infty)$ for some $R > 0$, when do we have

$$Z^d(t) = \int_{\mathbb{R}^d} \frac{1}{\|u\|^{(d-1)/2}} K(t \cdot \frac{u}{\|u\|}, \|u\|)dB(u)$$

as an extension of the $Z(t)$ to a stochastic process in $\mathbb{R}^d$, $d > 1$ with the same covariance properties as $Z(t)$? We notice that the extension works for the case of fractional Brownian motion because fractional Brownian motion has stationary increments. Then it makes sense to consider a Gaussian process $Z(t) = \int_0^\infty K(t,u)dB(u)$ with stationary increments. That is we consider the covariance function of $Z(t)$. First we assume $Z(0) = 0$. Then

$$\mathbb{E}[(Z(s) - Z(t))^2] = \mathbb{E}(Z(s)^2) - 2Cov(Z(s), Z(t)) + \mathbb{E}(Z(t)^2),$$

So

$$\begin{aligned} Cov(Z(s), Z(t)) &= \frac{1}{2}(\mathbb{E}(Z^2(s)) - \mathbb{E}[(Z(s) - Z(t))^2] + \mathbb{E}(Z^2(t)) \\ &= \frac{1}{2}(\mathbb{E}(Z^2(s)) + \mathbb{E}(Z^2(t)) - \mathbb{E}(Z^2(s-t))) \end{aligned}$$

So we start with considering

$$Var(Z(t) - Z(s)) = 2g(t-s)$$

for some function g defined on $\mathbb{R}$. This corresponds to assuming that Z has stationary increments. Indeed, let $\{Z(t) : t \in I\}$, $I \subset \mathbb{R}$ be a one dimensional Gaussian process. Assume $Z(t) = \int_0^\infty K(t,u)dB(u), t \in \mathbb{R}$. Define

$$g(t) = \int_0^\infty (K(t,u))^2 du.$$

Then the covariance function of Z is

$$\begin{aligned} Cov(Z(s)Z(t)) &= \frac{1}{2}(\mathbb{E}(Z^2(s)) + \mathbb{E}(Z^2(t)) - \mathbb{E}(Z^2(s-t))) \\ &= \frac{1}{2}(\int_0^\infty (K(t,u))^2 du + \int_0^\infty (K(s,u))^2 du - \int_0^\infty (K(s-t,u))^2 du \\ &= 1/2(g(t) + g(s) - g(s-t)) \end{aligned}$$

Since $g(t-s) = \frac{1}{2}Var(Z(t) - Z(s))$, we have the following corollary.

Corollary 5.3.0.13. *g is symmetric, that is*

$$g(r) = g(-r)$$

Next we will define a Gaussian process in $\mathbb{R}^d$ in the same way we did to fractional Brownian motion. Now we are ready to state the Theorem as we mention earlier.

Theorem 5.3.0.26. *For each $t \in [-1,1]$, let $Z(t) = \int_0^\infty K(t,u)dB(u), t \geq 0$ be a one dimensional Gaussian process with stationary increments and $Z(0) = 0$. Then we have for all $t, \|t\| \leq 1$,*

$$Z_d(t) = \int_{\mathbb{R}^d} \frac{1}{\|u\|^{(d-1)/2}} K(t \cdot u, \|u\|) dB(u)$$

is a Gaussian process in the unit disk and the covariance function of the form

$$Cov(Z_d(t)Z_d(s)) = c_d(G_d(\|t\|) + G_d(\|s\|) - G_d(\|t-s\|))$$

where $G_d(u) = \int_{-1}^1 (1-v^2)^{\frac{d-3}{2}} g(uv)dv$, and c_d is a constant depending only on d.

Proof. By the Lemma, we obtain

$$
\begin{aligned}
\mathbb{E}(Z_d(t)Z_d(s)) &= \mathbb{E}\Big(\int_{\mathbb{R}^d} K(t\cdot u, \|u\|)\frac{1}{\|u\|^{(d-1)/2}}dB(u) \int_{\mathbb{R}^d} K(s\cdot u, \|v\|)\frac{1}{\|v\|^{(d-1)/2}}dB(v)\Big) \\
&= \int_{\mathbb{R}^d} K(t\cdot \frac{u}{\|u\|}, \|u\|)K(s\cdot \frac{u}{\|u\|}, \|u\|)\frac{1}{\|u\|^{d-1}}du \\
&= \int_{S^{d-1}}\int_0^{\infty} K(t\cdot\theta, r)K(s\cdot\theta, r)dr d\sigma(\theta) \\
&= \int_{S^{d-1}} (g(t\cdot\theta) + g(s\cdot\theta) - g((t-s)\cdot\theta))d\sigma(\theta) \\
&= \frac{|S^{d-1}|}{2}\int_{-1}^{1}(1-v^2)^{\frac{d-3}{2}}(g(\|t\|v) + g(\|s\|v) - g(\|t-s\|v))dv \\
&= \frac{|S^{d-1}|}{2}(G_d(\|t\|) + G_d(\|s\|) - G_d(\|t-s\|))
\end{aligned}
$$

