1852

ELASTIC STABILITY OF CYLINDRICAL SANDWICH SHELLS UNDER AXIAL AND LATERAL LOAD

(Report)
No. 1852

July 1955

This Report is One of a Series
Issued in Cooperation with the
ANC-23 PANEL ON SANDWICH CONSTRUCTION
of the Departments of the
AIR FORCE, NAVY, AND COMMERCE

FOREST PRODUCTS LABORATORY
MADISON 5, WISCONSIN

UNITED STATES DEPARTMENT OF AGRICULTURE
FOREST SERVICE

In Cooperation with the University of Wisconsin

ELASTIC STABILITY OF CYLINDRICAL SANDWICH SHELLS UNDER AXIAL AND LATERAL LOAD[1]

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Summary

A linear solution for the determination of the loads under which a cylindrical sandwich shell will buckle is presented. The facings of the sandwich cylinder are treated as cylindrical shells and the core as an orthotropic elastic body. The method of solution is of interest in that it is of sufficient generality to be applied to many problems in sandwich analysis. The characteristic determinant that represents the solution to the problem is solved numerically. Curves that show how the buckling load changes as the parameters of the problem change are given.

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[1]This report is one of a series prepared by the Forest Products Laboratory under U. S. Navy, Bureau of Aeronautics Order No. 01593.

[2]Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

Introduction

Sandwich construction is a result of the search for a strong, stiff, and yet light weight material. It is usually made by gluing relatively thin sheets of a strong material to the faces of relatively thick but light weight, and often weak, material. The outer sheets are called "facings" and the inner layer is called the "core."

Such a layered system presents difficult design problems. What is offered here is a straightforward method for dealing with some of these problems.

The problem to which the method is applied is that of the elastic stability of a sandwich cylinder under uniform external lateral load and uniform axial load.

Notation

$r,\ \theta,\ z$	radial, tangential, and longitudinal coordinates, respectively
a	radius to middle surface of outer facing
b	radius to middle surface of inner facing
t	thickness of each facing
ℓ	length of cylinder
E	modulus of elasticity of facings
μ	Poisson's ratio of facings
G	modulus of rigidity of facings
E_c	modulus of elasticity of core in direction normal to facings
$G_{r\theta}$	modulus of rigidity of core in $\underline{r\theta}$ plane
G_{rz}	modulus of rigidity of core in $\underline{rz}$ plane
q	intensity of uniform external lateral loading
k	$\dfrac{1}{1 + \dfrac{b}{a} - \dfrac{Et \log \dfrac{b}{a}}{E_c\, a}}$
σ_r	normal stress in core in radial direction
$\tau_{r\theta},\ \tau_{rz}$	transverse shear stresses in core
$u,\ v,\ w$	radial, tangential, and longitudinal displacements, respectively
n	number of waves in circumference of buckled cylinder
m	number of half waves in length of buckled cylinder

λ	$\frac{m\pi a}{\ell}$
$\delta_{n\theta}$	$\frac{E_c}{2G_{r\theta}} - \frac{n^2}{2}$
δ_z	$\frac{E_c}{G_{rz}}$
N_θ, N_z, $N_{\theta z}$	normal forces and shear force per unit length of facing
Q_θ, Q_z	transverse shear forces per unit length of facing
M_θ, M_z	bending moments per unit length of facing
$M_{z\theta}$, $M_{\theta z}$	twisting moments per unit length of facing
R, θ, Z	surface forces per unit area of facing
β	$\frac{E_c a (1 - \mu^2)}{Et}$
ϕ_1	$\frac{qa (1 - \mu^2)}{Et}$
ϕ_2	$\frac{N_z (1 - \mu^2)}{Et}$
α	$\frac{t^2}{12a^2}$
α'	$\frac{t^2}{12b^2}$
log	natural logarithm
A, B, C, D, K, L, A', B', A'', B''	arbitrary constants

note -- any of the above terms that appear with a prime (as N_z') refer to the inner facing.

Mathematical Analysis

As previously stated, the core is relatively weak. Because of the high strength of the facings the core need carry little tension or compression except in a direction perpendicular to the facings. The facings are able to resist shearing deformation in their plane and it is necessary only that the core be able to resist shear in the radial direction in planes perpendicular to the facings. In this analysis the core is considered to be an orthotropic elastic body. It is unable to resist deformations other than those just mentioned. This assumption makes it possible to determine explicitly how the stresses vary throughout the thickness of the core.

The facings are treated as shells.

Interdependance of the core and the facings is gained by equating their displacements at the interfaces. To simplify the analysis the core is assumed to extend to the middle surface of each facing.

Figure 1 shows the cylinder and the coordinates that are used.

Prebuckling Stresses

Before buckling occurs the cylinder is in a state of uniform compression. The axial load is carried by the facings since the core material is assumed to be incapable of carrying load in this direction. With facings of like material the stress is the same in both facings.

If, in addition, the facings have the same thickness, then the loading per unit length of facing, N_z or N_z', will be the same. This means that for a total load P,

$$2\pi a\, N_z + 2\pi b\, N_z' = P.$$

The calculation of stresses due to the lateral pressure is a problem in rotational symmetry. Differential elements of the core and of the facings are shown in figure 2.

Summing forces in the radial direction gives for the core

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r}{r} = 0,$$

for the outer facing

$$aq - a\,(\sigma_r)_{r=a} - N_\theta = 0,$$

and for the inner facing

$$b\,(\sigma_r)_{r=b} - N_\theta' = 0.$$

Since $\sigma_r = E_c \dfrac{\partial u}{\partial r}$,

$$N_\theta = Et\left(+\frac{u}{a}\right)_{r=a}, \text{ and}$$

$N_\theta' = Et\left(+\dfrac{u}{b}\right)_{r=b}$, these equations can be solved for σ_r, N_θ and N_θ'. The results are*

$$\sigma_r = q\,\frac{a}{r}\,k$$

$$N_\theta = qa\,(1 - k), \text{ and}$$

*For a more detailed derivation of these terms see Reference 1.

$N_\theta' = qak$ where

$$k = \frac{1}{1 + \frac{b}{a} - \frac{Et \log \frac{b}{a}}{E_c a}}$$

As P and q increase, N_z, N_z', N_θ, N_θ', and σ_r also increase. Eventually a condition may be reached where a slight increase in load causes the cylinder to lose its state of uniform compression and buckle as a result of elastic instability. This buckling is assumed to cause only a small change in the stress distribution. These small changes will now be considered.

Buckling Stresses

The Core

A free body diagram of an element of the core is shown in figure 3.

Neglecting terms which are products of more than three differentials, a summation of forces in the r, θ, and z direction gives

$$\sigma_r + r\frac{\partial\sigma_r}{\partial r} + \frac{\partial\tau_{r\theta}}{\partial\theta} + r\frac{\partial\tau_{rz}}{\partial z} = 0 \qquad (1)$$

$$r\frac{\partial\tau_{ra}}{\partial r} + 2\tau_{r\theta} = 0 \qquad (2)$$

$$\tau_{rz} + r\frac{\partial\tau_{rz}}{\partial r} = 0 \qquad (3)$$

Equation (2) may be integrated to give

$$\tau_{r\theta} = \frac{B}{r^2} f_1(\theta) f_1(z) \qquad (4)$$

Equation (3) may be integrated to give

$$\tau_{rz} = \frac{A}{r} f_2(\theta) f_2(z) \qquad (5)$$

σ_r, $\tau_{r\theta}$ and τ_{rz} as defined in terms of u, v, and w are

$$\sigma_r = E_c \frac{\partial u}{\partial r} \qquad (6)$$

$$\tau_{r\theta} = G_{r\theta} \left[\frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \right] \qquad (7)$$

$$\tau_{rz} = G_{rz} \left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right] \qquad (8)$$

It is convenient to assume the displacements u, v and w in the form

$$u = f_1(r) \cos n\theta \cos \frac{\lambda}{a} z \qquad (9)$$

$$v = f_2(r) \sin n\theta \cos \frac{\lambda}{a} z \qquad (10)$$

$$w = f_3(r) \cos n\theta \sin \frac{\lambda}{a} z \qquad (11)$$

This form will permit a unique determination of $f_1(r)$, $f_2(r)$, and $f_3(r)$; assumes upon buckling n circumferential waves and m longitudinal half waves; results in zero displacements in the radial and circumferential directions at the ends; and imposes no moment upon the facings at the ends.

