

An Improved Method of Moments Estimator for TOA Based Localization

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Abstract—Estimation accuracy and computational complexity are two major areas of consideration for localization system design. For time-of-arrival based systems, the 1st-order method of moments (MOM) least-square (LS) estimator is simple to implement, but its performance is much worse than that of some computationally more complex estimators such as the MOM weighted LS (WLS) and nonlinear weighted LS (NLLS-WLS) estimators. In this paper, we develop an improved 1st-order MOM estimator to efficiently utilize the variances of the range measurements and the target position that, as the NLLS estimator, has a performance also approaching the Cramm er-Rao lower bound but is much simpler than MOM-WLS and NLLS-WLS estimators.

Index Terms—Location estimation and method of moments.

I. INTRODUCTION

Location estimation algorithms play a key role in many applications, such as GPS navigation [1] and ultra-wideband indoor localization [2]–[5]. Main areas of consideration for such algorithms include accuracy and complexity [6], [7].

When errors exist in the observed distances between a target and the anchors, the geometric technique will not work well [3], [8]. In this case, two types of least-square (LS) estimators are widely used [3]: one can be viewed as a method of moments (MOM) estimator [9]–[11], which is computationally efficient and easy to implement; the other one is a non-linear least-square (NLLS) estimator [1], [11], which performs better than the MOM estimator through iteration.

The variances of the errors of the observed distances between a target and different anchors are in general unknown and vary. For such cases, weighted LS (WLS) algorithms such as MOM-WLS and NLLS-WLS can be applied. However, these algorithms are computationally very complex, making them difficult to implement in practice.

In this paper, we develop an improved MOM estimator which efficiently utilizes the information of the range measurement noise variance and the target position; in terms of localization accuracy, this estimator also approaches the Cramm er-Rao lower bound (CRLB), like the NLLS-WLS estimator; in terms of computational complexity, this estimator is only slightly more complex than the original MOM-LS estimator and is much simpler than the MOM-WLS and NLLS-WLS estimators. The signal model and the conventional MOM-LS method are described briefly in Sec. II. The improved 1st-order MOM estimator is developed and analyzed in detail in Sec. III.

II. PRELIMINARIES

A. System Model

Let θ be the unknown target location to be estimated by using M anchors (v_1, \dots, v_M). The range measurement from

the m th anchor to the target is written as

$$r_m(i) = d_m + b_m + n_m(i), \quad i = 1, \dots, N; m = 1, \dots, M \quad (1)$$

where $d_m = \|\theta - v_m\|$ is the true distance between the m th anchor and the target, i is the index of the observation set, N is the total number of observation sets, b_m is a positive offset caused by non-line-of-sight (NLOS) propagation, and $n_m(i)$ is the range measurement error, which will be called the noise component sometimes in the rest of the paper for convenience. Typically, for a particular anchor, $n_m(i), i = 1, \dots, N$, are modeled as independent and identically distributed zero-mean Gaussian random variables; for different anchors, the variances of $n_m(i), m = 1, \dots, M$ are different, that is, $n_m(i) \sim \mathcal{N}(0, \sigma_m^2)$.

NLOS propagation often exists between the target and anchors. For links with severe NLOS conditions such that the range measurements contain very little information about the target’s location, there are existing techniques to identify them, and a common way is to discard these links in the localization process [12]–[14]. For less severe NLOS links, NLOS mitigation algorithms can be used to estimate the NLOS offset [1], [15] to construct more accurate range measurements. For clarity, in the derivation and analysis the residual errors after NLOS offset removal will be lumped into the noise term, since it does not affect the development of the algorithm. Simulation will be provided to assess how the proposed algorithm works with NLOS links.

