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Title Introduction to the Tensor Analysis of Electrical Networks

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The conventional thesis describes some research or investigation with its conclusions expressed in technical and learned language, designed to convey information to the advanced scholar. In this thesis, an attempt has been made to reverse the process; tensor analysis, a subject ordinarily used for graduate material, is brought out of the obscure and highly theoretical realm, and focused into elementary and practical terms.

The subject matter has been presented in such a way that the sophomore student in electrical engineering may appreciate the usefulness of this mathematical tool for solving network problems; the ability of the student of sophomore level to grasp this subject was tested during the last two weeks of the winter term and the first week of spring term, 1943, at the Oregon State College School of Engineering with very encouraging results.

The engineer begins his mathematical study by accepting a number system. He finds its manipulations useful but needs something more generalized to express physical laws which can be used for any number of numerical substitutions. Thus, algebra was accepted and a letter stood for any number desired to fulfill the requirements of a specific case. A step upward was acknowledged as the student passed from the specific to the general in mathematical language.

Now a further generalization is proposed--the step from algebraic analysis to tensor analysis, to be brought into the electrical engineering curriculum during the latter part of the regular college sophomore year. The class time required for a workable understanding of this mathematical tool is estimated at from three to six term hours of quarter-system length. The actual time required will be clarified by further experimentation.

The introductory chapter discusses the place tensor analysis should have in the mathematical training of the student in electrical engineering. A brief history is included along with a few of the needs for the replacement of ordinary algebraic processes. The results of the teaching of two courses in the application of tensors to electrical engineering are discussed.

The manipulations of matrix algebra are explained in the second chapter along with the concept of the primitive or original circuit. In Chapter 3, Kirchhoff's laws are reviewed and a comparison made with the mesh or connected impedance matrix and the results obtained by application of Kirchhoff's voltage and current equations. The significance of the components of this matrix are discussed in some detail.

The fundamental transformation tensor, or, more specifically, the connection matrix, "C", which mathematically connects the circuit is discussed in the following chapter; in Chapter 5, the impedance reduction formulas are derived and demonstrated.

With the background of tensor operations explained in the preceding chapters, the student is ready to use the primitive impedance matrix for mathematically carrying through mutual impedance effects between coils of a network. Finally, Chapter 7 presents a discussion of the junction-pair concept which is important in the later study of vacuum tube circuits and also finds application in particular types of electrical circuits.

INTRODUCTION
TO THE
TENSOR ANALYSIS
OF
ELECTRICAL NETWORKS

by

ROBERT ALBERT BRUNS

A THESIS

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PREFACE

The conventional thesis describes some research or investigation with its conclusions expressed in technical and learned language, designed to convey information to the advanced scholar. In this thesis, an attempt has been made to reverse the process; tensor analysis, a subject ordinarily used for graduate material, is brought out of the obscure and highly theoretical realm, and focused into elementary and practical terms.

The subject matter has been presented in such a way that the sophomore student in electrical engineering may appreciate the usefulness of this mathematical tool for solving network problems; the ability of the student of sophomore level to grasp this subject was tested during the last two weeks of the winter term and the first week of spring term, 1943, at the Oregon State College School of Engineering with very encouraging results.

To Gabriel Kron of the General Electric Co. goes the credit for transforming the classical mathematical study of tensor analysis into useful, engineering terms. His books and articles were the only ones found which deal directly with the subject matter described herein.

The author wishes to express his thanks to William Huggins, Research Associate at Oregon State College, for his helpful suggestions in the preparation of this thesis. Also, appreciation is expressed to Hendrik Oorthuys for his helpful advice in working out the basic idea for the thesis and to Professors F. O. McMillan and H. B. Cockerline, all of the Oregon State College Department of Electrical Engineering faculty, for their kind cooperation and valuable aid.

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CHAPTER I

INTRODUCTION

The science of engineering requires the manipulative tool of mathematics for maximum output in minimum time. For example, the calculation of the cost of 15 electric motors at $18.50 apiece would be time-consuming unless the manipulation of multiplication was known. Also, through the algebra, equations can be developed which satisfy any given set of numerical substitutions for some electrical network. A combination of the Laplace Transform and Heaviside's Operational calculus is expedient in dealing with electrical transients.

The engineer begins his mathematical study by accepting a number system. He finds its manipulations useful but needs something more generalized to express physical laws which can be used for any number of numerical substitutions. Thus, algebra was accepted and a letter stood for any number desired. A step upward was acknowledged as the student passed from the specific to the general in mathematical language.

Now a further generalization is proposed--the step from algebraic analysis to tensor analysis, to be brought into the electrical engineering curriculum during the latter part of the regular college sophomore year.

Before 1930, most books dealing with this subject of tensor or spinor analysis were written by mathematicians; hence, the engineering world was unable to recognize the significance of a very practical and useful tool camouflaged under such titles as absolute differential calculus, differential geometry and topology. As a matter of fact, the mathematical science of tensor analysis dates back to 1869; but only recently its applications to engineering, as well as to other practical sciences, have been discovered.

Perhaps the work of Ricci and Levi-Civita was most instrumental in bringing tensor analysis to the forefront. In 1916, Einstein contributed his ideas of relativity to the new study which was about to define its own dimensions. Subsequently, Euclidean geometry no longer restricted the mathematical analysis of an engineering problem. It is interesting to note that much of the mathematics still in vogue is limited by the three physical dimensions of space; tensor analysis pulls itself out of this rut and uses as many dimensions as are needed to solve a given problem.

For the ordinary network problem, algebraic quantities are assigned to each unknown and the solution lies in a set of simultaneous equations. When these become greater than three, the process becomes burdensome and it is difficult to keep the substitutions in mind. Also, the handling of electrical networks containing mutual effects between coils by ordinary algebraic methods becomes quite a chore. All in all, there is a demand at present for a shorter method of attack for an electrical network other than the tedious and lengthy processes involving ordinary algebraic notation.

In solving a given network, the basic theory employed is often buried beneath a maze of algebraic equations and substitutions. When the student completes a rather complicated network problem, he may have the answer but frequently has lost sight of the means by which that solution was obtained. Thus, the mathematics itself becomes the thing sought after rather than the fundamental engineering concepts involved. Perhaps one of the greatest needs in engineering education today is that of inspiring creative thinking on the part of the student. But how can this be done when the student must spend a great portion of his time trying to manipulate an unwieldy conglomeration of algebraic symbols which, in themselves, have little or no meaning but simply supply a means to an end?

This brings up another need for a replacement of ordinary algebraic notation and processes. Each time a problem in electrical engineering is undertaken, the student must go back to the beginning and go through the painful process of rebuilding the same algebraic framework all over again. With the use of the transformation tensor, it is possible to take a general network and carry it through to the stage from which special types of circuits may be evolved. Also the unknown quantities desired may be transferred from one part of the circuit to another by a simple matrix manipulation. In this manner, the original processes need not be repeated and that time previously spent in doing "setting up" exercises may be consumed more profitably in investigating the possibilities of a given network.

Furthermore, tensor analysis provides a standard mathematical approach for electrical problems where previously a variety of special attacks held sway. Now it is possible to write the primitive quantities for the most general network, the vacuum tube and the rotating machine using the same mathematical concepts in each case. The connected network, the vacuum tube circuit and the generator supplying a load can all be analyzed using the same theory of transformation. Also, design characteristics can be investigated using the common impedance reduction formulas and elemental positions of the active matrices may carry anything from a simple number to a Heaviside representation of the unit step.

The dual nature of electrical measurements and physical phenomena becomes more apparent and useful as the beginning student progresses in the applications of tensor analysis. Since this is discussed in some detail in Chapter 7 of this thesis, it will be mentioned here only as a further advantage of tensor analysis.

Finally, tensor analysis lends itself to tremendous future expansion. Gabriel Kron has pointed out the versatility of this mathematics and has indicated the similarity between the tensor equations used in hydrodynamics, electrodynamics, optics and elasticity. The following books are suggested for further study:

Kron, Gabriel. A Short Course in Tensor Analysis for Electrical Engineers. G. E. Series. John Wiley & Sons, Inc. 1942. New York.

Kron, Gabriel. Tensor Analysis of Networks.

G. E. Series. John Wiley & Sons, Inc.
1939. New York.

There has been some rather colorful criticism in recent years of the generalization of tensor theory; the beginning student is urged to try it out for himself. It has been the experience of the author that it stimulates investigation into phases of engineering which otherwise might have been overlooked.

This thesis is written for the primary purpose of proposing the use of a universal mathematical system called tensor analysis in the engineering colleges as a basic part of the undergraduate curriculum. It is presented in a form which should be understandable to the college sophomore in electrical engineering who has completed his basic course in physics and has become acquainted with the fundamental electrical laws of Ohm and Kirchhoff.

It is the belief of this author that tensor analysis satisfies many of the needs pointed out. It provides a systematic method of writing simultaneous equations by assigning components of geometric entities to the coefficients involved. New symbols are introduced by which a whole network can be drawn into one unit or vector. Just as complex notation draws the real and imaginery axes into one unit, so the matrix (the mathematical tool of tensor analysis) unifies n-axes into a single quantity. This will be more readily understood as the student progresses in the study of tensor analysis.

Only the most elementary consideration of tensor analysis is covered in this thesis; however, a bibliography has been attached for further reference. Perhaps a more suitable title for the material covered herein, from a mathematical standpoint, might be "The Applica- of Matrix Algebra to Electrical Networks," but the subject matter does provide an introduction to the use of tensor theory.

A possible reaction from the beginning student may be that the solutions of some of the simpler problems contained in this thesis could be worked more quickly by the ordinary algebraic methods. This is probably true in many cases but the student will undoubtedly find it profitable to use the method prescribed regardless of the simplicity of the problem so that he may equip himself for later studies. The initial step of learning matrix algebra must be taken more or less on faith, just as it was necessary for the beginner to learn the manipulations of algebra before he could apply it profitably.

Engineers and mathematicians must work hand in hand for optimum results in the engineering field. In grade school, the prospective engineer took up the study of the number system and numbers were used in the manipulations of arithmetic. Later in high school, he was introduced to a shortcut called algebra, where letters were used to designate numbers without particular regard to their values; also, the axes of space and their relationships to physical objects were investigated in the study of geometry. Now a third step is suggested: to take up the study of matrices so that equations may be handled without regard to their particular sizes. Just as algebra was used to organize the number system and provide shortcuts, now tensor notation has been developed for the purpose, among other things, of organizing the algebra into a more compact and useful form.

Now the question may be asked: can an advanced subject such as tensor analysis be absorbed by college students to the point where it becomes of use in a reasonable length of instructional time? In answer to this question, the results of the teaching of two courses in the Department of Electrical Engineering at Oregon

First of all, a two-term course was given to senior students involving four quarter-system term hours. After the first term's work, the students were able to apply transformation theory to complicated linear networks where mutual effects existed between coils. Also, the impedance reduction formulas were applied to simplify given networks, and the magnetic and dielectric circuits were investigated using matrix notation. During the second term, Heaviside's Operational calculus in conjunction with the Laplace transform was introduced and later used in matrices to study transient behavior of series-parallel circuits; Campbell-Foster notation supplemented this investigation. Then the junction-pair concept was studied and the students became acquainted with the need for two approaches to the solution of networks (see Chapter 7). Finally, as an incentive for further study, the tensor application to vacuum tubes and vacuum-tube circuits was studied and those interested commenced the more involved subject of rotating machinery. The interest was evidenced by the fact that some students wrote their reports on such subjects as transmission-line regulation and synchronous machinery using the tensor concepts, equations and manipulations.

Secondly, an experimental three-week course was given to sophomores in electrical engineering at the same institution. The results appeared to be very satisfactory. In the first place, much interest was shown by the students themselves as evidenced by several articles which appeared in school papers announcing the course. The matrix operations were learned the first week and at the end of four hours of computation, the majority of the students could perform the basic manipulations as outlined in Chapter 2 with ease. In the second week, Kirchhoff's Laws and the transformation tensor were introduced much in the same manner as is done in Chapters 2 and 3 of this thesis. Finally, the impedance reduction formulas and mutual effects were studied during the last week. The students were actually investigating networks using tensor analysis at the end of this brief study period and many were continuing work in this line.

Having observed the results of these courses, this author believes that the sophomore student is capable and willing to learn the applications of tensor analysis to electrical networks. Furthermore, the class time required for a workable understanding of this mathematical tool is estimated at from three to six term hours of quarter-system length. Further experimentation will clarify the time required.

The next chapter begins with a brief study of matrix algebra from a practical standpoint only. It is necessary for the student to learn these manipulations before continuing with the actual study of tensor analysis.

CHAPTER II

MATRIX ALGEBRA

1. Need for Matrix Algebra.

The study of tensor analysis hinges on the system of matrix algebra. Before the student can perform tensor operations, he must be familiar with the fundamental manipulations of matrices.

2. Description of a Matrix.

A matrix,[1] in itself, has no definite significance in a physical sense. It is a means by which a desired result is obtained. Just as our number system was devised to expedite value exchanges, the matrix is now suggested to alleviate the burden of handling lengthy algebraic equations.

3. Kinds of Matrices:

There are three types of matrices within the scope of this text: (a) the 0-matrix, (b) the 1-matrix and (c) the 2-matrix.

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[1] A matrix is defined in Elementary Matrices by Frazer, Duncan and Collar thus: "Matrices are sets of numbers or other elements which are arranged in rows and columns as in a double entry table and which obey certain rules of addition and multiplication." Cambridge University Press. London. 1938.

The 0-matrix has but one quantity involved and is without dimensions. It may be illustrated by the scalar, power, thus:

$$P = \boxed{36} \text{ watts.} \tag{1}$$

The 1-matrix consists of a row arranged in the following manner:

$$I = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{|c|c|c|c|} \hline i_a & i_b & i_c & i_d \\ \hline \end{array} \end{array} \text{ amps.} \tag{2}$$

The 2-matrix is arranged in the form of a rectangle. The number of elements, or separate boxes, is determined by the product of the number of rows and the number of columns. A special case of the 2-matrix is the following square matrix which might represent the impedance of a network:

$$Z = \begin{array}{c|c|c|c|c|} & a & b & c & d \\ \hline a & Z_{aa} & Z_{ab} & Z_{ac} & Z_{ad} \\ \hline b & Z_{ba} & Z_{bb} & Z_{bc} & Z_{bd} \\ \hline c & Z_{ca} & Z_{cb} & Z_{cc} & Z_{cd} \\ \hline d & Z_{da} & Z_{db} & Z_{dc} & Z_{dd} \\ \hline \end{array} \text{ ohms.} \tag{3}$$

The notation used here (that is, the symbols a, b, c and d labeling rows and columns) is called direct notation and will be used in this text. Index notation has many uses in later studies but will not be included here.[2]

4. <u>The Primitive Network.</u>

This refers to the original circuit which can be represented by matrices. Consider the electrical network shown in Fig. 1.

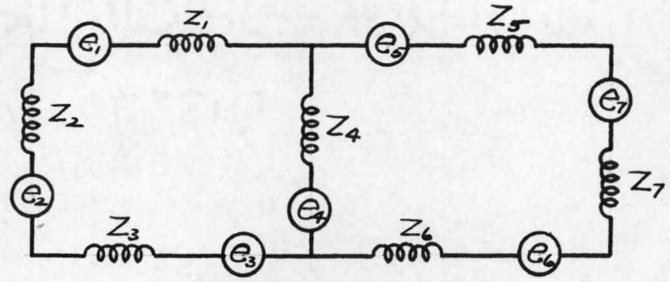

Figure 1

To analyze this circuit using tensor analysis, each coil with its series generator is disconnected from all others and studied individually. The result of this dissection is shown in Fig. 2 and is called

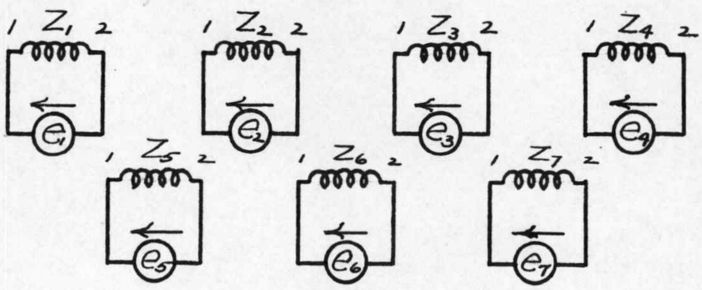

Figure 2

the primitive network. This original network is the

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[2]See Chapter 7, page 174, <u>Tensor Analysis of Networks</u> by Gabriel Kron. G.E. Series. John Wiley & Sons, Inc. New York. 1939.

simplest network which can be formed from the given number of coils. Many different types of circuits could be derived from it.