From a consideration of equations (4), (5), (7), (8), (9), (10) and (11) it is clear that

$$f_1(\theta)\, f_1(z) = \sin n\theta \cos \frac{\lambda}{a} z \qquad \text{and}$$

$$f_2(\theta)\, f_2(z) = \cos n\theta \sin \frac{\lambda}{a} z, \qquad \text{so that}$$

$$\tau_{r\theta} = \frac{B}{r^2} \sin n\theta \cos \frac{\lambda}{a} z \quad \text{and} \tag{12}$$

$$\tau_{rz} = \frac{A}{r} \cos n\theta \sin \frac{\lambda}{a} z. \tag{13}$$

Substituting equation (9) into (6) and then equations (6), (12) and (13) into equation (1) gives

$$E_c \frac{\partial f_1(r)}{\partial r} + E_c r \frac{\partial^2 f_1(r)}{\partial r^2} + \frac{nB}{r^2} + \frac{\lambda}{a} A = 0 \tag{14}$$

which upon integration shows that

$$f_1(r) = C + D \log r + A' r + B' \frac{1}{r} \tag{15}$$

Equations (9), (10) and (12) are substituted into equation (7) to give

$$\frac{B}{r^2} = G_{r\theta} \left[\frac{n}{r} (C + D \log r + A' r + \frac{B'}{r}) + \frac{\partial f_2(r)}{\partial r} - \frac{f_2(r)}{r} \right], \tag{16}$$

from which

$$f_2(r) = Fr + Cn + Dn(1 + \log r) + A' n\, r \log r + \frac{B'' n}{r}. \tag{17}$$

Equations (9), (11) and (13) are substituted into equation (8) to give

$$\frac{A}{r} = G_{rz} \left[C + D \log r + A' r + B' \frac{1}{r} + \frac{\partial f_3(r)}{\partial r} \right], \tag{18}$$

from which

$$f_3(r) = K + A'' (r^2 + \log r) + Cr + Dr(\log r - 1)$$
$$+ B \log r. \tag{19}$$

It is convenient to have the constants of $f_1(r)$, $f_2(r)$ and $f_3(r)$ in non-dimensional form. Redefining the constants the following form is obtained.

$$u = (Aa + Br + C\frac{a^2}{r} + Da \log\frac{r}{a})\cos n_\theta \cos\frac{\lambda}{a}z \qquad (20)$$

$$v = [-Ana + Bnr \log\frac{r}{a} + C\frac{a^2}{nr}\,\delta_{n\theta} - Dan(\log\frac{r}{a} + 1) +$$

$$Fr]\sin n\theta \cos\frac{\lambda}{a}z \qquad (21)$$

$$w = [A\lambda r + Ba\lambda(\frac{r^2}{2a^2} - \frac{\delta_z}{\lambda^2}\log\frac{r}{a}) + C\lambda a \log\frac{r}{a} + D\lambda r$$

$$(\log\frac{r}{a} - 1) + La]\cos n\theta \sin\frac{\lambda}{a}z \qquad (22)$$

The Facings

Free body diagrams of a facing element showing the forces and moments are shown in figures 4 and 5. It is necessary, in this type of problem, to include components of forces which result from elastic deformation of the element. The geometry of the situation is such that it is difficult to write equations of equilibrium. It is safest to use results obtained from a mathematical theory of thin shells. Such theory, as developed by Osgood and Joseph (ref. 2), when applied to cylindrical shells yields, for the outer facing at r = a, the following equations

$$\Sigma F_z = 0 = a \frac{\partial N_z}{\partial z} + \frac{\partial N_{\theta z}}{\partial \theta} + N_{z\theta} \frac{\partial^2 w}{\partial z \partial \theta} - N_\theta \left(\frac{\partial^2 v}{\partial z \partial \theta} + \frac{\partial u}{\partial z} \right) - a\, Q_z \frac{\partial^2 u}{\partial z^2} - Q_\theta \left(\frac{\partial^2 u}{\partial z \partial \theta} - \frac{\partial v}{\partial z} \right) + aZ \tag{23}$$

$$\Sigma F_\theta = 0 = a \frac{\partial N_{z\theta}}{\partial z} + \frac{\partial N_\theta}{\partial \theta} - Q_z \left(\frac{\partial^2 u}{\partial z \partial \theta} - \frac{\partial v}{\partial z} \right) + Q_\theta \left(1 - \frac{1}{a} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{a} \frac{\partial v}{\partial \theta} \right) - N_z \frac{\partial^2 w}{\partial z \partial \theta}$$

$$+ N_{\theta z} \left(\frac{\partial^2 v}{\partial z \partial \theta} + \frac{\partial u}{\partial z} \right) + a\, \theta \tag{24}$$

$$\Sigma F_r = 0 = a \frac{\partial Q_z}{\partial z} + \frac{\partial Q_\theta}{\partial \theta} + a\, N_z \frac{\partial^2 u}{\partial z^2} + N_{\theta z} \left(\frac{\partial^2 u}{\partial z \partial \theta} - \frac{\partial v}{\partial z} \right) + N_{z\theta} \left(\frac{\partial^2 u}{\partial z \partial \theta} - \frac{\partial v}{\partial z} \right)$$

$$- N_\theta \left(1 - \frac{1}{a} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{a} \frac{\partial v}{\partial \theta} \right) + a\, R \tag{25}$$

$$\Sigma M_z = 0 = a \frac{\partial M_{z\theta}}{\partial z} - \frac{\partial M_\theta}{\partial \theta} + M_z \frac{\partial^2 w}{\partial z \partial \theta} - M_{\theta z} \left(\frac{\partial^2 v}{\partial z \partial \theta} + \frac{\partial u}{\partial z} \right) - a\, Q_\theta - a\, Q_z \frac{\partial v}{\partial z} \tag{26}$$

$$\Sigma M_\theta = 0 = a \frac{\partial M_z}{\partial z} + \frac{\partial M_{\theta z}}{\partial \theta} - M_{z\theta} \frac{\partial^2 w}{\partial z \partial \theta} - M_\theta \left(\frac{\partial^2 v}{\partial z \partial \theta} + \frac{\partial u}{\partial z} \right) + a\, Q_z + Q_\theta \frac{\partial w}{\partial \theta} \tag{27}$$

$$\Sigma M_r = 0 = M_z \left(\frac{\partial^2 u}{\partial z \partial \theta} - \frac{\partial v}{\partial z} \right) - M_\theta \left(\frac{\partial^2 u}{\partial \theta \partial z} - \frac{\partial v}{\partial z} \right) + a\, M_{z\theta} \frac{\partial^2 u}{\partial z^2} - M_{\theta z} \left(1 - \frac{1}{a} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{a} \frac{\partial v}{a\theta} \right) -$$

$$a(N_{z\theta} - N_{\theta z}) - a\,(N_z - N_\theta) \left(\frac{\partial v}{\partial z} + \frac{1}{a} \frac{\partial w}{\partial \theta} \right) \qquad (28)$$

As is customary in such problems the stretching of the middle surface is taken into account by substituting in equations (23) to (28)

$N_z\ (1 + \epsilon_\theta)$ for N_z,

$N_\theta\ (1 + \epsilon_z)$ for N_θ,

and multiplying the surface forces by

$(1 + \epsilon_\theta)\ (1 + \epsilon_z)$.

In these expressions

$$\epsilon_\theta = \frac{1}{a}\frac{\partial v}{\partial \theta} + \frac{u}{a} \quad \text{and}$$

$$\epsilon_z = \frac{\partial w}{\partial z}.$$

N_z and N_θ of equations (23) to (28) are replaced by $\left(\frac{P}{2\pi\ (a + b)} + \Delta N_z\right)$ and $[qa\ (1 - k) + \Delta N_\theta]$, and in the corresponding equations for the inner facing N_z' and N_θ' are replaced by $\left(\frac{P}{2\pi\ (a + b)} + \Delta N_z'\right)$ and $(qak + \Delta N_\theta')$. This is necessary because the forces in the buckled shell are the prebuckling forces plus the forces due to buckling. The ΔN_z, $\Delta N_z'$, ΔN_θ, and $\Delta N_\theta'$ are the forces due to buckling which are later to be expressed in terms of displacements.

