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The divergent k-plane transform of a function f on an n-dimensional real vector space V is the function $Df(a,\alpha) = D_a f(\alpha)$ which assigns to each point $a \in V$ and each $\alpha \in G_k(V)$ the integral of f over the translate of $\pi(\alpha)$ passing through a. Here $\pi(\alpha)$ is the non-oriented k-dimensional subspace of V associated with α and $G_k(V)$ the Grassmann manifold of unit k-vectors on V. It is generally assumed that $f \in L_0^2(\Omega)$, where Ω is a bounded open subset of V, and that a is outside the closure of Ω.

It is shown that under these conditions $D_a f \in L^2(G_k(V))$, and the adjoint is calculated. If $D_a f$ is known for infinitely many sources a, this determines f uniquely, while for finitely many sources f is essentially arbitrary. Exact and approximate inversion formulas are derived.

Some formulas for integration on the Grassmannian may have independent interest.

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THE DIVERGENT K-PLANE TRANSFORM

I. INTRODUCTION

The problem of recovering a function on R^n, $n \geq 2$, from its integrals over k-dimensional planes has been treated by many authors, beginning with Radon in 1917 [9]. Intuitively, the parallel k-plane transform of a measurable function on R^n is the function Pf which assigns to each k-dimensional subspace π of R^n and each point x'' in the subspace $\pi^\perp$ perpendicular to π the integral of f over the translate of π through x''. If $k = n - 1$, Pf is commonly called the Radon transform of f.

In this thesis, the following definition is used. Let V be an n-dimensional real vector space with an inner product, let $G_k(V)$ be the Grassmann manifold of unit k-vectors on V, and for each $\alpha \in G_k(V)$ let $\pi(\alpha)$ be the non-oriented k-dimensional subspace of V associated with α. Then

$$Pf(\alpha,x'') = P_\alpha f(x'') = \int_{\pi(\alpha)} f(x' + x'')\, dx',$$

where dx' is the k-dimensional Hausdorff measure on $\pi(\alpha)$, $\alpha \in G_k(V)$ and $x'' \in \pi(\alpha)^\perp$.

Interest in this transform increased dramatically in the 1970's due to the introduction of Computed Tomography (CT) and, more recently, of Nuclear Magnetic Resonance Tomography (NMR). The object of CT and NMR is to reconstruct a two- or three-dimensional density function from the values of its integrals over lines or planes (see Herman [6] for a summary).

In CT, it soon turned out to be advantageous to measure the integrals not over parallel lines, but rather over lines passing through a common source point a. This lead to the investigation of another operator D_a. If f is a function on R^n, then intuitively $D_a f$ is the function on the unit sphere S^{n-1} whose value at a point $\theta \in S^{n-1}$ is the integral of f over the line passing through a with direction θ.

In a more general setting, the <u>divergent</u> <u>k-plane</u> <u>transform</u> of a function f is defined by

$$Df(a,\alpha) = D_a f(\alpha) = \int_{\pi(\alpha)} f(a+x)\,dx,$$

where $a \in V$ and $\alpha \in G_k(V)$.

Another transform related to D is the <u>spherical</u> <u>k-plane</u> <u>transform</u> S defined by

$$Sf(\alpha) = \int_{S^{n-1} \cap \pi(\alpha)} f(\theta)\,d\theta,$$

where f is a function on S^{n-1} and $\alpha \in G_k(V)$.

Some useful treatments of the parallel k-plane transform can be found in Helgason [5] and Solmon [14]. The spherical k-plane transform appears, e.g., in Helgason [5]. The divergent line transform (k = 1) is described, e.g., in Hamaker/Smith/Solmon/Wagner [4].

The main purpose of this thesis is the investigation of the divergent k-plane transform for arbitrary k. Because of the many connections between D, P and S, a number of results on the parallel and spherical k-plane transforms are included. Sometimes these are needed in later proofs, in other cases they are mentioned only for comparison with similar results for D.

Chapter II. contains the background necessary for the later chapters. Most of it is fairly standard, but the geometrical description of the metric on $G_k(V)$ and some of the integration formulas on the Grassmannian are not readily available in the literature.

Chapter III. describes some possible domains and associated range spaces. The emphasis lies with square-integrable functions rather than with possible generalizations to differentiable functions or distributions. For square-integrable functions with compact support, the adjoint is calculated.

For the remaining chapters it is assumed that $f \in L_0^2(\Omega)$, where Ω is a bounded open subset of V with closed convex hull Ω.

Chapter IV contains uniqueness and non-uniqueness theorems. It is proved that f is uniquely determined, if $D_a f$ is known for any infinite set of sources bounded away from $\hat{\Omega}$. If $D_a f$ is known for finitely many sources only and f is infinitely differentiable, then essentially nothing can be said about the behaviour of f inside Ω.

Finally, chapter V. contains exact and approximate inversion formulas to recover f from Pf or Df.

II. BASIC FORMULAS

1. k-Vectors

This section contains a brief review of certain aspects of k-vectors needed in the following. Details and proofs can be found in many books, e.g. Greub [3].

Let V be a real vector space of dimension n with an inner product. If $v_1,\ldots,v_k$ is a collection of elements of V, $[v_1,\ldots,v_k]$ is the subspace spanned by these vectors.

The (contravariant) skew-symmetric tensors of rank k over V form a real vector space $\Lambda^k(V)$, where $\Lambda^1(V)$ is identified with V, and $\Lambda^0(V) = R$ by definition. The inner product on V induces an inner product (and thus also a norm) on $\Lambda^k(V)$.

If $\alpha \in \Lambda^k(V)$, $\beta \in \Lambda^m(V)$, their exterior product $\alpha\wedge\beta$ is in $\Lambda^{k+m}(V)$. If $\alpha \in R = \Lambda^0(V)$, $\alpha\wedge\beta = \beta\wedge\alpha = \alpha\beta$.

Non-zero products $v_1\wedge\ldots\wedge v_k$, $v_j \in V$, are called <u>k-vectors</u>. If $\alpha = v_1\wedge\ldots\wedge v_k$ is a k-vector, then $||\alpha||$ is the volume of the parallelepiped spanned by $v_1,\ldots,v_k$.

<u>Lemma (1.1)</u> If $\alpha = v_1\wedge\ldots\wedge v_k$ and $\beta = w_1\wedge\ldots\wedge w_m$, then $||\alpha\wedge\beta|| \leq ||\alpha||\ ||\beta||$. If the subspaces $[v_1,\ldots,v_k]$ and $[w_1,\ldots,w_m]$ are orthogonal, then $||\alpha\wedge\beta|| = ||\alpha||\ ||\beta||$.

If π is a k-dimensional subspace of V, an <u>orientation of π</u> is an equivalence class of bases of π. Two bases $\{v_1,\ldots,v_k\}$ and $\{w_1,\ldots,w_k\}$ determine the same orientation if the map $T:\pi \to \pi$ which maps v_j into w_j has a positive determinant. An <u>oriented subspace</u> of V is a subspace together with an orientation.

If $\alpha = v_1 \wedge \ldots \wedge v_k$ and $\beta = w_1 \wedge \ldots \wedge w_k$ are k-vectors, then α is a positive multiple of β if and only if $[v_1,\ldots,v_k] = [w_1,\ldots,w_k]$ and the two bases determine the same orientation. Thus, there is a one-to-one correspondence between k-vectors of unit length and oriented k-dimensional subspaces of V.

For each k-vector α, let $\pi(\alpha)$ be the non-oriented subspace associated with α. Then $\pi(\alpha) = \pi(-\alpha)$.

If ρ is a unit n-vector (there are two of them), there is a unique (n-k)-vector $\alpha^{\perp}$ of unit length with

$$\alpha \wedge \alpha^{\perp} = \rho.$$

The relationship between the associated subspaces is

$$\pi(\alpha^{\perp}) = \pi(\alpha)^{\perp}.$$

2. The Grassmann Manifold

The unit k-vectors in $\Lambda^k(V)$ form a set $G_k(V)$ called the <u>Grassmann manifold</u> of oriented k-dimensional subspaces of V. In particular, $G_0(V) = \{1, -1\}$ and $G_1(V) = S^{n-1}$ (the unit sphere in V).

The norm on $\Lambda^k(V)$ makes $G_k(V)$ into a metric space. It is shown in Whitney [17] that $G_k(V)$ with the topology defined by this metric admits the structure of an k(n-k)-dimensional analytic manifold.

Let α_0 be a fixed unit m-vector, $m \leq k$. The Grassmann manifold $G_{k-m}(\pi(\alpha_0)^\perp)$ is an (k-m)(n-k)-dimensional submanifold of $G_{k-m}(V)$. By (1.1), the map $\alpha' \to \alpha' \wedge \alpha_0$, $\alpha' \in G_{k-m}(\pi(\alpha_0)^\perp)$, is an isometry, so its range $\Gamma_k(\alpha_0) = \{\alpha \in G_k(V)\colon\ \alpha = \alpha' \wedge \alpha_0,\ \alpha' \in G_{k-m}(\pi(\alpha_0)^\perp)\}$ is an (k-m)(n-k)-dimensional submanifold of $G_k(V)$. $\alpha \in \Gamma_k(\alpha_0)$ if and only if $\pi(\alpha)$ contains $\pi(\alpha_0)$ as a subspace.

The remainder of this section is devoted to a geometric interpretation of the metric on $G_k(V)$. In the following theorems, E will always be the orthogonal projection of V onto $\pi(\alpha_2)$, restricted to $\pi(\alpha_1)$.

<u>Lemma (2.1)</u> Let $\alpha_1, \alpha_2 \in G_k(V)$. Then

$$|\langle \alpha_1, \alpha_2 \rangle| = J(E),$$

where J(E) is the Jacobian of E.

<u>Proof:</u> See Whitney [17], section I.15.∎

Remark: If T is a linear transformation from one k-dimensional vector space π into another and $\{v_j\}$ is a basis of π, the Jacobian J(T) is defined as the ratio of the volumes of the parallelepipeds spanned by $\{Tv_j\}$ and $\{v_j\}$.

