AN ABSTRACT OF THE THESIS OF

JOHN J. DRANCHAK for the degree of DOCTOR OF PHILOSOPHY

in MATHEMATICS presented on April 22, 1976

Title: PROPERTIES AND INTEGRAL REPRESENTATIONS OF DISTRIBUTIONS OF OPERATORS

Abstract approved: Redacted for Privacy

Stuart M. Newberger

An operator-valued integral representation is obtained for positive distributions of operators defined on certain classes of $K\{M_p\}$ spaces. Hermitian bilinear, translation-invariant, positive-definite distributions of operators are characterized as Fourier transforms of tempered operator-valued measures $E(\cdot)$ on the ring generated by the compact subsets of R^n. Any such operator distribution B admits the unique representation

$$B_{(\phi,\psi)} = \int \hat{\phi}\bar{\hat{\psi}}\,dE, \quad (\phi,\psi) \in \mathcal{D}(R^n) \times \mathcal{D}(R^n),$$

where $E(\cdot)$ is the tempered operator measure, the hat denotes the Fourier transform and the bar denotes complex conjugation.

Finally, there is obtained a representation of multiplicatively positive distributions of operators on the space Z of entire analytic functions of exponential type on C^n.

Properties and Integral Representations of Distributions of Operators

by

John Joseph Dranchak

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

June 1976

APPROVED:

Redacted for Privacy

Associate Professor of Mathematics
in charge of major

Redacted for Privacy

Chairman of Department of Mathematics

Redacted for Privacy

Dean of Graduate School

Date thesis is presented April 22, 1976

Typed by Clover Redfern for John J. Dranchak

ACKNOWLEDGMENT

I am indebted to Professor Stuart M. Newberger for directing my attention to the papers which led to the subject of this thesis. I am particularly grateful for Professor Newberger's patience with me during the research phase of my work, for his many helpful suggestions and for his careful reading of the final version, as well as the numerous intermediate forms, of the manuscript.

A note of appreciation is extended also to the faculty and graduate students of the Department of Mathematics. The openness, interest and encouragement offered by my colleagues is remembered with heartfelt thanks.

My deepest appreciation, gratitude and a special feeling of love go to my wife, Mary Ann. Were it not for her encouragement, devotion and sacrifice, my work on this thesis would have been utterly impossible. Our children, David James and Kathryn Sue, are remembered with love for absorbing their father's frequent absence from home while engaged in the preparation of this thesis. All of my work, both visible and invisible, is dedicated to my family.

I wish to express my thanks to the typist, Mrs. Clover Redfern, for her excellent help in preparing the final copy.

TABLE OF CONTENTS

PROPERTIES AND INTEGRAL REPRESENTATIONS OF DISTRIBUTIONS OF OPERATORS

INTRODUCTION

The problem of characterizing linear functionals defined on various spaces and possessing certain properties is by no means new. The extension theorems of Hahn-Banach [10], the F. Riesz Representation Theorem [11] and the theorem of S. Bochner [11] on the representation of complex-valued continuous positive-definite functions are among the more important theorems in classical analysis dealing with the general problem of characterizing and representing certain classes of functions and the duals of function spaces.

The theorems mentioned above have extensive application in various areas of mathematics and have been shown to admit various degrees of generalization (see for example, [6], pp. 152, 157, 219, 220, and [8]). The theorem that a positive continuous linear functional is a positive Radon measure and the associated integral representation of such (and more general) functionals carries over to scalar-valued distributions on open subsets of Euclidean n-space R^n. S. Bochner's theorem on the characterization and integral representation of continuous positive-definite functions has likewise been generalized to distributions by L. Schwartz [6, p. 157]. More recently B. Kritt [8] extended some of these results on positive and positive-definite distributions to distributions defined on an open subset Ω of R^n and taking their values in the Banach algebra of bounded linear operators on a complex Hilbert space. Kritt showed that a positive distribution of operators on an open subset Ω of R^n is given by an operator-valued integral relative to a positive operator valued measure on the ring generated by the compact subsets of R^n, and if the positive operator distribution on R^n is tempered then so is the associated operator measure. In the same paper Kritt showed that a positive-definite distribution of operators on R^n is the Fourier transform of a tempered positive operator-valued measure and that such an operator distribution has a unique integral representation with respect to the operator measure.

The first result in this thesis (Chapter 2) is an extension of the first of the above mentioned theorems of Kritt to the extensive class of test-function spaces $K\{M_p\}$, containing $D(R^n)$ as a dense subspace, first introduced by Gel'fand and Shilov [4, p. 255] in connection with the study of the problem of classes of uniqueness of the solution of the Cauchy problem for systems of partial differential equations. The balance of this thesis (Chapters 3 and 4) is addressed to the problem of establishing for operator-valued distributions results which hold for certain classes of scalar-valued bilinear distributions and for scalar-valued multiplicatively positive distributions--theorems which have immediate application to generalized random processes [6,p.237].

Many important transformations in analysis are multiplicative: $T_{\phi_1\phi_2} = T_{\phi_1} T_{\phi_2}$ for all elements in the domain of T. The distributions of operators that are studied here are not required to be multiplicative. On the other hand the range of these distributions will always be assumed to consist of bounded operators.

1. BACKGROUND AND PRELIMINARIES

This chapter is devoted to a brief discussion of the spaces, concepts and preparatory lemmas leading to, and culminating in, the development of a certain operator-valued integral which plays a prominent role in the first theorem in this thesis.

A. The Spaces $K\{M_p\}$

The concept of a fundamental space is central in the development of the theory of generalized functions. Indeed a generalized function is, by definition, an element in the topological dual of a fundamental space.

Let C denote the field of complex numbers and let Φ be a linear space over C of complex-valued functions defined on a non-empty subset X of R^n. Thus, if ϕ and ψ are functions in Φ and λ and μ are any complex numbers, then the function $\lambda\phi + \mu\psi$ belongs to Φ. In particular, the zero function $\theta(x) = 0$, $x \in X$, belongs to Φ.

<u>Definition 1</u>. Φ is a <u>fundamental space</u> provided that:

(1) Φ is either a complete countably normed space, or it is a countable union of complete countably normed spaces. In the latter case Φ is given the inductive limit topology.

(2) If $\phi_m \to \theta$ in Φ, then $\phi_m(x_0) \to 0$ in C for any $x_0 \in X$.

All of our work involves functions and the calculus of functions in some fundamental space Φ. Since these functions are defined on subsets of R^n we summarize the notation used in the n-dimensional calculus.

A multi-index α is an ordered n-tuple of nonnegative integers α_j:

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n).$$

To each multi-index α there corresponds a differential operator

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n}$$

where

$$D_j = \frac{\partial}{\partial x_j}, \quad 1 \leq j \leq n.$$

The order of D^α is the nonnegative integer $|\alpha|$ defined by

$$|\alpha| = \sum_{j=1}^{n} \alpha_j .$$

If $|\alpha| = 0$, that is, if

$$\alpha = 0 = (0, 0, \ldots, 0),$$

then we set $D^{\alpha}\phi = \phi$. If α, β are n-dimensional multi-indices, then $\alpha \leq \beta$ means that $\alpha_j \leq \beta_j$, $1 \leq j \leq n$,

$$\alpha \pm \beta = (\alpha_1 \pm \beta_1, \ldots, \alpha_n \pm \beta_n),$$

$$\alpha! = \alpha_1! \alpha_2! \ldots \alpha_n!,$$

$$\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2} \ldots \binom{\alpha_n}{\beta_n},$$

where

$$\binom{\alpha_j}{\beta_j} = \frac{\alpha_j!}{\beta_j!(\alpha_j - \beta_j)!}, \quad 1 \leq j \leq n.$$

If $x = (x_1, x_2, \ldots, x_n)$ is a point in R^n, the <u>Euclidean norm</u> $|x|$ of x is defined by

$$|x| = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}.$$

If $x, y \in R^n$, the <u>inner product</u> is denoted $x \cdot y$ and is defined by

$$x \cdot y = \sum_{j=1}^{n} x_j y_j.$$

Finally, if $x \in R^n$ and α is an n-dimensional multi-index, the <u>monomial</u> x^{α} is defined by

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} .$$

Rather than deal with arbitrary fundamental spaces Φ we restrict our attention to a class of fundamental spaces which, on the one hand, is of sufficient generality to include the fundamental spaces of classical distribution theory as particular cases and to which, on the other hand, many of the theorems of distribution theory can be extended. These spaces are the fundamental spaces $K\{M_p\}$ which we now proceed to define.

Let M_p, $p = 0, 1, 2, \ldots,$ be a nondecreasing sequence of functions defined on R^n and bounded below by the function which is identically equal to one on R^n. Thus

$$1 \leq M_0 \leq M_1 \leq M_2 \leq \cdots \leq M_p \leq M_{p+1} \leq \cdots .$$

<u>We will assume that each M_p is continuous on all of R^n and that $M_p(x) < \infty$, $p \geq 0$, at each point x in R^n.</u>

<u>Definition 2.</u> The set $K\{M_p\}$ consists of all $C^\infty(R^n)$ complex-valued functions ϕ which, for every multi-index α and for $0 \leq p < \infty$, satisfy the following condition:

$M_p D^\alpha \phi$ is a continuous bounded function on R^n.

The linear space $K\{M_p\}$ is given the structure of a complete countably normed topological vector space by the family of norms $\|\phi\|_p$ defined, for each $\phi \in K\{M_p\}$ and $0 \leq p < \infty$, by

$$\|\phi\|_p = \max_{|\alpha| \leq p} \sup_{x \in R^n} M_p(x)|(D^\alpha \phi)(x)| . \tag{1}$$

For any ϕ in $K\{M_p\}$,

$$\|\phi\|_0 \leq \|\phi\|_1 \leq \|\phi\|_2 \leq \cdots \leq \|\phi\|_p \leq \|\phi\|_{p+1} \leq \cdots ,$$

and

$$\|\phi\|_p < \infty, \quad 0 \leq p < \infty.$$

Note that each ϕ in $K\{M_p\}$ is, together with its derivatives of all orders, bounded on R^n.

<u>Definition 3</u>. A complete countably normed space X is called <u>perfect</u> [3, p. 15] if the bounded subsets of X are relatively sequentially compact; equivalently, since X is a metric space, the bounded subsets of X are relatively compact (the bounded subsets of X have compact closure).

<u>Proposition</u>. $K\{M_p\}$ is a complete countably normed space [3, p. 30].

<u>Proposition</u>. Suppose that corresponding to any integer $p \geq 0$ there is an integer $p' > p$ such that

$$\lim_{|x|\to\infty} \left(\frac{M_p}{M_{p'}}\right)(x) = 0 .$$

Then $K\{M_p\}$ is <u>perfect</u> [3, p. 31].

In all discussions relating to $K\{M_p\}$ spaces in thie thesis we will assume that these spaces satisfy the condition of the preceding Proposition and are therefore perfect spaces.

<u>Convergence in $K\{M_p\}$ Spaces</u>

<u>Definition 4</u>. A sequence $\{\phi_n\}$ of functions defined on R^n is <u>properly convergent</u> if the sequence $\{D^\alpha \phi_n\}$ converges uniformly on bounded subsets of R^n for every multi-index α.

<u>Definition 5</u>. The sequence of functions $\{\phi_m\}$ in $K\{M_p\}$ is said to be <u>M_p-bounded</u> if to each p, $0 \leq p < \infty$, there corresponds a positive constant c_p such that $\|\phi_m\|_p \leq c_p$ for $m = 1, 2, 3, \ldots$.

<u>Proposition</u>. If $\{\phi_m\}$ is an M_p-bounded sequence such that $\phi_m \to \theta$ properly, then $\|\phi_m\|_p \to 0$ for any $p \geq 0$; that is, $\phi_m \to \theta$ in $K\{M_p\}$.

If $\{\phi_m\} \subset K\{M_p\}$ is an M_p-bounded sequence such that $\phi_m \to \phi_0$ properly, then ϕ_0 is in $K\{M_p\}$ and $\phi_m \to \phi_0$ in $K\{M_p\}$ [3, p. 31].

Proposition. If $M_p < \infty$, $0 \leq p < \infty$, then the linear space $C_0^\infty(R^n)$ of infinitely differentiable functions with compact support in R^n is dense in $K\{M_p\}$ [3, p. 32].

Examples of $K\{M_p\}$ Spaces

1. The Space $\mathcal{S}$. The space $\mathcal{S}$ consists of all $C^\infty(R^n)$-functions ϕ which together with their derivatives of all orders, decrease more rapidly at infinity than any power of $|x|^{-1}$. For example, $\exp(-|x|^2)$ and all functions in $C_0^\infty(R^n)$ belong to $\mathcal{S}$. If ϕ is in $\mathcal{S}$, $\|\phi\|_{\alpha,\beta}$ is defined by

$$\|\phi\|_{\alpha,\beta} = \sup_{x \in R^n} |x^\beta (D^\alpha \phi)(x)| ,$$

where α and β are multi-indices and

$$x^\beta = \prod_{j=1}^{n} x_j^{\beta_j} .$$

$\|\phi\|_{\alpha,\beta} < \infty$ for each ϕ in $\mathcal{S}$.