All forces, moments, and twists other than the prebuckling forces are considered to be small quantities resulting from the buckling. The displacements u, v and w, and their derivatives, are also small quantities resulting from the buckling. In equations (24) to (28) products of any two such small quantities are neglected. Equation (26) is solved for Q_θ and equation (27) for Q_z. The results are substituted into equations (23), (24) and (25). This gives:

$$\Sigma F_z = 0 = a\frac{\partial(\Delta N_z)}{\partial z} + \frac{\partial N_{\theta z}}{\partial \theta} - qa(1-k)\left(\frac{\partial^2 v}{\partial z \partial \theta} + \frac{\partial u}{\partial z}\right) + a\,Z \tag{29}$$

$$\Sigma F_\theta = 0 = a\frac{\partial N_{z\theta}}{\partial z} + \frac{\partial(\Delta N_\theta)}{\partial \theta} + \frac{\partial M_{z\theta}}{\partial z} - \frac{1}{a}\frac{\partial M_\theta}{\partial \theta} - N_z\frac{\partial^2 w}{\partial z \partial \theta} + a\,\theta \tag{30}$$

$$\Sigma F_r = 0 = -a\frac{\partial^2 M_z}{\partial z^2} - \frac{\partial^2 M_{\theta z}}{\partial \theta \partial z} + \frac{\partial^2 M_{z\theta}}{\partial z \partial \theta} - \frac{1}{a}\frac{\partial^2 M_\theta}{\partial \theta^2} + a\,N_z\frac{\partial^2 u}{\partial z^2} - qa(1-k)$$

$$\left(1 - \frac{1}{a}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{a}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z}\right) + \Delta N_\theta + a\,R\left(1 - \frac{1}{a}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{a}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z}\right) = 0 \tag{31}$$

(These equations are for the outer facing. A similar set is obtained for the inner facing.) Into equations (29), (30), and (31) expressions for the forces, moments and twists in terms of the displacements (ref. 3) are substituted. The surface forces

$R = qa - (q\frac{a}{r}k + \Delta\sigma_r)_{r=a}$ (for outer facing) where $q\frac{a}{r}k$ is the prebuckling stress and $\Delta\sigma_r$ the stress due to buckling,

$\theta = -(\tau_{r\theta})_{r=a}$ (for outer facing), and

$Z = -(\tau_{rz})_{r=a}$ (for outer facing),

are also expressed in terms of u, v and w and substituted into the three equations.

This leads to three equations in terms of u, v, and w for the outer facing and three similar equations for the inner facing. The equations for the outer facing are:

$$\Sigma F_z = 0 = a^2 \frac{\partial^2 w}{\partial z^2} + \frac{1-\mu}{2} \frac{\partial^2 w}{\partial \theta^2} + a \frac{1+\mu}{2} \frac{\partial^2 v}{\partial z \partial \theta} + a\mu \frac{\partial u}{\partial z} - a\,\phi_1 (1-k) \left(\frac{\partial^2 v}{\partial z \partial \theta} + \frac{\partial u}{\partial z} \right)$$

$$- a^2 \frac{(1-\mu^2)}{Et} G_{rz} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) \tag{32}$$

$$\Sigma F_\theta = 0 = \frac{1+\mu}{2} a \frac{\partial^2 w}{\partial z \partial \theta} + (1+\alpha) \frac{\partial^2 v}{\partial \theta^2} + \frac{1-\mu}{2} a^2 \frac{\partial^2 v}{\partial z^2} + \alpha (1-\mu) a^2 \frac{\partial^2 v}{\partial z^2} + \frac{\partial u}{\partial \theta} - a^2 \alpha$$

$$\frac{\partial^3 u}{\partial^2 z \partial \theta} - \alpha \frac{\partial^3 u}{\partial \theta^3} - a\,\phi_2 \frac{\partial^2 w}{\partial z \partial \theta} - a^2 \frac{1-\mu^2}{Et} G_{r\theta} \left[\frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \right] \tag{33}$$

$$\Sigma F_r = 0 = -a\mu \frac{\partial w}{\partial z} - \frac{\partial v}{\partial \theta} - u + \alpha \frac{\partial^3 v}{\partial \theta^3} + (2-\mu)\,\alpha a^2 \frac{\partial^3 v}{\partial \theta \partial z^2} - \alpha a^4 \frac{\partial^4 u}{\partial z^4} - \alpha \frac{\partial^4 u}{\partial \theta^4}$$

$$- 2\alpha a^2 \frac{\partial^4 u}{\partial \theta^2 \partial z^2} + a^2 \phi_2 \frac{\partial^2 u}{\partial z^2} - a\,\phi_1 (1-k) \left(1 - \frac{1}{a} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{a} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} \right) -$$

$$a^2 \frac{(1-\mu^2)}{Et} \left(qa - aq\,k - E_c \frac{\partial u}{\partial r} \right) \left(1 + \frac{1}{a} \frac{\partial v}{\partial \theta} + \frac{u}{a} + \frac{\partial w}{\partial z} \right). \tag{34}$$

To achieve proper interaction, between the core and the facings, the displacements of the middle surfaces of the facings are set equal to the displacements of the core at $r = a$ and $r = b$.

Thus displacements u, v and w in equations (32), (33), and (34) are replaced by equations (20), (21), and (22) with r made equal to a. In this manner three equations in six arbitrary constants (A, B, C, D, L, and F) are written for the outer facing. In a like fashion three equations are written for the inner facing. The coefficients of the six arbitrary constants are shown in the form of a determinant on the following page.

