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By general approximation methods, within the framework of functional analysis, algorithms are developed for numerically determining the solution $K(x, t)$ of the integral equation

$$K(x, t) + F(x, t) + \int_0^x F(s, t)K(x, s)ds = 0 ,$$

and the solution $\frac{dK(x, x)}{dx}$ of the integro-differential equation

$$\frac{dK(x, x)}{dx} + \frac{dF(x, x)}{dx} + K(x, x)F(x, x) + \int_0^x F(s,x) \frac{\partial K(x, s)}{\partial x} ds = 0.$$

These functions are essential in the solution of typical inverse problems of mathematical physics. In addition, applications are presented of the inverse Sturm-Liouville theory to problems of mathematical physics. Furthermore, asymptotic results are developed for an approximate spectral function.

Numerical Aspects of the Inverse Sturm-Liouville Problem

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NUMERICAL ASPECTS OF THE INVERSE STURM-LIOUVILLE PROBLEM

I. INTRODUCTION

1.0 Objectives

The fact that the implications of the Inverse Sturm-Liouville Theory are far-reaching has been the motivating force behind this dissertation.

The objectives of this dissertation are, first, to resolve certain mathematical problems which restrict the practical utility of the Inverse Sturm-Liouville Theory, and secondly, to indicate the significance of this inverse theory to other fields.

In Section 1.1 the inverse Sturm-Liouville problem will be stated; in Section 1.2 a brief resume of the historical development of the problem will be given, and in Section 1.3 the principal theory of this dissertation will be formulated.

1.1 Definition

Determining $q(x)$ of the differential equation

$$y'' + (\lambda - q(x))\, y = 0, \quad 0 \leq x < \infty, \tag{1.0}$$

with the initial conditions,

$$(1.0') \qquad y'(0) = h, \quad y(0) = 1$$

for a given spectral function $\rho(\lambda)$, where h is a real number, is referred to as the inverse Sturm-Liouville problem. $\rho(\lambda)$ is the spectral function of (1.0)-(1.0') if $\rho(\lambda)$ is a nondecreasing function on $-\infty < \lambda < \infty$ such that

$$\int_{-\infty}^{\infty} E^2(\lambda)d\rho(\lambda) = \int_0^{\infty} f^2(x)dx$$

where

$$E(\lambda) = \lim_{N\to\infty} E_N(\lambda),$$

and

$$E_n(\lambda) = \int_0^n f(x)\varphi(x,\lambda)dx,$$

for any function f, which is square integrable in the Lebesque sense on $0 < x < \infty$ (see Reference [4]), and $\varphi(x, \lambda)$ is the unique solution of (1.0)-(1.0'). For Equation (1.0) on $0 \leq x \leq \pi$ with the boundary conditions

$$(1.0'') \qquad y'(0) - hy(0) = 0, \quad y(\pi) + Hy'(\pi) = 0,$$

the spectral function $\rho(\lambda)$ equals $\sum_{\lambda_n<\lambda} \frac{1}{a_n}$, where $\{\lambda_i\}_{i=0}^{\infty}$ and $\{\varphi_i(x)\}_{i=0}^{\infty}$ are the eigenvalues and eigenfunctions, respectively, of the system (1.0)-1.0''), and $a_n = \int_0^{\pi} \varphi_n^2(x)dx$ (see Gelfand and

Levitan, [4]).

1.2 Background

The general theory for solving this problem has evolved during the last 40 years. In 1929, V.A. Ambartsumyan produced the first result in this field [1]. He proved the following theorem:

"let $\lambda_0, \lambda_1, \ldots$ denote the eigenvalues of the system:

$$y'' + \{\lambda - q(x)\}y = 0, \quad 0 \leq x \leq \pi, \tag{1.1}$$

$$y(0) = y(\pi) = 0, \tag{1.2}$$

where $q(x)$ is a real, continuous function on the interval. If $\lambda_n = n^2$ $(n = 0, 1, \ldots)$, then $q(x) = 0$."

The second important step in the general development of the theory was made by G. Borg in 1945 [2, p. 1-98]. His main result is:

"let $\lambda_0, \lambda_1, \ldots,$ denote the eigenvalues of Equation (1.1) under the boundary conditions, (h and H are finite, real numbers),

$$y'(0) - hy(0) = 0 \tag{1.3}$$

$$y'(\pi) + Hy(\pi) = 0, \tag{1.4}$$

and let $\mu_0, \mu_1, \ldots,$ be the corresponding eigenvalues of (1.1) under the boundary conditions (1.4) and

(1.5) $$y'(0) - h_1 y(0) = 0, \quad (h_1 \neq h).$$

Then the sequences $\{\lambda_n\}$ and $\{\mu_n\}$, $(n = 0, 1, \ldots)$, uniquely determine the function $q(x)$ and the numbers h, h_1, and H."

In 1950, V. Marchenko made the next important advance in the theory by showing that if the spectral function of (1.0)-(1.0') (or (1.0")) is given, then $q(x)$ and the constant(s) h (or h and H) are unambiguously determined [7].

In 1951 Gelfand and Levitan presented the first effective method for constructing $q(x)$ of (1.0)-(1.0') (or (1.0")) from its spectral function, as well as giving necessary and sufficient conditions for a monotonic function $\rho(\lambda)$ to be the spectral function of (1.0)-(1.0') (or (1.0")).

In 1964 Levitan and Gasymov [6, p. 1-63] published a paper which refines the 1951 paper of Gelfand and Levitan and also attains a new result for a variation of boundary conditions. A brief synopsis of this work will be presented in Appendix C.

1.3 Scope

In determining $q(x)$ of the system (1.0)-(1.0') or (1.0") from the spectral function $\rho(\lambda)$ by means of the Gelfand-Levitan theory, it is necessary to determine the function $K(x, t)$ and its derivative

$dK(x,x)/dx$ ([4] or [6]), for

$$q(x) = +2\frac{dK(x,x)}{dx},$$

where

$$K(x,t) + F(x,t) + \int_0^x F(s,t)K(x,s)ds = 0, \quad 0 \leq t \leq x.$$

$$F(x,t) = \lim_{N\to\infty} \int_{-\infty}^{N} \cos\sqrt{\lambda}\,x \cos\sqrt{\lambda}\,t\,d\sigma(\lambda),$$

and

$$\sigma(\lambda) = \begin{cases} \rho(\lambda), & \lambda < 0 \\ \rho(\lambda) - \frac{2}{\pi}\sqrt{\lambda}, & \lambda \geq 0. \end{cases}$$

Accordingly, in Chapters II and III algorithms are constructed for solving K and dK/dx numerically. In Chapter IV, the significance of this theory is discussed. In particular, an application of the theory is given in 4.1 which is relevant to the field of medicine, i.e. an indirect method of determining the elasticity of a flexible tube is suggested. This example is indicative of the truly diverse applicability of this theory. In 4.2, applications of the theory to various types of problems will be indicated, and in 4.3 the interrelationships between the physical systems of Section 4.2 and the Gelfand-Levitan theory are explicitly given. For the inverse problem on $[0,\pi]$, it is shown in Chapter V, that if--for sufficiently large N and appropriate growth conditions--the first N eigenvalues λ_i and normalizing constants a_i $(a_i = \int_0^{\pi} \varphi_i^2(x)dx)$ are known, meaningful results can be derived. In Appendix A, the integral equation of Chapter II is solved by various iterative methods. In Appendix B, a brief synopsis of the theory of Kantorovich, upon which Chapters II and III are based, is presented. In Appendix C, a brief synopsis of the Gelfand-Levitan theory is given.

II. ALGORITHM FOR $K(x, t)$

2.0 Introduction

In this chapter an algorithm is developed for numerically determining the solution $K(s,t)$ of the integral equation

$$K(s,t) + F(s,t) + \int_0^s F(x,t)K(s,x)dx = 0, \quad 0 \le t \le s \le R < \infty, \tag{2.0}$$

in which F is a given continuous function on $0 \le t \le s \le R$. Within the framework of the Gelfand-Levitan theory, F is even absolutely continuous in both variables and K is as smooth as F (see [4] or [6]). Results of this chapter are based upon Kantorovich's general theory of approximation methods [5].

The most important result to be developed in this chapter, for our purposes, is as follows:

<u>For a given</u> $\varepsilon > 0$, <u>there exists an equipartition</u> Δ_η <u>of</u> $[0, R]$ <u>such that for each</u> r, $0 \le s \le r \le R$, <u>there exists a readily constructed continuous, piecewise differentiable function</u> $\tilde{K}_r^n$ <u>such that</u> $|K(r,s) - \tilde{K}_r^n(s)| < \varepsilon$, $0 \le s \le r$, <u>for each equipartition</u> Δ_n <u>of</u> $[0, r]$, <u>providing the partition</u> Δ_n <u>is at least as fine as the partition</u> Δ_η; <u>if, in addition,</u> F <u>of Equation (2.0) satisfies a Lipschitz condition in each variable, then</u>

$$K(r, s) = \tilde{K}^n_r(s) + O(\Delta),$$

providing the equipartition Δ_n of $[0, r]$ is at least as fine as the equipartition $\Delta_{\eta\eta}$ of $[0, R]$, where Δ denotes the length of the subpartitions of $\Delta_{\eta\eta}$.

The procedure followed in Sections 2.2 and 2.3 in developing an algorithm for K of Equation (2.0) is briefly sketched in Section 2.1. Sufficient conditions for constructing an algorithm for $K(s, t)$ of Equation (2.0) for fixed s are developed in Section 2.2. Then based upon Section 2.2, the results of this chapter are developed in Section 2.3.

2.1 Kantorovich Approach

Let $x(r, s)^*$ be the solution of

(2.1) $$x(r, s) + \int_0^r h(s, t)x(r, t)dt = y(r, s), \quad \text{for} \quad 0 \leq s \leq r \leq R;$$

(or equivalently)

(2.1) $$Kx_r(s) = x_r(s) + Hx_r(s) = y_r(s)$$

in the Banach Space X_r of real continuous functions on $[0, r]$. Now suppose there exists a linear transformation φ^n_r that maps Equation (2.1) of X_r into the system

$$(2.2) \qquad x_r(t_j) + \sum_{k=1}^{n} \Delta h(t_j, t_k) x_r(t_k) = y_r(t_j), \quad j = 1, 2, \ldots, n,$$

of the n-dimensional vector space $\overline{X}_r^n$. In order to determine whether the solution (if it exists) of Equation (2.2) satisfactorily approximates the solution of (2.1), it is necessary to answer questions of the following type:

1. Is the "reduced" linear system (2.2) of algebraic equations uniquely solvable?
2. If (2.2) is uniquely solvable, then to what degree of accuracy does its solution actually represent the solution of Equation (2.1)?
3. For a given $\varepsilon > 0$, is it possible to determine a linear system of equations (i.e., a system (2.2)) such that its solution does not vary from the exact solution of Equation (2.1) by more than ε uniformly?

Questions of this type are readily answered within the Kantorovich theory under rather general conditions. Consequently, it will be necessary to familiarize the reader somewhat with the Kantorovich framework, and secondly to show how this theory ties in with the particular problem under consideration.

First, with respect to the Kantorovich framework, suppose $\exists$ a linear operation P_n projecting X_r on to $\widetilde{X}_r^n$ (a complete subspace of X_r) $\ni$ $P_n^2 = P_n$. In the space $\tilde{X}_r^n$, consider the equation

$$(2.1') \qquad \tilde{x}_r(s) + \tilde{H}\tilde{x}_r(s) = P_n y_r(s),$$

where $\tilde{H}$ is a linear operator on $\tilde{X}_r^n$. If the following conditions are satisfied Equation (2.1') and its solution $\tilde{x}_r^*$ will be referred to as the approximate equation and solution, respectively, of Equation (2.1).

I. (Condition that H and $\tilde{H}$ be neighboring operations) For every $\tilde{x}_r \in \tilde{X}_r^n$,

$$\|P_n H\tilde{x}_r - \tilde{H}\tilde{x}_r\| \leq \eta \|\tilde{x}_r\|.$$

II. (Condition for elements of the form Hx, $x \in X_r$, to be approximated by elements of $\tilde{X}_r^n$) For every $x \in X_r$, $\exists$ $\tilde{x} \in \tilde{X}_r^n$ $\ni$

$$\|Hx - \tilde{x}\| \leq \eta_1 \|x\|.$$

III. (Condition for close approximation of y_r of Equation (2.1)) $\exists$ $\tilde{y}_r \in \tilde{X}_r^n$ $\ni$

$$\|y_r - \tilde{y}_r\| \leq \eta_2 \|y_r\|,$$

where η_2 may be dependent on y_r (see Kantorovich [5]).

If these conditions are satisfied, questions of the type mentioned on page 9 are readily resolved [5]).

Secondly, with respect to the system under consideration (i. e., Equation (2.1) of X_r and (2.2) of $\overline{X}_r^n$), suppose there exists a subspace $\tilde{X}_r^n$ of X_r isomorphic to $\overline{X}_r^n$, where $\varphi_{0,r}^n$ is the function mapping $\tilde{X}_r^n$ isomorphically onto $\overline{X}_r^n$ and φ_r^n is a linear extension of $\varphi_{0,r}^n$ mapping X_r onto $\overline{X}_r^n$ such that $\varphi_r^n = \varphi_{0,r}^n P_n$. Therefore,

$$P_n = (\varphi_{0,r}^n)^{-1} \varphi_r^n.$$

In view of the isomorphism between $\tilde{X}_r^n$ and $\overline{X}_r^n$, Equation (2.1') can be transformed into an equivalent equation in $\overline{X}_r^n$ (and vice versa). This is accomplished by substituting $\tilde{x}_r = (\varphi_{0,r}^n)^{-1}\overline{x}_r$ in (2.1'), and applying the operation $\varphi_{0,r}^n$ to both sides.

$$\therefore \qquad \overline{x}_r + \varphi_{0,r}^n \tilde{H}(\varphi_{0,r}^n)^{-1}\overline{x}_r = \varphi_0 P_n y_r.$$

Letting

$$\overline{H} = \varphi_{0,r}^n \tilde{H}(\varphi_{0,r}^n)^{-1},$$

$$\overline{x}_r + \overline{H}\,\overline{x}_r = \varphi_r^n y_r.$$

Hence, the Kantorovich conditions on page 10 can be expressed in terms of the system under consideration (i. e., Equation (2.1) of X_r

and (2.2) of $\overline{X}_r^n$) by means of the mapping functions φ_r^n and $\varphi_{0,r}^n$. Therefore, under appropriate conditions (see page 10) involving $\overline{X}_r^n$, X_r, $\tilde{X}_r^n$, the mapping functions φ_r^n, $\varphi_{0,r}^n$, and the linear operator H of Equation (2.1) it is possible to answer all questions of the type stated on page 9 (see Kantorovich [5]). Consequently, in order to satisfactorily approximate the solution of Equation (2.0), it is sufficient to properly construct mapping functions and spaces in the above context such that all conditions of the Kantorovich theory are satisfied. In Section 2.2, this is accomplished for the solution $K(s,t)$ of Equation (2.0) for fixed s.

2.2 Algorithm Development

Mappings and spaces are constructed in this section for $K(s,t)$ of Equation (2.0) for fixed s such that the Kantorovich theory is applicable.

Let $\Delta_n(r) = \{\Delta_i\}_{i=1}^n$ be an equipartition of $[0, r]$, where

$$\Delta_i = \{s \mid \tau_{i-1} < s < \tau_i,\ \tau_i - \tau_{i-1} = \frac{r}{n},\ \tau_0 = 0\}, \quad i = 1, 2, \ldots, n,$$

and $\Delta (= \frac{r}{n})$ is the length of the interval Δ_i, $i = 1, \ldots, n$. Let t_i be the midpoint of Δ_i, $(i = 1, \ldots, n)$. Transform Equation (2.1)

$$x_r(s) + Hx_r(s) = y_r(s)$$

into

$$(2.2) \qquad x_r(t_j) + \sum_{k=1}^{n} \Delta h(t_j, t_k) x_r(t_k) = y_r(t_j), \quad (j = 1, 2, \ldots, n),$$

by requiring that Equation (2.1) be satisfied at the points t_i and that the integral $\int_0^r z(t)dt$ be replaced by the finite sum

$$\sum_{k=1}^{n} z(t_k)\Delta_k \left(= \sum_{k=1}^{n} z(t_k)\Delta\right).$$

Equation (2.2) may be expressed in the form:

$$\bar{K}_r^n \bar{x}_r = \bar{x}_r + \bar{H}_r^n \bar{x}_r = \varphi_r^n y_r,$$

where

$$\varphi_r^n y_r = (y_r(t_1), y_r(t_2), \ldots, y_r(t_n)) \in \bar{X}_r^n$$

and

$$\bar{H}_r^n = \begin{pmatrix} \Delta h(t_1, t_1) & \cdots & \Delta h(t_1, t_n) \\ \Delta h(t_2, t_1) & \cdots & \\ \vdots & & \\ \Delta h(t_n, t_1) & \cdots & \Delta h(t_n, t_n) \end{pmatrix}.$$

<u>Definition</u>. For the given equipartition $\Delta_n(r) = \{\Delta_i\}_1^n$, let $\tilde{X}_r^n$ be a subspace of X_r such that if $\tilde{x}_r \in \tilde{X}_r^n$, then $\tilde{x}_r(t_k) = \xi_k$, $\tilde{x}_r = (\varphi_{0,r}^n)^{-1} \bar{x}_r$ where $\bar{x}_r = (\xi_1, \xi_2, \ldots, \xi_n) \in \bar{X}_r^n$ and $x_r(t) = \xi_1$ for $t \in \Delta_1$.

Hence $\varphi_{0,r}^n$ maps $\tilde{X}_r^n$ isomorphically onto $\bar{X}_r^n$, and φ_r^n is a linear extension of $\varphi_{0,r}^n$. In brief then, the questions introduced on page 9 will be resolved within the framework of the spaces X_r, $\tilde{X}_r^n$ and $\bar{X}_r^n$, where X_r is the Banach space of continuous functions on $[0, r]$, $\tilde{X}_r^n$ a Banach space of continuous functions linear on each subinterval Δ_i, and $\bar{X}_r^n$, a vector space isomorphic to $\tilde{X}_r^n$. Diagrammatically

$$\begin{tikzcd} X_r \arrow[dr, "\varphi_r^n"] & \\ \tilde{X}_r^n \arrow[r, "\varphi_{0,r}^n"'] & \bar{X}_r^n \end{tikzcd}$$

where $\varphi_{0,r}^n$ is a function mapping $\tilde{X}_r^n$ isomorphically onto $\bar{X}_r^n$, and φ_r^n is a linear extension of $\varphi_{0,r}^n$ mapping X_r onto $\bar{X}_r^n$.

For the given equipartition $\Delta_n(r)$ of $[0, r]$ (with the length of $\Delta_i = \Delta$, $i = 1, \ldots, n$), let P_n be a projection operator mapping X_r onto $\tilde{X}_r^n$, $P_n^2 = P_n$.

$$\tilde{K}_r^n \tilde{x}_r^n = \tilde{x}_r^n + \tilde{H}_r^n \tilde{x}_r^n = P_n y_r,$$

with solution $\tilde{x}_r^{n*}$, will be called the <u>approximate equation and solution</u>, respectively, of

$$Kx_r = x_r + Hx_r = y_r,$$

if

I. There exists an η_r^{Δ} such that for each $\tilde{\tilde{x}}_r \in \tilde{\tilde{X}}_r^n$,

$\| P_n H\tilde{\tilde{x}}_r - \tilde{\tilde{H}}_r^n \tilde{\tilde{x}}_r \| \leq \eta_r^{\Delta} \| \tilde{\tilde{x}}_r \|$.