If there were no mutual effects between any of the coils of the primitive network and only self-impedance quantities existed, the primitive vector for Z would be

$$Z = \begin{array}{c|c|c|c|c|c|c|c|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline
1 & Z_1 & & & & & & \\ \hline
2 & & Z_2 & & & & & \\ \hline
3 & & & Z_3 & & & & \\ \hline
4 & & & & Z_4 & & & \\ \hline
5 & & & & & Z_5 & & \\ \hline
6 & & & & & & Z_6 & \\ \hline
7 & & & & & & & Z_7 \\ \hline
\end{array} \; . \qquad (4)$$

All spaces in (4) that are blank represent zero values of impedance. This type of matrix is called a "diagonal" matrix, since all elements except those along the diagonal are zero, thus indicating that only self impedances exist in the circuit to be connected. If mutual impedances were present, it would be a simple matter to fill these mutual elements as will be demonstrated in Chapter 6.

Perhaps this is the principal selling point of tensor analysis: it provides a standard system and method of attack which can be used for practically all types of network problems, regardless of their complexities. By starting with the simple networks, the correlation between past knowledge and future possibilities can be established.

The currents flowing in each separate coil of Fig. 2 may be written as components of the single vector, I, thus:

$$I = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline \multicolumn{1}{|c|}{i_1} & \multicolumn{1}{c|}{i_2} & \multicolumn{1}{c|}{i_3} & \multicolumn{1}{c|}{i_4} & \multicolumn{1}{c|}{i_5} & \multicolumn{1}{c|}{i_6} & \multicolumn{1}{c|}{i_7} \\ \hline \end{array} \text{ amps.} \quad (5)$$

The voltage sources in series with each coil can be combined into the single 1-matrix thus:

$$E = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline \multicolumn{1}{|c|}{e_1} & \multicolumn{1}{c|}{e_2} & \multicolumn{1}{c|}{e_3} & \multicolumn{1}{c|}{e_4} & \multicolumn{1}{c|}{e_5} & \multicolumn{1}{c|}{e_6} & \multicolumn{1}{c|}{e_7} \\ \hline \end{array} \text{ volts.} \quad (6)$$

If there were only one voltage impressed on the circuit of Fig. 1, for example at the point of e_1, the voltage vector for this specific case would be:

$$E_s = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline \multicolumn{1}{|c|}{e_1} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{0} \\ \hline \end{array} \text{ volts.} \quad (7)$$

Now it will be necessary to add, subtract, multiply and divide these matrices in order to accomplish the results desired for a rigorous circuit analysis.

5. Addition and Subtraction.

Addition and subtraction are performed in a similar manner to that of complex numbers. Such operations can be performed only with matrices of the same order; that is, with the same number of elements flanked by identical notations.

The addition and subtraction of 0-matrices (see Art. 3) is simply the familiar operation performed on ordinary scalar quantities thus:

$$3 + 2 - 7 + 4 = 2.$$

Two 1-matrices, however, are added or subtracted by performing the operation on each component separately. As an example, consider the two currents:

$$I_1 = \begin{array}{cccc} A & B & C & D \\ \hline \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{-6} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \text{ amps.} \qquad (8)$$

$$I_2 = \begin{array}{cccc} A & B & C & D \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{0} \\ \hline \end{array} \text{ amps.} \qquad (9)$$

Then

$$I_1 + I_2 = \begin{array}{cccc} A & B & C & D \\ \hline \multicolumn{1}{|c|}{2+1} & \multicolumn{1}{c|}{4+3} & \multicolumn{1}{c|}{-6+6} & \multicolumn{1}{c|}{3+0} \\ \hline \end{array}$$

$$= \begin{array}{cccc} A & B & C & D \\ \hline \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \text{ amps.} \qquad (10)$$

Also,

$$I_1 - I_2 = \begin{array}{cccc} A & B & C & D \\ \boxed{2-1} & \boxed{4-3} & \boxed{-6-6} & \boxed{3-0} \end{array}$$

$$= \begin{array}{cccc} A & B & C & D \\ \boxed{1} & \boxed{1} & \boxed{-12} & \boxed{3} \end{array} \quad \text{amps.} \qquad (11)$$

Two-matrices are added by adding components as follows:

$$Z_1 = \begin{array}{c|ccc|} & P & Q & R \\ P & 2 & -4 & 6 \\ Q & 1 & 1 & -2 \\ R & 4 & -6 & 2 \end{array} . \qquad (12)$$

$$Z_2 = \begin{array}{c|ccc|} & P & Q & R \\ P & 3 & 5 & 6 \\ Q & 0 & 0 & -2 \\ R & 9 & 8 & -7 \end{array} . \qquad (13)$$

$$Z_1 + Z_2 = \begin{array}{c|ccc|} & P & Q & R \\ P & 2+3 & -4+5 & 6+6 \\ Q & 1+0 & 1+0 & -2-2 \\ R & 4+9 & -6+8 & 2-7 \end{array}$$

$$= \begin{array}{c|ccc|} & P & Q & R \\ P & 5 & 1 & 12 \\ Q & 1 & 1 & -4 \\ R & 13 & 2 & -5 \end{array} . \qquad (14)$$

Subtraction, obviously, is done by subtracting corresponding components.

6. Multiplication.

The order of the 1- or 2-matrix is not as important in this operation. A 0-matrix, when multiplied by another 0-matrix, entails a simple arithmetical multiplication.

Multiplication of a 0-matrix by a 1- or 2-matrix involves the multiplication of each component of the dimensional matrix by the single quantity of the 0-matrix. An example is shown:

$$(2)\cdot \begin{array}{c|c|c|} & a & b \\ \hline a & 3 & 1 \\ \hline b & 0 & 6 \\ \hline \end{array} = \begin{array}{c|c|c|} & a & b \\ \hline a & 6 & 2 \\ \hline b & 0 & 12 \\ \hline \end{array} \quad . \qquad (15)$$

Multiplication of a 1-matrix by another 1-matrix introduces what is called the arrow rule. These arrows show the order in which successive components or elements are multiplied. As an example, consider the multiplication of current times voltage to obtain the scalar, power. Let

$$E = \begin{array}{|c|c|c|} \multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{3} \\ \hline 3 & 7 & 4 \\ \hline \end{array} \quad . \qquad (16)$$

And

$$I = \begin{array}{|c|c|c|} \multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{3} \\ \hline 8 & 0 & 9 \\ \hline \end{array} \quad . \qquad (17)$$

Then,

$$P = E \cdot I = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 3 & 7 & 4 \end{bmatrix} \end{array} \cdot \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} 8 \\ 0 \\ 9 \end{bmatrix}$$

$$= \boxed{24 + 0 + 36} = 60 \quad . \tag{18}$$

Note the rule here: the first term of such a multiplication is in the horizontal position while the second is in the vertical position, giving a result which is a 0-matrix--in this case, a scalar. This adjustment must be made to make the arrow rule work.

Another important thing about notation in this example is that the 1-2-3 axes drop out, the ones coincident with the arrow. This is a general rule for multiplication of matrices: those axes which are coincident with the proper positions of the arrows (these axes must always be the same) do not appear in the final product.

As a further example, consider the multiplication of a 2-matrix and a 1-matrix

$$E = Z \cdot i = \begin{array}{c|ccc} & a & b & c \\ \hline a & 3 & -1 & 2 \\ b & 4 & 6 & 8 \\ c & 9 & -10 & 0 \end{array} \quad \cdot \quad \begin{array}{c|c} a & 1 \\ b & 2 \\ c & 2 \end{array}$$

$$= \begin{array}{c} a \\ b \\ c \end{array}\begin{bmatrix} 3\cdot1+-1\cdot2+2\cdot2 \\ 4\cdot1+6\cdot2+8\cdot2 \\ 9\cdot1+-10\cdot2+0\cdot2 \end{bmatrix} = \begin{array}{c} a \\ b \\ c \end{array}\begin{bmatrix} 5 \\ 32 \\ -11 \end{bmatrix} \text{ volts.} \quad (19)$$

Again the so-called dummy axes or "arrow" axes disappear leaving only the single axis a-b-c in the result. An example of the multiplication of two 2-matrices is shown:

$$Z\cdot C = \begin{array}{c|ccc} & a & b & c \\ \hline a & 3 & -1 & 2 \\ b & 4 & 6 & 8 \\ c & 9 & -10 & 0 \end{array} \cdot \begin{array}{c|cc} & p & q \\ \hline a & 1 & 0 \\ b & 0 & 1 \\ c & 1 & -1 \end{array}$$

$$= \begin{array}{c|cc} & p & q \\ \hline a & 5 & -3 \\ b & 12 & -2 \\ c & 9 & -10 \end{array} . \quad (20)$$

Corresponding components have been multiplied and the products added according to the arrow rule. For example, to obtain the element, $(Z\cdot C)_{ap}$,[3] in the product whose value is (5), the (a) row of the first matrix, Z, is multiplied by the (p) column of the second matrix, C, in the order of the arrows:

$$(Z\cdot C)_{ap} = (3)(1) + (-1)(0) + (2)(1) = 5. \quad (21)$$

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[3]In the subscript notation, such as ap, the first letter refers to the row and the second to the column.

Similarly $(Z \cdot C)_{bq}$ could be solved thus:

$$(Z \cdot C)_{bq} = (4)(0) + (6)(1) + (8)(-1) = -2. \quad (22)$$

7. Transposition.

This operation involves the mechanical manipulation of the positions of the components of a given matrix. As an example, let

$$C = \begin{array}{c|c|c|c|} & a & b & c \\ \hline 1 & A & B & C \\ \hline 2 & D & E & F \\ \hline 3 & G & H & I \\ \hline \end{array} \; . \quad (23)$$

Now the transpose of C (written C_t) is, by definition, an exchange of rows and columns. So the transpose of the C in (23) is written

$$C_t = \begin{array}{c|c|c|c|} & 1 & 2 & 3 \\ \hline a & A & D & G \\ \hline b & B & E & H \\ \hline c & C & F & I \\ \hline \end{array} \; . \quad (24)$$

8. Determinants.

A brief review of determinants is necessary before division of matrices is possible.

A basic difference between a matrix and a determinant is that a determinant represents just one number or scalar; whereas, a matrix stands for a set of numbers or elements arranged in a definite order which may geometrically portray some physical picture.

For example, the determinant,

$$\begin{vmatrix} 3 & 2 & 1 \\ 6 & 4 & 3 \\ 0 & 2 & 1 \end{vmatrix}, \tag{25}$$

represents the number, -6; it is evaluated thus:

$$\begin{aligned} &(3)(4)(1) + (2)(3)(0) + (1)(6)(2) \\ &- (0)(4)(1) - (2)(3)(3) - (1)(6)(2) = -6. \end{aligned} \tag{26}$$

On the other hand, the matrix,

$$Z = \begin{array}{c|c|c|c|} & a & b & c \\ \hline a & Z_{aa} & Z_{ab} & Z_{ac} \\ \hline b & Z_{ba} & Z_{bb} & Z_{bc} \\ \hline c & Z_{ca} & Z_{cb} & Z_{cc} \\ \hline \end{array}, \tag{27}$$

may represent the various impedances existing in an electrical circuit where mutual impedances are present. Perhaps another way of expressing the difference between a determinant and a matrix is that a matrix has geometric dimensions while a determinant has none.

9. Co-Factor.

This term must be explained before inverse calculations can be made. Another term, the minor, will be introduced first, however. The minor of a quantity is the value of the determinant formed from the matrix determinant after the row and the column corresponding to the element in question have been removed from the matrix. An example may clarify this definition. In the following matrix, it is desired to know the minor of the AA element of value, 6:

$$\begin{array}{c|ccc|} & A & B & C \\ \hline A & 6 & 0 & 8 \\ B & 2 & 1 & 5 \\ C & 4 & 3 & 2 \\ \hline \end{array} \quad . \qquad (28)$$

To get the minor of this component, strike out the row and column shown by the dotted lines:

$$\begin{array}{c|ccc|} & A & B & C \\ \hline A & \cancel{6} & \cancel{0} & \cancel{8} \\ B & \cancel{2} & 1 & 5 \\ C & \cancel{4} & 3 & 2 \\ \hline \end{array} \quad . \qquad (29)$$

The value of the remaining determinant is:

$$\begin{vmatrix} 1 & 5 \\ 3 & 2 \end{vmatrix} = (2)(1) - (3)(5) = -13. \qquad (30)$$

So the minor of the AA element is -13 and can be put in the new matrix in the corresponding position. The minors of the other elements are calculated in a similar fashion to give:

	A	B	C
A	-13	-16	2
B	-24	-20	18
C	-8	14	6

. (31)

Now to get the co-factors, it is necessary to adjust the signs of the minors just obtained in (31) to fit the following pattern of unit multiplication:

+	−	+	....	−
−	+	−	....	+
+	−	+	....	−
....	....	....	....	....
−	+	−	....	+

. (32)

The signs of the various elements in (31) are changed where necessary to give what is called the "adjoint" matrix as follows:

	A	B	C
A	-13	16	2
B	24	-20	-18
C	-8	-14	6

. (33)

10. Division.

Division by a 0-matrix involves dividing each component of the given matrix by the number of the 0-matrix. However, division by a 1- or 2-matrix is not defined as such. Rather, the inverse of the divisor is multiplied by the matrix to be divided. For example, a familiar division is written

$$i = (Z^{-1})\cdot(E). \tag{34}$$

11. Inverse Calculation.

There are two methods for calculating the inverse of a matrix. Kron's method is outlined on page 29 of his book, "Tensor Analysis of Networks."[4] A refinement of this method has the advantage of not requiring as much determinant calculation as Kron's method. Also, the steps are convenient for later calculations. It will be presented here.

Suppose it is desired to calculate the value of current, i, in (34). The following impedance matrix is given:

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[4]See Chapter I of Tensor Analysis of Networks by Gabriel Kron. G.E. Series. John Wiley & Sons, Inc. New York. 1939.

$$Z = \begin{array}{c|c|c|c|} & A & B & C \\ \hline A & 3 & 1 & 5 \\ \hline B & 6 & 2 & 2 \\ \hline C & 1 & 0 & 4 \\ \hline \end{array} \quad . \tag{35}$$

To make the proper substitution, it is necessary to take the inverse of the Z of (35). The steps are as follows:

(1) Write Z_t (called transpose of Z) by interchanging rows and columns (see Art. 7):

$$Z_t = \begin{array}{c|c|c|c|} & A & B & C \\ \hline A & 3 & 6 & 1 \\ \hline B & 1 & 2 & 0 \\ \hline C & 5 & 2 & 4 \\ \hline \end{array} \quad . \tag{36}$$

(2) Write the adjoint[5] matrix of Z_t which is called M. Each component or element of M is calculated by writing its co-factor (see Art. 9):

$$M = \begin{array}{c|c|c|c|} & A & B & C \\ \hline A & 8 & -4 & -8 \\ \hline B & -22 & 7 & 24 \\ \hline C & -2 & 1 & 0 \\ \hline \end{array} \quad . \tag{37}$$

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[5]For further investigation of the adjoint matrix, refer to <u>Elementary Matrices</u> by Frazer, Duncan and Collar. Cambridge Press. London. 1938.

