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Title ASYMPTOTIC EXPANSIONS AND GENERAL BEHAVIOR OF THE MODIFIED BESSEL FUNCTION OF THE THIRD KIND OF IMAGINARY ORDER

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The behavior of the modified Bessel function of the third kind with pure imaginary order, $K_{i\nu}(z)$ is investigated. It is proved that $K_{i\nu}(\frac{\nu}{p})$ is a positive monotone decreasing convex function of p for $0 < p \leq 1$ and it is oscillating boundedly for $p > 1$, having countably infinite number of zeros. The three Debye-type series of the function are derived for $z = \frac{\nu}{p}$, with $p < 1$, $p > 1$ and $p \approx 1$ while $|z-\nu| = o(z^{1/3})$, respectively. In the transitional region, $|z-\nu| = O(z^{1/3})$, the Nicholson series of $K_{i\nu}(z)$ is obtained for $\operatorname{Re} z > 0$ and an asymptotic power series in terms of ν is given for the zeros of the function. Furthermore, the Debye-type series of the first and second kind $(p < 1, \quad p > 1)$ are presented for the derivative. In the second part the uniform asymptotic expansions of $K_{i\nu}(\nu z)$ and of $K'_{i\nu}(\nu z)$ are constructed for z lying in a domain which contains the sector $\operatorname{Re} z \geq 0$, $z \neq 0$. Also asymptotic

expansions of the zeros of the function and of its derivative are given which are uniform with respect to the enumeration of the zeros. The zeros lie asymptotically on the real axis.

ASYMPTOTIC EXPANSIONS AND GENERAL BEHAVIOR OF THE MODIFIED BESSEL FUNCTION OF THE THIRD KIND OF IMAGINARY ORDER

by

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INTRODUCTION

The asymptotic behavior of solutions of Bessel's differential equation

$$\frac{d^2w}{dz^2} + \frac{1}{z}\frac{dw}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) w = 0 \qquad (1)$$

especially of the four Bessel functions $J_\nu(z)$, $Y_\nu(z)$, $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$ (for notations see for instance [8, 26, 32, 34]) roughly be subdivided into the following classes [26, 32 Chapters 7,8].

(a) Expansions of the Hankel type. These give the behavior for large argument z while the order ν is fixed.

(b) Expansions for large order ν while the argument z is fixed. This is the simplest case. It requires only the definition of the function above by a series in ascending powers of the variable combined with Stirling's asymptotic series for the Gamma function.

(c) The Debye series. They consist of three separate expansions due to the fact that the differential equation (1) has a turning point at $z = \nu$. These series are commonly called the Debye series of the first and second kind and the so called "Ausnahmereihe" or B-type series. They hold for the case that both the order ν and the argument z are large, such that $\frac{\nu}{z} = p$ is a fixed constant. (Usually and in the investigations here it is assumed that ν and z are real and positive.) The series of the first and second kind are valid for $p < 1$ and $p > 1$ respectively. The series for $p < 1$ and $p > 1$ become useless as $p \rightarrow 1$. The "Ausnahmereihe" covers the case p close to unity such that $|z-\nu| = o(z^{1/3})$. It is clear that the three Debye series do not cover the entire real axis satisfactorily.

(d) Expansions of the Nicholson type, also valid for p close to unity such that $|z-\nu| = O(z^{1/3})$. This case is often referred to as the transitional case.

(e) Uniform asymptotic expansions. While the asymptotic representations (a)-(d) are obtained from the integral representations of the functions these expansions are based upon differential equation methods. The uniform asymptotic expansions yield approximations to an arbitrarily high order in ν^{-1} in a range of the argument which includes the transition point and is independent of the order ν, where ν is restricted to a certain sector of the complex plane.

In recent times expansions of the type (c), (d) and (e) have become of great importance in the problems involving the so called "creeping" wave phenomenon [13, 14, 16].

The Debye series of the Bessel functions were derived by Debye in 1909 [6] using the method of steepest descent which was originated by Riemann and developed by Debye. The simplest among the expressions of the type (d) are the Nicholson's formulas [32, p. 249]. They were obtained by Nicholson (1910) using the "heuristic" stationary phase method originated by Stokes (1883) and Kelvin (1887) for which at the time of their derivation no mathematical justification could be given. However, in later contributions by van der Corput (1934, 1936)[5] a rigorous theory of the principle of the stationary phase was developed. This theory has been simplified by Erdélyi (1956)[9, 10, 11] and Braun (1956) in [2], and has been applied by Bijl for the Bessel functions in 1937 [1]. The Nicholson formulas, apart from the fact that their method of derivation is a questionable one represent asymptotic expressions rather than asymptotic expansions. Asymptotic expansions for the transitional region have been given by Fok (1934)[12] under the condition $\sqrt{|\nu^2 - z^2|} \sim \nu^{2/3}$, where z and ν are real. (A summary of Fok's method is given in [29 Volume 3.1, Section 152, p. 553]). For complex argument $(-2\pi < \arg z < \pi)$ and complex order asymptotic expansions of the type (d) have been rigorously derived by Schoebe in 1954 for the Bessel functions [28]. The leading term of these expansions are the Nicholson formulas.

The derivation of the uniform asymptotic expansions are based on the idea that approximately identical differential equations have approximately identical solutions. Since the time of the first published investigations of this problem (Carlini, Green and Liouville, 1837) many papers have been published on the asymptotic solution of differential equations for large values of a parameter. We refer only to some of the fundamental contributions of Langer [18, 19], Cherry [3, 4], Olver [23, 25], and Thorne [30], and note that the most complete theory which can be applied to Bessel's differential equation was developed by Olver in 1954 [23, 25]. Olver applied this theory to obtain uniform asymptotic expansions of the Bessel functions of complex argument z and complex order ν with the restriction $|\arg\nu| < \frac{\pi}{2}$ [24].

The investigation of the zeros of functions of large order has been difficult using the expansions (a)-(d). The only explicit series based on the Hankel series has been given by McMahon in 1895 for the very large zeros [26, 32] and by Schoebe in 1954 for the transitional region using the Nicholson series [28]. Other series could be obtained by the reversion of the Debye expansions but if the enumeration of the zeros is fixed these expansions are not valid for infinitely large order. Using the uniform asymptotic expansions obtained in

[24] Olver derived an asymptotic series uniformly valid with respect to all the zeros in 1954.

All these investigations center around solutions of Bessel's differential equation (1) which is fundamental in the theory of diffraction of waves of time harmonic character. But in more recent times diffraction problems of pulses (i. e. not necessarily of time harmonic character) have become much more important. It is clear of course that the time harmonic case is only a special case of the pulse case when the pulse function is simply chosen to be time harmonic. But in the pulse case not (1) but the modified Bessel differential equation

$$\frac{d^2w}{dz^2} + \frac{1}{z}\frac{dw}{dz} - \left(1 - \frac{\nu^2}{z^2}\right) w = 0 \tag{2}$$

with its particular solution

$$w = K_{i\nu}(z) \tag{3}$$

describes the phenomena of this kind. Moreover this modified Hankel function $K_{ix}(y)$ is the kernel of an integral transform, called Lebedev transform, with the pair of inversion formulas [20] :

$$g(y) = \int_0^\infty f(x) K_{ix}(y) dx$$

$$f(x) = 2\pi^{-2} x \, sh(\pi x) \int_0^\infty K_{ix}(y) y^{-1} f(y) dy \tag{4}$$

which have become increasingly important in a number of wave propagation problems. See e. g. [17, 21, 22, 33].

With the exception of the contribution of Friedlander (1954)[15] little information is available about the behavior of $K_{i\nu}(z)$ when both ν and z are large. In the work just referred to, Langer's differential equation method [18] is applied and only an asymptotic formula is given. Since the differential equation (2) has a turning point at $\nu = z$, it has to be expected as in the case of solutions of (1) that asymptotic expansions of the type (a)-(e) exist. Again the types (c)-(e) are the most important ones. We shall call the types (c) and (d) the series of the Debye type and the Nicholson type, respectively. It seems that the only published result about the zeros of $K_{i\nu}(z)$ can be found in a work by Pólya (1926) [27] where it is shown that $K_{i\nu}(z)$ has infinite many real zeros. An asymptotic formula is given in the above quoted paper by Friedlander for the zeros of

$$\frac{dK_{i\nu}(z)}{dz}.$$

In the first part of this paper we derive the three Debye type series of $K_{i\nu}(z)$ for real order and positive argument. Because of the failure of these expansions in the transitional region we give the Nicholson series of the function in Section 1. 4; this expansion is valid for real order and complex argument also, if $\mathrm{Re}\, z > 0$, and $|z-\nu| = 0(z^{1/3})$. The leading term of the Nicholson series is the

Nicholson formula obtained by the principle of stationary phase.

Using the integral representation of the function we prove in Section 1.6 that $K_{i\nu}(\frac{\nu}{p})$ is a positive, monotone decreasing convex function of the real order ν for any p such that $0 < p \leq 1$; for $p > 1$ it is an oscillating function of ν and it has countably infinite number of zeros. The proof of this theorem gives an independent verification of Pólya's result mentioned above.

In Section 1.5 an asymptotic power series is given for the zeros of $K_{i\nu}(z)$ in terms of ν for the transitional region; it is shown that the zeros are lying asymptotically on the real axis in this region. Part 1 concludes by the presentation of the Debye series of the first and second kind for $\frac{d}{dz}K_{i\nu}(z)$, $(z > 0)$.

In the second part of this work using Olver's theory a uniform asymptotic expansion is constructed for $K_{i\nu}(\nu z)$ and for $\frac{d}{dz}K_{i\nu}(\nu z)$ for large positive order in a domain of the complex argument containing the non-negative half of the z-plane. It has been shown also that the expansions derived in Part 1, (and the series of the type (a)) can be deduced from the uniform expansions; this fact is of course a consequence of the uniformity of these series.

In Section 2.4 we prove that $K_{i\nu}(z)$ has only simple zeros, moreover the positive zeros lie asymptotically in $0 < z < 1$. It is also deduced that the zeros of $\frac{d}{dz}K_{i\nu}(z)$ are located asymptotically on the real axis. Finally asymptotic expansions of the zeros of

$K_{i\nu}(z)$ and $\frac{d}{dz} K_{i\nu}(z)$ are obtained based on the uniform asymptotic expansions of the functions. These expansions are uniform with respect to the enumeration of the zeros. The expansion of the zeros of $K_{i\nu}(z)$ is a power series in ν^{-2}. The coefficients in both series are transcendental functions which can be pretabulated.

PART I. RESULTS OBTAINED FROM THE INTEGRAL REPRESENTATION

1.1 Preliminaries

The function $K_{i\nu}(z)$ satisfies the differential equation

$$\frac{d^2w}{dz^2} + \frac{1}{z}\frac{dw}{dz} - \left(1 - \frac{\nu^2}{z^2}\right) w = 0, \tag{1.1}$$

and can be represented as [8, 26, 32].

$$2K_{i\nu}(z) = \int_{-\infty}^{\infty} \exp(-z\operatorname{ch} t - i\nu t)dt, \quad \operatorname{Re} z > 0 . \tag{1.2}$$

The investigations of the function in the first part of this paper are based on (1.2). We will apply the method of steepest descent (see e.g. [7, Section 5]) which is based on the following

Theorem (Laplace method).

Let the functions $g(x)$ and $h(x)$ be analytic at $x = a$, and the function $H(x) = h(a) - h(x)$ be real and non-negative in the real interval $a \leq x \leq b$; furthermore let $h'(a) = h''(a) = \cdots = h^{(m-1)}(a) = 0$, $h^{(m)}(a) \neq 0$ $(m \geq 1)$. If we can substitute $\tau = [H(x)]^{1/m}$ in the integral

$$f(s) = \exp\{h(a)s\} \int_a^b g(x) \exp\{-sH(x)\}\, dx , \tag{1.3}$$

then there exists an asymptotic expansion of the following form:

$$\int_a^b g(x)\exp\{sh(x)\}dx \sim \frac{1}{m} \exp\{h(a)\cdot s\} \sum_{k=0}^{\infty} a_k \frac{\Gamma\left(\frac{k+1}{m}\right)}{s^{\frac{k+1}{m}}} , \tag{1.4}$$

with

$$a_k = \frac{1}{k!}\left\{\frac{d^k}{dx^k}\left[g(x)\left(\frac{x-a}{[h(a)-h(x)]^{1/m}}\right)^{k+1}\right]\right\}_{x=a}, \qquad (1.5)$$

for $s \to \infty$ in $|\arg s| \leq \psi < \frac{\pi}{2}$.

If the functions $g(x)$ and $h(x)$ are analytic in a domain D which contains the path of integration, moreover if there exist a curve C lying in D such that $H = h(a) - h(x)$ is real and $H \geq 0$ along C, then by Cauchy's theorem we may deform the path of integration into the curve C, and by substituting $\tau = (H)^{1/m}$ in the integral (1.3) we obtain a line integral along a real path. The so called method of steepest descent is based upon this deformation of the path of integration and its resumé can be given as follows. First we determine the saddlepoints a_i for (1.3), i.e. the points where $h'(x) = 0$. Then we construct the path of steepest descent along which $\operatorname{Im} h(x) = \operatorname{Im} h(a_i)$ starting out of the saddlepoints and choose those curves along which $\operatorname{Re} h(x)$ reaches a maximum at a_i (Debye curves). Next try to deform the original path of integration into one of these curves. Now the premissae of the Laplace method are fulfilled so the integral can be evaluated asymptotically using (1.4) and (1.5).

1.2 The Debye-type Series of $K_{i\nu}(z)$

Using the method of the steepest descent to obtain the asymptotic expansion of $K_{i\nu}(z)$ for large real order and large real variable $z = x$ with constant $\frac{\nu}{x} \equiv p$ it turns out that we have to distinguish three different cases according to $p < 1$, $p > 1$, or $p \approx 1$.

Substituting $x = \frac{\nu}{p}$ in (1. 2) we obtain:

$$2K_{i\nu}\left(\frac{\nu}{p}\right) = \int_{-\infty}^{\infty} \exp\nu\left(it - \frac{\operatorname{ch} t}{p}\right) dt \tag{1.6}$$

Using the notation of the method of steepest descent in the quoted form, we have for $t = \xi + i\eta$, (ξ, η real)

$$h(t) = it - \frac{\operatorname{ch} t}{p} = -\eta - \frac{1}{p}\operatorname{ch}\xi\cos\eta + i\left[\xi - \frac{1}{p}\operatorname{sh}\xi\sin\eta\right], \tag{1.7}$$

$$g(t) \equiv 1 .$$

The condition for the saddlepoints $h'(t) = 0$ leads to the system of equations

$$\operatorname{ch}\xi\sin\eta = p, \quad \operatorname{sh}\xi\cos\eta = 0. \tag{1.8}$$

In order to find the solution of (1. 8) we have to distinguish two cases.

1) For $p \leq 1$ the complete solution of (1. 8) is

$a = i(\beta \pm 2k\pi)$, $\beta = \arcsin p$, $k = 0, 1, 2, \cdots$.

2) For $p \geq 1$ the complete solution is

$a^{+} = \alpha + i(\frac{\pi}{2} \pm 2k\pi)$, $a^{-} = -\alpha + i(\frac{\pi}{2} \pm 2k\pi)$,

where $\alpha = \log\left[p + \sqrt{p^2 - 1}\right]$, $k = 0, 1, 2, \cdots$.

In the sequel it will turn out that the case $p \approx 1$ has to be investigated separately.

1.21 The Debye type series of $K_{i\nu}(\frac{\nu}{p})$ for $p < 1$

Though this expansion can be derived by a standard application of the method of steepest descent we present the calculations because some of the results obtained in the derivation are needed to obtain further properties of the function.

It will be sufficient to confine our attention to the saddlepoint $a = i\beta$, $\beta = \arcsin p$. The condition $\operatorname{Im} h(x) = \operatorname{Im} h(a)$ with

$$h(t) = i\xi - \eta - \frac{\operatorname{ch}\xi \cos\eta}{\sin\beta} - i\,\frac{\operatorname{sh}\xi \sin\eta}{\sin\beta} \equiv X(\xi, \eta) + iY(\xi, \eta),$$

and $h(a) = -(\beta + \cot\beta)$ gives the equation of the path of steepest descent (Debye curve) C :

$$\sin\eta = \sin\beta \frac{\xi}{\operatorname{sh}\xi}, \quad 0 \le \eta \le \beta \le \frac{\pi}{2}. \tag{1.9}$$

Indeed on this curve $H(t) \equiv h(a) - h(t) \Big|_{\text{on } C} \ge 0$ since $H''(a) = \cot\beta > 0$, hence H as a function of ξ has a minimum at $(0, \beta)$, moreover $H(i\beta) = 0$. The curve C can be decomposed into two parts : C_2 and C_1 respectively which go from a to $-\infty$ and a to $+\infty$ respectively. The Debye curve is shown in Figure 1.1.