II. There exists an $\eta_{1,r}^{\Delta}$ such that for each $x_r \in X_r$, there exists an $\tilde{\tilde{x}}_r \in \tilde{\tilde{X}}_r^n$ such that $|Hx_r - \tilde{x}_r^n \| \leq \eta_{1,r}^{\Delta} \| x_r \|$.

III. For each $y_r \in X_r$, there exists a $\tilde{y}_r^n \in \tilde{\tilde{X}}_r^n$ such that $\| y_r - \tilde{y}_r^n \| \leq \eta_{2,r}^{\Delta} \| y_r \|$, where $\eta_{2,r}^{\Delta}$ may depend on y_r.

It should be noted that $\tilde{H}_r^n$ is a linear operator on $\tilde{X}_r^n$, and if $x \in X_r$ or $\tilde{\tilde{X}}_r^n$, then $\|x\| = \max_{0 \leq s \leq r} |x(s)|$, and if $x \in \overline{X}_r^n$, then $\|x\| = \max_{i=1,\ldots,n} |\xi_i|$, where $x = (\xi_1, \xi_2, \ldots, \xi_n)$.

In accordance with the Kantorovich theory, if η_r^{Δ}, $\eta_{1,r}^{\Delta}$ and $\eta_{2,r}^{\Delta}$ can be made sufficiently small, if K^{-1} exists, and if the three conditions (I, II, III) are satisfied, then $\tilde{x}_r^{n*}$ (the approximate solution, see page 14) exists [5]. Furthermore, if η_r^{Δ}, $\eta_{1,r}^{\Delta}$ and $\eta_{2,r}^{\Delta}$ converge to zero as $n \to \infty$ in $\Delta_n(r)$ (or equivalently, $\Delta \to 0$), the equipartition of $[0, r]$, then $\| x_r^* - \tilde{\tilde{x}}_r^n \| \to 0$ as $n \to 0$, where x_r^* is the solution of $Kx_r = y_r$ [5].

Since the problem of interest directly involves $\overline{X}_r^n$ rather than $\tilde{X}_r^n$, it will be necessary to interpret Condition I in the setting of $\overline{X}_r^n$. To this end, we proceed as follows:

Writing $\varphi_r^n = \varphi_{0,r}^{\ \ n} P_n$, we see that $P_n = (\varphi_{0,r}^{\ \ n})^{-1} \varphi_r^n$. Since $\tilde{x}_r^n + \tilde{H}_r^n \tilde{x}_r^n = P_n y_r$, there exists an $\bar{x}_r^n \in \overline{X}_r^n$ such that

$$\tilde{x}^n_r = (\varphi^n_{0,r})^{-1}\bar{x}^n_r, \qquad \bar{x}^n_r + \varphi^n_{0,r}\tilde{H}^n_r(\varphi^n_{0,r})^{-1}\bar{x}^n_r = \varphi^n_r y_r .$$

Hence

$$\bar{x}^n_r + \bar{H}^n_r\bar{x}^n_r = \varphi^n_r y_r, \qquad \text{so} \quad \bar{H}^n_r = \varphi^n_{0,r}\tilde{H}^n_r(\varphi^n_{0,r})^{-1} .$$

If we replace Condition I by the Condition

$$\text{Ia} \qquad \|\varphi^n_r H\tilde{x}^n_r - \bar{H}^n_r\varphi^n_{0,r}\tilde{x}^n_r\| \le \bar{\eta}^\Delta_r \|\tilde{x}^n_r\| ,$$

then by letting $\eta^\Delta_r = \|\varphi^{n^{-1}}_{0,r}\|\bar{\eta}^\Delta_r$ Condition I is satisfied, for

$$\begin{aligned}
\|P_n H\tilde{x}^n_r = \tilde{H}^n_r\tilde{x}^n_r\| &= \|(\varphi^n_{0,r})^{-1}\varphi^n_r H\tilde{x}^n_r - \tilde{H}^n_r\tilde{x}^n_r\| \\
&= \|(\varphi^n_{0,r})^{-1}\varphi^n_r H\tilde{x}^n_r - (\varphi^n_{0,r})^{-1}\bar{H}^n_r\varphi^n_{0,r}\tilde{x}^n_r\| \\
&\le \|(\varphi^n_{0,r})^{-1}\|\,\|\varphi^n_r H\tilde{x}^n_r - \bar{H}^n_r\varphi^n_{0,r}\tilde{x}^n_r\| \\
&\le \|(\varphi^n_{0,r})^{-1}\|\bar{\eta}^\Delta_r\|\tilde{x}^n_r\| \le \eta^\Delta_r\|\tilde{x}^n_r\| .
\end{aligned}$$

In summary we have the following:

<u>Problem</u>. $Kx(r,s) = x(r,s) + \int_0^r h(t,s)x(r,t)dt = y(r,s)$;

(equivalently) $Kx_r = x_r(s) + Hx_r(s) = y_r(s)$.

<u>Domain</u>. $0 \le s \le r \le R$.

<u>Assumption</u>. r fixed $(0 < r \le R)$ and equipartition $\Delta_n(r) = \{\Delta_i\}_1^n$

of $[0, r]$; Δ = length of Δ_i, $i = 1, 2, \ldots, n$.

<u>Reduced Problem (Algebraic System).</u> t_k the midpoint of Δ_k,

$$x(r, t_j) + \sum_k \Delta h(t_j, t_k) x(r, t_k) = y(r, t_j), \quad (j = 1, 2, \ldots, n);$$

$$\text{(equivalently)} \quad \overline{K}_r^n \overline{x}_r^n = \overline{x}_r^n + \overline{H}_r^n \overline{x}_r^n = \overline{y}_r,$$

where

$$\overline{x}_r^n \in \overline{X}_r^n, \quad \overline{y}_r \in \overline{X}_r^n.$$

$\overline{H}_r^n$: An $n \times n$ matrix whose element in the i-th row, j-th column is of the form $\Delta h(t_j, t_i)$.

X_r : Banach Space of continuous functions on $[0, r]$.

$\widetilde{X}_r^n$: Banach Space of continuous functions linear on each interval Δ_i, $i = 1, 2, \ldots, n$, such that if $\tilde{x} \in \widetilde{X}_r^n$, then $\tilde{x}(t_k) = \xi_k$, where $\tilde{x} = (\varphi_{0,r}^n)^{-1} \overline{x}$, $\overline{x} = (\xi_1, \xi_2, \ldots, \xi_n) \in \overline{X}_r^n$, and $\tilde{x}(t) = \xi_1$ for $t \in \Delta_1$.

<u>Lemma 1.</u> For each equipartition $\Delta_n(r) = \{\Delta_i\}_1^n$ of $[0, r]$ there exists an $\overline{\eta}_r^\Delta = r\omega_t^r(\frac{\Delta}{2})$ such that for each $\tilde{x}_r \in \widetilde{X}_r^n$

$$\| \varphi_r^n H \tilde{x}_r - \overline{H}_r^n \varphi_{0,r}^n \tilde{x}_r \| \leq \eta_r^\Delta \| \tilde{x}_r \|,$$

where $\omega_t^r(\frac{\Delta}{2})$ is the <u>modulus of t-continuity</u> of $h(t, s)$:

$$\omega_t^r\left(\frac{\Delta}{2}\right) = \sup\left\{\left|h(t+\delta, s)-h(t, s)\right|,\ 0 \le s \le r,\ 0 \le t \le r,\ |\delta| \le \frac{\Delta}{2}\right\};$$

Δ is the length of the interval Δ_i, $i = 1, 2, \ldots, n$; h, φ_r^n, $\varphi_{0,r}^n$, H, and $\overline{H}_r^n$ are discussed on page 17.

Proof. Suppose z is a function in X_r, whose modulus of continuity does not exceed $\omega(\delta)$. Let $\tilde{x} \in \tilde{X}_r^n$,

$$\left|\int_0^r z(t)\tilde{x}_r(t)dt - \sum_k \Delta z(t_k)\tilde{x}_r(t_k)\right|$$

$$= \left|\sum_k \int_{\Delta_k} [z(t)-z(t_k)]\tilde{x}_r(t)dt + \Delta\sum_k \left[\frac{1}{\Delta}\int_{\Delta_k} \tilde{x}_r(t)dt - \tilde{x}_r(t_k)\right]z(t_k)\right|, \frac{1}{\Delta}\int_{\Delta_k}\tilde{x}_r(t)dt$$

$$= \tilde{x}_r(t_k).$$

$$\Rightarrow \qquad \left|\int_0^r z(t)\tilde{x}_r(t)dt - \sum_k \Delta z(t_k)\tilde{x}_r(t_k)\right|$$

$$= \left|\sum_k \int_{\Delta_k} [z(t)-z(t_k)]\tilde{x}_r(t)dt\right| \le r\,\omega\left(\frac{\Delta}{2}\right)\|\tilde{x}_r\|.$$

Let $z(t) = h(t_j, t)$ $\Rightarrow$

$$\left|\int_0^r h(t_j, t)\tilde{x}_r(t)dt - \sum_k \Delta h(t_j, t_k)\tilde{x}_r(t_k)\right| \le r\,\omega_t^r\left(\frac{\Delta}{2}\right)\|\tilde{x}_r\|$$

$$\Rightarrow \qquad \|\varphi_r^n H\tilde{x}_r - \overline{H}_r^n \varphi_{0,r}^n \tilde{x}_r\| \le \overline{\eta}_r^\Delta \|\tilde{x}_r\|.$$

<u>Lemma 2</u>. Given an equipartition $\Delta_n(r) = \{\Delta_i\}_1^n$ of $[0, r]$, for each $x_r \in X_r$ there exists an $\tilde{x}_r \in \tilde{X}_r^n$ such that $\|Hx_r - \tilde{x}_r\| \leq \eta_{1,r}^{\Delta}\|x_r\|$, where

$$\omega_s^r(\Delta) = \sup \{|h(t, s+\delta) - h(t, s)|,\ 0 \leq t \leq r,\ 0 \leq s \leq r,\ |\delta| \leq \Delta\},$$

$$\eta_{1,r}^{\Delta} = r\omega_s^r(\Delta), \quad Hx_r(s) = \int_0^r h(t, s)x_r(t)dt,$$

and Δ is the length of Δ_i, $i = 1, 2, \ldots, n$.

<u>Proof</u>. Suppose $z(s) \in X_r$; let $\tilde{z}(s) \in \tilde{X}_r^n$, where $\tilde{z}$ is a piecewise linear function whose values at the points $\tau_1, \tau_2, \ldots,$ (i.e., $\tau_i - \tau_{i-1} = \frac{r}{n}$, $\tau_0 = 0$), coincide with the corresponding values of the function z.

Suppose $\tau_j \leq s \leq \tau_{j+1}$, $j = 1, 2, \ldots, n-1$. Therefore

$$|z(s) - \tilde{z}(s)| = |z(s) - [(\tau_{j+1} - s)z(\tau_j) + (s - \tau_j)z(\tau_{j+1})]\frac{1}{\Delta}|$$

$$\leq \frac{1}{\Delta}[|(\tau_{j+1} - s)(z(s) - z(\tau_j)| + |(s - \tau_j)(z(s) - z(\tau_{j+1})|].$$

Since $z(s) = [(\tau_{j+1} - s)z(s) + (s - \tau_j)z(s)]\frac{1}{\Delta}$,

$$|z(s) - \tilde{z}(s)| \leq \frac{1}{\Delta}(\tau_{j+1} - \tau_j)\omega(\Delta) = \omega(\Delta),$$

where $\omega(\Delta)$ = the modulus of continuity of $z(s)$.

If $0 \leq s \leq \tau_1$, then

$$|z(s)-\tilde{z}(s)| = |z(s)-z(\tau_1)| \le \omega(\Delta).$$

Consider $z(s) - z(s')$, for $z = Hx$, $x \in X_r$,

$$|z(s)-z(s')| \le \int_0^r |h(s,t) - h(s',t)|\,|x(t)|\,dt.$$

Therefore

$$|z(s)-z(s')| \le r\omega_s^r(\delta)\|x\|,$$

which implies for arbitrary s,

$$|z(s)-\tilde{z}(s)| \le r\omega_s^r(\Delta)\|x\| \quad \text{for} \quad z = Hx,\ x \in X_r$$

$$=> \qquad \|Hx_r-\tilde{z}\| \le \eta_{1,r}\|x_r\|, \quad \text{where} \quad \eta_{1,r} \stackrel{\Delta}{=} r\omega_s^r(\Delta).$$

Lemma 3. Given an equipartition $\Delta_n(r) = \{\Delta_i\}_1^n$ of $[0, r]$, for each $y_r \in X_r$ there exists a $\tilde{y} \in \tilde{X}_r^n$ such that

$$\|y_r-\tilde{y}\| \le \eta_{2,r}\|y_r\|, \quad 0 \le r \le R,$$

where

$$\eta_{2,r} \stackrel{\Delta}{=} \frac{1}{\|y_r\|}\,\bar{\omega}^{(r)}(\Delta),$$

$$\bar{\omega}^{(r)}(\Delta) = \sup\{|y(r,s+\delta)-y(r,s)|,\ 0 \le s \le r,\ |\delta| \le \Delta\},$$

and Δ is the length of the partition Δ_i, $i = 1, 2, \ldots, n$.

Proof. In the proof of Lemma 2, it was shown that for each $z \in X_r$, there exists a $\tilde{z} \in \tilde{X}_r^n$ such that $|z(s)-\tilde{z}(s)| \le \omega(\Delta)$, where ω was the modulus of continuity of z. By letting $\eta_{2,r}^{\Delta} = \frac{1}{\|y\|}\bar{\omega}^{r}(\Delta)$ implies that $|y(s)-\tilde{y}(s)| \leq \bar{\omega}^{(r)}(\Delta) = \eta_{2,r}^{\Delta}\|y\|$ for arbitrary s. Hence, $\|y-\tilde{y}\| \leq \eta_{2,r}^{\Delta}\|y\|$.

2.3 Results

Theorem 2.1. There exists an equipartition $\Delta_{\mu} = \{\Delta_i'\}_1^{n'}$ of $[0, R]$ such that for each equipartition $\Delta_n(r) = \{\Delta_i\}_1^n$ of $[0, r]$, $0 < r \leq R$, at least as fine as Δ_{μ}, (i.e., $\Delta \leq \Delta'$)

$$\bar{K}_r^n \bar{x}_r^n = \bar{y}_r^n$$

is uniquely solvable for all $\bar{y}_r^n \in \bar{X}_r^n$; also,

$$\|x(r, s)-(\varphi_{0,r}^{n})^{-1}\bar{x}_r^n\| \leq \bar{\bar{P}}_{\Delta'}\|x(r, s)\|,$$

where Δ' is the length of Δ_i', $i = 1, 2, \ldots, n$; Δ is the length of Δ_i, $i = 1, 2, \ldots, n$; $\bar{\bar{P}}$ is defined below, and the remaining notation is discussed on page 17.

$$\bar{\bar{P}}_{\Delta} = (1+\bar{\varepsilon}_R^{\Delta})\bar{\eta}_R^{\Delta}\frac{(1+\eta_{1,R}^{\Delta})M_1}{1-\bar{\bar{q}}_R^{\Delta}} + \bar{\varepsilon}_R^{\Delta}(1+\frac{M_1 M(1+\eta_{1,R}^{\Delta})}{1-\bar{\bar{q}}_R^{\Delta}}),$$

where $M \geq \sup_{0<r\leq R}\{\|K^{-1}\|\}$, $M_1 \geq \sup_{0<r\leq R}\{\|K\|\}$,

$$\bar{q}_R^{\Delta} = [\bar{\eta}_R^{\Delta}(1+\eta_{1,R}^{\Delta}) + \eta_{1,R}^{\Delta}M]M_1,$$

$$\bar{\varepsilon}_R^\Delta = \eta_{1,R}^\Delta + \bar{\eta}_{2,R}^\Delta M.$$

<u>Proof of Theorems 1 and 2.</u> $\bar{\eta}_r^\Delta \leq \bar{\eta}_R^\Delta$, for $0 \leq r \leq R$, since $\bar{\eta}_r^\Delta = r\omega_t^r(\frac{\Delta}{2})$. Also, $\eta_{1,r}^\Delta \leq \eta_{1,R}^\Delta$, for $0 \leq r \leq R$, since $\eta_{1,r}^\Delta = r\omega_s^r(\Delta)$. In applying the Kantorovich theory [5], it is necessary to make $q_r^\Delta < 1$, where

$$q_r^\Delta = [\bar{\eta}_r^\Delta(1+\eta_{1,r}^\Delta) + \eta_{1,r}^\Delta \| (\varphi_{0,r}^n)^{-1} \varphi_r^n K \|] \| K^{-1} \| .$$

Therefore

$$q_r^\Delta \leq (\bar{\eta}_R^\Delta(1+\eta_{1,R}^\Delta) + \eta_{1,R}^\Delta M)M_1 .$$

Let

$$\bar{\bar{q}}_R^\Delta = (\bar{\eta}_R^\Delta(1+\eta_{1,R}^\Delta) + \eta_{1,R}^\Delta M)M_1 ,$$

therefore

$$q_r^\Delta \leq \bar{\bar{q}}_R^\Delta \quad \text{for} \quad 0 \leq r \leq R.$$

Let $\Delta_\mu = \{\Delta_i'\}_1^{n'}$ be an equipartition of $[0, R]$, with the length of the subinterval $\Delta_i' = \Delta'$, $(i = 1, 2, \ldots, n)$, such that $\bar{\bar{q}}_R^{\Delta'} < 1$. Therefore $\bar{\bar{q}}_R^{\Delta'} < 1$ for $\Delta_\mu \Rightarrow$

$$\bar{K}_R^{n'} \bar{x}_R^{n'} = \bar{y}_R$$

is solvable for all $\bar{y}_R \in \bar{X}_R^{n'}$ (see Appendix B), and all finer equipartitions of $[0, R]$. Hence $[\bar{K}_r^{n(\Delta_{\mu'})} \bar{x}_r^{n(\Delta_{\mu'})} = \bar{y}_r]$ is solvable for all $\bar{y}_r \in X_r^{n(\Delta_{\mu'})}$, where $\Delta_{\mu'}$ is an equipartition of $[0, r]$ at least as fine as Δ_μ.

Let

$$\bar{\omega}(\Delta) = \sup_{s,\, r} \{|y(r, s+\delta) - y(r, s)|,\ 0 \leq s \leq r \leq R,\ |\delta| \leq \Delta\}$$

and

$\|y\|_{min} = \min_{0 \leq r \leq R} \{ \max_{0 \leq s \leq r} (|y(r, s)|)$, deleting those valuse of r

such that $r \in (r_0 - \delta, r_0 + \delta)$, where δ is a given, positive, aribtrarily small number and $y(r_0, s) = 0$ for $0 \leq s \leq r_0$, which implies, $x^*(r_0, s)$, the solution of the equation $Kx(r_0, s) = y(r_0, s)$, $0 \leq s \leq r_0$, is identically zero.

In particular, since $x(r, s)$ is a continuous function, for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|x(r, s)| < \varepsilon$ for $|r - r_0| \leq \delta$, $0 \leq s \leq r$. We will refer to this deleted set as the "exceptional set." Therefore

$$\eta_{2,r}^{\Delta} = \frac{\bar{\omega}^{(r)}(\Delta)}{\|y_r(s)\|} \Rightarrow \eta_{2,r}^{\Delta} \leq \frac{\bar{\omega}(\Delta)}{\|y\|_{min}} = \eta_{2,R}^{\Delta}$$

providing $r \notin$ exceptional set for $0 \leq r \leq R$.

Noting that the two theorems, plus the corollary and theorems to follow in this section must be qualified with respect to the "exceptional set."