Let

$$M_p(x) = \sup_{|k| \leq p} |x^k| .$$

Then $M_p \leq M_{p+1}$ and

$$\|\phi\|_p = \sup_{\substack{x \\ |\alpha| \leq p}} M_p(x) |(D^\alpha \phi)(x)| < \infty$$

for each p. Thus the space $\mathcal{S}$ of rapidly decreasing functions (at ∞) is realized as a particular $K\{M_p\}$ space.

2. The Spaces S_α and $S_{\alpha,A}$. The space S_α, $(\alpha \geq 0)$, consists of all $C^\infty(R^n)$-functions ϕ satisfying inequalities of the form $|x^k (D^q \phi)(x)| \leq C_q A^k k^{k\alpha}$, where the constants C_q and A depend on ϕ [4, p. 169]. For $\alpha = 0$, $S_0 = \mathcal{D}_K(R^n)$, the space of $C^\infty(R^n)$ functions with support in the fixed compact set $K = \{x \in R^n: |x| \leq A\}$. If $\alpha > 0$, the definition of S_α may be formulated another way. S_α, $(\alpha > 0)$, consists of all $C^\infty(R^n)$-functions ϕ which satisfy inequalities of the form

$$|(D^q \phi)(x)| \leq C_q \exp(-a|x|^{1/\alpha}),$$

where the constants C_q and a depend on ϕ [4, 172].

S_α can be realized as the union of countably normed spaces. Let $S_{\alpha,A}$ denote the set of functions ϕ in the space S_α for which the following inequalities hold:

$$|x^k (D^q \phi)(x)| \leq C_{q\bar{A}} \bar{A}^k k^{k\alpha},$$

where $\bar{A}$ is any constant greater than the constant A in $S_{\alpha,A}$. Thus

$$|x^k(D^q\phi)(x)| \leq C_{q\delta}(A+\delta)^k k^{ka}$$

holds for any $\delta > 0$. Referring to the second formulation of the spaces S_a, $(a > 0)$, given above, it is clear that $S_{a,A}$ consists of those functions ϕ which satisfy the inequalities

$$|(D^q\phi)(x)| \leq C'_{q\delta} \exp[-(a-\delta)|x|^{1/a}]$$

for $\delta > 0$ [4, p. 176]. When $a = 0$, $S_{0,A}$ consists of those $C^\infty(R^n)$ functions with compact suppose in $\{x \in R^n: |x| \leq A\}$.

Let

$$M_p(x) = \exp[a(1-\frac{1}{p})|x|^{1/a}], \quad p = 2,3,\ldots \,.$$

Then $M_p \leq M_{p+1}$, $p \geq 2$, and

$$\|\phi\|_p = \sup_{\substack{x \\ |q| \leq p}} M_p(x)|(D^q\phi)(x)| < \infty, \quad p \geq 2.$$

Thus the space $S_{a,A}$ belongs to the family of $K\{M_p\}$ spaces. In particular $S_{a,A}$ is a perfect, complete, countably normed space.

3. <u>The Spaces W_M and $W_{M,a}$</u>. Let μ be a continuous increasing C^∞-function on $[0,\infty)$ such that $\mu(0) = 0$ and define the function M on $[0,\infty)$ by

$$M(x) = \int_0^x \mu(t)dt \; .$$

M is an increasing, continuous, C^∞ convex function on $[0,\infty)$ with $M(0) = 0$ and $\lim_{x \to \infty} M(x) = \infty$. M is extended to R^1 by defining $M(x) = M(-x)$ for $x < 0$.

Let W_M denote the set of all $C^\infty(R^1)$-functions ϕ such that

$$|(D^q\phi)(x)| \leq C_q \exp(-M(ax)) \; ,$$

where the constants C_q and a depend in general on ϕ. W_M is clearly a linear space. A sequence $\{\phi_n\}$ of functions in W_M is said to converge to zero if $\{D^q\phi_n\}$ converges uniformly for all q on any finite interval in R^1 and if in addition

$$|(D^q\phi_n)(x)| \leq C_q \exp(-M(ax)) \; ,$$

where the constants C_q and a are independent of n.

Let $W_{M,a}$ denote the set of all those functions ϕ in W_M for which

$$|(D^q\phi)(x)| \leq C_q \exp[-M(\bar{a}x)]$$

holds, where the constant $\bar{a}$ is arbitrary but less than a. Thus, if ϕ is in $W_{M,a}$, then

$$|(D^q\phi)(x)| \leq C_q \exp[-M[(a-\delta)x]], \quad q = 0, 1, 2, \ldots$$

for any fixed $\delta > 0$. Let

$$M_p(x) = \exp(M[a(1-\frac{1}{p})x]), \quad p = 2, 3, \ldots .$$

Then $M_p \leq M_{p+1}$, $p \geq 2$, and the functions ϕ in $W_{M,a}$ are precisely those for which

$$\|\phi\|_p = \sup_{\substack{x \\ |q| \leq p}} M_p(x)|(D^q\phi)(x)| < \infty$$

for all $p \geq 2$. Hence, $W_{M,a}$ is a $K\{M_p\}$ space. Moreover, due to the convexity of the functions M, the space $W_{M,a}$ is perfect. Of course W_M is a complete countably normed space [5, pp. 2,3].

B. Operator-Valued Measures

Let $H = (H, <,>)$ be a complex Hilbert space, $H \neq \{0\}$, with norm $\| \|$ induced by the positive-definite sesquilinear form $<,>$ on $H \times H$ by

(1) $$\|\xi\| = \sqrt{<\xi, \xi>} \quad \text{for every} \quad \xi \in H.$$

Let $B = (B(H), \| \|)$ denote the Banach algebra of all bounded linear operators on H. If T is in $B(H)$, the norm $\|T\|$ of T is defined by

(2) $$\|T\| = \sup\{\|T\xi\| : \xi \in H, \|\xi\| \leq 1\} .$$

$B(H)$ is partially ordered by the relation "$\leq$" defined on the set of Hermitean operators by

(3) $$S \leq T$$

if and only if $T - S = A$ is a positive operator, meaning that $<A\xi, \xi> \geq 0$ for every $\xi \in H$. We write $A \geq 0$ to denote that A is positive. Thus if $S \in B(H)$, $T \in B(H)$, then $S \leq T$ if and only if $<S\xi, \xi> \leq <T\xi, \xi>$ for every vector $\xi \in H$.

Let X be any nonempty set and let R be a ring of subsets of X.

Definition 1. A positive operator-valued measure E on X is a set function

(4) $$M \rightsquigarrow E(M): R \to B(H)$$

which satisfies the following conditions:

(i) $E(M) \geq 0$ for every $M \in R$.

(ii) $E(M \cup N) = E(M) + E(N)$ whenever $M \in R$, $N \in R$ and $M \cap N = \emptyset$.

(iii) If $\{M_j\}$ is a sequence in R such that $i \neq j \Rightarrow M_i \cap M_j = \emptyset$ and $\bigcup_{j=1}^{\infty} M_j = M \in R$, then

$$E(M) = \sum_{j=1}^{\infty} E(M_j) \quad \text{(pointwise)}.$$

Condition (i) states that $E(M)$ is a positive operator on H for every M in R. Condition (ii) expresses the additivity of $E(\cdot)$ which, together with (i) implies that $E(\cdot)$ is monotone: if $M, N \in R$ and $M \subset N$, then $E(M) \leq E(N)$.

Condition (iii), expresses the countable additivity of E.

A positive operator-valued measure will henceforth be referred to as a PO-measure [2, p. 5].

The following result shows that facts concerning PO-measures can be obtained by working with scalar-valued measures.

Proposition 1. Let R be a ring of subsets of $X \neq \phi$, and $E(\cdot)$ a positive operator-valued set function on R. Then $E(\cdot)$ is a PO-measure on X if and only if, for every vector $\xi \in H$, the set function

$$M \rightsquigarrow \mu_\xi(M) : R \to C,$$

defined by

$$\mu_\xi(M) = \langle E(M)\xi, \xi \rangle, \tag{5}$$

is a measure [2, pp. 8, 9].

Thus each PO-measure $E(\cdot)$ on X gives rise to a family $\{\mu_\xi : \xi \in H\}$ of positive measures on X indexed by the set of vectors in the underlying Hilbert space.

The next result gives conditions under which a family $\{\mu_\xi : \xi \in H\}$ of finite positive measures, indexed by the vectors of a Hilbert space H, is generated by a PO-measure [2, pp. 9, 10].

<u>Proposition 2</u>. Let $X \neq \phi$, R a ring of subsets of X, $H \neq \{0\}$ a complex Hilbert space, and $\{\mu_\xi : \xi \in H\}$ a family of finite measures on X. Then there exists a PO-measure $E(\cdot)$ on X such that

$$\mu_\xi(M) = \langle E(M)\xi, \xi\rangle \qquad \text{for all } \xi \in H$$

and all $M \in R$, if and only if:

(a) $[\mu_{\xi+\eta}(M)]^{1/2} \leq [\mu_\xi(M)]^{1/2} + [\mu_\eta(M)]^{1/2}$

(b) $\mu_{c\xi}(M) = |c|^2 \mu_\xi(M)$

(c) $\mu_{\xi+\eta}(M) + \mu_{\xi-\eta}(M) = 2\mu_\xi(M) + 2\mu_\eta(M)$ for all vectors $\xi, \eta \in H$, all complex numbers c and all M in R, and for each M in R there is a constant k_M such that

(d) $\mu_\xi(M) \leq k_M \|\xi\|^2$ for all $\xi \in H$.

$E(\cdot)$ is then <u>uniquely determined</u> by $\mu_\xi(M) = \langle E(M)\xi, \xi\rangle$.

Let $X = R^n$, R the ring generated by the compact subsets of R^n, and $E(\cdot)$ a PO-measure on R^n.

<u>Definition 2</u>. $E(\cdot)$ is <u>tempered</u> [8, pp. 865, 866] if and only if there is a positive integer p and a positive number K such that for every vector $\xi \in H$,

(6) $$\int (1+|x|^2)^{-p} d\mu_\xi(x) \leq K\|\xi\|^2 ,$$

where μ_ξ is the Borel measure given on R by

$$\mu_\xi(M) = \langle E(M)\xi, \xi\rangle$$

for all $M \in R$. $|x|$ denotes the Euclidean norm of $x \in R^n$: $x = (x_1, x_2, \ldots, x_n)$,

$$|x| = \left(\sum_{j=1}^{n} (x_j)^2 \right)^{1/2} .$$

<u>Definition 3</u>. $E(\cdot)$ is $\{M_p\}$-tempered if and only if there is a nonnegative integer p and a positive constant c such that

(7) $$\int (M_p)^{-1} d\mu_\xi \leq c\|\xi\|^2 \quad \text{for all } \xi \text{ in } H.$$

The function M_p occurs in a defining sequence for a given $K\{M_p\}$ space.

C. Bounded Sesquilinear Forms on $H \times H$

Let $H \neq \{0\}$ be a complex Hilbert space.

Definition 1. A function $f: H \times H \to C$ is sesquilinear if and only if:

(a) $f(y, x) = \overline{f(x, y)}$, where the bar denotes complete conjugation.

(b) $f(x+y, z) = f(x, z) + f(y, z)$

(c) $f(cx, y) = cf(x, y)$, $x, y \in H$, $c \in C$

(d) $f(x, x) \geq 0$ for all $x \in H$.

(e) $f(x, x) = 0$ only if $x = 0$.

Thus, for fixed y, f is linear in x, and for fixed x, f is conjugate linear in y, the latter meaning that

$$f(x, y+z) = f(x, y) + f(x, z),$$

and

$$f(x, cy) = \bar{c}f(x, y).$$

We shall make frequent use of the following standard uniqueness result on $B(H)$.

Proposition 1. If $T \in B(H)$ and $\langle T\xi, \xi \rangle = 0$ for every

$\xi \in H$, then $T = 0$ (the zero operator).

Corollary. If $S \in B(H)$, $T \in B(H)$ and $<S\xi, \xi> = <T\xi, \xi>$ for every $\xi \in H$, then $S = T$.

Proposition 2. If $f: H \times H \to C$ is sesquilinear and bounded, in the sense that

$$\sup\{|f(x, y)| : \|x\| = \|y\| = 1\} = M < \infty,$$

then there exists a unique $T \in B(H)$ that satisfies

$$f(x, y) = <Tx, y>, \quad x \in H, \quad y \in H.$$

We have $\|T\| = M$.