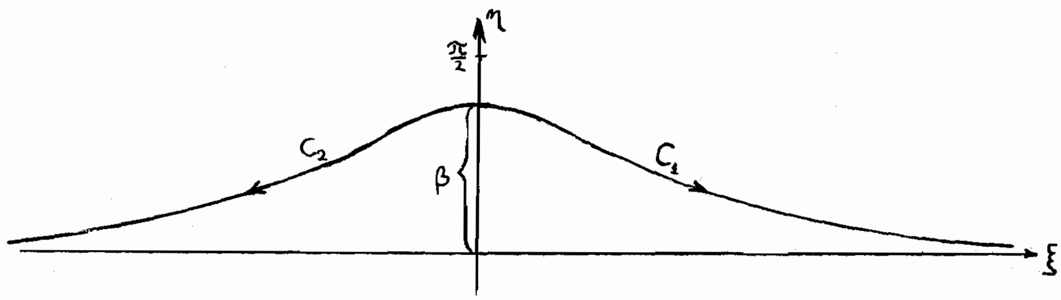

Figure 1. 1

Since $h''(t) = -\frac{\operatorname{ch} t}{\sin\beta} \neq 0$, $m = 2$. Therefore we have on C

$\tau^2 = h(a) - h(t) = X(0,\beta) - X(\xi,\eta) = \xi^2[\frac{1}{2!}X''(0,\beta) + \cdots]$, and

so $\tau = \pm\xi\sqrt{\frac{1}{2!}X''(0,\beta) + \cdots}$, where the positive (negative) sign holds for $\xi > 0$, ($\xi < 0$). By Cauchy's theorem we can deform the path of integration into C, so

$$2K_{i\nu}\left(\frac{\nu}{\sin\beta}\right) = e^{\nu h(a)}\int_{-\infty}^{\infty}\exp\{-\nu[h(a)-h(t)]\}dt =$$

$$= e^{\nu h(a)}\left\{\int_{C_1} - \int_{C_2}\right\}\exp\{-\nu[h(a)-h(t)]\}dt\,.$$

The asymptotic expansions of the integral over C_1 and C_2 are :

$$\int_{C_1}\exp\{-\nu[h(a)-h(t)]\}dt \sim \frac{1}{2}\sum_{k=0}^{\infty} a_k\cdot\frac{\Gamma(\frac{k+1}{2})}{\nu^{\frac{k+1}{2}}}\,,$$

$$\int_{C_2}\exp\{-\nu[h(a)-h(t)]\}dt \sim \frac{1}{2}\sum_{k=0}^{\infty} b_k\cdot\frac{\Gamma(\frac{k+1}{2})}{\nu^{\frac{k+1}{2}}}\,,$$

where a_k and b_k are given by the Lagrange-Bürmann formula (1.5). It is easy to see that $a_{2k} = -b_{2k}$, and $a_{2k+1} = b_{2k+1}$.

Therefore we have

$$2K_{i\nu}\left(\frac{\nu}{\sin\beta}\right) \sim e^{-\nu(\beta+\cot\beta)} \sum_{k=0}^{\infty} a_{2k} \frac{\Gamma\left(\frac{2k+1}{2}\right)}{\nu^{\frac{2k+1}{2}}} ,$$

where

$$a_{2k} = \frac{1}{(2k)!} \left\{ \frac{d^{2k}}{dt^{2k}} \left(\frac{t-a}{T}\right)^{2k+1} \right\} =$$

$$= \frac{1}{(2k)!} \left\{ \frac{d^{2k}}{dt^{2k}} \left[\frac{\mathrm{ch}\, a}{p.\ 2!} + \frac{\mathrm{sh}\, a}{p.\ 3!}(t-a) + \cdots \right]^{-k-\frac{1}{2}} \right\}_{t=a} .$$

The first four coefficients are :

$$a_0 = \sqrt{2\tan\beta}, \quad a_2 = -a_0^3 \left[\frac{5}{24} \cdot \frac{a_0}{4} + \frac{1}{8} \right],$$

$$a_4 = a_0^5 \left[\frac{3}{128} + \frac{77}{576} \cdot \frac{a_0^4}{4} + \frac{385}{3456} \frac{a_0^8}{16} \right], \tag{1.10}$$

$$a_6 = -a_0^7 \left[\frac{5}{1024} + \frac{1521}{25600} \cdot \frac{a_0^4}{4} + \frac{17017}{138240} \frac{a_0^8}{16} + \frac{17017}{248832} \frac{a_0^{12}}{64} \right].$$

This series can be written in the form :

$$K_{i\nu}(\nu \cdot \mathrm{cosec}\beta) \sim$$

$$\sqrt{\frac{\pi}{2\nu\cos\beta}}\, e^{-\nu(\beta+\cot\beta)} \sum_{m=0}^{\infty} (-)^m \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})} \left(\frac{2}{\cot\beta}\right)^m \frac{A_m}{\nu^m} , \tag{1.11}$$

where $A_0 = 1, \quad A_1 = \frac{1}{8} + \frac{5}{24}\tan^2\beta,$

$$A_2 = \frac{3}{128} + \frac{77}{576}\tan^2\beta + \frac{385}{3456}\tan^4\beta,$$

$$A_3 = \frac{5}{1024} + \frac{1521}{25600}\tan^2\beta + \frac{17017}{138240}\tan^4\beta + \frac{17017}{248832}\tan^6\beta, \cdots .$$

We see that this series becomes useless as $\beta \to \frac{\pi}{2}$ i. e. as $p \to 1$.

1. 22 The Debye-type series of $K_{i\nu}(\frac{\nu}{p})$ for $p > 1$

We use the method of steepest descent with some modification.

It is sufficient to consider the following two stationary points :

$$a = \alpha + i\frac{\pi}{2}, \quad b = -\alpha + i\frac{\pi}{2}, \quad \text{where} \quad \alpha = \mathrm{ch}^{-1} p, \quad \alpha > 0 .$$

In this case

$$h(t) = it - \frac{\mathrm{ch}\, t}{\mathrm{ch}\, \alpha} = -\left(\frac{\mathrm{ch}\,\xi \cos\eta}{\mathrm{ch}\,\alpha} + \eta\right) + i\left(\xi - \frac{\mathrm{sh}\,\xi \sin\eta}{\mathrm{ch}\,\alpha}\right) \equiv$$

$$\equiv X(\xi, \eta) + iY(\xi, \eta),$$

and $\quad h(a) = -\frac{\pi}{2} + i(\alpha - \mathrm{th}\,\alpha), \quad h(b) = -\frac{\pi}{2} - i(\alpha - \mathrm{th}\,\alpha).$

The condition $\mathrm{Im}\,h(t) = \mathrm{Im}\,h(a)$ leads to the equation :

$$\sin\eta = \frac{\mathrm{ch}\,\alpha(\xi - \alpha + \mathrm{th}\,\alpha)}{\mathrm{sh}\,\xi}, \qquad (1.\,12)$$

which becomes meaningless in the neighborhood of $\xi = 0$, where

$$\left| \frac{\operatorname{ch}\alpha\,(\xi - \alpha + \operatorname{th}\alpha)}{\operatorname{sh}\xi} \right| > 1 \ .$$

However we will show that (1. 12) with $0 \leq \eta \leq \pi$ defines a Debye curve for

$$\int_{\omega}^{\infty} \exp \nu \left(it - \frac{\operatorname{ch} t}{\operatorname{ch}\alpha}\right) dt \ ,$$

where $\omega = \alpha - \operatorname{th}\alpha$.

The function

$$y(\xi) = \frac{\operatorname{ch}\alpha}{\operatorname{sh}\xi} (\xi - \alpha + \operatorname{th}\alpha)$$

decreases from one to zero as ξ increases from α to $+\infty$, and in this case η decreases from $\frac{\pi}{2}$ to zero, or increases from $\frac{\pi}{2}$ to π; we denote these curves by C_2^1 and C_2^2 respectively. Moreover $y'(\xi) > 0$ for $\xi < \alpha$, and $y'(\xi) < 0$ for $\xi > \alpha$ and $y'(\alpha) = 0$, hence y has its only maximum at $\xi = \alpha$ on (ω, ∞). Thus y increases from zero to one as ξ increases from ω to α, and in this case η increases from zero to $\frac{\pi}{2}$ or decreases from π to $\frac{\pi}{2}$; we denote these curves by C_1^1 and C_1^2 respectively.

So the equation (1. 12) defines the following four curves for $\omega \leq \xi < \infty$, $0 \leq \eta \leq \pi$:

$$C_1^i \cup C_2^j, \quad i = 1, 2, \quad j = 1, 2 \ .$$

The graphs are shown in Figure 1.2.

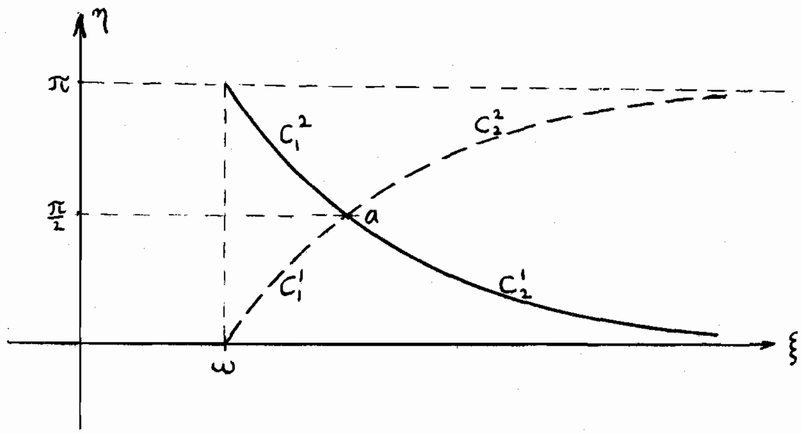

Figure 1.2

Now we show that $H = h(a) - h(t) \geq 0$ on $C = C_1^2 \cup C_2^1$. For $\xi \in [\omega, a]$ and $\eta \in [\frac{\pi}{2}, \pi]$ H is non-negative since $H \cdot \mathrm{ch}\, a = \mathrm{ch}\,\xi \cos\eta + (\eta - \frac{\pi}{2})\,\mathrm{ch}\, a \geq 0$ if and only if

$$\frac{\mathrm{ch}\, a}{\mathrm{ch}\,\xi} \geq \frac{\cos\eta}{\eta - \frac{\pi}{2}},$$

and the last inequality is obvious.

Next we prove that $H > 0$ on C_2^1 i.e. for $\xi > a$ and $0 \leq \eta < \frac{\pi}{2}$. Clearly $H > 0$ for sufficiently large $\xi > a$. For sufficiently small $\xi - a = \Delta\xi$ and $\frac{\pi}{2} - \eta = \Delta\eta$ H is positive, since

$$H = \frac{1}{\mathrm{ch}\, a}[\mathrm{ch}\, a\, \mathrm{ch}\Delta\xi + \mathrm{sh}\, a\, \mathrm{sh}\Delta\xi]\sin\Delta\eta - \Delta\eta \approx$$

$$\approx [1 + \frac{(\Delta\xi)^2}{2} + \cdots + \mathrm{th}\, a\, \mathrm{sh}\Delta\xi]\Delta\eta - \Delta\eta > 0$$

for sufficiently small $\Delta\xi > 0$ and $\Delta\eta > 0$.

Using indirect proof we show that $H \geq 0$ for any $\xi > \alpha$ on C_2^1.

Assume that $H < 0$ for some $\xi > \alpha$. Then H as a continuous function of ξ must have at least two zeros. The assumption $H = 0$ implies that

$$\frac{\operatorname{ch}\xi}{\operatorname{ch}\alpha} = \frac{-\eta + \frac{\pi}{2}}{\cos\eta} ,$$

but the curves $\frac{\operatorname{ch}\xi}{\operatorname{ch}\alpha} = \phi$ and $\frac{-\eta + \frac{\pi}{2}}{\cos\eta} = \psi$ have at most one common point as it can be seen on the graph in Figure 1.3.

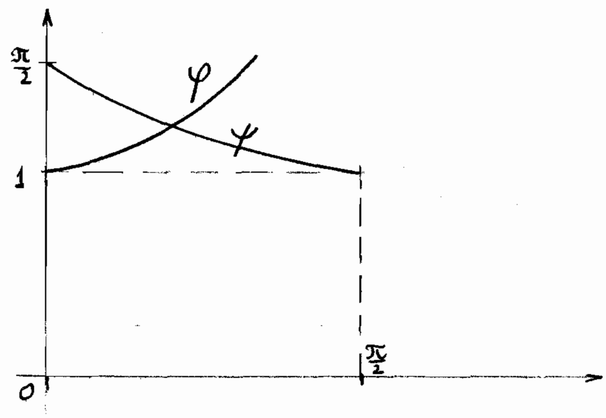

Figure 1.3

It follows that the curve $\overline{C}$, which can be obtained by reflecting C with respect to the η-axis, has similar properties as the curve C, and $\overline{C}$ is a path of steepest descent for

$$\int_{-\omega}^{-\infty} \exp \nu \left(it - \frac{\operatorname{ch} t}{\operatorname{ch}\alpha}\right) dt .$$

Connecting the endpoint $(-\omega, \pi)$ of $\overline{C}$ with the endpoint (ω, π) of C by a line segment parallel to the real axis, we can replace the path of integration along the real line by the curve $K = \overline{C} \cup L \cup C$, indicated in Figure 1.4.

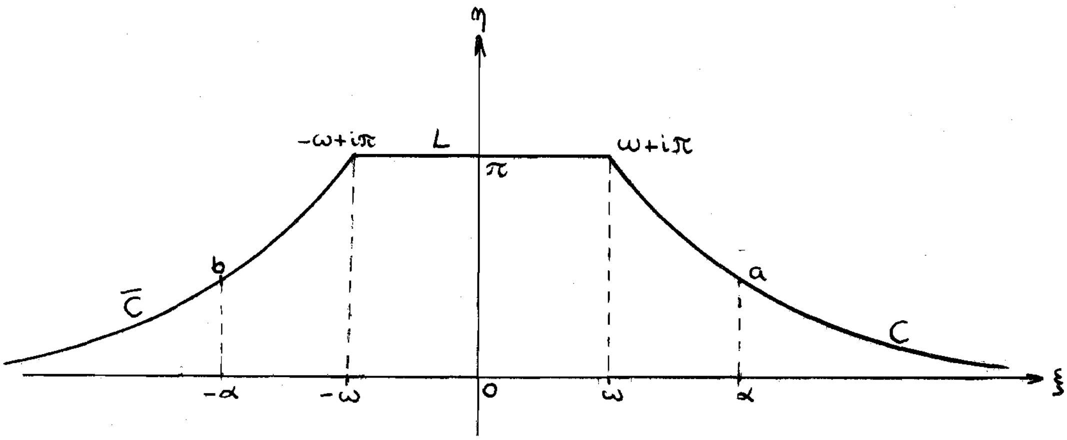

Figure 1.4

The contribution of the integral along the line L can be estimated as follows:

$$\left| \int_L \exp \nu \left(iz - \frac{\operatorname{ch} z}{\operatorname{ch} a}\right) dz \right| \leq \int_{-\omega}^{\omega} \exp \nu \left(\frac{\operatorname{ch} t}{\operatorname{ch} a} - \pi\right) dt <$$

$$< 2\omega \exp \nu (1 - \pi) = o(\exp(1-\pi)\nu) . \tag{1.13}$$

The asymptotic expansion of the function defined by

$$\tilde{K}_{i\nu}\left(\frac{\nu}{\operatorname{ch} a}\right) = \left\{ \int_{\overline{C}} + \int_{C} \right\} \exp \nu\, h(t) dt \equiv I_{\overline{C}} + I_C$$

can be given by the method of steepest descent. Since

$h''(a) = -i\,\text{th}\,a \neq 0, \quad m = 2,$ therefore

$$\mathcal{T} = \{h(a) - h(t)\}^{\frac{1}{2}} \Big|_{\text{on } C} = \pm (\xi - a). \left\{ \frac{X''(a, \frac{\pi}{2})}{2} + \cdots \right\}^{\frac{1}{2}},$$

where the positive sign holds if $\xi > a$, and the negative sign holds for $\omega \leq \xi < a$. Similarly

$$\overline{\mathcal{T}} = \{h(b) - h(t)\}^{\frac{1}{2}} \Big|_{\text{on } \overline{C}} = \pm (\xi + a) \left\{ \frac{X''(-a, \frac{\pi}{2})}{2!} + \cdots \right\}^{\frac{1}{2}},$$

where the positive sign holds if $\xi > (-a)$, and the negative sign holds if $\xi < (-a)$. Decomposing the line integral I_C as

$$I_C = \left\{ -\int_a^{\omega + i\pi} + \int_a^{\infty} \right\} \exp \nu\, h(t)\, dt,$$

the corresponding asymptotic expansions are:

$$\int_a^{\infty} \exp \nu\, h(t)\, dt \sim \frac{1}{2} e^{-\frac{\pi}{2}\nu} \exp\{i\nu(a - \text{th}\,a)\} \sum_{k=0}^{\infty} a_k^{+} \cdot \frac{\Gamma(\frac{k+1}{2})}{\nu^{\frac{k+1}{2}}},$$

and

$$\int_a^{\omega + i\pi} \exp \nu\, h(t)\, dt \sim \frac{1}{2} e^{-\frac{\pi}{2}\nu} \exp\{i\nu(a - \text{th}\,a)\} \sum_{k=0}^{\infty} b_k^{+} \cdot \frac{\Gamma(\frac{k+1}{2})}{\nu^{\frac{k+1}{2}}},$$

where a_k^{+} and b_k^{+} are given by (1. 5). It is easy to see that $a_{2k}^{+} = -b_{2k}^{+}$, $a_{2k+1}^{+} = b_{2k+1}^{+}$, therefore

$$I_C \sim e^{-\frac{\pi}{2}\nu} \exp\{i\nu\,(\alpha - \mathrm{th}\,\alpha)\} \cdot \sum_{k=0}^{\infty} a_{2k}^{+} \frac{\Gamma(\frac{2k+1}{2})}{\nu^{\frac{2k+1}{2}}} .$$

Similarly

$$I_{\overline{C}} \sim e^{-\frac{\pi}{2}\nu} \exp\{i\nu(\mathrm{th}\,\alpha - \alpha)\} \cdot \sum_{k=0}^{\infty} a_{2k}^{-} \frac{\Gamma(\frac{2k+1}{2})}{\nu^{\frac{2k+1}{2}}} ,$$

where a_{2k}^{-} can be calculated again by (1. 5).

