AN ABSTRACT OF THE THESIS OF

Andrew M. Svesko for the degree of Honors Baccalaureate of Science in Mathematics and Physics presented on May 31st, 2013. Title: Writing *A Detailed Introduction to String Theory*

Abstract approved: ______________________________

Albert Stetz

String theory, one of the more popular approaches to quantizing gravity, is a highly complex theory, involving high level mathematics and physics. But the basic ideas of string theory are not inaccessible, even to the undergraduate. This document acts as report to my thesis project, the writing of *A Detailed Introduction to String Theory*, an undergraduate themed text intended to demystify the basics of string theory. In this report the motivation for, and process of, writing are thoroughly discussed. Excerpts from the text are included. This present document should be viewed as the thesis itself, while the text should be viewed as the 'thesis project'. For a PDF copy of the complete text, visit

http://people.oregonstate.edu/~sveskoa/introductiontostringtheory.pdf.

Writing *A Detailed Introduction to String Theory*

by

Andrew M. Svesko

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Honors Baccalaureate of Science in Mathematics and Physics

Presented May 31st, 2013

Honors Baccalaureate of Science in Mathematics and Physics thesis of Andrew M. Svesko presented on May 31st, 2013

APPROVED:

Mentor, representing College of Science

Committee Member, representing College of Science

Committee Member, representing College of Science

Chair of Physics Department

Dean, University Honors College

I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Andrew M. Svesko, Author

ACKNOWLEDGMENTS

This project was a tremendous undertaking, one which required the assistance and feedback of several of my peers. I would like to thank all of those who gave such feedback, including the students of 2012-2013 Physics thesis class. In particular, I would like to thank my mentor Professor Albert Stetz for giving me the idea of writing this text in the first place, supporting my efforts, and providing me with deeper insight when needed. I would also like to recognize Teal Pershing who has acted as an editor, figure designer, student, and friend through the entire process. I can honestly say that this text would not be the same without their help.

Contents

Writing *A Detailed Introduction to String Theory*

Andy Svesko

June 3, 2013

Introduction

In high school I had a vision. I held that if I could understand the mathematics behind physics problems, the physical interpretation would naturally follow. Rather quickly I realized the naivety of my perspective toward physics. After I purchased texts on relativity, quantum mechanics, and string theory, it was apparent that I held an ill-conceived approach to physics, as nearly each text I bought was almost unreadable. The content that lay within each text was no different than a foreign language. Although I didn't understand the material in each text, I found what was needed to comprehend each of these physical theories; finding the road to understanding string theory was a long one.

After enrolling at Oregon State University to study both math and physics, I continued to study the texts I collected as a high school student. It became clear in my independent studies that the texts I referred to had a specific audience in mind, graduate students, and advanced ones at that. Searching for the ultimate source on string theory, I began filling my personal library with a plethora of texts on the subject, some of which became more accessible as my academic career continued, however still out of reach. And the reason for this was simple: each text assumed that the reader had background in quantum field theory, general relativity, and supersymmetry, along with a fair understanding of a variety of branches of mathematics. Of course, each of these subjects require background in other physical theories and mathematical tools. The task of learning string theory was a daunting one; at times I felt like Sisyphus. Just as I reached a pinnacle in my studies, I would come across a new but crucial detail necessary for understanding the basics of string theory; the boulder I pushed would roll back down the hill. During this process I could only dream of finding that sole text which had everything I needed to know about string theory. I was aiming for a shortcut; a text which would not only provide the details of string physics, but would also present the necessary background material. Of course I would not find such a text, as such a text had not been written yet, particularly at the level an undergraduate could hope to understand.

This is the point of my thesis project. Specifically, the project consisted of drafting a text which presents a detailed introduction on string theory to the reader new to the subject. The document here is an article specifying the motivation for, and process of, writing the text. The text, *A Detailed Introduction to String Theory* [1], will also be compared with the present literature availble on the market, marking the similarities and differences between this text and the alternatives. An outline of what is to be discussed is given below.

At the time this text is being written, there are three main graduate level books on the topic of superstring theory, and one text aimed at undergraduates. The graduate texts include the two volume series *Superstring Theory* by Green, Schwarz, and Witten (GSW) [2],[3]; *String Theory and M-Theory: A Modern Introduction* by Becker, Becker, and Schwarz (BBS) [4], and the two volume series *Bosonic String Theory and Superstring Theory* by Polchinski [5], [6]. The text aimed for undergraduates is Zwiebach's *A First Course in String Theory* [7]. As I have been independently studying this subject for quite some time, I have found that each of these sources have both benefits and deficiencies. Firstly, the graduate texts make it difficult for the undergraduate to follow by assuming the reader has a formidable background in both physics and mathematics. Moreover, as the texts are graduate level, they rarely detail the results, leaving most of the derivations to the reader, a serious problem for someone new to the subject. Nonetheless, these texts summarize the more difficult topics of the subject, and include fairly modern avenues for research in string theory and, more broadly, quantum gravity.

Alternatively, Zwiebach's text is particularly thorough in discussing the bosonic string, making it a widely accessible book for undergraduates, especially those aiming for self-study. The cost of this however is that it does not provide a rigorous treatment of superstrings, nor does it go into detail on the subjects which are necessary for further study in string theory, marking its deficiency. In this sense, after a student completes a first reading of Zwiebach's text, they will only have the tools to understand the first chapter of the mainstream graduate level texts, limiting the reader's ability to grasp fundamental topics, let alone advanced topics.

The overarching goal of my text [1] is to resolve this issue. With *A Detailed Introduction to String Theory* I aim to bridge the material presented in Zwiebach's text and the graduate level texts on string theory. In a sense, my work presents only a few new topics in string theory. A student studying string theory, depending on their level, can already obtain necessary information from other resources, including the ones above, in which case my text gives nothing new. What is different about this text is the presentation of the subject. Synthesizing the material already available, this book also augments the current information with additional computational details to aid the reader as they study this text. As Zwiebach provides a suitable approach toward the fundamentals of bosonic string theory and D-branes, much of the first half of this text closely follows his methods. In the second half of this text, for the more advanced material, particularly the work done on superstrings, much of the material was synthesized from the three mainstream texts as well as other books and online sources. For the reader who is interested in this subject, it is strongly encouraged that they review the source materials on which my text is based as the authors of those texts and papers have a far better understanding of the subject than the author of [1] does. In short, my work aims to be a sole resource on the basics of string theory: a detailed introduction assuming minimal background knowledge on the part of the reader.

Chapter 1

Motivation

My project involves the writing of another 'textbook' on string theory. The reason for this is that through my own studies I found that most of the texts out there do little to make the material accessible to an undergraduate, and for those who pursue self-study. It is my goal to resolve this issue, creating a document which provides a thorough introduction to string theory aimed for the undergraduate, as well as a document that could be used for self-study. In order to observe how my own text differs from the other books already on the market, a brief summary of the content in these texts is necessary. But before reviewing the standard texts, a brief introduction to string theory is necessary.

1.1 The History of String Theory

At the turn of the 19th century, it was thought that the studies of physics were coming to a close. Newtonian mechanics described the motion of everything from the trails of the comets to the trajectories of falling apples. Maxwell's equations elegantly summarized the relationship between stationary and moving charges, and even going as far as providing a mathematical treatment of electromagnetic radiation. The work completed by Boltzmann lent insight to chemical kinetics and the laws of thermodynamics, supplying a near complete understanding of chemical reactions and the microscopic world. Physicists at the time found that the natural world seemed to have fewer and fewer mysteries. Even Lord Kelvin, purportedly, in 1900 announced to the British Association of the Advancement of Science that "[T]here is nothing new to be discovered in physics now. All that remains is more and more precise measurement" [1]. Whether Kelvin said it or not is unclear, but the same tone was felt among many of his contemporaries, and they couldn't have been more wrong.

The 20th century gave rise to two big pillars of physics: relativity theory and quantum mechanics. In 1905 a bright young physicist going by Albert Einstein developed his theory of *special relativity*, a work which revolutionized the laws of physics and our entire conception of reality. Space and time were no longer the rigid objects that Newtonian mechanics required. In order to provide an accurate description of the universe, absolute space and absolute time were thrown out to be replaced by a malleable structure known as space-time. For the laws of physics to remain invariant, our notion of time and space had to become dynamic. Ten years after his special theory of relativity, Einstein published his *general theory of relativity*, correcting Newton's ideas about gravity. There Einstein showed that one is to interpret gravity as the curvature of space-time, and that the universe has a structure to a type of malleable fabric.

If the physics community wasn't disturbed by relativity, they were almost certainly horrified quantum mechanics. Not long after Einstein's special theory of relativity, Max Planck, among others (including Einstein) was able to show that at a microscopic level dynamics of systems include uncertainty. That is, quantum mechanics is inherently non-deterministic. Moreover, physical observable quantities such as position, momentum, and energy become discretized, losing continuous structure as given in classical physics, including relativity theory.

As quantum mechanics and special relativity came on the scene around the same time, the natural progression was to proceed in unifying both theories, a theory of relativistic quantum mechanics, *quantum field theory* (QFT). For starters, QFT posited the existence of antiparticles, particles which have the same mass but have opposite additive quantum numbers. For example, if the positron, or anti-electron, were to come into contact with the electron, the two would abruptly annihilate each other, leaving only a radiation signature behind. As time went on, quantum field theory was felt to be the correct candidate in accurately describing the entirety of interacting particles, supplying a deeper understanding of the microscopic world.

Just as physicists had unified quantum mechanics and special relativity with quantum field theory, the natural expectation was to unify quantum mechanics with general relativity. To their dismay however, a successful theory of quantum gravity was out of reach. For technical reasons, the issue lies with *renormalization theory*, which when applied to the quantization of the gravitational field leads to unphysical results. In short, general relativity and quantum mechanics describe different realms of physics. General relativity describes the macroscopic world; the motion of planets, galaxies, and the evolution of the cosmos. Alternatively, quantum mechanics specializes in describing the microscopic world: the motion of elementary particles, particle decay,and the fundamental properties of matter. The theories, by themselves, perform well in their respective realms, but in the microscopic limit, general relativity breaks down, and the instant quantum mechanics attempts to absorb gravity, the theory blows up.

For several decades now theorists have been working to achieve a theory of quantum gravity; to develop a coherent framework in which quantum mechanics and gravity can coexist. There are several approaches in accomplishing this, but in a general sense there are three main approaches: (1) String Theory, being the most popular at this point; (2) Loop Quantum Gravity (LQG), being the second most popular, and (3) Causal Dynamical Triangulation (CDT), which is closely related to (2). In general LQG and CDT are quite different from string theory, specifically in their approach. LQG and CDT are considered to be branches of *canonical quantum gravity*, theories born from a relativist's perspective, in which theorists elevate space and time to quantum operators, i.e. quantize space-time itself. The reason for this is in general relativity space and time are dynamical entities. In this way theorists are able to build quantum space-time, in which gravity would then emerge from. The precise difference between LQG and CDT is somewhat subtle (it involves the fact that in CDT the space-time manifold is granularized in such a way that causality is preserved [8],[9]) but subtle enough that they have become their own distinct approaches. These alternate routes however are quite different from string theory, which in a historical sense was born from a particle physicist's perspective.

Historically, string theory was first introduced in the 1960's as an attempt to understand the strong nuclear force through interacting hadrons. It turned out that a theory based on one-dimensional extended objects, referred to as strings, solved some of the issues that the point-like behavior of hadronic interactions introduced. The crucial idea of using strings is that specific particles would correspond to oscillation modes (quantum states) of a string, much like the oscillation modes on a violin string producing a variety of musical notes. With the string description, a single one-dimensional object retains the ability to explain the differences of the myriad of observed hadrons [3]. In the early 1970's however, another theory, *Quantum Chromodynamics* (QCD), was developed to describe the strong nuclear force. With this development, as well as technical issues of using strings, string theory fell out of favor with the masses of the physics community. Some, however, weren't ready to abandon the elegance string theory seemed to offer.

In 1974, around the same time QCD was being fine tuned, physicists Julius Wess and Bruno Zumino developed a solution to eliminate tachyons from the models of particle physics. Not only did the theory eliminate undesired tachyons it also provided a symmetry between bosons and fermions. Such a symmetry is formally known as *supersymmetry*. Not long after the presentation of global supersymmetry, work was done in extending the theory to *local supersymmetry* or *supergravity*, an extension which includes general relativity.

The year 1984 became known as the first superstring revolution amongst theoretical physicists. Due to the importance of supersymmetry, it is expected that string theory should contain local supersymmetry. The revolution was marked by the discovery that to maintain quantum mechanical consistency with a ten-dimensional supersymmetry requires one of two possible Lie algebras: $SO(32)$ and $E_8 \times E_8$. It turns out that the superstring formalism gives rise to five distinct theories: type I, type IIA, type IIB, and two Heterotic string theories. This realization that there were five different string theories posited an intuitive problem: if there is only one universe, why are there so many theories? In the late 1980s, a property known as T-duality was found to relate the type II theories and the two heterotic theories, suggesting that they shouldn't be viewed as distinct theories [3]. But what of the other theories?

The mid-1990s for theoretical physicists has become known as the second superstring revolution. Similar to T-duality, another type of duality, called S-duality was discovered, which relates the type I theory and one of the heterotic string theories and the type II B theory to itself [3]. Remarkably, it was found that if an 11th dimension was introduced, a quantum theory called M-theory emerges. Together with the S and T dualities, the five superstring theories and 11-dimensional supergravity are connected by a web of dualities. That is, each theory separately can be viewed as different corners of the same theory, a single model of string theory describing quantum gravity. To this date however, there is not yet a complete or compelling enough formulation of M-theory. As it stands, the physics community is waiting for the upcoming theoretical physicists to either complete the description, or abandon M-theory altogether.

String theory is not without its issues. The number one problem with string theory, and every approach to quantum gravity for that matter, is that it lacks predictive power. That is to say, so far string theory is unable to provide any verifiable claims. This problem should be taken seriously as any good science requires the ability to make predictions about the physical world. A related issue is the fact that string theory requires the existence of extra dimensions and supesymmetry. If an experiment were to come out providing evidence that our physical world does not include supersymmetry or extra dimensions, string theory would almost certainly fall apart as it hinges on these two requirements. Alternatively, LQG and CDT do not require the existence of extra dimensions or supersymmetry, however LQG and CDT can include these features if need be ([10],[11]).

In short, string theory is one of three main candidates to a quantum theory of gravity. Since string theory can also be viewed as an extension of the standard model, it is also considered to be a candidate for a unified field theory. LQG and CDT are only concerned with gravity and are therefore not candidates of a unified field theory. Despite its shortcomings, string theory has much to offer, making it a highly active area of research and the most popular approach toward quantum gravity.

With this brief summary of string theory, one is now equipped with enough terminology to review the basic elements of the current standard texts on the subject.

1.2 The Standard

There are numerous sources for an introduction to string theory, but there are a few works which have become the standard documents for this subject. These texts include the two volume series *Superstring Theory* by Green, Schwarz, and Witten (GSW) ([2],[3]); *String Theory and M-Theory: A Modern Introduction* by Becker, Becker, and Schwarz (BBS) [4]; the two volume series *Bosonic String Theory and Superstring Theory* by Polchinski ([5],[6]), and lastly Barton Zwiebach's *A First Course in String Theory* [7]. Although these texts may be the standard, by all means there are a variety of other texts out there on the subject, some of which are better than others. Regardless, here I will only review the standard texts, as it is these texts I wish to bridge.

First consider the two volume series GSW ([2],[3]). The first volume of this series has a copyright date of 1987, coming out a year after the so-called *first string theory revolution*, and is therefore somewhat dated. A variety of special topics including D-branes, black holes, and M-theory, where some of the most exciting research is currently taking place, are no where to be seen. GSW is meant, however, to give a 'brief' summary of the results of string theory up to that point in history, most of which largely focused on the mathematical framework necessary for string theory, and the string theory models which included superstrings. For this reason, GSW is indispensable, as it was the first major text on the subject to come out. This does not mean, however, GSW is an ideal text for learning string theory, especially from the perspective of an undergraduate. GSW assumes a background in a variety of other fields. Rarely are the results in the text derived explicitly, leaving most of the computations for the reader to do in the privacy of their own home. Of course, GSW is a Cambridge Monograph on mathematical physics, and it is therefore written for an advanced audience, one assumed to have experience in all of these fields.

The other standard graduate texts aren't much better in this respect. *String Theory and*

M-Theory [4], being written after the second string theory revolution, is certainly far more modern than GSW ([2],[3]) and includes several exciting topics currently being researched. BBS [4] is also a graduate level textbook and includes several worked out examples and homework problems, making it accessible to a wider, albeit still limited, audience. Despite all of this, from a pedagogical standpoint, BBS fails in the sense that it does not provide explicit derivations of many of the results. Additionally, BBS is notorious for skipping several steps in the 'worked out' examples, making it difficult for the reader new to the subject to approach the material and fully grasp the mathematics and physics of string theory.