where $G_d(u) = \int_{-1}^{1}(1-v^2)^{\frac{d-3}{2}}g(uv)dv$. ☐

Next we consider the case $Z(0) = 0$. Then the covariance function of Z is

$$
\begin{aligned}
\mathbb{E}(Z(t)Z(s)) &= \frac{1}{2}(\mathbb{E}(Z^2(t)) + \mathbb{E}(Z^2(s)) - \mathbb{E}((Z(t) - Z(s))^2) \\
&= \frac{1}{2}(\mathbb{E}(Z^2(t)) + \mathbb{E}(Z^2(s)) - (\mathbb{E}((Z(t-s) - Z(0))^2) \\
&= \frac{1}{2}(\mathbb{E}(Z^2(t)) + \mathbb{E}(Z^2(s)) - (\mathbb{E}(Z^2(t-s)) + \mathbb{E}(Z^2(0)) - 2\mathbb{E}(Z(t-s)Z(0)) \\
&= \frac{1}{2}(g(t) + g(s) - g(t-s) - g(0)) + \mathbb{E}(Z(t-s)Z(0))
\end{aligned}
$$

Then we have the following theorem.

Theorem 5.3.0.27. *For each $t \in [-1,1]$, let $Z(t) = \int_0^{\infty} K(t,u)dB(u), t \geq 0$ be a one dimensional Gaussian process with stationary increments. Then we have for all $t, \|t\| \leq 1$,*

$$Z_d(t) = \int_{\mathbb{R}^d} \frac{1}{\|u\|^{(d-1)/2}} K(t\cdot u, \|u\|)dB(u)$$

is a Gaussian process in the unit disk and the covariance function of the form

$$Cov(Z_d(t)Z_d(s)) = c_d(G_d(\|t\|) + G_d(\|s\|) - G_d(\|t-s\|) - G_d(0)) + c_d H_d(\|t-s\|)$$

where $G_d(u) = \int_{-1}^{1}(1-v^2)^{\frac{d-3}{2}}g(uv)dv$, $H_d(u) = \int_{-1}^{1}(1-v^2)^{\frac{d-3}{2}}(\mathbb{E}(Z(\|t-s\|v)Z(0)))dv$ and c_d is a constant depending only on d.

Proof. We consider

$$\begin{aligned}
\mathbb{E}(Z_d(t)Z_d(s)) &= \mathbb{E}(\int_{\mathbb{R}^d} K(t\cdot u, \|u\|)\frac{1}{\|u\|^{(d-1)/2}}dB(u) \int_{\mathbb{R}^d} K(s\cdot u, \|v\|)\frac{1}{\|v\|^{(d-1)/2}}dB(v)) \\
&= \int_{\mathbb{R}^d} K(t\cdot \frac{u}{\|u\|}, \|u\|)K(s\cdot \frac{u}{\|u\|}, \|u\|)\frac{1}{\|u\|^{d-1}}du \\
&= \int_{S^{d-1}}\int_0^\infty K(t\cdot\theta, r)K(s\cdot\theta, r)drd\sigma(\theta) \\
&= \int_{S^{d-1}} 1/2(g(t\cdot\theta) + g(s\cdot\theta) - g((t-s)\cdot\theta) - g(0)) + \mathbb{E}(Z((t-s)\cdot\theta)Z(0))d\sigma(\theta) \\
&= \frac{|S^{d-1}|}{2}\int_{-1}^1 (1-v^2)^{\frac{d-3}{2}}(g(\|t\|v) + g(\|s\|v) - g(\|t-s\|v) - g(0)) \\
&\quad +\mathbb{E}(Z(\|t-s\|v)Z(0))dv \\
&= \frac{|S^{d-1}|}{2}(G_d(\|t\|) + G_d(\|s\|) - G_d(\|t-s\|) - G_d(0)) + |S^{d-1}|H_d(\|t-s\|)
\end{aligned}$$

where $G_d(u) = \int_{-1}^1 (1-v^2)^{\frac{d-3}{2}}g(uv)dv$ and $H_d(u) = \int_{-1}^1 (1-v^2)^{\frac{d-3}{2}}(\mathbb{E}(Z(\|t-s\|v)Z(0)))dv$.

□

Since Z has stationary increments, Z_d is as the following corollary.