$a\,[-\mu\lambda+\lambda\phi_1(1-k)]-$ $an\,[-\frac{1+\mu}{2}n\lambda+n\lambda\phi_1(1-k)]+$ $\lambda a\,[-\lambda^2-\frac{1-\mu}{2}n^2]$	$a\,[-\mu\lambda+\lambda\phi_1(1-k)]+$ $\frac{a\lambda}{2}[-\lambda^2-\frac{1-\mu}{2}n^2]+$ $\frac{a}{\lambda}\beta$	$a\,[-\mu\lambda+\lambda\phi_1(1-k)]+$ $\frac{a}{n}\delta_{n0}[-\frac{1+\mu}{2}n\lambda+n\lambda\phi_1(1-k)]$	$-an\,[-\frac{1+\mu}{2}n\lambda+n\lambda\phi_1(1-k)]-$ $\lambda a\,[-\lambda^2-\frac{1-\mu}{2}n^2]$	$a\,[-\frac{1+\mu}{2}n\lambda+n\lambda\phi_1(1-k)]$	$a\,[-\lambda^2-\frac{1-\mu}{2}n^2]$
$a\,[-\mu\lambda\frac{b}{a}+\lambda\frac{b}{a}\phi_1 k]-$ $na\,[-\frac{1+\mu}{2}n\lambda\frac{b}{a}+n\lambda\frac{b}{a}\phi_1 k]+$ $\lambda b\,[-\lambda^2\frac{b^2}{a^2}-\frac{1-\mu}{2}n^2]$	$b\,[-\mu\lambda\frac{b}{a}+\lambda\frac{b}{a}\phi_1 k]+$ $nb\log\frac{b}{a}[-\frac{1+\mu}{2}n\lambda\frac{b}{a}+n\lambda\frac{b}{a}\phi_1 k]+$ $a\lambda\,[\frac{b^2}{2a^2}-\frac{\delta_z}{\lambda^2}\log\frac{b}{a}]\,[-\lambda^2\frac{b^2}{a^2}-\frac{1-\mu}{2}n^2]-\frac{b}{\lambda}\beta$	$\frac{a^2}{b}[-\mu\lambda\frac{b}{a}+\lambda\frac{b}{a}\phi_1 k]+$ $\frac{1}{n}\frac{a^2}{b}\delta_{n0}[-\frac{1+\mu}{2}n\lambda\frac{b}{a}+n\lambda\frac{b}{a}\phi_1 k]+$ $\lambda a\log\frac{b}{a}[-\lambda^2\frac{b^2}{a^2}-\frac{1-\mu}{2}n^2]$	$a\log\frac{b}{a}[-\mu\lambda\frac{b}{a}+\lambda\frac{b}{a}\phi_1 k]$ $an\,(\log\frac{b}{a}+1)[-\frac{1+\mu}{2}n\lambda\frac{b}{a}+n\lambda\frac{b}{a}\phi_1 k]+$ $\lambda b\,(\log\frac{b}{a}-1)[-\lambda^2\frac{b^2}{a^2}-\frac{1-\mu}{2}n^2]$	$b\,[-\frac{1+\mu}{2}n\lambda\frac{b}{a}+n\lambda\frac{b}{a}\phi_1 k]$	$a\,[-\lambda^2\frac{b^2}{a^2}-\frac{1-\mu}{2}n^2]$
$a\,[-n-\alpha n^3-\alpha n\lambda^2]-$ $na\,[-\frac{1-\mu}{2}\lambda^2-n^2-\alpha n^2-(1-\mu)\alpha\lambda^2]+$ $\lambda a\,[-\frac{1+\mu}{2}n\lambda+\lambda n\phi_2]$	$a\,[-n-\alpha n^3-\alpha n\lambda^2]+$ $\frac{a\lambda}{2}[-\frac{1+\mu}{2}n\lambda+\lambda n\phi_2]$	$a\,[-n-\alpha n^3-\alpha n\lambda^2]+$ $\frac{a}{n}\delta_{n0}[-\frac{1-\mu}{2}\lambda^2-n^2-\alpha n^2(1-\mu)\alpha\lambda^2]$ $\frac{a}{n}\beta$	$-an\,[-\frac{1-\mu}{2}\lambda^2-n^2-\alpha n^2-(1-\mu)\alpha\lambda^2]-$ $\lambda a\,[-\frac{1+\mu}{2}n\lambda+\lambda n\phi_2]$	$a\,[-\frac{1-\mu}{2}\lambda^2-n^2-\alpha n^2-(1-\mu)\alpha\lambda^2]$	$a\,[-\frac{1+\mu}{2}n\lambda+\lambda n\phi_2]$
$a\,[-n-\alpha' n^3-\alpha'\lambda^2\frac{b^2}{a^2}n]-$ $na\,[-\frac{1-\mu}{2}\lambda^2\frac{b^2}{a^2}-n^2-\alpha' n^2-(1-\mu)\alpha'\lambda^2\frac{b^2}{a^2}]+$ $\lambda b\,[-\frac{1+\mu}{2}n\lambda\frac{b}{a}+\lambda\frac{b}{a}n\phi_2]$	$b\,[-n-\alpha' n^3-\alpha'\lambda^2\frac{b^2}{a^2}n]+$ $nb\log\frac{b}{a}[-\frac{1-\mu}{2}\lambda^2\frac{b^2}{a^2}-n^2-\alpha' n^2-(1-\mu)\alpha'\lambda^2\frac{b^2}{a^2}]+$ $a\lambda\,[\frac{b^2}{2a^2}-\frac{\delta_z}{\lambda^2}\log\frac{b}{a}]\,[-\frac{1+\mu}{2}n\lambda\frac{b}{a}+\lambda\frac{b}{a}n\phi_2]$	$\frac{a^2}{b}[-n-\alpha' n^3-\alpha'\lambda^2\frac{b^2}{a^2}n]+$ $\frac{1}{n}\frac{a^2}{b}\delta_{n\theta}[-\frac{1-\mu}{2}\lambda^2\frac{b^2}{a^2}-n^2-\alpha' n^2-(1-\mu)\alpha'\lambda^2\frac{b^2}{a^2}]+$ $\lambda a\log\frac{b}{a}[-\frac{1+\mu}{2}n\lambda\frac{b}{a}+\lambda\frac{b}{a}n\phi_2]-\frac{a}{n}\beta$	$a\log\frac{b}{a}[-n-\alpha' n^3-\alpha'\lambda^2\frac{b^2}{a^2}n]-$ $an\,(\log\frac{b}{a}+1)[-\frac{1-\mu}{2}\lambda^2\frac{b^2}{a^2}-n^2-\alpha' n^2-(2-\mu)\alpha'\lambda^2\frac{b^2}{a^2}]+$ $\lambda b\,(\log\frac{b}{a}-1)[-\frac{1+\mu}{2}n\lambda\frac{b}{a}+\lambda\frac{b}{a}n\phi_2]$	$b\,[-\frac{1-\mu}{2}\lambda^2\frac{b^2}{a^2}-n^2-\alpha' n^2-(1-\mu)\alpha'\lambda^2\frac{b^2}{a^2}]$	$a\,[-\frac{1+\mu}{2}n\lambda\frac{b}{a}+\lambda\frac{b}{a}n\phi_2]$
$a\,[-1-\alpha\lambda^4-\alpha n^4-2\alpha\lambda^2n^2-\lambda^2\phi_2+(1-k)\phi_1-(1-k)n^2\phi_1]-$ $na\,[-n-\alpha n^3-(2-\mu)\alpha n\lambda^2]+$ $\lambda a\,[-\mu\lambda]$	$a\,[-1-\alpha\lambda^4-\alpha n^4-2\alpha\lambda^2n^2-\lambda^2\phi_2+(1-k)\phi_1-(1-k)n^2\phi_1]+$ $\frac{a\lambda}{2}[-\mu\lambda]-$ $a\beta$	$a\,[-1-\alpha\lambda^4-\alpha n^4-2\alpha\lambda^2n^2-\lambda^2\phi_2+(1-k)\phi_1-(1-k)n^2\phi_1]+$ $\frac{a}{n}\delta_{n\theta}[-n-\alpha n^3-(2-\mu)\alpha n\lambda^2]+$ $a\beta$	$-an\,[-n-\alpha n^3-(2-\mu)\alpha n\lambda^2]-$ $\lambda a\,[-\mu\lambda]-$ $a\beta$	$a\,[-n-\alpha n^3-(2-\mu)\alpha n\lambda^2]$	$a\,[-\mu\lambda]$
$a\,[-1-\alpha'\lambda^4\frac{b^4}{a^4}-\alpha' n^4-2\alpha'\lambda^2\frac{b^2}{a^2}n^2-\lambda^2\frac{b^2}{a^2}\phi_2+k\phi_1-kn^2\phi_1]-$ $na\,[-n-\alpha' n^3-(2-\mu)\alpha' n\lambda^2\frac{b^2}{a^2}]+$ $\lambda b\,[-\mu\lambda\frac{b}{a}]$	$b\,[-1-\alpha'\lambda^4\frac{b^4}{a^4}-\alpha' n^4-2\alpha'\lambda^2\frac{b^2}{a^2}n^2+k\phi_1-kn^2\phi_1]+$ $nb\log\frac{b}{a}[-n-\alpha' n^3-(2-\mu)\alpha' n\lambda^2\frac{b^2}{a^2}]+$ $a\lambda\,[\frac{b^2}{2a^2}-\frac{\delta_z}{\lambda^2}\log\frac{b}{a}]\,[-\mu\lambda\frac{b}{a}]+$ $+\frac{b^2}{a}\beta$	$\frac{a^2}{b}[-1-\alpha'\lambda^4\frac{b^4}{a^4}-\alpha' n^4-2\alpha'\lambda^2\frac{b^2}{a^2}n^2-\lambda^2\frac{b^2}{a^2}\phi_2+k\phi_1-kn^2\phi_1]+$ $\frac{1}{n}\frac{a^2}{b}\delta_{n\theta}[-n-\alpha' n^3-(2-\mu)\alpha' n\lambda^2\frac{b^2}{a^2}]+$ $\lambda a\log\frac{b}{a}[-\mu\lambda\frac{b}{a}]-$ $a\beta$	$a\log\frac{b}{a}[-1-\alpha'\lambda^4\frac{b^4}{a^4}-\alpha' n^4-2\alpha'\lambda^2\frac{b^2}{a^2}n^2-\lambda^2\frac{b^2}{a^2}\phi_2+k\phi_1-kn^2\phi_1]-$ $an\,(\log\frac{b}{a}+1)[-n-\alpha' n^3-(2-\mu)\alpha' n\lambda^2\frac{b^2}{a^2}]+$ $\lambda b\,(\log\frac{b}{a}-1)[-\mu\lambda\frac{b}{a}]+$ $b\beta$	$a\,[-n-\alpha' n^3-(2-\mu)\alpha' n\lambda^2\frac{b^2}{a^2}]$	$a\,[-\mu\lambda\frac{b}{a}]$

It is possible to find simultaneous values of ϕ_1 and ϕ_2 for which these six equations will be satisfied for any values of the arbitrary constants. Mathematically this means that for such a combination of loads the deflections are indeterminate. The shell becomes elastically unstable and the loads that bring about this condition are called critical loads.