D. The Integral of a $K\{M_p\}$-Function with Respect to an $\{M_p\}$-Tempered PO-Measure

Let R be the ring generated by the class of compact subsets of R^n and let $E(\cdot)$ be a PO-measure defined on R. For every vector ξ in the nonzero complex Hilbert space $(H, <,>)$, let μ_ξ be the positive Borel measure on R given by

(1) $$\mu_\xi(M) = <E(M)\xi, \xi> \quad \text{for every} \quad m \in R.$$

Since $E(M)$ is a positive operator, the Cauchy-Schwarz inequality yields

$$(2) \qquad \langle E(M)\xi, \xi \rangle = |\langle E(M)\xi, \xi \rangle| \leq \|E(M)\| \, \|\xi\|^2$$

for all M in R and all ξ in H.

Referring to (1) above, we have

$$(2a) \qquad \mu_\xi(M) \leq \|E(M)\| \, \|\xi\|^2 \quad \text{for every } M \in R$$

and all $\xi \in H$.

Recall that a PO-measure $E(\cdot)$ on R is said to be <u>$\{M_p\}$-tempered</u> if and only if there is an integer $p \geq 0$ and a positive constant A such that

$$(3) \qquad \int [M_p(x)]^{-1} d\mu_\xi \leq A \|\xi\|^2 \quad \text{for all } \xi \in H.$$

<u>Definition 1.</u> Let $E(\cdot)$ be an $\{M_p\}$-tempered PO-measure on R. A function f is <u>$E(\cdot)$-integrable</u> if and only if f is μ_ξ-integrable for each $\xi \in H$.

<u>Definition 2.</u> Let $I(E)$ denote the set of all $E(\cdot)$-integrable functions. Thus $f \in I(E)$ if and only if f is Borel measurable and

$$\int |f| \, d\mu_\xi < \infty$$

for all $\xi \in H$.

Lemma 1. $\chi_M \in I(E)$ for every $M \in R$.

Proof. The characteristic function χ_M of the measurable set $M \in R$ is an R-measurable function on R^n.

$$\int \chi_M d\mu_\xi = \mu_\xi(M) = \langle E(M)\xi, \xi \rangle$$

$$\leq \|E(M)\| \, \|\xi\|^2 < \infty$$

for all $\xi \in H$. Hence, $\chi_M \in I(E)$. The preceding lemma implies that $I(E)$ contains every simple function.

Lemma 2. $K\{M_p\} \subset I(E)$.

Proof. Let $\phi \in K\{M_p\}$. Since $E(\cdot)$ is an $\{M_p\}$-tempered PO-measure, there is an integer $p \geq 0$ and a positive number A such that

$$\int (M_p)^{-1} d\mu_\xi \leq A \|\xi\|^2 \quad \text{for all} \quad \xi \in H.$$

We have

$$\int |\phi| d\mu_\xi = \int M_p |\phi| M_p^{-1} d\mu_\xi$$

$$\leq \int \Big(\sup_{x \in R^n} \sup_{|q| \leq p} M_p(x) |D^q \phi(x)| \Big) (M_p)^{-1}(x) d\mu_\xi$$

$$= \|\phi\|_p \int (M_p)^{-1} d\mu_\xi \quad \leq$$

$$\leq \|\phi\|_p A \|\xi\|^2 < \infty \quad \text{for all} \quad \xi \in H.$$

Thus

$$\phi \in K\{M_p\} \Rightarrow \int |\phi| d\mu_\xi < \infty$$

for all $\xi \in H$. Hence, $\phi \in I(E)$. Q.E.D.

Let $\phi \in I(E)$. For every pair ξ, η of vectors from H, define $L_{\xi,\eta}$ by

$$L_{\xi,\eta}(\phi) = \frac{1}{4}\left[\int \phi d\mu_{\xi+\eta} - \int \phi d\mu_{\xi-\eta} + i\int \phi d\mu_{\xi+i\eta} - i\int \phi d\mu_{\xi-i\eta}\right]. \tag{4}$$

For each fixed pair ξ, η of vectors in H, $L_{\xi,\eta} : I(E) \to C$ is clearly linear.

<u>Lemma 3</u>. For each fixed pair of vectors $\xi, \eta \in H$,

$$L_{\xi,\eta}(\chi_M) = \langle E(M)\xi, \eta\rangle \quad \text{for all} \quad M \in R. \tag{5}$$

<u>Proof</u>.

$$\begin{aligned} L_{\xi,\eta}(\chi_M) &= \frac{1}{4}\left[\int \chi_M d\mu_{\xi+\eta} - \int \chi_M d\mu_{\xi-\eta} + i\int \chi_M d\mu_{\xi+i\eta} - i\int \chi_M d\mu_{\xi-i\eta}\right] \\ &= \frac{1}{4}\left[\mu_{\xi+\eta}(M) - \mu_{\xi-\eta}(M) + i\mu_{\xi+i\eta}(M) - i\mu_{\xi-i\eta}(M)\right] \\ &= \frac{1}{4}\Big[\langle E(M)(\xi+\eta), \xi+\eta\rangle - \langle E(M)(\xi-\eta), \xi-\eta\rangle \\ &\qquad + i\langle E(M)(\xi+i\eta), \xi+i\eta\rangle - i\langle E(M)(\xi-i\eta), \xi-i\eta\rangle\Big] = \end{aligned}$$

$$= \frac{1}{4}\Big[<E(M)\xi, \xi> + <E(M)\xi, \eta> + <E(M)\eta, \xi> + E(M)\eta, \eta>$$

$$- <E(M)\xi, \xi> - <E(M)\xi, -\eta> - <E(M)(-\eta), \xi> - <E(M)(-\eta), -\eta>$$

$$+ i<E(M)\xi, \xi> + i<E(M)\xi, i\eta> + i<E(M)(i\eta), \xi>$$

$$+ i<E(M)(i\eta), i\eta> - i<E(M)\xi, \xi> - i<E(M)\xi, -i\eta>$$

$$- i<E(M)(-i\eta), \xi> - i<E(M)(-i\eta), -i\eta>\Big]$$

$$= \frac{1}{4}\{4<E(M)\xi, \eta>\} = <E(M)\xi, \eta>.$$

Thus $L_{\xi,\eta}(\chi_M) = <E(M)\xi, \eta>$ for every pair of vectors $\xi, \eta \in H$ and all $M \in R$.

It is clear from the definition of $L_{\xi,\eta}(\phi)$ in (4) that if $f, f_n \in I(E)$, $n = 1, 2, \ldots$, and $\int f_n d\mu_\xi \to \int f d\mu_\xi$ for every $\xi \in H$, then $L_{\xi,\eta}(f_n) \to L_{\xi,\eta}(f)$ for every pair of vectors $\xi, \eta \in H$.

<u>Definition 3.</u> A linear form L on $I(E)$ is <u>quasicontinuous</u> [2, p. 22] if, whenever $f, f_n \in I(E)$, $n = 1, 2, \ldots$, and $0 \leq f_n \uparrow f$, then $L(f_n) \to L(f)$.

Any finite linear combination of quasicontinuous linear forms on $I(E)$ is a quasicontinuous linear form on $I(E)$.

<u>Lemma 4</u> [2, p. 23]. For every pair of vectors ξ, η in H, the linear form $L_{\xi,\eta}$ on $I(E)$ given by (4) is quasicontinuous.

This is a simple consequence of the monotone convergence theorem.

Lemma 5 [2, pp. 23, 24]. Let $\mathcal{L}$ be a linear form on $I(E)$ and let ξ, η be in H. Then

(6) $$\mathcal{L}(\phi) = L_{\xi, \eta}(\phi) \quad \text{for all} \quad \phi \in I(E)$$

if and only if

(i) $\mathcal{L}$ is quasicontinuous

(ii) $\mathcal{L}(\chi_M) = \langle E(M)\xi, \eta\rangle$ for all $M \in R$ where $L_{\xi, \eta}(\phi)$ is as in (4).

Lemma 6. For each vector $\xi \in H$,

(7) $$L_{\xi, \xi}(\phi) = \int \phi d\mu_\xi$$

Proof. Let $L(\phi) = \int \phi d\mu_\xi$ for every $\phi \in I(E)$, where ξ is a fixed vector in H. L is clearly linear on $I(E)$ and the monotone convergence theorem implies that L is quasicontinuous. Moreover,

$$L(\chi_M) = \int \chi_M d\mu_\xi = \mu_\xi(M) = \langle E(M)\xi, \xi\rangle$$

for all $M \in R$. By Lemmas 3 and 5, $L = L_{\xi, \xi}$ on $I(E)$; that is,

$$\int \phi d\mu_\xi = L_{\xi, \xi}(\phi) \quad \text{for all} \quad \xi \in H \quad \text{and all} \quad \phi \in I(E).$$

Lemma 7. For each fixed function $\phi \in I(E)$, the functional $(\xi, \eta) \rightsquigarrow L_{\xi,\eta}(\phi) : H \times H \to C$ is sesquilinear; that is,

(a) $L_{\xi_1+\xi_2,\eta}(\phi) = L_{\xi_1,\eta}(\phi) + L_{\xi_2,\eta}(\phi)$

(b) $L_{c\xi,\eta}(\phi) = cL_{\xi,\eta}(\phi)$

(c) $L_{\xi,\eta_1+\eta_2}(\phi) = L_{\xi,\eta_1}(\phi) + L_{\xi,\eta_2}(\phi)$

(d) $L_{\xi,c\eta}(\phi) = \bar{c}L_{\xi,\eta}(\phi)$.

In addition,

(e) $L_{\eta,\xi}(\phi) = \overline{L_{\xi,\eta}(\bar{\phi})}$.

The proof is as in [2, pp. 24, 25].

We now direct our attention to $K\{M_p\}$. Recall that $K\{M_p\} \subset I(E)$, (Lemma 2), and

$$\|\phi\|_p = \sup_{x \in R^n} \sup_{|q| \leq p} M_p(x)|(D^q\phi)(x)|, \quad p = 0, 1, 2, \ldots .$$

For every fixed $\phi \in K\{M_p\}$, we have a mapping $(\xi, \eta) \rightsquigarrow L_{\xi,\eta}(\phi) : H \times H \to C$ where $L_{\xi,\eta}(\phi)$ is the sesquilinear form on $H \times H$ given by (4).

Lemma 8. For every $\phi \in K\{M_p\}$, $(\xi,\eta) \rightsquigarrow L_{\xi,\eta}(\phi) : H \times H \to C$ is a bounded sesquilinear form.

Proof. Recall that $E(\cdot)$ is an $\{M_p\}$-tempered PO-measure. Thus, there is a positive integer p and a positive number C such that

$$\int (M_p)^{-1} d\mu_\lambda \leq C\|\lambda\|^2$$

for every vector λ in H. With this positive integer p we have

$$L_{\xi,\eta}(\phi) = \frac{1}{4}\left[\int \phi d\mu_{\xi+\eta} - \int \phi d\mu_{\xi-\eta} + i\int \phi d\mu_{\xi+i\eta} - i\int \phi d\mu_{\xi-i\eta}\right]$$

$$= \frac{1}{4}\left[\int M_p \phi (M_p)^{-1} d\mu_{\xi+\eta} - \int M_p\phi(M_p)^{-1} d\mu_{\xi-\eta}\right.$$

$$\left. + i\int M_p\phi(M_p)^{-1} d\mu_{\xi+i\eta} - i\int M_p\phi(M_p)^{-1} d\mu_{\xi-i\eta}\right],$$

so that

$$|L_{\xi,\eta}(\phi)| \leq \int M_p|\phi|(M_p)^{-1} d\mu_{\xi+\eta} + \int M_p|\phi|(M_p)^{-1} d\mu_{\xi-\eta}$$

$$+ \int M_p|\phi|(M_p)^{-1} d\mu_{\xi+i\eta} + \int M_p|\phi|(M_p)^{-1} d\mu_{\xi-i\eta}$$

$$\leq \int \left(\sup_x \sup_{|q|\leq p} M_p(x)|(D^q\phi)(x)|\right)(M_p)^{-1}(x) d\mu_{\xi+\eta}$$

$$+ \int \left(\sup_x \sup_{|q|\leq p} M_p(x)|(D^q\phi)(x)|\right)(M_p)^{-1}(x) d\mu_{\xi-\eta}$$

$$+ \int \left(\sup_x \sup_{|q|\leq p} M_p(x)|(D^q\phi)(x)\right)(M_p)^{-1}(x) d\mu_{\xi+i\eta} \quad +$$

$$+\int\left(\sup_x \sup_{|q|\le p} M_p(x)\,|(D^q\phi)(x)|\right)(M_p)^{-1}(x)\,d\mu_{\xi-i\eta}$$

$$= \|\phi\|_p\left[\int (M_p)^{-1}d\mu_{\xi+\eta} + \int (M_p)^{-1}d\mu_{\xi-\eta} + \int (M_p)^{-1}d\mu_{\xi+i\eta} + \int (M_p)^{-1}d\mu_{\xi-i\eta}\right]$$

$$\le C\|\phi\|_p\{\|\xi+\eta\|^2+\|\xi-\eta\|^2+\|\xi+i\eta\|^2+\|\xi-i\eta\|^2\}$$

$$= 4C\|\phi\|_p(\|\xi\|^2+\|\eta\|^2).$$

Therefore

$$\|L_{\xi,\eta}(\phi)\| = \sup\{|L_{\xi,\eta}(\phi)|,\ \xi\in H,\ \eta\in H,\ \|\xi\|\le 1,\ \|\eta\|\le 1\}$$

$$= \sup\{4C\|\phi\|_p(\|\xi\|^2+\|\eta\|^2),\ \|\xi\|\le 1,\ \|\eta\|\le 1\}$$

$$\le 8C\|\phi\|_p < \infty.$$

Thus for each $\phi \in K\{M_p\}$, $|L_{\xi,\eta}(\phi)| \le 8C\|\phi|_p\|\xi\|\,\|\eta\|$ for every ξ, η in H.