B. First Order MOM Estimator

For 3-D localization, the square of the expectation of $r_m(i)$ given by Eq. (1) is written as

$$\begin{aligned} \bar{r}_m^2 &= \{E(r_m(i))\}^2 \\ &= (x - x_m)^2 + (y - y_m)^2 + (z - z_m)^2. \end{aligned} \quad (2)$$

Without loss of generality, let v_1 be the reference position. Subtracting it from Eq. (2) yields

$$\bar{r}_{m,1}^2 = 2x(x_m - x_1) + 2y(y_m - y_1) + 2z(z_m - z_1) \quad (3)$$

where $\bar{r}_{m,1}^2 = -\bar{r}_m^2 + \bar{r}_1^2 + x_m^2 + y_m^2 + z_m^2 - x_1^2 - y_1^2 - z_1^2$, $m = 2, \dots, M$. We further define

$$\bar{r}_{M1} = [\bar{r}_{2,1}^2, \bar{r}_{3,1}^2, \dots, \bar{r}_{M,1}^2]^T, \quad (4)$$

$$\mathbf{H}_{M1} = 2 \begin{bmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ \vdots & \vdots & \vdots \\ x_M - x_1 & y_M - y_1 & z_M - z_1 \end{bmatrix}. \quad (5)$$

In vector-matrix form, \bar{r}_{M1} is written as

$$\bar{r}_{M1} = \mathbf{H}_{M1}\theta.$$

The LS estimator is expressed as

$$\hat{\theta}_{M1} = \mathbf{H}_{LS}\bar{r}_{M1} \quad (6)$$

where $\mathbf{H}_{LS} = (\mathbf{H}_{M1}^T \mathbf{H}_{M1})^{-1} \mathbf{H}_{M1}^T$.

A 2-D localization case is simulated with the z -axis fixed at zero, since it is easy to illustrate the results clearly with 2-D positioning. Fig. 1 shows the positions of the four anchors ($\{\pm 1, \pm 1\}$) and 25 targets ($x, y = [-0.6, -0.3, 0, 0.3, 0.6]$) with indices $[1, 2, \dots, 25]$. The estimated target positions obtained by using the MOM-LS estimator are also shown in Fig. 1. The mean-square errors (MSE) of the estimated position and the CRLB are shown in Fig. 2.

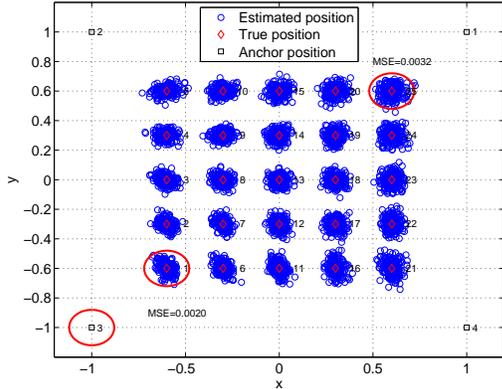


Fig. 1. Layout of anchor positions and target positions for simulation, together with the estimated target positions by employing the MOM-LS estimator ($\mathbf{v}_{\text{ref}} = [-1, -1]$, noise variances are randomly chosen as $\sigma^2 = [0.063, 0.141, 0.070, 0.061]$, and $N = 50$).

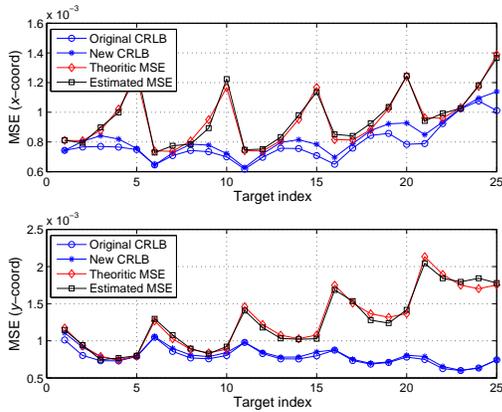


Fig. 2. MSE of the MOM-LS estimator and CRLB ($\mathbf{v}_{\text{ref}} = [-1, -1]$, noise variances are randomly chosen as $\sigma^2 = [0.063, 0.141, 0.070, 0.061]$, and $N = 50$).

The following observations are made from these results:

- 1) The MSE curve for all the target locations is not smooth; for targets that are close to the reference position, their MSE values are much smaller than those of the targets away from the reference position.
- 2) For most target locations, the MSE is much larger than the CRLB.