Since

$$2K_{i\nu}(\frac{\nu}{\mathrm{ch}\,\alpha}) = I_L + \widetilde{K}_{i\nu}(\frac{\nu}{\mathrm{ch}\,\alpha}) ,$$

and

$$\widetilde{K}_{i\nu}(\frac{\nu}{\mathrm{ch}\,\alpha}) = I_C + I_{\overline{C}} \sim o\left(e^{-\frac{\pi}{2}\nu}\right), \qquad (1.\,14)$$

by the estimation of I_L we have

$$2K_{i\nu}(\frac{\nu}{\mathrm{ch}\,\alpha}) \sim I_C + I_{\overline{C}} .$$

The first coefficients are given by

$$a_0^{\pm} = \sqrt{2\mathrm{cth}\,\alpha} \cdot \exp(\mp i\frac{\pi}{4}) ;$$

the higher coefficients a_{2k}^{+}, a_{2k}^{-} can be obtained from (1. 10) by the substitution $a_0 = a_0^{+}$ and $a_0 = a_0^{-}$ respectively.

The expansion can be written in the following form:

$$K_{i\nu}(\nu \operatorname{sech} a) \sim$$

$$e^{-\frac{\pi}{2}\nu} \cdot \sqrt{\frac{2\pi}{\nu \operatorname{th} a}} \cdot \left\{ \cos\theta \sum_{k=0}^{\infty} (-)^k \frac{\Gamma(2k+\frac{1}{2})}{\Gamma(\frac{1}{2})} \left(\frac{2}{\operatorname{th} a}\right)^{2k} \frac{A_{2k}}{\nu^{2k}} + \right.$$

$$\left. \sin\theta \sum_{k=0}^{\infty} (-)^{k+1} \frac{\Gamma(2k+\frac{3}{2})}{\Gamma(\frac{1}{2})} \left(\frac{2}{\operatorname{th} a}\right)^{2k+1} \frac{A_{2k+1}}{\nu^{2k+1}} \right\},$$

where $\theta = \nu(a - \operatorname{th} a) - \frac{\pi}{4}$,

$$A_0 = 1 \;, \quad A_1 = \frac{1}{8} - \frac{5}{24} \coth^2 a \;,$$

$$A_2 = \frac{3}{128} - \frac{77}{576} \coth^2 a + \frac{385}{3456} \coth^4 a \;, \tag{1.15}$$

$$A_3 = \frac{5}{1024} - \frac{1521}{25600} \coth^2 a + \frac{17017}{138240} \coth^4 a - \frac{17017}{248832} \coth^6 a, \; \cdots .$$

It is clear that this series becomes useless as $\operatorname{th} a \to 0$ i.e. as $p \to 1$.

Thus there is a gap between the two Debye-type series in the neighborhood of $p = 1$. This gap can be filled partially by a third Debye-type series called "Ausnahmereihe" or B-type series. This expansion can be obtained again by the method of steepest descent with the assumption $|\nu - z| = o(z^{1/3})$.

1. 23 The asymptotic expansion of $K_{i\nu}(z)$ for $p \approx 1$ and $|\nu - z| = o(z^{1/3})$

We present here the "Ausnahmereihe" the verification of which is given in section 1. 4.

$$K_{i(z-\gamma)}(z) \sim \frac{1}{3} e^{-\frac{\pi}{2}\nu} \sum_{k=0}^{\infty} (-)^k \Gamma\left(\frac{k+1}{3}\right) \sin\frac{2\pi(k+1)}{3} B_k(-i\gamma) e^{-i\frac{\pi}{2}k} \left(\frac{6}{z}\right)^{\frac{k+1}{3}}$$

where $\operatorname{Re} z > 0$, $\gamma = z - \nu$, $|\nu - z| = o(z^{1/3})$ and

$$B_0(y) = 1, \quad B_1(y) = y \tag{1. 16}$$

$$B_2(y) = \frac{y^2}{2} - \frac{1}{10}, \quad B_3(y) = \frac{y^3}{6} - \frac{1}{15} y \ , \ \cdots .$$

For higher coefficients see [28, p. 301].

It is clear that the three Debye-type series do not give a satisfactory approximation over the positive real axis.

1. 3 An Approximate Formula Valid in the Transitional Region (Formula of the Nicholson-type)

We follow Nicholson's method which is based on Kelvin's principle.

We write

$$K_{i\nu}(x) e^{\frac{\pi}{2}\nu} = \int_0^{\infty} \cos(\nu t - x \operatorname{sh} t)\, dt .$$

The major contribution of the integral comes from the stationary points i. e. from the points where

$$\frac{d}{dt}(\nu t - x\,\mathrm{sh}\,t) = 0 .$$

If $\mathrm{ch}\,t \approx 1$ then $t \approx 0$, hence $\mathrm{sh}\,t \approx t + \frac{t^3}{6}$.

Therefore using (A. 3)

$$e^{\frac{\pi}{2}\nu} K_{i\nu}(x) \sim \int_0^\infty \cos\left[x\frac{t^3}{6} + (x-\nu)t\right] dt = \pi\left(\frac{2}{x}\right)^{1/3} \mathrm{Ai}\left(-\sqrt[3]{\frac{2}{x}}\cdot(\nu - x)\right) \tag{1. 17}$$

A rigorous justification of this formula is given in the following section where we obtain (1. 17) as the leading term of an expansion called series of the Nicholson type.

1. 4 The Nicholson Series for $K_{i\nu}(z)$

We intend to use Schöbe's results [28] to obtain the asymptotic expansion for $K_{i\nu}(z)$, $\mathrm{Re}\, z > 0$, when $|z - \nu| = O(z^{1/3})$.

Schöbe's method is based on the method of steepest descent and on the basic ideas involved in the proof of Watson's lemma; the starting point of the investigations is the Sommerfeld representation of $H_\nu^{(2)}(z)$, i. e.

$$H_\nu^{(2)}(z) = \frac{i}{\pi}\int_C \exp(-z\,\mathrm{sh}\,u + \nu u)\, du , \tag{1. 18}$$

where C is the path from $-\infty + i(\pi - \sigma)$ to $+\infty + i\sigma$ with $-2\pi < \arg z < \pi$ and

$$|\sigma + \arg \mathbf{z}| < \frac{\pi}{2} . \tag{1.19}$$

Following the principle of the method of steepest descent the path of integration will be led through the stationary point with constant phase and using the substitution suggested by the method of steepest descent we define

$$h(a) - h(t) = \tau^3 ,$$

corresponding to the fact that $h(t) = it + \mathrm{ch}\, t$ has a zero of order three at the stationary point $(0, -\frac{\pi}{2})$.

Consider

$$2K_{i\nu}(z) = \int_{-\infty}^{\infty} \exp\,(-z\,\mathrm{ch}\,t - i\nu t)\,dt , \qquad \mathrm{Re}\,\mathbf{z} > 0. \tag{1.20}$$

Substitute $t = u - i\frac{\pi}{2}$ and $\nu = z - \gamma$, then

$$2K_{i(z-\gamma)}(z) = e^{-\nu\frac{\pi}{2}} \int_{-\infty + i\frac{\pi}{2}}^{+\infty + i\frac{\pi}{2}} \exp[\, iz(\mathrm{sh}\,u - u) + i\gamma u]\; du.$$

It is easy to see that the path of integration of this integral can be deformed into the path C passing through the stationary point $(0, 0)$ as indicated in Figure 1.5.

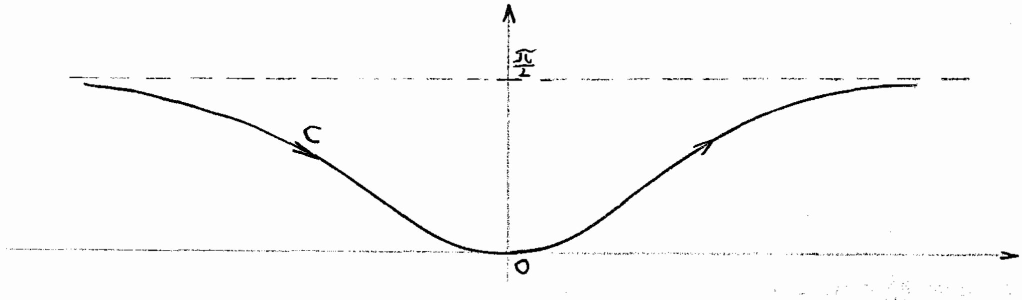

Figure 1.5 u-plane.

Hence

$$2K_{i(z-\gamma)}(z) = e^{-\nu\frac{\pi}{2}} \int_C \exp[iz(\operatorname{sh} u - u) + i\gamma u]\, du \,. \qquad (1.21)$$

This integral is the same as (1.18) since it satisfies the requirement (1.19) with $\sigma = \frac{\pi}{2}$ and $|\arg z| < \frac{\pi}{2}$. Note that in (1.21) the integral represents $\frac{\pi}{i} H^{(2)}_{i(\gamma-z)}(-iz)$.

Using (1.21) it is possible to get asymptotic expansions of $K_{i\nu}(z)$ from the asymptotic expansions of $H^{(2)}_{i(\gamma-z)}(-iz)$. In particular we obtain Debye's "Ausnahmereihe" from the corresponding series of $H^{(2)}_{i(\gamma-z)}(-iz)$ derived by the method of steepest descent as it is given e.g. in [32] or we can use Schöbe's result which gives a recurrence relation for the polynomial $B_k(a)$ which are of fundamental importance in computing the terms of the asymptotic series. We note that the Debye "Ausnahmereihe" and the Nicholson series of $H^{(2)}_{\nu}(z)$ derived in [28] hold for $-2\pi < \arg z < \pi$ and for any complex ν. Hence using (1.21) and the series given in [28, p. 272] or [32, p. 247] we can easily verify (1.16).

To obtain the Nicholson series for $K_{i\nu}(z)$ we apply (1.21) and (37) in [28]. We can express the Nicholson series in terms of $K_{1/3}(w)$ and $K_{-2/3}(w)$ or in terms of Airy functions $Ai(w)$. Since the Airy functions extensively tabulated (e.g. British Association Mathematical Tables Part-Volume B, The Airy Integral, Cambridge 1946) we present the expansion in terms of $Ai(w)$.

Thus after some reductions we obtain the Nicholson series for $K_{i\nu}(z)$ with $\operatorname{Re} z > 0$, and $z - \nu = O(z^{1/3})$:

$$K_{i\nu}(z) \sim$$

$$\sim \pi e^{-\frac{\pi}{2}\nu}\left(\frac{2}{z}\right)^{1/3}\left\{\sum_{k=0}^{m}(-)^k p_k(\xi)\left(-\frac{iz}{2}\right)^{-\frac{2k}{3}}\right\} Ai\left(\rho e^{-\frac{2\pi i}{3}}\right) + \\ + O\left(z^{-\frac{2m+2}{3}}\right), \tag{1.22}$$

where

$$\xi = i\gamma\left(-\frac{iz}{2}\right)^{-1/3}$$

$$\rho = \sum_{r=0}^{m}(-)^r q_r(\xi)\left(-\frac{iz}{2}\right)^{-\frac{2r}{3}},$$

and $p_k(\xi)$ and $q_r(\xi)$ are polynomials of ξ:

$$p_0(\xi) = 1 \quad , \qquad q_0(\xi) = \xi \quad ,$$
$$p_1(\xi) = \frac{1}{15}\xi \quad , \qquad q_1(\xi) = \frac{1}{60}\xi^2 \quad ,$$
$$p_2(\xi) = \frac{13}{1260}\xi^2 \quad , \qquad q_2(\xi) = \frac{2}{1575}\xi^3 + \frac{1}{140} \quad ,$$
$$\cdots \qquad \cdots \quad ,$$

for higher polynomials see page 290 in [28].

The first term of this series gives the asymptotic formula of the Nicholson type (1. 17) which was derived by the principle of stationary phase.

1. 5 Asymptotic Expansion for the Zeros of $K_{i\nu}(z)$ in the Transitional Region

Based on the asymptotic series of $K_{i\nu}(z)$ we can derive an asymptotic expansion for its zeros.

We show that the zeros of $K_{i\nu}(z)$ in the transitional region lie asymptotically on the real axis and we present an expansion for them.

We have seen that the leading term of the Nicholson series is (1. 17). Using this formula we show that any zero of $K_{i\nu}(z)$ in the region $|z - \nu| = O(\nu^{1/3})$ must correspond to a zero of $Ai\left((z-\nu)(\frac{2}{z})^{1/3}\right)$. Let k_ν be a zero of $K_{i\nu}(z)$, i. e. $K_{i\nu}(k_\nu) = 0$, and let $k_\nu = f(\nu)$ be a continuous function of ν. (It follows from the integral representation that such a function exists.) If $\nu \to \infty$ so that $|k_\nu - \nu| = O(\nu^{1/3})$, then it follows by (1. 17) that $Ai(q_\nu) \to 0$ since

$q_\nu \equiv (k_\nu - \nu)(\frac{2}{k_\nu})^{1/3}$ is bounded. It is known [24, 32] that Ai(z) has only real and negative zeros. Hence

$$(k_\nu - \nu)(\frac{2}{k_\nu})^{1/3} = [\,|k_\nu| \exp(\arg k_\nu) - \nu]\,(\frac{2}{|k_\nu|})^{1/3} \exp(-\frac{i}{3}\arg k_\nu)$$

must be a negative real number as $n \to \infty$, i.e. $\arg k_\nu \to 0$ as $n \to \infty$.

By the reversion of the Nicholson series of the function $H_\nu^{(2)}(z)$ an asymptotic expansion is derived in [28] for its zeros. The s-th zero $h_{\nu,s}$ of $H_\nu^{(2)}(z)$ is given by

$$h_{\nu,s} \sim \nu - \sum_{k=0}^{\infty} s_k(a_s)(\frac{\nu}{2})^{-\frac{2k-1}{3}}$$

with $s_0(a) = a$, $s_1(a) = -\frac{3a^2}{20}$, $s_2(a) = -\frac{a^3+10}{1400}$, ...,

where $a = a_s$ is the s-th zero of

$$\mathrm{Ai}\left(-\gamma(\frac{z}{2})^{-1/3} \exp(-\frac{2\pi i}{3})\right) \; ;$$

for higher coefficients see [28, p. 295].

Using the expansion above and (1.21) we obtain after some reduction the asymptotic expansion for the s-th zero of $K_{i\nu}(z)$ in $|z - \nu| = O(\nu^{1/3})$ for $\mathrm{Re}\, z > 0$.

$$k_{\nu,s} \sim \nu + \sum_{r=0}^{\infty} (-)^r s_r(a_s)\left(\frac{\nu}{2}\right)^{-\frac{2r-1}{3}} , \qquad (1.23)$$

where a_s is the s-th zero of $\mathrm{Ai}\left(\gamma\left(\frac{z}{2}\right)^{-1/3}\right)$, $\gamma = z-\nu$; and the coefficients s_r are the same as above.