Lemma (2.2) Let α_1, $\alpha_2 \in G_k(V)$ with $||\alpha_1-\alpha_2|| < 2^{1/2}$, and let $J = J(E)$. Then

$$||\alpha_1-\alpha_2|| = [2(1-J)]^{1/2}.$$

Proof: $$||\alpha_1-\alpha_2||^2 = \langle\alpha_1-\alpha_2,\alpha_1-\alpha_2\rangle$$

$$= \langle\alpha_1,\alpha_1\rangle - 2\langle\alpha_1,\alpha_2\rangle + \langle\alpha_2,\alpha_2\rangle = 2(1-\langle\alpha_1,\alpha_2\rangle).$$

From the condition $||\alpha_1-\alpha_2|| < 2^{1/2}$ it follows that $\langle\alpha_1,\alpha_2\rangle > 0$, so $\langle\alpha_1,\alpha_2\rangle = J$ by (2.1).∎

Corollary (2.3) E is non-singular if $||\alpha_1-\alpha_2|| < 2^{1/2}$.

Theorem (2.4) Let α_1, $\alpha_2 \in G_k(V)$ with $||\alpha_1-\alpha_2|| < 2^{1/2}$. Then

$$||Ex-x|| \leq ||x||\ ||\alpha_1-\alpha_2||, \quad x \in \pi(\alpha_1).$$

Proof: For any $x \in \pi(\alpha_1)$, choose $x_2,\ldots,x_k \in \pi(\alpha_1)$, $||x_j|| = 1$, such that $x, x_2,\ldots, x_k$ are mutually orthogonal. Then if $J = J(E)$,

$$|J| = \frac{||Ex\wedge Ex_2\wedge\ldots\wedge Ex_k||}{||x\wedge x_2\wedge\ldots\wedge x_k||} = \frac{1}{||x||}\,||Ex\wedge\ldots\wedge Ex_k||$$

$$\leq \frac{1}{||x||}\,||Ex||\ldots||Ex_k|| \text{ by (1.1).}$$

Since $||Ex_j|| \leq 1$,

$$||Ex|| \geq |J|\ ||x||.$$

Ex and x-Ex are orthogonal, thus

$$||x||^2 = ||Ex||^2 + ||x-Ex||^2,$$

and

$$||x-Ex||^2 = ||x||^2 - ||Ex||^2 \leq ||x||^2(1-|J|^2)$$

$$\leq ||x||^2\ 2(1-J) = ||x||^2\ ||\alpha_1-\alpha_2||^2. \blacksquare$$

3. Measure and Integration

The metric defined in the preceding section induces a finite Hausdorff area measure $d\alpha$ on $G_k(V)$. This measure is invariant under orthogonal transformations of V (since the metric is) and is usually normalized so that the measure of the entire Grassmannian becomes

$$|G_k(V)| = \frac{|S^{n-1}||S^{n-2}| \dots |S^{n-k}|}{|S^{k-1}||S^{k-2}| \dots |S^1|}, \qquad k \geq 2,$$

$$|G_0(V)| = 2,$$

$$|G_1(V)| = |S^{n-1}|,$$

where S^k denotes the k-dimensional unit sphere in V, and $|S^k|$ its k-dimensional measure. As an isometry, the map $\alpha' \to \alpha' \wedge \alpha_0$ is measure-preserving between $G_{k-m}(\pi(\alpha_0)^{\perp})$ and $\Gamma_k(\alpha_0)$.

Lemma (3.1) If h is continuous on $G_k(V)$, then

$$H(\theta) = \int_{\Gamma_k(\theta)} h(\alpha)\, d\alpha$$

is continuous on S^{n-1}.

Proof: Pick any θ_1, $\theta_2 \in S^{n-1}$. Let U be the rotation on V that carries θ_1 into θ_2 and acts as the identity on $[\theta_1,\theta_2]^{\perp}$. U extends to an isometry from $\Gamma_k(\theta_1)$ onto $\Gamma_k(\theta_2)$.

If $\alpha \in \Gamma_k(\theta_1)$, then

$$\begin{aligned} \alpha &= \alpha' \wedge \theta_1, \\ U\alpha &= \alpha' \wedge \theta_2, \end{aligned} \qquad \alpha' \in G_{k-1}(\pi(\theta_1 \wedge \theta_2)^{\perp})$$

so

$$\|\alpha - U\alpha\| = \|\alpha' \wedge \theta_1 - \alpha' \wedge \theta_2\|$$

$$= \|\alpha' \wedge (\theta_1 - \theta_2)\| = \|\theta_1 - \theta_2\| \quad \text{by (1.1).}$$

Now

$$|H(\theta_1)-H(\theta_2)| = \left| \int_{\Gamma_k(\theta_1)} h(\alpha)\, d\alpha - \int_{\Gamma_k(\theta_2)} h(\alpha)\, d\alpha \right|$$

$$\leq \int_{\Gamma_k(\theta_1)} |h(\alpha) - h(U\alpha)| \, d\alpha.$$

By the uniform continuity of h it is possible to find a δ for any $\varepsilon > 0$ so that $|h(\alpha) - h(U\alpha)| < \varepsilon$ whenever $||\alpha - U\alpha|| = ||\theta_1 - \theta_2|| < \delta$.∎

<u>Theorem (3.2)</u> If h is non-negative, measurable and defined almost everywhere on $G_k(V)$, then

(a) For almost every $\theta \in S^{n-1}$ the restriction of h to $\Gamma_k(\theta)$ is measurable and defined almost everywhere on $\Gamma_k(\theta)$,

(b) The function $H(\theta) = \int_{\Gamma_k(\theta)} h(\alpha) \, d\alpha$ is measurable on S^{n-1}, and

(c)
$$\int_{G_k(V)} h(\alpha) \, d\alpha = \frac{1}{|S^{k-1}|} \int_{S^{n-1}} \int_{\Gamma_k(\theta)} h(\alpha) \, d\alpha \, d\theta \qquad (3.3)$$

<u>Remark:</u> In subsequent integration formulas the analogs of (a) and (b) in theorem (3.2) are considered as implicit parts of the formulas and are not stated explicitly. The proof of the present theorem, typical of the others, is given in full detail. Subsequent proofs that are basically repetitions are omitted.

<u>Proof:</u> By lemma (3.1), the right-hand side of (3.3) defines a continuous linear form on the space of all continuous functions on

$G_k(V)$, hence a finite regular Borel measure on $G_k(V)$. This measure is obviously rotation invariant, so it is $d\alpha$ up to a constant factor. The constant is determined by setting $h = 1$. This gives (3.3) when h is continuous.

If h is the characteristic function of an open set in $G_k(V)$, h is the limit everywhere of an increasing sequence of continuous functions $\{h_j\}$. Therefore the restriction of h to $\Gamma_k(\theta)$ is measurable for every θ. The functions H_j defined by

$$H_j(\theta) = \int_{\Gamma_k(\theta)} h_j(\alpha)\, d\alpha$$

form an increasing sequence of continuous functions on S^{n-1}, bounded above by the constant $|\Gamma_k(\theta)|$, and converging pointwise to the function H, which is therefore measurable. By the dominated convergence theorem, H is integrable over S^{n-1}, and

$$\int_{S^{n-1}} H(\theta)\, d\theta = \lim \int_{S^{n-1}} H_j(\theta)\, d\theta$$

$$= \lim \int_{S^{n-1}} \int_{\Gamma_k(\theta)} h_j(\alpha)\, d\alpha\, d\theta = |S^{k-1}| \lim \int_{G_k(V)} h_j(\alpha)\, d\alpha$$

$$= |S^{k-1}| \int_{G_k(V)} h(\alpha)\, d\alpha.$$

This proves the theorem when h is the characteristic function of an open set.

Let h be the characteristic function of a G_δ. Then h is the limit everywhere of a decreasing sequence $\{h_j\}$ of characteristic functions of open sets. The above argument proves the theorem when h is the characteristic function of a G_δ.

Let h be the characteristic function of a set N of measure zero. N is contained in a G_δ set A of measure zero with characteristic function h_A. From above, H_A is integrable over S^{n-1} and has integral zero, so H_A is zero almost everywhere. This shows that the integral of h_A over $\Gamma_k(\theta)$ is zero almost everywhere, hence that the restriction of h_A to $\Gamma_k(\theta)$ is zero almost everywhere on $\Gamma_k(\theta)$, for almost every θ. Since N is a subset of A, this is also true for the function h. Thus, for almost every θ, $h = 0$ almost everywhere on $\Gamma_k(\theta)$.

By the regularity of $d\alpha$, any measurable set E is a difference $E = A - N$, where A is a G_δ and $N \subset A$ has measure zero. Hence $h_E = h_A - h_N$, so the theorem is proved for characteristic functions of measurable sets, hence for simple functions.

An arbitrary non-negative measurable function is the pointwise limit almost everywhere of an increasing sequence $\{h_j\}$ of simple functions. The same arguments used above for the characteristic function of an open set now yield the statement of the theorem.∎

Corollary (3.4) If h is integrable on $G_k(V)$, then h is integrable over $\Gamma_k(\theta)$ for almost every $\theta \in S^{n-1}$.

Lemma (3.5) If f is continuous with compact support on V, then

$$F(\alpha) = \int_{\pi(\alpha)} f(x)\, dx$$

is continuous on $G_k(V)$.

Proof: Pick α_1, $\alpha_2 \in G_k(V)$ with $||\alpha_1-\alpha_2|| < 2^{1/2}$, and let E be the orthogonal projection of V onto $\pi(\alpha_2)$, restricted to $\pi(\alpha_1)$. E is non-singular by (2.3), so

$$|F(\alpha_1) - F(\alpha_2)| = \left| \int_{\pi(\alpha_1)} f(a+x)\, dx - \int_{\pi(\alpha_2)} f(a+y)\, dy \right|$$

$$= \left| \int_{\pi(\alpha_1)} f(a+x)\, dx - \int_{\pi(\alpha_1)} f(a+Ex)\, J(E)\, dx \right|$$

$$\leq \int_{\pi(\alpha_1)} |f(a+x) - f(a+Ex)|\, dx + \int_{\pi(\alpha_1)} |f(a+Ex)|\, (1-J(E))\, dx$$

By (2.4) and the uniform continuity of f the first integral can be made arbitrarily small by taking α_1, α_2 sufficiently close together. (2.2) yields the same for the second integral.∎

<u>Corollary (3.6)</u> If f is continuous on S^{n-1}, then

$$F(\alpha) = \int_{S^{n-1} \cap \pi(\alpha)} f(\theta)\, d\theta$$

is continuous on $G_k(V)$.