In accordance with the Kantorovich Theory [5], (see Appendix B), for a given equipartition $\Delta_n(r) = \{\Delta_i\}_1^n$ of $[0, r]$, with Δ the length of Δ_i, $i = 1, 2, \ldots, n$,

$$\overline{P}_{r,\Delta} = (1+E_r^\Delta)\,\overline{\eta}_r^\Delta\,\|(\varphi_{0,r}^n)^{-1}(\overline{K}_r^n)^{-1}\| + E_r^\Delta(1+\|(\varphi_{0,r}^n)^{-1}(\overline{K}_r^n)^{-1}\varphi_r^n K\|).$$

$$E_r^\Delta \le \eta_{1,r}^\Delta + \eta_{2,r}^\Delta\,\|K\| \le \eta_{1,R}^\Delta + \overline{\eta}_{2,R}^\Delta, \quad r \notin \text{(Exceptional set)}.$$

Let $\overline{E}_R^\Delta = \eta_{1,R}^\Delta + \overline{\eta}_{2,R}^\Delta$, therefore

$$\overline{P}_{r,\Delta} \le (1+\overline{E}_R^\Delta)\,\overline{\eta}_R^\Delta\,\|\overline{K}_r^{n^{-1}}\| + \overline{E}_R^\Delta(1+\|\overline{K}_r^{n^{-1}}\|\,\|K\|).$$

Suppose $\Delta_\mu = \{\Delta_i'\}_1^{n'}$ is the equipartition of $[0, R]$ with the length of the subinterval $\Delta_i' = \Delta'$, $i = 1, 2, \ldots, n'$, and that $\Delta_n(r)$ is at least as fine as Δ_μ, i.e., $\Delta \le \Delta'$. Through use of the Kantorovich Theory (see Appendix B), we directly obtain the following estimate of $(\overline{K}_r^n)^{-1}$:

$$\|\overline{K}_r^{n^{-1}}\| \le \frac{(1+\eta_{1,r}^\Delta)M_1}{1-q_r^\Delta} \le \frac{(1+\eta_{1,R}^\Delta)M_1}{1-\overline{\overline{q}}_R^\Delta};$$

$$\|K\| \le M_1 \quad \text{for} \quad 0 \le r \le R.$$

Also, $\overline{E}_R^\Delta \le \overline{E}_R^{\Delta'}$ since $\eta_{1,R}^\Delta \le \eta_{1,R}^{\Delta'}$.

Therefore

$$\bar{P}_{r,\Delta} \le (1+\bar{E}_R^{\Delta'})\bar{\eta}_R^{\Delta'} \frac{(1+\eta_{1,R}^{\Delta'})M_1}{1-\bar{\bar{q}}_R^{\Delta'}} + \bar{E}_R^{\Delta'}\left(1+\frac{(1+\eta_{1,R}^{\Delta})M_1 M}{1-\bar{\bar{q}}_R^{\Delta'}}\right),$$

since $\bar{\bar{q}}_R^{\Delta} \le \bar{\bar{q}}_R^{\Delta'} \Rightarrow \bar{P}_{r,\Delta} \le \bar{\bar{P}}_{\Delta'}$;

$$\|x_r(s)-(\varphi_{0,r}^{n})^{-1}\bar{x}_r\| \le \bar{P}_{r,\Delta}\|x_r\| \le \bar{\bar{P}}_{\Delta'}\|x_r\|$$

from which Theorem (1) directly follows.

Since $\|x_r\| \le \|(\varphi_{0,r}^{n})^{-1}\bar{x}_r\| + \|x_r-(\varphi_{0,r}^{n})^{-1}\bar{x}_r\|$,

$$\|x_r-(\varphi_{0,r}^{n})^{-1}\bar{x}_r\| \le \bar{\bar{P}}_{\Delta'}\|(\varphi_{0,r}^{n})^{-1}\bar{x}_r\| + \|x_r-(\varphi_{0,r}^{n})^{-1}\bar{x}_r\|\bar{\bar{P}}_{\Delta'}.$$

Therefore

$$(\|x_r-(\varphi_{0,r}^{n})^{-1}\bar{x}_r)(1-\bar{\bar{P}}_{\Delta'}) \le \bar{\bar{P}}_{\Delta'}\|(\varphi_{0,r}^{n})^{-1}\bar{x}_r\|.$$

By selecting the equipartition $\Delta_{\mu\mu}$ such that $P_{\Delta''} < 1$, Theorem (2) directly follows.

$$\|x_r-(\varphi_{0,r}^{n})^{-1}\bar{x}_r\| \le \frac{\bar{\bar{P}}_{\Delta''}}{1-\bar{\bar{P}}_{\Delta''}}\|(\varphi_{0,r}^{n})^{-1}\bar{x}_r\| \quad .$$

<u>Corollary</u>. Under the hypothesis of Theorem 2.1 or 2.2

$$\|(\varphi_{0,r}^{n})^{-1}\bar{x}_r^n\| \le (1+\eta_{1,R}^{\Delta})\frac{M_1\|\bar{y}_r^n\|}{1-\bar{\bar{q}}_R^{\Delta}} \quad .$$

Proof. Since

$$\bar{K}_r^n \bar{x}_r^n = \bar{y}_r^n = \varphi_r^n y_r$$

and

$$\bar{x}_r^n = (\bar{K}_r^n)^{-1} \bar{y}_r^n ,$$

$$(\varphi_{0,r}^n)^{-1} \bar{x}_r^n = (\varphi_{0,r}^n)^{-1} (\bar{K}_r^n)^{-1} \bar{y}_r^n .$$

Therefore

$$\| x_r^n \| \leq \| (\bar{K}_r^n)^{-1} \| \, \| \bar{y}_r^n \| ,$$

where

$$\| (\bar{K}_r^n)^{-1} \| \leq \frac{(1+\eta_{1,\hat{R}}) M_1}{1-\bar{\bar{q}}_{\hat{R}}} .$$

Theorem 2.3. For the solution $\bar{x}_r^{n*}$ of

$$\bar{K}_r^n \bar{x}_r^n = \varphi_r^n y_r , \quad 0 \leq r \leq R, \quad 0 \leq s \leq r,$$

which is an approximate solution of the equation

$$K x_r(s) = y_r(s), \quad 0 \leq r \leq R,$$

whose solution is x_r^*,

$$\lim_{n \to \infty} \| x_r^* - (\varphi_{0,r}^n)^{-1} x_r^{n*} \| = 0;$$

in addition, there exist constants $\bar{Q}$, $\bar{Q}_1$, and $\bar{Q}_2$ such that

$$\|x_r^*(s)-(\varphi_{0,r}^{n})^{-1}\bar{x}_r^{n*}\| \leq \bar{Q}\eta_R^{\Delta'} + Q_1\eta_{1,R}^{\Delta'} + \bar{Q}_2\bar{\eta}_{2,R}^{\Delta'},$$

where the symbols have exactly the same meaning as given in Theorem 2.1.

Proof. From the results of Theorem 2.1, these results readily follow since $\bar{\eta}_R^{\Delta} \to 0$ as $\Delta \to 0$, $\eta_{1,R}^{\Delta} \to 0$ as $\Delta \to 0$, and $\eta_{2,R}^{\Delta} \to 0$ as $\Delta \to 0$.

Since the kernel $h(s,t)$ of concern is actually the function $F(s,t)$ of Spectral theory, which is at least absolutely continuous in each variable, let us now assume $h(s,t)$ is absolutely continuous in each variable. Furthermore, let us assume $h(s,t)$ is lipschitzian in each variable This implies that there exists an α such that

$$|h(s+\sigma,t)-h(s,t)| \leq \alpha|\sigma|, \quad \text{for all } s,t \text{ in } [0,R].$$

$$|h(s,t+\sigma)-h(s,t)| \leq \alpha|\sigma|, \quad \text{for all } s,t \text{ in } [0,R].$$

$$\therefore \quad \omega_s^R(\delta) \leq \alpha|\delta| \ .$$

$$\omega_t^R(\delta) \leq \alpha|\delta| \Rightarrow \bar{\eta}_R^{\Delta} \leq R\alpha\frac{\Delta}{2}, \quad \eta_{1,R}^{\Delta} \leq R\alpha\Delta,$$

and

$$\bar{\eta}_{2,R}^{\Delta} = \frac{\bar{\omega}(\Delta)}{\|y\|_{min}} \leq \beta\Delta \quad ,$$

for partition Δ_μ with the length of its subintervals $\Delta_i = \Delta$.

Theorem 2.4. If in addition to the hypothesis of Theorem 2.3, the kernel $h(s,t)$ satisfies a lipschitz condition in each variable, then

$$\|x(r,s)-\tilde{x}_r^n\| = O(\Delta)$$

for all r such that $0 < r \leq R$ providing the equipartition is at least as fine as Δ_μ.

Proof. $\Box$.

III. ALGORITHM FOR q(x)

3.0 Definition of Problem

In determining $q(x)$ of the Sturm-Liouville differential equation

(3.0) $$y'' - q(x)y + \lambda y = 0, \quad 0 \leq x \leq R,$$

from its spectral function, it has been shown by Gelfand-Levitan (see [4] or [6]) that

$$q(x) = +2\,\frac{dK(x,x)}{dx},$$

where

$$K(x,t) + F(x,t) + \int_0^x F(s,t)K(x,s)ds = 0, \quad 0 \leq t \leq x \leq R.$$

In this chapter an algorithm for numerically solving for $dK(x,x)/dx$ is developed, where

(3.1) $$\frac{dK(x,x)}{dx} + \frac{dF(x,x)}{dx} + F(x,x)K(x,x) + \int_0^x F(s,x)\frac{\partial K}{\partial x}(x,s)dx = 0.$$

This algorithm is based primarily upon Chapter II. However, this time it will be necessary to assume that F of Equation (2.0) is in C^1 rather than just in C^0.

The main result to be developed in this chapter is the following:

Given $\varepsilon > 0$, there exists an equipartition $\Delta_{\eta\eta}$ of $[0, R]$ such that for each x, $0 \leq x \leq R$, there exists a readily constructed continuous, piecewise differentiable function $G_x^n(s)$ on $0 \leq s \leq x$ such that

$$\left|\frac{dK(x,x)}{dx} - G_x^n(x)\right| < \varepsilon$$

providing the equipartition Δ_n of $[0, x]$ associated with $G_x^n(s)$ is at least as fine as $\Delta_{\eta\eta}$.

In Section 3.1, it is shown to be sufficient to develop algorithms for $\partial K/\partial x$ and $\partial K/\partial t$ rather than for $dK(x,x)/dx$ of Equation (3.1). In Sections 3.2 and 3.3 algorithms are developed for $\partial K/\partial t$ and $\partial K/\partial x$, respectively, based primarily upon Chapter II. In Section 3.4 the results of this chapter are obtained by combining the results of Sections 3.2 and 3.3.

3.1 Simplification

Since

$$\frac{dK(x,t)}{dx} = \left(\frac{\partial K(x,t)}{\partial x} + \frac{\partial K}{\partial t}\frac{dt}{dx}\right) \quad \text{along} \quad t = f(x),$$

it follows that

$$\frac{d}{dx}K(x,x) = \left[\frac{d}{dx}(K(x,t))\right]_{t=x} = \left(\frac{\partial K}{\partial x} + \frac{\partial K}{\partial t}\right)\Big|_{t=x}.$$

Therefore, determining $dK(x,x)/dx$ of Equation (3.1) reduces to determining $\partial K/\partial x|_{t=x}$ and $\partial K/\partial t|_{t=x}$.

Differentiating Equation (2.0) yields:

$$\frac{\partial K(x,t)}{\partial t} = -\frac{\partial F(x,t)}{\partial t} - \int_0^x \frac{\partial F(s,t)}{\partial t} K(x,s)ds, \tag{3.2}$$

and

$$\frac{\partial K(x,t)}{\partial x} = -\frac{\partial F(x,t)}{\partial x} - F(x,t)K(x,x) - \int_0^x F(s,t)\frac{\partial K(x,s)}{\partial x} ds. \tag{3.3}$$

It follows directly that $\partial K/\partial t$ of Equation (3.2) is unique and continuous. Also, since $\partial F/\partial x$ is continuous by hypothesis, it follows from the spectral theory [6, p. 14] that $\partial K/\partial x$ is unique and continuous.

3.2 Algorithm for $\partial K/\partial t$

Let $\partial K^*/\partial t$ be the unique solution of Equation (3.2).

$$\frac{\partial K^*(x,t)}{\partial t} = -\frac{\partial F(x,t)}{\partial t} - \int_0^x \frac{\partial F(s,t)}{\partial t} K^*(x,s)ds.$$

$$= -\frac{\partial F(x,t)}{\partial t} - \int_0^x \frac{\partial F(s,t)}{\partial t} (K^*(x,s) - \tilde{K}_x^{n*}(s))ds$$

$$- \int_0^x \frac{\partial F}{\partial t} K_x^{n*} ds.$$

For fixed x, let

$$K_x^n(t)^* = \frac{\partial F(x,t)}{\partial t} - \int_0^x \frac{\partial F}{\partial t} K_x^{n*}(s)ds,$$

where $K_x^{n*}(s)$ is the approximate solution of $K(x, s)$ of Equation (2.0), for fixed x, $0 \le s \le x \le R$, as developed in Chapter II (see page 7), and we denote the unique solution of Equation (2.0) by K^*.

Therefore,

$$\frac{\partial K^*}{\partial t} - \tilde{K}_x^{n*} = \int_0^x \frac{\partial F(s,t)}{\partial t} (\tilde{K}_x^n(s)^* - K^*(x,s))ds \, ;$$

$$\left| \frac{\partial K^*(x,t)}{\partial t} - \tilde{K}_x^n(t)^* \right| \le \varepsilon_1 M_4 = \varepsilon_2, \quad \text{for} \quad 0 \le t \le x,$$

where

$$M_4 = \max_{0 \le t \le x \le R} \int_0^x \left| \frac{\partial F(s,t)}{\partial t} \right| ds.$$

Hence, for a given $\varepsilon_1 > 0$, there exists a continuous function $\tilde{K}_x^n(t)^*$ for each x, $0 \le x \le R$, such that

$$\left| \frac{\partial K^*(x,t)}{\partial t} - \tilde{K}_x^n(t)^* \right| \le \varepsilon_2, \quad 0 \le t \le x, \ \varepsilon_2 = \varepsilon_1 M_4,$$

and

$$\left| \frac{\partial K^*(x,t)}{\partial t} - \tilde{K}_x^n(t)^* \right| \le \varepsilon_1 M_4 = \varepsilon_2, \quad \text{for} \quad 0 \le t \le x.$$

In particular, for a given $\varepsilon_1 > 0$ there exists an equipartition Δ_μ of $[0, R]$ such that for each equipartition $\Delta_n(x)$ of $[0, x]$ at least as fine as Δ_μ a continuous piecewise differentiable function $\tilde{K}_x^n(s)^*$, $0 \le s \le x$, can be constructed (as shown in Chapter II) such that for each x, $0 < x \le R$, $|K^* - \tilde{K}_x^{n*}| \le \varepsilon_1$, $0 \le t \le x$, and

$$\left|\frac{\partial K^*(x,t)}{\partial t} - \tilde{\tilde{K}}_x^{n*}(t)\right| \le \varepsilon_2, \quad 0 \le t \le x, \quad 0 < x \le R, \quad \varepsilon_2 = \varepsilon_1 M_4.$$

3.3 Algorithm for $\partial K/\partial x$

For Equation (3.3)

$$(3.3) \qquad \frac{\partial K(x,t)}{\partial x} = -\frac{\partial F(x,t)}{\partial x} - F(x,t)K(x,x) - \int_0^x F(s,t)\frac{\partial K(x,s)}{\partial x}\,ds;$$

let $G^* = \partial K^*/\partial t$ denote its unique solution. Therefore

$$G^* = \frac{\partial F(x,t)}{\partial x} - F(x,t)K^*(x,x) - \int_0^x F(s,t)G^*(x,s)\,ds.$$

Fix x, let

$$(3.4) \qquad \hat{G}_x^n(t) = -F(x,t)[K^*(x,x) - \tilde{K}_x^{n*}(x)] - \int_0^x F(s,t)\hat{G}_x^n(s)\,ds.$$

For fixed x, consider the equation:

$$(3.5) \qquad \tilde{G}_x^n(t) = -\frac{\partial F}{\partial x} - F(x,t)\tilde{K}_x^n(x)^* - \int_0^x F(s,t)\tilde{G}_x^n(s)\,ds\,.$$

In Equation (3.4), divide through by $K^*(x,x) - \tilde{K}_x^{n*}(x)$:

$$\frac{\hat{G}_x^n(t)}{K^*(x,x) - \tilde{K}_x^{n*}(x)} = -F(x,t) - \frac{1}{(K^*(x,x) - \tilde{K}_x^{n*}(x))}\int_0^x F(s,t)G_x^n(s)\,ds.$$

Noting that Equation (3.3) cannot be numerically solved directly for $\partial K/\partial x$, since $K(x,x)$ is typically unknown. Therefore

$$\frac{\hat{G}_x^n(t)}{K^*(x,x)-\tilde{K}_x^{n*}(x)} = -F(x,t) - \int_0^x F(s,t)\left[\frac{\hat{G}_x^n(s)}{K^*(x,x)-\tilde{K}_x^{n*}(x)}\right]ds .$$

However

$$K(x,t) = -F(x,t) - \int_0^x F(s,t)K(x,s)ds,$$

for each fixed x, has the unique solution $K^*(x,t)$. Hence,

$$\frac{\hat{G}_x^n}{K^*(x,x)-\tilde{K}_x^{n*}(x)} = K^*(x,t).$$

Let $\hat{G}_x^{n*}$ denote the unique solution of Equation (3.4) for the given equipartition $\Delta_n(x)$ of $[0,x]$. Therefore for each x, there exists a unique function $\hat{G}_x^n(t)^*$ satisfying Equation (3.4) for each appropriate equipartition $\Delta_n(x) = \{\Delta_i\}_1^n$ of $[0,x]$ (i.e., $\Delta \leq \Delta'$, see page 7); it will be assumed that $K^*(x,x) \neq \tilde{K}_x^{n*}(x)$, for if $K^*(x,x) = \tilde{K}_x^{n*}(t)$, then $G(x,t) = \tilde{G}_x^n(t)$ of (3.5). Hence, the following analysis holds.

For fixed x, consider $G^*(x,t) - \hat{G}_x^n(t)$, where $G^*(x,t) = \dfrac{\partial K^*}{\partial x}$ is the solution of Equation (3.3):

$$(3.3) \qquad G^*(x,t) = \frac{\partial F(x,t)}{\partial x} - F(x,t)K^*(x,x) - \int_0^x F(s,t)G^*(x,s)ds;$$

$$(3.4) \qquad \hat{G}_x^{n*}(t) = -F(x,t)[K^*(x,x)-K_x^{n*}(x)] - \int_0^x F(s,t)\hat{G}_x^{n*}(s)ds.$$

Therefore,

$$G^*(x,t) - \hat{G}_x^n(t)^* = \frac{-\partial F}{\partial x} - F(x,t)\tilde{K}_x^n(x) - \int_0^x F(x,t)[G^* - \hat{G}_x^n]ds.$$

Let

$$\tilde{G}_x^n(t)^* = G^*(x,t) - \hat{G}_x^n(t).$$

Therefore $\tilde{G}_x^{n*}(t)$ is the unique solution of Equation (3.5):

$$(3.5) \qquad \tilde{G}_x^n(t)^* = -\frac{\partial F(x,t)}{\partial x} - \tilde{K}_x^{n*}(x)F(x,t) - \int_0^x F(s,t)\tilde{G}_x^n(s)^* ds.$$

We will now show that for a given $\varepsilon_3 > 0$, there exists an equipartition $\Delta_{\mu\mu\mu}$ of $[0, R]$ such that for each x,

$$\|\hat{G}_x^n\| \le \varepsilon_3, \quad (\text{i.e.}, \ \|G^* - \hat{G}_x^n\| \le \varepsilon_3),$$

for all equipartitions of $[0, x]$ at least as fine as $\Delta_{\mu\mu\mu}$.