Combining Lemmas 7 and 8, for each fixed $\phi \in K\{M_p\}$, $(\xi,\eta)\rightsquigarrow L_{\xi,\eta}(\phi): H \times H \to C$ is a <u>bounded sesquilinear form</u>. By Proposition 2 of Sec. C, there exists a <u>unique bounded linear operator</u> S on H such that

(8) $$L_{\xi,\eta}(\phi) = \langle S\xi, \eta\rangle \quad \text{for all vectors } \xi,\eta \in H.$$

Definition 5. The unique bounded linear operator S on H such that $L_{\xi,\eta}(\phi) = \langle S\xi, \eta\rangle$ for any given fixed $\phi \in I(E)$ and all pairs ξ, η of vectors in H will be denoted by

$$\int \phi dE. \tag{9}$$

Thus, to each $\phi \in K\{M_p\}$ there corresponds a unique bounded linear operator $\int \phi dE$ on H such that

$$\left\langle \left(\int \phi dE\right)\xi, \eta \right\rangle = L_{\xi,\eta}(\phi) \tag{10}$$

for every pair ξ, η of vectors in H. By Lemma 6, we have

$$\int \phi d\mu_\xi = \left\langle \left(\int \phi dE\right) \xi, \xi \right\rangle \quad \text{for all} \quad \xi \in H. \tag{11}$$

E. Basic Properties of $\int \phi dE$

(1) The map $\phi \rightsquigarrow \int \phi dE : K\{M_p\} \to B(H)$ is linear. Moreover,

$$\int \bar{\phi} dE = \left(\int \phi dE \right)^*,$$

where $*$ denotes the adjoint. The proof is as in [2, p. 28].

(2) For some integer $p \geq 0$, there is a positive constant C such that $\| \int \phi dE \| \leq C \|\phi\|_p$ for every $\phi \in K\{M_p\}$.

Proof. Let ϕ be any real-valued function in $K\{M_p\}$. Since $E(\cdot)$ is an $\{M_p\}$-tempered PO-measure, there is an integer $p \geq 0$ and a positive constant k such that

$$\int (M_p)^{-1} d\mu_\xi \leq k\|\xi\|^2 \quad \text{for all} \quad \xi \in H.$$

Set

$$A = \int \phi dE \qquad (A = A^*).$$

Then

$$<A\xi, \xi> = L_{\xi,\xi}(\phi) = \int \phi d\mu_\xi$$

$$= \int (M_p)\phi(M_p)^{-1} d\mu_\xi .$$

Therefore

$$|<A\xi, \xi>| = \left| \int (M_p)\phi(M_p)^{-1} d\mu_\xi \right|$$

$$\leq \int (M_p)|\phi|(M_p)^{-1} d\mu_\xi$$

$$\leq \int \sup_{\substack{x \\ |q| \leq p}} M_p(x)|(D^q\phi)(x)|(M_p)^{-1}(x) d\mu_\xi$$

$$\leq \|\phi\|_p \int (M_p)^{-1}(x) d\mu_\xi \leq k\|\phi\|_p \|\xi\|^2$$

Thus

$$\|A\| \le k\|\phi\|_p .$$

That is,

$$\left\|\int \phi dE\right\| \le k\|\phi\|_p .$$

Finally, if $\phi = \phi_1 + i\phi_2$, then $\|\phi_1\|_p \le \|\phi\|_p$ and $\|\phi_2\|_p \le \|\phi\|_p$, so that

$$\left\|\int \phi dE\right\| = \left\|\int (\phi_1 + i\phi_2) dE\right\| = \left\|\int \phi_1 dE + \int i\phi_2 dE\right\|$$
$$\le 2k(\|\phi_1\|_p + \|\phi_2\|_p) \le 4k\|\phi\|_p .$$

In any case, then,

$$\left\|\int \phi dE\right\| \le C\|\phi\|_p$$

for some $p \ge 0$, some positive constant C and all ϕ in $K\{M_p\}$.

(3) If $\phi, \phi_n \in K\{M_p\}$, $n = 1, 2, \ldots,$ and $\phi_n \to \phi$ in $K\{M_p\}$, then $\left\|\int \phi_n dE - \int \phi dE\right\| \to 0$. That is, $\int \phi_n dE \to \int \phi dE$ uniformly.

<u>Proof</u>. Let $\phi, \phi_n \in K\{M_p\}$, $n = 1, 2, \ldots,$ and suppose that $\phi_n \to \phi$ in $K\{M_p\}$. Then $\|\phi_n - \phi\|_p \to 0$ as $n \to \infty$ for $0 \le p < \infty$. Hence, for some p,

$$\left\| \int \phi_n dE - \int \phi dE \right\| = \left\| \int (\phi_n - \phi) dE \right\|$$

$$\leq 4k \| \phi_n - \phi \|_p$$

(by (2) above). Thus,

$$\left\| \int \phi_n dE - \int \phi dE \right\| \to 0 \quad \text{as} \quad n \to \infty .$$

2. OPERATOR-VALUED DISTRIBUTIONS

The principal notion in this chapter is that of an operator-valued distribution. The scalar distributions of L. Schwartz are discussed briefly as motivation for the definition of distributions of operators on various fundamental spaces of test functions. Examples of nonmultiplicative as well as multiplicative distributions of operators are given. Integral representation theorems for positive and positive-definite distributions of operators due to Kritt [8] are stated. The chapter concludes with the proof of our first theorem which generalizes a result on the integral representation of positive distributions of operators on $\mathcal{D}(R^n)$ to positive distributions of operators on the broad class of fundamental spaces $K\{M_p\}$.

A. Distributions of Operators

The space $C_0^\infty(\Omega)$ of infinitely differentiable functions with compact support in a nonempty open subset Ω of R^n is fundamental in the theory of distributions. $C_0^\infty(\Omega)$ is a linear space under the usual definition of pointwise addition of functions and multiplication of a function by a scalar. If K is any nonempty compact subset of Ω, let $\mathcal{D}_K(\Omega)$ denote the set of the $C_0^\infty(\Omega)$-functions with support in K. The family of seminorms on $\mathcal{D}_K(\Omega)$, defined by

$$p_{K,m}(\phi) = \sup_{\substack{x \in K \\ |\alpha| \leq m}} \{|(D^{\alpha}\phi)(x)|\}, \quad m = 0, 1, 2, \ldots ,$$

induces a locally convex topology on $\mathcal{D}_K(\Omega)$ such that, if $K_1 \subset K_2$, then the topology on $\mathcal{D}_{K_1}(\Omega)$ is the same as the relative topology on $\mathcal{D}_{K_1}(\Omega)$ as a subspace of $\mathcal{D}_{K_2}(\Omega)$. Let K_i, $i = 1, 2, \ldots,$ be a sequence of compact subsets of Ω such that $K_i \subset \overset{\circ}{K}_{i+1}$ and $\bigcup_{i=1}^{\infty} K_i = \Omega$. $\overset{\circ}{K}_{i+1}$ denotes the interior of K_{i+1}. The topological vector space $\mathcal{D}(\Omega)$ of L. Schwartz is then the strict inductive limit of the spaces $\mathcal{D}_{K_i}(\Omega)$. The elements of $\mathcal{D} = \mathcal{D}(\Omega) = C_0^{\infty}(\Omega)$ are called test functions. We let $\mathcal{D}' = \mathcal{D}'(\Omega)$ denote the dual space of all continuous linear maps from $\mathcal{D}(R^n)$ to the field of complex numbers C with C carrying the usual topology of the complex plane. Thus $\mathcal{D}' = (\mathcal{D}(\Omega), C)$ is the space of distributions of Schwartz.

Let $H = (H, <, >)$ be a complex Hilbert space, $H \neq \{0\}$, with norm $\| \|$ induced by the positive-definite sesquilinear form $<, >$ on H by $\|\xi\| = \sqrt{<\xi, \xi>}$, $\xi \in H$. Let $B(H) = (B(H), \| \|)$ denote the Banach algebra of all bounded linear operators on H. $B(H)$ is given the <u>uniform operator topology</u> induced by the norm $\| \|$ defined by $\|T\| = \sup\{\|T\xi\| : \xi \in H, \|\xi\| \leq 1\}$ for $T \in B(H)$. Thus if $T_n, T \in B(H)$, $n = 1, 2, \ldots,$ then $\|T_n - T\| \to 0$ if and only if $\|T_n\xi - T\xi\| \to 0$ uniformly on the closed unit ball $\{\xi \in H: \|\xi\| \leq 1\}$.

Let $\mathcal{D}'_H(\Omega) = L(\mathcal{D}(\Omega), B(H))$ denote the set of all continuous linear maps from $\mathcal{D}(\Omega)$ to $B(H)$. If $T \in \mathcal{D}'_H$, the "value" in $B(H)$ of T at $\phi \in \mathcal{D}(\Omega)$ is the unique bounded linear operator on H denoted by T_ϕ. $\mathcal{D}'_H$ is clearly a linear space with

$$(S+T)_\phi = S_\phi + T_\phi$$

$$(\alpha T)_\phi = \alpha(T_\phi) ,$$

for all $S, T \in \mathcal{D}'_H$, all $\phi \in \mathcal{D}(\Omega)$, and all complex numbers α. As is usually the case, the value of the operator T_ϕ at the vector $\xi \in H$ will be denoted by $T_\phi \xi$.

Definition 1. An element T in $\mathcal{D}'_H(\Omega)$ is called a distribution of operators in Ω or a distribution of operators on $\mathcal{D}(\Omega)$.

There corresponds to each vector $\xi \in H$ a unique continuous scalar-valued function on $\mathcal{D}(\Omega)$ determined by $<T_\phi \xi, \xi>$, $\phi \in \mathcal{D}(R^n)$. We thus obtain, for each ξ in H, a Schwartz distribution $T^\xi \in \mathcal{D}'(\Omega)$,

$$T^\xi : \mathcal{D}(\Omega) \to C$$

by

$$T^\xi(\phi) = <T_\phi \xi, \xi> \quad \forall \quad \phi \in \quad (R^n)$$

It is customary in the scalar setting to indicate the value of T^ξ at ϕ by (T^ξ, ϕ). Thus for any given $\xi \in H$, we have a $T^\xi \in \mathcal{D}'(\Omega)$ defined by

$$(T^\xi, \phi) = \langle T_\phi \xi, \xi \rangle, \quad \text{for all} \quad \phi \in \mathcal{D}(\Omega).$$

The point of all this is that a distribution of operators on Ω gives rise to a family of Schwartz distributions indexed by the set of vectors in the underlying Hilbert space. Henceforth the term <u>scalar-distribution</u> refers to the elements of $\mathcal{D}'(\Omega) = \mathcal{D}'_C(\Omega)$, where C is the Hilbert space of complex numbers with the usual inner product. Our main concern is with <u>distributions of operators in R^n</u>.

A scalar distribution T is positive if $(T, \phi) \geq 0$ whenever $\phi \in \mathcal{D}(\Omega)$ is nonnegative.

<u>Definition 2.</u> A distribution of operators T in Ω is <u>positive</u> if the scalar distribution T^ξ is positive for each $\xi \in H$. Thus, T is positive iff T_ϕ is a positive operator whenever ϕ is nonnegative:

$$\phi \geq 0 \Rightarrow \langle T_\phi \xi, \xi \rangle \geq 0 \quad \forall \quad \xi \in H.$$

A scalar distribution T is positive-definite if $(T, \phi * \phi^*) \geq 0$ for each $\phi \in \mathcal{D}(\Omega)$, where ϕ^* is defined by $\phi^*(x) = \overline{\phi(-x)}$, the bar denoting complex conjugation, and $\phi * \phi^*$ is the convolution of ϕ and ϕ^*:

$$\phi*\phi^*(x) = \int_{R^n} \phi(y)\phi^*(x-y)dy \ .$$

Definition 3. A distribution of operators T in R^n is positive-definite if the scalar distribution T^ξ is positive-definite for every $\xi \in H$. Thus, T is positive-definite iff for every $\phi \in \mathcal{D}(R^n)$,

$$<T_{\phi*\phi^*}\xi, \xi> \geq 0 \quad \text{for each} \quad \xi \in H.$$

Definition 4. A distribution of operators T in R^n is tempered if T is continuous on $\mathcal{D}(R^n)$ in the relative topology on $\mathcal{D}(R^n)$ as a subspace of $\mathcal{S}(R^n)$. Thus the tempered distributions of operators are just those operator distributions T which have continuous extensions S to $\mathcal{S}(R^n)$. These extensions are unique since $\mathcal{D}(R^n)$ is dense in $\mathcal{S}(R^n)$.