Polchinski's two volume series *String Theory* ([5],[6]) is certainly different from BBS [4] and GSW ([2],[3]) as far as approach is concerned, but it also expects a lot from the reader. One of the major difficulties of Polchinski's text is that it uses the path integral approach as the main quantization procedure, a hurdle for many who are only mildly familiar with the path integral formulation of quantum theory. Certainly Polchinski's text is unique, however it is a difficult text to get through for first time reader, let alone the undergraduate eager to learn string theory.

After reviewing each of the graduate level texts, a theme emerges. In similar spirit as GSW ([2],[3]), nearly all of the graduate texts on string theory begin first with a brief history of string theory, and then spend a chapter, sometimes two, on the various quantization procedures of the bosonic string. The bosonic string is important because it is in many ways a toy model, one which is not realistic as the model does not include fermions, but it introduces several of the ideas imperative to the various superstring theories. Therefore, the bosonic string is crucial, as it paves the way toward the more realistic, and more complicated, superstring theories. The issue with the standard graduate texts is that the bosonic string is typically covered in a single dense chapter. In other words, the model which inspires the physics of superstrings is only lightly covered, leaving the student confused about the importance of the bosonic string.

In this respect alone, Barton Zwiebach's undergraduate themed text, *A First Course in String Theory* [7], sets itself apart from the standard graduate texts. Published around the same time as BBS [4], Zwiebach remains as one of the more modern texts on string theory. Dissimilar to BBS and the other standard texts however, Zwiebach assumes that the reader has had minimal experience with special relativity, Lagrangian mechanics, quantum mechanics, and the other core subjects. Zwiebach builds from the ground up, from classical strings all of the way to the quantization of the open and closed bosonic string. D-branes and other special topics are also considered, however typically only with bosonic strings. Zwiebach indeed includes a chapter on superstrings, and mentions their importance throughout the text, however the chapter is rather qualitative, making it hard for the student to fully grasp the essential tenets of supersymmetry as well as superstring theory. This is Zwiebach's greatest downfall. Zwiebach is particularly successful at being a first course in the subject, however, due to the lack of material on supersymmetry and superstrings, the student is likely to only be prepared to read the material on the bosonic string in the graduate texts, which is only the first chapter.

Simply put, the transition from Zwiebach's undergraduate text to the standard graduate level books is not a smooth one. Indeed, the transition from undergraduate to graduate is unlikely to be smooth as well, however it is in my opinion that work can be done to make the transition from Zwiebach to the graduate texts easier. The gap evident in the standard textbooks can be bridged.

1.3 The Theoretical Minimum and Bridging the Gap

While I was still in high school, I began scouring the internet for online lectures on relativity and quantum mechanics. I was looking for anything that wouldn't completely dumb the material down, however still at a level I could hope to understand. Doing my search I came across string theorist Leonard Susskind's two year lecture series titled *The Theoretical Minimum* [12] that was being filmed and uploaded to youtube. The goal of the series was to provide a foundation in the physical theories necessary to understand more advanced topics in theoretical physics, particularly string theory. The courses ranged from classical field theory to general relativity to supersymmetry, and ended with a two term series on string theory. Susskind's lectures indeed were what I was looking for, a method of learning advanced physics topics without having to be enrolled in the Stanford graduate program.

Not long after reviewing all of Susskind's lectures, I discovered Zwiebach's text [7]. Despite my satisfaction with his work, I felt that the book could go into more detail on other crucial subjects necessary to fully appreciate superstring theory, while still keeping the content at a level that an undergraduate could understand. I saw that had Zwiebach spent some time fleshing out the physics of general relativity or quantum field theory, the more esoteric topics laid forth in the standard graduate texts could be brought to a level that an undergraduate could understand.

This is precisely the aim of my project; to bridge the gap that exists between Zwiebach's detailed text on bosonic string theory, and the graduate level texts. And the best way I think that this can be achieved is to write a textbook in similar spirit as Susskind's theoretical minimum. To form this bridge I have put together what I call the 'crash course chapters' to act as background information that allow the reader to continue on with the main subject. Essentially, the layout of the book is to provide as much of the building blocks of string theory as possible until it is absolutely necessary that an important and fundamental physical theory is reviewed, such as quantum field theory. In this way, as the reader moves through the text, not only do they learn the basics of string theory, but also become acquainted with the other fields of physics comprising the 'theoretical minimum', providing, in principle, an easier transition from Zwiebach [7] to the standard graduate texts ([2]-[6]).

Chapter 2

Methodology

2.1 Learning while Writing

In its initial conception, the primary objective of this thesis project was to investigate the basics of string theory, and then provide a summary on the physics of black holes in the context of string theory. The project quickly changed as my mentor, Professor Stetz, suggested that the best way to learn a subject is to write a book about it. And thus I began writing a textbook.

One of the major challenges of writing this text has been striking a balance between the learning and writing processes. But this balance provides the basis of the methodology I used to complete this project. The first task was to learn the subject. As mentioned above, I had begun perusing a variety of texts on the fields which string theory builds itself from years prior to the conception of this thesis project. Moreover, the summer before I began writing the text I worked through Zwiebach's *A First Course in String Theory* [7]. Using this background, gathering resources for the project was a relatively simple task. I reviewed a plethora of texts, papers, and online lectures to gain an overview, while working through the entirety of Zwiebach's text, as well as selected exercises of BBS [3]. Along with homework exercises, I would work through all of the derivations from Zwiebach and the other texts explicitly, as to understand nearly all of the mathematical and physical details of each calculation.

I filled a multitude of notebooks, pulling from various resources, learning the introductory elements of string theory in a variety of ways. Using these notes I began constructing my own text. In this sense, my thesis project is not much more than a well organized compilation of notes. It is also from this perspective that I have been able to determine the strengths and weaknesses of each of the standard texts, allowing me to elaborate on details not given in the graduate texts, and enhance the derivations given by Zwiebach. Of course, credit is given where it is due, as many of the derivations shown are not original. Rather, by utilizing the notes from a range of texts and papers, I have been able to construct a detailed introduction to string theory.

2.2 Lecturing while Writing

A textbook with the aim of reaching undergraduates should, naturally, be tried by the intended audience. Certainly, as an undergraduate myself, I have heard and understood the majority of complaints most undergraduates have about courses and the textbooks used in those courses. However, as the author of an introductory text on string theory, I am no longer an arbitor for what is deemed as the 'undergraduate level'. Students that have not had the extensive training I have had in this subject must be the individuals in which this text is tested against. Only then can my thesis project be viewed as a success.

For this reason it is necessary that the text be tested on my fellow peers. So far, I have only had the opportunity of lecturing to a single individual. During the summer I began lecturing on the basics of supersymmetry to a single student. Not only did I learn the subject better coming from the persepective of a lecturer, I was also able to determine where the text needed pedagogical improvements; finding gaps within the presentation of the material. Based on these few lectures, I made some changes in my writing, looking to fill in these holes with either more explicit mathematical computations, or with more physical examples. All in all, the mix of lecturing while writing has offered me a chance to determine whether the material as presented in the text is sufficient as is, or whether it needs improvements. Again, I have only been able to lecture to one of my fellow peers, and only on a single topic, a poor sample size to base results off of, however still insightful. One of the long term goals of this project is to teach an entire class using the written text as a core set of notes, thereby allowing me to determine how the book may be augmented.

Chapter 3

Results: *A Detailed Introduction to String Theory*

The end result: an organized collection of notes providing a detailed survey of string theory. Ranging from the historical origins to the description of black holes, this collection of notes has been arranged such that the reader is given a thorough tour of the basics, as well as an introduction to more advanced topics in string theory. All of the results presented in this text have indeed been presented before, and therefore do not distinguish it from the currently available texts on the market. We will not focus on these aspects of the text. Rather, there are a number of features that distinguish this set of notes from what is currently available. Here we will emphasize these features as they are what act as the bridge between the standard texts.

3.1 The Track System: How to read this book

As the author of the text, naturally my preference would be for the entire book to be read. However this is unfeasible for the student new to the material, and the amount of material in the text itself would take more than one semester to get through if this book was going to be used as the core resource. For that reason a "track system" has been devised to guide the reader based on their level. To understand this system let us consider two types of potential readers, avoiding all of the possible combinations of sub-types: the undergraduate senior, and the first-second year graduate student. For the undergraduate senior, it is assumed they have had roughly a year's worth of quantum mechanics, and have had core classes in electromagnetism, classical mechanics (with an emphasis on both the Lagrangian and Hamiltonian formalisms), thermal physics, mathematical methods, and a minimal amount of special relativity. Also, any amount of experience with modern algebra, differential geometry, and topology help, but are not required, as these subjects are discussed when necessary. For the first-second year graduate, it is expected they have all of the above requirements as well as more mathematical methods (including a little bit group theory and complex variables), and some experience with field theory and gauge transformations.

Before we get to discussing the details of the two track systems, a brief summary of each chapter is necessary. Chapter 1 is primarily concerned with giving a historical introduction to string theory as well as the unit system to be used throughout the text. In chapter 2 a review of special relativity, and an introduction to index notation for tensors and vectors is given. Chapter 3 is a review of Lagrangian mechanics, an introduction to field theory, and an introduction to the classical dynamics of non-relativistic strings. Chapters 4–6 give a detailed introduction to relativistic strings, including conserved quantities. Chapter 7, the first of the "crash course" chapters, is an introduction to quantum field theory, particularly focusing on free scalar fields and free spinor fields. Chapter 8 quantizes the relativistic point particle using light-cone coordinates. Chapters 9–11 give the three main approaches to quantizing the bosonic string in order of increasing difficulty: light-cone quantization, covariant quantization, and BRST quantization (an approach which makes use of conformal field theory and so-called BRST symmetry); the second 'part' of the text begins with chapter 11. A survey of D-branes in both bosonic string theory and, briefly, superstring theory, is given in chapter 12. Chapter 13 details T-duality in bosonic string theory and qualitatively outlines the basics of compactification. The second of the "crash course" chapters, chapter 14, gives a dense introduction to supersymmetry (this chapter is also supplemented by two appendices on notation and Grassmann calculus). A brief introduction to superstring theory is given in chapter 15. Chapter 16 reviews some results from thermal physics and gives an introduction to the thermodynamics of strings, mainly the bosonic string; chapter 16 also marks the beginning of the 'third' and final part of the text. In chapter 17 an introduction to elements of differential geometry, including differential forms and covariant differentiation, is given. Chapters 18–19 introduce the tenets of general relativity, and provide a quick, albeit dense, survey of black holes as viewed in general relativity and semi-classical gravity; chapter 18 is the third "crash course" chapter. Chapter 20 provides an introduction to black holes as interpreted in string theory. The final chapter, chapter 21, provides a critique of string theory, and a qualitative introduction to the other approaches to quantum gravity. With that, let's move on to the details of the track system.

The track devised for the undergraduate senior is as follows:

For a single semester course, chapters 1–3, excluding section (3.8); chapters 4–7, excluding sections (7.8)-(7.10); chapters 8–10, and conclude with the final chapter, chapter 21. For a year long course, the teacher might choose to include the sections first excluded, or could go through chapters 12–13, skip chapters 14–15, and then move on to chapter 16. From here chapters 17–20 flow together, as they all pertain to general relativity and black holes in string theory and are relatively self-contained, however this set of chapters might be too much for the first time reader.

Alternatively, the track devised for the first-second year graduate is as follows:

For a single semester course, chapters 2-4, excluding sections (2.1)-(2.3); chapters 4–7 excluding sections (7.8)-(7.10); chapters 8–13, 16, and conclude with chapter 21. For a year long course, include sections (7.8)-(7.10) as well as chapters 14–15. If the students have a background in general relativity, then chapters 17–20 could be tackled without too much difficulty.

It should be pointed out that the two track systems are not strikingly different from one another. The prime decision on creating the track system is to come up with a guide for students with particular experience and abilities. For a student who has not had too much experience with group theory, complex variables, or quantum fields, the chapters which rely heavily on these mathematical and physical concepts should not be stressed on the first time through. The student who is experienced with these subjects, could very well handle the material presented in this text. Moreover, a student who has gone through the first track, would then be prepared to tackle the second track, reading through the material previously omitted.

Aside from these two track systems, it should also be noted that this text also has the intention of acting as a guide for self-study. This was in fact one of the chief aims of writing this book: to create a document which may be used by teachers or by the individual willing to study the subject independently. Of course, a teacher experienced with the material would prove far more beneficial than simply reading the text on one's own.

In summary, my text [1] may be read in which ever way the reader or lecturer prefers. However, as the author and a student who undertook the process of learning this material, it is my belief that the devised track systems would prove to be the most effective means of learning the material for the first time. Most of all, once the student is experienced with the contents of this text, they will be far better prepared for the advanced standard works on the subject.

3.2 "A Crash Course in..."

The point of string theory is to provide a unified description of all the forces and particles in nature. In a sense, string theory is the culmination of several fields of physics that appear to be incomplete in some way; that is, to come up with a theory that goes beyond the present models of modern physics. Part of the issue of learning string theory is that one should be familiar with each of these models, or previous modes of thought. It was found that this is also the crucial difficulty in moving from Zwiebach's text to the graduate level standards: Zwiebach [7] assumes very little background and the graduate texts assume too much. Moreover, Zwiebach does little to provide details on background in other fields that are absolutely essential in understanding string theory. As noted earlier, this is one of the main problems *A Detailed Introduction to String Theory* [1] aims to resolve. In short, this text includes several " crash course" chapters on background that is fundamental to the student continuing on. Here we will examine excerpts from these "crash course" chapters and observe how this text bridges the current undergraduate books with the graduate level standards.

How these crash course chapters were organized is also important. One goal of this book is to introduce the basic elements of string theory without cutting too many corners and with a text of finite length. Keeping this in mind, several chapters would be written on topics in string theory, avoiding the formalism of physical theories, such as quantum field theory, until absolutely necessary. From the perspective of the author, the book focuses solely on strings as much as it can until a concept from an 'old' field is needed: e.g. scalar fields. From the perspective of the reader, the text reads like a book on string theory, with the occasional detour toward other fundamental fields of physics. Due to this feature, only a few "crash course" chapters are included in the overall structure of text, maintaining a strong focus on string theory, but using details from other theories to aid in the discussion.

In this section the topics and goals of each of the "crash course" chapters is given, along with short excerpts. Appendix A gives longer excerpts from these chapters elucidating how these chapters were structured. For a more complete view of these chapters, the entire document, *A Detailed Introduction to String Theory*, can be found amongst the other references in the bibliography [1].

3.2.1 Quantum Field Theory

Historically, string theory was an attempt at describing the interaction of gluons. In this sense, string theory in its original form belongs to the realm of particle physics. Of course, the interaction of gluons, governed by the strong nuclear force, is described by another physical theory, *Quantum Chromodynamics* (QCD), pushing string theory to the side for a time. Despite its mathematical evolution, string theory has its primitive roots in particle physics, where quantum field theory reigns supreme. Therefore, to have an understanding of string theory (and most fields of modern theoretical physics for that matter), one should be fairly experienced with quantum field theory.

For this reason, the first of the "crash course" chapters is one pertaining to quantum fields. In no way is this chapter meant to be a thorough introduction to quantum fields; rather it is intended as a brief discussion of the very basics. In this chapter the reader will not find any real discussion on interacting fields, Feynman calculus, or the various methods of field quantization. Instead the focus is on free fields, particularly free scalar fields, though there is some detail on free spinor fields and the Dirac equation. Ultimately the point of this chapter is to provide some basic insight into the physics of quantum field theory without doing away with all of the mathematics.