<u>Proof:</u> Define $g(x) = \rho(|x|)\, |x|^{1-k} f(x/|x|)$, where $\rho(|x|) = 0$ if $|x|$ is outside the interval $[1,2]$, and $\int \rho(|x|)dx = 1$. g is clearly continuous, and

$$F(\alpha) = \int_{S^{n-1} \cap \pi(\alpha)} f(\theta)\, d\theta$$

$$= \int_1^2 \int_{S^{n-1} \cap \pi(\alpha)} \rho(t)\, f(\theta)\, d\theta\, dt$$

$$= \int_{\pi(\alpha)} \rho(|x|)\, |x|^{1-k} f(x/|x|)\, dx = G(\alpha),$$

which is continuous by (3.5).∎

<u>Lemma (3.7)</u> If g is non-negative, measurable and defined almost everywhere on S^{n-1}, then

$$\int_{S^{n-1}} g(\theta)\, d\theta = \frac{|S^{n-1}|}{|S^{k-1}||G_k(V)|} \int_{G_k(V)} \int_{S^{n-1} \cap \pi(\alpha)} g(\theta)\, d\theta\, d\alpha$$

$$= \frac{|S^{n-1}|}{|S^{n-k-1}||G_k(V)|} \int_{G_k(V)} \int_{S^{n-1} \cap \pi(\alpha)^{\perp}} g(\theta)\, d\theta\, d\alpha$$

<u>Proof:</u> By the preceding corollary, the first double integral on the right defines a continuous linear form on the space $C(S^{n-1})$, hence a finite regular Borel measure on S^{n-1}. This measure is obviously rotation invariant, and there is only one up to a constant factor. The constant is determined by setting $g = 1$. The theorem is then extended to non-negative measurable functions as in (3.2).

The proof of the second equality is identical.∎

<u>Lemma (3.8)</u> If g is non-negative, measurable and defined almost everywhere on V, then,

$$\int_V g(x)\, dx = \frac{|S^{n-1}|}{|S^{k-1}||G_k(V)|} \int_{G_k(V)} \int_{\pi(\alpha)} |x'|^{n-k} g(x')\, dx'\, d\alpha$$

$$= \frac{|S^{n-1}|}{|S^{n-k-1}||G_k(V)|} \int_{G_k(V)} \int_{\pi(\alpha)^{\perp}} |x'|^{k} g(x')\, dx'\, d\alpha$$

<u>Proof:</u> Using polar coordinates in V and in $\pi(\alpha)$ and theorem (3.7),

$$\int_V g(x)\, dx = \int_{S^{n-1}} \int_0^{\infty} t^{n-1} g(t\theta)\, dt\, d\theta$$

$$= \frac{|S^{n-1}|}{|S^{k-1}|\,|G_k(V)|} \int_{G_k(V)} \int_{S^{n-1}\cap\pi(\alpha)} \int_0^\infty t^{n-1}\, g(t\theta)\, dt\, d\theta\, d\alpha$$

$$= \frac{|S^{n-1}|}{|S^{k-1}|\,|G_k(V)|} \int_{G_k(V)} \int_{S^{n-1}\cap\pi(\alpha)} \int_0^\infty t^{n-k}\, t^{k-1}\, g(t\theta)\, dt\, d\theta\, d\alpha$$

$$= \frac{|S^{n-1}|}{|S^{k-1}|\,|G_k(V)|} \int_{G_k(V)} \int_{\pi(\alpha)} |x'|^{n-k}\, g(x')\, dx'\, d\alpha.$$

The proof of the second part is identical.∎

4. Riesz Potentials

The <u>Riesz</u> <u>kernel</u> R_k is defined by

$$R_k(x) = c(n,k)\ |x|^{k-n}, \qquad 0 < k < n,$$

where

$$c(n,k) = \frac{\Gamma((n-k)/2)}{2^k \pi^{n/2} \Gamma(k/2)}.$$

This function comes up in various places in computed tomography. In this thesis, it appears, e.g., in lemma (III.2.1), which states that

$$\int_{G_k(V)} D_a f(\alpha)\, d\alpha = c\, R_k * f(a).$$

The following two lemmas are adapted from [11].

Lemma (4.1) If ρ is integrable and $(1+|x|)^n \rho$ is bounded, then

$$|R_k * \rho(x)| \leq c\, (1+|x|)^{k-n} \left(||\rho||_{L^1} + ||(1+|x|)^n \rho||_{L^\infty} \right).$$

Proof: Assume $\rho \geq 0$, and let $M = ||(1+|x|)^n \rho||_{L^\infty}$.

It will be shown first that

$$R_k * \rho(x) \leq c\, (||\rho||_{L^1} + M),$$

then that

$$R_k * \rho(x) \leq c\, |x|^{k-n} (||\rho||_{L^1} + M).$$

Together the two prove the lemma.

For the first estimate,

$$\frac{1}{c(n,k)} R_k * \rho(x) = \int_V |x-y|^{k-n} \rho(y)\, dy$$

$$= \int_{|x-y| \leq 1} |x-y|^{k-n} \rho(y)\, dy + \int_{|x-y| \geq 1} |x-y|^{k-n} \rho(y)\, dy$$

$$\le M \int_{|z|\le 1} |z|^{k-n}\, dz \;+\; ||\rho||_{L^1} \;\le\; c\,(||\rho||_{L^1} + M).$$

Because of the first estimate, the second estimate only needs to be proved for $|x| \ge 1$.

$$\frac{1}{c(n,k)} R_k * \rho(x) \;=\; \int_{|y|\le|x|/2} |x-y|^{k-n}\,\rho(y)\,dy$$

$$+ \int_{|y|\ge 2|x|} |x-y|^{k-n}\,\rho(y)\,dy \;+\; \int_{|x|/2\le|y|\le 2|x|} |x-y|^{k-n}\,\rho(y)\,dy$$

$$\le (|x|/2)^{k-n}\,||\rho||_{L^1} \;+\; |x|^{k-n}\,||\rho||_{L^1}$$

$$+\; |x|^{k-n} \int_{1/2\le|z|\le 2} |\theta-z|^{k-n}\,\rho(|x|z)\,|x|^{n}\,dz,$$

where $\theta = x/|x|$, $z = y/|x|$.

Since $\rho(|x|z)|x|^n = \rho(|x|z)(|x|z)^n z^{-n} \le 2^n M$, the last integral is bounded by

$$2^n M|x|^{k-n} \int_{|w|\le 3} |w|^{k-n} dw. \blacksquare$$

In the following lemma, $\mathcal{P}^k$ is the space of Bessel potentials [1]. It is similar to the Sobolev space $\mathcal{H}^k$, but the functions are defined more precisely than almost everywhere. Functions in $\mathcal{P}^k$ are defined except on sets of 2k-capacity zero. (For any $\varepsilon > 0$, a set of 2k-capacity 0 has $(n-2k+\varepsilon)$-dimensional measure zero).

<u>Lemma (4.2)</u>

(a) If $(1+|x|)^{k-n}f \in L^1$, then R_k*f is defined almost everywhere and is locally integrable. Moreover, if $\rho \in L^1$ and $(1+|x|)^n\rho$ is bounded,

$$|\langle R_k*f,\rho\rangle| = \left| \int_V R_k*f(x)\ \bar{\rho}(x)\ dx \right|$$

$$\leq c||(1+|x|)^{k-n}f||_{L^1}(||\rho||_{L^1} + ||(1+|x|)^n\rho||_{L^\infty}).$$

(b) If in addition $f \in L^2_{loc}$, then R_k*f is defined except on a set of 2k-capacity zero and is in $\mathcal{P}^{2k}_{loc}$; if $k > n/2$, R_k*f is continuous, as points have positive capacity.

(c) If $(1+|x|)^{k-n}f \notin L^1$ and $f \geq 0$, then R_k*f is undefined everywhere.

<u>Proof:</u> (a) $$\left| \int_V R_k*f(x)\ \bar{\rho}(x)\ dx \right|$$

$$= \left| \int_V \int_V R_k(x-y)\ f(y)\ dy\ \bar{\rho}(x)\ dx \right|$$

$$\leq \int_V |f(y)| \; | \int_V R_k(x-y)\, \bar{\rho}(x)\, dx \;|\; dy$$

$$= \int_V |f(y)| \; |R_k * \rho(y)| \; dy$$

$$\leq c \;\; (||\rho||_{L^1} + ||(1+|x|)^n \rho||_{L^\infty}) \int_V |f(y)| \; (1+|x|)^{k-n} \; dx$$

The rest of (a) now follows by choosing ρ to be the characteristic function of a bounded measurable set.

(b) This is proved in [1].

(c) $|y-x| \leq |y| + |x| \leq 1 + |x|$, if $|y| \leq 1$,

$\leq |y|(1+|x|)$, if $|y| \geq 1$,

so $|y-x| \leq \max(1,|y|)\;(1+|x|)$, and

$$R_k * f(y) \;=\; \int_V |y-x|^{k-n}\, f(x)\, dx$$

$$\geq \min(1,|y|^{k-n}) \int_V (1+|x|)^{k-n}\, f(x)\, dx,$$

which is infinite for all y.∎

5. Spherical Harmonics

Most of this section consists of a brief review of some needed facts about spherical harmonics. A more detailed treatment can be found e.g. in Seeley [10].

Let P_m be the space of homogeneous polynomials of degree m on V, and let H_m consist of the restrictions to S^{n-1} of harmonic homogeneous polynomials of degree m. H_m is a real vector space of dimension $(2m+n-2)(m+n-3)!/(m!(n-2)!)$. The functions in H_m are called <u>spherical harmonics of degree m</u>.

In spherical coordinates the Laplace operator on V is given by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta',$$

where Δ' is the Laplace operator on S^{n-1}. H_m is the space of eigenfunctions of Δ' with eigenvalue $-m(m+n-2)$, and $L^2(S^{n-1})$ is the orthogonal direct sum of the H_m. Thus, if $\{h_{m,j}\}$, $1 \leq j \leq \dim(H_m)$, form a basis of H_m, then every $f \in L^2(S^{n-1})$ can be uniquely written as

$$f(\theta) = \sum_m \sum_j a_{m,j}\, h_{m,j}(\theta).$$

For fixed m and each $e \in S^{n-1}$, there is a unique function $Z_e^m \in H_m$ such that

$$\int_{S^{n-1}} h(\theta)\ Z_e^m(\theta)\ d\theta \quad = \quad h(e) \tag{5.1}$$

for any $h \in H_m$. $Z_e^m(\theta)$ depends only on the angle between e and θ and is called the <u>zonal harmonic of order m with pole e</u>. Z_e^m is even or odd about the hyperplane $e^{\perp}$, depending on whether m is even or odd.