(3) Calculate the determinant of Z by finding the value of any diagonal term of the product of M • Z thus:

$$M \cdot Z = \begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 8 & -4 & -8 \\ -22 & 7 & 24 \\ -2 & 1 & 0 \end{bmatrix} \end{array} \cdot \begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 3 & 1 & 5 \\ 6 & 2 & 2 \\ 1 & 0 & 4 \end{bmatrix} \end{array}$$

$$= \begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} \begin{array}{c} 24-24-8 \\ = -8 \end{array} & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -8 \end{bmatrix} \end{array} . \qquad (38)$$

Therefore,[6]

$$\Delta = -8 .$$

(4) Write the inverse matrix as the reciprocal of the determinant times M thus:

$$Z^{-1} = \frac{1}{\Delta} \cdot M = -\frac{1}{8} \begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 8 & -4 & -8 \\ -22 & 7 & -5 \\ 4 & 1 & 0 \end{bmatrix} \end{array} . \qquad (39)$$

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[6] A check on the accuracy of calculations is provided by the resulting matrix of (38). If the diagonal terms are all equal and all non-diagonal terms equal to zero, the work is correct up to this point.

CHAPTER III

THE NETWORK MATRIX

Chapter 1 dealt with the mathematical approach to the subject of tensor analysis. This, undoubtedly, introduced many concepts which might be confusing if no further steps were taken to show the tie between past knowledge and this new study. Consequently, certain laws familiar to the beginning electrical engineer will be considered first; then their significance in tensor notation will be illustrated.

12. Kirchhoff's Laws.

There are two important laws first stated by Kirchhoff which are fundamental in electrical network theory: (1) the algebraic[7] sum of the currents flowing into any point in a network is zero, and (2) the algebraic[7] sum of the products of the current and resistance[8] in each of the conductors around any closed path in a network is equal to the algebraic sum of the emfs.

An illustration of the use of these laws may be

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[7]This term can be changed to vector sum when a.c. values are used.

[8]This term can be changed to impedance when a.c. is involved.

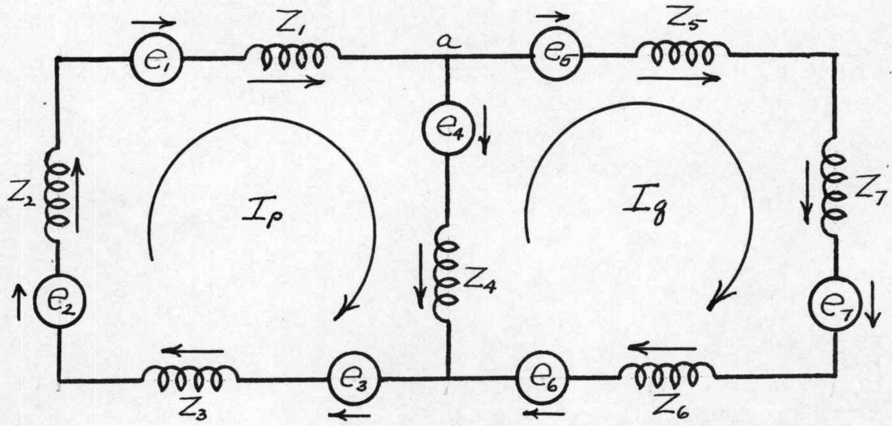

Figure 3.

based on the circuit shown in Fig. 1 of Chapt. 2. It is reproduced in Fig. 3 with I_p and I_q being the currents flowing in the two assumed meshes.[9] Assumed directions of current flow are shown by the straight arrows for coil currents and circular arrows for mesh currents. (See Art. 4).

Using Kirchhoff's first law, all currents flowing into point (a) on the circuit diagram of Fig. 3 must be equal to zero or,

$$I_p - I_q - I_4 = 0.$$

$$I_4 = I_p - I_q , \tag{40}$$

where I_4 is the current flow through Z_4.

Kirchoff's second law may be demonstrated by

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[9]Although there is a generator in series with each coil here, any other case might allow any of these to be removed; then the corresponding voltage element for the removed generator would be equal to zero.

tracing the voltages around the P mesh shown by the circular arrow labeled P in Fig. 3, thus:

$$e_1 + e_2 + e_3 + e_4 = Z_1 I_p + Z_2 I_p + Z_3 I_p + Z_4(I_p - I_q). \qquad (41)$$

A second equation is obtained by going around the Q mesh but here it is necessary to watch the directions of the voltages more carefully. Going around the Q mesh in a clockwise direction, starting at point (a), gives

$$e_5 + e_6 + e_7 - e_4 = Z_5 I_q + Z_7 I_q + Z_6 I_q - Z_4(I_p - I_q). \qquad (42)$$

The sign is minus on the last term since the direction of I_q is opposite to that of I_4 as originally assumed. Now write the two simultaneous equations thus:

$$e_1 + e_2 + e_3 + e_4 = (Z_1 + Z_2 + Z_3 + Z_4)I_p - Z_4(I_q). \qquad (43)$$

$$e_7 + e_6 + e_5 - e_4 = -(Z_4)I_p + (Z_7 + Z_6 + Z_5 + Z_4)I_q. \qquad (44)$$

I_p and I_q are the Kirchhoff currents which can be solved since there are two simultaneous equations involved.

13. Superposition Theory.

From the mathematical development presented in

Art. 12 and the circuit diagram of Fig. 3, it is possible to postulate that the coil currents can be represented by mesh currents for ordinary linear networks. As shown in equation (40) and by the arrows in Fig. 3, these mesh currents may not actually exist in all parts of the circuit. For instance, consider the actual current flow through the impedance, Z_4; its value is neither I_p or I_q but is evaluated by a combination of the two.

In other words, it is possible to consider a portion of a circuit (a mesh) alone, forgetting the remainder of the circuit for the time. The system of tensor analysis enables the engineer to use this superposition theory by beginning at the primitive network which contains the basic, indivisible physical units from which the network will be constructed. Just as a chemist finds it convenient to break known solutions called compounds into their separate elements, the engineer can attack a given complex network more easily if he breaks it up into simple components, forming the primitive network.

14. <u>The Matrix Equation</u>.

The basic matrix equation which follows Ohm's law is

$$\bar{e}' = \bar{Z}' \cdot \bar{I}'.\text{[10]} \tag{45}$$

The dashes over the letters indicate that they represent matrix quantities; but hereafter, the dash will be omitted.

For example, the Kirchhoff equations for the circuit of Fig. 3,

--Loop P--

$$e_1 + e_2 + e_3 + e_4 = (Z_1 + Z_2 + Z_3 + Z_4)I_p - (Z_4)I_q, \tag{46}$$

--Loop Q--

$$e_7 + e_6 + e_5 - e_4 = -(Z_4)I_p + (Z_7 + Z_6 + Z_5 + Z_4)I_q, \tag{47}$$

can be written in the form of matrices as follows:

$$e' = \begin{array}{c|c|} \cline{2-2} p & e_1 + e_2 + e_3 + e_4 \\ \cline{2-2} q & e_7 + e_6 + e_5 - e_4 \\ \cline{2-2} \end{array}, \tag{48}$$

$$I' = \begin{array}{c|c|} \cline{2-2} p & I_p \\ \cline{2-2} q & I_q \\ \cline{2-2} \end{array}, \tag{49}$$

$$Z' = \begin{array}{c|c|c|} & p & q \\ \cline{2-3} p & Z_1 + Z_2 + Z_3 + Z_4 & -Z_4 \\ \cline{2-3} q & -Z_4 & Z_7 + Z_6 + Z_5 + Z_4 \\ \cline{2-3} \end{array}. \tag{50}$$

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[10] These matrices are primed for reasons which will be explained in Chapter 4. Briefly, they stand for mesh quantities.

Substitution in equation (45) of the quantities in equations (48), (49) and (50) gives

$$\begin{array}{c} p \\ q \end{array}\begin{bmatrix} e_1+e_2+e_3+e_4 \\ e_7+e_6+e_5-e_4 \end{bmatrix} = \begin{array}{cc} & \begin{array}{cc} p & \qquad\qquad q \end{array} \\ \begin{array}{c} p \\ q \end{array} & \begin{bmatrix} Z_1+Z_2+Z_3+Z_4 & -Z_4 \\ -Z_4 & Z_7+Z_6+Z_5+Z_4 \end{bmatrix} \end{array} \cdot \begin{array}{c} p \\ q \end{array}\begin{bmatrix} I_p \\ I_q \end{bmatrix} \tag{51}$$

Multiply the right-hand side of equation (51) according to the rules of multiplication in Chapter II and equate components on either side of the equation[11] which gives the resulting equations:

$$e_1 + e_2 + e_3 + e_4 = (Z_1 + Z_2 + Z_3 + Z_4)I_p - (Z_4)I_q, \tag{52}$$

$$e_7 + e_6 + e_5 - e_4 = -(Z_4)I_p + (Z_7 + Z_6 + Z_5 + Z_4)I_q. \tag{53}$$

This compares the old method of writing algebraic equations with the new shortcut method of matrix notation.

15. Meaning of Matrix Components.

Consider the network of Fig. 3 redrawn in Fig. 4 with switches (a) and (b) as shown. All the generators have been gathered together into two large generators

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[11]This is similar to equating reals and imaginaries in vector analysis.

for each mesh so that

$$e_p = e_1 + e_2 + e_3 + e_4, \tag{54}$$

$$e_q = e_7 + e_6 + e_5 - e_4. \tag{55}$$

The arrows on the separate coils show the assumed directions of the network current flowing in that particular coil. The circular arrows show the assumed Kirchhoff currents or mesh currents as to direction.

The impedance matrix has already been established from the mesh standpoint thus:

$$Z' = \begin{array}{c|c|c|} & p & q \\ \hline p & Z_1 + Z_2 + Z_3 + Z_4 & -Z_4 \\ \hline q & -Z_4 & Z_7 + Z_6 + Z_5 + Z_4 \\ \hline \end{array} \;. \tag{56}$$

Now the establishment of the physical significance of each of these components is desirable. The

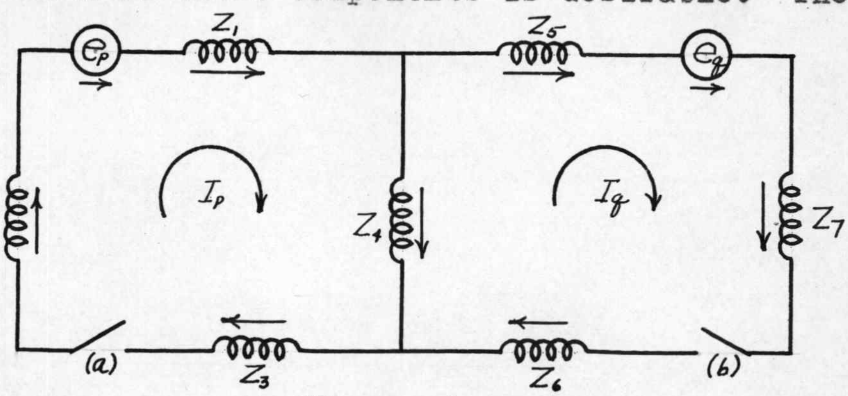

Figure 4.

PP element, that is, the quantity, $Z_{pp} = (Z_1 + Z_2 + Z_3 + Z_4)$, represents the self-impedance around the P mesh. Numerically, it is equal to the sum of the voltage drops around the P mesh when unit current (or a current equal to one ampere) is flowing from the generator, E_p. During this operation, switch (a) is closed and switch (b) left open in Fig. 4. The (qp) element in the left-hand, lower corner, equal to $(-Z_4)$, is the voltage drop seen by the Q mesh under these conditions. Note that the sign is reversed since the assumed direction of I_p is opposite to that of I_q in the common path through Z_4.

Now when the Q column is considered, switch (a) is opened and switch (b) is closed. The Z_{qq} element is numerically equal to the voltage drop around the Q mesh when unit current flows from the generator, E_q. The Z_{pq} element is the voltage drop seen by the P mesh when this unit current flows in the Q mesh. It is called a mutual-impedance component, the same name as that given to the Z_{qp} element previously considered.[12] The generator, E_p, may be considered as being shorted out in which case the voltage appearing across the switch contacts at (a) would be numerically equal to the mutual Z_{pq} element.

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[12] Note that the two mutual elements have equal values. This is typical of static bilinear networks and can be used as a check on the arithmetic under these conditions.

Another way of picturing the significance of the separate elements is as follows: Z_{pp} represents the voltage drop around the P mesh when unit current is supplied from E_p, with generator E_q being taken out of the Q mesh thus leaving two open ends there. The open-circuit voltage appearing across these terminals in the Q mesh is the numerical value of Z_{qp}.

CHAPTER IV

THE TRANSFORMATION TENSOR

The student has now seen a simple electrical network handled in two ways. In Chapter 2, the primitive Z for the network of Fig. 1 was written thus:

$$Z = \begin{array}{c|ccccccc|} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & Z_1 & & & & & & \\ 2 & & Z_2 & & & & & \\ 3 & & & Z_3 & & & & \\ 4 & & & & Z_4 & & & \\ 5 & & & & & Z_5 & & \\ 6 & & & & & & Z_6 & \\ 7 & & & & & & & Z_7 \\ \hline \end{array} \quad . \tag{57}$$

This conforms with equation (4) and mathematically portrays the electrical properties of the isolated coils of Fig. 2, Chapt. 2.

In Chapt. 3, equation (50) was written for the same network but along different axes thus:

$$Z' = \begin{array}{c|c|c|} & p & q \\ \hline p & Z_1 + Z_2 + Z_3 + Z_4 & -Z_4 \\ \hline q & -Z_4 & Z_7 + Z_6 + Z_5 + Z_4 \\ \hline \end{array} \quad . \tag{58}$$

Now the question is: if these matrices stand for the impedances of the same network, how can they be derived from each other? Or, what intermediary matrix or matrices must be introduced so that one can be obtained from the other through manipulation rules outlined in Chapter 2?

16. Definition of a Tensor.

A tensor is a matrix,[13] subject to a definite law of transformation, which has axes that ordinarily mathematically portray a definite physical entity. This transformation of axes is brought about mathematically by means of the so-called "transformation matrix, C."

17. The Transformation Matrix, C.

The key for changing from one set of axes to another is the connection matrix, C. This contains the coefficients of the new variables in terms of the old, or the "components of the transformation tensor along the given reference frames." This C ordinarily contains either units or zeros since the elements represent just the coefficients of the variables rather than the variables themselves.[14]

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[13] See Chapt. 2, Art. 2.

[14] The mathematical representation is:

$$C^{x}_{x'} = \frac{\partial i^{x}}{\partial i^{x'}}$$

The transformation matrix is set up in terms of the currents for most purposes. As an example, in the network which has been considered thus far, the old currents or coil currents from the primitive network of Fig. 2 were written:

$$I = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline \multicolumn{1}{|c|}{I_1} & \multicolumn{1}{c|}{I_2} & \multicolumn{1}{c|}{I_3} & \multicolumn{1}{c|}{I_4} & \multicolumn{1}{c|}{I_5} & \multicolumn{1}{c|}{I_6} & \multicolumn{1}{c|}{I_7} \\ \hline \end{array} \text{ amps.} \qquad (59)$$

The mesh currents of Art. 13, which might represent the new currents, were determined by assuming two mesh currents in the matrix,

$$I' = \begin{array}{cc} p & q \\ \hline \multicolumn{1}{|c|}{I_p} & \multicolumn{1}{c|}{I_q} \\ \hline \end{array} . \qquad (60)$$

Now it is possible to construct a set of equations according to Kirchhoff's first law and the theory of superposition (see Art. 13) by expressing the old currents in terms of the new (see Fig. 3):

$$\begin{array}{ccccc} \underline{\text{Old Currents}} & & \underline{\text{New Currents}} & & \\ I_1 & = & I_p & & \\ I_2 & = & I_p & & \\ I_3 & = & I_p & & \\ I_4 & = & I_p & - & I_q \\ I_5 & = & & & I_q \\ I_6 & = & & & I_q \\ I_7 & = & & & I_q \end{array} \qquad (61)$$

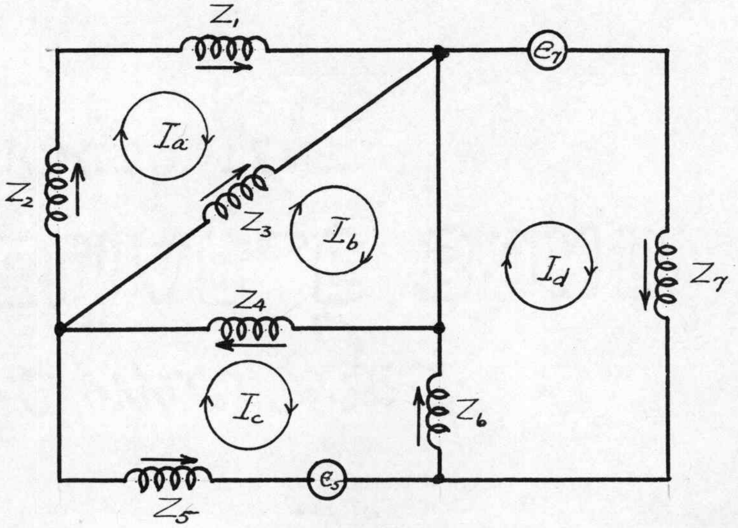

Figure 4.