One particular feature of this chapter is the relation given between quantum fields and light-cone coordinates. Since much of this text considers dealing with the light-cone quantization procedure of the string, it is imperative that the reader witness the relationship between light-cone coordinates and the different types of quantum fields (specific to this chapter are scalar fields, photon fields, and gravitational fields). Below is a brief excerpt exemplifying the relation between scalar fields and light-cone coordinates [1]:

> It will be useful later on to consider scalar fields in terms of light-cone coordinates. Let $\vec{x}_T$ denote a vector whose components are transverse coordinates x^I
>
> $$\vec{x}_T = (x^2, x^3, ...x^d) \tag{3.1}$$
>
> The collection of space-time coordinates then becomes $(x^+, x^-, \vec{x}_T)$. Using light-cone coordinates, the Klein-Gordon equation is written as
>
> $$\left(-2\frac{\partial}{\partial x^+}\frac{\partial}{\partial x^-} + \frac{\partial}{\partial x^I}\frac{\partial}{\partial x^I} - m^2\right)\phi(x^+, x^-, \vec{x}_T) = 0 \tag{3.2}$$
>
> To simplify, we Fourier transform the spatial dependence of the field, changing x^- into p^+ and x^I into p^I. Similarly then,
>
> $$\vec{p}_T = (p^2, p^3, ...p^d) \tag{3.3}$$
>
> The Fourier transform of the field is given by [7]:

$$\phi(x^+, x^-, \vec{x}_T) = \int \frac{dp^+}{2\pi} \int \frac{d^{D-2}\vec{p}_T}{(2\pi)^{D-2}} e^{-ix^- p^+ + i\vec{x}_T \cdot \vec{p}_T} \phi(x^+, p^+, \vec{p}_T) \tag{3.4}$$

Substituting this into the Klein-Gordon equation, (7.81), we find

$$\left(-2\frac{\partial}{\partial x^+}(-ip^+) - p^I p^I - m^2\right) \phi(x^+, p^+, \vec{p}_T) = 0$$

Dividing by $2p^+$ we find

$$\left(i\frac{\partial}{\partial x^+} - \frac{1}{2p^+}(p^I p^I + m^2)\right) \phi(x^+, p^+, \vec{p}_T) = 0 \tag{3.5}$$

To make things even simpler, let's introduce a new time parameter, τ, which is related to x^+ in the following way:

$$x^+ = \frac{p^+ \tau}{m^2} \tag{3.6}$$

allowing us to write

$$\left(i\frac{\partial}{\partial \tau} - \frac{1}{2m^2}(p^I p^I + m^2)\right) \phi(\tau, p^+, \vec{p}_T) = 0 \tag{3.7}$$

Before moving on, let's briefly consider what we have just shown. We started with the Klein-Gordon equation, a second order equation in space and time, and rewrote it using light-cone coordinates to yield a first order differential equation in light-cone time. This expression appears to have the same structure as the Schrödinger equation, a feature we will exploit in the next chapter.

Moving right along, to describe quantum states of scalar fields in light-cone coordinates we label the oscillators (creation/annihilation operators) with p^+ and $\vec{p}_T$. Single particle states are then constructed via

$$a^\dagger_{p^+, \vec{p}_T} |0\rangle \tag{3.8}$$

In light-cone coordinates, one can show that the momentum operator becomes three operators [7]:

$$\hat{p}^+ = \sum_{p^+, \vec{p}_T} p^+ a^\dagger_{p^+, \vec{p}_T} a_{p^+, \vec{p}_T} \tag{3.9}$$

$$\hat{p}^I = \sum_{p^+, \vec{p}_T} p^I a^\dagger_{p^+, \vec{p}_T} a_{p^+, \vec{p}_T} \tag{3.10}$$

$$\hat{p}^- = \sum_{p^+, \vec{p}_T} p^- a^\dagger_{p^+, \vec{p}_T} a_{p^+, \vec{p}_T} \tag{3.11}$$

If we use the mass-shell condition $(p^2 + m^2) = 0$, and the momentum in light-cone coordinates, yielding $p^2 = (-p^+ p^- - p^- p^+ + p^I p^I)$, we find that

$$2p^+ p^- = (p^I p^I + m^2) \to p^- = \frac{1}{2p^+}(p^I p^I + m^2)$$

which allows us to rewrite the above as

$$\hat{p}^- = \sum_{p^+, \vec{p}_T} \frac{1}{2p^+}(p^I p^I + m^2) a^\dagger_{p^+, \vec{p}_T} a_{p^+, \vec{p}_T} \tag{3.12}$$

> These expansions are important in the analysis of the relativistic quantum point particle which we consider in the next chapter.

This is the final section of the chapter concerning scalar fields. As the reader discovers, the basic procedure of determining single particle states in light-cone coordinates is identical to the procedure given before; the only real difference is using different coordinates. More interesting is the brief comparison between the Klein-Gordon equation written using light-cone coordinates, and the Schrödinger equation-like structure. This relationship is used later on when the reader learns how to quantize the relativistic point particle using light-cone coordinates.

All in all, the prime focus of this chapter is the quantization of free scalar fields. They are the simplest to deal with mathematically, and also lend insight into the physics of quantum fields, particles, and antiparticles, topics which are not really addressed in either Zwiebach [7] or the other standard texts ([2]-[6]). On this basis, "A Crash Course in Quantum Field Theory" is one of the first chapters which exemplifies the bridging nature of this text.

3.2.2 Supersymmetry

Bosonic string theory, despite its elegance, lacks (at least) one fundamental ingredient: fermions. A theory without fermions is a theory without matter. In order to be considered a semi-plausible theory, let alone a theory which describes all possible types of interactions, string theory must include a description of fermions. The way this is achieved is to use supersymmetry (SUSY), which associates every boson with a fermion, and every fermion with a boson. The marriage of bosonic string theory with supersymmetry is known as superstring theory, and to have any hopes in understanding it, it is imperative one have a fair background with supersymmetry, leading to the second "crash course chapter".

To make the material accessible, details on interacting fields, non-Abelian gauge fields, and the quantitative structure of the minimal supersymmetric standard model (MSSM), are all avoided. Rather the chapter focuses heavily on developing Lorentz invariant spinor quantities, understanding SUSY transformations, and developing the supersymmetric algebra. Moreover, there are basically two approaches to studying supersymmetry: one being from a more canonical approach, building the supersymmetric algebra from studying left chiral spinors; the other being the more elegant superspace formalism. Superstring theories are typically written using this latter formalism. For this reason, after some time is spent developing the SUSY algebra in the canonical way, the reader is introduced to superspace. The superspace formalism involves the inclusion of superfields which must be dealt with using Grassmann calculus. To aid this "crash course" chapter, two appendices are included in the text to give further information on van der Waerden notation and Grassmann calculus.

One particular feature of this chapter which sets itself apart from other texts is the discussion surrounding the set up of a 'simple' supersymmetric Lagrangian. Motivated by [13], the method of constructing a supersymmetric Lagrangian was aided by a 'guess' type approach, as can be observed below [1]:

> ...The simplest supersymmetric theory we may consider is composed of two massless fields, one that is bosonic and one that is fermionic. For our purposes, we will consider a Weyl spinor and a complex scalar field. Since we are only concerned with left-chiral spinors, we start with the Lagrangian density
>
> $$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^\dagger + \chi^\dagger i \bar{\sigma}^\mu \partial_\mu \chi \tag{3.13}$$
>
> We recognize that this is the correct starting point as it is simply the sum of the usual scalar field part we are now familiar with from studying quantum scalar fields, and the appropriate part of the Dirac Lagrangian. We must now introduce SUSY transformations that will leave this Lagrangian invariant. Since SUSY transformations are non-trivial, let's spend some time to understand how they are constructed.
>
> Let us consider the transformation of the scalar field, as this will be the easier one to deal with first. We will consider a transformation that is proportional to some space-time independent, infinitesimal parameter ζ. By space-time independent we mean that the parameter is not a function of space-time, thereby vanishing under a derivative, $\partial_\mu \zeta = 0$. If we think back to our earlier discussion on gauge transformations, we recognize that the transformation for our scalar field is *global*, contrary to *local* transformations which do depend on space-time. In short, we will only consider global SUSY transformations. Some call this *rigid supersymmetry.* Had we decided to examine local SUSY transformations, where ζ does depend on space-time, we would be forced to introduce a gauge field that has the properties of a graviton. Theories with local SUSY invariance are *supergravity theories*, and are a modern research avenue for theoretical physicists studying various approaches to quantizing gravity [13].
>
> Let us assume that the variation of the scalar field is proportional to the Weyl spinor χ. The reason for this choice is because scalar fields, which describe spin-0 bosons, should transform into fermionic fields. That is the fundamental consequence of SUSY. Therefore, we consider the transformation
>
> $$\phi \to \phi' = \phi + \delta\phi \tag{3.14}$$
>
> with $\delta\phi \approx \zeta\chi$. Notice we have not written an 'equal' sign. This is because when we write down transformations we must make sure that both sides of the equation have the same *dimension*, and behave the same way under Lorentz transformations. Considering this second requirement first, since ϕ is a scalar field, we must build a Lorentz invariant out of ζ and χ. Since χ is a Weyl spinor, we are forced to make ζ a Weyl spinor. Hence, the constant parameter of our global SUSY transformation is infact a Weyl spinor independent of space-time, and, by convention, we declare ζ to be a left-chiral spinor. Luckily, we already know of a Lorentz invariant constructed from left-chiral spinors: the spinor dot product between the two spinors, $\zeta \cdot \chi$. We are then tempted to write the SUSY transformation of the scalar field as

$$\delta\phi = \zeta \cdot \chi \tag{3.15}$$

To be certain however, we must check that the dimensions of both sides match up. In natural units we have $c = \hbar = 1$. Moreover, recall that the action S is the integral over all four dimensional space of the Lagrangian density $\mathcal{L}$, and is dimensionless (since we are working with natural units). In such a system, we only have one independent dimension left, that of energy, or mass M. We say that mass has dimension 1 (M^1). On the other hand, since $c = 1$, length L and time T have the same dimension of M^{-1}, since $\hbar = 1$. What this means is for the action to remain dimensionless, we require that the Lagrangian density have dimension M^4 (coming from the fact that the action is an integral over four dimensional space-time). Since gradients ∂_μ have dimension M, we can read off the dimensions of the scalar field ϕ and spinor field χ from looking at the Klein-Gordon and Dirac Lagrangian densities, yielding

$$[\phi] = M \qquad [\chi] = M^{\frac{3}{2}}$$

In order to make the dimensions of our SUSY transformation work out, we require that $[\zeta] = M^{-\frac{1}{2}}$. Let us now move on to the SUSY transformation of the Weyl spinor χ. We would like the transformation to be linear in the infinitesimal parameter ζ and either ϕ or $\phi^\dagger$, since dealing with non-linear transformations is much too difficult. For reasons which will become clear shortly, we will use the complex scalar field, allowing us to make our first guess for the SUSY transformation of χ:

$$\delta\chi \approx C\zeta\phi^\dagger$$

where C is some constant yet to be determined. Just as before, we must ensure that this guess has both sides transforming in the same way under Lorentz transformations. The left-hand side would transform as a left-chiral field, and since $\phi^\dagger$ is a scalar field, the right-hand side also transforms as a left-chiral spinor. There is a problem however: the dimensions don't match up: the left-hand side has a dimension of $M^{\frac{3}{2}}$, while the dimensions of the right-hand side are $M^{\frac{1}{2}}$, meaning we need to raise the dimension by M. To keep a linear transformation, we introduce ∂_μ which has dimension M. Therefore, our next guess is

$$\delta\chi \approx C\zeta\partial_\mu\phi^\dagger$$

The dimensions are certainly correct, however introducing ∂_μ caused a discrepancy in the Lorentz properties of both sides. In short, the indices don't match, meaning we must apply another object such that it contracts with ∂_μ. It must also be an object that is dimensionless. A natural choice is one of the Pauli matrices σ^μ or $\bar{\sigma}^\mu$. For reasons which will become clear momentarily, we choose $\bar{\sigma}^\mu$, and make another guess

$$\delta\chi \approx C(\partial_\mu\phi^\dagger)\bar{\sigma}^\mu\zeta$$

But this is still incorrect! Recall that $\bar{\sigma}^\mu\partial_\mu\chi$ transforms like a right-chiral spinor. This is exactly our same issue since, although the derivative is acting on $\phi^\dagger$, it does not actually affect the behavior of the expression under a Lorentz transformation. Meaning that the right-hand side transforms like a right-chiral spinor while the left hand side still transforms like a left-chiral spinor. Luckily, this problem can be easily resolved.

Remember that we showed $-i\sigma^2\eta^{\dagger T}$ transforms as a left-chiral spinor. Applying this to our latest guess, we have

$$\delta\chi = -i\sigma^2(C(\partial_\mu\phi^\dagger)\bar{\sigma}^\mu\zeta)^{\dagger T}$$

Since the operation of $\dagger T$ on anything other than a quantum field, including ζ, is nothing more than simple complex conjugation $*$, we have

$$\delta\chi = -i\sigma^2 C^*(\partial_\mu\phi)\bar{\sigma}^{\mu *}\zeta^* = -C^*(\partial_\mu\phi)i\sigma^2\bar{\sigma}^{\mu *}\zeta^*$$

Then, if we use the fact that $\bar{\sigma}^{\mu *} = \bar{\sigma}^{\mu T}$ and $(\sigma^2)^2 = I$ we find

$$\delta\chi = -C^*(\partial_\mu\phi)i\sigma^2\bar{\sigma}^{\mu T}\sigma^2\sigma^2\zeta^*$$

Lastly, using $\sigma^2\bar{\sigma}^{\mu T}\sigma^2 = \sigma^\mu$ we find that the correct SUSY transformation for χ is

$$\delta\chi = -C^*(\partial_\mu\phi)\sigma^\mu i\sigma^2\zeta^* \tag{3.16}$$

It remains to be seen whether we may choose a value for C. The easiest way to go about doing so is ensure that our Lagrangian is indeed invariant under the SUSY transformations we have developed, (14.58) and (14.59). Let us vary our Lagrangian density, (14.56):

$$\delta\mathcal{L} = \partial_\mu(\delta\phi)\partial^\mu\phi^\dagger + \partial_\mu\phi\partial^\mu\delta\phi^\dagger + (\delta\chi^\dagger)i\bar{\sigma}^\mu\partial_\mu\chi + \chi^\dagger i\bar{\sigma}^\mu\partial_\mu\delta\chi$$

$$= \partial_\mu(\delta\phi)\partial^\mu\phi^\dagger + \partial_\mu\phi\partial^\mu(\delta\phi)^\dagger + (\delta\chi)^\dagger i\bar{\sigma}^\mu\partial_\mu\chi + \chi^\dagger i\bar{\sigma}^\mu\partial_\mu\delta\chi$$

where we made use of $\delta\phi^\dagger = (\delta\phi)^\dagger$ and $\delta\chi^\dagger = (\delta\chi)^\dagger$. Taking Hermitian conjugates of our transformations (14.58) and (14.59), we find that

$$(\delta\phi)^\dagger = \bar{\chi}\cdot\bar{\zeta} = \chi^\dagger(i\sigma^2)\zeta^* \qquad (\delta\chi)^\dagger = C(\partial_\mu\phi^\dagger)\zeta^T i\sigma^2\sigma^\mu \tag{3.17}$$

If we substitute everything in, we find that the variation of the Lagrangian is

$$\delta\mathcal{L} = (\partial^\mu\chi^\dagger)i\sigma^2\zeta^*\partial_\mu\phi - (\partial^\mu\phi^\dagger)\zeta^T(i\sigma^2)\partial_\mu\chi$$
$$+C(\partial_\nu\phi^\dagger)\zeta^T(i\sigma^2)\sigma^\nu i\bar{\sigma}^\mu\partial_\mu\chi - C^*\chi^\dagger i\bar{\sigma}^\mu\sigma^\nu(\partial_\mu\partial_\nu\phi)(i\sigma^2\zeta^*)$$

As an exercise, the reader will prove that

$$\sigma^\mu\bar{\sigma}^\nu\partial_\mu\partial_\nu\chi = \partial^\mu\partial_\mu\chi = \Box\chi \tag{3.18}$$

which will help us show that the Lagrangian is invariant under our SUSY transformations. To see this explicitly, consider the sum of the first and last term of our variation above:

$$(\partial^\mu\chi^\dagger)i\sigma^2\zeta^*\partial_\mu\phi - iC^*\chi^\dagger i\sigma^2\zeta^*\Box\phi$$

where we moved $\Box\phi$ around to the end of the second term since it is not a matrix quantity. We can make these two terms cancel if we use integration by parts, allowing us to get the derivatives to act on the same fields. Applying integration by parts to the first term, keeping in mind that ζ is space-time independent, we find that the above becomes

$$-\chi^\dagger i\sigma^2\zeta^*\Box\phi - iC^*\chi^\dagger i\sigma^2\zeta^*\Box\phi$$

At this point it is easy to see that indeed these two terms cancel as long as $C = -i$. Completing a similar analysis shows that the second and third term of the variation also cancel when $C = -i$. In summary, we have introduced SUSY transformations

$$\delta\phi = \zeta\cdot\chi \qquad \delta\phi^\dagger = \bar{\zeta}\cdot\bar{\chi} \tag{3.19}$$

$$\delta\chi = -i(\partial_\mu\phi)\sigma^\mu i\sigma^2\zeta^* \qquad \delta\chi^\dagger = -i(\partial_\mu\phi^\dagger)\zeta^T i\sigma^2\sigma^\mu \tag{3.20}$$

that leave the Lagrangian

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^\dagger + \chi^\dagger i \bar{\sigma}^\mu \partial_\mu \chi$$

invariant. We have completed our first supersymmetric theory! However we are not done yet. As we will see, when we determine the supersymmetric algebra we will be forced to introduce a new field to maintain consistency. Let us proceed to examining the supersymmetric algebra now...

Although this model is overly simplistic, it allows the reader to become familiar with the techniques of constructing a lagrangian in SUSY, techniques which will be used later on in the chapter.