<u>Lemma (5.2)</u> If m is even and $\alpha \in G_k(e^{\perp})$, then

$$\int_{S^{n-1} \cap \pi(\alpha)} Z_e^m(\theta)\ d\theta \quad \neq \quad 0.$$

<u>Proof:</u> For fixed $e \in S^{n-1}$, an arbitrary point $\theta \in S^{n-1}$ can be described by coordinates (r,σ), where r is the (geodesic) distance from θ to e and σ is the intersection of the great circle through θ and e with the (n-2)-sphere perpendicular to e. In these coordinates,

$$\Delta' \quad = \quad \frac{\partial^2}{\partial r^2} \quad + \quad (n-1)\ \cot r\ \frac{\partial}{\partial r} \quad + \quad \sin^{-2} r\ \Delta'',$$

where Δ'' is the Laplace operator on the (n-2)-sphere perpendicular to e.

Since Z_e^m depends only on r in these coordinates, Z_e^m satisfies the differential equation

$$h'' + (n-1)\ \cot r\ h' = -m(m+n-2)h.$$

Z_e^m is even about $r = \pi/2$, so $(Z_e^m)'(\pi/2) = 0$. Since Z_e^m is not the zero solution, $Z_e^m(\pi/2) \neq 0$. The lemma follows directly from this, since Z_e^m is constant on $S^{n-1} \quad \pi(\alpha)$.∎

<u>Theorem (5.3)</u> If H is a subspace of H_m invariant under orthogonal transformations, then either $H = \{0\}$ or $H = H_m$.

<u>Proof:</u> Choose a non-zero $h \in H$ and a point e with $h(e) \neq 0$. Define $\bar{h}$ as the function whose value at (r,σ) is the average of h over all points with distance r from e. As an average of orthogonal transformations of h, the new function remains in H, and it is not zero because the value at e is unchanged. Since $\bar{h}$ depends only on r, it must be a non-zero multiple of Z_e^m. By (5.1), $g(e) = 0$ for any $g \in H^{\perp}$ (orthogonal complement in H_m). Since $H^{\perp}$ is also invariant under orthogonal transformations, it follows that g must be identically zero, so $H^{\perp} = 0$.∎

The following theorem is standard for bounded operators.

<u>Theorem (5.4)</u> Let T be a closed linear operator with dense domain on the space $L^2(S^{n-1})$ of complex square-integrable functions on S^{n-1}. If T commutes with orthogonal transformations, then each H_m is in the domain of T, and on H_m, T is a multiple of the identity.

<u>Proof:</u> To say that T commutes with orthogonal transformations means that whenever U is a unitary operator arising from an orthogonal transformation of the sphere, so

$$UT \subset TU. \tag{5.5}$$

This implies that the intersection of the domain of T with H_m is invariant, hence is either 0 or H_m.

Suppose the latter, and let T_{ml} be the restriction of T to H_m, followed by the projection on H_l. The null space and range are invariant, so either T_{ml} is identically zero or it is one-to-one and onto, in which case m and l must be equal, as different H_m have different dimensions. It follows that $T(H_m) \subset H_m$, since the H_m span L^2. Because of the invariance of eigenspaces of T, T must be a multiple of the identity on H_m. This proves the theorem when the domain of T has a non-zero intersection with H_m.

By (5.5), TU has dense domain, from which it follows that $(TU)^* \supset U^*T^*$. Because U is bounded, $(UT)^* = T^*U^*$. Therefore, $T^*U^* = (UT)^* \supset (TU)^* \supset U^*T^*$. Replacement of U by U^* gives $UT^* \subset T^*U$, hence

$$UT^*T \subset T^*TU.$$

This implies that each projection E in the resolution of the identity for the self-adjoint operator T^*T commutes with U. By what has been proved for bounded operators, E must be a multiple of the identity on each H_m, therefore either 0 or 1 on each H_m, which implies that H_m is in the domain of T^*T, hence in the domain of T.∎

III. The Transforms P, D and S

1. Definitions

Let f be a measurable function on V. If α is a k-vector, the <u>parallel k-plane transform</u> of f is defined by

$$Pf(\alpha,x'') = P_\alpha f(x'') = \int_{\pi(\alpha)} f(x'+x'')\,dx', \quad x'' \in \pi(\alpha)^\perp,$$

whenever the Lebesgue integral exists. The k-vector α is called the <u>x-ray direction</u>.

For a fixed point $x \in V$, the <u>divergent k-plane transform</u> of f is defined by

$$Df(\alpha,x) = D_x f(\alpha) = \int_{\pi(\alpha)} f(x+x')\,dx'$$

whenever the Lebesgue integral exists. The point x is called the <u>x-ray source</u>.

The transforms P_α and D_x are related by

$$D_x f(\alpha) = P_\alpha f(E_{\alpha^\perp}x), \tag{1.1}$$

where $E_{\alpha^\perp}x$ is the orthogonal projection of x onto $\pi(\alpha^\perp) = \pi(\alpha)^\perp$.

If f is a measurable function on S^{n-1}, the <u>spherical k-plane transform</u> of f is defined by

$$Sf(\alpha) = \int_{S^{n-1} \cap \pi(\alpha)} f(\theta)\, d\theta$$

whenever the Lebesgue integral exists.

2. Domains

Let Ω be a bounded open set in V with closure $\bar{\Omega}$. $C_0(\Omega)$ and $L_0^2(\Omega)$ are, respectively, the spaces of continuous and square integrable functions defined on V and zero outside Ω. The fiber bundle $T = T(G_k(V))$ is defined by

$$T = \{(\alpha, x''): \alpha \in G_k(V),\ x'' \in \pi(\alpha)^{\perp}\}.$$

T has a natural measure μ so that

$$\int_T f\, d\mu = \int_{G_k(V)} \int_{\pi(\alpha)^{\perp}} f(\alpha, x'')\, dx''\, d\alpha.$$

In the following theorems and proofs, most constants will be denoted by c. The value of c need not be the same from one line to the next.

<u>Lemma (2.1)</u> If f is non-negative, measurable and defined almost everywhere on V, then $D_a f$ is measurable, and

$$\int_{G_k(V)} D_a f(\alpha)\, d\alpha = \frac{|S^{k-1}|\ |G_k(V)|}{c(n,k)\ |S^{n-1}|} R_k * f(a)$$

where $R_k = c(n,k)\ |x|^{k-n}$ is the Riesz kernel defined in section II.4.

<u>Proof:</u> By (II.3.5), $D_a f$ is continuous if f is continuous and has compact support. The proof that $D_a f$ is measurable proceeds like the proof of theorem (II.3.2).

Now

$$\int_{G_k(V)} D_a f(\alpha)\, d\alpha = \int_{G_k(V)} \int_{\pi(\alpha)} f(a+x')\, dx'\, d\alpha$$

$$= \int_{G_k(V)} \int_{\pi(\alpha)} |x'|^{n-k}\, |x'|^{k-n} f(a+x')\, dx'\, d\alpha$$

$$= \frac{|S^{k-1}|\ |G_k(V)|}{|S^{n-1}|} \int_V |x|^{k-n} f(a+x)\, dx$$

by corollary (II.3.8).■

<u>Theorem (2.2)</u>

(a) If $f \in L_0^p(\Omega)$, $1 \leq p \leq \infty$, then $D_a f \in L^p(G_k(V))$ for all sources a outside $\bar{\Omega}$, and for such a,

$$||D_a f||_{L^p(G_k(V))} \leq c\ ||f||_{L_0^p(\Omega)} .$$

(b) If $(1+|x|)^{k-n}f \in L^1$ and $f \in L^2_{loc}$, then for all sources a outside a set of 2k-capacity zero, $D_a f(\alpha)$ is defined for almost every α, and $D_a f$ is integrable on $G_k(V)$. If $k > n/2$, this is true for any a.

<u>Proof:</u> (a) For $p < \infty$, $$|D_a f(\alpha)|^p = \left| \int_{\pi(\alpha)} f(a+x)\, dx \right|^p$$

$$\leq \left(\int_{\pi(\alpha)} \chi_\Omega(a+x)\, dx \right)^{p/q} \int_{\pi(\alpha)} |f(a+x)|^p\, dx$$

$$\leq c \int_{\pi(\alpha)} |f(a+x)|^p\, dx = c\, D_a |f|^p(\alpha),$$

where χ_Ω is the characteristic function of Ω and $1/p + 1/q = 1$. Thus

$$||D_a f||^p_{L^p(G_k(V))} = \int_{G_k(V)} |D_a f(\alpha)|^p\, d\alpha$$

$$\leq c \int_{G_k(V)} D_a |f|^p(\alpha)\, d\alpha = c\, R_k * |f|^p(a).$$

$$= c \int_V |a-x|^{k-n}\, |f|^p(x)\, dx$$

$$\leq c\, [\mathrm{dist}(a,\Omega)]^{k-n}\, ||f||^p_{L^p_0(\Omega)},$$

where dist(a,Ω) is the distance from a to Ω.

The case $p = \infty$ is easy to check.

(b) follows from (2.1) and (II.4.2).∎

Theorem (2.3)

(a) If $f \in L_0^p(\Omega)$, $1 \leq p \leq \infty$, then $P_\alpha f \in L_0^p(\pi(\alpha)^\perp)$ for all $\alpha \in G_k(V)$, $Pf \in L_0^p(T)$, and

$$||P_\alpha f||_{L_0^p(\pi(\alpha)^\perp)} \leq c\,||f||_{L_0^p(\Omega)},$$

$$||Pf||_{L_0^p(T)} \leq c\,||f||_{L_0^p(\Omega)}.$$

(b) If $(1+|x|)^{k-n} f \in L^1$, then Pf is defined almost everywhere on T and is locally integrable, thus for almost every $\alpha \in G_k(V)$, $P_\alpha f$ is defined for almost everywhere on $\pi(\alpha)^\perp$ and is locally integrable.