III. IMPROVED 1ST ORDER MOM ESTIMATOR

A. Analysis with Noise Considered

In the derivation of the 1st-order MOM-LS estimator in Sec. II-B, the effect of noise is neglected because of the process of taking expectation over the range measurements.

Taking time average of $r_m(i)$, rather than expectation as in the normal 1st-order MOM estimator, with noise kept in the analysis, we have

$$\hat{r}_m = \|\boldsymbol{\theta} - \mathbf{v}_m\| + \frac{1}{N} \sum_{i=1}^N n_m(i). \quad (7)$$

The square of \hat{r}_m is obtained as

$$\hat{r}_m^2 = d_m^2 + 2 \frac{d_m}{N} \sum_{i=1}^N n_m(i) + \left(\frac{1}{N} \sum_{i=1}^N n_m(i) \right)^2. \quad (8)$$

It is reasonable to assume that compared to the actual distance d_m , the noise component is small. This allows us to obtain an approximated value of \hat{r}_m^2 , by ignoring the third term, $\left(\frac{1}{N} \sum_{i=1}^N n_m(i) \right)^2$, as

$$\hat{r}_m^2 \approx d_m^2 + 2 \frac{d_m}{N} \sum_{i=1}^N n_m(i). \quad (9)$$

Let \mathbf{v}_1 be the reference position. Subtracting \hat{r}_1^2 from Eq. (9) yields

$$\hat{r}_{m,1}^2 = 2x(x_m - x_1) + 2y(y_m - y_1) + 2z(z_m - z_1) + \hat{n}_1 - \hat{n}_m \quad (10)$$

where

$$\begin{aligned} \hat{r}_{m,1}^2 &= -\hat{r}_m^2 + \hat{r}_1^2 + x_m^2 + y_m^2 + z_m^2 - x_1^2 - y_1^2 - z_1^2, \\ \hat{n}_m &= 2 \frac{d_m}{N} \sum_{i=1}^N n_m(i) \sim \mathcal{N}(0, 4d_m^2 \sigma_m^2 / N). \end{aligned}$$

The covariance matrix is expressed as

$$\mathbf{C}_{M1} = \frac{4}{N} (\text{diag}([d_2^2 \sigma_2^2, \dots, d_M^2 \sigma_M^2]) + d_1^2 \sigma_1^2 \mathbf{1} \mathbf{1}^T) \quad (11)$$

where $\text{diag}(\cdot)$ denotes a diagonal matrix and $\mathbf{1}$ denotes the $(M-1) \times 1$ vector with all elements equal to 1.

The new CRLB with a squaring step in the MOM estimation for the layout in Fig. 1 is shown in Fig. 2; it is observed that the squaring step virtually has no effects on noise, since the new CRLB is very close to the original CRLB.

The covariance matrix in Eq. (11) is not an identity matrix, and the WLS algorithm can be applied for best performance. However, since Eq. (11) contains the unknown variable $\boldsymbol{\theta}$, WLS method requires iteration [16] to estimate the variances and the target position, which makes it computationally a lot more complex than the LS algorithm (Sec. III-D).

B. Improved Approach

Let us re-examine the MSE of LS estimates:

$$\text{MSE}(\hat{\boldsymbol{\theta}}_{M1}) = \mathbf{H}_{LS} \mathbf{C}_{M1} \mathbf{H}_{LS}^T. \quad (12)$$

Eqs. (5), (11), and (12) show that the MSE depends on the reference anchor position as well as $d_m^2 \sigma_m^2$, $m = 1, 2, \dots, M$.