After the student has constructed this set of equations, the thought required to analyze such a network is completed--the rest is simply mechanical manipulation. It is apparent that these equations could be developed with ease from the most complicated type of circuit. For practice, the student may make up a mesh circuit like that shown in Fig. 4. The table here would be:

$$
\begin{array}{lclllll}
\underline{\text{Old Currents}} & & \underline{\text{New Currents}} & & & & \\
I_1 & = & I_a & & & & \\
I_2 & = & I_a & & & & \\
I_3 & = & -I_a & + I_b & & & \\
I_4 & = & & I_b & - I_c & & \\
I_5 & = & & & I_c & & \\
I_6 & = & & & I_c & + I_d & \\
I_7 & = & & & & + I_d &
\end{array}
\qquad (62)
$$

This illustrates the importance of keeping signs straight when writing the equations. With practice, the student will become accustomed to writing these coil currents in terms of mesh currents. The direction assumed is not important if one is certain to be consistent after these initial assumptions have been made.

The transformation matrix is simply constructed by referring to the coefficients of the quantities in the tables. For example, the C for the circuit of Fig. 3 from Table 61 is

$$C = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{|c|c|} \multicolumn{1}{c}{p} & \multicolumn{1}{c}{q} \\ \hline 1 & \\ \hline 1 & \\ \hline 1 & \\ \hline 1 & -1 \\ \hline & 1 \\ \hline & 1 \\ \hline & 1 \\ \hline \end{array} \quad . \qquad (63)$$

This matrix, in reality, is a shorthand form of the algebraic equations written in (61).

The C for the circuit of Figure 4 can also be constructed from the table of (62):

$$C = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \left[\begin{array}{cccc} 1 & & & \\ 1 & & & \\ -1 & 1 & & \\ & 1 & -1 & \\ & & -1 & \\ & & -1 & 1 \\ & & & 1 \end{array}\right] \end{array} . \qquad (64)$$

It is interesting to note that the primitive Z's for these two circuits would be identical since both have seven coils and no mutual impedances between coils exist. Thus the distinction between the two circuits in tensor notation is brought about by the different connection matrices of (63) and (64).

18. Transformation Formulas.

The old currents can now be written in terms of the new by the following equation:

$$I = C \cdot I' \quad . \qquad (65)$$

To coordinate this relationship with the manipulations of Chapt. 2, substitute the quantities of Art. 17 in (65) and perform the multiplication:

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline I_1 & I_2 & I_3 & I_4 & I_5 & I_6 & I_7 \end{array} = \begin{array}{c|cc} & p & q \\ \hline 1 & 1 & \\ 2 & 1 & \\ 3 & 1 & \\ 4 & 1 & -1 \\ 5 & & 1 \\ 6 & & 1 \\ 7 & & 1 \end{array} \cdot \begin{array}{c|c} p & I_p \\ q & I_q \end{array},$$

$$= \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline I_p & I_p & I_p & I_p - I_q & I_q & I_q & I_q \end{array}. \tag{66}$$

Equating components just as reals and unreals can be equated in vector analysis:

$$\begin{aligned} I_1 &= I_p \\ I_2 &= I_p \\ I_3 &= I_p \\ I_4 &= I_p - I_q \\ I_5 &= \phantom{I_p - {}} I_q \\ I_6 &= \phantom{I_p - {}} I_q \\ I_7 &= \phantom{I_p - {}} I_q \quad . \end{aligned} \tag{67}$$

This checks table (61), thus verifying the process.

The old coil voltages can also be written in terms of the new mesh voltages by applying the following

$$e' = C_t \cdot e. \tag{68}$$

Substituting the quantities of the illustrative example:

$$e' = \begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{matrix} p \\ q \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 & & & \\ & & & -1 & 1 & 1 & 1 \end{bmatrix} \end{array} \cdot \begin{array}{cc} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix} & \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{bmatrix} \end{array}$$

$$= \begin{array}{cc} \begin{matrix} p \\ q \end{matrix} & \begin{bmatrix} e_1 + e_2 + e_3 + e_4 \\ e_7 + e_6 + e_5 - e_4 \end{bmatrix} \end{array} \text{ volts,} \tag{69}$$

or

$$e_p = e_1 + e_2 + e_3 + e_4. \tag{70}$$

$$e_q = e_7 + e_6 + e_5 - e_4. \tag{71}$$

This checks equations (54) and (55) of Chapt. 3, based on Kirchhoff's laws.

Finally, experimentation with the laws of manipulation will reveal that

$$Z' = C_t \cdot Z \cdot C. \tag{72}$$ [15]

This is one of the most important relationships of all in changing from old quantities to new. Primed quantities are always the new; unprimed, the old. Taking the Z of equation (4), Art. 4 for the primitive network and substituting it in (72),

$$Z' = \begin{array}{c|ccccccc|} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline p & 1 & 1 & 1 & 1 & & & \\ q & & & & -1 & 1 & 1 & 1 \\ \hline \end{array} \cdot \begin{array}{c|ccccccc|} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & Z_1 & & & & & & \\ 2 & & Z_2 & & & & & \\ 3 & & & Z_3 & & & & \\ 4 & & & & Z_4 & & & \\ 5 & & & & & Z_5 & & \\ 6 & & & & & & Z_6 & \\ 7 & & & & & & & Z_7 \\ \hline \end{array} \cdot C$$

$$= \begin{array}{c|ccccccc|} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline p & Z_1 & Z_2 & Z_3 & Z_4 & & & \\ q & & & & -Z_4 & Z_5 & Z_6 & Z_7 \\ \hline \end{array} \cdot \begin{array}{c|cc|} & p & q \\ \hline 1 & 1 & \\ 2 & 1 & \\ 3 & 1 & \\ 4 & 1 & -1 \\ 5 & & 1 \\ 6 & & 1 \\ 7 & & 1 \\ \hline \end{array}$$

$$= \begin{array}{c|c|c|} & p & q \\ \hline p & Z_1 + Z_2 + Z_3 + Z_4 & -Z_4 \\ \hline q & -Z_4 & Z_4 + Z_5 + Z_6 + Z_7 \\ \hline \end{array} . \tag{73}$$

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[15] See page 104, <u>Tensor Analysis of Networks</u> by Gabriel Kron. G.E. Series. John Wiley & Sons, Inc. 1939. New York.

which checks the Z' of equation (50).

19. Summary.

So the three important equations for changing from old to new quantities are:

$$I = C \cdot I' , \tag{74}$$

$$e' = C_t \cdot e , \tag{75}$$

$$Z' = C_t \cdot Z \cdot C . \tag{76}$$

These should be memorized by the student. It is important that the order be kept as shown for these equations. They form the basis for nearly all calculations in obtaining circuit values in tensor analysis.

CHAPTER V

IMPEDANCE REDUCTION FORMULAS

20. Elimination of Meshes.

The engineer is often interested in circuit values for only one part of a circuit without regard to the remainder of the circuit. Also, the mathematician may wish to obtain only one or two unknowns from four, five or more simultaneous equations. For one, two or three simultaneous equations, algebra conveniently lends itself for solution. But for more than three or four equations, the algebra becomes burdensome. The student in electrical engineering may wish to select two or three unknowns from six or seven simultaneous voltage equations without having to handle the whole group twice.

Matrix algebra provides the impedance reduction formulas which allow the student to eliminate certain meshes or other circuit axes by combining their effects on the retained meshes or elements, without disturbing the accuracy of the calculations. Mathematically, it is possible to solve for any unknowns desired without going through the extensive work required by ordinary algebraic processes.

21. Impedance Reduction Formulas.

Thus far, two sets of axes have been described in this treatise:

$$e, \ I, \ Z, \quad \text{(primitive or original network)} \tag{77}$$

$$e', \ I', \ Z', \quad \text{(the mesh or connected network).} \tag{78}$$

Now a third set of quantities can be added which describe axes of the meshes retained after the effect of those eliminated has been accounted for through the impedance reduction formula stated in (80):

$$e'', \ I'', \ Z''. \tag{79}$$

In Chapt. 4, equations (74), (75) and (76) made it possible to transform the quantities of (77) into those of (78). Now it is possible to write the quantities of (79) in terms of those of (78) thus making a further transformation. This is done by application of the impedance reduction,

$$Z'' = Z'_1 - Z'_2 \, Z'^{-1}_4 \, Z'_3 \, . \tag{80}$$

The corresponding voltage reduction formula which will prove useful is:

$$e'' = e'_1 - Z'_2 \, Z'^{-1}_4 \, e'_2 \, . \tag{81}$$

The quantity, I", is initially assumed.

The derivation[16] of equations (80) and (81) is obtained by taking the equation,

$$e = Z \cdot I \,, \tag{82}$$

and making the arbitrary substitution,[17]

$$\begin{array}{|c|}\hline e_1 \\ \hline e_2 \\ \hline\end{array} = \begin{array}{|c|c|}\hline Z_1 & Z_2 \\ \hline Z_3 & Z_4 \\ \hline\end{array} \cdot \begin{array}{|c|}\hline I_1 \\ \hline I_2 \\ \hline\end{array} \,. \tag{83}$$

The axes are unlabeled. Assume that the current, I_2, and the impressed voltage, e_2, are the quantities in the mesh to be eliminated. Equation (83) is written algebraically

$$e_1 = Z_1\, I_1 + Z_2\, I_2 \;, \tag{84}$$

$$e_2 = Z_3\, I_1 + Z_4\, I_2 \;. \tag{85}$$

To eliminate I_2 from (85),

$$Z_4\, I_2 = e_2 - Z_3\, I_1 \,,$$

$$I_2 = Z_4^{-1}\,(\, e_2 - Z_3\, I_1 \,). \tag{86}$$

---

[16] See A Short Course in Tensor Analysis by Gabriel Kron. General Electric Series. John Wiley & Sons, Inc. New York. 1942. Pages 15-16.

[17] The primes are omitted in this derivation for reasons of convenience.

Substituting (86) into (84),

$$\begin{aligned} e_1 &= Z_1 I_1 + Z_2 Z_4^{-1} (e_2 - Z_3 I_1), \\ &= Z_1 I_1 + Z_2 Z_4^{-1} e_2 - Z_2 Z_4^{-1} Z_3 I_1, \\ &= Z_2 Z_4^{-1} e_2 + (Z_1 - Z_2 Z_4^{-1} Z_3) I_1, \quad (87) \end{aligned}$$

from which

$$e_1 - Z_2 Z_4^{-1} e_2 = (Z_1 - Z_2 Z_4^{-1} Z_3) I_1 . \quad (88)$$

Adopting the proper primes for mesh quantities (see footnote (17)), (88) can be written

$$e_1' - Z_2' Z_4'^{-1} e_2' = (Z_1' - Z_2' Z_4'^{-1} Z_3') I_1' . \quad (89)$$

This can be represented by the new axes thus:

$$e'' = Z'' \cdot I'' , \quad (90)$$

where I'' is the retained current, in this case, I_1'. Then,

$$Z'' = Z_1' - Z_2' Z_4'^{-1} Z_3' , \quad (91)$$

and

$$e'' = e_1' - Z_2' Z_4'^{-1} e_2' . \quad (92)$$

Some of the uses of equations (91) and (92), called impedance and voltage reduction formulas respectively, will now be demonstrated.

22. Solving Ordinary Simultaneous Equations.

Suppose four simultaneous equations are given which are representative of some network, and it is desired to find only the unknown current, i_1. The equations follow:

$$3i_1 + i_2 - 2i_3 + 4i_4 = 15, \tag{93}$$

$$-i_1 - 3i_2 + 3i_3 - i_4 = -2, \tag{94}$$

$$2i_1 + 2i_2 - i_3 - 2i_4 = -5, \tag{95}$$

$$4i_1 + i_2 + 4i_3 + 3i_4 = 30. \tag{96}$$

These equations can be written in matrix form to conform with the basic equation, $E' = Z' I'$, thus:

$$e' = \begin{array}{cccc} 1' & 2' & 3' & 4' \\ \hline \multicolumn{1}{|c|}{15} & \multicolumn{1}{c|}{-2} & \multicolumn{1}{c|}{-5} & \multicolumn{1}{c|}{30} \\ \hline \end{array} = \begin{array}{cc} 1 & 2 \\ \hline \multicolumn{1}{|c|}{e_1} & \multicolumn{1}{c|}{e_2} \\ \hline \end{array}\ , \tag{97}$$

$$i' = \begin{array}{cccc} 1' & 2' & 3' & 4' \\ \hline \multicolumn{1}{|c|}{i_1} & \multicolumn{1}{c|}{i_2} & \multicolumn{1}{c|}{i_3} & \multicolumn{1}{c|}{i_4} \\ \hline \end{array} = \begin{array}{cc} 1 & 2 \\ \hline \multicolumn{1}{|c|}{i_1} & \multicolumn{1}{c|}{i_2} \\ \hline \end{array}\ , \tag{98}$$

$$Z' = \begin{array}{c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{1'} & \multicolumn{1}{c}{2'} & \multicolumn{1}{c}{3'} & \multicolumn{1}{c}{4'} \\ \cline{2-5} 1' & 3 & 1 & -2 & 4 \\ \cline{2-5} 2' & -1 & -3 & 3 & -1 \\ \cline{2-5} 3' & 2 & 2 & -1 & -2 \\ \cline{2-5} 4' & 4 & 1 & 4 & 3 \\ \cline{2-5} \end{array} = \begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} \\ \cline{2-3} 1 & Z_1 & Z_2 \\ \cline{2-3} 2 & Z_3 & Z_4 \\ \cline{2-3} \end{array}\ . \tag{99}$$

This introduces a new concept--that of compound matrices,--which is a matrix within a matrix. As an example, Z' is shown divided into four sections in (99) and then relabeled. For the elements of the second matrix in (99):

$$Z_1 = \begin{array}{c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{1'} \\ \cline{2-2} 1' & 3 \\ \cline{2-2} \end{array} \quad , \tag{100}$$

$$Z_2 = \begin{array}{c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{2'} & \multicolumn{1}{c}{3'} & \multicolumn{1}{c}{4'} \\ \cline{2-4} 1' & 1 & -2 & 4 \\ \cline{2-4} \end{array} \quad , \tag{101}$$

$$Z_3 = \begin{array}{c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{1'} \\ \cline{2-2} 2' & -1 \\ \cline{2-2} 3' & 2 \\ \cline{2-2} 4' & 4 \\ \cline{2-2} \end{array} \quad , \tag{102}$$

$$Z_4 = \begin{array}{c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{2'} & \multicolumn{1}{c}{3'} & \multicolumn{1}{c}{4'} \\ \cline{2-4} 2' & -3 & 3 & -1 \\ \cline{2-4} 3' & 2 & -1 & -2 \\ \cline{2-4} 4' & 1 & 4 & 3 \\ \cline{2-4} \end{array} \quad . \tag{103}$$