Since

$$\frac{\hat{G}_x^n(t)}{K^*(x,x) - \tilde{K}_x^{n*}(x)} = K^*(x,t),$$

$$\left| \frac{\hat{G}_x^n(t)}{K^*(x,x) - K_x^n(x)^*} - \tilde{K}_x^n(t)^* \right| \le \varepsilon_1, \quad 0 \le t \le x.$$

Hence,

$$|\hat{G}_x^n(t) - \tilde{K}_x^n(t)^*(K^*(x,x) - \tilde{K}_x^n(x)^*)| \le \varepsilon_1 |K^*(x,x) - \tilde{K}_x^{n*}(x)|.$$

$$|\hat{G}_x^n(t) - \tilde{K}_x^{n*}(t)(K^*(x,x) - \tilde{K}_x^n(x)^*)| \le \varepsilon_1^2.$$

Therefore

$$|\hat{G}_x^n(t)| - |\tilde{K}_x^n(t)^*|\,|K^*(x,x)-\tilde{K}_x^n(x)^*| \le \varepsilon_1^2 .$$

$$|\hat{G}_x^n(t)| \le \varepsilon_1^2 + \varepsilon_1 |\tilde{K}_x^n(t)^*| = \varepsilon_1(\varepsilon_1 + |\tilde{K}_x^n(t)^*|).$$

For fixed x,

$$\| G^*(x,t) - G_x^{n*}(t) \| \le \varepsilon_1(\varepsilon_1 + \|\tilde{K}_x^{n*}\|),$$

where $G^*(x,t) = \dfrac{\partial K^*(x,t)}{\partial x}$ is the solution of (3.3):

$$\frac{\partial K(x,t)}{\partial x} + \frac{\partial F(x,t)}{\partial x} + \int_0^x F(s,t)\frac{\partial K(x,s)}{\partial x}\,ds + F(x,t)K(x,x) = 0,$$

and $\tilde{G}_x^{n*}$ is the solution of (3.5):

$$\tilde{G}_x^n(t) = -\frac{\partial F(x,t)}{\partial x} - F(x,t)\tilde{K}_x^n(x)^* - \int_0^x F(s,t)\tilde{G}_x^n(s)\,ds.$$

From the above result, the assertion given on page 30 directly follows, since $\|\tilde{K}_x^{n*}\| \le \|K(x,t) - \tilde{K}_x^{n*}\| + \|K(x,t)\|$, and ε_1 is independent of x.

By letting $y_x(t) = -\dfrac{\partial F(x,t)}{\partial x} - F(x,t)K_x^n(x)^*$, Equation (3.5) can be written in the form:

$$\tilde{G}_x^n(t) + \int_0^x F(s,t)\tilde{G}_x^n(s)\,ds = y_x(t).$$

This is the same type of integral equation whose numerical solution was considered in Chapter II; the only difference is that in Chapter II we were primarily interested in the case where $y_x(t) = F(x,t)$; whereas in this case

$$y_x(t) = -\frac{\partial F(x,t)}{\partial x} - F(x,t)\tilde{K}_x^n(x)^*.$$

Therefore, in order to apply the results of Chapter II to Equation (3.5), it will be sufficient to show that the three conditions cited on page 15 are satisfied. However, since the equations are the same except for the nonhomogeneous term $y_x(t)$, it is sufficient to check the conditions that involve $y_x(t)$. Therefore, it suffices to show that for each $y_x(t)$ there exists a $\tilde{y}_x(t)$ such that

$$\| y_x - \tilde{y}_x \| \leq \eta_{2,x}^{\Delta} \| y_x \|, \quad 0 \leq x \leq R, \quad \text{(see page 20)}.$$

Since

$$y_x(t) = -\frac{\partial F(x,t)}{\partial x} - F(x,t)K_x^n(x)^*,$$

which is continuous, the above condition is shown by following exactly the same line of reasoning as presented in Chapter II on this item.

Hence, for a given $\varepsilon_4 > 0$, and a sufficiently fine equipartition Δ_m of $[0,R]$, for each equipartition Δ_n of $[0,x]$, $0 < x \leq R$, at least as fine as Δ_m, there exists a continuous piecewise differentiable function $\approx{G}_x^{m*}$ on $[0,x]$ such that

$$\|\widetilde{G}^{n*}_x - \widetilde{\widetilde{G}}^{m*}_x\| \le \varepsilon_4 .$$

3.4 Results

For each fixed x:

$$\frac{\partial K^*(x,t)}{\partial t} = \left(\frac{\partial K^*(x,t)}{\partial t} - \widetilde{\widetilde{K}}^n_x(t)^*\right) + \widetilde{\widetilde{K}}^n_x(t)^* ;$$

$$\frac{\partial K^*(x,t)}{\partial x} = \left(\frac{\partial K^*(x,t)}{\partial x} - \widetilde{G}^{n*}_x(t)\right) + \widetilde{G}^n_x(t)^* .$$

Therefore, since

$$\frac{dK(x,x)}{dx} = \frac{dK^*(x,t)}{dx}\bigg|_{t=x} = \left(\frac{\partial K^*(x,t)}{\partial x} + \frac{\partial K^*(x,t)}{\partial t}\right)_{t=x},$$

we obtain

$$\frac{dK^*(x,t)}{dx}\bigg|_{t=x} - \left(K^n_x(t)+G^n_x(t)^*\right)\Big|_{t=x} = \left(\frac{\partial K^*(x,t)}{\partial t} - \widetilde{\widetilde{K}}^n_x(t)^*\right)_{t=x}$$

$$+ \left(\frac{\partial K^*}{\partial x} - \widetilde{G}^n_x(t)^*\right)_{t=x} .$$

$$\left|\frac{dK^*(x,x)}{dx} - (\widetilde{\widetilde{K}}^n_x(x) + \widetilde{G}^{n*}_x(x)\right| \le \varepsilon_1 M_4 + \varepsilon_1(\varepsilon_1 + \|\widetilde{K}^{n*}_x\|) \le M\varepsilon_1 = \bar{\varepsilon}$$

uniformly in x, providing for each x, its associated equipartition $\Delta_n(x)$ is at least as fine as both equipartitions Δ_μ (see page 32) and $\Delta_{\mu\mu\mu}$ (see page 35), where

$$\|K^* - \widetilde{K}^n_x\| \le \varepsilon_1, \quad \widetilde{\widetilde{K}}^n_x(t)^* = -\frac{\partial F(x,t)}{\partial t} - \int_0^x \frac{\partial F(x,t)}{\partial t}\,\widetilde{K}^n_x(x)^*\,ds$$

and

$$(3.5) \qquad \tilde{G}^n_x(t) = -\frac{\partial F(x,t)}{\partial x} - F(x,t)\tilde{K}^n_x(x)^* - \int_0^x F(s,t)\tilde{G}^n_x(s)ds.$$

Since $\tilde{K}^{n*}_x$ is known, $\tilde{\tilde{K}}^{n*}_x$ is readily determined. However, $\tilde{G}^{n*}_x$ can only be obtained by solving the integral Equation (3.5).

As mentioned earlier, Equation (3.5) can be numerically solved by the method developed in Chapter II. In this light, (see page 37), for fixed x:

$$\frac{\partial K^*(x,t)}{\partial x} - \tilde{\tilde{G}}^m_x(t)^* = \left(\frac{\partial K^*(x,t)}{\partial x} - \tilde{G}^{n*}_x(t)\right) + \left(\tilde{G}^n_x(t)^* - \tilde{\tilde{G}}^m_x(t)^*\right);$$

$$\left|\frac{\partial K^*(x,t)}{\partial x} - \tilde{\tilde{G}}^m_x(t)^*\right| \le \left|\frac{\partial K^*(x,t)}{\partial x} - \tilde{G}^n_x(t)^*\right| + \left|\tilde{G}^n_x(t)^* - \tilde{\tilde{G}}^{m*}_x(t)\right|;$$

$$\left|\frac{\partial K^*(x,t)}{\partial x} - \tilde{G}^{n*}_x(t)\right| \le \varepsilon_1(\varepsilon_1 + \|\tilde{K}^n_x\|).$$

$$|\tilde{G}^{n*}_x - \tilde{\tilde{G}}^m_x(t)^*| \le \varepsilon_4 .$$

For fixed x:

$$\left|\frac{dK^*(x,t)}{\partial x} - (\tilde{\tilde{K}}^{n*}_x(t) + \tilde{\tilde{G}}^m_x(t)^*)\right|$$

$$\le \left|\left(\frac{\partial K^*(x,t)}{\partial t} - \tilde{\tilde{K}}^{n*}_x(t)\right| + \left|\left(\frac{\partial K^*(x,t)}{\partial x} - \tilde{\tilde{G}}^m_x(t)^*\right|$$

$$\le \left|\frac{\partial K^*(x,t)}{\partial t} - \tilde{\tilde{K}}^n_x(t)^*\right| + \left|\frac{\partial K(x,t)}{\partial x} - \tilde{G}^{n*}_x(t)\right| + \left|\tilde{G}^{n*}_x(t) - \tilde{\tilde{G}}^{m*}_x(t)\right|$$

$$\le M_4\varepsilon_1 + \varepsilon_1(\varepsilon_1 + \|\tilde{K}^n_x\|) + \varepsilon_4$$

$$\le M\varepsilon_1 + \varepsilon_4$$

$$\therefore \quad \left|\frac{dK(x,x)}{dx} - \tilde{\tilde{K}}_x^n(x)^* - \tilde{\tilde{G}}_x^{m*}\right| \leq M\varepsilon_1 + \varepsilon_4 \quad \text{for} \quad 0 < x \leq R .$$

The results of Sections 3.2 and 3.3 were based upon the assumptions that

(a) there does not exist an r_0 such that $F(r_0, s) = 0$ for $0 < s \leq r_0 \leq R$, and

(b) that there does not exist an x_0 such that

$$\frac{\partial F(x_0, t)}{\partial x} - F(x_0, t)\tilde{K}_{x_0}^{n*}(x_0) = 0 \quad \text{for} \quad 0 < t \leq x_0 \leq R.$$

If (a) does not hold, one must proceed as in Chapter II (see page 23). If (b) does not hold, then in determining $\tilde{\tilde{G}}_x^m$ the same type of qualifications must be imposed as were imposed in constructing $\tilde{K}_x^m$ under similar conditions in Chapter II (see page 15).

IV. APPLICATION OF BASIC THEORY

4.0 Introductory Statement

The significance, in the sciences, of the inverse theory will be indicated in this chapter. In Section 4.1 an application of the theory is given, which is relevant to the field of medicine; in particular, an indirect method of determining the elasticity of a flexible tube is suggested. In section 4.2 the application of the inverse theory to a few problems of mathematical physics is indicated, and in Section 4.3 the interrelationships between the equations describing the systems of Section 4.2 and the inverse theory are given.

4.1 Specific Application

Consider a one-dimensional flow through a slightly flexible tube of an inviscid liquid with constant density and velocity $v(x, t)$, which is constant over each cross sectional area. Let $F(x, t)$ denote the total flow through the cross section, and $f(x, t)$ the cross sectional area. Furthermore, assume that $f(x, t)$ is a linear function of the pressure $P(x, t)$, i.e.,

$$f(x, t) = f_0(1+k(x)P(x, t)) = f_0 + f_0 k(x)P(x, t),$$

where $k(x)$ is the proportionality constant of elasticity.

In this section it will be assumed that all variables are sufficiently smooth, in particular, twice continuously differentiable.

A first order linearized theory will be assumed, in particular, with respect to the equation

$$F(x, t) = \rho v(x, t) f(x, t) = \rho v f_0 + \rho v f_0 k(x) P(x, t),$$

the magnitude of the second order term vP will be assumed to be negligible in comparison to the magnitude of v. Hence,

$$F(x, t) = \rho v f_0 .$$

By conservation of mass

$$\frac{\partial F}{\partial x} + \rho \frac{\partial f}{\partial t} = H(x, t),$$

where $H(x, t)$ = source density (fluid mass added per unit length per unit time).

$$\therefore \quad \rho f_0 \frac{\partial v}{\partial x} + \rho f_0 k(x) \frac{\partial P}{\partial t} = H(x, t).$$

$$\therefore \quad \frac{\partial v}{\partial x} + k \frac{\partial P}{\partial t} = \frac{H}{\rho f_0} . \tag{4.0}$$

By conservation of momentum (one dimensional Euler Equation),

$$\rho \frac{Dv}{Dt} = -\frac{\partial P}{\partial x} .$$

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} .$$

Disregarding the second order term $v \frac{\partial v}{\partial x}$ implies

(4.1) $$\frac{\partial P}{\partial x} + \rho \frac{\partial v}{\partial t} = 0.$$

From Equations (4.0)-(4.1) it follows that

(4.2) $$\frac{\partial}{\partial x}\left(\frac{1}{k}\frac{\partial v}{\partial x}\right) - \rho \frac{\partial^2 v}{\partial t^2} - \tilde{H} = 0,$$

where

$$\tilde{H} = \frac{\partial}{\partial x}\left(\frac{H}{\rho f_0 k}\right).$$

The elasticity of the tube is expressed by the function $k(x)$. Therefore, the problem of determining the elasticity by indirect means entails determining the function $k(x)$ by indirect means. The function $k(x)$ can be determined by indirect means through utilization of the Inverse Sturm-Liouville Theory. In this regard, let us consider a specific example.

Now suppose we wish to determine $k(x)$ for a flexible tube of length π, where the end $(x = 0)$ is closed and the liquid flows from the other end $(x = \pi)$ into a reservoir. Also, let the source density $H(x,t)$ be of the form $G(x) \sin \omega t$. Therefore, from Equation (4.2), the flow in this tube is described by the equation

(4.3) $$\rho v_{tt} = \left(\frac{1}{k} v_x\right)_x + J(x) \sin \omega t$$

where

$$J(x) = \frac{d}{dx}\left(\frac{G(x)}{f_0 \rho k(x)}\right).$$

At $x = 0$,

(4.4) $$v(0,t) = 0.$$

It will be assumed that an incremental change in the internal pressure of the reservoir is proportional to an incremental change in the volume of the reservoir (i.e., $dP = CdV_{(volume)}$). Hence,

$$\frac{dP(\pi,t)}{dt} = f_0 C v(\pi,t),$$

since

$$\frac{dV_{(volume)}}{f_0 dt} = v(\pi,t).$$

Therefore, by Equation (4.0),

$$v_x(\pi,t) + hv(\pi,t) = 0,$$

where

$$h = k(\pi)Cf_0.$$

Suppose $v(x,t) = U(x)\sin\omega t$, then Equation (4.3) reduces to

(4.5) $$\left(\frac{1}{k}U'\right)' + \omega^2 \rho U + J = 0.$$

(4.6) $$U(0) = 0, \quad U'(\pi) + hU(\pi) = 0.$$

For (4.5)-(4.6) there exists the eigenvalues $\{\lambda_i\}_0^\infty$ and the normalized real-valued eigenfunctions $\{\bar{\varphi}_i(x)\}_0^\infty$ such that

$$\left(\frac{1}{k}\bar{\varphi}_i'\right)' + \lambda_i \rho \bar{\varphi}_i(x) = 0, \quad \bar{\varphi}_i(0) = 0, \quad \bar{\varphi}_i'(\pi) + h\bar{\varphi}_i(\pi) = 0.$$

$$J(x) = \rho \sum a_i \bar{\varphi}_i(x), \quad a_i = (J, \bar{\varphi}_i).$$

$$U(x) = \sum b_i \bar{\varphi}_i(x).$$

$$\sum b_i \left(\frac{\bar{\varphi}_i'(x)}{k(x)}\right)' + \omega^2 \rho \sum b_i \bar{\varphi}_i(x) + \rho \sum a_i \bar{\varphi}_i(x) = 0.$$

Therefore,

$$\sum \left[b_i(-\lambda_i \rho + \omega^2 \rho) + a_i \rho\right] \bar{\varphi}_i(x) = 0.$$

$$\Rightarrow \qquad b_i = \frac{a_i}{\lambda_i - \omega^2}.$$

$$\therefore \qquad U(x) = \sum \frac{(J, \bar{\varphi}_i)\bar{\varphi}_i(x)}{\lambda_i - \omega^2}.$$

Therefore,

$$v(x, t) = \sum_{i=0}^{\infty} \frac{(J, \bar{\varphi}_i)\bar{\varphi}_i(x) \sin \omega t}{\lambda_i - \omega^2}.$$

Noting that this mathematical solution is singular at $\omega = \lambda_i$ $(i = 0, 1, \ldots)$ as a result of linearizing the equations of the actual physical system.

By varying the frequency ω of the driver mechanism it is possible to determine the resonances of the system (in particular, the eigenvalues λ_i). Exactly how this example ties into the Gelfand-Levitan Theory will be clarified in 4.3.

This example is applicable to the field of medicine, in particular, in determining indirectly the elasticity of an artery, which is a measure of the degree of arteriosclerosis.

4.2 Indicated Applicability

Now let us briefly look at a few additional mathematical models, which describe numerous problems of mathematical physics, to which the Gelfand-Levitan Theory can be applied.

Determining $f(x)$ (assuming $f \in C^0$) of the equation

(4.7) $$U_{xx} = U_{tt} + f(x)U(x,t), \text{ (with suitable boundary conditions)},$$

reduces, by separation of variables $(U = X(x)T(t))$, to determining $f(x)$ of the differential equation

(4.8) $$X''(x) - f(x)X + \lambda X = 0, \text{ (with appropriate boundary conditions)}.$$

$f(x)$ can be determined, for example, if in addition to knowing the eigenvalues λ_i $(i = 0, 1, 2, \ldots)$, the constants c_i and d_i $(i = 0, 1, 2, \ldots)$[1] can be determined (see Gelfand and Levitan, [4] or [6]), where $c_i = \int \tilde{\varphi}_i^2(\xi)d\xi$, and $\tilde{\varphi}_i$'s are the eigenfunctions such

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[1] a_i of references [4] and [6] is determined from c_i and d_i.

that $\tilde{\varphi}_i(0) = d_i$, or if, in addition to knowing the eigenvalues λ_i $(i = 0, 1, \ldots)$ under one set of boundary conditions, one boundary condition can be changed and the eigenvalues μ_i of the altered system determined (Theorem C8, Appendix C).

Determining $\rho(x)$ (assuming $\rho \in C^2$) of

(4.9) $$\rho(x) W_{tt} = W_{xx}, \text{ (with suitable boundary conditions),}$$

can be reduced (by separation of variables, $W = XT$) to determining $\rho(x)$ of the differential equation

(4.10) $$X'' + \lambda \rho X(x) = 0, \text{ (with appropriate boundary conditions),}$$

which, in turn, can be transformed into a differential equation of the form

(4.11) $$y''(t) - q(t)y(t) + \lambda y(t) = 0,$$

to which the Gelfand-Levitan Theory is applicable. For example, $\rho(x)$ of (4.9) can be determined indirectly by means of the Gelfand-Levitan Theory if, in addition to knowing the eigenvalues λ_i of (4.10), $\rho(0)$, $\rho'(0)$,

$$\ell = \int \rho^{1/2}(\xi) d\xi, \quad \rho \in C^2, \quad \psi_n(0), \quad \text{and} \quad \gamma_n \ (n = 0, 1, 2, \ldots)$$

can be determined, where $\gamma_n = \int \rho \psi_n^2$, and ψ_n's are a set of eigenfunctions of (4.10), or if addition to knowing the eigenvalues λ_i of (4.10), $\rho(0)$, $\rho'(0)$, $\ell = \int \rho^{1/2}$ and $\rho \in C^2$, it is possible to change one boundary condition and determine the new set of eigenvalues for the system.