With the notion of a tempered distribution of operators in hand we use the usual definition of the Fourier transformation of a tempered distribution of operators.

Definition 5. The Fourier transform of a tempered distribution of operators T is that distribution of operators $\hat{T}$ defined by

$$\hat{T}_\phi = T_{\hat{\phi}} \quad \text{for every} \quad \phi \in \mathcal{D}(R^n).$$

$\hat{T}$ is tempered and therefore has a unique extension to $\mathcal{S}(R^n)$.

Definition 6. A distribution of operators T in R^n is said to be $\{M_p\}$-tempered if T is continuous in the relative topology on $\mathcal{D}(R^n)$ as a subspace of $K\{M_p\}$. Since $\mathcal{D}(R^n)$ is dense in $K\{M_p\}$, the $\{M_p\}$-tempered distributions of operators are just those that have continuous extensions to $K\{M_p\}$. These extensions are unique. In case $M_p(x) = (1+|x|)^p$, the $K\{M_p\}$ space is $\mathcal{S}(R^n)$ and M_p-tempered specializes to tempered as defined earlier.

Definition 7. Let T be a distribution of operators. T is multiplicative if and only if $T_{\phi_1\phi_2} = T_{\phi_1} T_{\phi_2}$ for every pair ϕ_1, ϕ_2 in $\mathcal{D}(\Omega)$.

Note. The product $T_{\phi_1} T_{\phi_2}$ of the operators T_{ϕ_1} and T_{ϕ_2} is defined in the usual way by composition:

$$(T_{\phi_1} T_{\phi_2})\xi = T_{\phi_1}(T_{\phi_2}\xi) \text{ for all } \xi \in H.$$

The multiplicative distributions of operators are the operator-valued algebra homomorphisms on $\mathcal{D}(\Omega)$. The spectral theory of normal operators is concerned with the case of multiplicative and positive distributions of operators [2]. We will be concerned in this thesis with distributions of operators that are not required to satisfy the condition of being multiplicative.

B. Examples

1. Let $S \in \mathcal{D}'(R^n)$, $B \in B(H)$. Then $T = S \otimes B$ defines an element in $\mathcal{D}'_H(R^n)$ by the equation

$$T_\phi = (S \otimes B)_\phi = (S, \phi)B.$$

T is obviously a distribution of operators.

In particular, if $B = I$, the identity operator on H, then the mapping $\phi \rightsquigarrow T_\phi : \mathcal{D}(R^n) \to B(H)$, defined by $T_\phi = \phi(0)I$ clearly belongs to $\mathcal{D}'_H(R^n)$. For a given vector $\xi \in H$, T^ξ defined by $(T^\xi, \phi) = \langle T_\phi \xi, \xi \rangle$ belongs to $\mathcal{D}'_C(R^n)$. Since $(T^\xi, \phi) = \langle \phi(0)\xi, \xi \rangle = \phi(0)\|\xi\|^2$, choosing ξ from the unit sphere in H leads to $(T^\xi, \phi) = \phi(0) = (\delta, \phi)$, where δ is the Dirac measure on R^n. Thus this special case of the first example, namely, $T = \delta \otimes I$, is a Dirac distribution of operators. More generally, $\delta \otimes B$ for any $B \in B(H)$ is called a Dirac distribution of operators. Note that $T = S \otimes B$ is a positive element in $\mathcal{D}'_H(R^n)$ if $S \in \mathcal{D}'_C(R^n)$ is positive and B is a positive operator.

2. (a). Let $x \rightsquigarrow A(x): R^n \to B(H)$ be continuous and define $T: \mathcal{D}(R^n) \to B(H)$ by $T_\phi = \int \phi(x)A(x)dx$, where the integral is in the sense of Riemann.

(b). Let $\phi_\xi(x) = A(x)\xi$ be continuous into H for any ξ.

Define T by

$$<T_\phi \eta, \xi> = \int \phi(x) <A(x)\eta, \xi> dx .$$

In connection with the above examples see also [1, pp. 409, 413], [7, p. 268] and [9, p. 364].

3. The distribution of operators defined by $T_\phi f = \phi f$ for every $f \in L^2(R^1)$ is clearly <u>multiplicative</u>. Moreover,

$$<T_\phi f, f> = <\phi f, f> = \int (\phi f)\overline{f} = \int \phi |f|^2 \geq 0$$

whenever $\phi \geq 0$. Thus T is a positive distribution of operators.

We now give in some detail examples of specific distributions of operators, most of which are nonmultiplicative, some of which are positive and others positive-definite.

4. Consider the map $\phi \rightsquigarrow T_\phi : \mathcal{D}(R^n) \to B(L^2(R^n))$ given by $T_\phi f = \phi * f$ for every f in $L^2(R^n)$. We have, for every $f, g \in L^2$ and all complex numbers α,

$$\begin{aligned} T_\phi(\alpha f+g) &= \phi*(\alpha f+g) \\ &= \phi*(\alpha f) + \phi*g = \alpha(\phi*f) + \phi*g \\ &= \alpha T_\phi f + T_\phi g . \end{aligned}$$

Thus, T is linear and T_ϕ is clearly a linear operator on $L^2(R^n)$ for every $\phi \in \mathcal{D}(R^n)$.

For any $\phi \in \mathcal{D}$,

$$\|T_\phi\| = \sup_{\|f\|_{L^2} \le 1} \|\phi * f\|_{L^2}$$

$$\le \sup_{\|f\|_{L^2} \le 1} \|\phi\|_{L^1} \|f\|_{L^2}$$

$$\le \|\phi\|_{L^1} = \int_{R^n} |\phi(x)|\,dx < \infty.$$

Thus T_ϕ is a bounded operator on $L^2(R^n)$ for each $\phi \in \mathcal{D}(R^n)$ and we have $\|T_\phi\| \le \|\phi\|_{L^1}$. Let K be an arbitrary compact subset of R^n.

$$\|T_\phi\| = \sup_{\|f\|_{L^2} \le 1} \|T_\phi f\|_{L^2} = \sup_{\|f\|_{L^2} \le 1} \|\phi * f\|_{L^2}$$

$$\le \sup_{\|f\|_{L^2} \le 1} \|\phi\|_{L^1} \|f\|_{L^2} \le \|\phi\|_{L^1}$$

$$= \int_{R^n} |\phi(x)|\,dx = \int_{\text{supp}\,\phi} |\phi(x)|\,dx \le \int_K |\phi(x)|\,dx$$

$$\le \int_K \sup_{\substack{x \in K \\ 0 \le \alpha \le m}} |D^\alpha \phi(x)|\,dx = \int_K p_{K,m}(\phi)\,dx$$

$$= |K|_n \, p_{K,m}(\phi),$$

where $|K|_n$ is the n-dimensional Lebesgue measure of the set K. Set $|K|_n = C_K$. Thus, $\|T_\phi\| \leq C_K p_{K,m}(\phi)$. Hence, T is continuous on every space $\mathcal{D}_K(R^n)$. Since $\mathcal{D}(R^n)$ is the inductive limit of the spaces $\mathcal{D}_K(R^n)$ as K varies throughout the family of all compact subsets of R^n, T is continuous on $\mathcal{D}(R^n)$ and is therefore a <u>distribution of operators</u>. Since $(\phi_1\phi_2)*f \neq (\phi_1 * \phi_2)*f$ in general,

$$T_{\phi_1\phi_2} f = (\phi_1\phi_2)*f \neq (\phi_1 * \phi_2)*f$$

$$= \phi_1 * (\phi_2 * f) = T_{\phi_1}(T_{\phi_2} f) = T_{\phi_1} T_{\phi_2} f .$$

Thus T is <u>nonmultiplicative</u>.

Finally, for all $\phi \in \mathcal{D}(R^n)$ and any $f \in L^2(R^n)$,

$$<T_{\phi*\phi^*} f, f>_{L^2} = <(\phi*\phi^*)*f, f>_{L^2}$$

$$= <\widehat{(\phi*\phi^*)*f}, \hat{f}>_{L^2} = <\widehat{\phi*\phi^*}\hat{f}, \hat{f}>_{L^2}$$

$$= \int (\widehat{\phi*\phi^*}\hat{f})\overline{\hat{f}} = (2\pi)^{n/2} \int \hat{\phi}\overline{\hat{\phi}}\hat{f}\overline{\hat{f}}$$

$$= (2\pi)^{n/2} \int |\hat{\phi}|^2 |\hat{f}|^2 \geq 0 .$$

Hence, T is <u>positive-definite</u>.

5. Let the map $\phi \rightsquigarrow T_\phi : \mathcal{D}(R^1) \to B(L^2(R^1))$ be defined by $T_\phi f(t) = t^n \phi(t) f(t)$ for every $f \in L^2(R^1)$, where n is a nonnegative integer. If $n = 0$, this is just Example 3. Let $n > 0$. Then

$$\begin{aligned} T_{\alpha\phi_1+\phi_2} f(t) &= t^n(\alpha\phi_1+\phi_2)(t)f(t) \\ &= t^n(\alpha\phi_1(t)+\phi_2(t))f(t) \\ &= \alpha t^n\phi_1(t)f(t) + t^n\phi_2(t)f(t) \\ &= \alpha T_{\phi_1} f(t) + T_{\phi_2} f(t) . \end{aligned}$$

Thus T is <u>linear.</u>

For each ϕ, T_ϕ is clearly a linear operator on L^2 and

$$\|t^n\phi f\|^2_{L^2} = \int_{R^1} |t^n\phi f|^2 dt$$

$$\leq \sup_{t\in K} |t^n\phi|^2 \int_K |f|^2 dt \leq \sup_{t\in K} |t^n\phi|^2 \int_{R^1} |f|^2 dt.$$

Thus

$$\|t^n\phi(t)f(t)\|^2_{L^2} \leq C_K \|f\|^2_{L^2} ,$$

where

$$C_K = \sup_{t\in K} |t^n\phi(t)|^2 \geq 0 .$$

Therefore

$$\|T_\phi\| = \sup_{\|f\|_{L^2} \leq 1} \|t^n \phi f\|_{L^2} \leq \sqrt{C_K} < \infty,$$

so that T_ϕ is a <u>bounded</u> operator on $L^2(R^1)$ for every $\phi \in \mathcal{D}(R^1)$.

To establish the continuity of T, let $\phi_j \to \phi$ in $\mathcal{D}(R^1)$. This means that there is a compact subset K of R^1 such that $\operatorname{supp} \phi_j \subset K$ for all j, and for every nonnegative integer α, $D^\alpha \phi_j \to D^\alpha \phi$ uniformly on K. In particular, $|\phi_j - \phi| \to 0$ uniformly on K and therefore so does $|\phi_j - \phi|^2$. We have

$$\|t^n \phi_j f - t^n \phi f\|_{L^2}^2 = \int_{R^1} |t^n \phi_j f - t^n \phi f|^2 dt$$

$$= \int_K |\phi_j - \phi|^2 |t^n f|^2 dt \leq \sup_K |\phi_j - \phi|^2 \sup_K |t^n| \int |f|^2 dt$$

Hence

$$\|T_{\phi_j} - T_\phi\| \leq \sup_K |\phi_j - \phi|^2 \sup_K |t^n| \int |f|^2 dt$$

$\to 0$ as $j \to \infty$. Thus T is continuous, hence, a distribution of operators.

If n is <u>even</u>, T is a <u>positive</u> distribution of operators, since $n \geq 0$, n even, and $\phi \geq 0$ yield

$$<T_\phi f, f> = <t^n \phi f, f> = \int t^n \phi |f|^2 dt \geq 0$$

for all $f \in L^2(R^1)$.

Finally, T is <u>nonmultiplicative</u> for $n > 0$:

$$T_{\phi_1\phi_2} f(t) = t^n \phi_1(t)\phi_2(t)f(t) ,$$

whereas

$$T_{\phi_1}(T_{\phi_2} f(t)) = T_{\phi_1}(t^n \phi_2(t)f(t))$$

$$= t^n \phi_1(t)[t^n \phi_2(t)f(t)] = t^{2n} T_{\phi_1\phi_2} f(t)$$

$$\neq T_{\phi_1\phi_2} f(t) = t^n \phi_1\phi_2 f .$$

6. Let $\phi \rightsquigarrow T_\phi : \mathcal{D}(R^1) \to B(L^2(R^1))$ be defined by $T_\phi f = \hat{\phi} f$ for every $f \in L^2(R^1)$. Since the Fourier transform is linear, the linearity of T is clear. It is also clear that T_ϕ is a linear operator on $L^2(R^1)$ for each $\phi \in \mathcal{D}(R^1)$. Noting that $\hat{\phi}_1 * \hat{\phi}_2 \neq \hat{\phi}_1 \hat{\phi}_2$ in general, we have

$$T_{\phi_1\phi_2} f = \widehat{\phi_1\phi_2} f = (2\pi)^{-n/2} (\hat{\phi}_1 * \hat{\phi}_2) f$$

$$\neq (\hat{\phi}_1 \hat{\phi}_2) f = T_{\phi_1}(\hat{\phi}_2 f) = T_{\phi_1}(T_{\phi_2} f) = T_{\phi_1} T_{\phi_2} f .$$

Thus T is <u>nonmultiplicative</u>.