In short, "A Crash Course in Supersymmetry" and its supplemental appendices provide a brief , although dense, introduction to supersymmetry. When compared to Zwiebach, *A Detailed Introduction to String Theory* includes far more details on supersymmetry and superstring theory in the sense that a thorough discussion is given on the elements of SUSY. Zwiebach avoids these details to make the chapter on superstrings more readily accessible to the undergraduate. Unfortunately, for this reason when the reader moves on to more advanced texts suchs as BBS, they are immediately lost as it is assumed SUSY is a prerequisite for the standard graduate level books. In this way, "A Crash Course in Supersymmetry" is another example of bridging the gap between Zwiebach and the other standard texts.

3.2.3 Differential Geometry and General Relativity

String theory is the culmination of theories of the known forces, including gravity. Although considered to be part of the realm of classical field theories, the most 'modern' theory of the gravitational force is in the language of Einstein's general relativity (certainly, there exist 'semi-classical' theories of gravity, but these are really just extensions of Einstein's original theory). Gravity therefore enters string theory through Einstein's general relativity. Thus a student of string theory needs to be familiar with the fundamentals of general relativity, yielding yet another necessary prerequisite, and leading to another "crash course" chapter.

Instead of a single chapter devoted to all of the mechanics of general relativity, the "crash course" chapter was broken up into three separate chapters: "Elements of Differential Geometry", "A Crash Course in General Relativity", and "Black Holes in General Relativity". The first of these chapters is to lay down the groundwork for the majority of the mathematics used in classical general relativity, including tensor calculus, differential forms, and determining the various curvature tensors. The second is the actual "crash course" chapter which develops the physical intuition of general relativity and connects it to the mathematics developed in the previous chapter. The third chapter is concentrated solely on various black hole solutions, extending the work completed in the previous two chapters. All in all, the focus of the first two chapters is to provide the reader with a brief, though detailed summary of the important elements of general relativity. The third chapter in the sequence pays close attention to particular solutions to Einstein's field equations, the black hole solutions, and develops a context which is assumed as background in the following chapter.

A particular feature of "Elements of Differential Geometry" is its focus on two methods for computing curvature. The first is to use tensor calculus and the covariant derivative. Learning how to compute curvature in this way is often the conventional approach when one learns the mathematics of general relativity. The issue is that this approach is a rather tedious way to calculate the components of the Riemann curvature tensor. For this reason another, perhaps, more pragmatic route using differential forms is given. To aid in the introduction of the use of differential forms and to become familiar with exterior differentiation, some time is spent doing explicit calculations using the Hodge star operator and the wedge product; calculations which cannot be found in any other standard text on the subject. Below is an excerpt where the Laplacian in spherical coordinates is derived:

> The Hodge star operator also allows us to define the three famous vector calculus operators: div, grad, curl. Given a 1-form field F, one can prove that the divergence and curl of F are given by
>
> $$\nabla \cdot F = *(d*F) \qquad \nabla \times F = *dF \tag{3.21}$$
>
> Moreover, given any scalar function f, one can show that the gradient of f and Laplacian of f can be written as
>
> $$\nabla f \cdot d\vec{r} = df \qquad \Delta f = *d*(df) \tag{3.22}$$
>
> From these identities it is straightforward to prove that, as a consequence of the Poincaré lemma $d^2 = 0$, we obtain the familiar rules
>
> $$\nabla \cdot (\nabla \times F) = 0 \qquad \nabla \times (\nabla \cdot F) = 0 \tag{3.23}$$
>
> Conversely, if one assumed these rules, one could work backwards and show that the Poincaré lemma must hold.
>
> For concreteness, let us work out the Laplacian in spherical coordinates. First let us work out the gradient of some scalar function f in spherical coordinates. This isn't too bad since all we have to do is apply a total differential to f:
>
> $$df = \frac{\partial f}{\partial x^a} dx^a = \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \phi} d\phi$$
>
> which may be rewritten as

$$df = \frac{\partial f}{\partial r}dr + \frac{1}{r}\frac{\partial f}{\partial \theta}rd\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}r\sin\theta d\phi$$

Then, taking the Hodge dual gives us

$$*df = \frac{\partial f}{\partial r}r^2\sin\theta d\theta\wedge d\phi - \frac{1}{r}\frac{\partial f}{\partial \theta}r\sin\theta dr\wedge d\phi + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}rdr\wedge d\theta$$

Taking the exterior derivative of $*df$ yields

$$d{*}df = \left[\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}r^2\sin\theta\right)dr + \frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial r}r^2\sin\theta\right)d\theta + \frac{\partial}{\partial \phi}\left(\frac{\partial f}{\partial r}r^2\sin\theta\right)d\phi\right]\wedge d\theta\wedge d\phi$$

$$-\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f}{\partial \theta}r\sin\theta\right)dr + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial f}{\partial \theta}r\sin\theta\right)d\theta + \frac{\partial}{\partial \phi}\left(\frac{1}{r}\frac{\partial f}{\partial \theta}r\sin\theta\right)d\phi\right]\wedge dr\wedge d\phi$$

$$+\left[\frac{\partial}{\partial r}\left(\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}r\right)dr + \frac{\partial}{\partial \theta}\left(\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}r\right)d\theta + \frac{\partial}{\partial \phi}\left(\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}r\right)d\phi\right]\wedge dr\wedge d\theta$$

Since $\alpha\wedge\alpha = 0$, we can see that we will only get one non-zero term from each of these lines. In the first line, the only term we get is

$$\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}r^2\sin\theta\right)dr\wedge d\theta\wedge d\phi = \left[\frac{\partial^2 f}{\partial r^2}(r^2\sin\theta) + \frac{\partial f}{\partial r}(2r\sin\theta)\right]dr\wedge d\theta\wedge d\phi$$

$$= \left(\frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r}\right)r^2\sin\theta dr\wedge d\theta\wedge d\phi$$

In the second line, the only non-zero term is

$$-\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial f}{\partial \theta}r\sin\theta\right)d\theta\wedge dr\wedge d\phi = \left(\frac{\partial^2 f}{\partial \theta^2}\sin\theta + \frac{\partial f}{\partial \theta}\cos\theta\right)dr\wedge d\theta\wedge d\phi$$

$$= \left(\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2\sin\theta}\frac{\partial f}{\partial \theta}\cos\theta\right)r^2\sin\theta dr\wedge d\theta\wedge d\phi$$

Finally, in the third line, the only non-vanishing term is

$$\frac{\partial}{\partial \phi}\left(\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}r\right)d\phi\wedge dr\wedge d\theta = \frac{1}{r\sin\theta}r\frac{\partial^2 f}{\partial \phi^2}dr\wedge d\theta\wedge d\phi$$

$$= \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}r^2\sin\theta dr\wedge d\theta\wedge d\phi$$

Summing these results together and taking the Hodge dual yields

$$*d{*}df = \left(\frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r}\right) + \left(\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{cot\theta}{r^2}\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \tag{3.24}$$

which we recognize as the Laplacian given in spherical coordinates.

For more excerpts of these chapters, review appendix A, or even better, see [1].

One way in which Zwiebach [7] is able to reach a wider audience is he avoids most of the finer mathematical details of general relativity. Occasionally a metric other than the usual Minkowski metric is given, but one will not find any computations done with differential forms or covariant derivatives. This in itself is no real issue, as Zwiebach's essential goal is to provide the reader with a background in bosonic string theory. However, as one moves from Zwiebach's text to the graduate level texts, it is soon realized that the reader is assumed to be comfortable with the mathematical language of general relativity, making for a difficult transition from Zwiebach to these more advanced texts. *A Detailed Introduction to String Theory* [1], by including a few chapters devoted to the basics of general relativity, aims to ease this transition.

As a final note, it is important to point out that it is not the material in each of these "crash course" chapters which set this text apart from other textbooks. Rather what sets this text apart from the other standards is the fact that the liberty was taken to include this material within a textbook on string theory. Certainly one could learn much more about quantum field theory from Peskin and Schroeder [14], or general relativity from D'Inverno [15]. nut this would defeat one of the purposes of this text: replacing multiple books with a single document.

3.3 Exercises

Just like any other textbook, *A Detailed Introduction to String Theory* includes exercises at the end of each chapter. In general, the goal of homework exercises is twofold: to test the student's understanding of the material, and expand on the presented topics. The exercises given in this text followed the same criteria, yielding two different 'types' of homework problems. One type is for the reader to derive certain expressions given, forcing the reader to go through the details of some of the discussed computations; checking the reader's understanding of the mathematics involved. The second type of exercise is to expand on some of the details briefly or not at all discussed in the chapter. These exercises are usually longer than the first type, and typically has the reader to connect some of the concepts in string theory with ideas in other fields of mathematics and physics.

Here are examples of the two different types of exercises that are given. For example, in chapter four, "Classical Relativistic Strings", the first exercise asks the reader to derive a particular expression for the magnitude of the cross product between two vectors [1]:

1. Derive the expression $|x \times y| = \sqrt{(x \cdot x)(y \cdot y) - (x \cdot y)^2}$ using the conventional rules of dot products and cross products.

The motivation for this calculation was that the result is used during the derivation of the Nambu-Goto string action, and for the reader unfamiliar with the identity, completing this exercise allows them to prove the relation. Another example is from chapter seven, "A Crash Course in Quantum Field Theory". The first exercise asks the reader to derive the commutation relation between the creation and annihilation operators from the canonical commutation relations between the position and momentum operators.

The second type of homework exercise is meant to expand on the material given in the text. For example, in chapter eight, "Quantizing the Relativistic Point Particle", in exercise two the reader is asked to derive the given form of the Poisson bracket, and work out some of its interesting identities [1]:

2. In a sense, when we canonically quantize a classical theory we are really promoting our observables (dynamical variables) to operators, and insist that they satisfy an appropriate set of commutation relations that have a classical correspondence. Here we will explore this classical correspondence. (a) Let $\omega(p, q)$ be some function of the state variables p, q with no explicit time dependence. Show that the time variation of ω is

$$\frac{d\omega}{dt} = \sum_i \left(\frac{\partial \omega}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \right) \equiv \{\omega, \mathcal{H}\}$$

where $\mathcal{H}$ is the classical Hamiltonian, and where we have defined the *Poisson bracket* $\{\cdot, \cdot\}$.

The reader is then asked to compare these identities to the canonical commutation relations of ordinary quantum mechanics.

(b) Show that

$$\dot{q}_i = \{q_i, \mathcal{H}\} \qquad \dot{p}_i = \{p_i, \mathcal{H}\}$$

Compare this result to what was considered in problem 1. What can we say about Ehrenfest's theorem?

(c) Prove that

$$\{q_i, q_j\} = \{p_i, p_j\} = 0 \qquad \{q_i, p_j\} = \delta_{ij}$$

Compare this to the canonical commutation relations for the operators X_i and P_j. Write a brief statement describing what is meant by canonical quantization, i.e. how does one choose the canonical commutation relations?

In short, this exercise lends the reader insight into the method of 'canonical quantization'. Another example can be found in chapter nine, "Light Cone Quantization of the String". In this chapter the reader is introduced to the definition of a Lie algebra, and it is shown that the commutation relations of the Virasoro operators satisfy the definition. To augment this, in the third exercise the reader is asked to show that the Poisson bracket satisfies two of the properties necessary for a Lie algebra [1]:

3. A more complete definition for a Lie algebra is: L, a real or complex vector space with a law of composition given by the *bracket* $[X, Y]$, is a Lie algebra if the following are satisfied:

$$(i) \quad [X, Y] = -[Y, X]$$
$$(ii) \quad [X, aY + bZ] = a[X, Y] + b[X, Z]$$
$$(iii) \quad [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

(a) Check that the Virasoro operators for both the open and closed string satisfy property (ii).

(b) Determine whether the Poisson bracket given in the last chapter constitutes a Lie algebra (Hint: Don't bother showing property (iii), although it does hold. Rather check the first two properties.)

Ultimately, this exercise reveals that the Lie bracket need not be commutators, and also informs the reader that Lie algebras show up in various fields of physics.

Lastly, recall that *A Detailed Introduction to String Theory* was also designed for those who enjoy independent study. As one who studies other fields of physics independently, I have found that texts with hints or partly worked solutions to homework exercises can be extremely useful for going through the material on one's own. Remember that the goal of exercises is an educational one, and therefore a student who is unable to complete an exercise misses out on an educational opportunity. For this reason, the final appendix in the text gives partly worked solutions to all of the exercises, leaving most of the details up to the student, however acting as a guide for the student who is completely lost. All in all, the exercises coupled with the solutions further strengthen the reader's overall understanding of the material.

Chapter 4

Conclusion and Final Remarks

String theory, despite its popular interest, tends to remain a mystery to many young physics students. One reason for this is that learning string theory is at first a very daunting task, especially with the current standard texts on the subject. In light of this, Zwiebach has successfully written a "first course in string theory", one which allows the young physicist to approach the subject with minimal prior knowledge. Unfortunately, after reading his text, the student who wishes to move on to the core of string physics will have a difficult time doing so. To resolve this issue, I took on the task of writing *A Detailed Introduction to String Theory*, a text aimed for undergraduates (and also includes material for the beginning graduate student) with the hope of providing a bridge between Zwiebach's wonderful book and the current graduate level standards. One of the ways in which this text acts as a bridge is through the implementation of "crash course" chapters. These chapters provide the reader with brief introductions on material that is assumed to background in the graduate standards, such as quantum field theory, supersymmetry, and general relativity; providing many of the necessary ideas to continue on with string theory. Another way in which this book assists the student is through the inclusion of homework exercises provided with worked solutions.

In spite of the approach taken in *A Detailed Introduction to String Theory* [1], an imperative question remains: does this text succeed in its aim? There have only been a select few who have read through this book, making it difficult to determine how effective this approach really is. In short, a larger sample size must be used before any conclusive results on the effectiveness of this text can be given. One way in which I hope this can be accomplished is by making these extensive set of notes widely accessible and free of charge. I have also placed the text online so that anyone who wishes to read through them can at their own leisure. Moreover, the next step would be to lecture from these set of notes for the intended audience, yielding information on how they might be augmented or changed. Simply put, now that the writing part of this project is nearly complete, the more difficult task of teaching the material awaits.

A long term goal for future work is provide an undergraduate themed series on modern theoretical physics. So far, the loose outline includes a three volume text: Background, String Theory, and Quantum Gravity. The background volume would be the first book in the series and include material reviewing and expanding on many of the topics an undergraduate is assumed to know. This volume would also include “crash course” motivated parts, i.e. longer introductions to quantum field, SUSY, and general relativity. Other chapters would likely cover some of the more advanced topics in mathematics, such as Lie groups and topology. The second volume would essentially be *A Detailed Introduction to String Theory*, however with more material on modern research avenues. Lastly, the third volume would consist of an undergraduate introduction to canonical quantum gravity, a field which is certainly different to string theory in approach, however may in fact be connected in some way (so far it has not been determined if the two are compatible). Before this task is begun however, *A Detailed Introduction to String Theory* must be revised and lectured from.

As a final remark, a banal question is often raised: what if string theory is wrong? What if an experiment comes out showing explicitly that the universe is not higher dimensional, or that supersymmetry does not actually exist in our world, what is to become of string theory? Like all theories, string theory would have to be reimagined in some way; it would have to be reworked to see if the theory can accomplish its original goals without these ‘requirements’. If this is not possible, if string theory cannot be mended, then as good scientists we must take it upon ourselves to move on. This would not mean however that the efforts of string theory would have been fruitless; string theory has had tremendous influence on other fields of mathematical physics and several branches of pure mathematics. Similarly, regardless of whether string theory ends up being the “right” theory of everything, the student who reads this document is likely to learn the methods of modern theoretical physics, lessons which prove invaluable for any subject in physics and mathematics. All in all, whether string theory is found to be true or not, our perspective of physical reality will likely never be the same.

Appendix A

Excerpts from *A Detailed Introduction to String Theory*

A.1 Excerpt 1: Quantum Field Theory

Presented below are a few excerpts from "A Crash Course in Quantum Field Theory" from *A Detailed Introduction to String Theory.* To read the complete chapter, the reader is pointed to the entire text, which can be found in the references. To note, the majority of this chapter was based on material from [16]-[18].