Now it is possible to use (100), (101), (102) and (103) for substitution in equation (91) to solve for Z":

$$Z'' = \begin{array}{c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{1'} \\ \cline{2-2} 1' & 3 \\ \cline{2-2} \end{array} - \begin{array}{c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{2'} & \multicolumn{1}{c}{3'} & \multicolumn{1}{c}{4'} \\ \cline{2-4} 1' & 1 & -2 & 4 \\ \cline{2-4} \end{array} \cdot \begin{array}{c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{2'} & \multicolumn{1}{c}{3'} & \multicolumn{1}{c}{4'} \\ \cline{2-4} 2' & 5 & -13 & -7 \\ \cline{2-4} 3' & -8 & -8 & -8 \\ \cline{2-4} 4' & 9 & 15 & -3 \\ \cline{2-4} \end{array} \cdot \left(\frac{-1}{48}\right) \cdot Z_3 \tag{104}$$

$$= \begin{array}{c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{1'} \\ \cline{2-2} 1' & 3 \\ \cline{2-2} \end{array} - \begin{array}{c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{2'} & \multicolumn{1}{c}{3'} & \multicolumn{1}{c}{4'} \\ \cline{2-4} 1' & 57 & 63 & -3 \\ \cline{2-4} \end{array} \left(\frac{-1}{48}\right) \cdot \begin{array}{c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{1'} \\ \cline{2-2} 2' & -1 \\ \cline{2-2} 3' & 2 \\ \cline{2-2} 4' & 4 \\ \cline{2-2} \end{array} \tag{105}$$

$$= \frac{1}{48} \cdot \left[\begin{array}{cc} & 1' \\ 1' & \boxed{144} \end{array} + \begin{array}{cc} & 1' \\ 1' & \boxed{57} \end{array} \right] = \begin{array}{cc} & 1'' \\ 1'' & \boxed{\frac{201}{48}} \end{array} . \qquad (106)$$

Then e" can be calculated by substituting[18] the quantities of (97) and (99) in (92):

$$e'' = \begin{array}{c} 1' \\ \boxed{15} \end{array} - \begin{array}{c|c|c|c} & 2' & 3' & 4' \\ \hline 1' & 57 & 63 & -3 \end{array} \cdot \begin{array}{c|c} 2' & -2 \\ \hline 3' & -5 \\ \hline 4' & 30 \end{array} \cdot \left(\frac{-1}{48}\right) \qquad (107)$$

$$= \begin{array}{c} 1' \\ \boxed{15} \end{array} - \begin{array}{c} 1' \\ \boxed{-519} \end{array} \cdot \left(\frac{-1}{48}\right) \qquad (108)$$

$$= \frac{1}{48} \left[\begin{array}{c} 1' \\ \boxed{720} \end{array} - \begin{array}{c} 1' \\ \boxed{519} \end{array} \right] = \begin{array}{c} 1'' \\ \boxed{\frac{201}{48}} \end{array} . \qquad (109)$$

With the values of both e" and Z" known, it is possible to solve for the required current, i_1:

$$i'' = i_1 = \frac{e''}{Z''} = Z''^{-1} \cdot e''$$

$$= \begin{array}{cc} & 1'' \\ 1'' & \boxed{\frac{48}{201}} \end{array} \cdot 1'' \boxed{\frac{201}{48}} = 1'' \ \boxed{1} . \qquad (110)$$

$$\therefore \ i_1 = 1. \qquad (111)$$

---

[18] It is desirable to retain the value of the product of $Z'_2 Z'^{-1}_4$ from impedance reduction calculations for use here.

If it were desired to obtain the values of two of the unknown currents, the Z_1 of (99) would have been a 4-element 2-matrix. The result, then, would contain two simultaneous equations which could have been readily solved. Sets of simultaneous equations can be broken up by steps using these compound matrices and reduction formulas; there is no longer a practical limitation to the number of simultaneous equations which can be solved in a systematic manner.

23. Calculation of Network Impedances Between Terminals.

Suppose it is necessary to know the impedance of the network shown in Fig. 5; that is, the impedance which would be measured by an impedance bridge between points A and B with the generator open circuited. Resistance values are given for the impedances, but complex quantities could be used if desired. Furthermore, each element of the network might be matrix in character.

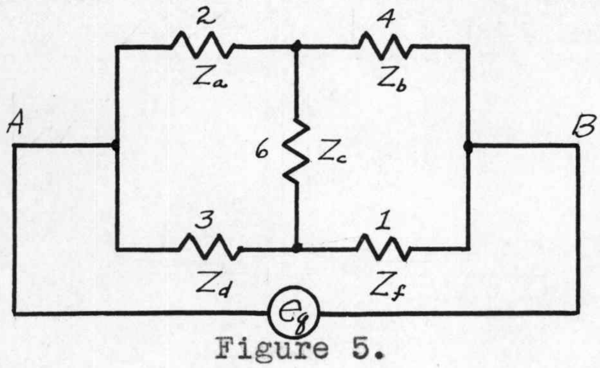


Figure 5.

The first step in tensor analysis of a problem of this kind is to assume current directions in the individual coils. This can be done by laying out the primitive network as demonstrated previously or by redrawing the circuit with arrows showing these assumed directions. The straight arrows in Fig. 6, then, represent the positive direction of current flow in the individual coils as assumed before the circuit was connected. After connection, the new mesh currents are indicated by the circular arrows--the meshes being labeled p, q and r. Assuming the existence of no mutual effects between coils, the primitive z is

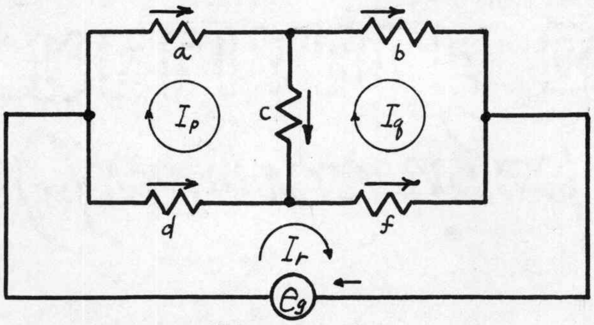


Figure 6.

$$Z = \begin{array}{c|cccccc|} & g & a & b & c & d & f \\ \hline g & 0 & & & & & \\ a & & 2 & & & & \\ b & & & 4 & & & \\ c & & & & 6 & & \\ d & & & & & 3 & \\ f & & & & & & 1 \\ \hline \end{array} \quad \text{ohms.} \qquad (112)$$

The connection matrix, C, which matches the old coil currents with the new mesh currents, by inspection is

$$C = \begin{array}{cc} & \begin{matrix} p & q & r \end{matrix} \\ \begin{matrix} g \\ a \\ b \\ c \\ d \\ f \end{matrix} & \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \\ 1 & -1 & \\ -1 & & 1 \\ & -1 & 1 \end{bmatrix} \end{array} . \tag{113}$$

The impedance matrix, Z', can now be calculated from equation (72):

$$Z' = C_t \cdot Z \cdot C$$

$$= \begin{array}{cc} & \begin{matrix} g & a & b & c & d & f \end{matrix} \\ \begin{matrix} p \\ q \\ r \end{matrix} & \begin{bmatrix} & 1 & & 1 & -1 & \\ & & 1 & -1 & & -1 \\ 1 & & & & 1 & 1 \end{bmatrix} \end{array} \cdot \begin{array}{cc} & \begin{matrix} g & a & b & c & d & f \end{matrix} \\ \begin{matrix} g \\ a \\ b \\ c \\ d \\ f \end{matrix} & \begin{bmatrix} 0 & & & & & \\ & 2 & & & & \\ & & 4 & & & \\ & & & 6 & & \\ & & & & 3 & \\ & & & & & 1 \end{bmatrix} \end{array} \cdot C$$

$$= \begin{array}{cc} & \begin{matrix} g & a & b & c & d & f \end{matrix} \\ \begin{matrix} p \\ q \\ r \end{matrix} & \begin{bmatrix} 0 & 2 & 0 & 6 & -3 & 0 \\ 0 & 0 & 4 & -6 & 0 & -1 \\ 0 & 0 & 0 & 0 & 3 & 1 \end{bmatrix} \end{array} \cdot \begin{array}{cc} & \begin{matrix} p & q & r \end{matrix} \\ \begin{matrix} g \\ a \\ b \\ c \\ d \\ f \end{matrix} & \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \\ 1 & -1 & \\ -1 & & 1 \\ & -1 & 1 \end{bmatrix} \end{array}$$

$$Z = \begin{array}{c|c|c|c|} & p & q & r \\ \hline p & 11 & -6 & -3 \\ \hline q & -6 & 11 & -1 \\ \hline r & -3 & -1 & 4 \\ \hline \end{array} \text{ ohms.} \tag{114}$$

This is the new mesh Z' which represents the impedances along the mesh axes. As the student becomes familiar with these concepts, he will be able to write out the Z' for a simple network such as this by inspection. For example, the value of Z_{pp} is the sum of the resistance values around the P circuit. Z_{qp} is the drop seen by the Q mesh when unit current flows in the P mesh; since I_q is reversed in respect to I_p in that C-leg, the sign must be minus, hence -6. The R mesh sees a drop of 3 volts in the opposite direction to I_r, so the Z_{rp} element is -3. The other elements could be written from inspection in a similar fashion.

Now to get the equivalent resistance of the circuit as seen by the generator, e_g; that is, to get the value of R by which the network of Fig. 5 may be replaced (see Fig. 7), apply the impedance reduction formula (91) and eliminate the P and Q meshes.

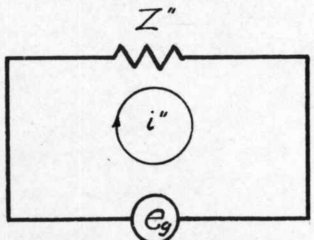


Figure 7.

So the mesh Z of (114) may be subdivided as follows:[19]

$$Z = \begin{array}{c|c|c|c|} & p & q & r \\ \hline p & 11 & -6 & -3 \\ \hline q & -6 & 11 & -1 \\ \hline r & -3 & -1 & 4 \\ \hline \end{array} = \begin{array}{|c|c|} \hline Z_4 & Z_3 \\ \hline Z_2 & Z_1 \\ \hline \end{array}. \qquad (115)$$

The student will note the rearrangement of subscripts to suit the equation (83); this illustrates the freedom of shifting indices as long as the relative order remains unchanged.

Substitution in (91) gives:

$$Z'' = Z_1 - Z_2 Z_4^{-1} Z_3$$

$$= \begin{array}{c|c|} & r \\ \hline r & 4 \\ \hline \end{array} - \begin{array}{c|c|c|} & p & q \\ \hline r & -3 & -1 \\ \hline \end{array} \cdot \begin{array}{c|c|c|} & p & q \\ \hline p & 11 & 6 \\ \hline q & 6 & 11 \\ \hline \end{array} \cdot \frac{1}{\triangle} \cdot Z_3, \qquad (117)$$

where

$$\triangle = (11)(11) - (-6)(-6) = 85. \qquad (118)$$

---

[19] Another way to handle this Z-matrix would be to rearrange rows and columns thus:

$$Z = \begin{array}{c|c|c|c|} & r & q & p \\ \hline r & 4 & -1 & -3 \\ \hline q & -1 & 11 & -6 \\ \hline p & -3 & -6 & 11 \\ \hline \end{array} = \begin{array}{|c|c|} \hline Z_1 & Z_2 \\ \hline Z_3 & Z_4 \\ \hline \end{array}. \qquad (116)$$

$$Z'' = \begin{array}{cc} & r \\ r & \boxed{4} \end{array} - \begin{array}{ccc} & p & q \\ r & -39 & -29 \end{array} \cdot \begin{array}{cc} & r \\ p & -3 \\ q & -1 \end{array} \cdot \frac{1}{85}$$

$$= \begin{array}{cc} & r \\ r & \boxed{4} \end{array} - \begin{array}{cc} & r \\ r & \boxed{+\frac{146}{85}} \end{array} = 4 - 1.72$$

$$= \quad 2.28 \text{ ohms.} \qquad (119)$$

The solution retains only the R mesh as the other two meshes drop out in the multiplication. Thus, by this mechanical process, ordinary delta-wye transformations are carried out.

24. Establishing Equivalent Voltages.

The problem of Art. 23 illustrates how an impedance network may be replaced by a single impedance.

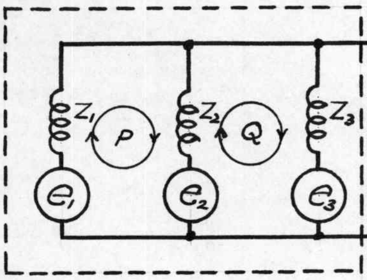


Figure 8.

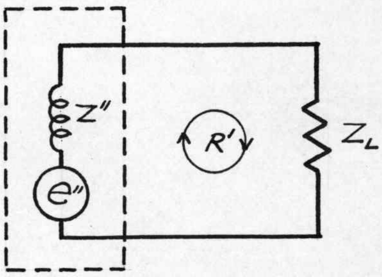


Figure 9.

However, it does not illustrate how voltages of a network may be replaced by a single voltage. Consider the circuit shown in Figure 8. Here there are three generators in parallel supplying a load, Z_L. The dotted line boxes in these power sources and Fig. 9 shows the equivalent circuit after the three generators have been replaced by one.

The procedure followed in Article 23 would give the value of Z" in Fig. 9. The P and Q meshes could be eliminated, leaving only the R mesh. The impedance around this mesh would equal the sum of Z" and Z_L, the former being the required impedance.

To get the equivalent voltage, it is necessary first of all to express the voltage in terms of the P, Q and R meshes thus:

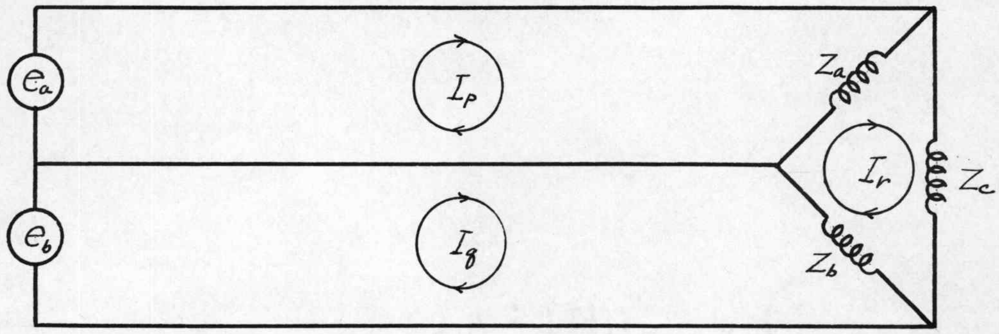
$$e = \begin{array}{ccc} p & q & r \\ \hline \multicolumn{1}{|c|}{e_p} & \multicolumn{1}{c|}{e_q} & \multicolumn{1}{c|}{e_r} \\ \hline \end{array} = \begin{array}{cc} 1 & 2 \\ \hline \multicolumn{1}{|c|}{e_1} & \multicolumn{1}{c|}{e_2} \\ \hline \end{array} , \qquad (120)$$

where the mesh voltages are obtained by (75). Then by applying equation (92), the resulting e" may be obtained. It is interesting to consider that Z_L might also represent some complicated network which has been replaced by a single impedance.

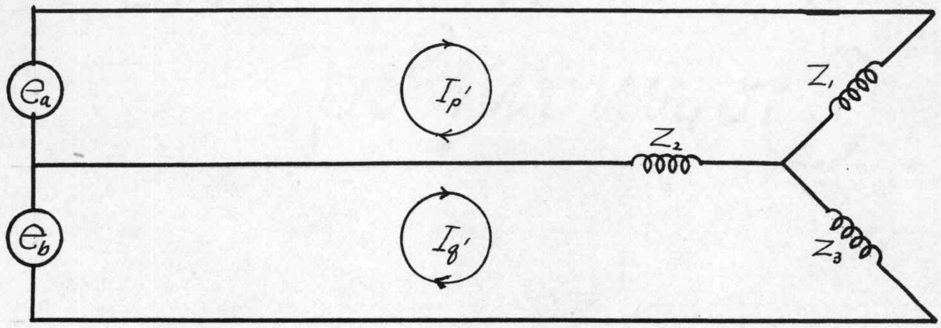
25. Delta-Wye Transformations.