Similarly, for example, $\rho(x)$ (assuming $\rho \in C^2$) of the equations

$$\rho(x)U_t = U_{xx}, \quad U_t = (\rho(x)U_x)_x, \quad \text{etc.}$$

can be determined indirectly by means of the Gelfand-Levitan Theory.

4.3 Interrelationships Between Indicated Systems and Basic Theory

The physical systems as described by the above equations will now be explicitly connected to the Gelfand-Levitan Theory.

The differential equation

$$X'' + \lambda\rho(x)X(x) = 0, \quad 0 \leq x \leq \pi, \tag{4.12}$$

with the boundary conditions

$$X'(0) + hX(0) = 0, \quad X'(\pi) + HX(\pi) = 0, \tag{4.13}$$

(4.12)-(4.13), can be reduced to the system:

$$Z''(r) + q(r)Z(r) + \lambda Z(r) = 0, \quad 0 \leq r \leq \ell, \tag{4.14}$$

with the boundary conditions

$$(4.15) \qquad Z'(0) + gZ(0) = 0, \quad Z'(\ell) + GZ(\ell) = 0,$$

where g and G are constants, by the appropriate transformation [3]; noting that the Gelfand-Levitan Theory is directly applicable to (4.14)-(4.15).

The following theorem clarifies the relation between (4.12)-(4.13) and (4.14)-(4.15).

Theorem 4.3.1. If ρ is known and $\rho''(x) \in L(0, \pi)$, then there exists a transformation setting up a 1:1 correspondence between the eigenfunctions of the system (4.14)-(4.15), where q of (4.14) is related to ρ of (4.12) in the following manner:

$$q(r) = -\frac{1}{\rho^{3/4}(x)} \frac{d^2}{dx^2}\left(\frac{1}{\rho^{1/4}(x)}\right),$$

and the constants g and G of (4.15) are related to h and H of (4.13) in the following way:

$$g = \left(h - \frac{\rho'(0)}{4\rho(0)}\right)\frac{1}{\rho^{1/2}(0)}, \qquad G = \left(H - \frac{\rho'(\pi)}{4\rho(\pi)}\right)\frac{1}{\rho^{1/2}(\pi)}$$

(In [2], similar results are used but never proven explicitly.)

Proof. Let

$$t = \int_0^x \rho^{1/2}(\xi)d\xi \quad \text{and} \quad \ell = \int_0^\pi \rho^{1/2}(\xi)d\xi .$$

For each eigenfunction $\psi_i(x)$ of (4.12)-(4.13) with its eigenvalue λ_i, consider the mapping $\varphi_i(r) = \rho^{1/4}(x)\psi_i(x)$, where r is the running dummy variable in the transform space. By direct substitution, it can be shown that

$$\varphi_i''(r) - q(r)\varphi_i(r) + \lambda_i\varphi_i(r) = 0, \quad 0 \le r \le \ell,$$

and that

$$\varphi_i'(0) + g\varphi_i(0) = 0, \quad \varphi_i'(\ell) + G\varphi_i(\ell) = 0,$$

where g and G are given by the above formulas. Conversely, suppose we have the eigenfunction $\varphi_j(r)$ of (4.14)-(4.15) with its eigenvalue λ_j; that is,

$$\varphi_j''(r) - q(r)\varphi_j(r) + \lambda_j\varphi_j(r) = 0, \quad 0 \le r \le \ell,$$

and

$$\varphi_j'(0) + g\varphi_j(0) = 0, \quad \varphi_j'(\ell) + G\varphi_j(\ell) = 0,$$

where

$$g = (h - \frac{\rho'(0)}{4\rho(0)})\frac{1}{\rho^{1/2}(0)}, \qquad G = (H - \frac{\rho'(\pi)}{4\rho(\pi)})\frac{1}{\rho^{1/2}(\pi)},$$

and q is constructed from ρ by the formula:

$$q(r) = -\frac{1}{\rho^{3/4}(x)}\frac{d^2}{dx^2}(\frac{1}{\rho^{1/4}(x)}).$$

Letting

$$\psi_j = \frac{\varphi_j(r)}{\rho^{1/4}(x)},$$

we see by direct substitution that $\psi_j(x)$ satisfies (4.12)-(4.13).

Theorem 4.3.2. If there exists a unique solution y^* of

$$q(\xi) = \frac{1}{y(\xi)} \frac{d^2}{d\xi^2} y(\xi), \quad 0 \le \xi \le \pi .$$

$$y(0) = \frac{1}{\alpha_0}, \quad y'(0) = -\beta_0, \quad q \in L,$$

such that y^* is strictly positive, then there exists a unique solution to the nonlinear differential equation

$$q\left(\int_0^x \frac{1}{Z^2(\xi)} d\xi\right) = -Z^3(x) \frac{d^2 Z}{dx^2}, \quad 0 \le x \le \ell ,$$

$$Z(0) = \alpha_0, \quad Z'(0) = \beta_0 .$$

Proof. For $q \in L$, it is well known (see Theory of Differential Equations by Coddington and Levinson) that there exists a unique solution to the differential equation

$$q(x) = \frac{1}{y(x)} \frac{d^2 y(x)}{dx^2}, \quad y(0) = \frac{1}{\alpha_0}, \quad y'(0) = -\beta_0.$$

(This proof is based upon ideas developed by Borg [2].)

Let

$$r = \int_0^x \frac{1}{y^*(\xi)^2} d\xi, \quad \text{and} \quad \ell = \int_0^\pi \frac{1}{y^*(\xi)^2} d\xi .$$

Definition of the function $\rho(r)$: $\rho(r) = +\left(\frac{dx}{dr}\right)^{1/2}$.

Noting that

$$\frac{dr}{dx} = \frac{1}{y^*(x)^2}$$

and

$$\rho(r) = y(x),$$

where

$$r = \int_0^x \frac{1}{y^*(\xi)^2}\, d\xi .$$

$$\therefore \qquad \frac{d}{dx}y(x) = \frac{d}{dx}\rho(r) = \frac{d\rho(r)}{dr}\,\frac{dr}{dx} = \frac{1}{\rho^2(r)}\,\frac{d\rho(r)}{dr} .$$

Also,

$$\frac{d^2y(x)}{dx^2} = \frac{d}{dx}\left(\frac{1}{\rho^2(r)}\,\frac{d\rho(r)}{dr}\right) = \frac{1}{\rho^2(r)}\left[\frac{d}{dr}\left(\frac{1}{\rho^2(r)}\,\frac{d\rho(r)}{dr}\right)\right]$$

$$= \frac{1}{\rho^2(r)}\left[\frac{1}{\rho^2(r)}\,\frac{d^2\rho}{dr^2} - \frac{2}{\rho^3(r)}\left(\frac{d\rho(r)}{dr}\right)^2\right].$$

Since

$$\frac{d^2}{dr^2}\left(\frac{1}{\rho(r)}\right) = -\frac{d}{dr}\left(\frac{\rho'(r)}{\rho^2(r)}\right) = -\frac{\rho''(r)}{\rho^2(r)} + \frac{2\rho'(r)^2}{\rho^3(r)} ,$$

$$\frac{d^2y(x)}{dx^2} = -\frac{1}{\rho^2(r)}\,\frac{d^2}{dr^2}\left(\frac{1}{\rho(r)}\right) .$$

Also,

$$y(x)q(x) = \frac{d^2y(x)}{dx^2}, \qquad y(0) = \frac{1}{\alpha_0}, \qquad y'(0) = -\beta_0, \qquad \Rightarrow$$

$$q(x) = -\frac{1}{\rho^3(r)}\,\frac{d^2}{dr^2}\left(\frac{1}{\rho(r)}\right), \qquad \rho(0) = \frac{1}{\alpha_0}, \quad \rho'(0) = -\frac{\beta_0}{\alpha_0^2} ,$$

where

$$r = \int_0^x \frac{1}{y^*(\xi)^2} d\xi .$$

Since $x = \int_0^r \rho^2(\xi)d\xi$ and $\rho^2(r)dr = dx$,

$$q(\int_0^r \rho^2(\xi)d\xi) = -\frac{1}{\rho^3(r)} \frac{d^2}{dr^2}(\frac{1}{\rho(r)}), \quad \rho(0) = \frac{1}{\alpha_0}, \quad \rho'(0) = -\frac{\beta_0}{\alpha_0^2} .$$

Letting $Z(r) = \frac{1}{\rho(r)}$,

$$q((\int_0^r \frac{1}{Z^2(\xi)})d\xi) = -Z^3(r) \frac{d^2 Z(r)}{dr^2}, \quad Z(0) = \alpha_0, \quad Z'(0) = \beta_0, \quad Z > 0.$$

Therefore, it remains to show that the solution of

$$q(\int_0^r \rho^2(\xi)d\xi) = -\frac{1}{\rho^3(r)} \frac{d^2}{dr^2}(\frac{1}{\rho(r)}), \quad \rho(0) = \frac{1}{\alpha_0}, \quad \rho'(0) = -\frac{\beta_0}{\alpha_0}$$

is unique.

Suppose two solutions, $\rho_1(x)$ and $\rho_2(x)$, exist.

Construct a function $\tilde{y}_1$, such that

$$+(\frac{dx}{dr})^{1/2} = \tilde{y}_1(x),$$

where

$$x = \int_0^r \rho_1^2(\xi)d\xi.$$

Since $\rho_1 > 0$, it directly follows that $\tilde{y}_1$ is well-defined.

$$\frac{dx}{dr} = \rho_1^2(r), \quad \tilde{y}_1(x)^2 = \frac{dx}{dr},$$

$$dr = \frac{dx}{\tilde{y}_1(x)^2}, \quad r = \int_0^x \frac{d\xi}{\tilde{y}_1(\xi)^2} .$$

$$\therefore \quad q(x) = -\frac{1}{\rho_1^3} \frac{d^2}{dr^2}\left(\frac{1}{\rho_1(r)}\right), \quad \rho_1(0) = \frac{1}{\alpha_0}, \quad \rho_1'(0) = -\frac{\beta_0}{\alpha_0^2} .$$

$$\frac{d}{dx} \tilde{y}_1(x) = \frac{1}{\rho_1^2(r)} \frac{d\rho_1(r)}{dr} .$$

$$\frac{d^2}{dx^2} \tilde{y}_1 = \frac{1}{\rho_1^2(r)} \left[\frac{1}{\rho_1^2(r)} \frac{d^2\rho_1}{dr^2} - \frac{2}{\rho^3(r)} \left(\frac{d\rho(r)}{dr}\right)^2\right].$$

$$\frac{d^2\tilde{y}_1(x)}{dx^2} = -\frac{1}{\rho_1^2(r)} \frac{d^2}{dr^2}\left(\frac{1}{\rho_1(r)}\right) .$$

$$q(x) = \frac{1}{\tilde{y}_1(x)} \frac{d^2}{dx^2} \tilde{y}_1(x), \quad \tilde{y}_1(0) = \frac{1}{\alpha_0}, \quad \tilde{y}_1'(0) = -\beta_0 .$$

With respect to $\rho_2 \;\exists\; \tilde{y}_2 \;\ni$

$$q(x) = \frac{1}{\tilde{y}_2} \frac{d^2}{dx^2} \tilde{y}_2(x); \quad \tilde{y}_2(0) = \frac{1}{\alpha_0}; \quad \tilde{y}_2'(0) = -\beta_0 .$$

Hence

$$\rho_2 = \rho_1 .$$

By combining the results of Theorems 4.3.1 and 4.3.2 with the

Gelfand-Levitan Theory, we obtain the following theorem.

Theorem 4.3.3. If

1. The hypotheses of Theorem 4.3.2 are satisfied.
2. The eigenvalues $\{\lambda_i\}_{i=0}^{\infty}$ and the normalizing constants $\{a_i\}_{i=0}^{\infty}$ of (4.14)-(4.15) are known, where $a_i = \int_0^{\ell} \psi_i^2(\xi)d\xi$ and ψ_i's are the eigenfunctions such that $\psi_i(0) = d_i$.
3. $\rho(0)$ and $\rho'(0)$ of (4.12)-(4.13) are known $(a_0 = \frac{1}{\rho(0)}$, $\beta_0 = -\rho'(0)$ of Theorem 4.3.2). Then $\rho(x)$, h, and H of (4.12)-(4.13) are uniquely determined.

The following facts pertaining to Theorem 4.3.3 should be noted:

1. $\rho(x)$ of (4.12)-(4.13) can be explicitly determined by means of the Gelfand-Levitan Theory combined with the results of Theorems 4.3.1 and 4.3.2.
2. The smoothness of $\rho(x)$ of (4.12)-(4.13) (i.e., $\rho^m \in L(0,\pi)$ for some m) can be determined by means of the Gelfand-Levitan Spectral Theory and Theorem 4.3.3 could have been phrased accordingly.
3. In the hypotheses of the theorem, it was assumed that $\rho(0)$ and $\rho'(0)$ of (4.12)-(4.13) were given; however, the same results could have been attained if $\rho(\xi_1)$ and $\rho(\xi_2)$ or $\rho'(\xi_2)$ were known instead of $\rho(0)$ and $\rho'(0)$, where

$\xi_1 \in [0, \pi]$, $\xi_2 \in [0, \pi]$.

4. Also, if one boundary condition can be changed and the eigenvalues μ_i $(i = 0, 1, 2, \ldots)$ determined for this altered system, then it is not necessary to know the a_i's and d_i's. Similarly,

$$(k(x)X'(x))' + \lambda X = 0, \quad 0 \leq x \leq \pi,$$

$$X'(0) + hX(0) = 0, \quad X'(\pi) + HX(\pi) = 0,$$

may be transformed into

$$U''(P) - q(P)U(P) + \lambda U(P) = 0, \quad 0 \leq P \leq \ell,$$

$$U'(0) + (h - \frac{k'(0)}{4k(0)})U(0) = 0, \quad U'(\pi) + (H - \frac{k'(\pi)}{4k(\pi)})U(\pi) = 0,$$

by the transformation

$$U(P) = k^{1/4}(x)X(x),$$

$$P = \int_0^x \frac{1}{k^{1/2}(\xi)}\, d\xi, \qquad \ell = \int_0^\pi \frac{1}{k^{1/2}(\xi)}\, d\xi,$$

$$q(P) = -\frac{1}{k^{1/4}(x)} \frac{d^2}{dP^2}(k^{1/4}(x)).$$

Hence, analogous results of Theorems 4.3.1, 4.3.2, and 4.3.3 directly follow.

V. RESULTS FOR AN APPROXIMATE SPECTRAL FUNCTION

5.0 Introductory Remarks

For the inverse Sturm-Liouville problem

$$y'' - q(x)y + \lambda y = 0, \quad y'(0) - hy(0) = 0, \quad y'(\pi) + Hy(\pi) = 0,$$

on the finite interval $0 \le x \le \pi$, it has been assumed in the preceding chapters that the complete infinite set of eigenvalues $\{\lambda_i\}$, $i = 0, 1, 2, \ldots$ and normalizing constants $\{\alpha_i\}$, $i = 0, 1, 2, \ldots,$ are known. However, for practical considerations, it is important to know whether F, K, and $q(x)$ of chapters two and three can be uniformly approximated if only the first N eigenvalues λ_i and normalizing constants α_i are known. In this chapter it will be shown that if $\lambda_0, \lambda_1, \ldots, \lambda_{N-1}$, $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ are known,

$$\sqrt{\lambda_n} = n + \frac{a_0}{n} + \frac{a_1}{n^3} + O\left(\frac{1}{n^4}\right) \quad \text{for all } n,$$

and

$$\alpha_n = \frac{\pi}{2} + \frac{b_0}{n^2} + O\left(\frac{1}{n^3}\right) \quad \text{for all } n,$$

then for sufficiently large N, there exist continuous functions $G_N(x, t)$, $K_N(x, t)$, and $q_N(x)$ such that

$$F = G_N + O\left(\frac{1}{N^2}\right), \quad K = K_N + O\left(\frac{1}{N^2}\right),$$

and

$$q(x) = q_N(x) + O(\frac{1}{N}),$$

where

$$F(x,t) + K(x,t) + \int_0^x F(s,t)K(x,s)ds = 0, \quad 0 \leq t \leq x \leq \pi,$$

and

$$q(x) = +\frac{2dK(x,x)}{dx} .$$

The functions F, K and q will be approximated in Sections 5. 1, 5. 2 and 5. 3, respectively. Results of this Chapter have necessitated the estimation of certain asymptotic series. Similar techniques have been used by B. M. Levitan, Izv. Akad. Nauk SSSR Ser. Mat. 28 (1964), 63-78.

Condition A.

1. λ_n, for $n = 0, \ldots, N-1$, and

   α_n, for $n = 0, \ldots, N-1$, are known.

2. The asymptotic conditions

$$\sqrt{\lambda}_n = n + \frac{a_0}{n} + \frac{a_1}{n^3} + O(\frac{1}{n^4})$$

and

$$\alpha_n = \frac{\pi}{2} + \frac{b_0}{n^2} + O(\frac{1}{n^3})$$

are satisfied for arbitrary n.

3. There exist constants c_1 and c_2 such that

$$c_2 < n^3 \left|\frac{\pi}{2} + \frac{b_0}{n^2}\right| \quad \text{for all} \quad n \geq N,$$

$$|\alpha_n - \frac{\pi}{2} - \frac{b_0}{n^2}| \le \frac{c_2}{n^3},$$

and

$$|\sqrt{\lambda}_n - n - \frac{a_0}{n} - \frac{a_1}{n^3}| \le \frac{c_1}{n^4}.$$

5.1 Approximation of $F(x, t)$

Theorem 5.1. Under the hypotheses of Condition A, there exists a continuous symmetric function $G_N(x, t)$ such that

$$F(x, t) = G_N(x, t) + O(\frac{1}{N^2}),$$

where

$$G_N = F_N(x, t) + \frac{1}{2}[f_N(x+t)+f_N(x-t)],$$

$$F_N(x, t) = \frac{\cos\sqrt{\lambda_0}x \cos\sqrt{\lambda_0}t}{\alpha_0} - \frac{1}{\pi} + \sum_{n=1}^{N-1}[\frac{\cos\sqrt{\lambda}_n x \cos\sqrt{\lambda}_n t}{\alpha_n} - \frac{2}{\pi}\cos nx \cos nt]$$

and

$$f_N(x) = -\frac{4b_0}{\pi^2}\sum_{n=N}^{\infty}\frac{\cos nx}{n^2(1+\frac{2b_0}{\pi n^2})} - \frac{x^2}{2}\sum_{n=N}^{\infty}\cos nx\,(\frac{a_0}{n}+\frac{a_1}{n^3})\left|\frac{2}{\pi} - \frac{4b_0}{\pi n^2(1+\frac{2b_0}{\pi n^2})}\right|$$

$$+\sum_{n=N}^{\infty} x \sin nx(\frac{a_0}{n}+\frac{a_1}{n^3})\left|\frac{4b_0}{\pi n^2(1+\frac{2b_0}{\pi n^2})} - \frac{2}{\pi}\right|.$$

Proof. For the inverse Sturm-Liouville problem on $[0, \pi]$, the function F can be expressed in the form (see [4] or [6]):

(5.0)
$$F(x,t) = \frac{1}{a_0}\cos\sqrt{\lambda_0}x\cos\sqrt{\lambda_0}t - \frac{1}{\pi} + \sum_{n=1}^{\infty}\{\frac{\cos\sqrt{\lambda_n}x\cos\sqrt{\lambda_n}t}{a_n} - \frac{2}{\pi}\cos nx\cos nt\},$$

where $0 \leq t \leq x \leq \pi$, $a_n > 0$.