After a brief outline, the chapter begins with short discussion on the Klein-Gordon equation and scalar fields [1]:

> The first attempts to merge relativity with quantum mechanics involved the relativistic generalization of the Schrödinger equation. Schrödinger himself actually came with this equation, known as the *Klein-Gordon* equation, however he abandoned it because it gave solutions with negative energy (which, as noted earlier, must stay), and gave the incorrect energy spectrum for hydrogen. It turns out the Klein-Gordon equation is successful in describing spin-0 bosons, and is therefore a tool we must become familiar with.
>
> In relativity, time and space are on equal footing. To make a relativistic wave equation, we seek to make the Schrödinger equation on equal footing with time and space. Recall that the Schrödinger equation may take the form
>
> $$i\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar^2}{2m}\vec{\nabla}^2 \cdot \psi + V\psi \tag{A.1}$$
>
> Immediately we see that space and time are not on equal footing; there is a first derivative in time while there is a second derivative in space. Therefore, we cannot start with the Schrödinger equation, but instead must use a different method. Recall

Einstein's famous formula

$$E^2 = p^2c^2 + m^2c^4 \tag{A.2}$$

Then, using the more general form of the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{E}\psi \tag{A.3}$$

we decide to promote the energy E to become an operator. That is, let

$$E \rightarrow i\hbar\frac{\partial}{\partial t} \tag{A.4}$$

Using (7.4) and the usual definition the quantum mechanical momentum operator, the Einstein relation for energy becomes

$$-\hbar^2\frac{\partial^2}{\partial t^2} = -\hbar^2c^2\vec{\nabla}^2 + m^2c^4 \tag{A.5}$$

Applying (7.5) to a function of space and time, $\phi(\vec{x}, t)$ and using natural units, we have

$$\frac{\partial^2\phi}{\partial t^2} - \vec{\nabla}^2\phi + m^2\phi = 0 \tag{A.6}$$

which also takes the form

$$(\Box + m^2)\phi = 0 \tag{A.7}$$

where

$$\Box \equiv \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$$

Since both $\Box$ and m^2 are scalars, the operator $(\Box + m^2)$ is a scalar as well. Therefore, the Klein-Gordon equation is said to apply to scalar fields, which have been found to represent spin-0 particles [18]. We may also write the Klein-Gordon equation as

$$(\partial_\mu\partial^\mu + m^2)\phi = 0 \tag{A.8}$$

As written, the Klein-Gordon equation describes a free particle. Therefore it has a classical plane-wave solution, namely

$$\phi(\vec{x}, t) = e^{-ip\cdot x}$$

Remember that when we are working in relativity p and x are actually 4-vectors and therefore the scalar product is given by

$$p \cdot x = p_\mu x^\mu = Et - \vec{p}\cdot\vec{x}$$

Notice then

$$\frac{\partial\phi}{\partial t} = \frac{\partial}{\partial t}e^{-i(Et-\vec{p}\cdot\vec{x})} = -iE\phi$$

$$\nabla\phi = \nabla e^{-i(Et-\vec{p}\cdot\vec{x})} = i\vec{p}\phi$$

Together, we find

$$(E^2 - \vec{p}^2 - m^2) = 0 \rightarrow E = \pm\sqrt{\vec{p}^2 + m^2} \tag{A.9}$$

where we have kept both signs of the energy for a reason...

...Equation (7.9) implies that the energy of the particle takes on both positive and negative energy states. The issue of negative energy states turns out to not be a problem, as it is also a consequence of imposing causality in our theory, which we desire.

There is another issue however which also troubled Schrödinger when he arrived to the Klein-Gordon equation. The Klein-Gordon equation leads to a negative probability density in the free particle case. To see this explicitly, consider one spatial dimension for simplicity and assume that the probability current takes the usual form [19]:

$$J = -i\phi^* \frac{\partial \phi}{\partial x} + i\phi \frac{\partial \phi^*}{\partial x} \tag{A.10}$$

Taking the spatial derivative of the probability current yields

$$\frac{\partial J}{\partial x} = -i\phi^* \frac{\partial^2 \phi}{\partial x^2} + i\phi \frac{\partial^2 \phi}{\partial x^2}$$

Using the Klein-Gordon equation in one dimension

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial t^2} + m^2 \phi$$

we find

$$\frac{\partial J}{\partial x} = -i\phi^* \frac{\partial^2 \phi}{\partial x^2} + i\phi \frac{\partial^2 \phi}{\partial x^2} = -i \left(\phi^* \frac{\partial^2 \phi}{\partial t^2} - \phi \frac{\partial^2 \phi^*}{\partial t^2} \right) \tag{A.11}$$

A fundamental result from quantum mechanics is that the probability density ρ and the probability current J satisfy the conservation of probability equation [19]:

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0 \tag{A.12}$$

Hence,

$$\frac{\partial \rho}{\partial t} = i \left(\phi^* \frac{\partial^2 \phi}{\partial t^2} - \phi \frac{\partial^2 \phi^*}{\partial t^2} \right)$$

Leading to

$$\rho = i \left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t} \right) \tag{A.13}$$

Using our plane wave solution, we find

$$\rho = i \left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t} \right) = 2E \tag{A.14}$$

But $E = \pm\sqrt{\vec{p}^2 + m^2}$. Therefore, allowing the negative energy solution to exist yields a negative probability density

$$\rho = -2\sqrt{\vec{p}^2 + m^2} < 0$$

which doesn't make any sense at all! A first step to solve this problem is to quantize the fields by promoting ϕ to become an operator.

Notice that the essential motivation of this section is to provide an introduction to the concept of a field, as well as an interesting consequence from keeping both signs of the energy (a first inkling of particle and antiparticle states). From here the chapter discusses the quantization procedure for a free scalar field:

The process of quantizing a field basically involves us imposing commutation relations. *Canonical quantization* refers to the process of imposing the fundamental commutation relations in position and momentum

$$[\hat{x}, \hat{p}] = i \tag{A.15}$$

A similar procedure holds for quantizing classical fields. This method is formally known as *second quantization*, although it is a misleading name. To quantize fields we must continue to place space and time on equal footing. In quantum field theory, momentum and position revert back to parameters, just as they were in ordinary classical mechanics, and instead we promote the fields to operators, imposing *equal time* commutation relations on fields and their conjugate momentum fields. The fields are operators in the sense that they act on quantum states to destroy or create particles, which is important since particle number is not fixed in relativity theory [18].

But changing the number of particles has its roots in the simple harmonic oscillator from ordinary quantum mechanics. Let's briefly review. Recall that the Hamiltonian for a simple harmonic oscillator in quantum mechanics is [18]:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 \tag{A.16}$$

Let us define the *creation* and *annihilation* operators (also known as the raising and lowering operators):

$$\hat{a} = \sqrt{\frac{m\omega}{2}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \tag{A.17}$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right) \tag{A.18}$$

Using the commutation relation of $\hat{x}$ and $\hat{p}$, as the reader will show, one finds

$$[\hat{a}, \hat{a}^\dagger] = 1 \tag{A.19}$$

In terms of the creation and annihilation operators, the Hamiltonian takes the form

$$\hat{H} = \omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) \tag{A.20}$$

We define the *number operator* as

$$\hat{N} = \hat{a}^\dagger\hat{a} \tag{A.21}$$

which satisfies the eigenvalue equation,

$$\hat{N}|n\rangle = n|n\rangle$$

The Hamiltonian can then be written as

$$\hat{H} = \omega(\hat{N} + \frac{1}{2}) \tag{A.22}$$

The eigenstates of the Hamiltonian satisfy

$$\hat{H}|n\rangle = \omega(n + \frac{1}{2})|n\rangle \tag{A.23}$$

Implying the eigenenergy is

$$E_n = \omega(n + \frac{1}{2}) \tag{A.24}$$

The annihilation operator has its name because it drops the eigenstate $|n\rangle$ by one unit in the following way:

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \tag{A.25}$$

Alternatively, the creation operator increases the eigenstate $|n\rangle$ by one unit as

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{A.26}$$

There is a lowest lying energy state in quantum mechanics called the ground state or *vacuum state*. We enforce the condition that the vacuum state is annihilated:

$$\hat{a}|0\rangle = 0 \tag{A.27}$$

But $\hat{a}^\dagger$ raises the energy of the system without limit. Therefore, we can obtain a generic state from the vacuum state as

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle \tag{A.28}$$

We call the collection of all states spanned by the states formed by operating on the vacuum state with any number of the creation operators a *Fock space* [18].

In quantum field theory, we take the notion of the number operator literally. The state $|n\rangle$ is not a state of a single particle, but rather a state of a field with n particles. The creation operator adds one particle to the field while the annihilation operator removes one particle from the field. Moreover, as we will see, the physical vacuum $|0\rangle$ has no particles present, however the fields remain, indicating that the vacuum state is not entirely void of everything.

Let's move on to quantizing the free scalar field. For now, consider a real scalar field that satisfies the Klein Gordon equation

$$\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi + m^2\phi = 0 \tag{A.29}$$

The free field solution of the Klein-Gordon equation is

$$\phi(x,t) = e^{-i(Et-px)}$$

If we use the wave number instead, then we let $E \to k^0 = \omega_k$ and $\vec{p} \to \vec{k}$, allowing us to write

$$\phi(x) = e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} \tag{A.30}$$

Doing this allows us to write the general solution of the Klein-Gordon equation in terms of a Fourier expansion [18]

$$\phi(x) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_k}}\left[\phi(\vec{k})e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} + \phi^*(\vec{k})e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})}\right] \tag{A.31}$$

We now promote the field $\phi(x)$ to become an operator by having $\phi(\vec{k}) \to \hat{a}(\vec{k})$ and $\phi^*(\vec{k}) \to \hat{a}^\dagger(\vec{k})$. Therefore, the field operator is given by

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_k}}\left[\hat{a}(\vec{k})e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} + \hat{a}^\dagger(\vec{k})e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})}\right] \tag{A.32}$$

To impose the commutation relations, we also require a conjugate momentum to the field. The Klein-Gordon Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu \partial^\mu \phi - \frac{1}{2}m^2\phi \tag{A.33}$$

The conjugate momentum of the field is then

$$\Pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi \tag{A.34}$$

A brief calculation yields the conjugate momentum operator:

$$\partial_0 \hat{\phi}(x) = \partial_0 \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_k}} \left[\hat{a}(\vec{k})e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} + \hat{a}^\dagger(\vec{k})e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})}\right]$$

$$= \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_k}} \left[\hat{a}(\vec{k})(-i\omega_k)e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} + \hat{a}^\dagger(\vec{k})(i\omega_k)e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})}\right]$$

$$= -i \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{\omega_k}{2}} \left[\hat{a}(\vec{k})e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} - \hat{a}^\dagger(\vec{k})e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})}\right] \tag{A.35}$$

The commutation relations we impose follow from the canonical commutation relations from non-relativistic quantum mechanics:

$$[\hat{x}_i, \hat{p}_j] = i\delta_{ij} \tag{A.36}$$

$$[\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0 \tag{A.37}$$

For fields we impose the *equal time* commutation relations:

$$[\hat{\phi}(x), \hat{\Pi}(y)] = i\delta(\vec{x} - \vec{y}) \tag{A.38}$$

$$[\hat{\phi}(x), \hat{\phi}(y)] = [\hat{\Pi}(x), \hat{\Pi}(y)] = 0 \tag{A.39}$$

We call these the equal time operators since although $\vec{x} \neq \vec{y}$, we assume that we looking at the fields at the same time, i.e. $x^0 = y^0$...

...Now that we know how to write the field operators in terms of the creation and annihilation operators, we can see how the operators act on the state of the fields. Due to our understanding of the simple harmonic oscillator from ordinary quantum mechanics, we already have an idea on how the operators behave. Let's begin by considering the vacuum state $|0\rangle$. Analogous to the harmonic oscillator, the vacuum state is destroyed by the annihilation operator

$$a(\vec{k})|0\rangle = 0 \tag{A.40}$$

where we use the wave vector $\vec{k}$ to notate our states. On the other hand

$$|\vec{k}\rangle = a^\dagger(\vec{k})|0\rangle \tag{A.41}$$

which describes a one-particle state. If we use multiple creation operators of different modes, we may construct a Fock space

$$|\vec{k}_1, \vec{k}_2, ...\vec{k}_n\rangle = a^\dagger(\vec{k}_1)a^\dagger(\vec{k}_2)...a^\dagger(\vec{k}_n)|0\rangle \tag{A.42}$$

The accurate interpretation of the action of creation operator is that each creation operator $a^\dagger(\vec{k}_i)$ creates a single particle with momentum $\hbar\vec{k}_i$ and energy $\hbar\omega_{\vec{k}_i}$ where

$$\omega_{\vec{k}_i} = \sqrt{\vec{k}_i^2 + m^2}$$

Alternatively, the annihilation operator destroys particles with the same momentum and energy.

We can construct the number operator from the creation and annihilation operators analogously to the number operator defined in non-relativistic quantum mechanics:

$$N(\vec{k}) = a^\dagger(\vec{k})a(\vec{k}) \tag{A.43}$$

The eigenvalues of the number operator, $n(\vec{k})$ are called *occupation numbers* and are integers, telling us how many particles there are of momentum $\vec{k}$ for a given state. Therefore (7.53) is a state consisting of n particles, with a single particle with momentum $\vec{k}_1$, a single particle of momentum $\vec{k}_2$ and so on. We can also have states where there are multiple particles of the same momentum. Consider for example the state

$$|\vec{k}_1, \vec{k}_1, \vec{k}_2\rangle = \frac{a^\dagger(\vec{k}_1)a^\dagger(\vec{k}_1)}{\sqrt{2}} a^\dagger(\vec{k}_2)|0\rangle$$

We may rewrite this state as

$$|\vec{k}_1, \vec{k}_1, \vec{k}_2\rangle = |n(\vec{k}_1)n(\vec{k}_2)\rangle$$

where $n(\vec{k}_1) = 2$ and $n(\vec{k}_2) = 1$. Therefore,

$$|n(\vec{k}_1)n(\vec{k}_2)\rangle = \frac{(a^\dagger)^{n(\vec{k}_1)}}{\sqrt{n(\vec{k}_1)!}} \frac{(a^\dagger)^{n(\vec{k}_2)!}}{\sqrt{n(\vec{k}_2)}}|0\rangle$$

In general, the Fock space takes the form [18]:

$$|n(\vec{k}_1)n(\vec{k}_2)...n(\vec{k}_n)\rangle = \prod_{j=1}^{n} \frac{(a^\dagger)^{n(\vec{k}_j)}}{\sqrt{n(\vec{k}_j)!}}|0\rangle \tag{A.44}$$

As written, the number $N(\vec{k})$ is actually a number *density.* To get the total number of particles, we integrate over all states in momentum space

$$\hat{N} = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) \tag{A.45}$$

Moreover, in terms of the number operator, it can be shown that the Hamiltonian and momentum take the form...

By the time the reader is at the level of this text they would have likely encountered the quantization procedure of the harmonic oscillator. Using this as an analogy, the reader can then see the basic idea behind quantizing free scalar fields, and, more interestingly, free scalar fields may interpreted as a collection of "harmonic oscillators".

In the rest of this section, and several sections after, various aspects of free scalar fields are explored, including charged scalar fields, briefly, scalar fields using light-cone coordinates, and, briefly, the Feynman propagator. Eventually, the reader is given an introduction to free spinor fields and the Dirac equation:

> One the issues with the Klein-Gordon equation is that it does not give the correct spectra of the hydrogen atom, which was one of the original reasons why Schrödinger abandoned the Klein-Gordon equation. Moreover, the Klein-Gordon equation included negative energy solutions. We were able show that these negative energy solutions correspond to antiparticles, but in 1928, the physics community was unaware of antiparticles. Paul Dirac approached the difficulties of the Klein-Gordon equation by inventing his own equation, which we will go on to discuss briefly here.
>
> The Klein-Gordon equation is second-order in space and time. Dirac went the other way and instead chose an equation which is first order in space and time. His reason for this is that it eliminated the negative probability density which appeared to plague the Klein-Gordon equation. Moreover, to maintain Lorentz covariance, space and time should be treated on equal footing, another reason for having both derivatives in space and time be of the same order. A possible candidate for a relativistic wave equation fitting this description is given by
>
> $$i\frac{\partial\psi}{\partial t} = -i\vec{\alpha}\cdot\frac{\partial\psi}{\partial\vec{x}} + \beta m\psi \tag{A.46}$$
>
> We would like the components of ψ to satisfy the Klein-Gordon equation so that the relativistic relation of energy and momentum for a free particle holds. In the non-relativistic case, ψ can have two components, spin up and spin down, thereby requiring that α_i and β be matrices, making the Dirac equation a matrix equation. If we apply the operators $E = i\frac{\partial}{\partial t}$ and $p_j = -i\frac{\partial}{\partial x}$ twice to the Dirac equation, the Klein-Gordon equation should fall out. If we demand this to be true, one can show that the matrices must satisfy [17]
>
> $$\{\alpha_i, \alpha_j\} = 2\delta_{ij} \tag{A.47}$$
>
> $$\{\alpha_i, \beta\} = 0 \tag{A.48}$$
>
> $$\beta^2 = 1 \tag{A.49}$$
>
> Moreover, since $\alpha_i^2 = 1$, the eigenvalues of α_i and β are ± 1. It follows then that α_i and β must be traceless:
>
> $$tr(\alpha_i) = tr(\alpha_i\beta^2) = tr(\beta\alpha_i\beta) = -tr(\beta^2\alpha_i) = -tr(\alpha_i) \to tr(\alpha_i) = 0$$
>
> Since the eigenvalues are ± 1 and the matrices are traceless, α_i and β must be even dimensional. If we picked two dimensions we would simply choose the Pauli-spin matrices. Our next option is dimension four with the *Dirac matrices*:
>
> $$\alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix} \tag{A.50}$$
>
> $$\beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \tag{A.51}$$
>
> where the σ_i's are the familiar Pauli-spin matrices. It turns out to be more convenient to work with the *gamma matrices* defined as $\gamma^0 = \beta$ and $\gamma^i = \beta\alpha_i$, and they satisfy
>
> $$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{A.52}$$

In terms of the gamma matrices, the Dirac equation is of the form

$$(i\partial\!\!\!/ - m)\psi = 0 \tag{A.53}$$

where we have employed Feynman's *slash* notation, $\partial\!\!\!/ \equiv \gamma^\mu \frac{\partial}{\partial x^\mu}$. Finally, so that we don't spend too much time on the Dirac equation, without proof, however well established, the Dirac equation in fact describes spin-$\frac{1}{2}$ particles, i.e. fermions.