We verify the continuity of T. Let $\operatorname{supp}\phi \subset K$.

$$\|\widehat{\phi f}\|_{L^2} = \langle \widehat{\phi f}, \widehat{\phi f}\rangle_{L^2}^{1/2}$$

$$= \langle \widehat{\widehat{\phi f}}, \widehat{\widehat{\phi f}}\rangle_{L^2}^{1/2}$$

$$= (2\pi)^{-n/2} \langle \hat{\hat{\phi}} * \hat{f}, \hat{\hat{\phi}} * \hat{f}\rangle_{L^2}^{1/2}$$

$$= (2\pi)^{-n/2} \langle \check{\phi} * \hat{f}, \check{\phi} * \hat{f}\rangle_{L^2}^{1/2} \qquad (\check{\phi}(x) = \phi(-x))$$

$$= \|\check{\phi} * \hat{f}\|_{L^2} \leq \|\check{\phi}\|_{L^1} \|\hat{f}\|_{L^2}$$

$$= \|\phi\|_{L^1} \|f\|_{L^2}$$

$$= \left(\int_{R^1} |\phi(x)|\,dx\right) \|f\|_{L^2}$$

$$\leq \left(\int_K \sup_{\substack{x \in K \\ 0 \leq |\alpha| \leq m}} |D^\alpha \phi(x)|\,dx\right) \|f\|_{L^2}$$

$$\leq p_{K,m}(\phi) |K|_1 \|f\|_{L^2},$$

where $|K|_1$ is the Lebesgue measure of the compact subset K of R^1 and $\phi \in \mathcal{D}_K$. Therefore

$$\|T_\phi\| = \sup_{\|f\|_{L^2} \le 1} \|T_\phi f\|_{L^2} = \sup_{\|f\|_{L^2} \le 1} \|\hat{\phi} f\|_{L^2}$$

$$\le \sup_{\|f\|_{L^2} \le 1} p_{K,m}(\phi) |K_1| \, \|f\|_{L^2}$$

$$\le C_K p_{K,m}(\phi)$$

where $C_K = |K|_1$. Thus T is continuous on each space $\mathcal{D}_K(R^1)$ and therefore T is continuous on $\mathcal{D}(R^1)$. Hence, T is a distribution of operators in R^1.

In addition, for every $\phi \in \mathcal{D}(R^1)$, we have

$$<T_{\phi * \phi^*} f, f> = \int \widehat{\phi * \phi^*} f \bar{f} = (2\pi)^{n/2} \int \hat{\phi} \bar{\hat{\phi}} f \bar{f}$$

$$= (2\pi)^{n/2} \int |\hat{\phi}|^2 |f|^2 \ge 0 \quad \text{for every} \quad f \in L^2(R^1),$$

so that T is positive-definite.

C. Integral Representations of Distributions of Operators

We shall make use of the following two propositions, due to B. Kritt [8], concerning positive and positive-definite distributions of operators.

Proposition 1 (Kritt). For every positive distribution T of operators in an open subset Ω of R^n there is a unique PO measure

$E(\cdot)$ on the ring R generated by the compact subsets of Ω such that

$$T_\phi = \int \phi dE$$

for every test function ϕ from $\mathcal{D}(\Omega)$. If in addition $\Omega = R^n$ and T is tempered, then so is $E(\cdot)$.

Recall that $E(\cdot)$ is tempered means that there is a positive integer p and a positive number A such that for every vector ξ from H,

$$\int (1+|x|^2)^{-p} d\mu_\xi(x) \leq A \|\xi\|^2,$$

where μ_ξ is the positive Borel measure, on the class of Borel sets in R^n, uniquely determined by $\mu_\xi(M) = \langle E(M)\xi, \xi\rangle$ for all M in R.

<u>Proposition 2</u> (Kritt). For every positive-definite distribution T of operators in R^n there is a unique tempered PO measure $E(\cdot)$ on the ring R generated by the compact subsets of R^n such that

$$T_\phi = \int \hat{\phi} dE \qquad \text{for every } \phi \in \mathcal{D}(R^n),$$

where $\hat{\phi}$ denotes the Fourier transform of ϕ.

See [8] for the definitions of the integrals occurring in Propositions 1 and 2.

The following result is our first theorem in this thesis and deals with a generalization of Kritt's Proposition 1 to a class of distributions of operators defined on $K\{M_p\}$ spaces.

<u>Theorem 1</u>. Let the space $K\{M_p\}$ satisfy the following conditions:

(a) The functions M_p are C^∞ in the complement of some common bounded neighborhood of the origin in R^n;

(b) for any nonnegative integer p there are numbers q and C_p such that if $0 \leq |k| \leq p$, then $|[(M_q)^{-1}]^{(k)}(x)| \leq C_p(M_p)^{-1}(x)$ in the complement of some neighborhood of zero in R^n.

Then to each positive distribution of operators T on $K\{M_p\}$ there corresponds a unique $\{M_p\}$-tempered positive operator-valued measure $E(\cdot)$ on the ring generated by the compact subsets of R^n such that

$$T_\phi = \int \phi dE \quad \text{for every } \phi \text{ in } K\{M_p\} .$$

<u>Proof</u>. Let T be a positive distribution of operators on $K\{M_p\}$. Thus, $\phi \rightsquigarrow T_\phi : K\{M_p\} \to B(H)$ is a positive continuous linear mapping. Since $\mathcal{D}(R^n) \subset K\{M_p\}$ and the topology on

$\mathcal{D}(R^n)$ is stronger than the relative topology on $\mathcal{D}(R^n)$ induced as a subspace of $K\{M_p\}$, T is continuous on $\mathcal{D}(R^n)$. T is obviously positive and linear on $\mathcal{D}(R^n)$. According to Proposition 1 (Kritt) there exists a unique PO-measure $E(\cdot)$ on the ring generated by the compact subsets of R^n such that

$$T_\phi = \int \phi dE \quad \text{for all} \quad \phi \text{ in } \mathcal{D}(R^n). \tag{1}$$

We will first show that $E(\cdot)$ is $\{M_p\}$-tempered and that therefore $\int \phi dE$ is defined for all ϕ in $K\{M_p\}$. The following discussion and lemmas prepare us to prove that $E(\cdot)$ is an $\{M_p\}$-tempered PO-measure.

Let $\psi \in \mathcal{D}(R^n)$, $0 \leq \psi \leq 1$, $\psi(x) = 1$ on $\{x: |x| \leq 1\}$ and $\psi(x) = 0$ on $\{x: |x| \geq 2\}$. Then $\operatorname{supp} \psi = \{x: |x| \leq 2\}$. Set $\psi_m(x) = \psi(x/m)$, $m = 1, 2, 3, \ldots$. Then $\psi_m(x) = 1$ on $\{x: |x| \leq m\}$, $\psi_m(x) = 0$ on $\{x: |x| \geq 2m\}$, and $\operatorname{supp} \psi_m \subset \operatorname{supp} \psi_{m+1}$ for all m. Thus the functions ψ_m "flatten out" with increasing m. Let

$$\sup_x \sup_{|q| \leq p} |(D^q\psi)(x)| = r_p \tag{2}$$

Then

$$\sup_x \sup_{|q| \leq p} (|D^q\psi_m)(x)| \leq r_p \quad \text{for every} \quad m. \tag{3}$$

Recall now that a neighborhood $U = U_{p,\eta}$ of zero in $K\{M_p\}$

is determined by a fixed nonnegative integer p and a positive number η and is defined by

(4) $$U = \{\phi \in K\{M_p\}: \sup_x \sup_{|q| \leq p} M_p(x)|(D^q\phi)(x)| < \eta\}.$$

We may assume, without loss of generality, that the neighborhood Q of zero in R^n in the complement of which all the M_p and C^∞ is a closed ball. Let $\mathcal{O}$ be an open neighborhood of Q and let $R > 0$ be such that $N = \{x: |x| \leq R\} \supset \mathcal{O}$. The set N is fixed in what follows. Fix a C^∞-function h, $0 \leq h \leq 1$, such that $h = 0$ in $\mathcal{O}$ and $h = 1$ in N^c, the complement of N. Then

(6) $$\sup_x \sup_{|q| \leq p} |(D^q h)(x)| = s_p < \infty.$$

With h and ψ as above define a sequence $\{\phi_m\}$ as follows:

(7) $$\phi_m(x) = Ah(x)\psi_m(x)(M_{q_0})^{-1}(x),$$

where A is a positive constant, $\psi_m(x) = \psi(x/m)$, $m = 1, 2, \ldots,$ and q_0 is chosen as in hypothesis (b) of the theorem. Note that $\phi_m \in \mathcal{D}(R^n)$ and $\phi_m \geq 0$ for each m.

<u>Lemma 1</u>. The positive constant A can be chosen small enough to ensure that $\phi_m \in U$ for every m.

Proof. Since $U = \{\phi \in K\{M_p\} : \|\phi\|_p < \eta\}$, we need to show that A can be chosen in such a way that

$$\|\phi_m\|_p = \sup_x \sup_{|q| \leq p} M_p(x) |(D^q \phi_m)(x)| < \eta .$$

We need the Leibnitz formula,

$$D^q(uv) = \sum_{\alpha \leq q} \binom{q}{\alpha} (D^{q-\alpha} u)(D^\alpha v) ,$$

with three factors uvw. We obtain

$$D^q(uvw) = D^q[(uv)w] = \sum_{\alpha \leq q} \binom{q}{\alpha} (D^{q-\alpha}(uv))(D^\alpha w)$$

$$= \sum_{\alpha \leq q} \binom{q}{\alpha} \left(\sum_{\beta \leq q-\alpha} \binom{q-\alpha}{\beta} (D^{q-\alpha-\beta} u)(D^\beta v) \right) (D^\alpha w) ,$$

which we apply to $\phi_m(x) = Ah(x)\psi(\frac{\psi}{m}) \frac{1}{M_{q_0}(x)}$, $x \in Q$, with $u(x) = h(x)$, $v(x) = \psi(\frac{x}{m})$ and $w(x) = \frac{1}{M_{q_0}(x)}$. For all q such that $|q| \leq p$, we have:

$$M_p(x) |(D^q \phi_m)(x)|$$

$$= M_p(x) \left| D^q (Ah(x)\psi(\frac{x}{m}) \frac{1}{M_{q_0}(x)} \right| = AM_p(x) \left| D^q (h(x)\psi(\frac{x}{m}) \frac{1}{M_{q_0}(x)}) \right| =$$

$$= AM_p(x)\left|\sum_{\alpha \leq q} \binom{q}{\alpha}\left(\sum_{\beta \leq q-\alpha} \binom{q-\alpha}{\beta}(D^{q-\alpha-\beta}h(x)(D^{\beta}\psi(\frac{x}{m}))\right)(D^{\alpha}\frac{1}{M_{q_0}(x)})\right|$$

$$\leq AM_p(x) \sum_{\alpha \leq q}\left|\binom{q}{\alpha}\left(\sum_{\beta \leq q-\alpha} \binom{q-\alpha}{\beta}(D^{q-\alpha-\beta}h(x)(D^{\beta}\psi(\frac{x}{m}))\right)(D^{\alpha}\frac{1}{M_{q_0}(x)})\right|$$

$$\leq AM_p(x) \sum_{\alpha \leq q} \binom{q}{\alpha}\left(\sum_{\beta \leq q-\alpha} \binom{q-\alpha}{\beta}|D^{q-\alpha-\beta}h(x)|\,|D^{\beta}\psi(\frac{x}{m})|\right)|D^{\alpha}\frac{1}{M_{q_0}(x)}|$$

$$\leq AM_p(x) \sum_{\alpha \leq q} \binom{q}{\alpha}\left(\sum_{\beta \leq q-\alpha} \binom{q-\alpha}{\beta}|D^{q-\alpha-\beta}h(x)|\,|D^{\beta}\psi(\frac{x}{m})|\right)\frac{C_p}{M_p(x)},$$

where we have just applied hypothesis (b) of the theorem. Therefore,

$$M_p(x)|D^q\phi_m(x)| \leq AC_p \sum_{\alpha \leq q} \binom{q}{\alpha}\left(\sum_{\beta \leq q-\alpha} \binom{q-\alpha}{\beta}|D^{q-\alpha-\beta}h(x)|\,|D^{\beta}\psi(\frac{x}{m})|\right)$$

Applying (4) and (6) above, we obtain

$$\|\phi_m(x)\|_p$$

$$= \sup_x \sup_{|q| \leq p} M_p(x)|D^q\phi_m(x)|$$

$$\leq AC_p \sup_x \sup_{|q| \leq p}\left(\sum_{\alpha \leq q} \binom{q}{\alpha}\left[\sum_{\beta \leq q-\alpha} \binom{q-\alpha}{\beta}|D^{q-\alpha-\beta}h(x)|\,|D^{\beta}\psi(\frac{x}{m})|\right]\right) \leq$$

$$\leq AC_p \left(\max_{|q| \leq p} \sum_{\alpha \leq q} \binom{q}{\alpha} \right) \left(\max_{|q-\alpha| \leq p} \sum_{\beta \leq q-\alpha} \binom{q-\alpha}{\beta} \right) s_p r_p$$

$$= AG_p ,$$

where G_p is the positive finite constant

$$G_p = C_p s_p r_p \left(\max_{|q| \leq p} \sum_{\alpha \leq q} \binom{q}{\alpha} \right) \left(\max_{|q-\alpha| \leq p} \sum_{\beta \leq q-\alpha} \binom{q-\alpha}{\beta} \right) .$$

Thus, $\| \phi_m \|_p \leq AG_p$ for all m. Given any $\eta > 0$ we may take $A = \eta / 2G_p$ which implies that $\| \phi_m \|_p < \eta$ for all m. Thus A can be chosen in such a way that $\phi_m \in U$ for all m. Q. E. D.