It can be shown that the impedance reduction formulas perform the operation of transforming a given network from a delta combination to a wye combination.

For example, consider the two three-phase circuits in Fig. 10: circuit (a) has a balanced delta-connected load and circuit (b) shows the equivalent wye-connected load which could replace that of (a) without changing the power consumed. The voltage sources in each case are identical.



Circuit (a)



Circuit (b)

Figure 10.

The mesh impedance tensor for circuit (a) is by inspection:

$$Z'_{(a)} = \begin{array}{c|c|c|c|} & p & q & r \\ \hline p & Z_a & 0 & -Z_a \\ \hline q & 0 & Z_b & -Z_b \\ \hline r & -Z_a & -Z_b & Z_a + Z_b + Z_c \\ \hline \end{array} \quad . \tag{121}$$

Now the R mesh may be eliminated from circuit (a) by applying the impedance reduction formula,

$$Z'' = Z'_1 - Z'_2 \, Z'^{-1}_4 \, Z'_3 \, , \tag{1220}$$

which allows the following substitution:

$$Z'' = \begin{array}{c|c|c|} & p & q \\ \hline p & Z_a & 0 \\ \hline q & 0 & Z_b \\ \hline \end{array} - \begin{array}{c|c|} & r \\ \hline p & -Z_a \\ \hline q & -Z_b \\ \hline \end{array} \cdot \begin{array}{c|c|} & r \\ \hline r & \dfrac{1}{Z_a + Z_b + Z_c} \\ \hline \end{array} \cdot \begin{array}{c|c|c|} & p & q \\ \hline r & -Z_a & -Z_b \\ \hline \end{array}$$

$$= \begin{array}{c|c|c|} & p & q \\ \hline p & Z_a - \dfrac{Z_a^2}{Z_a + Z_b + Z_c} & \dfrac{-Z_a Z_b}{Z_a + Z_b + Z_c} \\ \hline q & \dfrac{-Z_b Z_a}{Z_a + Z_b + Z_c} & Z_b - \dfrac{Z_b^2}{Z_a + Z_b + Z_c} \\ \hline \end{array} \quad . \tag{123}$$

By inspection, the mesh impedance tensor for circuit (b) can be written thus:

$$Z'_{(b)} = \begin{array}{c|c|c|} & p' & q' \\ \hline p' & Z_1 + Z_2 & -Z_2 \\ \hline q' & -Z_2 & Z_2 + Z_3 \\ \hline \end{array} \quad . \tag{124}$$

Since the power is invariant in the two circuits, the impedances looking along the same axes must be equal. Therefore, it is possible to equate the Z_{pp} element of (123) with the $Z_{p'p'}$ element of (124):

$$Z_1 + Z_2 = Z_a - \frac{Z_a^2}{Z_a + Z_b + Z_c} . \qquad (125)$$

Equating the Z_{qp} element with the $Z_{q'p'}$ element:

$$-Z_2 = \frac{-Z_b Z_a}{Z_a + Z_b + Z_c} . \qquad (126)$$

Then by substituting (126) into (125):

$$Z_1 = \frac{Z_a Z_c}{Z_a + Z_b + Z_c} , \qquad (127)$$

which is the formula for changing from delta to wye. Also it can be found that

$$Z_2 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c} , \qquad (128)$$

$$Z_3 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c} , \qquad (129)$$

26. <u>Solution of Three-Phase Unbalanced Wye Loads.</u>

The derivation of the formula which gives the voltage drop across a phase of an unbalanced three-phase wye load is possible with the use of the tensor analysis developed thus far. Consider the three-phase circuit shown in Fig. 11. Assume that the impedances are given and are not necessarily equal in either phase angle or magnitude. The voltages as given, E_a and E_b, may not be in terms of what the student is accustomed to.

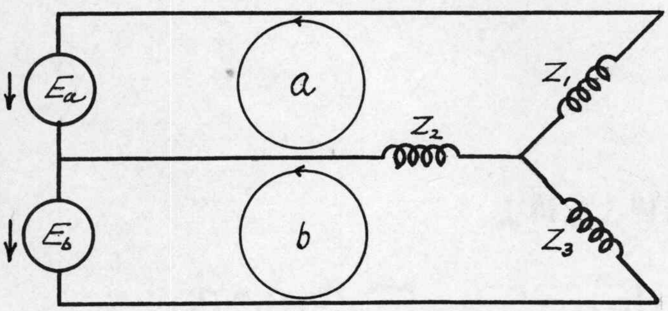


Figure 11.

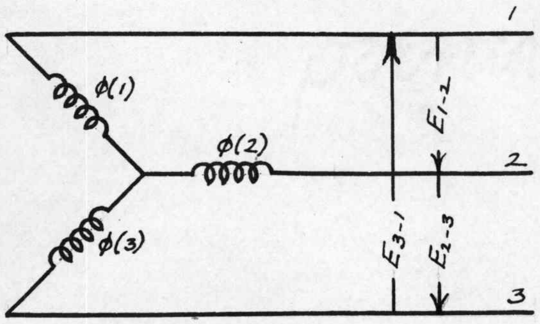


Figure 12.

For clarity, Figure 12 gives the usual three-phase voltages employed. The voltages of Figure 11 can be written in terms of those in Figure 12 by inspection as follows:

$$E_a = E_{1-2} , \tag{130}$$

$$E_b = E_{2-3} , \tag{131}$$

$$-E_a - E_b = E_{3-1} , \tag{132}$$

$$E_a + E_b = E_{1-3} . \tag{133}$$

Assuming meshes A and B as shown in Fig. 11, the connected Z' can be written from inspection as follows:

$$Z' = C_t \cdot Z \cdot C = \begin{array}{c|cc} & a & b \\ \hline a & Z_1 + Z_2 & -Z_2 \\ b & -Z_2 & Z_2 + Z_3 \end{array} . \tag{134}$$

Since it is desired to find the voltage drop across a phase of the load (for instance, e_1), the line current flowing into that phase is of importance. This line current is mesh current I_a, and is found by taking the inverse of (134) thus:

$$Y = Z^{-1} = \begin{array}{c|cc} & a & b \\ \hline a & Z_2 + Z_3 & Z_2 \\ b & Z_2 & Z_1 + Z_2 \end{array} \cdot \frac{1}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3} . \tag{135}$$

The voltages,

$$E = \begin{array}{cc} a & b \\ \hline E_a & E_b \end{array} , \tag{136}$$

are also known. Equations (135) and (136) are combined to give the current equation,

$$I = Y \cdot E , \tag{137}$$

from which

$$I_a = \frac{E_a(Z_2 + Z_3) + E_b Z_2}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3} \quad . \tag{138}$$

But

$$e_1 = I_a Z_1 = \frac{E_a(Z_2 + Z_3)Z_1 + E_b Z_2 Z_1}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3} . \tag{139}$$

This is the voltage drop across Z_1 in terms of known voltages and impedances. Replacing each impedance with its admittance $(Z = \frac{1}{Y})$, and simplifying:

$$e_1 = \frac{E_a Y_2 + (E_a + E_b) Y_3}{Y_1 + Y_2 + Y_3} \quad . \tag{140}$$

Writing the voltages of (136) in terms of those used in Fig. 12, employing equations (130), (131) and (133),

$$e_1 = \frac{E_{1-2} Y_2 + E_{1-3} Y_3}{Y_1 + Y_2 + Y_3} \quad . \tag{141}$$

This is the equation for the voltage drop across Z_1 in terms of given values of voltage and admittance.

With these reduction formulas, complicated networks can be simplified into their simplest form. As the student works sample problems using (91) and (92), he will become aware of the tremendous power of this new tool.

CHAPTER VI

MUTUAL EFFECTS

27. Omnipresence of Mutual Effects.

The physicist ordinarily analyses a single element or physical entity as it exists in space alone. He makes his investigations on the basis of the element itself and attributes its behavior to certain inherent peculiarities.

The engineer, then, must take these findings of the physicist concerning the theory of the element itself and proceed with his work--that of analysing the results of physical connection of a number of these components. Thus mutual effects gain considerable importance--their effects being determined by proximity of the various entities and the nature of the medium separating them.

In network theory, mutual effects exist when two independent circuits or electrical elements are in proximity to each other such that a change in the current or voltage of the first will cause an induced voltage or current, respectively, in the second. It is obvious that mutual effects exist between all elements in the universe, but only under special conditions are these worthy of investigation.

28. Balanced and Unbalanced Mutual Effects.

In the stationary circuit containing resistance, inductance and capacitance--the so-called static bilateral network--the mutual impedance between any two coils is the same regardless of whether it is taken from the first to the second coil, or from the second to the first. Thus, the primitive impedance tensor would be symmetrical in respect to the diagonal terms. A current in the first coil will induce an emf. in the second coil of the same magnitude as that induced in the first coil when a similar current appears in the second coil. Balanced mutual effects exist in practically all stationary, electrical networks which do not contain vacuum tubes or moving elements.

Unbalanced mutual effects between coils exist in electrical equipment where the relative positions of the coils change with respect to time. Rotating machinery is a notable example of unbalanced mutual impedances between coils since there is mechanical motion in the generation of emf. As a further example, vacuum-tube, mutual admittances between elements are unequal because of the movement of electrical charges. Copper-oxide circuits and crystal detectors are also examples of circuits having unbalanced mutual impedances or admittances.

29. Representation of Mutual Effects.

The absolute value of a mutual effect between two coils can be determined with a voltmeter and an ammeter. If direction is also desired, a wattmeter may be added.

In Fig. 13, there are two coils which are assumed to be wound on the same magnetic structure. If these coils had no mutual effects between them (such as the ones studied thus far), the primitive impedance matrix would be:

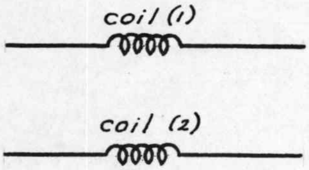


Figure 13.

$$Z = \begin{array}{c|c|c|} & 1 & 2 \\ \hline 1 & Z_1 & \\ \hline 2 & & Z_2 \\ \hline \end{array} \; . \qquad (142)$$

But since mutual effects are present, it is necessary to make a new set of measurements. In Fig. 14, the circuit for measuring the magnitude of the Z_{21} element is shown. With the a-c voltage source across coil (1), the rheostat in series is cut out until the ammeter, I, records one ampere. Then the reading of the volt-

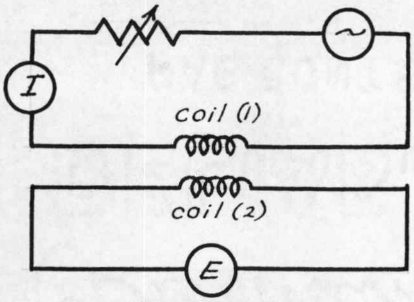


Figure 14.

meter, E, is the value which can be placed directly in the Z_{21} element position in (142). The sign before this mutual impedance may be either plus or minus; this can be determined by inserting a wattmeter with its current coil in the coil (1) circuit and its voltmeter coil in the coil (2) circuit. The wattmeter is connected so that it should read positive for the directions which have been assumed at the beginning, that is, the senses of voltages in the primitive circuit. Then if the meter reads negative, the sign before the mutual impedances will be negative; if positive power registers, the sign is positive.

The primitive circuit for two coils which have mutual effects between them is shown in Fig. 15. The coils are shown with an assumed direction, just as heretofore. The curved line connecting the two coils indicates the presence of a mutual impedance from (1) to (2) as well as from (2) to (1). The magnitude, it might be assumed, is Z_m, indicating that the wattmeter read positive for the connection of Fig. 14. Note that each time a Z-matrix is written,

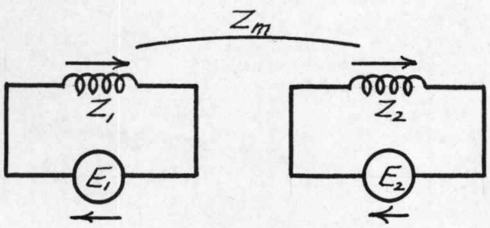


Figure 15.

each element may be considered as being a real number, a complex number, or even another matrix. The revised primitive Z for the network of Fig. 15 is:

$$Z = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & Z_1 & Z_m \\ 2 & Z_m & Z_2 \end{array} \quad . \tag{143}$$

30. Mutual Effects Between Two Coils.

Suppose the coils of Fig. 15 are connected in series such that the mutual effects are additive (see Fig. 16). The connection matrix for the coils and mesh shown is:

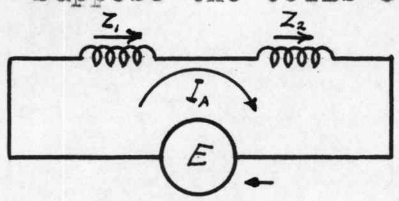


Figure 16.

$$C = \begin{array}{c|c} & A \\ \hline 1 & 1 \\ 2 & 1 \end{array} \quad . \tag{144}$$

From (143) and (144), the mesh Z becomes:

$$Z' = C_t \cdot Z \cdot C = \begin{array}{c|cc} & 1 & 2 \\ \hline A & 1 & 1 \end{array} \cdot \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & Z_1 & Z_m \\ 2 & Z_m & Z_2 \end{array} \cdot \begin{array}{c|c} & A \\ \hline 1 & 1 \\ 2 & 1 \end{array} \tag{145}$$

$$= \begin{array}{c|c} & A \\ \hline A & Z_1 + Z_2 + 2Z_m \end{array} \quad \text{ohms,}$$

defined along the (A) axis.

Now if the two coils are connected in series opposing (see Fig. 17), the connection matrix is:

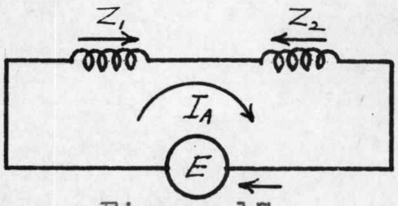


Figure 17.

$$C = \begin{array}{c|c|} & A \\ \hline 1 & 1 \\ \hline 2 & -1 \\ \hline \end{array} \; . \qquad (146)$$

The mesh Z, or Z' is:

$$Z' = C_t \cdot Z \cdot C = \begin{array}{c|c|c|} & 1 & 2 \\ \hline A & 1 & -1 \\ \hline \end{array} \cdot \begin{array}{c|c|c|} & 1 & 2 \\ \hline 1 & Z_1 & Z_m \\ \hline 2 & Z_m & Z_2 \\ \hline \end{array} \cdot \begin{array}{c|c|} & A \\ \hline 1 & 1 \\ \hline 2 & -1 \\ \hline \end{array}$$

$$= \begin{array}{c|c|} & A \\ \hline A & Z_1 + Z_2 - 2Z_m \\ \hline \end{array} \quad \text{ohms.} \qquad (147)$$

These simple derivations may give the student some idea of the possibilities offered by this method of analysis.

A rigorous treatment of the transformer is also possible. Laboratory measurements can be made so that the elements of the mesh Z-matrix or the Y-matrix (the admittance matrix) can be filled in from meter readings. Since the transformer offers a study in itself, it will not be introduced in this text. The student is referred to Kron's, "Tensor Analysis of Electrical Networks."[20]

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[20]See page 281. G.E. Series. John Wiley & Sons, New York. 1939.

31. The Bridge Circuit.

The student may now be aware of the power of this mathematical tool in analyzing an electrical circuit. Tensor analysis may also be used to derive useful equations which would ordinarily be burdensome using algebraic methods. Also, the thought carries through with each logical step and the student is able to get a firmer grasp on the fundamental concepts embodied in a particular problem.

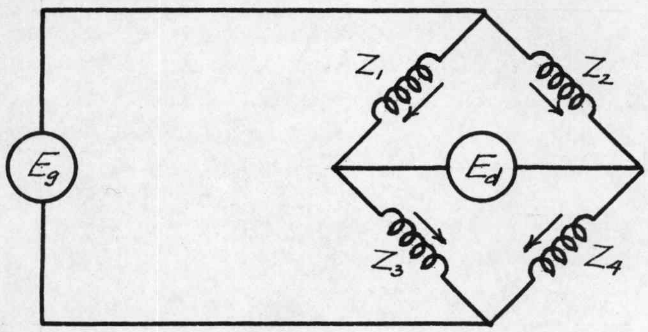

Figure 18.