Without resorting to iterations to estimate the covariance matrix (Eq. 11), the accuracy of the MOM-LS estimator can still be improved by selecting the optimal reference anchor position (\mathbf{v}_{ref}):

$$\mathbf{v}_{\text{ref}} |_{d_m^2 \sigma_m^2, m=[1, 2, \dots, M]} \underset{\text{argmin}}{\text{tr}} \left(\text{MSE}(\hat{\boldsymbol{\theta}}_{M1}) \right), \quad (13)$$

where $\text{tr}(\cdot)$ denotes the trace of a matrix and

$$\begin{aligned} & \text{tr} \left(\text{MSE}(\hat{\boldsymbol{\theta}}_{\text{M1}}) \right) \\ &= \frac{4}{N} \text{diagv} \left(\mathbf{H}_{\text{LS}}^T \mathbf{H}_{\text{LS}} \right)^T \left[d_2^2 \sigma_2^2, \dots, d_M^2 \sigma_M^2 \right]^T \\ & \quad + \frac{4d_1^2 \sigma_1^2}{N} \text{tr} \left(\mathbf{1}^T \mathbf{H}_{\text{LS}}^T \mathbf{H}_{\text{LS}} \mathbf{1} \right) \end{aligned} \quad (14)$$

where $\text{diagv}(\cdot)$ denotes the column vector formed by the main diagonal of its argument (a matrix).

So if the variance of the range measurements and target positions are known, Eqs. (13) and (14) can be used to find the optimal reference anchor position.

1) *Variance of the range measurements*: When there are many range measurements from each anchor (e.g. $N > 30$), the variance can be easily estimated from range measurements. If estimates of the range variances cannot be obtained, then they are assumed to be the same for all anchors.

2) *Target positions*: The unknown target position can be easily estimated from the initial MOM-LS estimates.

This analysis leads to the following improved 1st-order MOM-LS estimator:

- 1) MOM-LS: Randomly choose one anchor position as the reference position to estimate the target position ($\hat{\boldsymbol{\theta}}_{\text{LS}}$).
- 2) Optimizing: Use $\hat{\boldsymbol{\theta}}_{\text{LS}}$ to calculate the optimal anchor position with Eqs. (13) and (14). Then, use that anchor position as the reference position to estimate $\hat{\boldsymbol{\theta}}$.

C. Estimation Performance

The first term of Eq. (14) represents the MSE of the ideal LS estimator when the observed distances from the reference anchor position (\mathbf{v}_1) to the target is error free; the second term is the MSE caused by the noise.

Evaluating Eq. (14) requires the inverse of a non-diagonal matrix. For a specific anchor layout, we apply a coordinate transformation ($[x, y, z] \rightarrow [x', y', z']$) such that:

- 1) The reference anchor is at the origin ($\mathbf{v}_1 = [0, 0, 0]$).
- 2) $\frac{1}{M} \sum_m y'_m \approx 0$, $\frac{1}{M} \sum_m z'_m \approx 0$, $\frac{1}{M} \sum_m x'_m y'_m \approx 0$, $\frac{1}{M} \sum_m x'_m z'_m \approx 0$, $\frac{1}{M} \sum_m y'_m z'_m \approx 0$, assuming that there are a large number of anchors and anchor layout is optimized [17].

In this case, $\left(\mathbf{H}_{\text{M1}}^T \mathbf{H}_{\text{M1}} \right)^{-1}$ becomes a diagonal matrix. For Eq. (14), it is easy to show that

$$\begin{aligned} & \frac{4}{N} \text{tr} \left(\mathbf{H}_{\text{LS}}^T \mathbf{H}_{\text{LS}} \right)^T \underset{m}{\text{argmin}} (d_m^2 \sigma_m^2) \\ & \leq \frac{4}{N} \text{diagv} \left(\mathbf{H}_{\text{LS}}^T \mathbf{H}_{\text{LS}} \right)^T \left[d_2^2 \sigma_2^2, \dots, d_M^2 \sigma_M^2 \right]^T \\ & \leq \frac{4}{N} \text{tr} \left(\mathbf{H}_{\text{LS}}^T \mathbf{H}_{\text{LS}} \right)^T \underset{m}{\text{argmax}} (d_m^2 \sigma_m^2), \end{aligned} \quad (15)$$

and

$$\text{tr} \left(\mathbf{H}_{\text{LS}}^T \mathbf{H}_{\text{LS}} \right)^T = \frac{1}{\sum_m x'^2_m} + \frac{1}{\sum_m y'^2_m} + \frac{1}{\sum_m z'^2_m}. \quad (16)$$

Similarly

$$0 \leq \text{tr} \left(\mathbf{1}^T \mathbf{H}_{\text{LS}}^T \mathbf{H}_{\text{LS}} \mathbf{1} \right) = \left(\frac{\sum_m x'_m}{\sum_m x'^2_m} \right)^2. \quad (17)$$

From Eqs. (15), (16), and (17), it is found that the second term of Eq. (14) cannot be ignored.