<u>Lemma 2.</u> The PO-measure $E(\cdot)$ is $\{M_p\}$-tempered.

<u>Proof.</u> Since T is continuous on $K\{M_p\}$ there is a neighborhood U of zero in $K\{M_p\}$ such that

$$|\langle T_\phi \xi, \xi \rangle| \leq \| \xi \|^2 \text{ for all } \phi \text{ in } U$$

and all ξ in H.

Let $\{\phi_m\}$ be the sequence constructed above. Note that $\phi_m \geq 0$ for all m. By Lemma 1 we choose the positive constant A so that $\phi_m \in U$ for all m. By continuity, $|\langle T_\phi \xi, \xi \rangle| \leq \| \xi \|^2$ for all $\phi \in U$. In particular, $|\langle T_{\phi_m} \xi, \xi \rangle| \leq \| \xi \|^2$ for all m, and since $\phi_m \geq 0$, it follows that $\langle T_{\phi_m} \xi, \xi \rangle \leq \| \xi \|^2$ for all m

and for all ξ in H.

We need to establish the existence of a positive constant K such that

$$\int \frac{1}{M_{q_0}} d\mu_\xi \leq K \| \xi \|^2 \quad \text{for all } \xi \text{ in } H.$$

Recalling that $N = \{x \in R^n : |x| \leq R, R > 0\}$, we have, since $M_{q_0} \geq 1$,

$$\int_N \frac{1}{M_{q_0}} d\mu_\xi \leq \int_N 1 \, d\mu_\xi = \mu_\xi(N) = \langle E(N)\xi, \xi \rangle$$

$$\leq \| E(N) \| \, \| \xi \|^2 = k_N \| \xi \|^2 ,$$

where $k_N = \| E(N) \|$. Thus

$$\int_N \frac{1}{M_{q_0}} d\mu_\xi \leq k_N \| \xi \|^2 \quad \text{for all } \xi \text{ in } H.$$

On the complement N^c of N, we have:

$$A \int_{N^c} \frac{1}{M_{q_0}(x)} d\mu_\xi(x) = \int_{N^c} \frac{A}{M_{q_0}(x)} d\mu_\xi(x)$$

$$= \int_{N^c} Ah(x) \frac{1}{M_{q_0}(x)} d\mu_\xi(x) =$$

(since $h = 1$ on N^c)

$$= \int_{N^c} \lim_{m \to \infty} Ah(x)\psi\left(\frac{x}{m}\right) \frac{1}{M_{q_0}(x)} \, d\mu_\xi(x)$$

$$= \int_{N^c} \left(\lim_{m \to \infty} \phi_m(x)\right) d\mu_\xi(x)$$

$$\leq \lim_{m \to \infty} \int_{N^c} \phi_m(x) \, d\mu_\xi(x)$$

(by Fatou's lemma, since $\phi_m \geq 0$)

$$= \lim_{m \to \infty} \langle T_{\phi_m} \xi, \xi \rangle \leq \lim_{m \to \infty} \|\xi\|^2$$

(since $\phi_m \in U$)

$$= \|\xi\|^2 \qquad \text{for all } \xi \in H.$$

Thus we have

$$\int_{N^c} \frac{1}{M_{q_0}} \, d\mu_\xi \leq \frac{1}{A} \|\xi\|^2 .$$

Putting these last two facts together, we obtain

$$\int \frac{1}{M_{q_0}} \, d\mu_\xi = \int_N \frac{1}{M_{q_0}} \, d\mu_\xi + \int_{N^c} \frac{1}{M_{q_0}} \, d\mu_\xi$$

$$\leq k_N \|\xi\|^2 + \frac{1}{A} \|\xi\|^2$$

for all ξ in H. Thus

$$\int \frac{1}{M_{q_0}} \, d\mu_\xi \leq K \| \xi \|^2$$

for all ξ in H, where $K = k_N + \frac{1}{A}$ is a positive finite constant. This shows that $E(\cdot)$ is $\{M_p\}$-tempered. Q. E. D.

Since the PO-measure $E(\cdot)$ is $\{M_p\}$-tempered, the integral $\int \phi dE$ exists for every ϕ in $K\{M_p\}$.

To establish that $T_\phi = \int \phi dE$ for every ϕ in $K\{M_p\}$, let ϕ be an arbitrary element of $K\{M_p\}$. Since $\mathcal{D}$ is dense in $K\{M_p\}$, there is a sequence of functions ϕ_m in $\mathcal{D}$ converging to ϕ in the topology of $K\{M_p\}$. Thus $\| \phi_m - \phi \|_p \to 0$ as $m \to \infty$ for $1 \leq p < \infty$. We have

$$\| \int \phi_m dE - \int \phi dE \| = \| \int (\phi_m - \phi) dE \|$$

$$\leq 4c \| \phi_m - \phi \|_p \to 0 \quad \text{as} \quad m \to \infty, \ 1 \leq p < \infty.$$

(see page 31)

Hence

$$\| \int \phi_m dE - \int \phi dE \| \to 0 \, ;$$

that is,

$$\| T_{\phi_m} - \int \phi dE \| \to 0.$$

Therefore

$$\int \phi dE = \lim_{m \to \infty} T_{\phi_m} = T_\phi$$

by the continuity of $\phi \rightsquigarrow T_\phi$ from $K\{M_p\}$ into $(B(H), \| \|)$. This completes the proof of the theorem.

Corollary. Let a distribution of operators T on $K\{M_p\}$ satisfy $T_{\phi^2} \geq 0$ for every real ϕ. Then there is a unique tempered PO-measure $E(\cdot)$ on the ring generated by the class of compact subsets of R^n such that $T_\phi = \int \phi dE$ for all ϕ in $K\{M_p\}$.

Proof. The set of functions of the form $\phi\bar{\phi}$, $\phi \in K\{M_p\}$, is dense in the set of positive functions in $K\{M_p\}$ [6, pp. 150, 151]. With ϕ real, we have $\phi\bar{\phi} = \phi^2 \geq 0$. Thus T is a positive operator on a dense subset of the positive functions in $K\{M_p\}$. The result now follows by continuity and the theorem just proved.

3. BILINEAR DISTRIBUTIONS OF OPERATORS

We begin with a brief review of bilinear mappings on arbitrary topological vector spaces. This is followed by the definitions pertaining to bilinear distributions of operators. An application of Proposition 2 (Kritt) is shown to yield an operator-valued integral representation for bilinear distributions of operators of a certain type. The chapter culminates in the proof of an integral representation theorem for arbitrary positive-definite Hermitean bilinear translation--invariant distributions of operators.

A. Bilinear Mappings on Topological Vector Spaces

Let E, F, G be three topological vector spaces and

(1) $(x, y) \rightsquigarrow \Phi(x, y): E \times F \to G$ a bilinear mapping. Thus for every fixed $x_0 \in E$, the mapping

(2) $y \rightsquigarrow \Phi_{x_0}(x_0, y): \{x_0\} \times F \to G$ is linear, and for each fixed $y_0 \in F$ the mapping

(3) $x \rightsquigarrow \Phi^{y_0}(x, y_0): E \times \{y_0\} \to G$ is linear.

Definition 1. The bilinear mapping $\Phi: E \times F \to G$ is jointly continuous if for every neighborhood W of zero in G there is a neighborhood U of zero in E and a neighborhood V of zero in

F such that

$$x \in U \text{ and } y \in V \Rightarrow \Phi(x, y) \in W.$$

In case the vector spaces E, F, and G are locally convex the condition defining continuity of the bilinear map Φ takes the following form:

Definition 2. The bilinear mapping $\Phi: E \times F \rightarrow G$ is continuous if to every continuous seminorm r on G there are continuous seminorms p on E and q on F such that

$$r(\Phi(x, y)) \leq p(x)q(y)$$

for all $x \in E$ and $y \in F$ [11, pp. 420, 421].

In the following discussion we will be interested in the case where $E = F = \mathcal{D}(R^n)$, the Schwartz space of $C^\infty(R^n)$-complex-valued functions with compact support and $G = B(H)$, the ring of bounded linear operators on a complex Hilbert space H.

B. Positive-Definite Hermitian Bilinear Distributions of Operators

Let H be a nonzero complex Hilbert space and let $B(H)$ denote the space of all bounded linear operators on H. $B(H)$ is given the uniform operator topology. Let $\mathcal{D}(R^n)$ be the L. Schwartz space of test functions with the usual topology. Let B be the correspondence

(1) $$(\phi, \psi) \rightsquigarrow B(\phi, \psi) : \mathcal{D}(R^n) \times \mathcal{D}(R^n) \to B(H).$$

Definition 1. B is Hermitian bilinear if

(i) for each fixed $\psi \in \mathcal{D}(R^n)$, $\phi \to B(\phi, \psi)$ is linear and continuous, and

(ii) for each fixed $\phi \in \mathcal{D}(R^n)$, $\psi \to B(\phi, \psi)^*$ is linear and continuous, where $B(\phi, \psi)^*$ is the adjoint of the operator $B(\phi, \psi)$.

Definition 2. B is tempered if in (i) and (ii) above the continuity is in the relative topology on $\mathcal{D}(R^n)$ as a subspace of $\mathcal{S}(R^n)$.

Definition 3. B is positive-definite if $B(\phi, \phi)$ is a positive operator for every $\phi \in \mathcal{D}(R^n)$.

Let h be any vector in R^n and let $\tau_h : \mathcal{D}(R^n) \to \mathcal{D}(R^n)$ be defined by $(\tau_h \phi)(x) = \phi(x+h)$ for every $\phi \in \mathcal{D}(R^n)$. Clearly $\phi \in \mathcal{D}(R^n) \Rightarrow \tau_h \phi \in \mathcal{D}(R^n)$.

Definition 4. B is translation-invariant if for every pair ϕ, ψ of functions in $\mathcal{D}(R^n)$ and every vector h in R^n,

$$B(\tau_h \phi, \tau_h \psi) = B(\phi, \psi).$$

<u>Definition 5</u>. By a <u>distribution of operators on $\mathcal{D}(R^n) \times \mathcal{D}(R^n)$</u> we mean a bilinear jointly continuous mapping of $\mathcal{D}(R^n) \times \mathcal{D}(R^n)$ to B(H).

Let $\phi \rightsquigarrow T_\phi : \mathcal{D}(R^n) \to B(H)$ be any distribution of operators and set

(1) $$B(\phi, \psi) = T_{\phi * \psi^*}$$

for every pair of functions ϕ, ψ in $\mathcal{D}(R^n)$. Recall that ψ^* is defined by $\psi^*(x) = \overline{\psi(-x)}$, the bar denoting complex conjugation, and $\phi * \psi^*$ is the convolution of ϕ with ψ^*.

Let $\phi_1 = \tau_h \phi$, $\psi_1 = \tau_h \psi$, where h is any fixed vector in R^n. Since

$$\phi_1 * \psi_1^*(x) = \phi * \psi^*(x) \qquad [6, \text{ p. } 167],$$

B, as given by (1), is translation-invariant. We verify that B is Hermitian-bilinear.