With the Dirac equation we can also find the conserved current that it produces. However, we must also know the conjugate equation. Taking the complex conjugate of (7.92), we find

$$-i\frac{\partial\psi^\dagger}{\partial t} = i\vec{\alpha}\cdot\frac{\partial\psi^\dagger}{\partial\vec{x}} + m\beta\psi^\dagger \tag{A.54}$$

since α_i and β are Hermitian. To recover the form with the gamma matrices, we multiply the right hand side of (7.100) by $1 = \beta^2$, in turn causing $\psi^\dagger$ to be everywhere multiplied by β. We then define

$$\bar{\psi} = \psi^\dagger\beta = \psi^\dagger\gamma^0 \tag{A.55}$$

Therefore,

$$-i\frac{\partial\bar{\psi}}{\partial t}\gamma^0 = i\frac{\partial\bar{\psi}}{\partial\vec{x}}\cdot\vec{\gamma} + \bar{\psi}m \tag{A.56}$$

Or,

$$(i\partial\!\!\!/ + m)\bar{\psi} = 0 \tag{A.57}$$

To find the current, we multiply (7.99) by $\bar{\psi}$, multiply (7.103) by ψ and sum the two, yielding,

$$\partial_\mu\bar{\psi}\gamma^\mu\psi + \bar{\psi}\gamma^\mu\partial_\mu\psi = \partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0 \tag{A.58}$$

Thus, the conserved current is

$$j^\mu\bar{\psi}\gamma^\mu\psi \tag{A.59}$$

Moreover, the charge density is given by

$$j^0 = \psi^\dagger\psi \tag{A.60}$$

Remember, the Dirac equation describes spin-$\frac{1}{2}$ fermions, but ψ is 4-component column vector called a *spinor*. Since we forced the components of ψ to satisfy the Klein-Gordon equation, we, inevitably, introduced negative energy solutions, one that is spin-up and one that is spin-down. Therefore, the components of the spinor describes an electron with both spin states, and a positron (antielectron) with both spin states. Let's denote the positive energy solution by

$$\psi_+ = e^{-ip\cdot x}u(p) \tag{A.61}$$

where $u(p)$ is a 4-component spinor. Similarly, we denote the negative energy solution as

$$\psi_- = e^{ip\cdot x}v(p) \tag{A.62}$$

Subsituting both ψ_+ and ψ_- into the Dirac equation, we find

$$(p\!\!\!/ - m)u(p) = 0 \tag{A.63}$$

$$(p\!\!\!/ + m)v(p) = 0 \tag{A.64}$$

For particles at rest, $\vec{p} = 0$ and therefore $p^0 = E = m$. The positive energy solution then becomes

$$(\beta - 1)u(0) = 0 \tag{A.65}$$

Giving rise to two independent solutions:

$$u^1(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; u^2(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{A.66}$$

The solution $u^1(0)$ describes a positive energy particle which is spin up, while $u^2(0)$ describes a positive energy particle which is spin down. Similarly, for the negative energy solution we have

$$(\beta + 1)v(0) = 0 \tag{A.67}$$

giving rise to

$$v^1(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \; v^2(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{A.68}$$

where $v^1(0)$ describes a negative energy particle with spin up, and $v^2(0)$ describes a negative energy particle that is spin down. For particles not at rest, we solve (7.109) and (7.110) noting that $(\not{p} - m)(\not{p} + m) = p^2 - m^2 = 0$ such that $u^i(p) = N(\not{p} + m)u^i(0)$,where N is a normalization constant, satisfies (7.109), and $v^i(p) = N(-\not{p} + m)v^i(0)$ satisfies (7.110). With a bit of algebra, one can show that the resulting spinors for the positive and negative energy solutions to the Dirac equation take the form

$$u^1(p) = \begin{bmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_+}{E+m} \end{bmatrix}, \; u^2(p) = \begin{bmatrix} 0 \\ 1 \\ \frac{p_-}{E+m} \\ -\frac{p_z}{E+m} \end{bmatrix} \tag{A.69}$$

$$v^1(p) = \begin{bmatrix} \frac{p_z}{E+m} \\ \frac{p_+}{E+m} \\ 1 \\ 0 \end{bmatrix}, \; v^2(p) = \begin{bmatrix} \frac{p_-}{E+m} \\ -\frac{p_z}{E+m} \\ 0 \\ 1 \end{bmatrix} \tag{A.70}$$

where $p_\pm = p_x \pm ip_y$, and $E = (p^2 + m^2)^{\frac{1}{2}}$.

Dirac succeeded in producing a non-negative current density, getting rid of the negative probability densities that seemed to plague the Klein-Gordon equation, however, he could not get rid of the negative energy solutions. Faced with the problem of having an energy spectrum which is unbounded from below, Dirac supposed that all negative energy states, the *negative electron sea*, were filled. Since the equation describes fermions, by Pauli's exclusion principle no two electrons can occupy the same state. Therefore, the positive energy electrons are prevented from falling into the negative energy sea because there are no vacancies. Solving this issue implicitly assumed that one was no longer dealing with a one particle system. Moreover, the ground state with no positive energy electrons is no longer empty since the negative energy electrons would still be present. Rather, the *bare* vacuum, is the vacuum void of positive and negative energy electrons. It turns out however that the bare vacuum is unstable.

There are quantum fluctuations in the bare vacuum, making it unstable. Therefore, if we start with an empty vacuum, eventually a fluctuation will create a pair of electrons, one with negative energy, and one with positive energy, thereby causing the total energy of the new state to be zero. This system can lower its energy however by letting a particle pair "fall" to the bottom of the sea. This means that the bare vacuum is no longer empty, nor is the energy zero since it has been lowered. These fluctuations continue to occur until the negative energy sea is full. The fluctuations will no longer be able to create a zero energy pair because there will be no place to put the negative energy pair (the sea is "full"). This is what we call the *physical* vacuum, a vacuum which is stable against quantum fluctuations. What's more is the physical vacuum is not empty, as one might assume.

Particles and antiparticles can be distinguished by their charge. For scalar fields, we introduced a complex field ϕ^* to distinguish between particles and antiparticles. Suppose we did not want to describe spin-$\frac{1}{2}$ fermions with charge. This is analogous to our theory of real scalar fields. We therefore wish to make ψ real. This is possible if we work in a different representation of the gamma matrices. Instead we use the *Majorana* representation where we take

$$\gamma^0 = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix}, \ \gamma^1 = \begin{bmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{bmatrix} \tag{A.71}$$

$$\gamma^2 = \begin{bmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{bmatrix}, \ \gamma^3 = \begin{bmatrix} -i\sigma_3 & 0 \\ 0 & -i\sigma_3 \end{bmatrix} \tag{A.72}$$

If we multiply γ^μ by i, all four matrices are real. The Dirac equation then is

$$(i\partial\!\!\!/ - m)\psi = (i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{A.73}$$

which has solutions that are real. Therefore, with Majorana fermions, we cannot expect to distinguish between particles and antiparticles since now we are dealing with a real spinor field.

Finally, a last case to consider is when we are dealing with massless fermions. The Dirac equation is simply,

$$i\partial\!\!\!/\psi = 0 \tag{A.74}$$

By a proper choice of representation of the gamma matrices, we may decouple the 4-component spinor into 2-component spinors. Such a choice of representation is known as the *Weyl* representation or *chiral* representation, where the gamma matrices take the form

$$\gamma^0 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \ \gamma^i = \begin{bmatrix} i\sigma_i & 0 \\ 0 & -i\sigma_i \end{bmatrix} \tag{A.75}$$

When we restrict ψ to two components, we lose the antiparticles, but with a real four component we cannot distinguish the particles from antiparticles since there is no charge. We can distinguish particles from antiparticles through their *chirality* however. When the projection of spin of the particle onto the direction of motion is positive, we say that the particle has positive chirality. If the opposite hold, the particle has negative chirality. It turns out that particles and antiparticles have opposite chirality [17]. The operator which distinguishes chirality is the γ^5 matrix,

$$\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{A.76}$$

In the Weyl representation, the chirality operator takes the form

$$\gamma_5 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{A.77}$$

> If this is your first time seeing spinor fields, chirality, and helicity, don't be too alarmed, as when we quantize the bosonic string one doesn't need to work with fermions. However, in a later chapter when we examine the basics of *supersymmetry*, and move on to superstring theories these notions are fundamental, and will be examined in more detail there...

As can be observed by this final note at the bottom, the reader new to quantum fields is not expected to work too much with fermions and spinor fields. This is due to the fact that the text primarily deals with bosonic string theory, which is free of fermions. Of course, in the "Advanced Topics" section, superstrings are considered, and there the reader should be familiar with the Dirac equation and free spinor fields (note that the first track skips the sections on fermion fields as well as superstrings).

A.2 Excerpt 2: Supersymmetry

The bulk of this chapter is based around Patrick Labelle's *Supersymmetry Demystified* [13], a real tour de force seeking to provide a basic background on this subject. If the reader is interested in the topics presented in this chapter, they are urged to peruse Labelle's text. Another fair text on the subject, though a bit more advanced, is Aitchison's *Supersymmetry in Particle Physics* [20].

"A Crash Course in Supersymmetry", after beginning with a brief discussion on the history and physical motivation of supersymmetry (SUSY), Weyl spinors and the Dirac equation are reviewed [1]:

> In the chapter on quantum field theory, we found that the Dirac equation could be written in the form
>
> $$\gamma^\mu P_\mu \psi = m\psi \tag{A.78}$$
>
> where $P_\mu = i\partial_\mu$, and ψ is a four component *Dirac spinor*. Remember, the Dirac equation and spinor fields describe fermions. Moreover, recall the Dirac Lagrangian density
>
> $$\mathcal{L} = \bar{\psi}(\gamma^\mu P_\mu - m)\psi \tag{A.79}$$
>
> where $\bar{\psi} = \psi^\dagger \gamma^0$. We are going to be using dagger notation in this chapter to denote the Hermitian conjugate, as we will use the asterisk symbol to denote typical complex conjugation. As we mentioned before in the chapter on quantum field theory, there are alternative representations for the matrics γ^0 and $\vec{\gamma}$. When we study supersymmetry, we will make use of the representation
>
> $$\gamma^0 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \qquad \vec{\gamma} = \begin{bmatrix} 0 & -\vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix} \tag{A.80}$$

where I is the 2×2 identity matrix and $\vec{\sigma}$ are the *Pauli matrices* we are familiar with from ordinary quantum mechanics

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{A.81}$$

One property of the Pauli matrices is that the product of any two Pauli matrices is

$$\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k \tag{A.82}$$

where δ^{ij} is the Kronecker delta, and ϵ^{ijk} is the totally antisymmetric Levi-Civita tensor, having that $\epsilon^{123} = \epsilon^{231} = \epsilon^{312} = 1$, and $\epsilon^{321} = \epsilon^{213} = \epsilon^{132} = -1$, and has all other components equal to zero. Equation (14.5) allows us to compute the commutator and anticommutator between two Pauli matrices:

$$[\sigma^i, \sigma^j] = \sigma^i\sigma^j - \sigma^j\sigma^i = 2i\epsilon^{ijk}\sigma^k \tag{A.83}$$

$$\{\sigma^i, \sigma^j\} = \sigma^i\sigma^j + \sigma^j\sigma^i = 2\delta^{ij} \tag{A.84}$$

We can use (14.3) to define

$$\gamma^\mu = (\gamma^0, \vec{\gamma}) \tag{A.85}$$

Moreover, using the mostly minus convention of the Minkowski metric yields the covariant version of (14.3):

$$\gamma_\mu = \eta_{\mu\nu}\gamma^\nu = (\gamma^0, -\vec{\gamma}) \tag{A.86}$$

We had also defined the γ_5 matrix which is important for defining *chirality*

$$\gamma_5 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \tag{A.87}$$

The γ_5 matrix comes in handy when we define Weyl spinors. Let's do that now. First let us write the four component Dirac spinor in terms of two two component spinors. That is,

$$\psi = \begin{bmatrix} \eta \\ \chi \end{bmatrix} \tag{A.88}$$

These two component spinors, η and χ are called *Weyl spinors*. For reasons which will become clear shortly, we decompose the Dirac spinor into two spinors because the Weyl spinors separately transform under Lorentz transformations, which will help us build Lorentz invariants. More precisely, we say that a Dirac spinor is a reducible representation of the Lorentz group while Weyl spinors form an irreducible representation, which, in a sense, suggests that Weyl spinors are more fundamental than Dirac spinors.

It is important to note that if we set either Weyl spinor equal to zero in the Dirac spinor, we find the eigenstates of the γ_5 matrix. In particular,

$$\gamma_5 \begin{bmatrix} \eta \\ 0 \end{bmatrix} = + \begin{bmatrix} \eta \\ 0 \end{bmatrix} \qquad \gamma_5 \begin{bmatrix} 0 \\ \chi \end{bmatrix} = - \begin{bmatrix} 0 \\ \chi \end{bmatrix}$$

We call the eigenvalue of γ_5 the *chirality* of the spinor. A Weyl spinor with positive chirality is sometimes referred to as a *right-chiral spinor*, while a Weyl spinor with negative chirality is referred to as a *left-chiral spinor*. In that sense, we see that η is a right-chiral spinor and χ is a left-chiral spinor. We therefore sometimes denote η by η_R, and χ by χ_L. We will only use η to mean right-chiral spinors, and χ to mean left chiral spinors in this text, so we avoid using the subscript. As we will see in this chapter, left-chiral spinors, though mostly through convention, are of particular importance in supersymmetry.

We leave it to the reader to show that substituting (14.11) into the Dirac equation yields two coupled equations

$$(EI - \vec{\sigma} \cdot \vec{P})\eta = m\chi \qquad (EI + \vec{\sigma} \cdot \vec{P})\chi = m\eta \tag{A.89}$$

Before moving on, let us introduce some further notation that will prove useful later on. Let us first define

$$\sigma^\mu \equiv (I, \vec{\sigma}) \qquad \bar{\sigma}^\mu \equiv (I, -\vec{\sigma}) \tag{A.90}$$

Then, using the fact that $(\sigma^2)^2 = I$, we find

$$\sigma^2 \vec{\sigma} \sigma^2 = -\vec{\sigma}^* \qquad \sigma^2 \vec{\sigma}^T \sigma^2 = -\vec{\sigma}$$

from which gives us the useful identities

$$\sigma^2 (\sigma^\mu)^T \sigma^2 = \bar{\sigma}^\mu \qquad \sigma^2 (\bar{\sigma}^\mu)^T \sigma^2 = \sigma^\mu \tag{A.91}$$

Or, taking the transpose, and using $(\sigma^2)^T = -\sigma^2$, we find

$$\sigma^2 \sigma^\mu \sigma^2 = (\bar{\sigma}^\mu)^T \qquad \sigma^2 \bar{\sigma}^\mu \sigma^2 = (\sigma^\mu)^T \tag{A.92}$$

Another identity that is rather trivial in proving is

$$\bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu = \sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = 2\eta^{\mu\nu} \tag{A.93}$$

Finally, using the definitions of σ^μ and $\bar{\sigma}^\mu$, the coupled equations become

$$P_\mu \sigma^\mu \eta = m\chi \qquad P_\mu \bar{\sigma}^\mu \chi = m\eta \tag{A.94}$$

Moreover, we may write the gamma matrices γ^μ as

$$\gamma^\mu = \begin{bmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{bmatrix} \tag{A.95}$$

In terms of Weyl spinors, the Dirac Lagrangian can be written, as the reader will prove, in the form

$$\mathcal{L} = \eta^\dagger \sigma^\mu i\partial_\mu \eta + \chi^\dagger \bar{\sigma}^\mu i\partial_\mu \chi - m\eta^\dagger \chi - m\chi^\dagger \eta \tag{A.96}$$

As noted above, supersymmetry requires a fair understanding of fermions and the Dirac equation. It is for this reason that this "crash course" chapter is left out of the first time reader's track. Moreover, this chapter should not be attempted until after the sections on the Dirac equations and spinor fields in chapter seven are worked through.