1.6 A Theorem on the General Behavior of $K_{i\nu}(\frac{\nu}{p})$, $0 < p$

The asymptotic expansions obtained in the previous sections give at best approximate representations. In this section we are going to give additional information about $K_{i\nu}(\frac{\nu}{p})$ based on the integral representation of the function for any positive value of x and ν.

Theorem. $K_{i\nu}(\frac{\nu}{p})$ is a positive, monotone decreasing convex function of ν for any p such that $0 < p \leq 1$. For $p > 1$ it is an oscillating function of ν and it has countably infinite number of zeros.

Proof. Using Cauchy's theorem

$$2K_{i\nu}\left(\frac{\nu}{p}\right) = \int_{-\infty}^{\infty} \exp\{-\frac{\nu}{p}\operatorname{ch} t + i\nu t\}dt =$$

$$= \exp\{\nu h(a)\} \int_C \exp\{-\nu[h(a)-h(t)]\}dt ,$$

where $h(t) = it - \frac{1}{p}\operatorname{ch} t$, $a = i\beta = i \cdot \arcsin p$ is the stationary point (i.e. $h'(a) = 0$), and C is the curve along which

$\mathrm{Im}\,h(t) = \mathrm{Im}\,h(a)$. Along C the integrand is positive hence $K_{i\nu}(\frac{\nu}{p}) > 0$ for $0 < p \leq 1$.

It is easy to see that

$$2\frac{d}{d\nu}K_{i\nu}(\frac{\nu}{p}) = \int_{-\infty}^{\infty} h(t)\exp\{\nu h(t)\}dt =$$

$$= \exp\{\nu h(a)\}\int_C \exp\{-\nu[h(a)-h(t)]\}h(t)dt.$$

Let $t = \xi + i\eta$ and $\eta = f(\xi)$ be the equation of the curve C: $\sin\eta = \sin\beta\,\frac{\xi}{\mathrm{sh}\,\xi}$. Clearly $f(\xi) = f(-\xi)$.

It is easy to see that $h(\xi + i\eta) = \bar{h}(-\xi + i\eta)$, where $\bar{h}$ denotes the complex conjugate of h. Along C $dt = [1 + if'(\xi)]\,d\xi$ and $\mathrm{Im}\,h(t) = 0$. Decomposing C into the curves C_1 and C_2 which go from a to $+\infty$ and from a to $-\infty$ respectively, we can write:

$$\int_C h(t)\exp\{-\nu[h(a)-h(t)]\}dt =$$

$$= \left\{\int_{C_1} - \int_{C_2}\right\} h(\xi + if(\xi))\exp\{-\nu[h(a)-h(\xi+if(\xi))]\}\cdot[1+if'(\xi)]\,d\xi =$$

$$= 2\int_0^{\infty} h(\xi + if(\xi))\exp\{-\nu[h(a)-h(\xi+if(\xi))]\}d\xi < 0,$$

since $h(\xi + if(\xi)) \leq 0$ and $f'(\xi) < 0$ for $\xi > 0$.

By the same procedure :

$$2\frac{d^2}{d\nu^2}K_{i\nu}(\frac{\nu}{p}) =$$

$$= \int_{-\infty}^{+\infty} h^2(t)\exp\{\nu h(t)\}dt =$$

$$= \exp\{\nu h(a)\}\cdot\left\{\int_{C_1} - \int_{C_2}\right\} h^2(t)\exp\{-\nu[h(a)-h(t)]\}dt =$$

$$= 2\exp\{\nu h(a)\}\cdot\int_0^{\infty} h^2(\xi+if(\xi))\exp\{-\nu[h(a)-h(\xi+if(\xi))]\}d\xi > 0 .$$

For $p > 1$ replace the path of integration along the real line by the curve $K = C \cup \overline{C} \cup L$ indicated in Figure 1.4. Writing the equation of C in the form $\eta = f(\xi)$ we obtain:

$$2K_{i\nu}(\frac{\nu}{p}) = 2\cos\nu(\alpha-\operatorname{th}\alpha)\int_{\omega}^{\infty}\exp\{-\nu[h(a)-h(\xi+if(\xi))]\}d\xi -$$

$$- 2\sin\nu(\alpha-\operatorname{th}\alpha)\int_{\omega}^{\infty} f'(\xi)\exp\{-\nu[h(a)-h(\xi+if(\xi))]\}d\xi + E,$$

where $E = \int_{-\omega+i\pi}^{\omega+i\pi}\exp[\nu h(t)]\,dt$ and $f'(\xi) < 0$ for $\xi > 0$.

By the estimations given by (1.13) and (1.14) it follows that the extrema of the sum of the first two terms is greater than the corresponding value of E at any such value of ν where the sum of the first two terms attains its relative extrema. Hence $K_{i\nu}(\frac{\nu}{p})$ has infinite numbers of zeros. The number of zeros is countable since $K_{i\nu}(\frac{\nu}{p})$ is an analytic function of ν for $\mathrm{Re}\,\nu > 0$, and by a known theorem the zeros of an analytic function are isolated points. q. e. d.

We note that the fact that $K_{i\nu}(\frac{\nu}{p})$ has an infinite number of real zeros was first proved by Pólya [27] derived from properties of the roots of polynomials. It is clear that the same thoerem can be stated for $K_{i\nu x}(x)$ for $0 < \nu \leq 1$ and $\nu > 1$. (Substitute $\frac{\nu}{x}$ by x).

1.7 Debye-type Expansions of the First and Second Kind for the Derivative

Using the same procedure as in section 1.21 and 1.22 we can obtain the asymptotic expansions of $K'_{i\nu}(x) = \frac{d}{dx}K_{i\nu}(x)$ for $p < 1$ and $p > 1$.

It is easy to see that

$$K'_{i\nu}(x) = K'_{i\nu}(\frac{\nu}{p}) = -\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{ch}\,t \cdot \exp\{\nu \cdot [it - \frac{\mathrm{ch}\,t}{p}]\}\,dt \, .$$

The asymptotic expansion for $p < 1$

$$K'_{i\nu}(\frac{\nu}{p}) \sim -\frac{1}{2}\exp\{\nu(\beta - \cot\beta)\} \cdot \sum_{0}^{\infty} C_{2k} \frac{\Gamma(\frac{2k+1}{2})}{\nu^{\frac{2k+1}{2}}} \, ,$$

where

$$C_{2k} = \frac{1}{(2k)!} \frac{d^{2k}}{dt^{2k}} \left\{ \operatorname{ch} t \left[\frac{\operatorname{ch} a}{2!\, p} + \frac{\operatorname{sh} a}{3!\, p}(t-a) + \cdots \right]^{-k-\frac{1}{2}} \right\} \Bigg|_{t=a} .$$

The first three coefficients are

$$C_0 = \frac{2p}{a_0}, \quad C_2 = \frac{C_0}{2} a_0^2 \left[\frac{3}{4} + \frac{7}{48} a_0^4 \right],$$

$$C_4 = -\frac{C_0}{384} a_0^4 \left[15 + \frac{33}{2} a_0^4 + \frac{455}{144} a_0^8 \right],$$

where $a_0 = \sqrt{\dfrac{2p}{\operatorname{ch} a}}$.

We put this series into the following form:

$$K'_{i\nu}(\nu \operatorname{cosec} \beta) \sim$$

$$\sqrt{\frac{\pi}{2\nu \tan\beta}} \sin\beta\, e^{-\nu(\beta + \cot\beta)} \sum_0^\infty (-)^{k+1} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})} \left(\frac{2}{\cot\beta} \right)^k \frac{E_k}{\nu^k},$$

$$E_0 = 1, \quad E_1 = -\left(\frac{3}{8} + \frac{7}{24} \tan^2\beta \right) \tag{1.24}$$

$$E_2 = \frac{5}{128} + \frac{11}{64} \tan^2\beta + \frac{455}{3456} \tan^4\beta, \cdots .$$

The series for $p > 1$:

$$K'_{i\nu}(\nu \operatorname{sech} a) \sim$$

$$\sqrt{\frac{2\pi}{\nu \coth a}} \operatorname{ch} a \cdot \left\{ \sin\theta \sum_{0}^{\infty} (-)^k \frac{\Gamma(2k+\frac{1}{2})}{\Gamma(\frac{1}{2})} \left(\frac{2}{\operatorname{th} a}\right)^{2k} \frac{E_{2k}}{\nu^{2k}} + \right.$$

$$\left. + \cos\theta \sum_{0}^{\infty} (-)^{k+1} \frac{\Gamma(2k+\frac{3}{2})}{\Gamma(\frac{1}{2})} \left(\frac{2}{\operatorname{th} a}\right)^{2k+1} \frac{E_{2k+1}}{\nu^{2k+1}} \right\},$$

where $\theta = \nu(a - \operatorname{th} a) - \frac{\pi}{4}$,

$$E_0 = 1, \quad E_1 = \frac{3}{8} - \frac{7}{24} \coth^2 a, \tag{1.25}$$

$$E_2 = -\frac{5}{128} + \frac{11}{64} \coth^2 a - \frac{455}{3456} \coth^4 a, \dots .$$

Obviously these expansions fail to give approximations if $p \to 1$.

PART II. RESULTS OBTAINED FROM THE DIFFERENTIAL EQUATION

2.1 Preliminaries

In Part I we derived asymptotic expansions for $K_{i\nu}(\nu x)$ from the Sommerfeld integral representation for $x < 1$ and $x > 1$ when ν is large. We have seen that these series are useless near the transition point, and even if we use the Debye "Ausnahmereihe", the neighborhood of $x = 1$ is not satisfactorily covered. By the construction of the Nicholson series this gap was filled. But we have to use three different series to cover the entire real x-axis. We wish to have an asymptotic series which is uniform to an arbitrary high order in $1/\nu$ in the range $|\arg z| \leq \frac{\pi}{2}$, $z \neq 0$, of the complex argument z which includes the transitional region and is independent of ν. Such an asymptotic series should yield of course the Hankel, Debye and Nicholson type series as special cases.

To get a uniform representation for Bessel functions their differential-equations are used, not the Sommerfeld integral representations. The basic idea of this approach is that the solutions of approximately equal differential-equations are approximately equal.

There are many papers published on the general theory of asymptotic solutions of differential equations with transition point.

The most complete theory which yields a uniform asymptotic expansion is published by Olver [23, 25].

Olver applied his theory to the different Bessel functions, but his expansions cannot be used to obtain a uniform asymptotic expansion for $K_{i\nu}(z)$.

In the sequel we will show that Olver's theory can be used to derive a uniform asymptotic expansion for $K_{i\nu}(z)$ also, and we will construct such an expansion.

First we give a short resumé of the part of the theory which we want to apply.

Consider the differential equation

$$\frac{d^2 w}{dz^2} = [up(z) + q(z)]\, w \tag{2.1}$$

for large values of the parameter u, z is a complex variable lying in a certain simply connected domain D and $p(z)$, $q(z)$ are analytic functions of z and are independent of u. A point z_o is called transition point of (2.1) if $p(z_o) = 0$ or $p(z)$ or $q(z)$ has singularity at $z = z_o$. There are different cases considered, they are referred to as cases A, B, C and D. They occur respectively, when equation (2.1) has in D

A) no transition point

B) one transition point, a simple zero of $p(z)$

C) one transition point, a double pole of $p(z)$

D) one transition point, a simple pole of $p(z)$ and a double pole of $q(z)$.

As it turns out, for $K_{i\nu}(z)$ case B applies.

Next we give some additional properties of the Airy functions, which will be used in the sequel.

The functions defined by

$$P_1(z) = \mathrm{Ai}(z), \quad P_2(z) = \mathrm{Ai}(ze^{\frac{2}{3}i\pi}), \quad P_3(z) = \mathrm{Ai}(ze^{-\frac{2}{3}i\pi}), \tag{2. 2}$$

are also solutions of the differential equation of the Airy functions. Denoting the sectors

$$|\arg z| < \frac{\pi}{3}, \quad -\pi < \arg z < -\frac{\pi}{3}, \quad \frac{\pi}{3} < \arg z < \pi \tag{2. 3}$$

by S_1, S_2, and S_3 respectively, it follows from the asymptotic formula for the Airy function (A. 5), that $P_j(z)$ is exponentially small in S_j, $(j = 1, 2, 3)$, when $|z|$ is large.

To get the so called standard form of (2, 1) we introduce new independent variable ζ and dependent variable W with

$$W = \dot{z}^{-\frac{1}{2}} \cdot w, \quad \dot{z} \equiv \frac{dz}{d\zeta} , \tag{2. 4}$$

then (2.1) reduces to:

$$\frac{d^2W}{d\zeta^2} = \{u\dot{z}^2 p(z) + f(\zeta)\}W, \tag{2.5}$$

where

$$f(\zeta) \equiv \dot{z}^2 q(z) + \dot{z}^{\frac{1}{2}} \frac{d^2}{d\zeta^2}(\dot{z}^{-\frac{1}{2}}) = \dot{z}^2 q(z) + \frac{3\ddot{z}^2 - 2\dot{z}\dddot{z}}{4\dot{z}^2} \tag{2.6}$$

If the relation between z and ζ is given by $\dot{z}^2 p(z) = \zeta$, then equation (2.5) becomes

$$\frac{d^2W}{d\zeta^2} = [u\zeta + f(\zeta)]\, W \tag{2.7}$$

It follows that

$$\zeta = \left\{ \frac{3}{2} \int_{z_o}^{z} [p(z)]^{\frac{1}{2}} dz \right\}^{2/3}, \tag{2.8}$$

therefore near $z = z_o$ we have

$$\zeta = \text{const}\cdot (z - z_o) + (z - z_o)^2 O(1) . \tag{2.9}$$

If we let $f(\zeta) \equiv 0$, then we have $W = Ai(\zeta)$ a solution of (2.7). A possible solution for (2.7) is if $f(\zeta) \not\equiv 0$ the series

$$W(\zeta) = P(\zeta)\cdot\left\{ 1 + \sum_{s=1}^{\infty} \frac{A_s(\zeta)}{u^s} \right\} + \frac{P'(\zeta)}{u}\cdot \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{u^s}, \tag{2.10}$$

where the coefficients $A_s(\zeta)$ and $B_s(\zeta)$ are independent of u.

Term by term differentiation gives

$$W'(\zeta) = P(\zeta)\sum_{s=0}^{\infty}\frac{C_s(\zeta)}{u^s} + P'(\zeta)\left\{1+\sum_{s=1}^{\infty}\frac{D_s(\zeta)}{u^s}\right\} \tag{2.11}$$

and

$$W''(\zeta) = uP(\zeta)\sum_{j=0}^{\infty}\frac{E_s(\zeta)}{u^s} + P'(\zeta)\sum_{s=0}^{\infty}\frac{F_s(\zeta)}{u^s}\ , \tag{2.12}$$

where $$C_s = A'_s + gB_s,\quad D_s = A_s + B'_{s-1}\ , \tag{2.13}$$

and $$E_s = C'_{s-1} + gD_s = A''_{s-1} + gA_s + 2gB'_{s-1} + g'B_{s-1}\ ,$$

(2.14)

$$F_s = C_s + D'_s = 2A'_s + B''_{s-1} + gB_s,$$

primes denoting differentiation with respect to z.

Substituting (2.10) and (2.12) in (2.7) it follows that the differential equation is formally satisfied if

$$E_{s+1} = gA_{s+1} + fA_s,\quad F_{s+1} = gB_{s+1} + fB_s\ .$$

Substituting E_{s+1} and F_{s+1} from (2.14), we get

$$A''_s - fA_s + 2gB'_{s*} + g'B_s = 0,\ 2A'_{s+1} + B''_s - fB_s = 0\ .$$

Integrating these equations we obtain

$$B_s = \frac{1}{2}g^{-\frac{1}{2}}\int g^{-\frac{1}{2}}(fA_s - A''_s)dz,\quad A_{s+1} = -\frac{1}{2}B'_s + \frac{1}{2}\int fB_s dz\ . \tag{2.15}$$

From (2.15) follows that if we let $A_0 = 1$, then we can determine A_s and B_s recursively apart from a constant of integration.

This formal process would yield a solution of (2.7) if the series (2.10) converged uniformly, with respect to ζ; in this case the term by term differentiation would be justified. However in general this series is divergent and could represent only an asymptotic solution of the differential equation under certain conditions as it is given in Olver's theorem B [23, 25].