Corollary 5.3.0.14. *Z_d has stationary increments.*

Proof. For simplicity, we will drop the constant c_d. In order to show this we need to check

$$\mathbb{E}((Z_d(t_1) - Z_d(s))(Z_d(t_2) - Z_d(s))) = \mathbb{E}((Z_d(t_1 - s) - Z_d(0))(Z_d(t_2 - s) - Z_d(0))).$$

Indeed,

$$\begin{aligned}
\mathbb{E}((Z_d(t_1) - Z_d(s))(Z_d(t_2) - Z_d(s))) &= \mathbb{E}(Z_d(t_1)Z_d(t_2)) - \mathbb{E}(Z_d(t_1)Z_d(s)) - \mathbb{E}(Z_d(t_2)Z_d(s)) + \mathbb{E}(Z_d^2(s)) \\
&= G_d(t_1) + G_d(t_2) - G_d(t_1 - t_2) - G_d(0) + H_d(t_1 - t_2) \\
&\quad -(G_d(t_1) + G_d(s) - G_d(t_1 - s) - G_d(0) + H_d(t_1 - s)) \\
&\quad -(G_d(t_2) + G_d(s) - G_d(t_2 - s) - G_d(0) + H_d(t_2 - s)) \\
&\quad +2G_d(s) - 2G_d(0) + H_d(0) \\
&= H_d(t_1 - t_2) - H_d(t_1 - s) - H_d(t_2 - s) + H_d(0) \\
&\quad -G_d(t_1 - t_2) + G_d(t_1 - s) + G_d(t_2 - s) - G_d(0)
\end{aligned}$$

On the other hand

$$\begin{aligned}
\mathbb{E}((Z_d(t_1-s)-Z_d(0))(Z_d(t_2-s)-Z_d(0))) &= \mathbb{E}(Z_d(t_1-s)Z_d(t_2-s))-\mathbb{E}(Z_d(t_1-s)Z_d(0)) \\
&\quad -\mathbb{E}(Z_d(t_2-s)Z_d(0))+\mathbb{E}(Z_d^2(0)) \\
&= G_d(t_1-s)+G_d(t_2-s)-G_d(t_1-t_2)-G_d(0) \\
&\quad +H_d(t_1-t_2)-(G_d(t_1-s)+G_d(0)-G_d(t_1-s) \\
&\quad -G_d(0))+H_d(t_1-s))-(G_d(t_2-s)+G_d(0) \\
&\quad -G_d(t_2-s)-G_d(0)+H_d(t_2-s))+H_d(0) \\
&= H_d(t_1-t_2)-H_d(t_1-s)-H_d(t_2-s)+H_d(0) \\
&\quad -G_d(t_1-t_2)+G_d(t_1-s)+G_d(t_2-s)-G_d(0)
\end{aligned}$$

Therefore

$$\mathbb{E}((Z_d(t_1)-Z_d(s))(Z_d(t_2)-Z_d(s)))=\mathbb{E}((Z_d(t_1-s)-Z_d(0))(Z_d(t_2-s)-Z_d(0))).$$

□

Before we continue we would like to state a definition.

Definition 5.3.0.17. *A stochastic process $X(t), t\in\mathbb{R}^d$ is isotropic if for each rotational matrix $A\in\mathbb{R}^{d\times d}$.*

$$X(t) =^d X(At).$$

Notice that if X is a Gaussian process. Then this definition is equivalent to $\mathbb{E}(X(t)X(s))=\mathbb{E}(X(At)X(As))$. So we have the following corollary.

Corollary 5.3.0.15. *The Gaussian process Z_d as defined as above is isotropic.*

Proof.

$$
\begin{aligned}
\mathbb{E}(Z_d(At)Z_d(As)) &= \mathbb{E}(\int_{\mathbb{R}^d} K(At\cdot u, \|u\|)\frac{1}{\|u\|^{(d-1)/2}}dB(u)\int_{\mathbb{R}^d} K(As\cdot u, \|v\|)\frac{1}{\|v\|^{(d-1)/2}}dB(v))\\
&= \int_{\mathbb{R}^d} K(At\cdot\frac{u}{\|u\|}, \|u\|)K(As\cdot\frac{u}{\|u\|}, \|u\|)\frac{1}{\|u\|^{d-1}}du\\
&= \int_{S^{d-1}}\int_0^\infty K(At\cdot\theta, r)K(As\cdot\theta, r)drd\sigma(\theta)\\
&= \int_{S^{d-1}} 1/2(g(At\cdot\theta)+g(As\cdot\theta)-g(A(t-s)\cdot\theta)-g(0))\\
&\quad +\mathbb{E}(Z(A(t-s)\cdot\theta)Z(0))d\sigma(\theta)\\
&= \frac{|S^{d-1}|}{2}\int_{-1}^{1}(1-v^2)^{\frac{d-3}{2}}(g(\|At\|v)+g(\|As\|v)-g(\|A(t-s)\|v)-g(0))\\
&\quad +\mathbb{E}(Z(\|A(t-s)\|)v)Z(0))dv\\
&= \frac{|S^{d-1}|}{2}\int_{-1}^{1}(1-v^2)^{\frac{d-3}{2}}(g(\|t\|v)+g(\|s\|v)-g(\|(t-s)\|v)-g(0))\\
&\quad +\mathbb{E}(Z(\|t-s\|)v)Z(0))dv\\
&= \frac{|S^{d-1}|}{2}(G_d(\|t\|)+G_d(\|s\|)-G_d(\|t-s\|)-G_d(0))+|S^{d-1}|H_d(\|t-s\|)\\
&= \mathbb{E}(Z_d(t)Z_d(s))
\end{aligned}
$$

Hence Z_d is isotropic. □

The Ornstein-Uhlenbeck process satisfies this. First we shall give the definition of the Ornstein-Uhlenbeck process.