From here a fair amount of time is spent devoted to becoming familiar with the language of left chiral spinors; developing Lorentz transformations of spinors, the spinor 'dot product', charge conjugation, and massive spinors. Once all of these details have been discussed, the chapter shows the reader how to construct the simplest supersymmetric Lagrangian possible, as noted above. The real meat of the chapter is discussion on the SUSY charges, and the supersymmetric algebra formed by the commutation and anticommutation relations formed by these charges. No details are left out, providing the reader with a detailed guide to the mathematical structure of SUSY. Below is an excerpt of discussion on the SUSY algebra:

...We will now make use of (14.79) to find the algebra of supersymmetric charges. First of all, since we are using a two component spinor ζ and its complex conjugate ζ^*, we have a total of four charges, denoted as $Q_1, Q_2, Q_1^\dagger, Q_2^\dagger$, and may form a Weyl spinor, which we shall simply label as Q, and it has a Hermitian conjugate $Q^\dagger$. These are our supersymmetric charges which are often called *supercharges*.

We require that the argument in the exponential of our unitary operator be Lorentz invariant. What's more is we may also choose that Q is a left-chiral spinor, in which case we have the two possible Lorentz invariants

$$Q \cdot \zeta = Q(-i\sigma^2)\zeta \qquad \bar{Q} \cdot \bar{\zeta} = Q^\dagger i\sigma^2 \zeta^* \tag{A.97}$$

Using this in our unitary operator U that generates SUSY transformations is given by

$$U_\zeta = exp(iQ \cdot \zeta + i\bar{Q} \cdot \bar{\zeta}) \tag{A.98}$$

If we then apply (14.66), and our SUSY transformations given in (14.62) and (14.63), we find

$$[\zeta \cdot Q + \bar{\zeta} \cdot \bar{Q}, \phi] = -i\zeta \cdot \chi \qquad [\zeta \cdot Q + \bar{\zeta} \cdot \bar{Q}, \chi] = -i(\partial_\mu \phi)\sigma^\mu \sigma^2 \zeta^* \tag{A.99}$$

which imply

$$[\zeta \cdot Q, \phi] = -i\zeta \cdot \chi \qquad [\bar{\zeta} \cdot \bar{Q}, \chi] = -i(\partial_\mu \phi)\sigma^\mu \sigma^2 \zeta^* \tag{A.100}$$

$$[\bar{\zeta} \cdot \bar{Q}, \phi] = [\zeta \cdot Q, \chi] = 0 \tag{A.101}$$

Just as we did near the end of the last section, let us consider two successive SUSY transformations in the same way as before. This time β as the infinitesimal parameter of the second transformation, which is generated by $U_\beta = exp(iQ \cdot \beta + i\bar{Q} \cdot \bar{\beta})$. If we apply (14.79) to a scalar field ϕ our supersymmetric Lagrangian in (14.56), we find

$$\delta_\beta \delta_\zeta \phi - \delta_\zeta \delta_\beta = [[Q \cdot \zeta + \bar{Q} \cdot \bar{\zeta}, Q \cdot \beta + \bar{Q} \cdot \bar{\beta}], \phi]$$

$$= [[Q \cdot \zeta, Q \cdot \beta], \phi] + [[Q \cdot \zeta, \bar{Q} \cdot \bar{\beta}], \phi] + [[\bar{Q} \cdot \bar{\zeta}, Q \cdot \beta], \phi] + [[\bar{Q} \cdot \bar{\zeta}, \bar{Q} \cdot \bar{\beta}], \phi]$$

Let us first examine the right hand side and the commutators between the charges first. Using our results of the spinor dot product (14.28) and (14.34), the first commutator is just

$$[Q \cdot \zeta, Q \cdot \beta] = [Q(-i\sigma^2)\zeta, Q(-i\sigma^2)\beta] = -[Q_a(\sigma^2)^{ab}\zeta_b, Q_c(\sigma^2)^{cd}\beta_d]$$

As a warning to the reader, we remind ourselves that the ζ_i and β_i are Grassmann numbers while the Q are Grassmann operators, meaning that as we pass them through each other, we will pick up extra minus signs as they anticommute with each other. Therefore, the above becomes

$$-(\sigma^2)^{ab}(\sigma^2)^{cd}(Q_a\zeta_bQ_c\beta_d - Q_c\beta_dQ_a\zeta_b) = (\sigma^2)^{ab}(\sigma^2)^{cd}\zeta_b\beta_d(Q_aQ_c + Q_cQ_a)$$

$$= (\sigma^2)^{ab}(\sigma^2)^{cd}\zeta_b\beta_d\{Q_a, Q_c\} \tag{A.102}$$

Rather than commutation relations, we have instead found that anticommutators between the supercharges arise. However this could be expected as we are dealing with fermionic variables instead of ordinary commuting numbers. Using the same method, the three other commutators are:

$$[Q\cdot\zeta, \bar{Q}\cdot\bar{\beta}] = -(\sigma^2)^{ab}(\sigma^2)^{cd}\zeta_b\beta_d^*\{Q_a, Q_c^\dagger\} \tag{A.103}$$

$$[\bar{Q}\cdot\bar{\zeta}, Q\cdot\beta] = -(\sigma^2)^{ab}(\sigma^2)^{cd}\zeta_b^*\beta_d\{Q_a^\dagger, Q_c\} \tag{A.104}$$

$$[\bar{Q}\cdot\bar{\zeta}, \bar{Q}\cdot\bar{\beta}] = (\sigma^2)^{ab}(\sigma^2)^{cd}\zeta_b^*\beta_d^*\{Q_a^\dagger, Q_c^\dagger\} \tag{A.105}$$

In summary, we have

$$\delta_\beta\delta_\zeta\phi - \delta_\zeta\delta_\beta = [\mathcal{O}, \phi]$$

...

...Since α and β and their complex conjugates are completely arbitrary, we are led to four anticommutation relations [13]:

$$\{Q_a, Q_b\} = 0 \qquad \{Q_a^\dagger, Q_b^\dagger\} = 0 \tag{A.106}$$

$$\{Q_a, Q_c^\dagger\} = (\sigma^\mu)_{ac}P_\mu \qquad \{Q_a^\dagger, Q_c\} = (\sigma^\mu)_{ca}P_\mu \tag{A.107}$$

These last two anticommutators are of particular interest as it shows that the anticommutator of two supercharges yields space-time translations, further indicating the deep connection SUSY has with space-time transformations...

From here the reader is shown that an auxiliary field must introduced into the SUSY lagrangian developed previously in order for the SUSY algebra to be closed (a bit of ad hoc argument is used), and then the reader is shown how to physically interpret the results by observing the action of the SUSY charges on particular quantum states. It isn't until the last few sections does the reader become introduced to superspace and superfields. An excerpt of this material will not be given.

Altogether, between this "crash course" chapter and the two supplemental appendices, the reader is given a fairly detailed introduction to SUSY, giving them most of the necessary background to understand the basics of superstrings.

A.3 Excerpt 3: General Relativity

As discussed above, the "crash course" on general relativity can be really thought as being composed of three chapters. Each of these chapters augment each other, and each subsequent chapter is really continuation of the previous one. A majority of the material was based on McMahon [21], Carroll [22], D'Inverno [15], and my own notes collected from three courses on the subject.

The first of these chapters is "Elements of Differential Geometry", in which the reader learns the mathematical content of general relativity. Beginning with a quick review of a differentiable manifold, the reader is introduced to the notion of tensors, and covariant differentiation [1]:

> ...A problem arises however: when we compute the ordinary partial derivative of a tensor, we don't get a tensor back in general. Partial differentiation of tensors does not yield tensorial objects. To see this, consider the contravariant vector V'^a and let us differentiate this vector with respect to x'^c. We see that,
>
> $$\partial'_c V'^a = \frac{\partial}{\partial x'^c}\left(\frac{\partial x'^a}{\partial x^b}V^b\right) = \frac{\partial x^d}{\partial x'^c}\frac{\partial}{\partial x^d}\left(\frac{\partial x'^a}{\partial x^b}V^b\right)$$
> $$= \frac{\partial x'^a}{\partial x^b}\frac{\partial x^d}{\partial x'^c}\partial_d V^b + \frac{\partial^2 x'^a}{\partial x^b \partial x^d}\frac{\partial x^d}{\partial x'^c}V^b$$
>
> From our transformation law of tensors, we can easily show that a (1,1) tensor transforms as
>
> $$T'^a{}_{'c} = \frac{\partial x'^a}{\partial x^b}\frac{\partial x^d}{\partial x'^c}T^b{}_d$$
>
> Therefore we see that the first term above transforms like a (1,1) tensor, however the presence of the second term makes it so the resulting object is not tensorial since as a whole it does not transform as a tensor. All in all, a partial derivative of a tensor does not yield a tensor in general, a result which we strongly desire. To fix this problem, we end up introducing an auxiliary field onto the manifold, similar to the auxiliary field we introduced in the supersymmetric Lagrangian to make it so the SUSY algebra would close. In effect, we will come up with a new type of derivative, the *covariant derivative* which will allow us to differentiate tensors and get back tensors...
>
> ...Exchanging dummy indices by letting $\Gamma^c{}_{ba}V^b e_c \to \Gamma^b{}_{ca}V^c e_b$, we find
>
> $$\frac{\partial V}{\partial x^a} = \left(\frac{\partial V^b}{\partial x^a} + \Gamma^b{}_{ca}V^c\right)e_b \equiv \nabla_a V^b e_b \tag{A.108}$$
>
> where we have defined the *covariant derivative* to be
>
> $$\nabla_a V^b = \frac{\partial V^b}{\partial x^a} + \Gamma^b{}_{ca}V^c \tag{A.109}$$
>
> ...

Eventually the reader is introduced to the Riemann curvature tensor through the failure of covariant derivatives to commute:

...It is interesting to point out that, unlike ordinary partial differentiation, covariant differentiation is not commutative in general. For any general tensor $T^{a\cdots}{}_{b\ldots}$, we define its commutator as

$$\nabla_c \nabla_d T^{a\cdots}{}_{b\ldots} - \nabla_d \nabla_c T^{a\cdots}{}_{b\ldots} \tag{A.110}$$

To explicitly show that the commutator does not vanish in general, let us work out the case for some vector V^c. That is, we will work out

$$\nabla_a \nabla_b V^c - \nabla_b \nabla_a V^c$$

First recall that the covariant derivative of a contravariant vector is simply

$$\nabla_b V^c = \frac{\partial V^c}{\partial x^b} + \Gamma^c{}_{eb} V^e$$

Earlier we noted that the covariant derivative of a contravariant vector returns a tensor of rank $(1,1)$, therefore it is also useful to recall the covariant derivative of a $(1,1)$ tensor, namely

$$\nabla_c T^a{}_b = \partial_c T^a{}_b + \Gamma^a{}_{cd} T^d{}_b - \Gamma^d{}_{bc} T^a{}_d \tag{A.111}$$

Using this we see that

$$\nabla_a \nabla_b V^c = \nabla_a (\partial_b V^c + \Gamma^c{}_{eb} V^e)$$

Since the term inside is a $(1,1)$ tensor, we may use (17.23) yielding

$$\nabla_a \nabla_b V^c = \nabla_a (\partial_b V^c + \Gamma^c{}_{eb} V^e) = \partial_a (\partial_b V^c + \Gamma^c{}_{eb} V^e) + \Gamma^c{}_{ad} (\partial_b V^d + \Gamma^d{}_{eb} V^e)$$

$$-\Gamma^d{}_{ba} (\partial_d V^c + \Gamma^c{}_{ed} V^e)$$

Similarly,

$$\nabla_b \nabla_a V^c = \partial_b (\partial_a V^c + \Gamma^c{}_{ea} V^e) + \Gamma^c{}_{bd} (\partial_a V^d + \Gamma^d{}_{ea} V^e) - \Gamma^d{}_{ab} (\partial_d V^c + \Gamma^c{}_{ed} V^e)$$

Now we subtract the last two expressions term by term. Examining the first term in each expression, the difference is

$$\partial_a (\partial_b V^c + \Gamma^c{}_{eb} V^e) - \partial_b (\partial_a V^c + \Gamma^c{}_{ea} V^e) = \partial_a (\Gamma^c{}_{eb} V^e) - \partial_b (\Gamma^c{}_{ea} V^e)$$

$$= V^e (\partial_a \Gamma^c{}_{eb} - \partial_b \Gamma^c{}_{ea}) + \Gamma^c{}_{eb} \partial_a V^e - \Gamma^c{}_{ea} \partial_b V^e$$

For the other terms, we will assume that we are considering torsion free connections, therefore $\Gamma^a{}_{bc} = \Gamma^a{}_{cb}$. Subtracting these terms yields

$$\Gamma^c{}_{ad} (\partial_b V^d + \Gamma^d{}_{eb} V^e) - \Gamma^d{}_{ba} (\partial_d V^c + \Gamma^c{}_{ed} V^e) - \Gamma^c{}_{bd} (\partial_a V^d + \Gamma^d{}_{ea} V^e) + \Gamma^d{}_{ab} (\partial_d V^c + \Gamma^c{}_{ed} V^e)$$

$$= \Gamma^c{}_{ad} \partial_b V^d - \Gamma^c{}_{bd} \partial_a V^d + \Gamma^c{}_{ad} \Gamma^d{}_{eb} V^e - \Gamma^c{}_{bd} \Gamma^d{}_{ea} V^e$$

Relabeling indices and again using the fact we are considering torsion free connections, the above just becomes

$$\Gamma^c{}_{ea}\partial_b V^e - \Gamma^c{}_{eb}\partial_a V^e + \Gamma^c{}_{ad}\Gamma^d{}_{eb}V^e - \Gamma^c{}_{bd}\Gamma^d{}_{ea}V^e$$

Altogether, after some cancellation, we find that the commutator of V^c is

$$\nabla_a\nabla_b V^c - \nabla_b\nabla_a V^c = V^e(\partial_a\Gamma^c{}_{eb} - \partial_b\Gamma^c{}_{ea}) + \Gamma^c{}_{ad}\Gamma^d{}_{eb}V^e - \Gamma^c{}_{bd}\Gamma^d{}_{ea}V^e$$

$$= (\partial_a\Gamma^c{}_{db} - \partial_b\Gamma^c{}_{da} + \Gamma^c{}_{ae}\Gamma^e{}_{db} - \Gamma^c{}_{be}\Gamma^c{}_{da})V^d$$

Defining the term in the parentheses as the tensor

$$R^c{}_{dab} \equiv \partial_a\Gamma^c{}_{db} - \partial_b\Gamma^c{}_{da} + \Gamma^c{}_{ae}\Gamma^e{}_{db} - \Gamma^c{}_{be}\Gamma^e{}_{da} \tag{A.112}$$

we see that the commutator is

$$[\nabla_a, \nabla_b]V^c = \nabla_a\nabla_b V^c - \nabla_b\nabla_a V^c = R^c{}_{dab}V^d \tag{A.113}$$

The tensor $R^c{}_{dab}$ defined in (17.24) is known as the *Riemann tensor* or sometimes the *curvature tensor*...