It is assumed in this theorem that $f(\zeta)$ is a regular function in a simply connected open domain D the boundaries of which consist of a finite number of straight lines, if D is unbounded then as $|\zeta| \to \infty$ in D

$$f(\zeta) = O(|\zeta|^{-\frac{1}{2}-\alpha})$$

uniformly in $\arg \zeta$, here α is a positive constant.

The following notations and definitions are used:
Any simply connected domain lying wholly in D, the boundaries of which consist of a finite number of straight lines not intersecting the boundaries of D will be denoted by D'.
Furthermore it is supposed, that $\zeta = 0$ is an interior point of D'.
A curve along which $F(\zeta) = \text{constant}$ is called the level curve of the function $F(\zeta)$. In Figure (2.1) some level curves of $\exp\{-\frac{2}{3}\zeta^{3/2}\}$ are given, the values of $|\exp\{-\frac{2}{3}\zeta^{3/2}\}|$ are indicated on the curves.

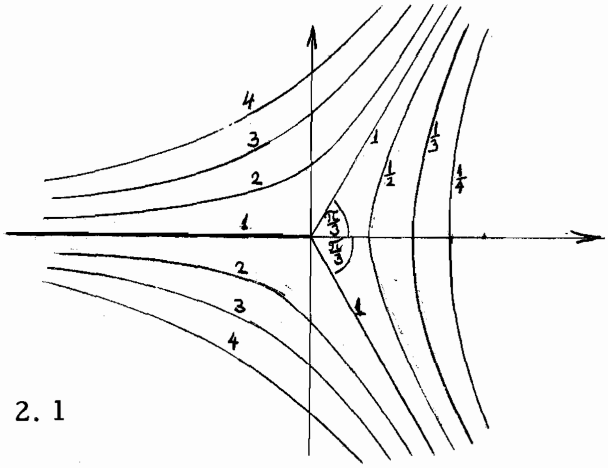

Figure 2. 1

The equation of the level curves of $\exp(-\frac{2}{3} z^{3/2})$ is given by

$$r^{3/2} \cos \frac{3}{2}\theta = \text{constant}, \quad \text{where } z \equiv re^{i\theta}. \qquad (2.16)$$

We denote an arbitrary fixed point of the region common to the domain D' and the sector S_j by a_j, $(j=1, 2, 3)$. If this common region is unbounded then a_j can be the point at infinity on a line lying in it.

D_j is the domain comprising those points z of D' for which

$$|\exp\{-\frac{2}{3}(\rho_j z)^{3/2}\}| \geq |\exp\{-\frac{2}{3}(\rho_j a_j)^{3/2}\}|, \quad \text{where } \rho_1 \equiv 1, \rho_2 \equiv e^{\frac{2}{3}i\pi}, \rho_3 \equiv e^{-\frac{2}{3}i\pi}.$$

and a contour can be found joining z and a_j which lies in D' and at most two of the sectors S_1, S_2, S_3, and does not cross the level curve of $\exp\{-\frac{2}{3}(\rho_j z)^{3/2}\}$ through z.

Furthermore $u^{1/3}$ is denoted by v.

<u>Theorem B.</u> Let sequences of functions $A_s(z)$, $B_s(z)$, $C_s(z)$ and $D_s(z)$ be defined by the relations $A_0(z) = 1$,

$$\left.\begin{aligned} B_s(z) &= \tfrac{1}{2} z^{-\frac{1}{2}} \int_0^z t^{-\frac{1}{2}} \{f(t)A_s(t) - A''_s(t)\} dt , \\ A_{s+1}(z) &= -\tfrac{1}{2} B'_s(z) + \tfrac{1}{2} \int f(z) B_s(z) dz + \text{constant} , \end{aligned}\right\} \quad (2.17)$$

$$C_s(z) = A'_s(z) + zB_s(z), \quad D_s(z) = A_s(z) + B'_{s-1}(z), \quad (2.18)$$

the arbitrary constants being subject to the condition that there exist a function $\phi(u)$ having the asymptotic expansion

$$\phi(u) \sim 1 + \sum_{s=1}^{\infty} \frac{A_s(c)}{u^s}$$

as $u \to \infty$, for a fixed point c in D'.

Then the equation

$$\frac{d^2 w}{dz^2} = \{uz + f(z)\} w \quad (2.19)$$

has solutions $W_j(z)$ $(j=1,2,3)$ such that if z lies in D_j

$$W_j(z) = P_j(vz) \left\{ 1 + \sum_{s=1}^{m} \frac{A_s(z)}{u^s} \right\} + \frac{P'_j(vz)}{v^2} \sum_{s=0}^{m-1} \frac{B_s(z)}{u^s} + \frac{\exp\{-\frac{2}{3}(\rho_j vz)^{3/2}\}}{1 + |vz|^{\frac{1}{4}}} O\left(\frac{1}{u^{m+\frac{1}{2}}}\right), \quad (2.20)$$

$$W_j'(z) = P_j(vz) \sum_{s=0}^{m-1} \frac{C_s(z)}{u^s} + vP_j'(vz) \left\{ 1 + \sum_{s=1}^{m} \frac{D_s(z)}{u^s} \right\} +$$

$$+ (1 + |vz|^{\frac{1}{4}}) \exp\{-\frac{2}{3}(\rho_j vz)^{3/2}\} O(\frac{1}{u^m}) , \qquad (2.21)$$

as $u \to \infty$, where the functions $P_j(z)$, $(j=1,2,3)$ are defined by (2.2) and the O's are uniform with respect to z. The solutions $W_j(z)$ are independent of m, m is an arbitrary unbounded integer.

2.2 Uniform Asymptotic Series for $K_{i\nu}(\nu z)$

The function $K_{i\nu}(z)$ satisfies the differential equation

$$z^2 w'' + zw' - (z^2 - \nu^2) w = 0. \qquad (2.22)$$

Elementary calculation shows that $w = \sqrt{z} \cdot K_{i\nu}(\nu z)$ is a solution of

$$\frac{d^2 w}{dz^2} = \left(\frac{z^2 - 1}{z^2} \nu^2 - \frac{1}{4z^2} \right) w . \qquad (2.23)$$

This equation has the form of (2.1) and has transition points at $z = 0$, $z = \pm 1$. Using Langer's transformation formulas (2.4; 2.8) given in the preceeding section we obtain with

$$\zeta\left(\frac{d\zeta}{dz}\right)^2 = \frac{z^2-1}{z^2} \tag{2.24}$$

$$W = \left(\frac{dz}{d\zeta}\right)^{-\frac{1}{2}} w \tag{2.25}$$

the following equation

$$\ddot{W} = \{\nu^2\zeta + f(\zeta)\}\, W , \quad (\dot{} \equiv \frac{d}{d\zeta}) , \tag{2.26}$$

where
$$f(\zeta) = \sqrt{\dot{z}}\ \frac{d^2}{d\zeta^2}\frac{1}{\sqrt{\dot{z}}} - \frac{1}{4}\frac{\dot{z}^2}{z^2} . \tag{2.27}$$

Integrating (2. 24) we have

$$\frac{2}{3}\zeta^{3/2} = \pm \int_1^z \frac{\sqrt{z^2-1}}{z}\, dz = \sqrt{z^2-1} - \operatorname{arc\,sec} z, \tag{2.28}$$

where we have taken the positive sign before the integral so that for real $z > 0$ real values of ζ should correspond.

This relation maps the z-plane into the ζ-plane; we denote this mapping by M. M can be decomposed into the mappings M_i, $i=1, 2, 3$, defined by:

$$M_1 : \ z \to \sigma , \quad \text{where} \quad \sigma = \operatorname{arc\,sec} z ,$$
$$M_2 : \ \sigma \to \rho , \quad \rho = \tan\sigma - \sigma ,$$
$$M_3 : \ \rho \to \zeta , \quad \zeta = \left(\frac{3}{2}\right)^{2/3} \rho^{2/3} .$$

The domain $0 < \arg z < \pi$ is mapped conformally by M_1 into the half strip S_1 of the σ-plane defined by $0 < \mathrm{Re}\,\sigma < \pi$, $\mathrm{Im}\,\sigma > 0$. M_2 maps conformally this half strip onto the region R_1 of the ρ-plane consisting of the half plane $\mathrm{Im}\,\rho > 0$ and of the half strip $-\pi < \mathrm{Re}\,\rho < 0$, $\mathrm{Im}\,\rho < 0$. M_3 maps conformally the region R_1 onto the domain Z_1 of the ζ-plane consisting of the sector $0 < \arg\zeta < \frac{2\pi}{3}$ and the curved strip bounded by the ray $\arg\zeta = \frac{2\pi}{3}$ by the real axis and by the curve whose parameteric equation is

$$\zeta = \left(\frac{3}{2}\right)^{2/3} (-\pi - it)^{2/3}, \quad 0 \leq t . \tag{2.29}$$

It follows by the reflection principle [31, p. 155] that the domains corresponding to the sector $-\pi < \arg z < 0$ can be obtained by reflecting the points of S_1, R_1, Z_1 with respect to the real axis. It is easy to see that the function $\zeta(z)$ describing M is a simple ("schlicht") function over $|\arg z| < \pi$, so the mapping M is 1-1. We can therefore consider z as a function of ζ which is again simple [31, p. 198] over the region Z bounded by the rays $\arg\zeta = \pm\frac{2\pi}{3}$, by the curve (2.29) and by its conjugate. Clearly $z = z(\zeta)$ is regular over Z.

The above described mappings are illustrated in Figures 2.2 - 2.5; the capitals with prime indicate the complex conjugate points.

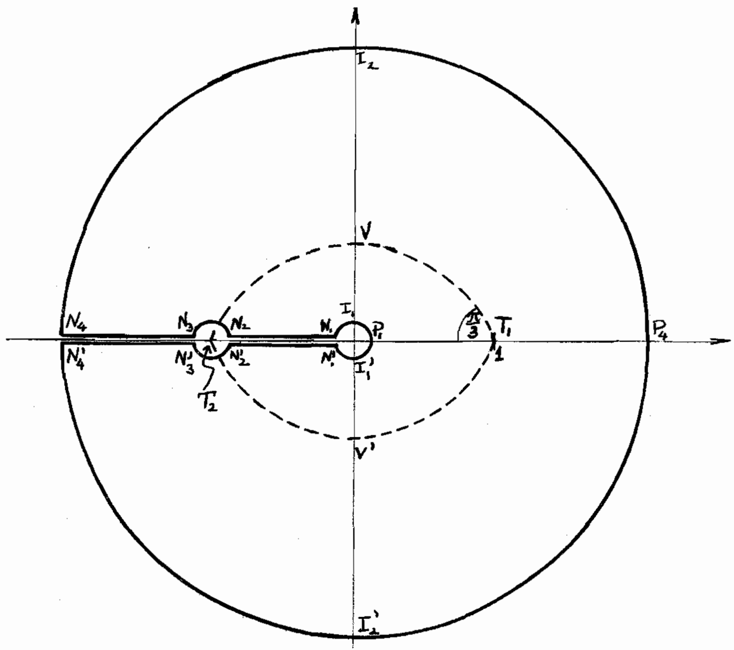

Figure 2. 2 z-plane.

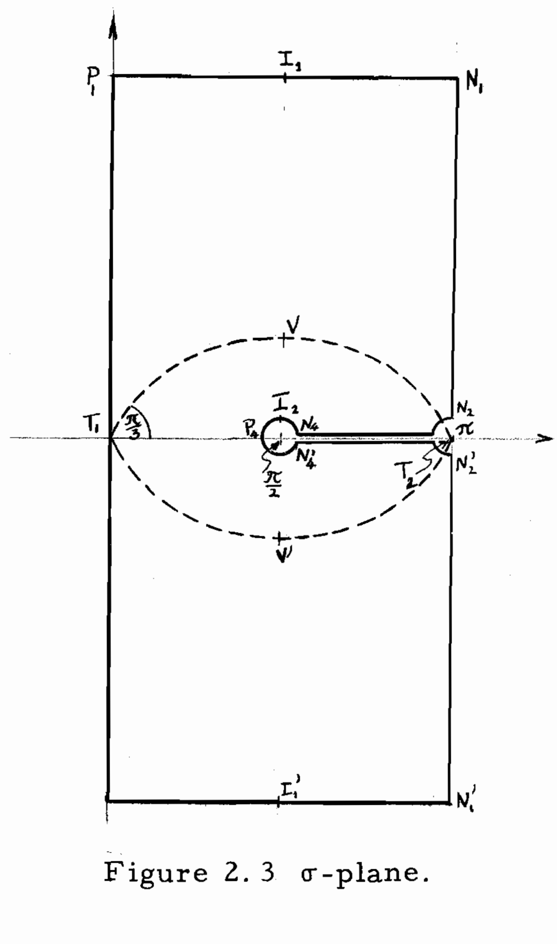

Figure 2. 3 σ-plane.

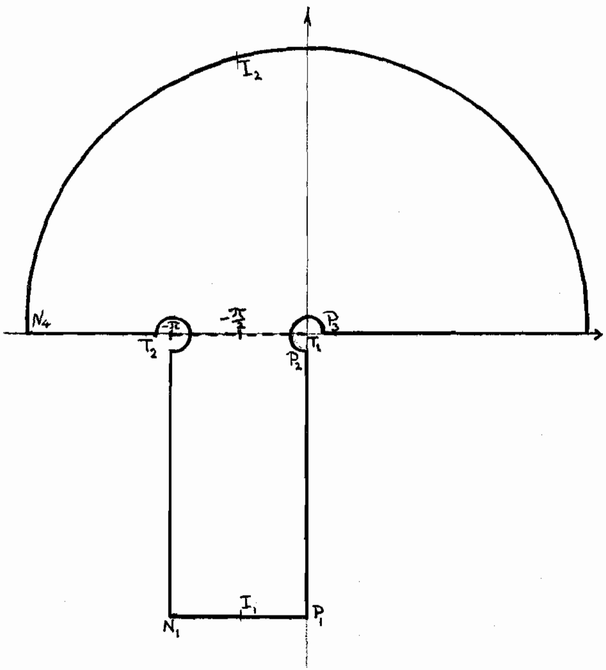

Figure 2.4a ρ-plane(Im z > 0).

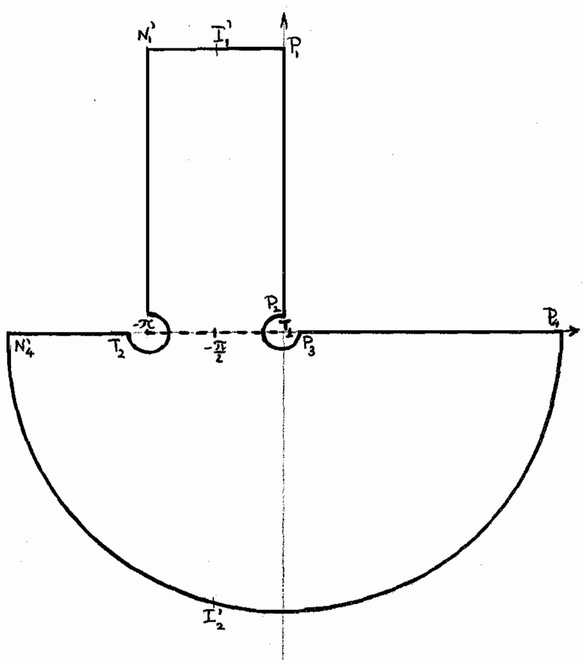

Figure 2.4b ρ-plane (Im z < 0).

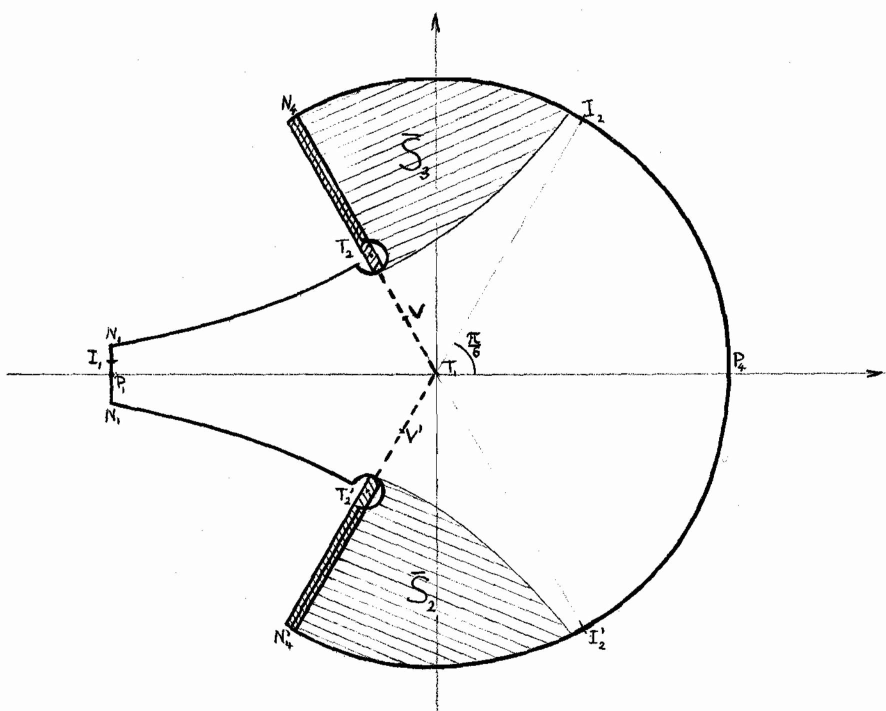

Figure 2.5 ζ-plane.