To demonstrate one of the applications of tensor analysis, refer to the ordinary bridge circuit of Fig. 18, where Z_1, Z_2, Z_3 and Z_4 represent the impedances of the four arms of the ordinary Wheatstone bridge, one of which is the unknown impedance. E_g represents the voltage source, assumed here to be without impedance, and E_d is the voltage that appears across the detector. By inspecting this circuit in the manner described in Art. 29, the primitive network can be drawn. Assume it to be as shown in Fig. 19. Here Z_g and Z_d are both zero; also the voltages E_1, E_2, E_3

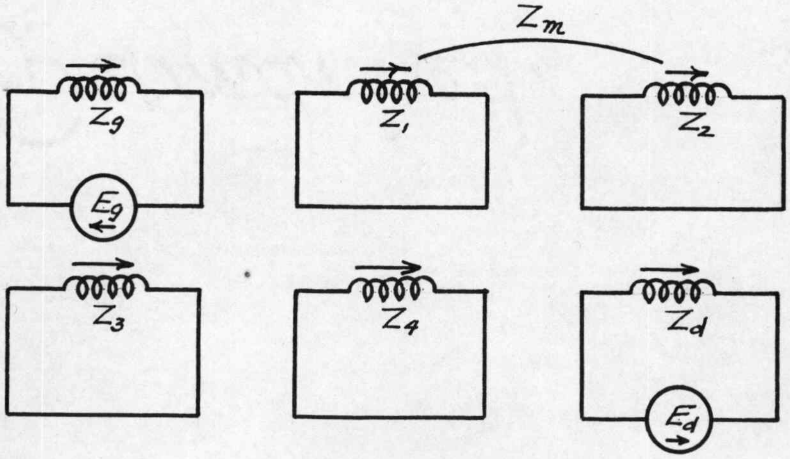

Figure 19.

and E_4 are zero since there are no impressed voltages on these coils. The network of Fig. 19, then, is the most general circuit of the coils and voltages considered in Figure 18.

The primitive or original quantities are all brought together into the Z-matrix thus:

$$Z = \begin{array}{c} \\ g \\ 1 \\ 2 \\ 3 \\ 4 \\ d \end{array} \begin{array}{c} \begin{array}{cccccc} g & 1 & 2 & 3 & 4 & d \end{array} \\ \left|\begin{array}{cccccc} 0 & & & & & \\ & Z_1 & Z_m & & & \\ & Z_m & Z_2 & & & \\ & & & Z_3 & & \\ & & & & Z_4 & \\ & & & & & 0 \end{array}\right| \end{array} . \qquad (148)$$

The primitive voltages and currents may be represented thus:

$$e = \begin{array}{cccccc} g & 1 & 2 & 3 & 4 & d \\ \hline \multicolumn{1}{|c|}{E_g} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{-E_d} \\ \hline \end{array} \, , \qquad (149)$$

$$i = \begin{array}{cccccc} g & 1 & 2 & 3 & 4 & d \\ \hline \multicolumn{1}{|c|}{I_g} & \multicolumn{1}{c|}{I_1} & \multicolumn{1}{c|}{I_2} & \multicolumn{1}{c|}{I_3} & \multicolumn{1}{c|}{I_4} & \multicolumn{1}{c|}{I_d} \\ \hline \end{array} \, . \qquad (150)$$

The student may wonder about the sign of E_d as shown in (149). Although it doesn't make any difference in the results whether this sign is taken positive or negative, it is helpful in working problems of this nature to distinguish between a voltage receiver and a voltage supplier, or a voltmeter and a voltage generator.

The next step is to interconnect the coils of Fig. 19 into the circuit shown in Fig. 20. Mesh currents, I_G, I_1 and I_D, are assumed to flow. Note that the currents through the generator and detector are both pure mesh currents; this is done to clarify the results. Mathematically, the interconnection into the network of Fig. 20 from the primitive is made by the use of the following C:

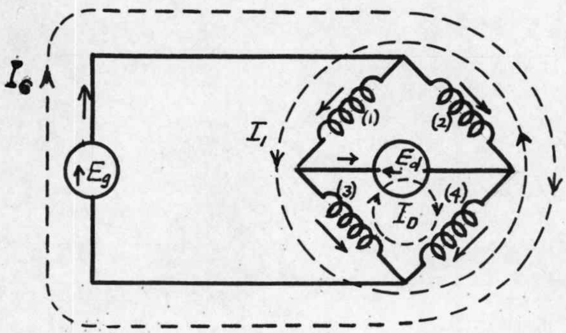

Figure 20.

$$C = \begin{array}{c} \\ g \\ 1 \\ 2 \\ 3 \\ 4 \\ d \end{array} \begin{array}{c} \begin{array}{ccc} G & 1 & D \end{array} \\ \left[\begin{array}{c|c|c} 1 & & \\ \hline & 1 & \\ \hline 1 & -1 & \\ \hline & 1 & -1 \\ \hline 1 & -1 & 1 \\ \hline & & 1 \end{array}\right] \end{array} . \qquad (151)$$

The calculations necessary to obtain the new mesh quantities, Z' and e', are shown:

$$Z' = C_t \cdot Z \cdot C$$

$$= \begin{array}{c|cccccc} & g & 1 & 2 & 3 & 4 & d \\ \hline G & 1 & & 1 & & 1 & \\ 1 & & 1 & -1 & 1 & -1 & \\ D & & & & -1 & 1 & 1 \end{array} \cdot \begin{array}{c|cccccc} & g & 1 & 2 & 3 & 4 & d \\ \hline g & 0 & & & & & \\ 1 & & Z_1 & Z_m & & & \\ 2 & & Z_m & Z_2 & & & \\ 3 & & & & Z_3 & & \\ 4 & & & & & Z_4 & \\ d & & & & & & 0 \end{array} \cdot C$$

$$= \begin{array}{c|cccccc} & g & 1 & 2 & 3 & 4 & d \\ \hline G & 0 & Z_m & Z_2 & 0 & Z_4 & 0 \\ 1 & 0 & Z_1-Z_m & Z_m-Z_2 & Z_3 & -Z_4 & 0 \\ D & 0 & 0 & 0 & -Z_3 & Z_4 & 0 \end{array} \cdot \begin{array}{c|ccc} & G & 1 & D \\ \hline g & 1 & & \\ 1 & & 1 & \\ 2 & 1 & -1 & \\ 3 & & 1 & -1 \\ 4 & 1 & -1 & 1 \\ d & & & 1 \end{array}$$

$$
= \begin{array}{c} \\ G \\ 1 \\ D \end{array}
\begin{array}{c}
\begin{array}{ccc} G & 1 & D \end{array} \\
\left[\begin{array}{c|c|c}
Z_2 + Z_4 & Z_m - Z_2 - Z_4 & Z_4 \\ \hline
Z_m - Z_2 - Z_4 & Z_1 + Z_2 - 2Z_m + Z_3 + Z_4 & -Z_3 - Z_4 \\ \hline
Z_4 & -Z_3 - Z_4 & Z_3 + Z_4
\end{array}\right]
\end{array} ; \qquad (152)
$$

$$
e' = C_t \, e = \begin{array}{c|cccccc}
 & g & 1 & 2 & 3 & 4 & d \\ \hline
G & 1 & & 1 & & 1 & \\
1 & & 1 & -1 & 1 & -1 & \\
D & & & & -1 & 1 & 1
\end{array}
\cdot
\begin{array}{c|c}
g & E_g \\
1 & 0 \\
2 & 0 \\
3 & 0 \\
4 & 0 \\
d & -E_d
\end{array}
$$

$$
= \begin{array}{c|c}
G & E_g \\
1 & 0 \\
D & -E_d
\end{array} . \qquad (153)
$$

The mesh e' of (153) could have been written from observation; but the mesh Z' of (152) is difficult to predict because of the mutual effects.

With these mesh quantities known, it is now possible to find the mesh currents by the use of the following matrix formula:

$$ i' = Z'^{-1} \cdot e' = y' \cdot e' . \qquad (154) $$

This involves solving for the inverse of Z'; first, the adjoint matrix (see equation 37, Art. 11) is obtained:

$$M = \begin{array}{c|c|c|c|} & G & 1 & D \\ \hline G & \begin{array}{c} Z_1Z_3+Z_2Z_3+Z_1Z_4 \\ +Z_2Z_4-2Z_m(Z_3+Z_4) \end{array} & \begin{array}{c} Z_2Z_3+Z_2Z_4 \\ -Z_m(Z_3+Z_4) \end{array} & \begin{array}{c} Z_2Z_3-Z_1Z_4 \\ +Z_m(Z_4-Z_3) \end{array} \\ \hline 1 & \begin{array}{c} Z_2Z_4+Z_2Z_3 \\ -Z_m(Z_3+Z_4) \end{array} & \begin{array}{c} Z_2Z_3+Z_2Z_4 \\ +Z_3Z_4 \end{array} & \begin{array}{c} Z_2Z_3+Z_3Z_4 \\ +Z_mZ_4 \end{array} \\ \hline D & \begin{array}{c} Z_2Z_3-Z_1Z_4 \\ +Z_m(Z_4-Z_3) \end{array} & \begin{array}{c} Z_2Z_3+Z_3Z_4 \\ +Z_mZ_4 \end{array} & \begin{array}{c} Z_1Z_2+Z_2Z_3 \\ +Z_3Z_4+Z_1Z_4 \end{array} \\ \hline \end{array} . \tag{155}$$

To calculate Δ, the following equation is used giving its value in each diagonal and zero in all other elements:

$$M \cdot Z = \begin{array}{c|c|c|c|} & G & 1 & D \\ \hline G & \Delta & 0 & 0 \\ \hline 1 & 0 & \Delta & 0 \\ \hline D & 0 & 0 & \Delta \\ \hline \end{array} , \tag{156}$$

where Δ =

$$Z_1Z_2Z_3 + Z_1Z_2Z_4 + Z_1Z_3Z_4 + Z_2Z_3Z_4$$
$$-2Z_mZ_3Z_4 - Z_m^2Z_3 - Z_m^2Z_4. \tag{157}$$

This matrix serves as a check on previous work. If all diagonal values are equal and all other elements equate to zero, the work previously done is correct. The Y-matrix (inverse of Z) for the bridge circuit is:

$$Y = \frac{1}{\Delta} \cdot \begin{array}{c|ccc} & G & 1 & D \\ \hline G & M_{GG} & M_{G1} & M_{GD} \\ 1 & M_{1G} & M_{11} & M_{1D} \\ D & M_{DG} & M_{D1} & M_{DD} \end{array} \quad . \tag{158}$$

where the M elements are shown in (155) and the value of Δ in (157).

When the bridge is balanced, the voltage, E_d, must be zero and no current will flow through the detector. Mathematically, for balance the following relations must be true:

$$E_d = 0, \tag{159}$$

$$I_d = 0. \tag{160}$$

From equation (154), it is possible to substitute (153) and (158) to give the following matrix expression:

$$\begin{array}{ccc} G & 1 & D \\ \hline I_G & I_1 & I_D \end{array} = \frac{1}{\Delta} \begin{array}{c|ccc} & G & 1 & D \\ \hline G & M_{GG} & M_{G1} & M_{GD} \\ 1 & M_{1G} & M_{11} & M_{1D} \\ D & M_{DG} & M_{D1} & M_{DD} \end{array} \cdot \begin{array}{c|c} G & E_g \\ 1 & 0 \\ D & -E_d \end{array}, \tag{161}$$

which can be written algebraically as three simultaneous equations,

$$I_g = \frac{E_g M_{GG}}{\Delta} - \frac{E_d M_{GD}}{\Delta}, \tag{162}$$

$$I_1 = \frac{E_g M_{1G}}{\Delta} - \frac{E_d M_{1D}}{\Delta}, \tag{163}$$

$$I_d = \frac{E_g M_{DG}}{\Delta} - \frac{E_d M_{DD}}{\Delta}; \tag{164}$$

substituting (159) and (160) in (164):

$$0 = \frac{E_g M_{DG}}{\Delta} \tag{165}$$

From this, it is necessary that

$$\frac{M_{DG}}{\Delta} = \frac{Z_2 Z_3 - Z_1 Z_4 + Z_m Z_4 - Z_m Z_3}{\Delta} = 0. \tag{166}$$

since E_g is a finite voltage source. As Δ is some finite value, the following expression must be true:

$$Z_2 Z_3 - Z_1 Z_4 + Z_m(Z_4 - Z_3) = 0. \tag{167}$$

From this, it can be seen that if the ratio arms, Z_4 and Z_3, were equal, the mutual effects would not hamper the accuracy of the bridge. For radio frequency measurements, this might be of importance.

To check the accuracy of (167), the value of Z_m may be set equal to zero. The resulting equation is the simple Wheatstone bridge equation,

$$Z_2Z_3 = Z_1Z_4,$$

$$\frac{Z_1}{Z_2} = \frac{Z_3}{Z_4} \quad . \qquad (168)$$

CHAPTER VII

JUNCTION PAIRS

32. The Dual Concept.

Recognition of the fundamental concept of duals is important in order that the engineer may more fully understand the possibility of mathematics as a tool for solving electrical networks. A constant, arbitrarily assigned, is ordinarily taken as the quotient of the cause and the effect; as an example,

$$Z = \frac{E}{I}, \tag{168}$$

where E and I are two quantities which can be measured.

Because of the nature of our electrical world, it has been found desirable to think of impressing generated voltages on low-impedance circuits with a resulting current flow. This is particularly true of networks whose prime purpose is to deliver large blocks of power through the medium of current.

The advent of vacuum tubes and circuits in recent years has introduced a new concept. Here it is desirable to consider that currents are established in the electrodes of a vacuum tube due to a voltage; whereas, in the transformer, voltage appears across the windings as a result of current flow.

The study of tensor analysis lends itself nicely to the study of duals because of its general, all-inclusive nature. The connection matrix, previously used to write old currents in terms of the new, may, in the dual concept, be used to write the old voltages of the primitive network in terms of the new connected voltages or junction pairs.

To define a "dual" would be pointless; a description serves better for such a concept. In this chapter, the electrical duals, voltage and current, will be discussed in the language of tensor analysis which has been developed previously in this thesis. It is interesting to note that these quantities are measured in a similar fashion; that is, they both manifest themselves in driving the meter indicator with the use of the same principle, but the ammeter has a low impedance in respect to the test circuit while the voltmeter has a very high impedance in comparison with the unknown. But the brain of man must have some concrete manner of comparing networks, so the dual assumption of voltage and current has been expressed. Similarly, man has created the duals of matter and energy, electric field and magnetic field, magnetic flux linkages and current, and many others. They may be thought of as dual manifestations of nature created by man to define some constant or comparison factor.

33. Thevenin's Theorem.

This theorem must be understood by the student before he can accept the accuracy of the dual concept of voltage and current. Thevenin has stated that any network as viewed from the terminals can be replaced by an impedance in series with a voltage where the impedance is the impedance of the network as measured from the terminals (with all internal voltage generators short-circuited), and the voltage is the voltage appearing across the terminals when open-circuited. For

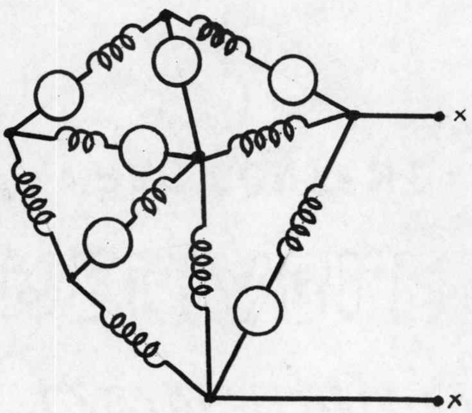

Figure 21.