Definition 5.3.0.18. *A mean zero Gaussian process $U(t), t \in \mathbb{R}$ is called Ornstein-Uhlenbeck process with parameter $\alpha > 0$ if its covariance function is of the form*

$$\mathbb{E}(U(t)U(s)) = \frac{1}{2\alpha}e^{-\alpha|t-s|}.$$

From Kaarakka [16], U can be written as

$$U(t) = e^{-\alpha t}B(\frac{e^{2\alpha t}}{2\alpha}) = e^{-\alpha t}\int_0^\infty 1_{[u\le\frac{e^{2\alpha t}}{2\alpha}]}dB(u), t\in\mathbb{R},$$

where B is the standard Brownian motion. Then we have the following corollary.

Corollary 5.3.0.16. *Let $U(t)$ be an Ornstein-Uhlenbeck process in $\mathbb{R}$. Then for $\|t\| \leq 1$,*

$$U_d(t) = e^{-\alpha t \cdot \frac{u}{\|u\|}} \int_{\mathbb{R}^d} 1_{[u \leq \frac{e^{2\alpha t \cdot \frac{u}{\|u\|}}}{2\alpha}]} \frac{1}{\|u\|^{(d-1)/2}} dB(u).$$

is a isotropic and Gaussian process with stationary increments in the unit disc with the covariance function of the form.

$$\mathbb{E}(U_d(t)U_d(s)) = \frac{|S^{d-1}|}{(2\alpha)} \int_{-1}^{1} (1-v^2)^{(d-3)/2} e^{-\alpha\|t-s\|v} dv$$

.

In the case d is odd, we have

$$\mathbb{E}(U_d(t)U_d(s)) = (e^{-\alpha\|t-s\|} P^d_{\|t-s\|}(1) - e^{\alpha\|t-s\|} P^d_{\|t-s\|}(-1)),$$

where

$$\begin{aligned} P^n_\beta(x) \quad = \quad & \frac{x^{2n}}{\beta} + \frac{2nx^{2n-1}}{\beta^2} + x^{2n-2}(\frac{2n(2n-1)}{\beta^3} - \frac{n}{\beta}) + x^{2n-3}(\frac{2n(2n-1)(2n-2)}{\beta^4} - \frac{n(2n-2)}{\beta^2}) \\ & + \cdots + x^{2n-2k}(\frac{(2n)!}{\beta^{2k+1}(2n-2k)!} - \frac{n(2n-2)!}{(\beta^{2k-1}(2n-2-2k)!} \\ & + \cdots + (-1)^k \frac{n(n-1)\cdots(n-k+1)}{\beta}) + \cdots + \frac{(2n)!}{\beta^{2n+1}} - \frac{n(2n-2)!}{\beta^{2n-1}} + \cdots + \frac{(-1)^n}{\beta} \end{aligned}$$

This is a good place to summarize what we already did before we continue to investigate for further generalization of representations of stochastic process. First we showed that if we have

$$B_H(t) = \int_0^\infty K(t,u)dB(u)$$

with $K(t,u)$ defined on $[-R,R] \times (0,\infty)$ then we can show that for $t \in \mathbb{R}^d$ with $\|t\| \leq R$ and $d > 1$,

$$B^d_H(t) = \int_{\mathbb{R}^d} \frac{1}{\|u\|^{(d-1)/2}} K(t \cdot \frac{u}{\|u\|}, \|u\|) dB(u)$$

is also a fractional Brownian motion in $\mathbb{R}^d$. Later we showed that if we have $Z(t) = \int_0^\infty K(t,u)dB(u)$ with the covariance function

$$Cov(Z(s)Z(t)) = g(|t|) + g(|s|) - g(|t-s|),$$

where g is bounded and Lebesgue integrable on $[-1,1]$ and $K(t,u)$ defined on $\mathbb{R}\times(0,\infty)$ for some $R>0$. Then for $t\in\mathbb{R}^d$,

$$Z_d(t) = {}_{\mathbb{R}^d} \frac{1}{\|u\|^{(d-1)/2}} K(t\cdot\frac{u}{\|u\|}, \|u\|)dB(u)$$

is also a Gaussian process in $\mathbb{R}^d$ with the covariance function of the form

$$Cov(Z_d(t)Z_d(s)) = c_d(G_d(\|t\|)+G_d(\|s\|)-G_d(\|t-s\|)).$$

For the next section, we will investigate on series representation of fractional Brownian motion.