In order to apply Olver's theorem we want to extend the region $Z_1 \cup Z_2$ onto the entire ζ-plane, and investigate how the remainder of the ζ-plane is mapped on the z-plane. First we observe that the segments T_1T_2 and T_1T_2' respectively are mapped onto the curves in the σ-plane given by the parametric equations:

$$\sigma = +\frac{\pi}{2} \pm \alpha + it,$$

with $0 \leq t \equiv \operatorname{Im}\sigma \leq t_0$ and $-t_0 \leq t \leq 0$ respectively, where $\alpha = \arccos\{\operatorname{ch} t[1 - \frac{1}{t} \cdot \operatorname{th} t]^{\frac{1}{2}}\}$, and t_0 is the positive root of $t = \coth t$ and $t_0 \sim 1.1997$. Furthermore these curves are mapped on the curves of the z-plane, whose parametric equations are (see Appendix, p. 71)

$$z = \pm(t \cdot \coth t - t^2)^{\frac{1}{2}} + i \cdot \operatorname{sign} t \cdot (t^2 - t \cdot \operatorname{th} t)^{\frac{1}{2}}, \qquad (2.30)$$

where t is varying in the same intervals as above. These curves intersect the imaginary axis of the z-plane at V and V' respectively with coordinates $\pm i(t_0^2 - 1)^{\frac{1}{2}} = \pm i\,0.663$, and are indicated by the broken lines in the figures. The images of V and V' on the ζ-plane are $(\frac{3}{4}\pi)^{2/3} e^{\frac{2\pi i}{3}}$, and $(\frac{3}{4}\pi)^{2/3} e^{-\frac{2\pi i}{3}}$ respectively. The mapping M maps the eyeshaped domain K of the z-plane bounded by the curves (2.30) onto the curved strip of the ζ-plane bounded by the curves $\zeta = (\frac{3}{2})^{2/3}(-\pi \mp it)^{2/3}$ with

$t \geq 0$ and by the rays $\arg\zeta = \pm\frac{2}{3}\pi$; this area is indicated on Figure 2.6 by $T_1T_2N_1N_1'T_2'T_1$.

Next we observe that for $m = 0, 1, 2, \cdots$ $\rho(\sigma + m\pi) = \rho(\sigma) - m\pi$, therefore the strip $m\pi \leq \mathrm{Re}\,\sigma \leq (m+1)\pi$, $\mathrm{Im}\,\sigma \geq 0$ is mapped onto the curved domain of the ζ-plane bounded by the ray $\zeta = e^{\frac{2\pi i}{3}}$ and by the curves $\zeta = (\frac{3}{2})^{2/3}[-m\pi - it]^{2/3}$ and $\zeta = (\frac{3}{2})^{2/3} \cdot [-(m+1)\pi - it]^{2/3}$, $t \geq 0$. The image of this curved domain on the z-plane is the lower or upper half of the eyeshaped domain K according to m is odd or even, since $z(\sigma + m\pi) = (-1)^m z(\sigma)$. Now it is easy to see that the sector $|\arg(-\zeta)| < \frac{\pi}{3}$ is mapped onto the domain K of the z-plane. Conversely for any z in K such that in $m\pi < \arg z < (m+1)\pi$, $m = 0, \pm 1, \pm 2, \cdots$, ζ lies in the curved domain bounded by the curves $\zeta = (\frac{3}{2})^{2/3}(-it - m\pi)^{2/3}$, $\zeta = (\frac{3}{2})^{2/3}[-it - (m+1)\pi]^{2/3}$, $0 \leq t$, and the ray $\arg\zeta = -\frac{2\pi}{3}$ for $m < 0$. The transformation from the z-plane to the ζ-plane is not 1-1; it can be represented by

$$\frac{2}{3}\zeta^{3/2} = \sqrt{z^2 - 1} + i\{\log(1 + \sqrt{1 - z^2}) - \mathrm{Ln}\, z\}, \tag{2.31}$$

where $\mathrm{Ln}\, z$ is a many-valued function whose imaginary part is $i \arg z$. [We used here the formula $\mathrm{arcsec}\, z = -i \cdot \log \frac{1 + \sqrt{1 - z^2}}{z}$].

By cutting the ζ-plane along the rays $\arg\zeta = \pm\frac{2\pi}{3}$ from $\zeta = (\frac{3\pi}{2})^{2/3} \exp\{\pm\frac{2\pi i}{3}\}$ to infinity we define a region D on which it will be shown that $f(\zeta)$ satisfies the requirements of Theorem B. It is easy to see that $z(\zeta)$ is regular over D. Carrying out the differentiations in (2. 27) we have

$$f(\zeta) = \frac{3\ddot{z}^2 - 2\dot{z}\dddot{z}}{4\dot{z}^2} - \frac{1}{4}\cdot\frac{\dot{z}^2}{z^2}, \tag{2.32}$$

and substituting $\dot{z}$, $\ddot{z}$ and $\dddot{z}$ (which can be calculated from (2. 24)) we obtain

$$f(\zeta) = \frac{5}{16\zeta^2} - \frac{\zeta z^2(z^2+4)}{4(z^2-1)^3} . \tag{2.33}$$

We observe that the only point D at which $z^2 = 1$ is $\zeta = 0$. But in the neighborhood of this point the first four terms of the Taylor's series of $z(\zeta)$, which can be deduced from (2.28) and (2.24), are:

$$z(\zeta) = 1 + 2^{-1/3}\zeta + \frac{3}{10}2^{2/3}\zeta^2 - \frac{1}{700}\zeta^3 + \zeta^4 O(1) . \tag{2.34}$$

Using (2. 34) we get from (2. 32) that

$$f(\zeta) = \frac{2^{1/3}}{70} + \zeta O(1) .$$

So $f(\zeta)$ is regular over D.

Next we show that if $|\zeta| \to \infty$ in the sector $|\arg(-\zeta)| < \frac{\pi}{3}$ then $f(\zeta) = O\left(\left|\frac{1}{\zeta^2}\right|\right)$. From (2. 31) we have for $|z| < 1$ that

$$\frac{2}{3}\zeta^{3/2} \sim i[\log 2 + 1 + \mathrm{Ln}\, z^{-1}] ,$$

hence $z^{-1} \sim \mathrm{const}\cdot\exp\{\frac{2}{3}|\zeta|^{3/2} e^{i\,3/2\,(\arg\zeta+\pi)}\}$, so $|z| \to \infty$ implies $\cos\frac{3}{2}(\arg\zeta+\pi) > 0$, i.e. $|\arg\zeta+\pi| = |\arg(-\zeta)| < \frac{\pi}{3}$. Therefore choosing ζ in the sector above, $|\zeta| \to \infty$ implies $|z| \to 0$, and conversely. From this we obtain that

$$f(\zeta) = \frac{5}{16\zeta^2} - \frac{\zeta z^2(z^2+4)}{4(z^2-1)^3} \sim \frac{5}{16\zeta^2} \quad \text{for } |\zeta| \to \infty \text{ in}$$

$|\arg(-\zeta)| < \frac{\pi}{3}$.

Furthermore we deduce from (2. 28) that for large $|z|$, $|\arg z| < \pi$

$$\frac{2}{3}\zeta^{3/2} = z + \frac{\pi}{2} + O(|\frac{1}{z}|) , \tag{2.35}$$

therefore $|z|\, e^{i\arg z} \sim \frac{2}{3}|\zeta|^{3/2} e^{2/3\, i\arg\zeta}$. Hence $|\zeta| \to \infty$ in $|\arg\zeta| < \frac{2\pi}{3}$ as $|z| \to \infty$ in the sector $|\arg z| < \pi$, and conversely. By (2. 35) we have in these sectors that $z = -\frac{\pi}{2} + \frac{2}{3}\zeta^{3/2} + O(|\zeta|^{-3/2})$, thus $f(\zeta) = O(|\frac{1}{\zeta}|^2)$ as $|\zeta| \to \infty$ in $|\arg\zeta| < \frac{2\pi}{3}$.

Hence $f(\zeta)$ satisfies the premissae of Olver's theorem for all $\zeta \in D$.

We define the domain D' on the ζ-plane by $D' = D - \overline{D'}$, where $\overline{D'}$ is the union of the half strips

$$\mathrm{Re}\{\zeta \exp(\mp \tfrac{2\pi i}{3})\} > (\tfrac{3}{2}\pi)^{2/3} - \delta , \quad |\mathrm{Im}\{\zeta \exp(\mp \tfrac{2\pi i}{3})\}| < \delta ,$$

here δ is an arbitrary small positive number. In Figure 2.6, $\overline{D'}$ is indicated by the densely shaded strips.

We choose the point a_1 at infinity on the positive real axis, the points a_3 and a_2 respectively at infinity and on the lower respectively upper boundary of the half strips defined above.

From the definition (p. 42) of the domain D_j, $j=1,2,3$, we see that D_1 coincides with D' ; $D_3 = D' - \overline{S}_3$, where $\overline{S}_3$ is the domain in the ζ-plane bounded by the ray $\arg \zeta = \frac{2\pi i}{3}$ and by the level curve of $F(\zeta) \equiv \exp\{-\frac{2}{3}\zeta^{3/2}\}$ passing through the point $\zeta = [(\frac{3}{2}\pi)^{2/3} - \delta]\, e^{\frac{2\pi i}{3}}$. Similarly $D_2 = D' - \overline{S}_2$, where $\overline{S}_2$ is the conjugate of $\overline{S}_3$. In Figure 2.5, $\overline{S}_3$, $\overline{S}_2$ are indicated by the shaded regions.

By Olver's theorem there exist solutions W_j of (2.26) in D_j, $j=1,2,3$, such that as $\nu \to \infty$

$$\left.\begin{aligned} W_j &= P_j(\xi)\left\{1 + \sum_{j=1}^{m} \frac{A_s(\zeta)}{\nu^{2s}}\right\} + \frac{P_j'(\xi)}{\nu^{4/3}} \cdot \sum_{s=0}^{m-1} \frac{B_s(\zeta)}{\nu^{2s}} + \\ &\quad \frac{\exp\{-\frac{2}{3}(\rho_j \xi)^{3/2}\}}{1 + |\xi|^{\frac{1}{4}}} \cdot O\left(\frac{1}{\nu^{2m+1}}\right), \\ \xi &= \nu^{2/3}\zeta , \end{aligned}\right\} \quad (2.36)$$

uniformly with respect to ζ. The coefficients $A_s(\zeta)$, $B_s(\zeta)$ are given by (2.17).

From (2.24) and (2.25) we get

$$K_{i\nu}(\nu z) = \sqrt[4]{\frac{\zeta}{z^2-1}}\ W(\zeta)\ , \qquad (2.37)$$

where W is a linear combination of W_j, j=1, 2, 3, and $\zeta \in \bigcap_{j=1}^{3} D_j$.
It follows from the Hankel expansion (A.9) that $K_{i\nu}(\nu z)$ is exponentially decreasing for large fixed ν when z real and $z \to +\infty$. In this case $\zeta \in S_1$ and $\zeta \to +\infty$, hence $W = \text{const.} W_1(\zeta)$ must be for all sufficiently large ν; we call this constant C.

We can determine C if we compare the uniform asymptotic expansion with any one of the asymptotic representations of $K_{i\nu}(\nu z)$ for large ν. It is easy to see that $P_1(\xi) = A_i(\nu^{2/3}\zeta)$ has asymptotically the same argument as the Airy function in the Nicholson formula for $z \approx 1$. Motivated by this fact we use the Nicholson formula to determine C. Using (1.17) we obtain from (2.37)

$$\lim_{z\to 1} C \cdot \sqrt[4]{\frac{\zeta}{z^2-1}}\ P_1(\xi) = C\cdot\lim_{z\to 1} \sqrt[4]{\frac{\zeta}{z^2-1}}\ \mathrm{Ai}(\nu^{2/3}\zeta) =$$

$$= C\cdot\lim_{z\to 1} \sqrt[4]{\frac{1}{z+1}}\ \mathrm{Ai}(\nu^{-2/3}(1-z)2^{1/3}) = C\cdot \frac{\mathrm{Ai}(0)}{2^{1/6}}\ .$$

Letting $z \to 1$ in the Nicholson formula we get

$$\lim_{z\to 1} K_{i\nu}(\nu z) \sim \lim_{z\to 1} \pi e^{-\frac{\pi}{2}\nu} \left(\frac{2}{\nu z}\right)^{1/3} \mathrm{Ai}\left(-\sqrt[3]{\frac{2}{\nu z}}\,(1-z)\nu\right) = \pi e^{-\frac{\pi}{2}\nu} \left(\frac{2}{\nu}\right)^{1/3} \mathrm{Ai}(0).$$

Therefore $C = \dfrac{\pi\sqrt{2}}{\nu^{1/3}}\, e^{-\frac{\pi}{2}\nu}$.

Thus
$$K_{i\nu}(\nu z) = \frac{\sqrt{2}\pi}{\nu^{1/3}}\, e^{-\frac{\pi}{2}\nu} \cdot \sqrt[4]{\frac{\zeta}{z^2-1}}\, W_1(\zeta) \quad . \tag{2.38}$$

The coefficient $B_0(\zeta)$ can be calculated from the integral (2.17), it is given by:

$$B_0(\zeta) = \frac{1}{2\sqrt{\zeta}} \int_0^{\zeta} \frac{1}{\sqrt{t}}\, f(t)dt = -\frac{5}{48\zeta^2} + \frac{1}{\sqrt{\zeta}}\,\frac{v}{8}\left(1 + \frac{5v^2}{3}\right) , \tag{2.39}$$

where $v = \dfrac{1}{\sqrt{z^2-1}}$.

The direct calculation of the coefficients for $s \geq 1$ is not practicable. We will use a method first suggested by Cherry [4] and applied by Olver [24] to obtain the coefficients for $J_\nu(\nu z)$.