Let

$$a_N(x) = \sum_{n=N}^{\infty}(\frac{\cos\sqrt{\lambda_n}x}{a_n} - \frac{2}{\pi}\cos nx).$$

Therefore,

(5.1)
$$F(x,t) = F_N(x,t) + \frac{1}{2}[a_N(x+t) + a_N(x-t)],$$

where

(5.2)
$$F_N(x,t) = \frac{\cos\sqrt{\lambda_0}x\,\mathrm{cox}\sqrt{\lambda_0}t}{a_0} - \frac{1}{\pi} + \sum_{n=1}^{N-1}[\frac{\cos\sqrt{\lambda_n}x\cos\sqrt{\lambda_n}t}{a_n} - \frac{2}{\pi}\cos nx\cos nt].$$

There exist functions g_n and h_n such that

(5.3)
$$\sqrt{\lambda_n} = n + \frac{a_0}{n} + \frac{a_1}{n^3} + g_n,$$

and

(5.4)
$$a_n = \frac{\pi}{2} + \frac{b_0}{n^2} - h_n,$$

where

$$|\sqrt{\lambda_n} - n - \frac{a_0}{n} - \frac{a_1}{n^3}| = |g_n| \leq \frac{c_1}{n^4},$$

and

$$|a_n - \frac{\pi}{2} - \frac{b_0}{n^2}| = |h_n| \leq \frac{c_2}{n^3}.$$

Let $\tilde{a}_n$ and $\tilde{b}_n$ be defined as follows:

(5.5) $$\tilde{a}_n = \frac{a_0}{n} + \frac{a_1}{n^3} + g_n,$$

and

(5.6) $$\tilde{b}_n = \frac{b_0}{n^2} - h_n.$$

Since, by Equation (5.1),

$$F(x,t) = F_N(x,t) + \frac{1}{2}[a_N(x+t)+a_N(x-t)],$$

to prove that $F(x,t)$ can be expressed in the form

$$F = (F_N(x,t) + \frac{1}{2}[f_N(x+t) + f_N(x-t)]) + O(\frac{1}{N^2})$$

reduces to appropriately estimating the terms of the series a_N:

$$a_N(x) = \sum_{n=N}^{\infty} (\frac{\cos\sqrt{\lambda_n}\, x}{\alpha_n} - \frac{2}{\pi}\cos nx).$$

In this regard, the factor $\frac{1}{\alpha_n}$ of the nth term of a_N can be expressed in the form:

$$\frac{1}{a_n} = \frac{2}{\pi} - \frac{4b_0}{\pi^2 n^2} \, \frac{1}{\left|1+\frac{2b_0}{\pi n^2}\right|} + \frac{h_n}{\left(\frac{\pi}{2}+\frac{b_0}{n^2}\right)^2 \left|1 - \frac{h_n}{\left(\frac{\pi}{2}+\frac{b_0}{n^2}\right)}\right|} .$$

Also,

$$\left| \frac{h_n}{\left(\frac{\pi}{2}+\frac{b_0}{n^2}\right)^2 \left|1 - \frac{h_n}{\frac{\pi}{2}+\frac{b_0}{n^2}}\right|} \right| \leq \frac{c_2}{n^3\left(\frac{\pi}{2}+\frac{b_0}{n^2}\right)^2 \left|1 - \frac{c_2}{n^3 \left|\frac{\pi}{2}+\frac{b_0}{n^2}\right|}\right|} ,$$

since

$$\frac{c_2}{n^3 \left|\frac{\pi}{2}+\frac{b_0}{n^2}\right|} < 1 \quad \text{for all} \quad n \geq N$$

by hypothesis. Therefore,

(5.7)

$$\frac{\cos\sqrt{\lambda_n}\, x}{a_n} = \frac{2}{\pi}\cos\sqrt{\lambda_n}\, x - \frac{4b_0 \cos\sqrt{\lambda_n}\, x}{\pi^2 n^2\left(1+\frac{2b_0}{\pi n^2}\right)} + \frac{h_n \cos\sqrt{\lambda_n}\, x}{\left(\frac{\pi}{2}+\frac{b_0}{n^2}\right)^2 \left|1 - \frac{h_n}{\left(\frac{\pi}{2}+\frac{b_0}{n^2}\right)}\right|} .$$

Since

$$\sqrt{\lambda_n} = n + \tilde{a}_n ,$$

by Equations (5.3) and (5.5), $\cos \sqrt{\lambda_n}\, x$ can be expressed in the form

$$\cos \sqrt{\lambda_n}\, x = \cos nx \cos \tilde{a}_n x - \sin nx \sin \tilde{a}_n x.$$

$$\cos\sqrt{\lambda_n}\,x = \cos nx + \cos nx \sum_{i=1}^{\infty} \frac{(-1)^i(\tilde{a}_n x)^{2i}}{(2i)!} - \sin nx \sin \tilde{a}_n x .$$

Therefore the nth term of the series a_N can be expressed in the form:

(5.8)

$$\left(\frac{\cos\sqrt{\lambda_n}\,x}{a_n} - \frac{2}{\pi}\cos nx\right) = -\frac{4b_0 \cos nx}{\pi^2 n^2\left(1+\frac{2b_0}{\pi n^2}\right)}$$

$$+ \cos nx\left(\sum_{i=1}^{\infty} \frac{(-1)^i(\tilde{a}_n x)^{2i}}{(2i)!}\right)\left(\frac{2}{\pi} - \frac{4b_0}{\pi^2 n^2\left(1+\frac{2b_0}{\pi n^2}\right)}\right)$$

$$+ \sin nx \sin \tilde{a}_n x\left(\frac{4b_0}{\pi^2 n^2\left(1+\frac{2b_0}{\pi n^2}\right)} - \frac{2}{\pi}\right)$$

$$+ \frac{h_n \cos\sqrt{\lambda_n}\,x}{\left(\frac{\pi}{2}+\frac{b_0}{n^2}\right)^2\left(1-\frac{h_n}{\left(\frac{\pi}{2}+\frac{b_0}{n^2}\right)}\right)} .$$

$$= I_{1n} + I_{2n} + I_{3n} + I_{4n} ,$$

respectively.

Now, let us consider I_{2n} of Equation (5.8)

$$(5.9)\qquad I_{2n} = \cos nx \sum_{i=1}^{\infty} \frac{(-1)^i(\tilde{a}_n x)^{2i}}{(2i)!} \left(\frac{2}{\pi} - \frac{4b_0}{\pi^2 n^2 (1+\frac{2b_0}{\pi n^2})}\right) .$$

Since

$$\sum_{i=1}^{\infty} \frac{(-1)^i(\tilde{a}_n x)^{2i}}{(2i)!} = -\frac{\tilde{a}_n x^2}{2} + \sum_{i=2}^{\infty} \frac{(-1)^i(\tilde{a}_n x)^{2i}}{(2i)!} ,$$

and by the definition of $\tilde{a}_n$ (Equation (5.5))

$$(\tilde{a}_n)^2 = (\frac{a_0}{n} + \frac{a_1}{n^3})^2 + \frac{g_n}{n}[ng_n + 2(a_0 + \frac{a_1}{n^2})],$$

it follows that

$$I_{2n} = -\frac{x^2}{2}(\frac{a_0}{n} + \frac{a_1}{n^3})^2 \cos nx[\frac{2}{\pi} - \frac{4b_0}{\pi n^2(1+\frac{2b_0}{\pi n^2})}]$$

$$-\frac{x^2}{2} \cos nx\, [\frac{g_n}{n}(ng_n + 2(a_0 + \frac{a_1}{n^2})][\frac{2}{\pi} - \frac{4b_0}{\pi n^2(1+\frac{2b_0}{\pi n^2})}]$$

$$+ \cos nx \sum_{i=2}^{\infty} \frac{(-1)^i(\tilde{a}_n x)^{2i}}{(2i)!} \left(\frac{2}{\pi} - \frac{4b_0}{\pi n^2(1+\frac{2b_0}{\pi n^2})}\right) .$$

$$\sum_{i=2}^{\infty} \frac{(-1)^i(\tilde{a}_n x)^{2i}}{(2i)!} = O(\frac{1}{n^4}) ,$$

since

$$(\tilde{a}_n)^4 = O(\frac{1}{n^4}),$$

and

$$\sum_{i=0}^{\infty} \frac{(-1)^i(\tilde{a}_n x)^{2i}}{(2i+4)!} = O(1).$$

Furthermore,

$$\frac{g_n}{n}(ng_n + 2(a_0 + \frac{a_1}{n^3}))\left(\frac{2}{\pi} - \frac{4b_0}{\pi n^2(1+\frac{2b_0}{\pi n^2})}\right) = O(\frac{1}{n^5}).$$

Hence,

$$I_{2n} = -\frac{x^2}{2}(\frac{a_0}{n} + \frac{a_1}{n^3})^2\left(\frac{2}{\pi} - \frac{4b_0}{\pi n^2(1+\frac{2b_0}{\pi n^2})}\right)\cos nx + O(\frac{1}{n^4}).$$

Similarly, let us consider the term I_{3n} of Equation (5.8).

$$I_{3n} = \sin nx \sin \tilde{a}_n x \left(\frac{4b_0}{\pi n^2(1+\frac{2b_0}{\pi n^2})} - \frac{2}{\pi}\right).$$

Since

$$\sin \tilde{a}_n x = \tilde{a}_n x + \tilde{a}_n^3 x^3 \sum_{i=0}^{\infty} \frac{(\tilde{a}_n x)^{2i}(-1)^{i+1}}{(2i+3)!},$$

$$\tilde{a}_n^3 = O(\frac{1}{n^3}),$$

and

$$\sum_{i=0}^{\infty} \frac{(\tilde{a}_n x)^{2i}(-1)^{i+1}}{(2i+3)!} = O(1),$$

it directly follows that

$$I_{3n} = x\left(\frac{a_0}{n}+\frac{a_1}{n^3}\right)\left(\frac{4b_0}{\pi n^2\left(1+\frac{2b_0}{\pi n^2}\right)} - \frac{2}{\pi}\right) \sin nx + O\left(\frac{1}{n^3}\right).$$

The term I_{4n} of Equation (5.8),

$$I_{4n} = \frac{h_n \cos\sqrt{\lambda_n}\, x}{\left(\frac{\pi}{2}+\frac{b_0}{n^2}\right)^2\left(1-\frac{h_n}{\left(\frac{\pi}{2}+\frac{b_0}{n^2}\right)}\right)},$$

reduces to

$$I_{4n} = O\left(\frac{1}{n^3}\right),$$

for it was shown on page 62 that

$$\frac{h_n}{\left(\frac{\pi}{2}+\frac{b_0}{n^2}\right)^2\left(1-\frac{h_n}{\frac{\pi}{2}+\frac{b_0}{n^2}}\right)} \leq \frac{c_2}{n^3\left(\frac{\pi}{2}+\frac{b_0}{n^2}\right)^2\left(1-\frac{c_2}{n^3\left|\frac{\pi}{2}+\frac{b_0}{n^2}\right|}\right)}.$$

Therefore, by letting,

$$f_N(x) = -\frac{4b_0}{\pi^2}\sum_{n=N}^{\infty}\frac{\cos nx}{\left(n^2(1+\frac{2b_0}{\pi n^2})\right)} - \frac{x^2}{2}\sum_{n=N}^{\infty}\cos nx\left(\frac{a_0}{n}+\frac{a_1}{n^3}\right)^2\left(\frac{2}{\pi}-\frac{4b_0}{\pi n^2(1+\frac{2b_0}{\pi n^2})}\right)$$

$$+\sum_{n=N}^{\infty} x\sin nx\left(\frac{a_0}{n}+\frac{a_1}{n^3}\right)\left(\frac{4b_0}{\pi n^2(1+\frac{2b_0}{\pi n^2})}-\frac{2}{\pi}\right),$$

we directly obtain the result

$$a_N = f_N + O(\frac{1}{N^2}) \; .$$

It directly follows from this result that

$$F(x, t) = G_N(x, t) + O(\frac{1}{N^2}) ,$$

where

$$G_N(x, t) = -F_N(x, t) - \frac{1}{2}\left[f_N(x+t)+f_N(x-t)\right].$$

5.2 Approximation of $K(x, t)$

Theorem 5. 2. Under the hypotheses of Condition A, there exists a continuous function $K_N(x, t)$ such that

$$K(x, t) = K_N(x, t) + O(\frac{1}{N^2}), \qquad 0 \le t \le x \le \pi .$$

Proof. The function F of Section 5. 1 is related to the function K by means of the integral equation

$$(5.10)\qquad K(x,t) + F(x,t) + \int_0^x F(s,t)K(x,s)ds = 0, \quad 0 \le t \le x \le \pi.$$

In Section 5.1 it was shown that G_N, a known continuous symmetric function, is related to the function F by means of the equation

$$F(x,t) = G_N(x,t) + O(\frac{1}{N^2}) .$$

Let

$$H_N = F - G_N,$$

and

$$G_N = F_N + R_N,$$

where

$$R_N = \frac{1}{2}[f_N(x+t)+f_N(x-t)].$$

Let $K_N(x,t)$, for large N, denote the unique solution of the equation

$$(5.11)\qquad G_N(x,t) + K_N(x,t) + \int_0^x G_N(s,t)K_N(x,s)ds = 0, \text{ (see [5, p. 547])}.$$

Let $\overline{G}_N$ denote the bounded linear operator such that

$$\overline{G}_N f(x,t) = \int_0^x G_N(s,t)f(x,s)ds.$$

Hence

$$K_N = -(I+\overline{G}_N)^{-1}G_N,$$

where

$$\|(I+\overline{G}_N)^{-1}\| \leq M.$$

By subtracting Equation (5.11) from Equation (5.10) we obtain the equation

$$(5.12) \qquad (F-G_N) + (K-K_N) + \int_0^x (FK-G_NK_N) = 0.$$

Therefore

$$(I+\overline{G}_N)(K-K_N) = -(H_N+\int_0^x H_NK).$$

Hence

$$K - K_N = (I+\overline{G}_N)^{-1}(-H_N-\int_0^x H_NK),$$

which implies

$$|K-K_N| \leq \|K-K_N\| \leq M\|-H_N-\int_0^x H_NK\|.$$

Consequently

$$K = K_N + O(\frac{1}{N^2}),$$

or

$$|K-K_N| \leq \frac{\alpha}{N^2}.$$

If $|\int_0^x H_NK|$ is replaced by $|\int_0^x H_N(K-K_N)| + |\int_0^x H_NK_N|$, then α can be explicitly calculated for sufficiently large N.

5.3 Approximation of $q(x)$

Theorem 5.3. Under the hypotheses of Condition A, there exists a continuous function $q_N(x)$ such that

$$q(x) = q_N(x) + O(\frac{1}{N}) .$$

Proof. By Chapter III, it is sufficient to show that there exist continuous functions ψ_1 and ψ_2 such that

$$\frac{\partial K}{\partial x} = \psi_1 + O(\frac{1}{N}),$$

and

$$\frac{\partial K}{\partial t} = \psi_2 + O(\frac{1}{N}) .$$

By point 2 of Condition **A** (see [6]) it follows that

$$\frac{\partial K(x,t)}{\partial t} + \frac{\partial F(x,t)}{\partial t} + \int_0^x \frac{\partial F(s,t)}{\partial t} K(x,s)ds = 0,$$

and that

$$\frac{\partial F}{\partial t} = \frac{\partial F_N}{\partial t} + \sum_{n=N}^{\infty} [\frac{-\sqrt{\lambda_n}}{\alpha_n} \cos\sqrt{\lambda_n}x \sin\sqrt{\lambda_n}t + \frac{2n}{\pi} \cos nx \sin nt],$$

$$= \frac{\partial F_N}{\partial t} + \frac{1}{2} [a'_N(x-t) - a'_N(x+t)],$$

where

$$a'_N(x) = \sum_{n=N}^{\infty} [\frac{\sqrt{\lambda_n}}{\alpha_n} \sin\sqrt{\lambda_n}x - \frac{2n}{\pi} \sin nx].$$

Noting that

$$a'_N(x-t) = \frac{\partial}{\partial t} a_N(x-t), \qquad a'_N(x+t) = -\frac{\partial}{\partial t} a_N(x+t).$$

As in Section 5.1,

$$\frac{1}{a_n} = \frac{2}{\pi} - \frac{4b_0}{\pi n^2}\,\frac{1}{1+\frac{2b_0}{\pi n^2}} + \frac{h_n}{\left(\frac{\pi}{2}+\frac{b_0}{n^2}\right)^2\left(1-\frac{h_n}{\left(\frac{\pi}{2}+\frac{b_0}{n^2}\right)}\right)} .$$

Since

$$\sqrt{\lambda_n} = n + \frac{a_0}{n} + \frac{a_1}{n^3} + g_n, \qquad |g_n| \le \frac{c_1}{n^4},$$

it directly follows that

$$\frac{\sqrt{\lambda_n}}{a_n} = \frac{2n}{\pi} + II_1 + g_n II_2 + h_n II_3,$$

where

$$II_1 = \frac{2a_0}{\pi n} + \frac{2a_1}{\pi n^3} - \frac{4b_0}{\pi^2}\left(\frac{1}{n}+\frac{a_0}{n^3}+\frac{a_1}{n^5}\right)\frac{1}{\left(1+\frac{2b_0}{\pi n^2}\right)},$$

$$II_2 = \frac{2}{\pi} - \frac{4b_0}{\pi^2 n^2}\,\frac{1}{\left(1+\frac{2b_0}{\pi n^2}\right)},$$

and

$$II_3 = \frac{n+\frac{a_0}{n}+\frac{a_1}{n^3}+g_n}{\left(\frac{\pi}{2}+\frac{b_0}{n^2}\right)^2\left(1-\frac{h_n}{\left(\frac{\pi}{2}+\frac{b_0}{n^2}\right)}\right)} .$$

Determination of $\left[\frac{\sqrt{\lambda_n}}{a_n}\sin\sqrt{\lambda_n}\,x - \frac{2n}{\pi}\sin nx\right]$:

$$\sin\sqrt{\lambda_n}\,x = \sin nx\cos\tilde{a}_n x + \cos nx\sin\tilde{a}_n x.$$

$$\cos \tilde{a}_n x = 1 + \sum_{i=1}^{\infty} \frac{(-1)^i (\tilde{a}_n x)^{2i}}{(2i)!} .$$

$$\sin nx \cos \tilde{a}_n x = \sin nx - x^2 \sin nx \left[\left(\frac{a_0}{n} + \frac{a_1}{n^3}\right)^2 + \frac{g_n}{n}\left(ng_n + 2\left(a_0 + \frac{a_1}{n^2}\right)\right)\right]$$

$$+ \sin nx \sum_{i=2}^{\infty} \frac{(-1)^i (\tilde{a}_n x)^{2i}}{(2i)!} .$$

As in Section 5.1,

$$\sum_{i=2}^{\infty} \frac{(-1)^i (\tilde{a}_n x)^{2i}}{(2i)!} = O\left(\frac{1}{n^4}\right) ,$$

and

$$\tilde{a}_n = O\left(\frac{1}{n}\right) .$$

Also,

$$\sin \tilde{a}_n x = \tilde{a}_n x + \tilde{a}_n^3 x^3 \sum_{i=0}^{\infty} \frac{(\tilde{a}_n x)^{2i} (-1)^{i+1}}{(2i+3)!} .$$

Hence,

$$\sin \sqrt{\lambda}_n x = \sin nx - x^2 \sin nx \left(\frac{a_0}{n} + \frac{a_1}{n^3}\right)^2 + x \cos nx \left(\frac{a_0}{n} + \frac{a_1}{n^3}\right)$$

$$+ \sin nx \sum_{i=2}^{\infty} \frac{(-1)^i (\tilde{a}_n x)^{2i}}{(2i)!} - x^2 \sin nx \, \frac{g_n}{n}\left(ng_n + 2\left(a_0 + \frac{a_1}{n^2}\right)\right)$$

$$+ x g_n \cos nx + (\cos nx) \tilde{a}_n^3 x^3 \sum_{i=0}^{\infty} \frac{(\tilde{a}_n x)^{2i} (-1)^{i+1}}{(2i+3)!} .$$