5.4. A series representation of fractional Brownian motion in $\mathbb{R}^d$

In this section we will discuss on series representation of fractional Brownian motion. We know that the Karhunen-Loéve expansion is one of the powerful tools for studying Gaussian process. However solving for eigenvalues and eigenfunctions are not always easy. With this reason the Karhunen-Loéve expansion of fractional Brownian motion in $\mathbb{R}^d$ is still not known. This section will show another way to express Gaussian processes as an infinite series, especially fractional Brownian motion. The idea of doing this can be summarized as follows.

Suppose we have $B_H(t) = {}_{B^d(0,1)} K(t,u)dB(u)$ is a fractional Brownian motion in a subset of $\mathbb{R}^d$. Then we can write $K(t,u) = {}_n c_n(t)f_n(u)$, where f_n is an othonormal basis for $L^2(B^d(0,1))$. Then this gives us $B_H(t) = {}_n c_n(t) {}_{B^d(0,1)} f_n(u)dB(u)$ as a series representation of fractional Brownian motion in a subset of $\mathbb{R}^d$. The key that allows us to do this is that the fractional Brownian motion has an integral representation which its support is compact. As we have seen earlier the Molchan-Golosov representation is an example of this. Before we continue we need the following lemma.

Lemma 5.4.0.11. *If $\{g_i(x), i \geq 1\}$ is an othonomal basis for $L^2((0,1), dx)$ then $\{g_i(x)/x^{(d-1)/2}, i \geq 1\}$ is also an orthonormal basis for $L^2((0,1), x^{d-1}dx)$.*

Proof. Assume $g_i, i \geq 1$ is an orthonomal basis for $L^2((0,1), dx)$. Then

$$\int_0^1 \frac{g_i(x)}{x^{(d-1)/2}} \frac{g_j(x)}{x^{(d-1)/2}} x^{d-1} dx = \int_0^1 g_i(x) g_j(x) dx = \delta_{i,j}.$$

Let $f \in L^2((0,1), x^{d-1}dx)$ and $\int_0^1 f(x) \frac{g_i(x)}{x^{(d-1)/2}} x^{d-1} dx = 0$ for all i. Then

$$\int_0^1 f(x) x^{(d-1)/2} g_i(x) dx = 0, \text{ for all } i.$$

Since $f \in L^2((0,1), x^{d-1}dx)$, $\int_0^1 f^2(x) dx = \int_0^1 (\frac{f(x)}{x^{(d-1)/2}})^2 x^{d-1} dx < \infty$.
That is $f \in L^2((0,1), dx)$. Since g_i is orthonormal basis for $L^2((0,1), dx)$, $f(x)x^{(d-1)/2} = 0$.
That is $f = 0$ a.e.
Hence $\{g_i(x)/x^{(d-1)/2}, i \geq 1\}$ is also an orthonormal basis for $L^2((0,1), x^{d-1}dx)$. □

Now let $\{g_i(r)\}$ be an orthonomal basis for $L^2((0,1), dx)$ and $\mathcal{H}_n^d$ be a set of spherical harmonics $\{\varphi_{n,k}(\theta)\}$ of degree n in $\mathbb{R}^d$, where $k = 1, ..., N(d,n)$ and $\theta \in S^{d-1}$ and $N(d,n) = (\frac{2n+d-2}{n+d-2}) \frac{(n+d-2)!}{n!(d-2)!}$ is the dimension of $\mathcal{H}_n^d$(Groemer [9]).
Then $\{\varphi_{n,k}, n \geq 0, k = 1, ..., N(d,n)\}$ forms an orthonormal basis for $L^2(S^{d-1}, \sigma)$.
Define

$$f_{n,k,i}(x) = \frac{g_i(\|x\|)}{\|x\|^{(d-1)/2}} \varphi_{n,k}(\frac{x}{\|x\|}).$$

We will show that $f_{n,k,i}$ is an orthonormal basis for a unit disk with respect to the Lebesǵue measure in $\mathbb{R}^d$ as follows.

Lemma 5.4.0.12. *If $\{g_i(r)\}$ is an orthonormal basis for $L^2((0,1), dx)$ and $\{\varphi_{n,k}(\theta)\}$ be a collection of spherical harmonics forming an orthonormal basis for $L^2(S^{d-1}, \sigma)$, then*

$$f_{n,k,i}(x) = \frac{g_i(\|x\|)}{\|x\|^{(d-1)/2}} \varphi_{n,k}(\frac{x}{\|x\|})$$

is also an orthonormal basis for a unit disk with respect to the Lebesǵue measure in $\mathbb{R}^d$.