The uniformity of the expansion implies that the elementary asymptotic expansions of $K_{i\nu}(\nu z)$ must be obtained if we substitute the corresponding expansion of the Airy function into (2.38). We assume that ζ is positive and fixed, therefore $\nu^{2/3}\zeta \to \infty$ as $\nu \to \infty$. Using the asymptotic expansions of $\mathrm{Ai}(z)$ and $\mathrm{Ai}'(z)$ given in the Appendix (A.5; A.6), we have

$$\mathrm{Ai}(\nu^{2/3}\zeta) \sim \frac{\exp(-\frac{2}{3}\nu\zeta^{3/2})}{2\nu^{1/6}\zeta^{1/4}\sqrt{\pi}} \sum_{s=0}^{\infty} (-)^s \frac{a_s}{\nu^s \zeta^{3/2\, s}} \,, \tag{2.40}$$

$$\mathrm{Ai}'(\nu^{2/3}\zeta) \sim -\frac{\nu^{1/6}\zeta^{1/4}}{2\sqrt{\pi}} \exp(-\frac{2}{3}\nu\zeta^{3/2}) \sum_{s=0}^{\infty} (-)^s \frac{b_s}{\nu^s \zeta^{3/2\, s}} \,, \tag{2.41}$$

in which

$$a_0 = 1, \qquad a_s = \frac{(2s+1)(2s+3)\ldots(6s-1)}{s!\,(144)^s} \,,$$

$$b_0 = 1, \qquad b_s = -\frac{6s+1}{6s-1}\, a_s \,.$$

Substituting these series into (2.38) the obtained expansion

$$K_{i\nu}(\nu z) \sim$$

$$\sqrt{\frac{\pi}{2\nu}} \cdot \frac{1}{\sqrt[4]{z^2-1}} \cdot \exp\{-\frac{\nu\pi}{2} - \frac{2}{3}\nu\zeta^{3/2}\} \cdot \left\{ \sum_{s=0}^{\infty} (-)^s \frac{a_s}{\nu^s \zeta^{3/2 s}} \sum_{s=0}^{\infty} \frac{A_s}{\nu^{2s}} - \right.$$

$$\left. - \sum_{s=0}^{\infty} (-)^s \frac{b_s}{\nu^s \zeta^{3/2 s}} \sum_{s=0}^{\infty} \frac{\zeta^{\frac{1}{2}} B_s}{\nu^{2s+1}} \right\} \tag{2.42}$$

must be equivalent to the Debye series (1.11), which can be written in the following form:

$$K_{i\nu}(\nu z) \sim \sqrt{\frac{\pi}{2\nu}} \cdot \frac{1}{\sqrt[4]{z^2-1}} \cdot \exp\{-\frac{\nu\pi}{2} - \frac{2}{3}\nu\zeta^{3/2}\} \cdot \sum_{s=0}^{\infty} (-)^s \frac{U_s}{\nu^s} \,,$$

where

$$U_0 = 1, \qquad U_1 = \frac{1}{8}v\left[\frac{5}{3}v^2 + 1\right],$$

$$U_2 = \left(\frac{9}{128} + \frac{7}{192}v^2 + \frac{385}{1152}v^4\right)v^2,$$

$$U_3 = \left(\frac{75}{1024} + \frac{4563}{5120}v^2 + \frac{17017}{9216}v^4 + \frac{85085}{82944}v^6\right)v^3, \ldots .$$

Comparing this expansion and (2. 42) we deduce the asymptotic equality:

$$\sum_{s=0}^{\infty}(-)^s\frac{U_s}{\nu^s} \sim \sum_{s=0}^{\infty}(-)^s\frac{a_s}{\nu^s\zeta^{3/2 s}}\sum_{s=0}^{\infty}\frac{A_s(\zeta)}{\nu^{2s}} - \sum_{s=0}^{\infty}(-)^s\frac{b_s}{\nu^s\zeta^{3/2 s}}\sum_{s=0}^{\infty}\frac{\zeta^{\frac{1}{2}}B_s(\zeta)}{\nu^{2s+1}}. \tag{2.43}$$

By equating the coefficients of $\frac{1}{\nu}$ we get the same formula for $B_0(\zeta)$ as in (2. 39). The asymptotic equality (2. 43) is formally the same as a corresponding equality in Olver's paper [24, p. 341] . Consequently we may use his formulae (6. 6) for $A_s(\zeta)$ and $B_s(\zeta)$:

$$A_s(\zeta) = \sum_{m=0}^{2s}(-)^m b_m\zeta^{-3/2 m}U_{2s-m}, \quad \zeta^{\frac{1}{2}}B_s(\zeta) = \sum_{m=0}^{2s+1}(-)^m a_m\zeta^{-3/2 m}U_{2s-m+1}.$$

[These formulae can be proved by induction using the relation

$$a_{2s}b_0 - a_{2s-1}b_1 + a_{2s-2}b_2 - \cdots + a_0b_{2s} = 0, \quad (s > 0),$$

which can be obtained from the Wronskian relation of $Ai(z)$ and $Bi(z) = e^{\frac{\pi}{6}i}P_2(z) + e^{-\frac{\pi}{6}i}P_3(z)$.] Thus $A_s(\zeta)$ and $\sqrt{\zeta}B_s(\zeta)$

are polynomials in $\zeta^{-3/2}$ and in v. The validity of these results for all values of $\zeta \in D$ follows by the principle of analytic continuation.

Thus the uniform asymptotic expansion of $K_{i\nu}(\nu z)$ for large ν is given by:

$$K_{i\nu}(\nu z) = \frac{\pi\sqrt{2}}{\nu^{1/3}} \exp\left(-\frac{\pi}{2}\nu\right) \cdot \left(\frac{\zeta}{z^2-1}\right)^{\frac{1}{4}} \cdot \left\{ \mathrm{Ai}(\xi)\left[1 + \sum_{s=1}^{m} \frac{A_s(\zeta)}{\nu^{2s}}\right] + \right.$$

$$\left. + \frac{\mathrm{Ai}'(\xi)}{\nu^{4/3}} \sum_{s=0}^{m-1} \frac{B_s(\zeta)}{\nu^{2s}} + \frac{\exp(-\frac{2}{3}\xi^{3/2})}{1+|\xi|^{\frac{1}{4}}} \; O\left(\frac{1}{\nu^{2m+1}}\right)\right\},$$

where $\xi = \nu^{2/3}\zeta$, $\frac{2}{3}\zeta^{3/2} = \sqrt{z^2-1} - \operatorname{arc\,sec} z$, and the coefficients are given by:

$$A_s(\zeta) = \sum_{m=0}^{2s} (-)^m b_m \zeta^{-3/2 m} U_{2s-m}, \quad \zeta^{\frac{1}{2}} B_s(\zeta) = \sum_{m=0}^{2s+1} (-)^m a_m \zeta^{-3/2 m} U_{2s-m+1},$$

where

$$a_0 = 1, \qquad a_s = \frac{(2s+1)(2s+3)\cdots(6s-1)}{s!(144)^s},$$

$$b_0 = 1, \qquad b_s = -\frac{6s+1}{6s-1} a_s,$$

and

$$U_0 = 1, \qquad U_1 = \frac{1}{8} v \left[\frac{5}{3} v^2 + 1 \right],$$

$$U_2 = v^2 \left[\frac{9}{128} + \frac{77}{192} v^2 + \frac{385}{1152} v^4 \right],$$

$$U_3 = v^3 \left[\frac{75}{1024} + \frac{4563}{5120} v^2 + \frac{17017}{9216} v^4 + \frac{85085}{82944} v^6 \right], \cdots,$$

with $v = \dfrac{1}{\sqrt{z^2 - 1}}$;

the expansion holds for $z \in R = \{z \,|\arg z| \le \frac{\pi}{2}\} \cup [K - \{0, -1\}]$ where K is the eyeshaped domain bounded by the curves (2.30).

We have seen that the uniform expansion yields the Nicholson formula and the Debye type series for $\frac{1}{z} = p < 1$. In the following we show that the Debye series for $p > 1$ can be obtained from the uniform expansion also and so we get another check of the asymptotic expansion. For brevity we consider only the first two terms of the expansions. We will use the following relations which can be easily verified by the transformation formulae on p. 45.

$$\operatorname{th} \alpha = \sqrt{1 - z^2}, \quad v = \frac{1}{\sqrt{z^2 - 1}} = -i \coth \alpha, \quad \alpha = \operatorname{ch}^{-1}\left(\frac{1}{z}\right);$$

for $\zeta < 0$, $\frac{2}{3}(-\zeta)^{3/2} = (-1)^{3/2} \frac{2}{3} \zeta^{3/2} = (-1)^{3/2} \rho = (-1)^{3/2} (\tan\sigma - \sigma)$,

and with $\sigma = 0 + i\sigma_2$, $\sigma_2 = \operatorname{ch}^{-1}\left(\frac{1}{z}\right) = \alpha$ we have

$$\frac{2}{3}(-\zeta)^{3/2} = \alpha - \operatorname{th} \alpha. \tag{2.44}$$

The first two terms of the expansion

$$K_{i\nu}(\nu z)\, e^{\frac{\pi}{2}\nu} \sim \frac{\pi\sqrt{2}}{\nu^{1/3}} \cdot \sqrt[4]{\frac{-\zeta}{1-z^2}} \left\{ Ai(\nu^{2/3}\zeta) + Ai'(\nu^{2/3}\zeta)\, \frac{B_0(\zeta)}{\nu^{4/3}} \right\}$$

Using the asymptotic expansions (A. 7) of Ai and (A. 8) of Ai' given in the appendix we have

$$Ai(\nu^{2/3}\zeta) \sim \frac{1}{\sqrt{\pi}\,\nu^{1/6}(-\zeta)^{1/4}} \left[\cos(\Omega - \frac{\pi}{4}) + \sin(\Omega - \frac{\pi}{4})\, \frac{5}{72}\, \frac{1}{\Omega}\right] ,$$

where $\Omega \equiv \frac{2}{3}\nu(-\zeta)^{3/2} = (\alpha - \text{th}\,\alpha)\nu$, so θ in (1. 15) equals to $\Omega - \frac{\pi}{4}$;

$$Ai'(\nu^{2/3}\zeta) \sim \frac{1}{\sqrt{\pi}}\, \nu^{1/6}\, (-\zeta)^{1/4} \cdot \cos(\Omega - \frac{3\pi}{4}) .$$

Using these relations we can write:

$$K_{i\nu}(\nu z)\, e^{\frac{\pi}{2}\nu} \sim$$

$$\sim \sqrt{\frac{2\pi}{\text{th}\,\alpha}}\, \frac{1}{\nu^{1/3}} \left\{ \frac{1}{\nu^{1/6}} \left[\cos(\Omega - \frac{\pi}{4}) + \sin(\Omega - \frac{\pi}{4})\, \frac{5}{72\Omega}\right] + \right.$$

$$\left. + \sqrt[4]{(-\zeta)^2}\, \frac{1}{\nu^{7/6}} \sin(\Omega - \frac{\pi}{4}) \left[-\frac{5}{48\zeta^2} + \frac{1}{8\sqrt{\zeta}} v\,(1 + \frac{5}{3} v^2)\right] \right\} =$$

$$= \sqrt{\frac{2\pi}{\text{th}\,\alpha}} \left[\frac{\cos\theta}{\sqrt{\nu}} + \frac{1}{\nu^{7/6}} \sin\theta \cdot \frac{1}{8\,\text{th}\,\alpha} (1 - \frac{5}{3}\coth^2\alpha)\right] ,$$

which is identical with the first two terms of the Debye series for $p > 1$ (p. 22).

Using the asymptotic expansions of the Airy function (A. 5), (A. 6) we can also easily verify that the uniform expansion yields as special case the Hankel series given in (A. 9) if we take ν fixed and ζ large and positive.

2.3 Uniform Asymptotic Expansion for the Derivative

As a consequence of Olver's theorem the expansion on p. 58 can be differentiated term by term with respect to ζ. Denoting the derivative of a function by prime, and using the notation

$$g(\zeta) = \left(\frac{\zeta}{z^2-1}\right)^{\frac{1}{4}} = \frac{1}{z}\left(\frac{dz}{d\zeta}\right)^{\frac{1}{2}} ,$$

we have

$$\nu \frac{dz}{d\zeta} K'_{i\nu}(\nu z) \sim \frac{g'(\zeta)}{g(\zeta)} K_{i\nu}(\nu z) +$$

$$+ c \cdot g(\zeta) \frac{Ai(\xi)}{\nu^{1/3}} \sum_{0}^{\infty} \frac{A_s(\zeta) + \zeta B_s(\zeta)}{\nu^{2s}} +$$

$$+ \frac{Ai'(\xi)}{\nu^{5/3}} \sum_{0}^{\infty} \frac{A_s(\zeta)\nu^2 + B'_s(\zeta)}{\nu^{2s}} ,$$

where $c = \pi \sqrt{2} \cdot e^{-\frac{\pi}{2}\nu}$.

The asymptotic expansion can be written in the following form

$$K'_{i\nu}(\nu z) \sim \frac{c}{zg(\zeta)} \left\{ \frac{Ai(\xi)}{\nu^{4/3}} \sum_{s=0}^{\infty} \frac{C_s}{\nu^{2s}} + \frac{Ai'(\xi)}{\nu^{2/3}} \sum_{s=0}^{\infty} \frac{D_s}{\nu^{2s}} \right\}, \tag{2.45}$$

where

$$C_s(\zeta) = y(\zeta)A_s(\zeta) + A'_s(\zeta) + \zeta B_s(\zeta) \ ,$$

$$D_s(\zeta) = y(\zeta)B_{s-1}(\zeta) + B'_{s-1}(\zeta) + A_s(\zeta) \ ,$$

and $\quad y(\zeta) = \frac{g'(\zeta)}{g(\zeta)}$.

By (2.33) it follows that $g(\zeta)$ is regular in D. Since

$$y(\zeta) = \frac{1 - 2z^2 g^6(\zeta)}{4\zeta}$$

it is easy to show by (2.33) that $y(\zeta)$ is also regular in D, hence so are $D_s(\zeta)$ and $C_s(\zeta)$. Therefore by Olver's theorem the expansion represents $K'_{i\nu}$ asymptotically in the same domain where the asymptotic series of $K_{i\nu}$ was obtained.

The explicit formulae for C_s and D_s can be derived similarly as earlier, namely by comparing the Debye series (1.24) and the asymptotic expansion (2.45).

Thus the asymptotic expansion of $K'_{i\nu}$ is

$$K'_{i\nu}(\nu z) \sim \frac{\sqrt{2}\pi e^{-\frac{\pi}{2}\nu}}{z} \left(\frac{\zeta}{z^2-1}\right)^{-\frac{1}{4}} \left\{ \frac{Ai(\xi)}{\nu^{4/3}} \sum_{0}^{\infty} \frac{C_s(\zeta)}{\nu^{2s}} + \right.$$

$$\left. + \frac{Ai(\xi)}{\nu^{2/3}} \sum_{0}^{\infty} \frac{D_s(\zeta)}{\nu^{2s}} \right\} , \tag{2.46}$$

where $\xi = \nu^{2/3}\zeta$ and the first three coefficients are given by

$$D_0 = 1$$

$$C_0 = \zeta^{\frac{1}{2}} V_1 - \frac{7}{48}\frac{1}{\zeta}$$

$$D_1 = -\frac{455}{4608\zeta^3} - \frac{5}{48}\frac{1}{\zeta^2} C_0 - V_2 ,$$

with

$$V_1 = -\left[\frac{3}{8} + \frac{7}{24} v^2\right] v$$

$$V_2 = \left[\frac{15}{128} + \frac{33}{64} v^2 + \frac{455}{1152} v^4\right] v^2 , \qquad v = \frac{1}{\sqrt{z^2-1}} ;$$

the region of validity is the same as in the expansion of $K_{i\nu}(\nu z)$.

Using the asymptotic expansions of the Airy function (A. 7) and (A. 8) we can verify that (2. 46) yields as a special case the Debye series (1. 25) if $\zeta < 0$, i. e. $z < 1$; so we can check (2.46). Again we consider only the first two terms of the expansion. We have

$$e^{\frac{\pi}{2}\nu} K'_{i\nu}(\nu z) \sim \frac{\sqrt{2}\pi}{z}\left(\frac{\zeta}{z^2-1}\right)^{-\frac{1}{4}} \left\{ \frac{Ai(\nu^{2/3}\zeta)}{\nu^{4/3}} C_0(\zeta) + \frac{Ai(\nu^{2/3}\zeta)}{\nu^{2/3}} \right\} =$$

$$= \frac{\sqrt{2}\pi}{z}\left(\frac{-\zeta}{1-z^2}\right)^{-\frac{1}{4}} \left\{ \frac{1}{\nu^{2/3}} \frac{\nu^{1/6}}{\sqrt{\pi}} (-\zeta)^{\frac{1}{4}} \left[\cos(\Omega - \frac{3\pi}{4}) - \right.\right.$$

$$\left.\left. - \sin(\Omega - \frac{3\pi}{4}) \cdot \frac{7}{72}\frac{1}{\Omega} + \frac{1}{\nu^{4/3}} \frac{1}{\sqrt{\pi}} \frac{C_0}{\nu^{1/6}(-\zeta)^{\frac{1}{4}}} \cos(\Omega - \frac{\pi}{4}) \right\} =$$

$$= \sqrt{\frac{\pi}{2\coth\alpha}} \frac{ch\,\alpha}{\sqrt{\nu}} \left\{ \sin(\Omega - \frac{\pi}{4}) - \frac{1}{\nu}\cos(\Omega - \frac{\pi}{4}) \cdot \frac{1}{th\alpha}\left(\frac{3}{8} - \frac{7}{24} ch^2\alpha\right) \right\} ,$$

which is the same as the first two terms in (1. 25).

2.4 Zeros of $K_{i\nu}(z)$ and $K'_{i\nu}(z)$

It follows from the differential equation of $K_{i\nu}(z)$, that the function has no repeated zeros except at the origin. Since if α was a repeated zero, then differentiating the differential equation successively with repsect to z we obtained that the derivatives of all order would vanish at α, hence by Taylor's theorem $K_{i\nu}(z)$ would be identically zero.

If $K_{i\nu}(\nu z) = 0$ and ν is large then it follows from the uniform asymptotic expansion that the corresponding values of ζ satisfy the following equation:

$$\mathrm{Ai}(\nu^{2/3}\zeta) + [1 + |\nu^{2/3}\zeta|^{\frac{1}{4}}]^{-1} \exp(-\frac{2}{3}\nu\zeta^{3/2}) \cdot O(\frac{1}{\nu}) = 0. \qquad (2.47)$$

Since $\mathrm{Ai}(z)$ has only real negative zeros, all zeros of $K_{i\nu}(\nu z)$ lie asymptotically on that segment of the real axis which lies inside of the eyeshaped domain K (Figure 2. 2).