Therefore,

$$- \frac{2n}{\pi} \sin nx + \frac{\sqrt{\lambda_n} \sin\sqrt{\lambda_n} x}{a_n}$$

$$= \quad II_1 (\sin nx - x^2 \sin nx (\frac{a_0}{n} + \frac{a_1}{n^3})^2 + x \cos nx (\frac{a_0}{n} + \frac{a_1}{n^3}))$$

$$+ II_1 \left(\sin nx \sum_{2}^{\infty} \frac{(-1)^i (\tilde{a}_n x)^{2i}}{(2i)!} + (\cos nx) \tilde{a}_n^3 x^3 \sum_{i=0}^{\infty} \frac{(\tilde{a}_n x)^{2i} (-1)^{i+1}}{(2i+3)!} \right)$$

$$+ g_n II_1 [- \frac{x^2}{n} (ng_n + 2(a_0 + \frac{a_1}{n^2}) + x \cos nx)] + g_n II_2 \sin \sqrt{\lambda_n} x$$

$$+ h_n II_3 \sin \sqrt{\lambda_n} x - \frac{2x^2}{\pi n} \sin nx (a_0 + \frac{a_1}{n^2})^2 + \frac{2}{\pi} x \cos nx (a_0 + \frac{a_1}{n^2})$$

$$+ \frac{2n}{\pi} \sin nx \sum_{i=2}^{\infty} \frac{(-1)^i (\tilde{a}_n x)^{2i}}{(2i)!} - \frac{2}{\pi} x^2 \sin nx \, g_n (ng_n + 2(a_0 + \frac{a_1}{n^2}))$$

$$+ \frac{2}{\pi} nx \, g_n \cos nx + \frac{2}{\pi} n \cos nx (\tilde{a}_n^3 x^3) \sum_{i=0}^{\infty} \frac{(\tilde{a}_n x)^{2i} (-1)^{i+1}}{(2i+3)!} .$$

Let

$$\tilde{II}_1 = \frac{2a_0}{\pi} + \frac{2a_1}{\pi n^2} - \frac{4b_0}{\pi^2} (1 + \frac{a_0}{n^2} + \frac{a_1}{n^4}) \frac{1}{\left(1 + \frac{2b_0}{\pi n^2}\right)} .$$

$$=> \qquad II_1 = \frac{\tilde{II}_1}{n} .$$

Let

$$e_n = \frac{\widetilde{II}_1}{n}\left(\sin nx - x^2 \sin nx\left(\frac{a_0}{n}+\frac{a_1}{n^3}\right)^2 + x\cos nx\left(\frac{a_0}{n}+\frac{a_1}{n^3}\right)\right)$$

$$-\frac{2}{\pi}\frac{x^2}{n}\sin nx\left(a_0+\frac{a_1}{n^2}\right)^2 + \frac{2}{\pi}x\cos nx\left(\frac{a_1}{n^2}\right).$$

Let

$$E_N = \sum_{n=N}^{\infty} e_n - \frac{a_0 x}{\pi} - \frac{2a_0 x}{\pi}\sum_{1}^{N-1}\cos nx$$

Therefore,

$$|a'_N - E_N| \leq III_1 + III_2 + III_3 + III_4 + III_5 + III_6 + III_7 + III_8 + III_9,$$

where

$$III_1 = \sum_{n=N}^{\infty} \frac{\widetilde{II}_1}{n}\sin nx \sum_{i=2}^{\infty}\frac{(-1)^i(\tilde{a}_n x)^{2i}}{(2i)!},$$

$$III_2 = \sum_{n=N}^{\infty} \frac{\widetilde{II}_1}{n}\cos nx\ \tilde{a}_n^3 x^3 \sum_{i=0}^{\infty}\frac{(\tilde{a}_n x)^{2i}(-1)^{i+1}}{(2i+3)!},$$

$$III_3 = \sum_{n=N}^{\infty} g_n \frac{\widetilde{II}_1}{n}\left[-\frac{x^2}{n}\left(ng_n + 2\left(a_0+\frac{a_1}{n^2}\right) + x\cos nx\right)\right],$$

$$III_4 = \sum_{n=N}^{\infty} g_n II_2 \sin\sqrt{\lambda_n}\,x,$$

$$III_5 = \sum_{n=N}^{\infty} h_n II_3 \sin\sqrt{\lambda_n}\,x,$$

$$III_6 = \sum_{n=N}^{\infty} \frac{2n}{\pi} \sin nx \left(\sum_{i=2}^{\infty} \frac{(-1)^i (\tilde{a}_n x)^{2i}}{(2i)!} \right) ,$$

$$III_7 = \frac{2}{\pi} x^2 \sum_{n=N}^{\infty} (\sin nx) g_n (n g_n + 2(a_0 + \frac{a_1}{n^2}) ,$$

$$III_8 = \frac{2}{\pi} x \sum_{n=N}^{\infty} n g_n \cos nx ,$$

$$III_9 = \frac{2}{\pi} \sum_{n=N}^{\infty} n \cos nx (\tilde{a}_n^3 x^3) \left(\sum_{i=0}^{\infty} \frac{(\tilde{a}_n x)^{2i} (-1)^{i+1}}{(2i+3)!} \right) .$$

It can be readily shown that

$$III_1 + III_2 + III_3 + III_4 + III_5 + III_6 + III_7 + III_8 + III_9 = O(\frac{1}{N}) .$$

Hence,

$$\frac{1}{2} [\frac{\partial}{\partial t} a_N(x-t) + \frac{\partial}{\partial t} a_N(x+t)] = \frac{1}{2} [E_N(x-t) - E_N(x+t)] + O(\frac{1}{N}) .$$

Let

$$\bar{F}_N = \frac{\partial F_N}{\partial t} + \frac{1}{2} [E_N(x-t) - E_N(x+t)].$$

Therefore,

$$\frac{\partial F}{\partial t} = \bar{F}_N + O(\frac{1}{N}) .$$

Since

$$\frac{\partial K(x,t)}{\partial t} = -\frac{\partial F(x,t)}{\partial t} - \int_0^x \frac{\partial F(s,t)}{\partial t} K(x,s)ds,$$

it directly follows that

$$\left|\frac{\partial K}{\partial t} + \bar{F}_N + \int_0^x \bar{F}_N K_N\right| \leq \left|\bar{F}_N - \frac{\partial F}{\partial t}\right| + \left|\int_0^x \bar{F}_N K_N - \frac{\partial F}{\partial t} K_N\right|$$

$$+ \left|\int_0^x \frac{\partial F}{\partial t} K_N - \frac{\partial F}{\partial t} K\right|,$$

where K_N is the function constructed in Section 5.2 such that

$$K = K_N + O(\frac{1}{N^2}) .$$

Consequently,

$$\frac{\partial K}{\partial t} = -\bar{F}_N - \int_0^x \bar{F}_N K_N + O(\frac{1}{N}) .$$

Determination of $\frac{\partial K(x,t)}{\partial x}$:

$$\frac{\partial K(x,t)}{\partial x} = -\frac{\partial F(x,t)}{\partial x} - F(x,t)K(x,x) - \int_0^x F(s,t) \frac{\partial K(x,s)}{\partial x} ds.$$

$$F(x,t) = F_N + \frac{1}{2}[a_N(x+t) + a_N(x-t)].$$

$$\frac{\partial F(x,t)}{\partial x} = \frac{\partial F_N(x,t)}{\partial x} + \frac{1}{2}[\frac{\partial a_N(x+t)}{\partial x} + \frac{\partial a_N(x-t)}{\partial x}] .$$

$$a_N(x+t) = \sum_{n=N}^{\infty} (\frac{\cos\sqrt{\lambda_n}(x+t)}{\alpha_n} - \frac{2}{\pi} \cos n(x+t)) .$$

$$\frac{\partial a_N(x+t)}{\partial x} = \sum_{n=N}^{\infty} \left(-\frac{\sqrt{\lambda_n}}{a_n} \sin \sqrt{\lambda_n}(x+t) + \frac{2n}{\pi} \sin n(x+t)\right)$$

$$a_N(x-t) = \sum_{n=N}^{\infty} \left(\frac{\cos\sqrt{\lambda_n}(x-t)}{a_n} - \frac{2}{\pi} \cos n(x-t)\right).$$

$$\frac{\partial a_N(x-t)}{\partial x} = \sum_{n=N}^{\infty} \left(-\frac{\sqrt{\lambda_n}}{a_n} \sin \sqrt{\lambda_n}(x-t) + \frac{2n}{\pi} \sin n(x-t)\right).$$

Let

$$a_N'(x) = \sum_{n=N}^{\infty} \left[+\frac{\sqrt{\lambda_n} \sin \sqrt{\lambda_n} x}{a_n} - \frac{2n}{\pi} \sin nx\right].$$

$$\frac{\partial a_N(x+t)}{\partial x} = -a_N'(x+t); \quad \frac{\partial a_N(x-t)}{\partial x} = -a_N'(x-t).$$

$$a_N'(x) = E_N(x) + O\left(\frac{1}{N}\right).$$

$$\frac{\partial a_N(x+t)}{\partial x} = -E_N(x+t) + O\left(\frac{1}{N}\right).$$

$$\frac{\partial a_N(x-t)}{\partial x} = -E_N(x-t) + O\left(\frac{1}{N}\right).$$

$$\frac{1}{2}\left[\frac{\partial a_N(x-t)}{\partial x} + \frac{\partial a_N(x+t)}{\partial x}\right] = -\frac{1}{2}\left[E_N(x+t) + E_N(x-t)\right] + O\left(\frac{1}{N}\right).$$

Let

$$\tilde{F}_N = -\left(\frac{\partial F_N}{\partial x} + \frac{1}{2}\left[E_N(x+t) + E_N(x-t)\right]\right).$$

Therefore

$$\frac{\partial F}{\partial x} = \tilde{F}_N + O(\frac{1}{N}), \qquad \frac{\partial F}{\partial t} = \bar{F}_N + O(\frac{1}{N}).$$

$$F = G_N + H_N, \quad \frac{\partial F}{\partial x} = \tilde{F}_N + \approx{F}_N, \quad H_N = O(\frac{1}{N^2}), \quad \approx{F}_N = O(\frac{1}{N}).$$

Let

$$L = \frac{\partial K}{\partial x}.$$

Hence

$$L + \int_0^x F(s,t)L(x,s)ds = -\frac{\partial F(x,t)}{\partial x} - F(x,t)K(x,x).$$

Consider

$$L_N + \int_0^x G_N(s,t)L_N(x,s)ds = -\tilde{F}_N - G_N(x,t)K_N(x,x).$$

Hence,

$$(L-L_N) + \int_0^x G_N(s,t)(L(x,s)-L_N(x,s))ds + \int_0^x H_N(s,t)L(x,s)ds$$

$$= -\approx{F}_N - F(x,t)K(x,x) + G_N(x,t)K_N(x,x).$$

Therefore

$$L-L_N = (I+\bar{G}_N)^{-1}\Big(-\int_0^x H_N L ds - \approx{F}_N$$

$$+ [(-F(x,t)K(x,x)+G_N(x,t)K(x,x)) - G_N(x,t)K(x,x)+G_N(x,t)K_N(x,x)]\Big),$$

$$\| L-L_N \| \le \| (I+\bar{G}_N)^{-1} \| \quad \| \int_0^x H_N L \| + \| \approx{F}_N \| + \| K \| \| F-G_N \| + \| G_N \| \| K-K_N \| \quad .$$

Therefore,

$$L = L_N + O(\frac{1}{N}).$$

Letting

$$q_N(x) = 2(L_N(x, x) + \overline{F}_N(x, x) + \int_0^x \overline{F}_N(s, x)K_N(x, s)ds),$$

it directly follows that

$$q = q_N + O(\frac{1}{N}) \quad \square .$$

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APPENDICES

APPENDIX A

Iterative Procedures[2/]

The problem we wish to consider being:

$$\text{(A. 1)} \qquad K(x,t) + \int_0^x K(x,s)F(s,t)ds = -F(x,t).$$

For practical considerations, let us restrict our attention to the finite region $0 \le t \le x \le a$, where a will be assumed sufficiently large.

Domain $\quad T = \{x \mid 0 \le x \le a\} \times \{t \mid 0 \le t \le x \le a\}$

Equation (A. 1) in the operator form: $f - Af = g$, where

$$Af(x,t) = -\int_0^x F(\xi,t)f(x,\xi)\,d\xi, \quad f = K, \quad g = -F.$$

We will assume $F(x,t)$ is the kernel of the said Integral Equation discussed in Chapter II. Consequently, at least continuous. Let us now consider (A. 1) in the B-space C_T. $g \in C_T$, and from the Gelfand Theory, we know that f is as smooth as g, hence, $f \in C_T$ [6]. A maps C_T into C_T. A is linear, which follows from the fact it is an integral operator. A is a bounded linear operator, for

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2/Similar interative techniques have been used by V. A. Marchenko, Trudy Moskov. Mat. Obsc. 2, 3-83 (1953).

$$\|Af\|_{C_T} = \max_{t,\, x \in T} \left| \int_0^x F(\xi, t) f(x, \xi) d\xi \right.$$

$$\|Af\|_{C_T} \leq \max_{t,\, x \in T} \int_0^x |F(\xi, t)|\, |f(x, \xi)|\, d\xi$$

$$\leq \|f\|_{C_T} \max_{t,\, x \in T} \int_0^a |F(\xi, t)|\, d\xi ,$$

where

$$\|f\|_{C_T} = \max_{x,\, t \in T} |f(x, t)| .$$

$\Rightarrow$

$$\|A\|_{C_T} \leq \max_{0 \leq t \leq a} \int_0^a |F(\xi, t)|\, d\xi .$$

Therefore A is a bounded linear operator from $C_T \to C_T$. Since by hypothesis a solution exists (a consequence of the Gelfand Theory) $\Rightarrow$ $(I-A)^{-1}$ exists; $f = (I-A)^{-1} g$.

$$f_n = g + Af_{n-1}, \quad n = 1, 2, 3, \ldots$$

$\therefore$ $$\|f - f_n\|_{C_T} \leq \|(I-A)^{-1}\|_{C_T} \|A^n\|_{C_T} \|f_1 - f_0\|_{C_T} .$$

$\therefore$ if $\|A\|_{C_T} \leq q < 1$, then $\|f - f_n\|_{C_T} \leq \dfrac{q^n}{1-q} \|f_1 - f_0\|_{C_T}$.

Since $f(x, t) = K(x, t)$; $g(x, t) = -F(x, t)$, we have that

$$K_n(x,t) + F(x,t) + \int_0^x F(\xi,t)K_{n-1}(x,\xi)d\xi = 0, \quad n = 1, 2, 3, \ldots$$

$$\|K-K_n\|_{C_T} \leq \|(I-A)^{-1}\|_{C_T} \|A^n\|_{C_T} \|K_1(x,t)-K_0(x,t)\|_{C_T}, \quad [5],$$

and if

$$\|A\|_{C_T} \leq q < 1,$$

then

$$\|K(x,t)-K_n(x,t)\|_{C_T} \leq \frac{q^n}{1-q} \|K_1(x,t)-K_0(x,t)\|,$$

in particular, if

$$q = \max_{0 \leq t \leq a} \int_0^a |F(\xi,t)|d\xi < 1.$$

Now, let us consider the problem in the space L_T.

$$f + Af = g; \quad f \text{ and } g \in L_T.$$

Let

$$\max_{0 \leq x \leq a} \int_0^a |F(x,t)|dt = M'$$

$$L_T \xrightarrow{A} L_T$$

$$\|f\|_{L_T} = \int_0^a dx \int_0^x |f(x,t)|dt$$

$$\|Af\| = \int_0^a dx \int_0^x \left| \int_0^x F(\xi,t)f(x,\xi)d\xi \right| dt$$

$$\leq \int_0^a dx \int_0^x \left(\int_0^x |F(\xi,t)|\,|f(x,\xi)|d\xi \right) dt$$

$$\int_0^x dt \int_0^x |F(\xi, t)| |f(x, \xi)| d\xi = \int_0^x d\xi \int_0^x |F(\xi, t)| |f(x, \xi)| dt$$

$$= \int_0^x |f(x, \xi)| d\xi \int_0^x |F(\xi, t)| dt$$

$$\leq M' \int_0^x |f(x, \xi) d\xi .$$

$$=> \qquad \|Af\|_{L_T} \leq M' \int_0^a dx \int_0^x |f(x, \xi)| d\xi = M' \|f\|_{L_T}$$

$$=> \qquad \|A\|_{L_T} \leq M'$$

$\therefore$ if $M' < 1$, then

$$\|K(x, t) - K_n(x, t)\|_{L_T} \leq \frac{(M')^n}{1 - M'} \|K_1(x, t) - K_0(x, t)\|_{L_T} .$$

Now, consider the problem in L^2.

$$f \in L_T^2, \quad g \in L_T^2$$

$$\|f\|_{L_T^2}^2 = \int_0^a dx \int_0^x |f(x, t)|^2 dt$$

$$\|Af\|^2 = \int_0^a dx \int_0^x |Af(x, t)|^2 dt; \quad Af(x, t) = \int_0^x F(\xi, t) f(x, \xi) d\xi$$

$$= \int_0^a dx \int_0^x [\, | \int_0^x F(\xi, t) f(x, \xi) d\xi | \,]^2 dt$$

$$\leq \int_0^a dx \int_0^x [\int_0^x |F(\xi, t)|^2 d\xi][\int_0^x |f(x, \xi)|^2 d\xi] dt \leq$$

$$\leq \int_0^a \int_0^x |f(x,\xi)|^2 d\xi dx \int_0^x \left(\int_0^x |F(\xi,t)|^2 d\xi\right) dt$$

$$\leq \int_0^a \left(\int_0^x |f(x,\xi)|^2 d\xi\right) dx \int_0^a \int_0^a |F(\xi,t)|^2 d\xi dt$$

$$\leq \|f\|^2_{L^2_T} \int_0^a \int_0^a |F(\xi,t)|^2 d\xi dt.$$

$$\therefore \qquad \|A\| \leq \left(\int_0^a \int_0^a |F(\xi,t)|^2 d\xi dt\right)^{1/2} = q.$$

$\therefore$ if $q < 1$, then

$$\|K(x,t) - K_n(x,t)\|_{L^2_T} \leq \frac{q^n}{1-q} \|K_1(x,t) - K_0(x,t)\|_{L^2_T}.$$

For the three spaces considered q can always be made strictly less than 1 by restricting the domain T (i.e., by restricting a).