Proof.

$$\begin{aligned}
\int_{B_d(0,1)} f_{n,k,i}(x) f_{m,l,j}(x)dx &= \int_{S^{d-1}} \int_0^1 g_i(r) g_j(r) dr \varphi_{n,k}(\theta)\varphi_{m,l}(\theta) d\theta \\
&= \delta_{(n,k),(m,l)}\delta_{i,j},
\end{aligned}$$

This is an orthonormal collection. Next we want to show if $f \in L^2(B^d(0,1), dx)$ with $\int_{B^d(0,1)} f(x) f_{n,k,i}(x)dx = 0$ for all n, k, i then $f \equiv 0$ a.e. If

$$\begin{aligned}
0 &= \int_{B_d(0,1)} f_{n,k,i}(x) f(x) dx \\
&= \int_{S^{d-1}} \int_0^1 f(r,\theta) r^{(d-1)/2} g_i(r) dr \varphi_{n,k}(\theta) d\theta
\end{aligned}$$

Since $f \in L^2((B_d(0,1), dx)$,

$$\int_0^1 f(r,\theta) g_i(r) r^{(d-1)/2} dr \in L^2(S^{d-1}, \sigma).$$

Since $\varphi_{n,k}$ is an orthonormal basis for $L^2(S^{d-1}, \theta)$,

$$\int_0^1 f(r,\theta) g_i(r) r^{(d-1)/2} dr = 0, \text{ for all } i$$

Hence $f(x) = 0$ a.e.

Therefore $f_{n,k,i}$ is an orthonormal basis for $L^2(B_d(0,1), dx)$. □

We can use this to develop series representation of fractional Brownian motion. Suppose $B_H(t) = \int_0^1 K(t,u) dB(u)$ is a fractional Brownian motion in $[-R, R]$, $R > 0$. Then we have

$$B_H(t) = \int_{B^d(0,1)} K(t \cdot \frac{u}{\|u\|}, \|u\|) \frac{1}{\|u\|^{(d-1)/2}} dB(u).$$

is a fractional Brownian motion in a disk of radius R in $\mathbb{R}^d$.

So for fixed t, we can write

$$K(t \cdot \frac{u}{\|u\|}, \|u\|) \frac{1}{\|u\|^{(d-1)/2}} = \sum_{n,k,i} c_{n,k,i}(t) \frac{g_i(\|u\|)}{\|u\|^{(d-1)/2}} \varphi_{n,k}(\frac{u}{\|u\|})$$

where g_i is orthonormal basis for $L^2((0,1),dx)$ and $\varphi_{n,k}$ is orthonormal basis for $L^2(S^{d-1})$ and $c_{n,k,i}(t) = \int_{B^d(0,1)} K(t \cdot \frac{u}{\|u\|}, \|u\|) \frac{1}{\|u\|^{(d-1)/2}} f_{n,k,i}(u)du$.

Thus

$$
\begin{aligned}
B_H(t) &= \int_{\mathbb{R}^d} K(t \cdot \frac{u}{\|u\|}, \|u\|) \frac{1}{\|u\|^{(d-1)/2}} dB(u) \\
&= \sum_{n,k,i} c_{n,k,i}(t) \int_{B_d(0,2)} \frac{g_i(\|u\|)}{\|u\|^{(d-1)/2}} \varphi_{n,k}(\frac{u}{\|u\|}) dB(u)
\end{aligned}
$$

Now we are trying to compute $c_{n,k,i}$. The key idea for the computation is the Funk-Hecke Theorem see [9].

Theorem 5.4.0.28. *If Φ is a bounded integrable function on $[-1,1]$ and $\varphi \in \mathcal{H}_n^d$, then $\Phi(u \cdot v)$ is (for any fixed $u \in S^{d-1}$) an integrable function on S^{d-1} and*

$$
\int_{S^{d-1}} \Phi(u \cdot v) \varphi(v) d\sigma(v) = \alpha_{d,n}(\Phi) \varphi(u)
$$

with

$$
\alpha_{d,n}(\Phi) = \sigma_{d-1} \int_{-1}^{1} \Phi(t) P_n^d(t)(1-t^2)^{\nu} dt
$$

where $\nu = (d-3)/2$, $\mathcal{H}_n^d$ is the space of all spherical harmonics of degree n and dimension d and $P_n^d(t) = (-1)^n 2^{-n} \prod_{i=1}^{n} (\nu+i)^{-1}(1-t^2)^{-\nu} \frac{d^n}{dt^n}(1-t^2)^{\nu+n}$ is the Legendre polynomial of dimension d and degree n.

Then we have the following theorem

Theorem 5.4.0.29. *Let $B_H(t) = \int_0^1 K(t,u)dB(u)$ be a fractional Brownian motion in $[-R,R]$, $R > 0$ and g_i be an orthonormal basis for $L^2((0,1),dx)$ and $\varphi_{n,k}$ be an orthonormal basis for $L^2(S^{d-1})$. Then we have*

$$
\begin{aligned}
B_H(t) &= \int_{B^d(0,1)} K(t \cdot \frac{u}{\|u\|}, \|u\|) \frac{1}{\|u\|^{(d-1)/2}} dB(u) \\
&= \sum_{n,k,i} c_{n,k,i}(t) \int_{B_d(0,1)} \frac{g_i(\|u\|)}{\|u\|^{(d-1)/2}} \varphi_{n,k}(\frac{u}{\|u\|}) dB(u)
\end{aligned}
$$

where $c_{n,k,i}(t) = \sigma_{d-1}(\int_{-1}^{1} \int_0^1 K(\|t\|u, r) g_i(r) dr P_n^d(u)(1-u^2)^{\nu} du) \varphi_{n,k}(\frac{t}{\|t\|})$, $\vartheta = (d-3)/2$ and P_n^d is the Legendre polynomial of dimension d and degree n.