From here the reader is introduced to parallel transport, geodesics, and how to quantitatively characterize the curvature of a manifold. The covariant derivative is often the conventional way of first learning the mathematics of computing space-time curvature in general relativity. There is a more practical approach through differential forms however. This alternative method was the way I first learned it, and it makes heavy use Cartan's structure equations. Due to its pragmatic use, and since differential forms show up in other contexts as well, the reader is introduced to this alternative method for computing curvature. After the wedge product, Hodge star operator, and Cartan's structure equations are properly defined, two example calculations are given: one being the ordnary 2-sphere, and the other being the general form of the Schwarzschild line element; introducing the reader early on to the vacuum solution, as well as breaking up the lengthy calculation:

...We start with the more general form of the Schwarzschild line element

$$ds^2 = -e^{2v(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{A.114}$$

from which we extract our basis of orthonormal 1-forms

$$\{\omega^{\hat{t}}, \omega^{\hat{r}}, \omega^{\hat{\theta}}, \omega^{\hat{\phi}}\} = \{e^{v(r)}dt, e^{\lambda(r)}dr, rd\theta, r\sin\theta d\phi\}$$

Using Cartan's first structure equation we may calculate the Ricci rotation coefficients. Starting with $\omega^{\hat{t}}$ we have

$$d\omega^{\hat{t}} + \Gamma^{\hat{t}}{}_{\hat{r}} \wedge \omega^{\hat{r}} + \Gamma^{\hat{t}}{}_{\hat{\theta}} \wedge \omega^{\hat{\theta}} + \Gamma^{\hat{t}}{}_{\hat{\phi}} \wedge \omega^{\hat{\phi}} = 0$$

$$= \frac{dv(r)}{dr} e^{v(r)} dr \wedge dt + \Gamma^{\hat{t}}{}_{\hat{r}} \wedge \omega^{\hat{r}} + \Gamma^{\hat{t}}{}_{\hat{\theta}} \wedge \omega^{\hat{\theta}} + \Gamma^{\hat{t}}{}_{\hat{\phi}} \wedge \omega^{\hat{\phi}} = 0$$

$$= \frac{dv}{dr} e^{-\lambda(r)} \omega^{\hat{r}} \wedge \omega^{\hat{t}} + \Gamma^{\hat{t}}{}_{\hat{r}} \wedge \omega^{\hat{r}} + \Gamma^{\hat{t}}{}_{\hat{\theta}} \wedge \omega^{\hat{\theta}} + \Gamma^{\hat{t}}{}_{\hat{\phi}} \wedge \omega^{\hat{\phi}} = 0 \tag{A.115}$$

For $\omega^{\hat{r}}$ we have

$$d\omega^{\hat{r}} + \Gamma^{\hat{r}}{}_{\hat{t}} \wedge \omega^{\hat{t}} + \Gamma^{\hat{r}}{}_{\hat{\theta}} \wedge \omega^{\hat{\theta}} + \Gamma^{\hat{r}}{}_{\hat{\phi}} \wedge \omega^{\hat{\phi}} = 0$$

$$= \frac{d\lambda(r)}{dr} e^{\lambda(r)} dr \wedge dr + \Gamma^{\hat{r}}{}_{\hat{t}} \wedge \omega^{\hat{t}} + \Gamma^{\hat{r}}{}_{\hat{\theta}} \wedge \omega^{\hat{\theta}} + \Gamma^{\hat{r}}{}_{\hat{\phi}} \wedge \omega^{\hat{\phi}} = \Gamma^{\hat{r}}{}_{\hat{t}} \wedge \omega^{\hat{t}} + \Gamma^{\hat{r}}{}_{\hat{\theta}} \wedge \omega^{\hat{\theta}} + \Gamma^{\hat{r}}{}_{\hat{\phi}} \wedge \omega^{\hat{\phi}} = 0 \tag{A.116}$$

We also have for $\omega^{\hat{\theta}}$

$$d\omega^{\hat{\theta}} + \Gamma^{\hat{\theta}}{}_{\hat{t}} \wedge \omega^{\hat{t}} + \Gamma^{\hat{\theta}}{}_{\hat{r}} \wedge \omega^{\hat{r}} + \Gamma^{\hat{\theta}}{}_{\hat{\phi}} \wedge \omega^{\hat{\phi}}$$

$$= \frac{e^{-\lambda(r)}}{r} \omega^{\hat{r}} \wedge \omega^{\hat{\theta}} + \Gamma^{\hat{\theta}}{}_{\hat{t}} \wedge \omega^{\hat{t}} + \Gamma^{\hat{\theta}}{}_{\hat{r}} \wedge \omega^{\hat{r}} + \Gamma^{\hat{\theta}}{}_{\hat{\phi}} \wedge \omega^{\hat{\phi}} = 0 \tag{A.117}$$

Lastly,

$$d\omega^{\hat{\phi}} + \Gamma^{\hat{\phi}}{}_{\hat{t}} \wedge \omega^{\hat{t}} + \Gamma^{\hat{\phi}}{}_{\hat{r}} \wedge \omega^{\hat{r}} + \Gamma^{\hat{\phi}}{}_{\hat{\theta}} \wedge \omega^{\hat{\theta}} = 0$$

$$= \frac{e^{-\lambda(r)}}{r} \omega^{\hat{r}} \wedge \omega^{\hat{\phi}} + \frac{cot\theta}{r} \omega^{\hat{\theta}} \wedge \omega^{\hat{\phi}} + \Gamma^{\hat{\phi}}{}_{\hat{t}} \wedge \omega^{\hat{t}} + \Gamma^{\hat{\phi}}{}_{\hat{r}} \wedge \omega^{\hat{r}} + \Gamma^{\hat{\phi}}{}_{\hat{\theta}} \wedge \omega^{\hat{\theta}} = 0 \tag{A.118}$$

...

...Using this same procedure we can find all of the other non-vanishing components of the Riemann tensor. Altogether we have

$$R^{\hat{t}}{}_{\hat{r}\hat{r}\hat{t}} = \left[\frac{d^2 v}{dr^2} + \left(\frac{dv}{dr} \right)^2 - \frac{d\lambda}{dr} \frac{dv}{dr} \right] e^{-2\lambda(r)} \tag{A.119}$$

$$R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} = R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} = \frac{d\lambda}{dr} \frac{e^{-2\lambda(r)}}{r} \qquad R^{\hat{t}}{}_{\hat{\phi}\hat{\phi}\hat{t}} = R^{\hat{t}}{}_{\hat{\theta}\hat{\theta}\hat{t}} = \frac{dv}{dr} \frac{e^{-2\lambda(r)}}{r} \qquad R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}} = \frac{1 - e^{-2\lambda(r)}}{r^2} \tag{A.120}$$

This calculation concludes "Elements of Differential Geometry". The next chapter is really "A Crash Course in General Relativity". Using the mathematics developed in the previous chapter, the energy-momentum tensor is introduced and explored before the reader is led to the two conventional derivations of Einstein's field equations. The first derivative is more of a physical approach, while the second is based on varying the Einstein-Hilbert action:

...For a rather simple guess approach to Einstein's field equations, recall the Poisson equation describing the Newtonian gravitation potential:

$$\nabla^2 \phi = 4\pi G\rho \tag{A.121}$$

If we look for a relativistic generalization of this equation, or rather a tensorial equation, we see may argue that the gravitational potential of Newtonian theory is to be replaced by the metric, as it is the metric which encodes the information of the gravitational field distribution. Moreover, the tensorial generalization of the mass density ρ is, as examined in the previous section, the energy-momentum tensor $T_{\mu\nu}$. Now notice that the left hand side of the Poisson equation has a second ordered differential operator acting on the field. This means in our generalization we would guess a second order differential operator on the metric. Now recall that in the explicit representation of the Riemann curvature tensor there are second order derivatives on the metric tensor. So a guess might be

$$R^{\mu}{}_{\nu\rho\sigma} \propto T_{\mu\nu}$$

Immediately we know this isn't right as the indices between both sides are not balanced. To get indices to match we contract the Riemann tensor, yielding the Ricci tensor. Therefore our next guess would be

$$R_{\mu\nu} = kT_{\mu\nu}$$

where k is some constant. Remember now that the energy-momentum tensor must obey the conservation law: $\nabla^\mu T_{\mu\nu} = 0$. If one recalls the Bianchi identity,

$$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} = 0$$

we can easily see that $\nabla^\mu R_{\mu\nu} \neq 0$. Contracting the Bianchi identity twice gives us

$$0 = g^{\nu\sigma} g^{\mu\lambda} \left(\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu}\right) = \nabla^\mu R_{\rho\mu} - \nabla_\rho R + \nabla^\nu R_{\rho\nu}$$

where we used metric compatibility. Rearranging leads us to conclude that

$$\nabla^\mu R_{\rho\mu} = \frac{1}{2}\nabla_\rho R$$

From here it is easy to see that in order for the conservation of $T_{\mu\nu}$ to be satisfied we must have

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = kT_{\mu\nu}$$

With a bit of work, one can show that the constant $k = 8\pi G$, which allows us to write out the correct form of Einstein's field equations:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{A.122}$$

...The great mathematician Hilbert showed that the simplest possible choice for a Lagrangian depending on the metric that is also a Lorentz scalar is just the Ricci scalar R, yielding the *Einstein-Hilbert action*

$$S = \int \sqrt{-g} R d^n x \tag{A.123}$$

As it happens, the Einstein-Hilbert action is the action which yields Einstein's field equations. To see this, let us vary the above action, which gives us three integrals:

$$\delta S = \int d^n x\sqrt{-g} g^{\mu\nu}\delta R_{\mu\nu} + \int d^n x\sqrt{-g} R_{\mu\nu}\delta g^{\mu\nu} + \int d^n x R\delta\sqrt{-g} \tag{A.124}$$

where we have written $R = g^{\mu\nu}R_{\mu\nu}$. Since we the metric tensor $g^{\mu\nu}$ is the dynamical variable here, we seek terms that strictly have the variation $\delta g^{\mu\nu}$. Thus, we don't have to do anymore work to the second integral as this term is already present. Let's focus on the more difficult term first, the first integral. Recall that the Ricci tensor is simply the contraction of the Riemann tensor, which we found to be written as

$$R^{\rho}{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\rho}{}_{\nu\mu} + \Gamma^{\rho}{}_{\lambda\sigma}\Gamma^{\sigma}{}_{\nu\mu} - \partial_\nu \Gamma^{\rho}{}_{\lambda\mu} - \Gamma^{\rho}{}_{\nu\sigma}\Gamma^{\sigma}{}_{\lambda\mu}$$

Therefore, when we vary the Ricci tensor, we can think of it as varying the Riemann tensor, which is done by varying the connection via the arbitrary variation

$$\Gamma^{\rho}{}_{\nu\mu} \to \Gamma^{\rho}{}_{\nu\mu} + \delta\Gamma^{\rho}{}_{\nu\mu}$$

The variation $\delta\Gamma^{\rho}{}_{\nu\mu}$ is actually a tensor, allowing us to take its covariant derivative, yielding

$$\nabla_\lambda(\delta\Gamma^{\rho}{}_{\nu\mu}) = \partial_\lambda \Gamma^{\rho}{}_{\nu\mu} + \Gamma^{\rho}{}_{\lambda\sigma}\delta\Gamma^{\sigma}{}_{\nu\mu} - \Gamma^{\sigma}{}_{\lambda\nu}\delta\Gamma^{\rho}{}_{\sigma\mu} - \Gamma^{\sigma}{}_{\lambda\mu}\delta\Gamma^{\rho}{}_{\nu\sigma}$$

From here it is straightforward but tedious to show that the variation of the Riemann tensor is simply

$$\delta R^{\rho}_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^{\rho}{}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\rho}{}_{\lambda\mu}) \tag{A.125}$$

This allows us to write the first integral in (18.20) as

$$\int d^n x\sqrt{-g} g^{\mu\nu}\left[\nabla_\lambda(\delta\Gamma^{\lambda}{}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\lambda}{}_{\lambda\mu})\right]$$

where we have contracted over ρ and λ to get the right Ricci tensor. Moreover, by metric compatibility and some minor relabeling of indices, the above becomes

$$\int d^n x\sqrt{-g}\nabla_\sigma\left[g^{\mu\nu}(\delta\Gamma^{\sigma}{}_{\mu\nu}) - g^{\mu\sigma}(\delta\Gamma^{\lambda}{}_{\lambda\mu})\right]$$

As written, we can see that we have the covariant divergence of some vector as it is integrated over a volume element. By Stoke's theorem then this integral is simply equal to a boundry contribution out at infinity, which we are free to set equal to zero. Hence, this integral vanishes and does not contribute to the overall variation of the Einstein Hilbert action [22]. Let's move on to the third integral. As one might recall, we have actually computed a variation similar to this one before, back when we varied the Polyakov action. To calculate this variation we need the identity

$$ln(det M) = Tr(ln M)$$

which is true for any square matrix M. Remembering that the variation acts like a derivative, the variation of this identity is

$$\frac{1}{det M}\delta(det M) = Tr(M^{-1}\delta M) \tag{A.126}$$

As an exercise, the reader will show that this is true for 2×2 matrices. Letting $g = det M$ and $M = g_{\mu\nu}$ it follows that

$$\delta g = g(g^{\mu\nu}\delta g_{\mu\nu})$$

since $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$. Now recall that $g^{\mu\nu}g_{\mu\nu} = constant$. This means that

$$\delta(g^{\mu\nu}g_{\mu\nu}) = \delta g^{\mu\nu}g_{\mu\nu} + g^{\mu\nu}\delta g_{\mu\nu} = 0 \Rightarrow g^{\mu\nu}\delta g_{\mu\nu} = -\delta g^{\mu\nu}g_{\mu\nu}$$

Therefore,

$$\delta g = g(g^{\mu\nu}\delta g_{\mu\nu}) = -g(g_{\mu\nu}\delta g^{\mu\nu})$$

Again using the fact that the variation acts like a derivative we find that

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \tag{A.127}$$

Altogether then, the variation of the Einstein-Hilbert action is

$$\delta S = \int d^n x\sqrt{-g}\left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right]\delta g^{\mu\nu} \tag{A.128}$$

By setting $\delta S = 0$, the only way this happens for arbitrary $\delta g^{\mu\nu}$ is if the integrand itself is zero, i.e.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0 \tag{A.129}$$

After these derivations, particular solutions to Einstein's field equations are explored, including the second portion of the calculation to derive the Schwarzschild line element, as well as arriving the Friedmann equations fundamental to basic cosmology. The Friedmann universe concludes the "crash course" chapter on general relativity. Beyond this chapter is a tour of the various classical black hole solutions, as well as an introduction to black hole thermodynamics, and the information paradox, as presented by [23]. The reason for this chapter was to give the reader a glimpse into one of string theory's biggest results: the statistical derivation of the Beckenstein-Hawking entropy formula, as shown by Strominger and Vafa in the mid-1990s [24]. An excerpt of this chapter will not be given.

References and Further Reading

[1] Svesko, Andy M. "A Detailed Introduction to String Theory." Thesis. Oregon State University, 2013. Dec. 2012. Web.
http : //people.oregonstate.edu/ ~ sveskoa/introductiontostringtheory.pdf

[2] Green, M., John H. Schwarz, and E. Witten. *Superstring Theory. Vol. I.* Cambridge: Cambridge UP, 1987. Print.

[3] Green, M., John H. Schwarz, and E. Witten. *Superstring Theory. Vol. II.* Cambridge: Cambridge UP, 1987. Print.

[4] Becker, Katrin, Melanie Becker, and John H. Schwarz. *String Theory and M-theory: A Modern Introduction.* Cambridge: Cambridge UP, 2007. Print.

[5] Polchinski, Joseph Gerard. *String Theory. Vol. I.* Cambridge, UK: Cambridge UP, 1998. Print.

[6] Polchinski, Joseph Gerard. *String Theory. Vol. II.* Cambridge, UK: Cambridge UP, 1998. Print.

[7] Zwiebach, Barton. *A First Course in String Theory.* New York: Cambridge UP, 2004. Print.

[8] J. Ambjorn, J. Jurkiewicz, R. Loll, "Causal Dynamical Triangulations and the Quest for Quantum Gravity", hep-th/1004.0352.

[9] Loll, Renate. “Quantum Origins of Space and Time.” Lecture. Quantum Origins of Space and Time. Perimeter Institute, Waterloo. 27 Dec. 2012. Renate Loll on the Quantum Origins of Space and Time. Youtube, 24 June 2011. Web. 27 Dec. 2012. http://www.youtube.com/watch?v=fv2gBjQ8xIo&list=PLF244E3ED0BCDAF23

[10] R. Gambini, J. Pullin. “Spin Foams”. *A First Course in Loop Quantum Gravity*. New York: Oxford UP, 2011. 149-56. Print.

[11] Rovelli, Carlo. “General Ideas and the Heuristic Picture”. *Quantum Gravity*. Cambridge, UK: Cambridge UP, 2004. 3-33. Print.

[12] Susskind, Leonard. “The Theoretical Minimum.”Lecture. The Theoretical Minimum. Stanford University. Summer 2011. The Complete Leonard Susskind Lectures. Ted Young, 11 Jan. 2011. Web. 1 Jan. 2013. http://tedyoung.me/2011/01/22/leonard-susskind-lectures/

[13] Labelle, Patrick. *Supersymmetry Demystified*. New York: McGraw-Hill, 2010. Print.

[14] Peskin, Michael Edward, and Daniel V. Schroeder. *An Introduction to Quantum Field Theory*. Reading, MA: Addison-Wesley, 1995. Print.

[15] D’Inverno, Ray. *Introducing Einstein’s Relativity*. Oxford [England: Clarendon, 1992. Print.

[16] Griffiths, David. “Quantum Electrodynamics.” Introduction to Elementary Particles. Wein- heim: Wiley- VCH, 2008. 225-35. Print.

[17] Hatfield, Brian F. *Quantum Field Theory of Point Particles and Strings*. Reading: Addison-Wesley, 1998. Print.

[18] McMahon, David. “Scalar Fields.” *Quantum Field Theory Demystified*. New York, NY: McGraw-Hill, 2008. N. pag. Print.

[19] Shankar, Ramamurti. “The Continuity Equation for Probability.” Principles of Quantum Mechanics. 2nd ed. New York: Plenum, 1994. 164-67. Print.

[20] Aitchison, Ian Johnston Rhind. *Supersymmetry in Particle Physics: An Elementary Introduction.* Cambridge: Cambridge UP, 2007. Print.

[21] McMahon, David. *Relativity Demystified.* New York: McGraw-Hill, 2006. Print.

[22] Carroll, Sean. *Spacetime and Geometry: An Introduction to General Relativity.* San Francisco: Addison Wesley, 2004. Print.

[23] Susskind, Leonard, and James Lindesay. “The Laws of Nature”. *An Introduction to Black Holes, Information and the String Theory Revolution: The Holographic Universe.* Hackensack, NJ: World Scientific, 2005. 69+. Print.

[24] A. Strominger and C. Vafa, hep-th/9601029 (1996).