Proof: (a) For $p < \infty$, $|P_\alpha f(x'')|^p = \left| \int_{\pi(\alpha)} f(x''+x')\,dx' \right|^p$

$$\leq c \int_{\pi(\alpha)} |f(x''+x')|^p\,dx'$$

as in the proof of (2.2)(a). Thus

$$||P_\alpha f||^p_{L^p(\pi(\alpha)^\perp)} = \int_{\pi(\alpha)^\perp} |P_\alpha f(x'')|^p \, dx''$$

$$\leq c \int_{\pi(\alpha)^\perp} \int_{\pi(\alpha)} |f(x''+x')|^p \, dx' \, dx'' = c \, ||f||^p_{L^p_0(\Omega)} .$$

To get the result for P, integrate over $G_k(V)$. Again, the case $p = \infty$ is easy.

(b) is proved in Solmon [15].■

<u>Theorem (2.4)</u> If $f \in L^p(S^{n-1})$, $1 \leq p \leq \infty$, then $Sf \in L^p(G_k(V))$, and

$$||Sf||_{L^p(G_k(V))} \leq c \, ||f||_{L^p(S^{n-1})} .$$

<u>Proof:</u> For $p < \infty$, $|Sf(\alpha)|^p = | \int_{S^{n-1} \cap \pi(\alpha)} f(\theta) \, d\theta \, |^p$

$$\leq c \int_{S^{n-1} \cap \pi(\alpha)} |f(\theta)|^p \, d\theta,$$

where $c = |S^{k-1}|^{p/q}$, $1/p + 1/q = 1$, so

$$||Sf||^p_{L^p(G_k(V))} = \int_{G_k(V)} |Sf(\alpha)|^p \, d\alpha$$

$$\leq c \int_{G_k(V)} \int_{S^{n-1} \cap \pi(\alpha)} |f(\theta)|^p \, d\theta$$

$$= c \, ||f||^p_{L^p(S^{n-1})}$$

by theorem (II.3.7).∎

<u>Theorem (2.5)</u> Let S^* be defined by

$$S^* f(\theta) = \int_{\Gamma_k(\theta)} f(\alpha) \, d\alpha.$$

If $f \in L^p(G_k(V))$, $1 \leq p \leq \infty$, then $S^* f \in L^p(S^{n-1})$, and

$$||S^* f||_{L^p(S^{n-1})} \leq c \, ||f||_{L^p(G_k(V))}.$$

<u>Proof:</u> Almost identical to (2.4), with obvious adaptations.∎

The various possible domains can be summarized as follows:

$$P_\alpha: \quad L^p_0(\Omega) \quad \to \quad L^p_0(\pi(\alpha)^\perp)$$

$$P: \quad L^p_0(\Omega) \quad \to \quad L^p_0(T)$$

$$D_a: \quad L^p_0(\Omega) \quad \to \quad L^p(G_k(V)), \qquad a \text{ outside } \bar{\Omega},$$

$$S: \quad L^p(S^{n-1}) \quad \to \quad L^p(G_k(V))$$

are bounded operators for all $1 \leq p \leq \infty$. P_α and P can also be regarded as unbounded operators on $L^2(V)$ with domain

$$\mathcal{D}_k = \{f \in L^2(V): (1+|x|)^{k-n} f \in L^1(V)\}.$$

3. Adjoints

Lemma (3.1) Let π be a fixed k-dimensional subspace of V, $\pi^\perp$ its orthogonal complement. If g is non-negative and measurable on $\pi^\perp$ and has support in $B^n \cap \pi^\perp$, then

$$\int_{S^{n-1}} g(E_{\pi^\perp}\theta)\, d\theta = |S^{k-1}| \int_{\pi^\perp} g(x)\, (1 - |x|^2)^{(k-2)/2}\, dx.$$

Proof: Let $B^n(r)$ be the ball of radius r around the origin, $B^n(1) = B^n$. Fix an arbitrary (k-1)-dimensional subspace π' of π. By first writing the integral over S^{n-1} as one over $\pi' + \pi^\perp$ and then applying Fubini's theorem,

$$\int_{S^{n-1}} g(E_{\pi^\perp}\theta)\, d\theta = 2 \int_{(\pi'+\pi^\perp) \cap B^n} g(E_{\pi^\perp}x)\, (1 - |x|^2)^{-1/2}\, dx$$

$$= 2 \int_{\pi^\perp \cap B^n} g(x') \int_{\pi' \cap B^n(y)} (y^2 - |x''|^2)^{-1/2}\, dx''\, dx',$$

where $y^2 = 1 - |x'|^2$. Setting $x'' = t\theta$, the inner integral in polar coordinates becomes

$$|S^{k-2}| \int_0^y t^{k-2} (y^2 - t^2)^{-1/2}\, dt.$$

With the substitutions $s = t/y = \cos v$ this equals

$$(1 - |x'|^2)^{(k-2)/2} |S^{k-2}| \int_0^1 s^{k-2} (1 - s^2)^{-1/2}\, ds$$

$$= (1 - |x'|^2)^{(k-2)/2} |S^{k-2}| \int_0^{\pi/2} \cos^{k-2} v\, dv$$

$$= \frac{\Gamma((k-1)/2)\,\Gamma(1/2)}{2\,\Gamma(k/2)} |S^{k-2}| (1 - |x'|^2)^{(k-2)/2}$$

$$= \frac{|S^{k-1}|}{2} (1 - |x'|^2)^{(k-2)/2}. \blacksquare$$

<u>Lemma (3.2)</u> Let ρ be non-negative and measurable on $[0,1]$, π a fixed k-dimensional subspace of V. Then

$$\int_{S^{n-1}} \rho(|E_{\pi^\perp}\theta|)\, d\theta = |S^{k-1}|\, |S^{n-k-1}| \int_0^1 \rho(t)\, t^{n-k-1} (1-t^2)^{(k-2)/2}\, dt$$

<u>Proof:</u> Using (3.1) and polar cordinates on $\pi^\perp$,

$$\int_{S^{n-1}} \rho(|E_{\pi^\perp}\theta|)\, d\theta$$

$$= |S^{k-1}| \int_{B^n \cap \pi^\perp} \rho(|x|)\,(1 - |x|^2)^{(k-2)/2}\, dx$$

$$= |S^{k-1}|\,|S^{n-k-1}| \int_0^1 t^{n-k-1}\,\rho(t)\,(1 - t^2)^{(k-2)/2}\, dt. \blacksquare$$

<u>Lemma (3.3)</u> Let ρ be non-negative and measurable on $[0,1]$, θ a fixed point on S^{n-1}. Then

$$\int_{G_k(V)} \rho(|E_{\alpha^\perp}\theta|)\, d\alpha$$

$$= \frac{|G_k(V)|\,|S^{k-1}|\,|S^{n-k-1}|}{|S^{n-1}|} \int_0^1 \rho(t)\, t^{n-k-1}\,(1-t^2)^{(k-2)/2}\, dt.$$

<u>Proof:</u> The first integral is clearly independent of θ, so

$$\int_{G_k(V)} \rho(|E_{\alpha^\perp}\theta|)\, d\alpha = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \int_{G_k(V)} \rho(|E_{\alpha^\perp}\theta|)\, d\alpha\, d\theta$$

$$= \frac{1}{|S^{n-1}|} \int_{G_k(V)} \int_{S^{n-1}} \rho(|E_{\alpha^\perp}\theta|)\, d\theta\, d\alpha$$

$$= \frac{|G_k(V)|}{|S^{n-1}|} \int_{S^{n-1}} \rho(|E_{\alpha^\perp}\theta|)\, d\theta,$$

since the inner integral is independent of α.∎

<u>Corollary (3.4)</u>

$$\int_{G_k(V)} |E_{\alpha^\perp}\theta|^{k-n+1}\, d\alpha = \frac{1}{2} |S^k|\, |G_k(\theta^\perp)|$$

<u>Proof:</u> By (3.3),

$$\int_{G_k(V)} |E_{\alpha^\perp}\theta|^{k-n+1}\, d\alpha = \frac{|G_k(V)|\, |S^{k-1}|\, |S^{n-k-1}|}{|S^{n-1}|} \int_0^1 (1 - t^2)^{(k-2)/2}\, dt$$

$$= \frac{|G_k(V)|\, |S^{k-1}|\, |S^{n-k-1}|}{|S^{n-1}|} \int_0^{\pi/2} \cos^{k-1} s\, ds$$

$$= \frac{|G_k(V)|\, |S^{k-1}|\, |S^{n-k-1}|}{|S^{n-1}|} \frac{\Gamma(k/2)\,\Gamma(1/2)}{2\,\Gamma((k+1)/2)}$$

$$= \frac{1}{2} |S^k|\, |G_k(\theta^\perp)| \quad \text{after simplification.}$$ ∎

For fixed $x \in V$, let T_x be defined by $T_x\alpha = x \wedge \alpha / \|x \wedge \alpha\|$, whenever $x \wedge \alpha \neq 0$. If $F: X \to Y$ and g is a function on Y, then $F^*g = g \circ F$.

<u>Lemma (3.5)</u> If h is continuous on $\Gamma_k(\theta)$, ρ non-negative and measurable on $[0,1]$, and

$$\int_{G_{k-1}(V)} \rho(|E_{\alpha^\perp}\theta|)\, d\alpha = c \int_0^1 \rho(t)\, t^{n-k}\, (1-t^2)^{(k-3)/2}\, dt = C < \infty,$$

then

$$\int_{\Gamma_k(\theta)} h(\alpha)\, d\alpha = \frac{|\Gamma_k(\theta)|}{C} \int_{G_{k-1}(V)} h(T_\theta\alpha)\, \rho(|E_{\alpha^\perp}\theta|)\, d\alpha.$$

<u>Proof:</u> Let

$$L(h) = \int_{G_{k-1}(V)} h(T_\theta\alpha)\, \rho(|E_{\alpha^\perp}\theta|)\, d\alpha.$$

$T_\theta\alpha$ is undefined for $\alpha \in \Gamma_{k-1}(\theta)$, but this is a submanifold of $G_{k-1}(V)$ of smaller dimension and therefore has measure zero. L is clearly a continuous linear form on $C(\Gamma_k(\theta))$.