Numerical Computations

A literal solution of the sixth order determinant for the eigenvalues is not feasible. A numerical solution, from which curves may be drawn, is possible if a digital computer is used. A CPC Model 2 was available to make computations. Even with the CPC the task seemed overwhclming. If, however, E_c is made infinite, some of thc terms of determinant vanish. The assumption that E_c is infinite is common in work with sandwich construction and has been found to give satisfactory results in most cases. The sixth order determinant with E_c made infinite is represented below:

$$\begin{array}{cccccc}
A_1 & B_1 & C_1 & 0 & F_1 & L_1 \\
A_2 & B_2 & C_2 & 0 & F_2 & L_2 \\
A_3 & 0 & C_3 & 0 & F_3 & L_3 \\
A_4 & B_4 & C_4 & 0 & F_4 & L_4 \\
A_5 & B_5 & C_5 & -\gamma & F_5 & L_5 \\
A_6 & B_6 & C_6 & +\frac{b}{a}\gamma & F_6 & L_6
\end{array}$$

This determinant is then reduced to a fourth order determinant shown below:

$$\begin{array}{cccc}
A_1 C_3 - C_1 A_3 & B_1 & C_1 L_3 - L_1 C_3 & F_1 L_3 - L_1 F_3 \\
A_2 C_3 - C_2 A_3 & B_2 & C_2 L_3 - L_2 C_3 & F_2 L_3 - L_2 F_3 \\
A_4 C_3 - C_4 A_3 & B_4 & C_4 L_3 - L_4 C_3 & F_4 L_3 - L_4 F_3 \\
(A_5 \frac{b}{a} + A_6)\, C_3\ - & B_5 \frac{b}{a}\ + & (C_5 \frac{b}{a} + C_6)\, L_3\ - & (F_5 \frac{b}{a} + F_6)\, L_3\ - \\
(C_5 \frac{b}{a} + C_6)\, A_3 & B_6 & (L_5 \frac{b}{a} + L_6)\, C_3 & (L_5 \frac{b}{a} + L_6)\, F_3
\end{array}$$

The determinant is then programmed for the CPC. A trial and error solution is made by substituting values of ϕ_1 or ϕ_2 until a value is found that will make the determinant zero. This was done by finding values on each side of zero and interpolating to find the eigenvalue.

Discussion of Results

Since the problem is solved by numerical methods the results are presented by the curves shown in figures 6, 7, 8 and 9. The values of $\frac{b}{a} = 0.97$ and $\frac{a}{t} = 1,000$ were used for all of the curves.

Figure 6 is a family of curves in which $-\phi_2$ is plotted against $\frac{\ell}{ma}$ for different values of n. In these curves the values $\frac{E}{G_{r\theta}} = 10,000$ and $\frac{E}{G_{rz}} = 1,000$ were used. Such a set of curves is used in the following manner.

Knowing $\frac{\ell}{a}$ of the cylinder one picks a value for m and n. A value of $-\phi_2$ is determined by reading above $\frac{\ell}{ma}$ on the corresponding n curve. This procedure is repeated until the lowest possible value of $-\phi_2$ is found. The axial load under which the cylinder will buckle can then be determined.

The curves of figure 7 differ from those of figure 6 as a result of making α and α' zero. This is equivalent to neglecting the bending stiffnesses of the facings. A comparison of the curves of figure 7 with those of figure 6 shows that for values of $\frac{\ell}{ma}$ greater than 0.15 there is little difference. It can be concluded that only for very short cylinders need the bending stiffnesses of the facings be considered. For $\frac{\ell}{ma}$ less than 0.15 the curves of figure 7 approach

$$-\phi_2 = \frac{G_{rz}\, a\, (1 - \mu^2)\, (1 - \frac{b}{a})^2}{Et \log \frac{b}{a}\, (1 + \frac{b}{a})} \,.$$

This value is obtained by making n and ℓ zero and expanding the determinant. Solving for N_z and replacing $\log \frac{b}{a}$ by the first term of its series expansion shows that

$$N_z = - \frac{(a - b)}{1 + \frac{b}{a}}\, G_{rz} \,.$$

The curves of figure 8 are the result of increasing G_{rz} and $G_{r\theta}$ tenfold. The value of $-\phi_2$ corresponding to

$$N_z = - \frac{(a - b)}{1 + \frac{b}{a}}\, G_{rz}$$

appears as a flattening of the curve in the region of $\frac{\ell}{ma} = 0.01$. For smaller values of $\frac{\ell}{ma}$ the curve rises due to the stiffness of the facings. For values of $\frac{\ell}{ma}$ greater than 0.1 the curves show a considerably lower buckling load.

From a comparison of figures 6 and 8 it appears that as G_{rz} is decreased the buckling load for all cylinders, except those long enough to fail as an Euler column, will approach

$$N_z = - \frac{(a - b)}{1 + \frac{b}{a}}\, G_{rz}$$

This limit has been recognized (ref. 4) as the critical load for shells with a low value of G_{rz}. It should be noted that this load depends only upon the thickness and the modulus of rigidity of the core.

Figure 9 shows curves of $-\phi_1$ plotted against $\frac{\ell}{ma}$ for different values of $n_1 \phi_2$ for these curves was taken to be

$$\phi_2 = \frac{\phi_1}{2\,(1 + \frac{b}{a})}$$

This represents the case for an end load equal to $q\pi a^2$. The situation is like that of a cylinder, with ends, under uniform pressure. The ends of course stiffen the cylinder, but, if the cylinder is not too short, reasonable results can be expected. Since ϕ_1 decreases as $\frac{\ell}{ma}$ is increased it must be concluded that the cylinder will buckle with $m = 1$. The critical pressure can be determined by reading ϕ_1 from the lowest n curve.

Conclusions

Although only a few curves were drawn it is apparent that this analysis is helpful in understanding the effect produced by a variation of the parameters that enter the problem. Further study is required before it will be known whether the actual buckling load may be predicted.

It is felt that the method by which this problem is solved can be applied with advantage to many problems of sandwich construction. Unfortunately in most cases a numerical solution will be required.

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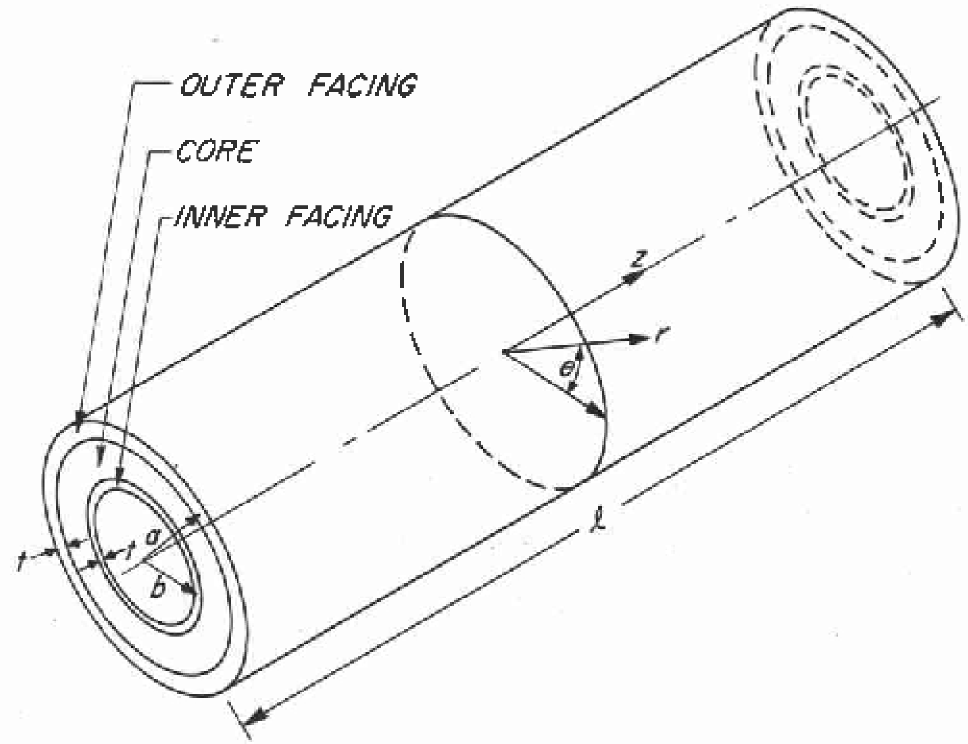

Figure 1. --Sandwich cylinder.