Proof. For $f \in L^2(B^d(0,1), dx)$, we have

$$\int_{B_d(0,1)} f_{n,k,i}(x) f(x) dx = \int_{S^{d-1}} \int_0^1 f(r,\theta) g_i(r) r^{(d-1)/2} dr \varphi_{n,k}(\theta) d\theta.$$

Now for fix $t \in \mathbb{R}^d$ with $|t| \leq R$, we replace $f(x)$ by $K(t \cdot \frac{x}{\|x\|}, \|x\|) \frac{1}{\|x\|^{(d-1)/2}}$. Then

$$\begin{aligned} c_{n,k,i}(t) &= \int_{B_d(0,1)} K(t \cdot \frac{u}{\|u\|}, \|u\|) \frac{1}{\|u\|^{(d-1)/2}} f_{n,k,i}(x) dx \\ &= \int_{S^{d-1}} \int_0^1 K(t \cdot \theta, r) g_i(r) dr \varphi_{n,k}(\theta) d\theta \end{aligned}$$

Thus for $x \in [-1,1]$ we define $\Phi_{t,i}$ by

$$\Phi_{t,i}(x) = \int_0^1 K(\|t\| x, r) g_i(r) dr$$

Hence by the Funk-Hecke theorem,

$$\begin{aligned} c_{n,k,i}(t) &= \int_{S^{d-1}} \Phi_{t,i}(\frac{t}{\|t\|} \cdot \theta) \varphi_{n,k}(\theta) d\theta \\ &= \alpha_{d,n}(\Phi_{t,i}) \varphi_{n,k}(\frac{t}{\|t\|}) \\ &= \sigma_{d-1}(\int_{-1}^1 \int_0^1 K(|t|u, r) g_i(r) dr P_n^d(u)(1-u^2)^{\vartheta} du) \varphi_{n,k}(\frac{t}{\|t\|}) \end{aligned}$$

where $\vartheta = (d-3)/2$. □

Example 2. *Molchan-Golosov representation satisfies the above theorem. That is when $0 < H < 1$ and $t \geq 0$,*

$$B_H(t) =^d C_H \int_0^t K(t,u) dB(u),$$

where

$$K(t,u) = \{(\frac{t}{u})^{H-1/2}(t-u)^{H-1/2} - (H - \frac{1}{2}) u^{1/2-H} \int_u^t x^{H-3/2}(x-u)^{H-1/2} dx\}$$

and C_H is a normalizing constant.

Then we define

$$B'_H(t) = B_H(t+1/2) - B_H(1/2) = \int_0^1 K(t+1/2, u) 1_{(0,t+1/2)} - K(1/2, u) 1_{(0,1/2)} dB(u)$$

is a fractional Brownian motion on $[-1/2, 1/2]$*. Then*

$$B_H(t) = \mathop{}_{B^d(0,1)} (K(t \cdot \frac{u}{\|u\|}, \|u\|) - K(1/2, \|u\|)) \frac{1}{\|u\|^{(d-1)/2}} dB(u)$$

is a fractional Brownian motion on a disk radius $1/2$ *whose series representation is of the form*

$$B_H(t) = \mathop{}_{n,k,i} c_{n,k,i}(t) \mathop{}_{B_d(0,1)} \frac{g_i(\|u\|)}{\|u\|^{(d-1)/2}} \varphi_{n,k}(\frac{u}{\|u\|}) dB(u),$$

where $c_{n,k,i}(t) = \sigma_{d-1}(\mathop{}_{-1}^{1} \mathop{}_{0}^{1} K(\|t\|u, r) g_i(r) dr P_n^d(u)(1-u^2)^{\nu} du) \varphi_{n,k}(\frac{t}{\|t\|})$, $\vartheta = (d-3)/2$ *and* P_n^d *is the Legendre polynomial of dimension* d *and degree* n*.*

6. CONCLUSION

In this thesis we obtain new representations of fractional Brownian motion in $\mathbb{R}^d, d \geq 2$ from extending the kenel appearing in the representations of fractional Brownian motions on $\mathbb{R}$. This extension can be summarized as follows:

Theorem 6.0.0.30. *Let $B_H(t) = \int_0^\infty K(t,u)dB(u)$ be a fractional Brownian motion with index H on the interval $[-1,1]$. If the function f defined as above satisfies the following conditions*

(1) For each t such that $\|t\| \leq 1$, $f(t,u) \in [-1,1]$ for all $u \in \mathbb{R}^d$,

(2) For all $t \in \mathbb{R}^d$, $u \in \mathbb{R}^d/\{0\}$, $f(t,u) = f(t, \frac{u}{\|u\|})$,

(3) $f(t,u) + f(s,u) = f(t+s,u)$,

(4) For $c \in [-1,1]$, $f(ct,u) = cf(t,u)$ for all $u \in \mathbb{R}^d$.