The first term of Eq. (14) can also be viewed as the MSE from the ideal LS estimator by removing the reference anchor position. For simplicity, all anchors are assumed equally important in determining the system performance when range measurements are noise free. Thus, the second term of Eq. (14) will dominate the performance. For the same reason, the values of Eq. (17) for different anchor positions will be approximately equal. Therefore, choosing $d_1^2 \sigma_1^2 = \underset{m}{\text{argmin}} (d_m^2 \sigma_m^2)$ will apparently decrease the overall MSE of the system. The maximum reduction in MSE with the proposed method is

$$\frac{1}{N} \left(\frac{\sum_m x'_m}{\sum_m x'^2_m} \right)^2 \left(\underset{m}{\text{argmax}} (d_m^2 \sigma_m^2) - \underset{m}{\text{argmin}} (d_m^2 \sigma_m^2) \right). \quad (18)$$

D. Computational Complexity

In the comparison, the number of multiplications is used to represent the computational complexity, and the multiplications required for estimating the noise variance are not included, since they are same for all methods.

Since \mathbf{H}_{LS} only depends on the anchor position, which can be calculated in advance, the MOM-LS estimator needs approximately $5M$ multiplications and no matrix inversion. The proposed estimator requires approximately $M^2 + 12M$ multiplications and no matrix inversion.

The MOM-WLS needs $2M$ multiplications to initialize. For each iteration, it requires $\approx 3M^2 + 10M$ multiplications and inversion of two matrices (3×3 and $(M-1) \times (M-1)$). In a typical scenario with 10 iterations, it requires $\approx 30M^2 + 102M$ multiplications and inversion of 20 matrices.

The NLLS-WLS estimator requires approximately $23M$ multiplications and inversion of one 3×3 matrix per iteration [1], [11]. In a typical scenario with 10 iterations, it requires $230M$ multiplications and inversion of 10 (3×3) matrices.

When M is small (e.g., $M < 10$), the proposed algorithm is computationally much more attractive than the NLLS-WLS and MOM-WLS estimators. When M is large (e.g., $M > 50$), we can use the second term of Eq. (14) to select the optimal reference anchor in the proposed algorithm, which results in $13M$ multiplications.

E. Simulation

Fig. 3 shows the MSEs of the proposed method, MOM-LS, and MOM-WLS. The MSE is the averaged value over 2000 runs. For all the target locations, the MSE with the proposed method is quite smooth, and is very close to the CRLB.

Comparison of MSE versus the average variance of the range measurements (σ_{avg}^2) of the proposed method, MOM-LS, MOM-WLS and NLLS-WLS is given in Fig. 4, where $\boldsymbol{\theta} = [0.6, 0.6]$. In the simulation, σ_{avg}^2 is first calculated according to the variance and the average distance between the target and the anchors. For each simulation, noise variance is randomly chosen between $[0.5\sigma_{\text{avg}}^2, 2\sigma_{\text{avg}}^2]$. For each σ_{avg}^2 , the MSE is the averaged value over 20000 runs. It is observed that the MSE of the proposed method is very consistent under different noise levels, and is very close to those of the MOM-WLS and NLLS-WLS.