Z M 106 298

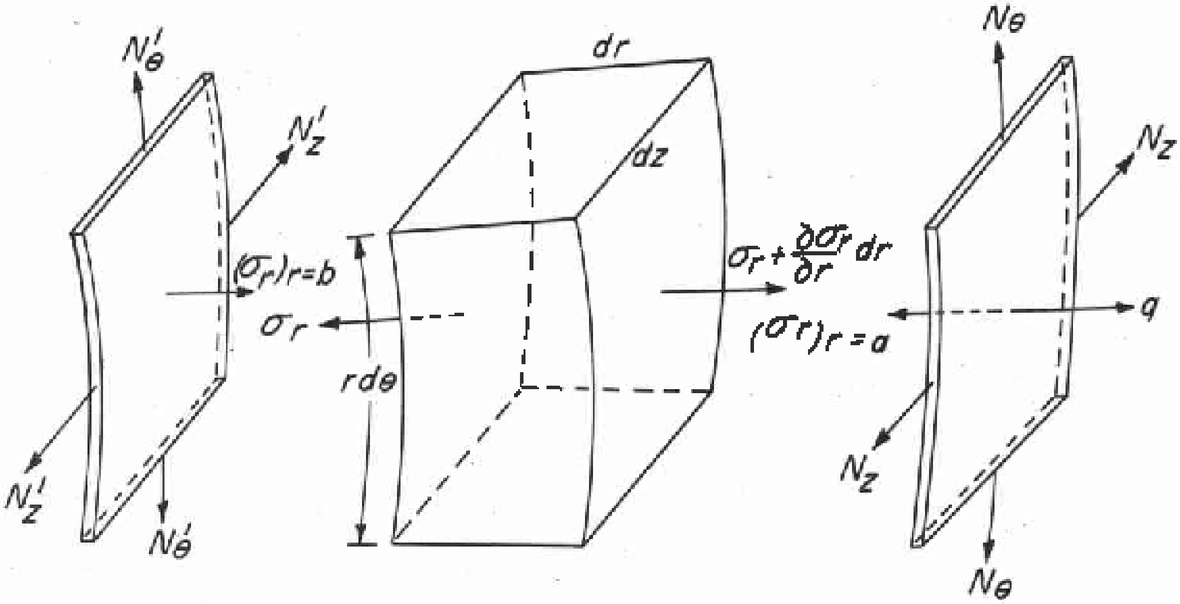

Figure 2. --Differential elements of core and facings before buckling.

Z M 106 299

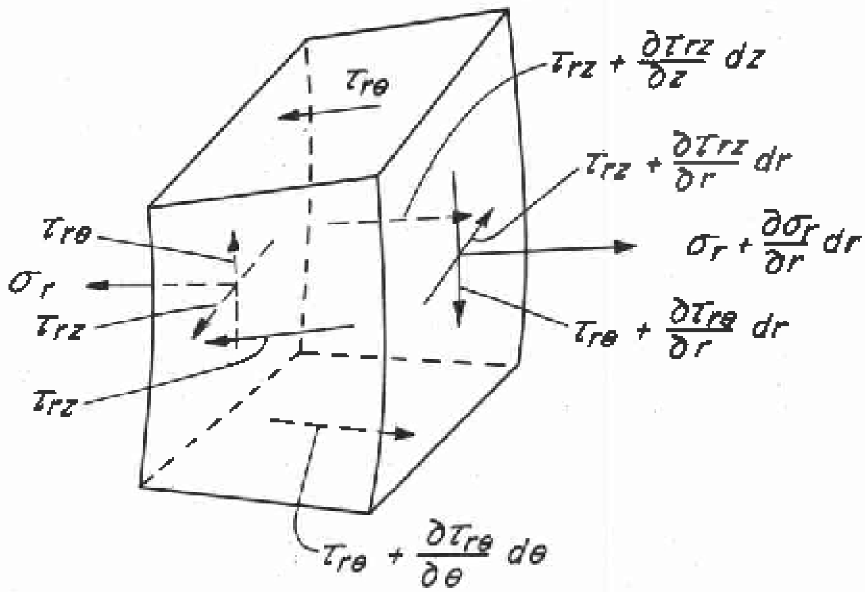

Figure 3. --Differential element of core.

Z M 106 300

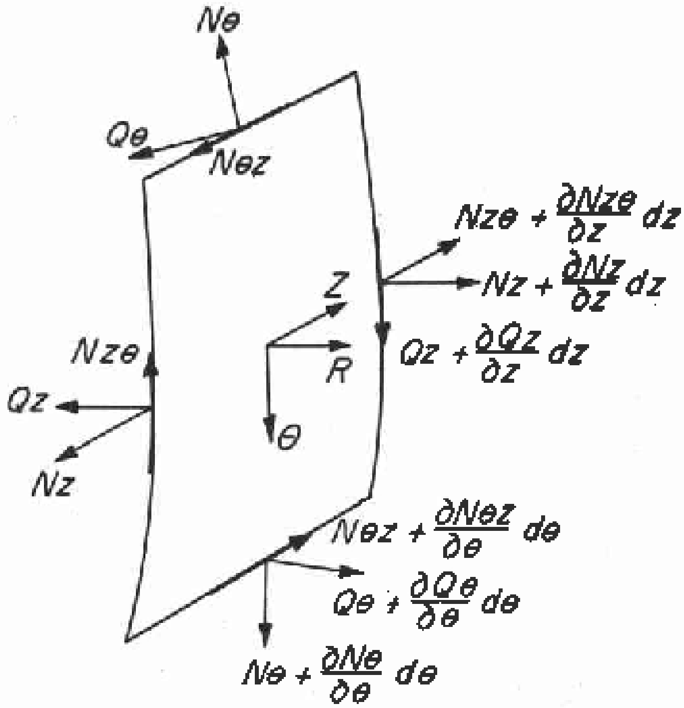

Figure 4.--Differential element of facing showing forces.

Z M 106 301

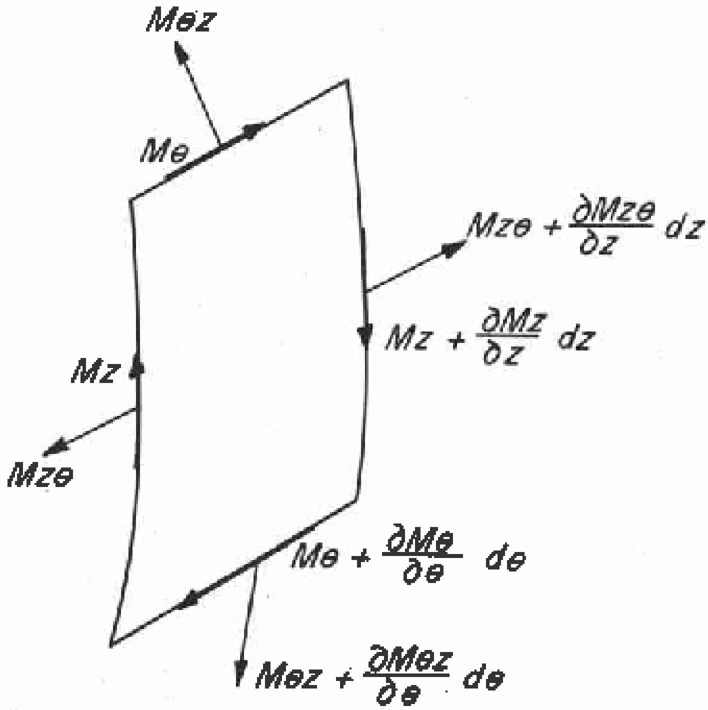

Figure 5. --Differential element of facing showing moments and twists.