(5) $\int_{S^{d-1}} |f(t,u)|^{2H} d\sigma(u) = C_H$ for all $\|t\| = 1$, where C_H is a constant.

then we have

$$B_H(t) = \int_{\mathbb{R}^d} K(f(t,u), \|u\|) \frac{1}{\|u\|^{(d-1)/2}} dB(u)$$

is a fractional Brownian motion with index H in a unit disk in $\mathbb{R}^d$.

We have seen some examples of f that satisfy all of the above conditions. We refer to Corollary 5.2.0.10, 5.2.0.11 and 5.2.0.12. So we have the following conjecture.

Conjecture 6.0.0.1. *If f satisfies all of the conditions in the theorem 6.0.0.30 then f must have either one of the form*

1) $f(u,v) =< u, \frac{v}{\|v\|} >$,

2) $f(u,v) =< u, \frac{v'_j}{\|v\|} >$ or

3) $f(u,v) = r < u, \frac{v}{\|v\|} >$,

where $v'_j = (v_1, ..., -v_j, ..., v_d)$ and $0 < |r| \leq 1$.

Since we know the conditions on f that allows the extension of fractional Brownian motion to work, we could use the same method to extend Gaussian processes with a certain kind of covariance function to Gaussian processes in higher dimension. That is

Theorem 6.0.0.31. *For each $t \in \mathbb{R}^d$, let $Z(t) = \int_0^\infty K(t,u)dB(u), t \geq 0$ be a one dimensional Gaussian process with stationary increments and assume*

$$g(t) = \mathbb{E}(Z^2(t))$$

is bounded and Lebesgue integrable on $[-1,1]$. Then we have for $t \in \mathbb{R}^d$,

$$Z_d(t) = \int_{\mathbb{R}^d} \frac{1}{\|u\|^{(d-1)/2}} K(t \cdot u, \|u\|)dB(u)$$

is a Gaussian process in $\mathbb{R}^d$ with isotropic property and the covariance function of the form

$$Cov(Z_d(t)Z_d(s)) = c_d(G_d(\|t\|) + G_d(\|s\|) - G_d(\|t-s\|))$$

where $G_d(u) = \int_{-1}^1 (1-v^2)^{\frac{d-3}{2}} g(uv)dv$, and c_d is a constant depending only on d.

The processes that we get from this method is also isotropic. From the result we obtained we could use it to develop a new series representation of fractional Brownian motion as follow:

Theorem 6.0.0.32. *Let $B_H(t) = \int_0^1 K(t,u)dB(u)$ be a fractional Brownian motion in $[-R,R]$, $R > 0$ and g_i be an orthonormal basis for $L^2((0,1),dx)$ and $\varphi_{n,k}$ be an orthomormal basis for $L^2(S^{d-1})$. Then we have*

$$\begin{aligned} B_H(t) &= \int_{B^d(0,1)} K(t \cdot \frac{u}{\|u\|}, \|u\|) \frac{1}{\|u\|^{(d-1)/2}} dB(u) \\ &= \sum_{n,k,i} c_{n,k,i}(t) \int_{B_d(0,1)} \frac{g_i(\|u\|)}{\|u\|^{(d-1)/2}} \varphi_{n,k}(\frac{u}{\|u\|}) dB(u) \end{aligned}$$

where $c_{n,k,i}(t) = \sigma_{d-1}(\int_{-1}^1 \int_0^1 K(\|t\|u,r)g_i(r)drP_n^d(u)(1-u^2)^\nu du)\varphi_{n,k}(\frac{t}{\|t\|})$, $\vartheta = (d-3)/2$ and P_n^d is the Legendre polynomial of dimension d and degree n.

A benefit of this series representation is ${}_{B_d(0,1)} \frac{g_i(\|u\|)}{\|u\|^{(d-1)/2}}\varphi_{n,k}(\frac{u}{\|u\|})dB(u)$ is i.i.d standard Gaussian process. Unfortunately, $c_{n,k,i}(t)$ may not be orthogonal. So searching for orthogonality of the functions in the expansion of fractional Brownian motion is an increasing interest. In 2005 Fourier expansion has been extended for fractional Brownian motion in $\mathbb{R}$ when $0 < H \leq 1/2$ by Istas [12] and when $1/2 \leq H < 1$ by Igloi [11]. This will be an interesting subject to investigate further in $\mathbb{R}^d$.

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