For NLOS cases, NLOS offsets described in [12] (Type II \sim IV NLOS error) are adopted. After NLOS mitigation [1], [15] and LOS range reconstruction, it is reasonable to assume that

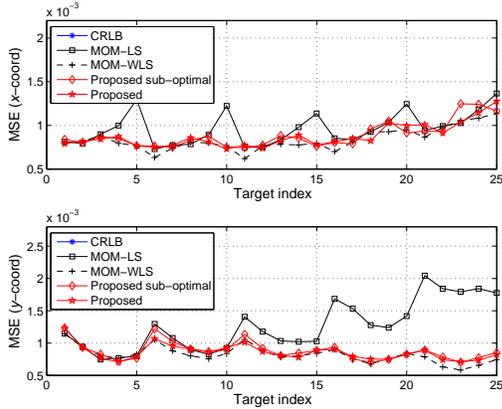


Fig. 3. Simulated MSE curves of the proposed algorithm, MOM-LS ($\mathbf{v}_{\text{ref}} = [-1, -1]$), and MOM-WLS. CRLB is also shown for reference (noise variances are randomly chosen as $\sigma^2 = [0.063, 0.141, 0.070, 0.061]$ and $N = 50$). For the proposed sub-optimal method, the second item of Eq. (14) is used to select the optimal reference anchor position.

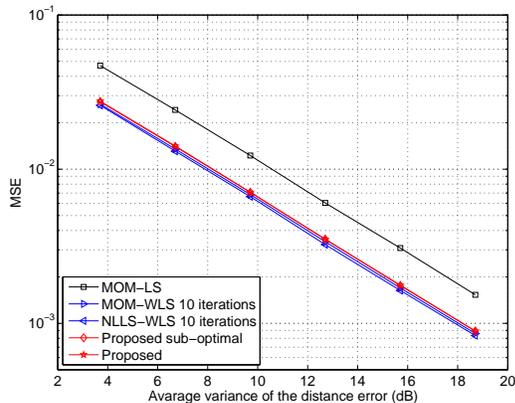


Fig. 4. MSE vs. σ_{avg}^2 of the proposed method, MOM-LS ($\mathbf{v}_{\text{ref}} = [-1, -1]$), MOM-WLS, and NLLS-WLS ($\theta = [0.6, 0.6]$).

the residual NLOS error in the range measurements is small (Type II). Fig. 5 shows the MSE of the range measurements with Type II NLOS offset. It is observed that the proposed method achieves a performance nearly identical to the MOM-WLS and NLLS-WLS algorithms, which is much better than the MOM method, but has a significantly lower computational complexity than MOM-WLS and NLLS-WLS methods.

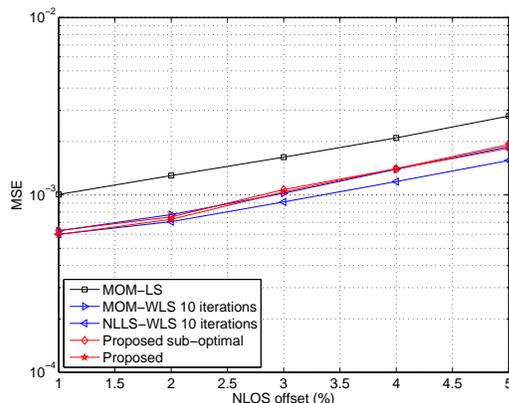


Fig. 5. MSE vs. NLOS offset of the proposed method, MOM-LS ($\mathbf{v}_{\text{ref}} = [-1, -1]$), MOM-WLS, and NLLS-WLS ($\theta = [0.6, 0.6]$, $\sigma^2 = 0.05$). The NLOS offset ($a\%$) denotes the random positive NLOS error with uniform distribution $b_m \sim \mathcal{U}(0, a\% \times d_m)$.

IV. CONCLUSIONS

We have analyzed the 1st-order MOM estimator when noise is taken into consideration. The analysis results have led to an improved MOM estimator with a slightly increased computational complexity compared to the conventional MOM-LS estimator but a performance approaching the CRLB. The MOM-WLS and NLLS-WLS estimators also achieve a performance close to the CRLB, but the complexity of the proposed scheme is a lot lower; for 3-D localization, the proposed method requires approximately $M^2 + 12M$, or $13M$ for large M , multiplications and no matrix inversion is needed, whereas NLLS-WLS requires about $230M$ multiplications and inversion of 10 matrices, assuming a typical value of 10 iterations to reach the close-to-the-CRLB performance.