Let U be any rotation in V with $U\theta = \theta$, and let the induced map from $G_{k-1}(V)$ into itself also be called U. Then $T_\theta U = UT_\theta$, thus $U^*T_\theta^* = T_\theta^*U^*$, so

$$L(U^*h) = \int_{G_{k-1}(V)} T_\theta^*h(U\alpha)\, \rho(|E_{\alpha^\perp}\theta|)\, d\alpha$$

$$= \int_{G_{k-1}(V)} T_\theta^* h(U\alpha)\ \rho(|E_{U\alpha^\perp}\theta|)\ d\alpha \ = \ L(h).$$

This shows that L is rotation-invariant and thus equal to the integral on the left, up to a constant. This constant is determined by setting $h = 1$.∎

Corollary (3.6) If h is continuous on $\Gamma_k(\theta)$, then

$$\int_{\Gamma_k(\theta)} h(\alpha)\ d\alpha = \frac{2}{|S^{k-1}|} \int_{G_{k-1}(V)} h(T_\theta\alpha)\ |E_{\alpha^\perp}\theta|^{k-n}\ d\alpha.$$

Theorem (3.7)

(a) Let $f \in L_0^2(\Omega)$, $h \in L^2(G_k(V))$, and let a be a point outside $\bar{\Omega}$. Then

$$\int_{G_k(V)} D_a f(\alpha)\ h(\alpha)\ d\alpha \ = \ H*f(a), \tag{3.8}$$

where

$$H(x) \ = \ |x|^{k-n} \int_{\Gamma_k(\frac{x}{|x|})} h(\alpha)\ d\alpha$$

(b) If D_a is considered as an operator on $L_0^2(\Omega)$ and a is a point outside $\bar{\Omega}$, the adjoint operator D_a^* is given by

$$D_a^* h(x) = |x-a|^{k-n} \int_{\Gamma_k(\frac{a-x}{|a-x|})} h(\alpha)\, d\alpha$$

for $h \in L^2(G_k(V))$ and $x \in \Omega$.

<u>Proof:</u> (a) By (3.6), the formula given for H is equivalent to

$$H(x) = \frac{2}{|S^{k-1}|} \int_{G_{k-1}(V)} h(T_x\alpha)\ |E_{\alpha^\perp}x|^{k-n}\, d\alpha.$$

This latter form will be used in the proof.

Assume first that both f and h are continuous and proceed by induction.

For k = 1, $\Gamma_1(x/|x|) = \{x/|x|, -x/|x|\}$, thus

$$H(x) = (h(x/|x|) + h(-x/|x|))\,|x|^{1-n},$$

a formula in Hamaker/Smith/Solmon/Wagner [4].

Assume now that the result is true for $G_k(V)$.

$$\int_{G_{k+1}(V)} D_a f(\alpha)\ h(\alpha)\, d\alpha = \int_{G_{k+1}(V)} h(\alpha) \int_{\pi(\alpha)} f(a+x')\, dx'\, d\alpha$$

$$= \frac{1}{|S^k|} \int_{S^{n-1}} \int_{\Gamma_{k+1}(\theta)} h(\alpha) \int_{\pi(\alpha)} f(a+x')\, dx'\, d\alpha\, d\theta$$

$$= \frac{1}{|S^k|} \int_{S^{n-1}} \int_{G_k(\theta^\perp)} h(\alpha' \wedge \theta) \int_{\pi(\alpha')} \int_{-\infty}^{\infty} f(a+x''+t\theta)\, dt\, dx''\, d\alpha'\, d\theta$$

$$= \frac{1}{|S^k|} \int_{S^{n-1}} \int_{G_k(\theta^\perp)} h(\alpha' \wedge \theta) \int_{\pi(\alpha')} P_\theta f(E_{\theta^\perp}a+x'')\, dx''\, d\alpha'\, d\theta$$

$$= \frac{1}{|S^k|} \int_{S^{n-1}} \int_{G_k(\theta^\perp)} h(\alpha' \wedge \theta)\, D_{E_{\theta^\perp}a} P_\theta f(\alpha')\, d\alpha'\, d\theta$$

$$= \int_{S^{n-1}} \int_{\theta^\perp} H_\theta(x')\, P_\theta f(E_{\theta^\perp}a-x')\, dx'\, d\theta$$

by induction, where

$$H_\theta(x') = \frac{2}{|S^{k-1}|\, |S^k|} \int_{G_{k-1}(\theta^\perp)} h(T_{x'}\alpha'' \wedge \theta)\, |E_{\alpha''^\perp}x'|^{k-n+1}\, d\alpha''.$$

Thus the last integral in the sequence above is equal to

$$\int_{S^{n-1}} \int_{\theta^\perp} H_\theta(x') \int_{-\infty}^{\infty} f(E_{\theta^\perp}a-x'-t\theta)\, dt\, dx'\, d\theta$$

$$= \int_{S^{n-1}} \int_{\theta^\perp} H_\theta(x') \int_{-\infty}^{\infty} f(a-x'-t\theta)\, dt\, dx'\, d\theta$$

$$= \int_{S^{n-1}} \int_{V} H_\theta(E_{\theta^\perp}x)\ f(a-x)\ dx\ d\theta \;=\; f*H(a),$$

where

$$H(x) \;=\; \int_{S^{n-1}} H_\theta(E_{\theta^\perp}x)\ d\theta$$

$$= \frac{2}{|S^{k-1}|\ |S^k|} \int_{S^{n-1}} \int_{G_{k-1}(\theta^\perp)} h(T_{E_{\theta^\perp}x}\alpha''\wedge\theta)\ |E_{\alpha''^\perp}E_{\theta^\perp}x|^{k-n+1}\ d\alpha''\ d\theta$$

Now $T_{E_{\theta^\perp}x}\alpha''\wedge\theta \;=\; T_x(\alpha''\wedge\theta)$, since both sides are positive multiples of $E_{\theta^\perp}x\wedge\alpha''\wedge\theta \;=\; x\wedge\alpha''\wedge\theta$ and have length 1. Furthermore, $E_{\alpha''^\perp}E_{\theta^\perp} \;=\; E_{(\alpha''\wedge\theta)^\perp}$, so the integral equals

$$\frac{2}{|S^{k-1}|\ |S^k|} \int_{S^{n-1}} \int_{G_{k-1}(\theta^\perp)} h(T_x(\alpha''\wedge\theta))\ |E_{(\alpha''\wedge\theta)^\perp}x|^{k-n+1}\ d\alpha''\ d\theta,$$

$$= \frac{2}{|S^{k-1}|\ |S^k|} \int_{S^{n-1}} \int_{\Gamma_k(\theta)} h(T_x\alpha)\ |E_{\alpha^\perp}x|^{k-n+1}\ d\alpha\ d\theta$$

$$= \frac{2}{|S^k|} \int_{G_k(V)} h(T_x\alpha)\ |E_{\alpha^\perp}x|^{k-n+1}\ d\alpha.$$

Now if $\{f_j\}$ is a sequence of functions in $C_0^2(\Omega)$ converging in L^2 to a function $f \in L_0^2(\Omega)$, then $D_a f$, $D_a f_j \in L^2(G_k(V))$ and $D_a f_j \to D_a f$ in L^2 by theorem (2.2)(a).

Likewise, if $\{h_j\}$ is a sequence of continuous functions converging in L^2 to a function $h \in L^2(G_k(V))$, then by theorem (2.5), $S^*h \in L^2(S^{n-1})$ and $S^*h_j \to S^*h$ in L^2.

Choose r so that Ω is contained inside the open ball B of radius r around a, and let H' be the restriction of H to B. Then

$$||H'||^2_{L_0^2(B)} = \int_0^r t^{n-1}\, t^{2(k-n))}\, ||S^*h||^2_{L^2(S^{n-1})}\, dt = c\, ||S^*h||^2_{L^2(S^{n-1})}$$

and likewise for the H_j'.

This shows that $H' \in L_0^2(B)$ and that $H_j' \to H'$ in L^2. Since $H*f(a) = H'*f(a)$ and $H_j*f(a) = H_j'*f(a)$, the integrals on both sides of (3.4) converge, and by Cauchy-Schwartz,

$$\int_{G_k(V)} D_a f_j(\alpha)\, h_j(\alpha)\, d\alpha \to \int_{G_k(V)} D_a f(\alpha)\, h(\alpha)\, d\alpha,$$

$$H'*f_j(a) \to H'*f(a).$$

(b) is now an easy consequence of (a).∎

<u>Theorem (3.9)</u>

(a) If $f \in L^2(S^{n-1})$, $h \in L^2(G_k(V))$, then

$$\int_{G_k(V)} Sf(\alpha)\, h(\alpha)\, d\alpha = \int_{S^{n-1}} f(\theta) \int_{\Gamma_k(\theta)} h(\alpha)\, d\alpha.$$

(b) If S is considered as an operator on $L^2(S^{n-1})$, its adjoint S^* is given by

$$S^*h(\theta) = \int_{\Gamma_k(\theta)} h(\alpha)\, d\alpha.$$

Proof: (a) Define $g(x) = |x|^{1-k} f(x/|x|)$ for $1 \leq |x| \leq 2$ and zero otherwise. Then

$$||g||^2_{L^2(V)} = \int_1^2 t^{n-1}\, t^{2(1-k)}\, ||f||^2_{L^2(S^{n-1})}\, dt$$

$$= c\, ||f||^2_{L^2(S^{n-1})},$$

so $g \in L^2(V)$. Since

$$D_0 g(\alpha) = \int_{\pi(\alpha)} g(x)\, dx = \int_1^2 t^{k-1}\, t^{1-k} \int_{S^{n-1} \cap \pi(\alpha)} f(\theta)\, d\theta\, dt$$

$$= Sf(\alpha),$$

it follows that

$$\int_{G_k(V)} Sf(\alpha)\, h(\alpha)\, d\alpha = \int_{G_k(V)} D_0 g(\alpha)\, h(\alpha)\, d\alpha$$

$$= \int_V H(-x)\, g(x)\, dx = \int_1^2 t^{n-1}\, t^{k-n}\, t^{1-k} \int_{S^{n-1}} H(-\theta)\, f(\theta)\, d\theta$$

$$= \int_{S^{n-1}} f(\theta) \int_{\Gamma_k(\theta)} h(\alpha)\, d\alpha\, d\theta,$$

since $H(\theta) = H(-\theta)$.

(b) follows directly from (a) and the known mapping properties of S and S^*.∎

The following theorem and corollary are proved in Solmon [15].

<u>Theorem (3.10)</u> If P is considered as an operator on $L_0^2(V)$, its formal adjoint $P^{\#}$ is given by

$$P^{\#}g(x) = \int_{G_k(V)} g(\alpha, E_{\alpha^\perp}x)\, d\alpha,$$

where $g \in L^2(T)$ and $E_{\alpha^\perp}$ is the orthogonal projection onto $\pi(\alpha)^\perp$.