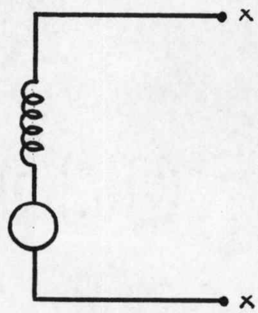

Figure 22.

example, consider the complicated circuit of Fig. 21 with the two leads xx being brought out. The small circles here represent generators in series with impedances. Figure 22 shows the equivalent network which will replace that of Fig. 21 by Thevenin's theorem.

By replacing each term of Thevenin's theorem with its electrical dual, the restatement is: any network as viewed from the terminals can be replaced by an admittance in parallel with a current where the ad-

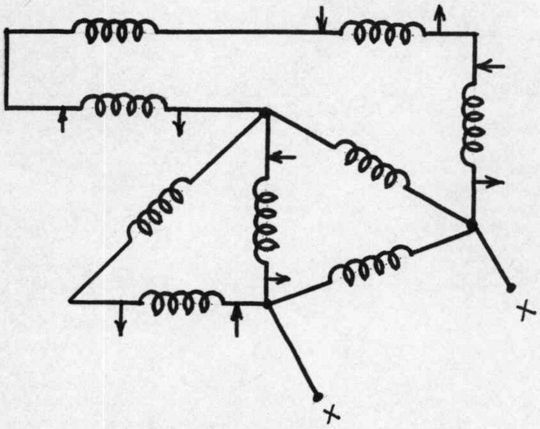

Figure 23.

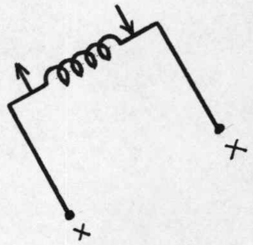

Figure 24.

mittance is the admittance of the network as measured from the terminals (with all internal current generators open-circuited), and the current is the current established across the terminals when short-circuited. As an example, the circuit of Fig. 23 may be replaced by the simple one of Fig. 24.

34. Reason for the Junction Pair Concept.

In Chapter 4, Article 17, the transformation matrix, C, was written from the coefficients of Kirchhoff's current equations. This connection tensor provided the step by which the original coil currents could be written in terms of the desired mesh currents when the currents and impedances of the network were known. However, the question concerning the possibilities of having voltages and admittances given naturally arises. With the method used thus far, the student must find the Z' for the network and then take the inverse in order to get the currents; this involves two sets of calculations. To simplify the solution of such a problem, the junction-pair concept is introduced.

As will be noted in later studies, certain types of circuits lend themselves more readily to the junction-pair idea. This will be discussed more fully in Article 37.

To write the transformation matrix, C, in Chapter 4, Kirchhoff's current equations were used. Now it will be possible to use his voltage equations to write the transformation matrix, A, which is associated with the junction-pair concept. This tensor changes old voltages or coil voltages into new voltages or those across arbitrarily chosen junction pairs. The dual of the word mesh is junction pair just as the dual of current is voltage. Also, just as currents were shown previously to have a direction, voltages may also be represented by arrows, indicating whether the voltage is being consumed or supplied by the network element.

A junction pair, as its name suggests, exists in a circuit wherever there are two junctions or connecting points describing a finite impedance lying between them in the circuit. For example, in Fig. 25, AB and AC are junction pairs in the ordinary π-filter circuit. Another illustration is in Fig. 26, the T-transmission line, where AB, AC and AD are the junction pairs of the network. It may be of value

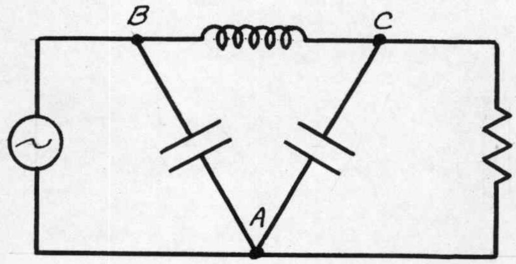

Figure 25.

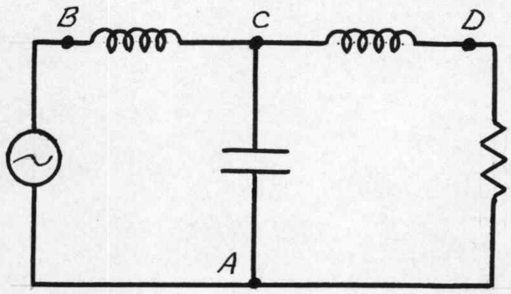

Figure 26.

for the student to note that the network of Fig. 25 has three meshes whereas that of Fig. 26 has but two. Conversely, the π-network has two junction pairs while the T-network has three. This may give the student a clue in regard to the selection of the type of approach to a given network solution. Obviously, the circuit of Fig. 25 would be more adaptable to the junction-pair approach while that of Fig. 26 lends itself to the mesh approach; in both cases the number of unknowns with which the student must deal has been reduced to a minimum.

To summarize: the mesh approach to a problem in tensor analysis embodies that which has been studied thus far--the assigning of convenient currents to a given circuit and solving for a mesh impedance tensor. Mesh voltages may then be solved for but solution of mesh currents requires an extra step. The junction-pair approach involves the assigning of convenient voltages across the available junction pairs of the given network and then solving for the currents flowing in and out of the bordering junctions. Here an extra step would be necessary in order to solve for the unknown voltages. The junction pair solution will now be taken up.

35. The Primitive Y.

Just as the primitive Z was illustrated in Chapter 2, Article 4, it is possible to illustrate the primitive admittance tensor, Y. Consider the ordinary π filter shown in Fig. 27. Here the power source is represented by a current generator and may be the output from some vacuum-tube circuit.

The electromagnetic generator may be thought of

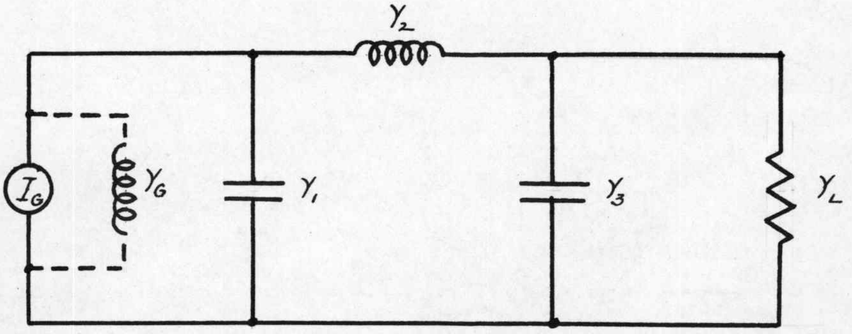

Figure 27.

as a voltage generator whose regulation is represented by an impedance in series. A vacuum tube, on the other hand is in reality a current generator with an admittance in parallel limiting its output. For perfect regulation in the former case, Z must be zero; for such a condition in the latter case, the impedance must be infinite, making the admittance equal to zero.

Referring again to the network of Fig. 27, the student will note that each coil has been defined by the admittance symbol, Y, where

$$Y = Z^{-1} \tag{169}$$

in each case.

The primitive network, drawn in Fig. 28, shows the arrows in opposing directions to indicate that the generator produces voltage while the admittance receives voltage. In the G circuit of Fig. 28, current can flow out at (a) if that same current is returned at (a').

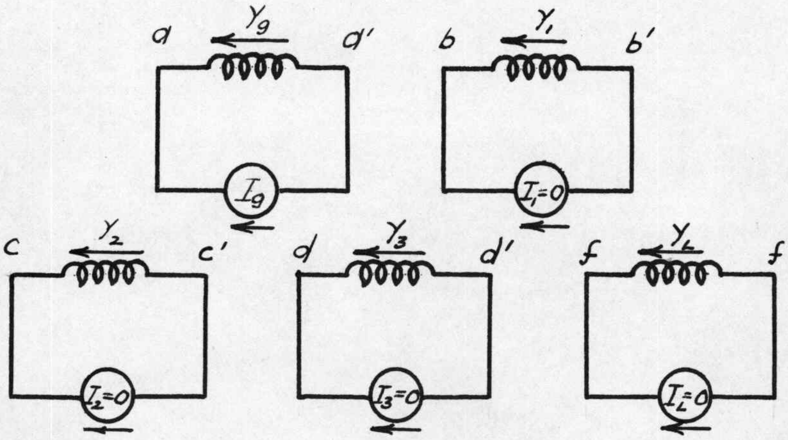

Figure 28.

The same is true for each of the other elemental networks; for example, current can be tapped at (b) to return to (b'), (c) to (c'), etc.

The primitive network of Figure 28 is mathematically represented thus:

$$i = \begin{array}{c} \begin{array}{ccccc} g & 1 & 2 & 3 & L \end{array} \\ \left[\begin{array}{c|c|c|c|c} I_g & 0 & 0 & 0 & 0 \end{array}\right] \end{array}, \tag{170}$$

$$Y = \begin{array}{c|c} & \begin{array}{ccccc} g & 1 & 2 & 3 & L \end{array} \\ \begin{array}{c} g \\ 1 \\ 2 \\ 3 \\ L \end{array} & \left[\begin{array}{c|c|c|c|c} Y_g & & & & \\ \hline & Y_1 & & & \\ \hline & & Y_2 & & \\ \hline & & & Y_3 & \\ \hline & & & & Y_L \end{array}\right] \end{array}, \tag{171}$$

$$e = \begin{array}{ccccc} g & 1 & 2 & 3 & L \\ \hline \boxed{E_g} & \boxed{E_1} & \boxed{E_2} & \boxed{E_3} & \boxed{E_L} \end{array} . \qquad (172)$$

The matrix equation involved is

$$i = Y \cdot e. \qquad (173)$$

To get the admittance matrix for the connected network of Figure 27, an equation paralleling that of the transformation tensor, C, in Chapter 4 is used:

$$Y' = A_t \cdot Y \cdot A. \qquad (174)$$

Here A is the connection matrix associated with the junction-pair concept.

36. The Connection Matrix, A.

After the primitive network has been established (see Fig. 28 and equations 170, 171 and 172), it is

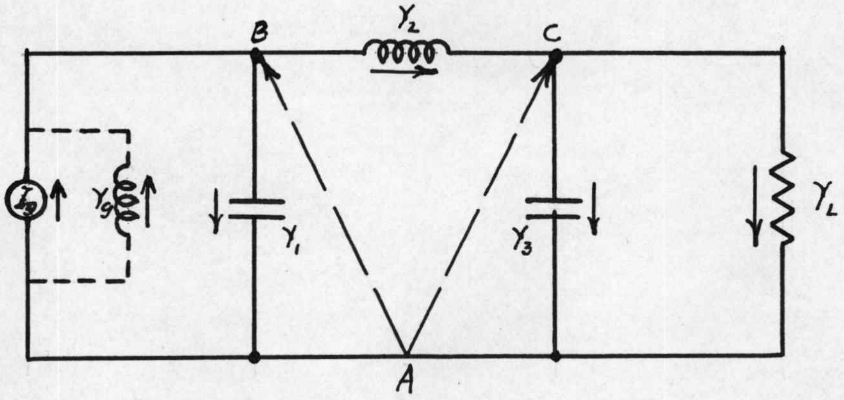

Figure 29.

necessary to assume junction-pair voltages in location and sense or direction. Just as mesh currents had to be assumed in dealing with the circuits previously studied, some initial assumptions must now be made before the mechanical process of connecting the circuit can be accomplished. In Fig. 29, the filter circuit of Fig. 28 is redrawn with arrows on each coil and current generator showing the assumed directions of the voltages in relation to each other in the circuit. Also the junction pairs AB and AC are assumed in the directions shown by the diagonal arrows. Let E_a represent AB and E_b, AC.

Writing Kirchoff's voltage equations around all possible closed paths (including the two fictious paths introduced to connect the circuit), the following equations result:

$$E_g + E_a = 0, \tag{175}$$

$$E_1 + E_a = 0, \tag{176}$$

$$E_2 + E_a - E_b = 0, \tag{177}$$

$$E_3 + E_b = 0, \tag{178}$$

$$E_L + E_b = 0. \tag{179}$$

In order to prevent difficulties arising in the use of arrows and signs, two rules are suggested: (1) voltages are added against the direction of the arrow when that voltage is taken across a current generator but added with the arrow when the voltage is taken across a power-receiving element and represents a voltage drop; (2) the sum of the voltages around any closed loop is equal to zero. As an example of the first rule, in going across I_g, the positive direction is downward against the arrow in Fig. 29 while the positive direction across Y_1 is downward with the arrow. Equations (175) to (179) illustrate the second rule.

To write the connection matrix, A, it is necessary to equate old voltages in terms of the new. The equation,

$$e = A \cdot e' , \qquad (180)$$

allows the following A matrix to be constructed from the resulting coefficients:

$$A = \begin{array}{c} \\ g \\ 1 \\ 2 \\ 3 \\ L \end{array} \begin{array}{cc} a & b \\ \hline -1 & \\ -1 & \\ -1 & 1 \\ & -1 \\ & -1 \end{array} . \qquad (181)$$

Applying equation (174) to (181) and (171):

$$Y' = \begin{array}{c|ccccc|} & g & 1 & 2 & 3 & L \\ \hline a & -1 & -1 & -1 & & \\ b & & & 1 & -1 & -1 \\ \hline \end{array} \cdot \begin{array}{c|ccccc|} & g & 1 & 2 & 3 & L \\ \hline g & Y_g & & & & \\ 1 & & Y_1 & & & \\ 2 & & & Y_2 & & \\ 3 & & & & Y_3 & \\ L & & & & & Y_L \\ \hline \end{array} \cdot A$$

$$= \begin{array}{c|ccccc|} & g & 1 & 2 & 3 & L \\ \hline a & -Y_g & -Y_1 & -Y_2 & 0 & 0 \\ b & 0 & 0 & Y_2 & -Y_3 & -Y_L \\ \hline \end{array} \cdot \begin{array}{c|cc|} & a & b \\ \hline g & -1 & \\ 1 & -1 & \\ 2 & -1 & 1 \\ 3 & & -1 \\ L & & -1 \\ \hline \end{array}$$

$$= \begin{array}{c|cc|} & a & b \\ \hline a & Y_g + Y_1 + Y_2 & -Y_2 \\ b & -Y_2 & Y_2 + Y_3 + Y_L \\ \hline \end{array} \text{ mhos.} \qquad (182)$$

Now to obtain the currents flowing into each junction pair, equation (173) is applied thus:

$$\begin{array}{|cc|} a & b \\ \hline I_a & I_b \\ \hline \end{array} = \begin{array}{c|cc|} & a & b \\ \hline a & Y_g + Y_1 + Y_2 & -Y_2 \\ b & -Y_2 & Y_2 + Y_3 + Y_L \\ \hline \end{array} \cdot \begin{array}{c|c|} \hline a & E_a \\ b & E_b \\ \hline \end{array} \qquad (183)$$

Equation (183) is algebraically expressed as:

$$I_a = E_a(Y_g + Y_1 + Y_2) - E_b Y_2, \qquad (184)$$

$$I_b = -E_a Y_2 + E_b(Y_2 + Y_3 + Y_L). \qquad (185)$$

If the voltages E_a and E_b are known, it is easy to solve for the currents, I_a and I_b. Since

$$e = A \cdot e' \tag{186}$$

from (180), the coil voltages can be calculated. Then

$$i = Y \cdot e, \tag{187}$$

so the currents flowing in each coil can be obtained. Also, the voltage drop across each element can be gotten from

$$E = Y^{-1} \cdot i. \tag{188}$$

By taking the inverse of the Y' received in (182), the Z' for the network could be obtained. This should not be confused with the Z' of Chapter 4 which was representative of mesh axes.

37. Selection of Attack.

There are two general rules which may help the student select the method of attack for a given network problem: (1) study the circuit to see whether it has fewer meshes or junction pairs and select the attack that offers the minimum number of necessary assumptions; (2) if the currents are known, select the mesh approach and if the voltages are known, select the junction-pair approach. If some voltages and other currents are given, it is desirable to approach the problem by a third method not within the scope of this text called the orthogonal network.

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