Z M 106 302

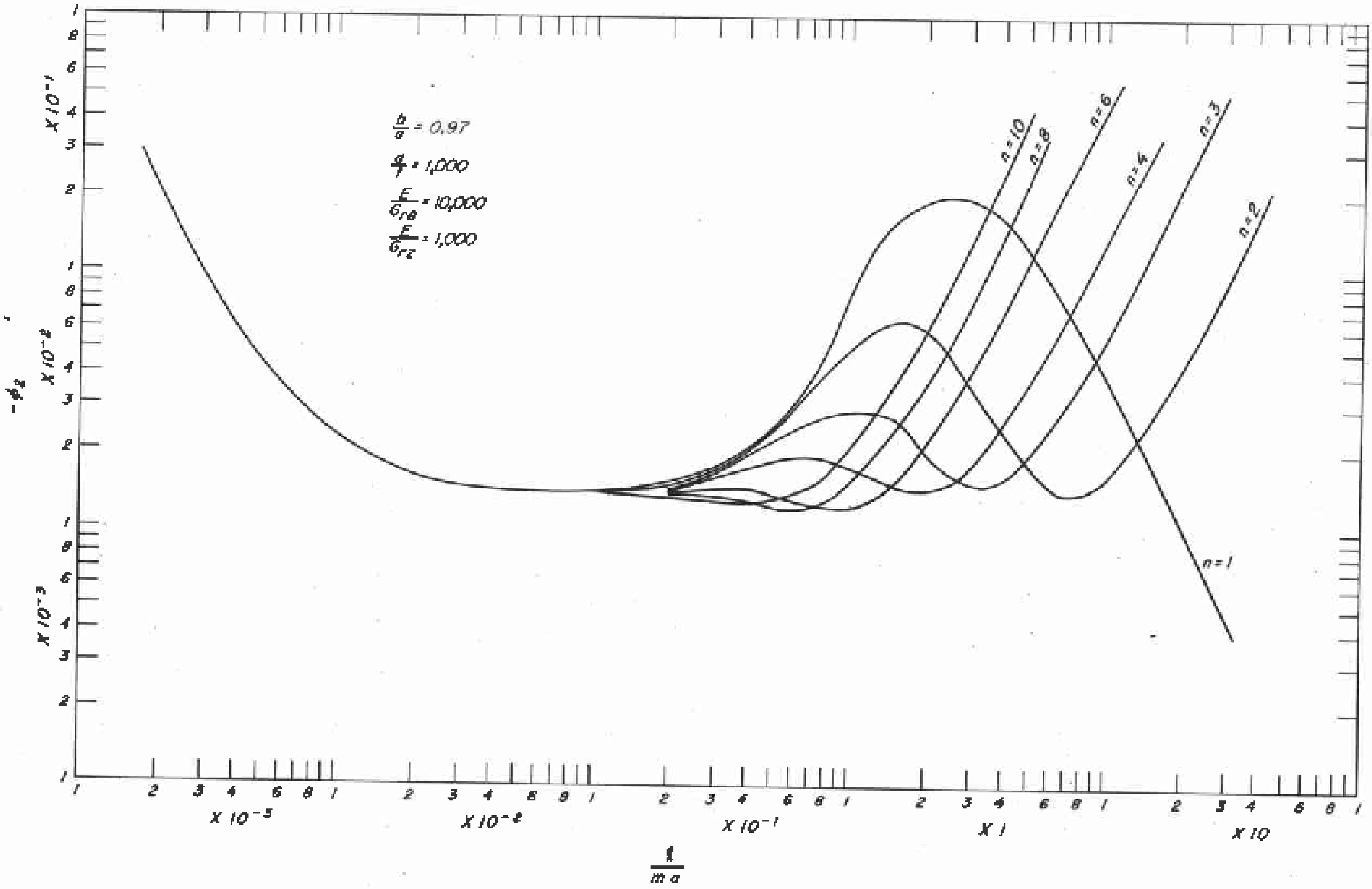

Figure 6. --Critical axial load in terms of ϕ_2 versus $\frac{\ell}{ma}$

Z M 106 303

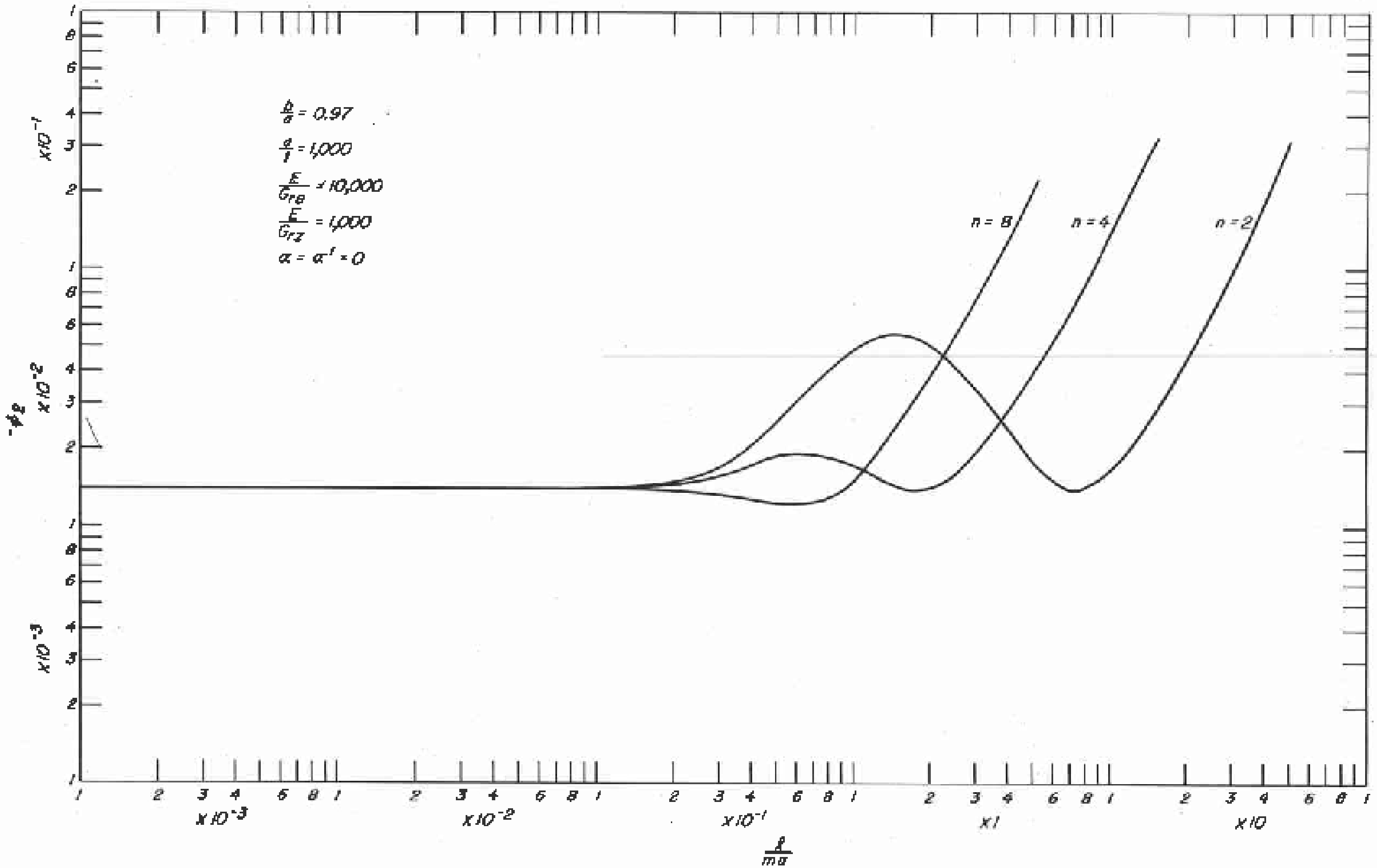

Figure 7. --Critical axial load in terms of ϕ_2 versus $\frac{\ell}{ma}$