For every $g \in L^2(T)$, $P^{\#}g$ is defined almost everywhere and is locally square integrable. Moreover, g is in the domain of the adjoint P^* of P if and only if $P^{\#}g$ is globally square integrable, in which case $P^{\#}g = P^*g$.

<u>Corollary (3.11)</u> If f is non-negative, measurable and defined almost everywhere in V, then

$$P^{\#}Pf = \frac{|S^{n-1}|}{c(n,k)\, |S^{k-1}|\, |G_k(V)|} R_k * f,$$

where R_k is the Riesz kernel defined in section (II.4).

IV. UNIQUENESS AND NON-UNIQUENESS THEOREMS

1. Uniqueness and Non-Uniqueness for D

Let Ω be a bounded open set in V with closure $\bar{\Omega}$ and closed convex hull $\hat{\Omega}$. $C_0(\Omega)$ and $L_0^2(\Omega)$ are the spaces of continuous and square integrable functions defined on V and zero outside Ω, respectively.

For $k \geq 0$, the operator D_a^k is defined by

$$D_a^k f(\theta) = \int_{-\infty}^{\infty} f(a+t\theta)\ |t|^k\ dt,$$

whenever the integral exists.

The proof of the following theorem is analogous to the proof of (III.2.2)(a).

Theorem (1.1) If $f \in L_0^p(\Omega)$, then $D_a^k f \in L^p(S^{n-1})$ for all a outside $\bar{\Omega}$, and for such a,

$$||D_a^k f||_{L^p(S^{n-1})} \leq c\ ||f||_{L_0^p(\Omega)} .$$

Theorem (1.2) If $f \in L_0^2(\Omega)$, a is a point outside $\bar{\Omega}$ and $\alpha \in G_k(V)$, then

$$D_a f(\alpha) = \frac{1}{2} S D_a^{k-1} f(\alpha).$$

<u>Proof:</u> By using polar coordinates in $\pi(\alpha)$,

$$D_a f(\alpha) = \int_{\pi(\alpha)} f(a+x)\, dx$$

$$= \int_{S^{n-1} \cap \pi(\alpha)} \int_0^\infty f(a+t\theta)\, t^{k-1}\, dt\, d\theta$$

$$= \frac{1}{2} \int_{S^{n-1} \cap \pi(\alpha)} \int_{-\infty}^\infty f(a+t\theta)\, |t|^{k-1}\, dt\, d\theta$$

$$= \frac{1}{2} S D_a^{k-1} f(\alpha). \blacksquare$$

<u>Theorem (1.3)</u> S is one-to-one on even functions in $L^2(S^{n-1})$.

<u>Proof:</u> By (II.5.4), S acts as a multiple of the identity on each H_m. To show that S is one-to-one on the set of even L^2-functions, it remains to show that this multiple is not zero for the H_m with m even. For this, it suffices to exhibit one function in each such H_m that does not get mapped into zero by S. By (II.5.2), the functions Z_e^m have this property, where e is arbitrary.$\blacksquare$

<u>Remark:</u> Clearly, S maps all odd functions into zero.

<u>Lemma (1.4)</u> If $f \in L_0^2(\Omega)$, if a is a point outside $\bar{\Omega}$ and $D_a f(\alpha) = 0$ for almost every $\alpha \in G_k(V)$, then $D_a^{k-1} f(\theta) = 0$ for almost every $\theta \in S^{n-1}$.

<u>Proof:</u> Immediate from (1.2) and (1.3).∎

<u>Theorem (1.5)</u> If $f \in L_0^2(\Omega)$ and A is an infinite set of points bounded away from $\hat{\Omega}$, then f is determined uniquely by $D_a f(\alpha)$ for $\alpha \in G_k(V)$ and $a \in A$.

<u>Proof:</u> By lemma (1.4), it suffices to show that f is determined uniquely by $D_a^{k-1} f(\theta)$ for $\theta \in S^{n-1}$ and $a \in A$. The proof of this is almost identical to the proof of theorem (5.1) in Hamaker/Smith/Solmon/Wagner [4], with theorem (3.19) in that paper in place of (3.2).∎

<u>Theorem (1.6)</u> Let $f_0 \in L_0^1(\Omega)$, let A be a finite set of sources outside Ω, let K be a compact set in the interior of the support of f_0, and let g be any integrable function on K.

Then there is a function $f \in L_0^1(\Omega)$ with the same shape as f_0, with $D_a f = D_a f_0$ for all $a \in A$, and $f = g$ on K.

<u>Proof:</u> Theorem (6.15) in Leahy/Smith/Solmon [7] shows that there is an f so that $D_a^k f(\theta) = D_a^k f_0(\theta)$ for all $\theta \in S^{n-1}$ and for all $k \geq 0$. The theorem then follows immediately from (1.2).∎

2. Uniqueness and Non-Uniqueness for P

The **Fourier transform** $\hat{f}$ of a function f is defined by

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_V f(x)\, e^{-i\langle x,\xi\rangle}\, dx.$$

The Fourier transform of P_α is given by

$$(P_\alpha f)^\wedge(\xi'') = (2\pi)^{k/2}\, \hat{f}(\xi''), \quad \xi'' \in \pi(\alpha)^\perp. \tag{2.1}$$

The following theorems are easy generalizations of (5.1) in Smith/Solmon/Wagner [13].

Theorem (2.2) Let $f \in L^1_0(V)$ and let $\{\alpha_1, \alpha_2, ..\}$ be a collection of unit k-vectors. If the subspaces $\pi(\alpha_j)^\perp$ of V are not contained in a proper algebraic variety, then f is uniquely determined by $P_{\alpha_j} f$.

Proof: Since f has compact support, its Fourier transform f extends to an entire function on C^n with a Taylor expansion

$$\hat{f}(\xi) = \sum_{m=0}^{\infty} p_m(\xi),$$

where $p_m(\xi)$ is a homogeneous polynomial of degree m.

If $P_{\alpha_j} f = 0$, then by (3.1), $\hat{f}$ vanishes on $\pi(\alpha_j)^\perp$. For any $\xi' \in \pi(\alpha_j)^\perp$,

$$f(t\xi') = \sum_{m=0}^{\infty} t^m p_m(\xi') = 0 \qquad \text{for all } t,$$

from which it follows that $p_m = 0$ on $\pi(\alpha_j)^{\perp}$. Since no non-zero p_m can vanish on all the $\pi(\alpha_j)^{\perp}$, $p_m = 0$, so $\hat{f} = 0$ and finally $f = 0$.∎

Theorem (2.3) Let $f_0 \in C_0^{\infty}(V)$, $\{\alpha_1, \alpha_2, ..\}$ a collection of unit k-vectors. If all of the subspaces $\pi(\alpha_j)^{\perp}$ of V are contained in a proper algebraic variety on V, if K is any compact set in the interior of the support of f_0, and if f_1 is any function in $C^{\infty}(V)$, there is a function $f \in C_0^{\infty}(V)$ so that

$$f = f_1 \quad \text{on } K,$$

$$P_{\alpha_j} f = P_{\alpha_j} f_0 \qquad \text{for all } j,$$

$$\operatorname{supp} f \subset \operatorname{supp} f_0.$$

Proof: Let Q be a polynomial that vanishes on all $\pi(\alpha_j)^{\perp}$. The theorem of Ehrenpreis-Malgrange on the existence of solutions to constant coefficient partial differential equations guarantees the existence of functions u_0 and u_1 in $C^{\infty}(V)$ so that

$$Q(D)u_k = f_k, \qquad k = 0,1.$$

Choose $\rho \in C_0^{\infty}(V)$ so that $\rho = 1$ in a neighborhood of K and ρ vanishes outside the support of f_0. Now let

$$v_k = Q(D)(\rho u_k), \qquad k = 0,1.$$

The last two formulas show that $v_k = f_k$ in a neighborhood of K, and $v_k = 0$ outside the support of f_0.

By (2.1),

$$(P_{\alpha_j} v_k)^\wedge(\xi') = (2\pi)^{k/2} Q(i\xi')(\rho u_k)^\wedge(\xi')$$

$$= 0 \quad \text{for} \quad \xi' \in \pi(\alpha_j)^\perp,$$

thus

$$P_{\alpha_j} v_k = 0 \qquad \text{for } k = 0,1 \text{ and all } j.$$

Finally, define $f = f_0 - v_0 + v_1$. Then $f = f_1$ in a neighborhood of K, supp $f \subset$ supp f_0 and $P_{\alpha_j} f = P_{\alpha_j} f_0$ for all j.

<u>Remark:</u> The theorem automatically applies if $\{\alpha_1,\dots,\alpha_N\}$ is a finite set (take $Q(\xi) = \langle\xi,\xi_1\rangle\langle\xi,\xi_2\rangle\ldots\langle\xi,\xi_N\rangle$, where $\xi_j \in \pi(\alpha_j)$, $\xi_j \neq 0$).

V. INVERSION FORMULAS

1. Exact Inversion Formulas

Throughout this chapter, let C be the constant

$$C = \frac{|S^{k-1}|\ |G_k(V)|}{c(n,k)\ \ |S^{n-1}|} = (2\pi)^k \frac{|S^{n-k-1}|\ |G_k(V)|}{|S^{n-1}|}.$$

The Riesz kernel $R_k = c(n,k)\ |x|^{k-n}$, $0 < k < n$, was defined in section (II.4). The constant $c(n,k)$ is chosen so that

$$\hat{R}_k(\xi) = (2\pi)^{-n/2}\ |\xi|^{-k},$$

where $\hat{R}_k$ is the Fourier transform of R_k, defined in section (IV.2).

The operator Λ is defined by

$$(\Lambda f)^\wedge(\xi) = |\xi|\ \hat{f}(\xi),$$

so that formally

$$\Lambda^k(R_k * f) = R_k * \Lambda^k f = f.$$

From (IV.2.1) it follows that formally

$$P\Lambda^k f = \Lambda^k P f,$$

This, together with (III.3.11), shows that

$$f = C\ \Lambda^k(P^* P f) = C\ \ P^* P \Lambda^k f = C\ \ P^* \Lambda^k P f.$$

Here and in general the Fourier transform and the operator Λ act on each $\pi()^{\perp}$, if f is a function defined on T.