REFERENCES

- [1] W. Kim, J. Lee, and G. Jee, "The interior-point method for an optimal treatment of bias in trilateration location," *IEEE Trans. Veh. Technol.*, vol. 55, no. 4, pp. 1291–1301, Jul. 2006.
- [2] M. G. D. Benedetto, T. Kaiser, A. F. Molisch, I. Oppermann, C. Politano, and D. Porcino, Eds., *UWB Communication Systems: A Comprehensive Overview*. New York, NY: Hindawi, 2006.
- [3] Z. Sahinoglu, S. Gezici, and I. Guvenc, *Ultra-Wideband Positioning Systems: Theoretical Limits, Ranging Algorithms, and Protocols*. Cambridge, UK: Cambridge University Press, 2008.
- [4] Y. Shen and M. Win, "On the accuracy of localization systems using wideband antenna arrays," *IEEE Trans. Commun.*, vol. 58, no. 1, pp. 270–280, Jan. 2010.
- [5] J. Shen, A. Molisch, and J. Salmi, "Accurate passive location estimation using TOA measurements," *IEEE Trans. Wireless Commun.*, vol. 11, no. 6, pp. 2182–2192, Jun. 2012.
- [6] B. Alavi and K. Pahlavan, "Modeling of the TOA-based distance measurement error using UWB indoor radio measurements," *IEEE Commun. Lett.*, vol. 10, no. 4, pp. 275–277, Apr. 2006.
- [7] C. Falsi, D. Dardari, L. Mucchi, and M. Z. Win, "Time of arrival estimation for UWB localizers in realistic environments," *EURASIP J. Advances in Signal Process.*, vol. 2006, pp. 1–13, Apr. 2006.
- [8] S. Gezici, "A survey on wireless position estimation," *Wireless Personal Commun.*, vol. 44, no. 3, pp. 263–282, Feb. 2008.
- [9] Y. T. Chan, H. Yau Chin Hang, and P. chung Ching, "Exact and approximate maximum likelihood localization algorithms," *IEEE Trans. Veh. Technol.*, vol. 55, no. 1, pp. 10–16, Jan. 2006.
- [10] K. Cheung, H. So, W.-K. Ma, and Y. Chan, "Least squares algorithms for time-of-arrival-based mobile location," *IEEE Trans. Signal Process.*, vol. 52, no. 4, pp. 1121–1130, Apr. 2004.
- [11] S. M. Kay, *Fundamentals of Statistical Processing, Volume I: Estimation Theory*. Englewood Cliffs, NJ: Prentice Hall, 1993.
- [12] E. Casas, A. Marco, J. Guerrero, and Falco, "Robust estimator for non-line-of-sight error mitigation in indoor localization," *EURASIP Journal on Applied Signal Processing*, pp. 1–8, 2006.
- [13] I. Guvenc and C.-C. Chong, "A survey on TOA based wireless localization and NLOS mitigation techniques," *IEEE Commun. Surveys Tuts.*, vol. 11, pp. 107–124, 3rd Quarter 2009.
- [14] S. Venkatesh and R. Buehrer, "NLOS mitigation using linear programming in ultrawideband location-aware networks," *IEEE Trans. Veh. Technol.*, vol. 56, no. 5, pp. 3182–3198, Sep. 2007.
- [15] M. Wylie and J. Holtzman, "The non-line of sight problem in mobile location estimation," in *Proc. IEEE Universal Personal*, vol. 2, 1996, pp. 827–831.
- [16] S. Colonnese, S. Rinauro, and G. Scarano, "Generalized method of moments estimation of location parameters: Application to blind phase acquisition," *IEEE Trans. Signal Process.*, vol. 58, no. 9, pp. 4735–4749, Sep. 2010.
- [17] D. Jourdan, D. Dardari, and M. Win, "Position error bound for UWB localization in dense cluttered environments," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 44, no. 2, pp. 613–628, Apr. 2008.