We denote the s-th zero of Ai(z) by a_s and the s-th zero of $K_{i\nu}(z)$ by $k_{\nu,s}$ and the corresponding ζ-value by ζ_s. If we write

$$\zeta_s = \alpha + \varepsilon ,$$

then by expanding $\mathrm{Ai}(z)$ at $z = a_s$ we obtain from (2. 47)

$$\alpha = \nu^{-2/3} a_s , \quad \varepsilon = O(\nu^{-5/3}) .$$

Hence $k_{\nu,s} = \nu z(\zeta_s) = \nu\left[z(\alpha) + \frac{z'(\alpha)}{1!}\varepsilon + \cdots\right]$,

or $$k_{\nu,s} = \nu z(\alpha) + O(\nu^{-2/3}) . \tag{2. 48}$$

An asymptotic expansion for $k_{\nu,s}$ may be found by following the same formal analysis given in Olver's paper for the zeros of $J_\nu(z)$ [24] . The uniform asymptotic expansion for $k_{\nu,s}$ is

$$k_{\nu,s} \sim \nu \sum_{r=0}^{\infty} \frac{p_r(\alpha)}{\nu^{2r}} , \tag{2. 49}$$

where $p_0 = z$, $p_1 = \alpha_1 z'$, $p_2 = \alpha_2 z' + \frac{1}{2}\alpha_1^2 z''$, $\cdots$, (2. 50)

and $\alpha_1 = -B_0$,

$$\alpha_2 = -[B_1 + \alpha_1 B_1^1 + \frac{1}{2}\alpha_1^2 B_0^2 + \frac{1}{6}\alpha_1^3 B_0^3] \tag{2. 51}$$

with $B_0^1 = 1$

$$A_r^{2m} = \frac{d}{d\zeta} A_{r-1}^{2m-1} + \zeta B_r^{2m-1} , \quad A_r^{2m+1} = \frac{d}{d\zeta} A_r^{2m} + \zeta B_r^{2m} ,$$

$$B_r^{2m} = A_r^{2m-1} + \frac{d}{d\zeta} B_r^{2m-1} , \quad B_r^{2m+1} = A_r^{2m} + \frac{d}{d\zeta} B_{r-1}^{2m} .$$

This expansion is valid in the cut ζ-plane D uniformly with respect to s and the error on terminating the expansion at the term

$\frac{p_r(\alpha)}{\nu^{2r-1}}$ is $O(\nu^{-2r-1})$. Thus the asymptotic formula (2, 48) has an error $O(\frac{1}{\nu})$ only.

It follows from Olver's theorem and from (2.46) that $K'_{i\nu}(\nu z) = 0$ implies that ζ must be such that it satisfies

$$Ai'(\nu^{2/3}\zeta) + (1 + |\nu^{2/3}\zeta|^{\frac{1}{4}})\exp(-\tfrac{2}{3}\nu\zeta^{3/2})\,O(\nu^{-2/3}) = 0 . \qquad (2.52)$$

Since the zeros of $Ai'(z)$ are all real and negative [24] the zeros of $K'_{i\nu}(z)$ lie asymptotically on the real axis. Denoting the s-th zero of $Ai'(z)$ by a'_s and the s-th zero of $K'_{i\nu}(z)$ by $k'_{\nu,s}$ and the corresponding ζ-value by $\overline{\zeta}_s$ and writing $\overline{\zeta}_s = \beta + \eta$ we have from the expansion (2.48) that

$$\beta = \nu^{-2/3} a'_s \quad , \quad \text{and} \quad \eta = O(\nu^{-4/3}) .$$

Hence $$k'_{\nu,s} = \nu z(\beta) + O(\nu^{-1/3}) .$$

Again by the same analysis given in [24] for $J_\nu(z)$ we define

$$C_r^{2m} = \frac{d}{d\zeta} C_r^{2m-1} + \zeta D_r^{2m-1} \; ; \quad C_r^{2m+1} = \frac{d}{d\zeta} C_{r-1}^{2m} + \zeta D_r^{2m}$$

$$D_r^{2m} = C_r^{2m-1} + \frac{d}{d\zeta} D_{r-1}^{2m-1} \, , \quad D_r^{2m+1} = C_r^{2m} + \frac{d}{d\zeta} D_r^{2m} \, ,$$

and we obtain the expansion:

$$k'_{\nu,s} \sim \nu \sum_{r=0}^{\infty} \frac{g_r(\beta)}{\nu^{2r}} \, , \qquad (2.53)$$

where the coefficients $g_r(\beta)$ given by the formulae (2.50) with the symbols p_r and α_r replaced by g_r and β_r respectively. In the place of (2. 51) (since $C_0^1 = \zeta$) we have

$$\beta_1 = -\zeta^{-1} C_0 , \quad \beta_2 = -\zeta^{-1}(C_1 + \beta_1 C_1^1 + \tfrac{1}{2}\beta_1^2 C_0^2 + \tfrac{1}{6}\beta_1^3 C_0^3) .$$

The error on curtailing the expansion (2. 53) at the term $\frac{g_r(\beta)}{\nu^{2r-1}}$ is $O(\nu^{-2/3 r - 1/3})$ uniformly with respect to s. For numerical calculations the coefficients $p_r(\zeta)$ and $g_r(\zeta)$ may be pretabulated.

Using some of the results deduced in this paper we give a qualitative graph of $K_{i\nu}(\nu z)$ as a function of the real $z > 0$ for fixed ν.

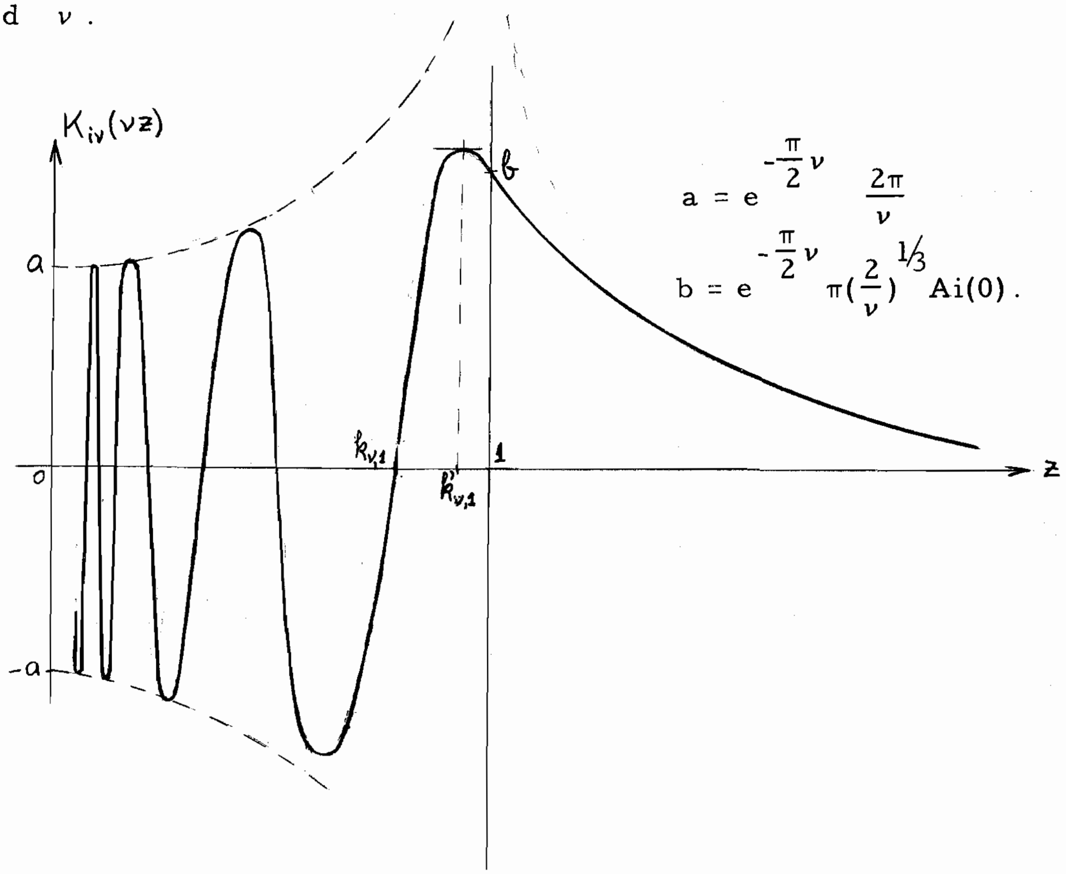

Figure 2. 6

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APPENDIX

APPENDIX

Some Properties of the Airy Function

The differential equation

$$w'' = zw \tag{A. 1}$$

can be reduced to the differential equation satisfied by the modified Bessel functions of order $\frac{1}{3}$.

Using the notation

$$u = \frac{2}{3} z^3$$

the two linearly independent solutions of the above differential equation

$$\begin{aligned} Ai(z) &= \frac{1}{\pi} \left(\frac{z}{3}\right)^{\frac{1}{2}} K_{1/3}(u) \\ Bi(z) &= \left(\frac{z}{3}\right)^{\frac{1}{2}} [I_{-1/3}(u) + I_{1/3}(u)] \end{aligned} \tag{A. 2}$$

are called the Airy functions of the first and second kind.

For real x the Airy functions can be represented as

$$\begin{aligned} Ai(x) &= \frac{1}{\pi} \int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt \\ Bi(x) &= \frac{1}{\pi} \int_0^\infty \left[\exp\left(-\frac{t^3}{3} + xt\right) + \sin\left(\frac{t^3}{3} + xt\right)\right] dt , \end{aligned} \tag{A. 3}$$

for complex x these integrals may be converted into contour integrals. $Ai(z)$ and $Bi(z)$ are both entire functions of z, and for real z they are real valued functions.

The following relation holds for $Ai(z)$ and $Bi(z)$:

$$Bi(z) = e^{1/6 \pi i} Ai(e^{2/3 \pi i} z) + e^{-1/6 \pi i} Ai(e^{-2/3 \pi i} z). \quad (A.4)$$

$Ai(z)$ and $Ai'(z)$ have a string of negative zeros and they have no zeros elsewhere [32, Section 15.7; 24]. The zeros of $Ai(z)$ and $Ai'(z)$ are extensively tabulated; the first 50 zeros of Ai and Ai' are given to 8 decimal places in the British Association Mathematical Tables (1946).

If $|z|$ is large then

$$Ai(z) \sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-u} L(-u), \quad |\arg z| < \pi, \quad (A.5)$$

$$Ai'(z) \sim \frac{1}{2\sqrt{\pi}} z^{\frac{1}{4}} e^{-u} M(-u), \quad |\arg z| < \pi; \quad (A.6)$$

and

$$Ai(-z) \sim \frac{1}{\sqrt{\pi}} z^{-\frac{1}{4}} \{\cos(u - \frac{\pi}{4}) P(u) + \sin(u - \frac{\pi}{4}) Q(u)\}, \quad (A.7)$$

$$Ai'(-z) \sim \frac{1}{\sqrt{\pi}} z^{\frac{1}{4}} \{\cos(u - \frac{3\pi}{4}) R(u) + \sin(u - \frac{3\pi}{4}) S(u)\}, \quad (A.8)$$

for $|\arg z| < \frac{2}{3}\pi$, where $L, M, P, Q, R, S,$ are defined below.

Let

$$u_s \equiv \frac{(2s+1)(2s+3)\cdots(6s-1)}{s!\,(216)^s}, \qquad v_s \equiv -\frac{6s+1}{6s-1}u_s,$$

then

$$L(u) \equiv \sum_0^\infty \frac{u_s}{u^s} = 1 + \frac{3.5}{1.216}\frac{1}{u} + \frac{5.7.9.11}{2!(216)^2}\frac{1}{u^2} + \cdots,$$

$$M(u) \equiv \sum_0^\infty \frac{v_s}{u^s} = 1 - \frac{3.7}{1.216}\frac{1}{u} - \frac{5.7.9.13}{2!(216)^2}\frac{1}{u^2} + \cdots,$$

$$P(u) \equiv \sum_0^\infty (-)^s \frac{u_{2s}}{u^{2s}} = 1 - \frac{5.7.9.11}{2!(216)^2}\frac{1}{u^2} + \cdots,$$

$$Q(u) \equiv \sum_0^\infty (-)^s \frac{u_{2s+1}}{u^{2s+1}} = \frac{3.5}{1!216}\frac{1}{u} - \frac{7.9.11.13.15.17}{3!(216)^3}\frac{1}{u^3} + \cdots,$$

$$R(u) \equiv \sum_0^\infty (-)^s \frac{v_{2s}}{u^{2s}} = 1 + \frac{5.7.9.13}{2!(216)^2}\frac{1}{u^2} - \cdots,$$

$$S(u) \equiv \sum_0^\infty (-)^s \frac{v_{2s+1}}{u^{2s+1}} = -\frac{3.7}{1!216}\frac{1}{u} + \frac{7.9.11.13.15.19}{3!(216)^3}\frac{1}{u^3} - \cdots.$$

These representations are taken from [24, p. 364].

The Parametric Equation of the Boundary of the Domain K

Consider the line segments T_1T_2 and T_1T_2' on the ρ-plane. For each point of these segments $-\pi \le \rho_1 \le 0$, $\rho_2 = 0$

$(\rho = \rho_1 + i\rho_2)$. These points are mapped on the σ-plane onto a set of points $\sigma_1 + it$, where

$$\rho_1 = \frac{\sin\sigma_1 \cos\sigma_1}{ch^2 t - \sin^2\sigma_1} - \sigma_1$$

$$\rho_2 = \frac{sh\,t\; ch\,t}{ch^2 t - \sin^2\sigma_1} - t \; . \qquad (*)$$

If we let $\sigma_1 = \frac{\pi}{2} \pm \alpha$, $0 \leq \alpha \leq \frac{\pi}{2}$, then since $\rho_2 = 0$

$$\alpha = \cos^{-1}\left\{ ch\,t \cdot \sqrt{1 - \frac{th\,t}{t}} \right\} .$$

So the parametric equation of the image of T_1T_2 and T_1T_2' on the σ-plane is

$$\sigma = \frac{\pi}{2} \pm \alpha + it \; .$$

By the transformation formula $z = \sec\sigma$ we have

$$\operatorname{Re} z = \frac{\cos\sigma_1 ch\,t}{\cos^2\sigma_1 ch^2 t + \sin^2\sigma_1 sh^2 t} = \frac{\pm t \sin\alpha}{sh\,t} =$$

$$= \pm (t\, cth\, t - t^2)^{\frac{1}{2}} , \quad \text{with } 0 \leq t \leq t_0 ,$$

where t_0 is the positive root of $cth\,t = t$. (From (*) with $\rho_2 = 0$ it is clear that t is maximal if $\sin^2\sigma_1 = 1$, $ch^2 t(1 - \frac{th\,t}{t}) = 0$, i.e. $t = cth\,t$, $t \neq 0$)

$$\operatorname{Im} z = \frac{\operatorname{sh} t \sin\sigma_1}{\cos^2\sigma_1 \operatorname{ch}^2 t + \sin^2\sigma_1 \operatorname{sh}^2 t} =$$

$$= -\frac{\operatorname{sh} t \operatorname{ch} t (1 - \frac{1}{t}\operatorname{th} t)^{\frac{1}{2}}}{\operatorname{ch}^2 t - \operatorname{ch}^2 t (1 - \frac{1}{t}\operatorname{th} t)} = -(t^2 - t\cdot\operatorname{th} t)^{\frac{1}{2}} \operatorname{sgn} t \ .$$

Therefore the parametric equation of the image of the segments T_1T_2 and T_1T_2' on the z-plane is given by

$$z = \pm[t \operatorname{cth} t - t^2]^{\frac{1}{2}} - i(t^2 - t\cdot\operatorname{th} t)^{\frac{1}{2}} \operatorname{sgn} t \ .$$

Near $\zeta = 0$ the first two terms of the Taylor series is $z(\zeta) = 1 + 2^{-1/3}\zeta + O(\zeta^2)$. Using this relation it can be easily seen that the slope of the tangent line of this curve at T_1 is $\pm\frac{2\pi}{3}$, and at T_2 it is $\pm\frac{\pi}{3}$.

The Hankel Series for $K_{i\nu}(z)$

This expansion can be obtained from the Hankel series of $K_{i\nu}(z)$ given in [8 , p. 23] .

$$K_{i\nu}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \left\{ \sum_{k=0}^{M-1} \frac{\Gamma(i\nu + \frac{1}{2} + k)}{k!\,\Gamma(i\nu + \frac{1}{2} - k)} \frac{1}{(2z)^k} + O(|z|^{-M}) \right\}, \qquad \text{(A. 9)}$$

where $-\frac{3\pi}{2} < \arg z < \frac{3\pi}{2}$.