Theorem (1.1)

(a) If $f \in L_0^2(V)$ and $|\xi|^{-k}\hat{f} \in L^2(V)$, then for almost every $x \in V$,

$$f(x) = C\ \Lambda^k \int_{G_k(V)} D_x f(\alpha)\ d\alpha$$

$$= C\ \Lambda^k \int_{G_k(V)} P_\alpha f(E_{\alpha^\perp} x)\ dx,$$

(b) If $f \in L_0^2(V)$ and $|\xi|^{k/2}\hat{f} \in L^2(V)$, then for almost every $x \in V$,

$$f(x) = C \int_{G_k(V)} \Lambda^k P_\alpha f(E_{\alpha^\perp} x)\ dx.$$

Remark: If $k < n/2$, the condition $|\xi|^{-k}\hat{f} \in L^2(V)$ in (a) is automatically satisfied.

Proof: (a) Both formulas are equivalent to

$$f = \Lambda^k(R_k * f).$$

By (II.4.2), $R_k * f$ is defined almost everywhere. The second condition guarantees that $R_k * f$ is in $L^2(V)$, and

$$(\Lambda^k(R_k*f))^\wedge(\xi) = |\xi|^k\,|\xi|^{-k}\,\hat{f}(\xi) = \hat{f}(\xi),$$

so $f = \Lambda^k(R_k*f)$ in $L^2(V)$ and therefore almost everywhere.

(b) By (II.3.8),

$$\|\,|\xi|^{k/2}\hat{f}\,\|_{L^2(V)} = \int_V |\xi|^k\,|\hat{f}|^2(\xi)\,d\xi$$

$$= c \int_{G_k(V)} \int_{\pi(\alpha)^\perp} |\xi|^{2k}\,|\hat{f}|^2(\xi)\,d\xi\,d\alpha < \infty,$$

so $|\xi|^k\hat{f} \in L^2(\pi(\alpha)^\perp)$ for almost every $\alpha \in G_k(V)$, which means that $\Lambda^k P_\alpha f$ is well-defined for almost every α, and $\Lambda^k Pf \in L^2(T)$.

If $g \in L^2_0(V)$, then

$$\langle Pg, \Lambda^k Pf\rangle_{L^2(T)} = \langle (Pg)^\wedge, (\Lambda^k Pf)^\wedge\rangle_{L^2(T)}$$

$$= \int_{G_k(V)} \int_{\pi(\alpha)^\perp} (P_\alpha g)^\wedge(\xi)\,\overline{(\Lambda^k Pf)^\wedge(\xi)}\,d\xi\,d\alpha$$

$$= (2\pi)^k \int_{G_k(V)} \int_{\pi(\alpha)^\perp} \hat{g}(\xi)\,\overline{|\xi|^k\,\hat{f}(\xi)}\,d\xi\,d\alpha = C \int_V \hat{g}(\xi)\,\overline{\hat{f}(\xi)}\,d\xi$$

$$= C\,\langle \hat{g}, \hat{f}\rangle_{L^2(V)} = C\,\langle g, f\rangle_{L^2(V)}.$$

This shows that $\Lambda^k Pf$ is in the domain of P^* and that

$$f = c\,P^*\Lambda^k Pf$$

almost everywhere.∎

The inversion formula for D is not quite satisfactory in this form. In any practical application, the point x is located inside the support of f, whereas $D_a f$ is measured for points a outside the support of f. It is desirable to rewrite the formula as an integration over a set of points surrounding the object.

<u>Lemma (1.2)</u> Let A be a sphere of radius r around 0, x a point not on A, and let g be non-negative and measurable on S^{n-1}. Then

$$\int_{S^{n-1}} g(\theta)\, d\theta = \frac{1}{r} \int_A g\left(\frac{a-x}{|a-x|}\right) |a-x|^{-n}\, |\langle a, a-x\rangle|\, da.$$

<u>Proof:</u> See Leahy/Smith/Solmon [7].∎

<u>Theorem (1.3)</u> If $f \in L_0^2(\Omega)$ and A is a sphere of radius r surrounding Ω, then for almost every $x \in \Omega$,

$$f(x) = \frac{|G_k(V)|}{r\, c(n,k)\, |S^{n-1}|} \Lambda^k \int_A D_a^* D_a f(x)\, |\langle a, a-x\rangle|\, da.$$

<u>Proof:</u> By (II.3.3),

$$\int_{G_k(V)} D_x f(\alpha)\, d\alpha = \frac{1}{|S^{k-1}|} \int_{S^{n-1}} \int_{\Gamma_k(\theta)} D_x f(\alpha)\, d\alpha\, d\theta$$

$$= \frac{1}{r\,|S^{k-1}|} \int_A \int_{\Gamma_k(\frac{a-x}{|a-x|})} D_x f(\alpha)\, d\alpha \quad |a-x|^{-n}\, |\langle a, a-x\rangle|\, da$$

$$= \frac{1}{r\,|S^{k-1}|} \int_A \int_{\Gamma_k(\frac{a-x}{|a-x|})} D_a f(\alpha)\, d\alpha \quad |a-x|^{-n}\, |\langle a, a-x\rangle|\, da$$

$$= \frac{1}{r\,|S^{k-1}|} \int_A D_a^* D_a f(x)\, |\langle a, a-x\rangle|\, da. \blacksquare$$

<u>Special Case:</u> For $k = 1$, $D_a^* D_a f(x) = 1/2\, |a-x|^{-n}\, \{D_a f(a-x/|a-x|) + D_a f(-(a-x)/|a-x|)\} = |a-x|^{-n}\, D_a f(a-x/|a-x|)$. (1.3) reduces to formula (3.11) in Leahy/Smith/Solmon [7]:

$$f(x) = \frac{1}{r\, c(n,1)}\, \Lambda^k \int_A D_a f(\tfrac{a-x}{|a-x|})\, |a-x|^{-n}\, |\langle a, a-x\rangle|\, da.$$

2. Approximate Inversion Formulas

In practice, none of the formulas presented in the preceding section is suitable for numerical inversion of D or P, because of the presence of Λ. The usual way to resolve this problem is to seek a reconstruction of $e*f$ instead of f, where e is an approximate δ-function.

Also, unless the integral of f is zero, $|\xi|^{-k}\hat{f}$ is not square-integrable at $\xi = 0$ for $k \geq n/2$, so it is not obvious how to interpret formula (1.1)(a) in this case.

<u>Lemma (2.1)</u> If e, f $\in L_0^2(V)$, $|\xi|^k e \in L^2(V)$, then

$$\Lambda^k(e*f) = \Lambda^k e*f.$$

<u>Proof:</u>
$$[\Lambda^k(e*f)]\,(\xi) = |\xi|^k\,(e*f)^\wedge(\xi)$$

$$= (2\pi)^{n/2}\,|\xi|^k\,\hat{e}(\xi)\,\hat{f}(\xi)$$

$$= (2\pi)^{n/2}\,(\Lambda^k e)^\wedge(\xi)\,\hat{f}(\xi) = (\Lambda^k e*f)^\wedge(\xi). \blacksquare$$

<u>Lemma (2.2)</u> If e, f $\in L_0^2(V)$, then

$$P_\alpha(e*f) = P_\alpha e*P_\alpha f.$$

<u>Proof:</u>
$$P_\alpha(e*f)(x'') = \int_{\pi(\alpha)} e*f(x''+x')\,dx'$$

$$= \int_{\pi(\alpha)} \int_V e(y)\,f(x''+x'-y)\,dy\,dx'$$

$$= \int_{\pi(\alpha)} \int_{\pi(\alpha)^\perp} \int_{\pi(\alpha)} e(y''+y')\,f(x''+x'-y''-y')\,dy'\,dy''\,dx'$$

$$= \int_{\pi(\alpha)^\perp} \int_{\pi(\alpha)} e(y''+y') \int_{\pi(\alpha)} f(x''+x'-y''-y')\,dx'\,dy'\,dy''$$

$$= \int_{\pi(\alpha)^{\perp}} \int_{\pi(\alpha)} e(y''+y') \int_{\pi(\alpha)} f(x''+x'-y'')\, dx'\, dy'\, dy''$$

$$= \int_{\pi(\alpha)^{\perp}} \int_{\pi(\alpha)} e(y''+y')\, P_\alpha f(x''-y'')\, dy'\, dy''$$

$$= \int_{\pi(\alpha)^{\perp}} P_\alpha e(y'')\, P_\alpha f(x''-y'')\, dy'' \quad = \quad P_\alpha e * P_\alpha f(x''). \blacksquare$$

<u>Theorem (2.3)</u> If e, $f \in L_0^2(V)$, $|\xi|^{k/2}\hat{e} \in L^2(V)$, then for almost every $x \in V$,

$$e*f(x) = \int_{G_k(V)} k*P_\alpha f(E_{\alpha^\perp} x)\, dx,$$

where

$$k = C\, \Lambda^k P_\alpha e.$$

<u>Proof:</u> By (1.1)(b), (2.1) and (2.2),

$$e*f = C\ P^*\Lambda^k P(e*f) = C\ P^*\Lambda^k(Pe*Pf)$$

$$= C\ P^*(\Lambda^k Pe*Pf) = P^*(k*Pf). \blacksquare$$

<u>Lemma (2.4)</u> If f, g are non-negative and measurable with support in the ball of radius r around 0, and A is the sphere of radius r around 0, then for almost every $\alpha \in G_k(V)$,

$$\int_{\pi(\alpha)^{\perp}} P_\alpha f(a'')\ g(a'')\ da''$$

$$= \frac{1}{r^{n-k+1}\ |S^{k-1}|} \int_A D_a f(\alpha)\ g(E_{\alpha^\perp} a)\ (1 - |E_{\alpha^\perp} a|^2)^{(2-k)/2}\ da.$$

Proof: Immediate from (III.1.1) and (III.3.1).∎

For $g(a) = k(E_{\alpha^\perp} x - a)$ this yields

Theorem (2.5) If e, $f \in L^2_0(\Omega)$, $|\xi|^{k/2}\hat{e} \in L^2(V)$, then for almost every $x \in \Omega$

$e*f(x)$

$$= \frac{1}{r^{n-k+1}\ |S^{k-1}|} \int_{G_k(V)} \int_A D_a f(\alpha)\ k(E_{\alpha^\perp}(x-a))\ (1-|E_{\alpha^\perp} a|^2)^{(2-k)/2}\ da\ d\alpha,$$

where k is the same function as in (2.3).

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