Assuming $F(x,t)$ (ϵC^1) is once continuously differentiable with respect to both variables, let us consider the problem in C^1_T.

$$F(s,t) + K(s,t) + \int_0^s F(\xi,t) K(s,\xi) d\xi$$

$$f - Af = g; \quad Af = \int_0^x F(\xi,t) f(x,\xi) d\xi$$

$$\|Af\|_{C'_T} = \max_{s,t\in T} \left|\int_0^s F(\xi,t)f(s,\xi)d\xi\right| + \max_{s,t\in T}\left|\frac{\partial}{\partial s}\int_0^s F(\xi,t)f(s,\xi)d\xi\right|$$

$$+ \max_{s,t\in T}\left|\frac{\partial}{\partial t}\int_0^s F(\xi,t)f(s,\xi)d\xi\right|$$

$$= I_1 + I_2 + I_3,$$

respectively.

$$I_1 = \max_{s,t\in T}\left|\int_0^s F(\xi,t)f(s,\xi)d\xi\right| \le \max_{s,t\in T}|f|\left(\int_0^s |F(\xi,t)|d\xi\right)$$

$$I_2 = \max_{s,t\in T}\left|F(s,t)f(s,s) + \int_0^s F(\xi,t)f_s(s,\xi)d\xi\right|$$

$$\le \max |F(s,t)|\max|(f(s,t)| + \max\int_0^s |F(\xi,t)|d\xi \,\max|f_s|$$

$$I_2 \le \max|f|(\max|F|) + \max|f_s(s,t)|\left(\max\int_0^s |F(\xi,t)|d\xi\right)$$

$$I_3 \le \max\int_0^s |F_t(\xi,t)|\,|f(s,\xi)|d\xi$$

$$\le \max\left|f(s,s)\int_0^s F_t(u,t)du - \int_0^s f_\xi(s,\xi)\left(\int_0^\xi F_t(u,t)du\right)d\xi\right.$$

$$\le \max|f|\,\max\left|\int_0^s F_t(u,t)du\right|$$

$$+ \max|f_t(s,t)|\,\max\int_0^s \left|\int_0^\xi F_t(u,t)du\right|d\xi .$$

$$\therefore \quad I_1 + I_2 + I_3 \leq \max|f|(\max|\int_0^s F_t(u,t)du| + \max|F(s,t)|$$

$$+ \max\int_0^s |F(\xi,t)|d\xi)$$

$$+ \max|f_s(s,t)| (\max\int_0^s |F(\xi,t)|d\xi)$$

$$+ \max|f_t(s,t)| \max\int_0^s |\int_0^\xi F_t(u,t)du|d\xi$$

Let

$$M = \max\{[\max|\int_0^s F_t(u,t)du| + \max|F(s,t)| + \max\int_0^s |F(\xi,t)|d\xi \ ,$$

$$\max\int_0^s |F(\xi,t)|d\xi \ , \quad \max\int_0^s |\int_0^\xi F_t(u,t)du|d\xi\,]\} \ .$$

$$\therefore \qquad I_1 + I_2 + I_3 \leq M(\max|f| + \max|f_s(s,t)| + \max|f_t|)$$

$$\|Af\|_{C_T^1} \leq \|f\|_{C_T^1} M$$

$$\Rightarrow \qquad \|A\|_{C_T^1} \leq M.$$

$\therefore$ if $M < 1$, we have that

$$\|K(x,t) - K_n(x,t)\|_{C_T^1} \leq \frac{M^n}{1-M} \|K_1(x,t) - K_0(x,t)\|_{C_T^1},$$

which implies that K_n converges uniformly to K, $\frac{\partial K_n(x,t)}{\partial t}$ converges uniformly to $\frac{\partial K(x,t)}{\partial t}$, and $\frac{\partial K_n(x,t)}{\partial x}$ converges uniformly to $\frac{\partial K(x,t)}{\partial x}$.

APPENDIX B

Kantorovich Method [5]

A brief synopsis of the results of Kantorovich [5] upon which Chapters II and III are based:

Let $\tilde{X}$ be a complete subspace of the normed space X, $\tilde{X} \subseteq X$. Let $\overline{P}$ be a linear operation such that

$$\overline{P}(X) = \tilde{X}; \quad \overline{P}^2 = \overline{P}$$

(B.1) In space X: $Kx = x - Hx = y$

(B.2) In space $\tilde{X}$: $\tilde{K}\tilde{x} = \tilde{x} - \tilde{H}\tilde{x} = \overline{P}y$

H and $\tilde{H}$ are linear operations in X and $\tilde{X}$, respectively. The following results are based upon the lemma:

Let V be a linear operation from the B-space X into the B-space Y and let there exist for every $y \in Y$ an $x \in X$ such that

$$\| V(x) - y \| \leq q \| y \|; \quad \| x \| \leq N \| y \|$$

where $q < 1$ and N are constants. Then the equation $V(x) = y$ has for every $y \in Y$, a solution $x \in X$ satisfying

$$\| x \| \leq \frac{N \| y \|}{1 - q}$$

Conditions.

I. For every $\tilde{x} \in \tilde{X}$, $\|\overline{P}H\tilde{x}-\tilde{H}\tilde{x}\| \leq \eta\|\tilde{x}\|$.

II. For every $x \in X$, there exists an $\tilde{x} \in \tilde{X}$ such that

$\|Hx-\tilde{x}\| \leq \eta_1 \|x\|$.

III. An element $\tilde{y} \in \tilde{X}$ exists such that $\|y-\tilde{y}\| \leq \eta_2 \|y\|$,

where η_2 may depend on y.

Theorem B1. If Conditions I and II are satisfied and K has a linear inverse, then if

$$q = |\lambda|[\eta(1+|\lambda|\eta_1) + \eta_1\|\overline{P}K\|]\|K^{-1}\| < 1,$$

(B.3) the equation $\tilde{K}\tilde{x} = \tilde{y}$ has a solution $\tilde{x}^*$ for any $\tilde{y} \in \tilde{X}$.
Also $\|\tilde{x}^*\| \leq \frac{N}{1-q}\|\tilde{y}\|$, where $N = (1+|\lambda|\eta_1)\|K^{-1}\|$.

Theorem B2. If Conditions I, II are satisfied, the linear operation $\tilde{K}^{-1}$ and the Equation (B.1) has a solution x^*, then

$$\|x-\tilde{x}^*\| \leq B\|x^*\|,$$

where x is a solution of the Equation (B.2) and

$$B = 2|\lambda|\eta\|\tilde{K}^{-1}\| + (\eta_1|\lambda|+\eta_2\|K\|)(1+\|\tilde{K}^{-1}\overline{P}K\|).$$

Theorem B3. If

1) K has a linear inverse

2) $\tilde{K}$ satisfies Condition A (for each $n = 1, 2, \ldots$)

3) for each $n = 1, 2, \ldots,$ the Conditions I, II and III are satisfied, and

$$\lim_{n\to\infty} \eta = 0, \quad \lim_{n\to\infty} \eta_1 \|\overline{P}\| = 0, \quad \text{and} \quad \lim_{n\to\infty} \eta_2 \|\overline{P}\| = 0$$

then the approximate equations are soluble for sufficiently large n and the sequence of approximate solutions converges to the exact solution:

$$\lim_{n\to\infty} \|x^* - \tilde{x}^*\| = 0$$

and there exists constants Q_0, Q_1, and Q_2 such that

$$\|x^* - \tilde{x}_n^*\| \le Q_0\eta + Q_1\eta_1\|\overline{P}\| + Q_2\eta_2\|\overline{P}\|.$$

<u>Condition A.</u> The existence of a solution of Equation (B.3) for every $\tilde{y} \in \tilde{X}$ implies its uniqueness.

As discussed in Chapter II, similar results can be derived between the space X and a space $\overline{X}$, which is isomorphic to $\tilde{X}$. In particular: Let φ_0 be a linear operation defining a 1:1 mapping on $\tilde{X}$, onto $\overline{X}$. Let φ be a linear extension of the operation φ_0 to the entirety of X. $\overline{P} : X \to \tilde{X}$; $\varphi_0(\tilde{X}) = \overline{X}$. Let $\varphi = \varphi_0\overline{P}$, $\overline{P} = \varphi_0^{-1}\varphi$, $\therefore$ $\tilde{x} = \varphi_0^{-1}\overline{x}$; Equation (B.2) transforms to

$$\overline{x} = \varphi_0\tilde{H}\varphi_0^{-1}\overline{x} = \varphi_0\overline{P}y, \qquad \overline{H} = \varphi_0\tilde{H}\varphi_0^{-1},$$

$$\overline{K}\overline{x} = \overline{x} - \lambda\overline{H}\overline{x} = \varphi_y, \qquad \overline{K} = \varphi_0\tilde{K}\varphi_0^{-1}$$

Condition Ia. For every $\tilde{x} \in \tilde{X}$, $\|\bar{H}\varphi_0\tilde{x} - \varphi H x\| \le \bar{\eta}\,\|\tilde{x}\|$.

Theorem B1a. If Ia and II are satisfied, and K^{-1} exists, then if

$$q = |\lambda|\,[\bar{\eta}(1+|\lambda|\eta_1)\|\varphi_0^{-1}\| + \eta_1\|\varphi_0^{-1}\varphi K\|]\,\|K^{-1}\| < 1.$$

(B.4) the equation $\bar{K}\bar{x} = \bar{y}$ has a solution $\bar{x}^*$ for every right-handside $y \in \bar{X}$, with

$$\|\bar{x}^*\| \le \frac{\bar{N}}{1-\bar{q}}\|\bar{y}\|, \quad \bar{N} = (1-|\lambda|\eta_1)\|\varphi_0\|\,\|\varphi_0^{-1}\|\,\|K^{-1}\|$$

Theorem B2a. If Ia, II, and III are satisfied, linear operator $\bar{K}^{-1}$ exists and the Equation (B.2) has solution x^*, then

$$\|x^* - \varphi_0^{-1}\bar{x}^*\| \le B\|x^*\|,$$

where

$$B = (1+\varepsilon)|\lambda|\bar{\eta}\|\varphi_0^{-1}\bar{K}^{-1}\| + \varepsilon(1+\|\varphi_0^{-1}\bar{K}^{-1}\varphi\bar{K}\|),$$

$$\varepsilon \le \eta_1|\lambda| + \eta_2\|K\|.$$

Theorem B3a. If the following conditions are satisfied

1) K has a linear inverse

2) $\bar{K}$ satisfies Condition A

3) Conditions Ia, II and III are satisfied for every n, $n = 1, 2, \ldots$, where

$$\lim_{n\to\infty}\bar{\eta}\|\varphi_0^{-1}\| = \lim_{n\to\infty}\eta_1\|\varphi_0^{-1}\varphi\| = \lim_{n\to\infty}\eta_2\|\varphi_0^{-1}\varphi\| = 0,$$

then the approximate Equation (B.4) is soluble for sufficiently large n and the sequence of approximate solutions converges to the exact solution. Also

$$\|x^{*}-\tilde{x}_n^{*}\| \leq \overline{Q}\eta\|\varphi_0^{-1}\| + \overline{Q}_1\eta_1\|\varphi_0^{-1}\varphi\| + \overline{Q}_2\eta_2\|\varphi_0^{-1}\varphi\|,$$

where $\tilde{x}_n^{*} = \varphi_0^{-1}\overline{x}_n^{*}$, and $\overline{Q}_1$, $\overline{Q}_2$, and $\overline{Q}_3$ are constants.

APPENDIX C

Synopsis of Basic Theory [6]

Let

(C.1) $$y'' - q(x)y = -\lambda y$$

with

(C.2) $$y'(0) - hy(0) = 0$$

where $0 \leq x < \infty$, $q(x)$ is a real integrable function; h is a real number. $\varphi(x, \lambda)$ denotes the solution of (C.1) with the initial conditions (C.2).

Theorem (Gelfand and Levitan) C1. Suppose $\varphi(x, \lambda)$ is the solution of (C.1) satisfying the initial conditions (C.2) and that $q(x)$ has m locally integrable derivatives. Then there exist functions $K(x, t)$ and $H(x, t)$, each have $m+1$ integrable derivatives with respect to each of the variables, such that

$$\varphi(x, \lambda) = \cos\sqrt{\lambda}x + \int_0^x K(x, t)\cos\sqrt{\lambda}t\,dt$$

$$\cos\sqrt{\lambda}x = \varphi(x, \lambda) + \int_0^x H(x, t)\varphi(t, \lambda)\,dt$$

$$K(x, x) = h + \frac{1}{2}\int_0^x q(t)\,dt, \qquad \left.\frac{\partial K(x, t)}{\partial t}\right|_{t=0} = 0,$$

$$\left(\frac{\partial H(x,t)}{\partial t} - hH(x,t)\right)\Big|_{t=0} = K(x,-t) = 0, \quad H(x,-t) = 0 \text{ for } t > 0;$$

$$K(x,t) = 0, \quad H(x,t) = 0 \quad \text{for} \quad t > x.$$

Further, if $M \geq 1$, then $K_{xx} - q(x)K = K_{tt}$ and

$$H_{xx} = H_{tt} - q(t)H.$$

Theorem (Gelfand and Levitan). Suppose $q(x)$ has m locally integrable derivatives and that

$$\sigma(\lambda) = \begin{cases} \rho(\lambda), & \lambda < 0 \\ \rho(\lambda) - \frac{2}{\pi}\sqrt{\lambda}, & \lambda \geq 0 \end{cases}$$

then the integral $\int_{-\infty}^{N} \cos\sqrt{\lambda}\, x\, d\sigma(\lambda)$ converges boundedly to a function $\Phi(x) = \chi(x, 0)$ in any finite range of values of x, as $N \to \infty$, where $\Phi(x)$ has $m+1$ integrable derivatives.

Theorem (Gelfand and Levitan) C2. The kernel $K(x,t)$ satisfies the integral equation

$$F(x,t) + K(x,t) + \int_0^x K(x,s)F(s,t)ds = 0, \quad 0 \leq t < x,$$

where

$$F(x,t) = \lim_{n\to\infty} \int_{-\infty}^{N} \cos\sqrt{\lambda}x \cos\sqrt{\lambda}t d\sigma(\lambda), \ \sigma(\lambda) = \begin{cases} \rho(\lambda), & \lambda < 0 \\ \rho(\lambda) - \frac{2}{\pi}\sqrt{\lambda}, & \lambda \geq 0. \end{cases}$$

Theorem (Gelfand and Levitan) C3. The integral equation

$$F(x, t) + K(x, t) + \int_0^x K(x, s)F(s, t)ds = 0$$

has a unique solution $K(x, t)$ for every fixed x.

Theorem (Gelfand and Levitan) C4. Suppose the kernel $K(x, t)$ satisfies the above integral equation. Then the function

$$Q(x, \lambda) = \cos \sqrt{\lambda}t + \int_0^x K(x, t) \cos \sqrt{\lambda}tdt$$

satisfies the differential equation $\varphi'' + \{\lambda - q(x)\}\varphi = 0$ and the initial conditions $\varphi(0, \lambda) = 1$, $\varphi'(0, \lambda) = K(0, 0) = -F(0, 0) = h$,

$$q(x) = + \frac{2dK(x, x)}{dx}.$$

Theorem (Gelfand and Levitan) C5. The monotonically increasing function $\rho(\lambda)$ is the spectral function of a boundary value problem of the type (C.1)-(C.2) with a function $q(x)$ (having m integrable derivatives) and a number h if and only if the following conditions are satisfied:

a. If $E(\lambda)$ is the cosine transform of an arbitrary function $f(x)$ of compact support in $L_2(0, \infty)$ and

$$\int_{-\infty}^{\infty} E^2(\lambda)d\rho(\lambda) = 0,$$

then $f(x) = 0$ (a.e.).

b. The limit

$$\Phi(x) = \lim_{N \to \infty} \int_{-\infty}^{N} \cos \sqrt{\lambda} x \, d\sigma(\lambda),$$

where

$$\sigma(\lambda) = \begin{cases} \rho(\lambda), & \lambda < 0 \\ \rho(\lambda) - \frac{2}{\pi}\sqrt{\lambda}, & \lambda \geq 0 \end{cases}$$

exists boundedly in every finite range of values of x and $\Phi(x)$ has $m+1$ integrable derivatives with $\Phi(0) = -h$.

<u>Theorem. Basic Inverse Sturm-Liouville Theorem on $[0, \pi]$ (Gelfand) C6.</u>

(C.1) $$-y'' + q(x)y = \lambda y$$

(C.3) $$y'(0) - hy(0) = 0, \quad y'(\pi) + Hy(\pi) = 0$$

The numbers $\{\lambda_n\}$ and $\{\alpha_n\}$ are the spectral characteristics of some boundary value problem (C.1)-(C.3) for $[0, \pi]$ with a function $q(x)$, where $q^{(m)}(x) \in L(0, \pi)$ iff the following asymptotic estimates hold:

$$\sqrt{\lambda_n} = n + \frac{a_0}{n} + O(\frac{1}{n}), \quad \alpha_n = \frac{\pi}{2} + o(\frac{1}{n})$$

where $\lambda_n \neq \lambda_m$ for $n \neq m$ and all the $\alpha_n > 0$, and if the function

$$F(x,t) = \frac{1}{\alpha_0}\cos\sqrt{\lambda_0}x\cos\sqrt{\lambda_0}t - \frac{1}{\pi} + \sum_1^{\infty}\left[\frac{\cos\sqrt{\lambda_n}x\cos\sqrt{\lambda_n}t}{\alpha_n} - \frac{2}{\pi}\cos nx\cos nt\right]$$

has integrable derivatives of order $m+1$ in the region $(0 \le x,t \le \pi)$. This implies that there exists a function $K(x,t)$ such that

$$F(x,t) + K(x,t) + \int_0^x K(x,s)F(s,t)ds = 0,$$

where $0 \le t \le x \le \pi$ for the kernel K; $K(x,t) = 0$ for $t > x$ and

$$q(x) = +2\,\frac{dK(x,x)}{dx}.$$

<u>Theorem (Gelfand and Levitan) C7.</u> If all the $\alpha_n > 0$, and

$$\sqrt{\lambda_n} = n + \frac{a_0}{n} + \frac{a_1}{n^3} + O\left(\frac{1}{n^4}\right),$$

$$\alpha_n = \frac{\pi}{2} + \frac{b_0}{n^2} + O\left(\frac{1}{n^3}\right),$$

where a_0, a_1, b_0 are constants, then there exists an absolutely continuous function $q(x)$ corresponding to the given λ_n and α_n.

$$a_0 = \frac{1}{\pi}\left[h + H + \frac{1}{2}\int_0^{\pi} q(t)dt\right]$$

Note: If $O(\frac{1}{n^3})$ for a_n can be replaced by $O(\frac{1}{n^4})$, then $q(x)$ has an absolutely continuous derivative.

The basis of Levitan's new result with respect to a variation of boundary conditions is the formula

$$a_n = \frac{h_1 - h}{\mu_n - \lambda_n} \prod_{n=0}^{\infty}{}' \frac{\lambda_k - \lambda_n}{\mu_k - \lambda_n}, \tag{C.4}$$

which gives an expression for the normalizing constants of a regular Sturm-Liouville operator in terms of two of its spectra. In addition, (C.4) gives a conditional solution of the inverse problem in terms of two spectra, for once we know the numbers $\{\lambda_n\}$ and $\{a_n\}$, we can define the spectral function by the formula

$$\rho(\lambda) = \sum_{\lambda_n < \lambda} \frac{1}{a_n}$$

and then form the operator by the prescription given by the basic Spectral Theory.

For the problem

$$-y'' + q(x)y = \lambda y, \tag{C.1}$$

with

$$y'(0) - h_1 y(0) = 0, \quad y'(\pi) + Hy(\pi) = 0 \tag{C.5}$$

there exists the set of eigenvalues $\{\lambda_i\}$. Similarly, for

(C.1) $$-y'' + q(x)y = \lambda y$$

with

(C.6) $$y'(0) - h_2 y(0) = 0, \quad y'(\pi) + Hy(\pi) = 0$$

there exists the set of eigenvalues $\{\mu_i\}$.

Theorem (Gasymov and Levitan) C8. Suppose that we are given two sequences of numbers $\{\lambda_n\}$ and $\{\mu_n\}$ $(n = 0, 1, 2, \ldots)$ satisfying the following conditions:

1. The numbers λ_n and μ_n interlace,
2. λ_n and μ_n satisfy the asymptotic estimates of Theorem C7, and $(a_0 \neq a_0')$ then there exist an absolutely continuous function $q(x)$ and numbers h_1, h_2, H such that $\{\lambda_n\}$ is the spectrum of the problem (C.1)-(C.5) and $\{\mu_n\}$ the spectrum of the problem (C.1)-(C.6); moreover $h_2 - h_1 = \pi(a_0' - a_0)$.