Z M 106 305

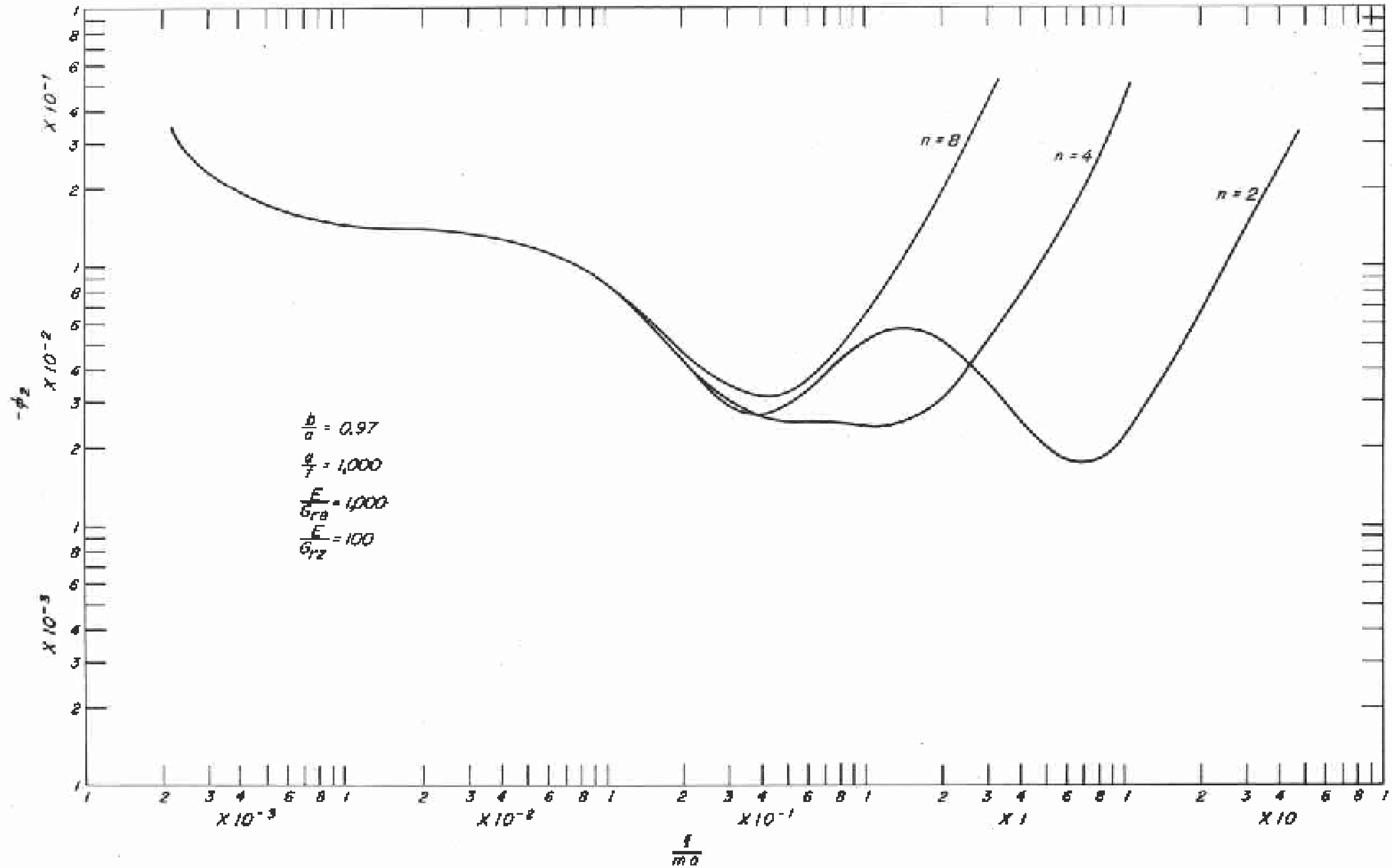

Figure 8. --Critical axial load in terms of ϕ_2 versus $\frac{f}{ma}$

Z M 106 304

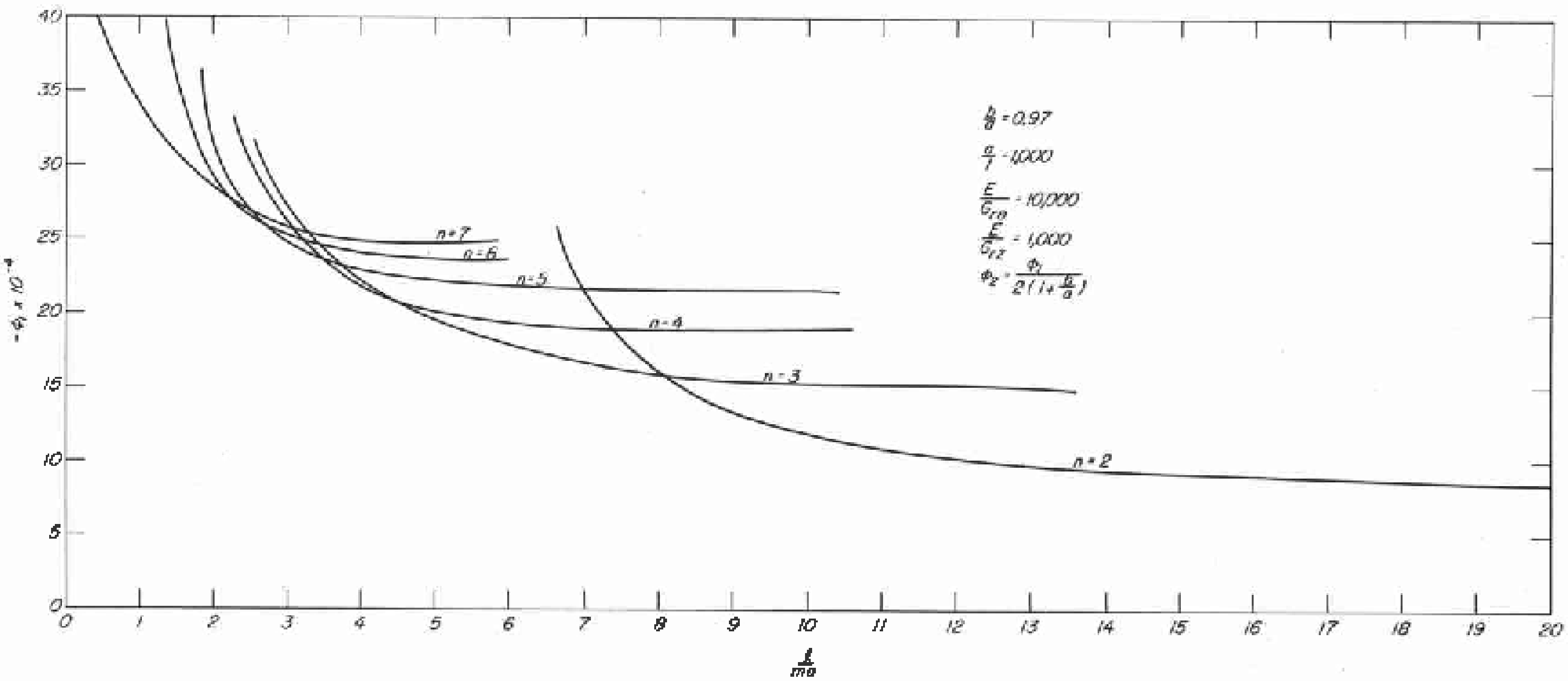

Figure 9. --Critical pressure in terms of ϕ_1 versus $\frac{\ell}{ma}$

Z M 106 306