Fix ψ and consider $\phi \to T_{\phi * \psi^*}$. We have

$$\begin{aligned} \alpha\phi_1 + \beta\phi_2 \to T_{(\alpha\phi_1 + \beta\phi_2) * \psi^*} &= T_{\alpha\phi_1 * \psi^* + \beta\phi_2 * \psi^*} \\ &= \alpha T_{\phi_1 * \psi^*} + \beta T_{\phi_2 * \psi^*}, \end{aligned}$$

since T is linear. Thus $B(\alpha\phi_1 + \beta\phi_2, \psi) = \alpha B(\phi_1, \psi) + \beta B(\phi_2, \psi)$ for

for each fixed ψ; that is, B is linear in its first argument. We have for ϕ and ψ in $\mathcal{D}_K(R^n)$

$$\|B(\phi,\psi)\| = \|T_{\phi*\psi^*}\| \leq C \sup_{\substack{x\in R^n \\ |s|\leq m}} |D^s(\phi*\psi^*)(x)|$$

$$= C \sup_{\substack{x\in R^n \\ |s|\leq m}} |\phi*D^s\psi^*(x)|$$

$$= C \sup_{\substack{x\in R^n \\ |s|\leq m}} \left|\int \phi(x)D^s\psi^*(x-y)dy\right|$$

$$\leq C \sup_{\substack{x\in R^n \\ |s|\leq m}} \int |\phi(x)|\,|D^s\psi^*(x-y)|\,dy$$

$$\leq C \sup_{\substack{x\in K \\ |s|\leq m}} |(D^s\phi)(x)| \sup_{\substack{x\in K \\ |s|\leq m}} |(D^s\psi\ (x)| \ \mathrm{meas}(K).$$

Thus $\|B(\phi,\psi)\| \leq \text{constant}\ p_K(\phi)p_K(\psi)$, where p_K denotes the $\mathcal{D}_K$-norm. Hence, B is continuous.

Now consider the map $\psi \to T_{\overline{\phi*\psi^*}}$. In view of the properties of convolution this map is clearly linear. By the linearity of T, we have $\phi \to B(\phi,\psi)^* = T_{\overline{\phi*\psi^*}}$ is linear in ψ for each fixed ϕ.

Putting this together we have the following result: if $\phi \rightsquigarrow T_\phi : \mathcal{D}(R^n) \to B(H)$ is <u>any</u> distribution of operators, then

$$(\phi,\psi) \rightsquigarrow B(\phi,\psi) = T_{\phi*\psi^*} : \mathcal{D}(R^n) \times \mathcal{D}(R^n) \to B(H)$$

is a translation-invariant Hermitian bilinear distribution of operators.

C. Integral Representation of Translation-Invariant Hermitian Bilinear Distributions of Operators

The following result is a corollary of Proposition 2 (Kritt) stated on page 48.

Proposition 1. Let $\phi \rightsquigarrow T_\phi : \mathcal{D}(R^n) \to B(H)$ be any positive-definite distribution of operators and set

$$B(\phi, \psi) = T_{\phi * \psi^*} \quad \text{for every} \quad (\phi, \psi) \in \mathcal{D}(R^n) \times \mathcal{D}(R^n).$$

Then B is positive-definite and

$$B(\phi, \psi) = (2\pi)^{n/2} \int \hat{\phi}\bar{\hat{\psi}} dE \quad \text{for all} \quad (\phi, \psi) \in \mathcal{D}(R^n) \times \mathcal{D}(R^n),$$

where $E(\cdot)$ is as in Theorem 2 of Kritt and the "hat" denotes Fourier transform.

The integral occurring here is the integral of a rapidly decreasing function with respect to a tempered PO measure as defined in [8, pp. 866, 867]. This same integral occurs in the main theorem developed in the next section.

Proof. $T_{\phi * \phi^*} = B(\phi, \phi)$ is a positive operator for all $\phi \in \mathcal{D}(R^n)$. Thus B is positive-definite. By Proposition 2 of Kritt, there is a unique PO-measure $E(\cdot)$ on the ring R generated by the compact subsets of R^n such that

$$T_{\phi*\psi^*} = \int \widehat{\phi*\psi^*}\,dE .$$

But $\widehat{\phi*\psi^*} = (2\pi)^{n/2}\hat{\phi}\bar{\hat{\psi}}$, and therefore

$$B(\phi, \psi) = (2\pi)^{n/2} \int \hat{\phi}\bar{\hat{\psi}}\,dE . \qquad \text{Q.E.D.}$$

We summarize the results of Sections B and C.

If $\phi \rightsquigarrow T_\phi : \mathcal{D}(R^n) \to B(H)$ is any distribution of operators in R^n, then the formula

$$B(\phi, \psi) = T_{\phi*\psi^*}$$

defines a translation-invariant Hermitian bilinear distribution of operators:

$$(\phi, \psi) \rightsquigarrow B(\phi, \psi) : \mathcal{D}(R^n) \times \mathcal{D}(R^n) \to B(H).$$

If, in addition, T is positive-definite, then B is positive-definite and has the unique representation

$$B(\phi, \psi) = (2\pi)^{n/2} \int \hat{\phi}\bar{\hat{\psi}}\,dE ,$$

where $E(\cdot)$ is a tempered PO-measure on the ring R generated by the compact subsets of R^n.

D. Main Theorem

The following result characterizes a class of bilinear distributions of operators as positive operator-valued measures. The proof involves the notion of a barrier sequence as in [6, pp. 161, 164].

Theorem 2. Let $(\phi, \psi) \rightsquigarrow B(\phi, \psi): \mathcal{D}(R^n) \times \mathcal{D}(R^n) \to B(H)$ be any continuous Hermitian bilinear translation-invariant positive-definite mapping. Then there is a unique tempered PO measure $E(\cdot)$ on the ring R generated by the class of compact subsets of R^n such that

$$(1) \qquad B(\phi, \psi) = \int \hat{\phi}\bar{\hat{\psi}} dE \quad \text{for all} \quad (\phi, \psi) \in \mathcal{D}(R^n) \times \mathcal{D}(R^n).$$

In other words, every Hermitian translation-invariant bilinear distribution of operators on $\mathcal{D}(R^n) \times \mathcal{D}(R^n)$ is uniquely represented by a tempered PO measure in the sense of Equation (1) above.

Proof. For each vector $\xi \in H$, the map

$$(\phi, \psi) \rightsquigarrow L_\xi(\phi, \psi): \mathcal{D}(R^n) \times \mathcal{D}(R^n) \to C$$

given by

$$(2) \qquad L_\xi(\phi, \psi) = \langle B(\phi, \psi)\xi, \xi \rangle$$

is a Hermitian bilinear translation-invariant positive-definite distribution. According to Theorem 6 [6, p. 169] there is a tempered positive measure μ_ξ such that

$$(3) \qquad L_\xi(\phi, \psi) = \int \hat{\phi}\overline{\hat{\psi}} d\mu_\xi \quad \text{for all} \quad \phi, \psi \in \mathcal{D}(R^n).$$

We thus obtain a family $\{\mu_\xi : \xi \in H\}$ of positive tempered measures in R^n.

Lemma 1. There is a positive integer p and a positive number K such that

$$\int (1+|x|^2)^{-p} d\mu_\xi \leq K\|\xi\|^2 \quad \text{for all} \quad \xi \in H.$$

Proof. Let $S = \{x \in R^n : |x| < 1\}$. Since B is continuous, there is a neighborhood $\mathcal{O}$ of the origin in $\mathcal{D}(S) \times \mathcal{D}(S)$ such that

$$|\langle B(\phi, \psi)\xi, \xi\rangle| \leq 1\|\xi\|^2 \quad \text{for all} \quad (\phi, \psi) \in \mathcal{O} = U \times V,$$

where U is a neighborhood of the origin in $\mathcal{D}(S)$ and so is V. In other words, $|\langle B(\phi, \psi)\xi, \xi\rangle| \leq \|\xi\|^2$ for all $\phi \in U$ and all

$\psi \in V$.

There is a barrier sequence [6, p. 161] α_m corresponding to U; that is, α_m is a sequence such that

$$\alpha_m \in C_0^\infty(U), \quad m = 1, 2, \ldots$$

$$\hat{\alpha}_m \geq 0 \quad \text{for all} \quad m$$

$$\lim_{m \to \infty} \hat{\alpha}_m(x) = \alpha_0(x) \quad \text{exists for all} \quad x \in U$$

$$\alpha_0(x) \geq A(1+|x|^2)^{-q_1-n-1},$$

where A is a positive constant, q_1 is a nonnegative integer and n is the number of variables. Similarly there is a sequence β_m corresponding to V such that

$$\beta^*_m \in C_0^\infty(V) \quad \text{for all} \quad m = 1, 2, \ldots$$

$$\widehat{\beta^*_m} = \overline{\hat{\beta}}_m \geq 0 \quad \text{for all} \quad m$$

$$\lim_{m \to \infty} \overline{\hat{\beta}}_m(x) = \beta_0(x) \quad \text{exists for all} \quad x \in V, \quad \text{and}$$

$$\beta_0(x) \geq B(1+|x|^2)^{-q_2-n-1},$$

where q_2 is a nonnegative integer and B is a positive constant. Recalling that μ_ξ is a positive measure for each ξ in H, we have

$$\int (1+|x|^2)^{-q_1-q_2-2n-2} d\mu_\xi = \left| \int (1+|x|^2)^{-q_1-q_2-2n-2} d\mu_\xi \right|$$

$$\leq \left| \frac{1}{AB} \int \alpha_0\beta_0(x) d\mu_\xi \right|$$

$$= \frac{1}{AB} \left| \int \lim_{m\to\infty} \hat{\alpha}_m \overline{\hat{\beta}}_m d\mu_\xi \right|$$

$$= \frac{1}{AB} \left| \int \underline{\lim}_{m\to\infty} \hat{\alpha}_m \overline{\hat{\beta}}_m d\mu_\xi \right|$$

$$\leq \frac{1}{AB} \left| \underline{\lim}_{m\to\infty} \int \hat{\alpha}_m \overline{\hat{\beta}}_m d\mu_\xi \right|$$

(Fatou's lemma)

$$= \frac{1}{AB} \left| \underline{\lim}_{m\to\infty} \langle B(\alpha_m, \beta_m^*)\xi, \xi\rangle \right|$$

$$\leq \frac{1}{AB} \left| \underline{\lim}_{m\to\infty} |\langle B(\alpha_m, \beta_m^*)\xi, \xi\rangle| \right|$$

$$\leq \frac{1}{AB} \left| \underline{\lim}_{m\to\infty} \|\xi\|^2 \right| = \frac{1}{AB} \|\xi\|^2$$

Let $p = q_1 + q_2 + 2n + 2$, $K = 1/AB$. Then

$$\int (1+|x|^2)^{-p} d\mu_\xi \leq K\|\xi\|^2$$

for all $\xi \in H$.

<u>Lemma 2</u>. B is <u>tempered</u>. That is, B is continuous on $\mathcal{D}(R^n) \times \mathcal{D}(R^n)$ in the relative topology induced by $\mathcal{S}(R^n) \times \mathcal{S}(R^n)$.

Proof. Let $p_1 = q_1 + n + 1$, $K_1 = 1/A$, $p_2 = q_2 + n + 1$, $K_2 = 1/B$, where q_1, q_2, A and B are as in Lemma 1 above. Let W_1 be the neighborhood of the origin in $\mathcal{S}(R^n)$ given by

$$W_1 = \{\phi \in \mathcal{S}: \sup_{x \in R^n} |(1+|x|^2)^{p_1}\phi(x)| \leq \frac{1}{2K_1}\},$$

and let W_2 be the neighborhood of the origin in $\mathcal{S}(R^n)$ given by

$$W_2 = \{\psi^* \in \mathcal{S}: \sup_{x \in R^n} |(1+|x|^2)^{p_2}\psi^*(x)| \leq \frac{1}{2K_2}\}.$$

With $\mathcal{F}^{-1}$ denoting the inverse Fourier transform, let

$$U_1 = \mathcal{F}^{-1}(W_1) \subset \mathcal{S}(R^n),$$

and

$$U_2 = \mathcal{F}^{-1}(W_2) \subset \mathcal{S}(R^n).$$

$\mathcal{O} = U_1 \times U_2$ is a neighborhood of the origin in $\mathcal{S} \times \mathcal{S}$. Let $(\phi, \psi) \in \mathcal{O} \cap \mathcal{D} \times \mathcal{D}$. Then

$$|L_\xi(\phi, \psi)| = |<B(\phi, \psi)\xi, \xi>|$$

$$= |\int \hat{\phi}\bar{\hat{\psi}} d\mu_\xi| \leq \int |\hat{\phi}\bar{\hat{\psi}}| d\mu_\xi$$

$$\leq \frac{1}{(2K_1)(2K_2)} \int (1+|x|^2)^{-(p_1+p_2)} d\mu_\xi \leq$$

$$\leq \frac{1}{4K_1K_2}(K_1K_2\|\xi\|^2) = \frac{1}{4}\|\xi\|^2 .$$

Thus

$$|L_\xi(\phi, \psi)| = |<B(\phi, \psi)\xi, \xi>| \leq \frac{1}{4}\|\xi\|^2 ,$$

and therefore

$$\|L_\xi(\phi, \psi)\| = \sup\{|L_\xi(\phi, \psi)| : \|\xi\| \leq 1\} \leq \frac{1}{4} .$$

The associated bilinear form $L_{\xi,\eta}(\phi, \psi) = <B(\phi, \psi)\xi, \eta>$ is symmetric and therefore

$$\|L_{\xi,\eta}(\phi, \psi)\| = \|L_\xi(\phi, \psi)\| = \frac{1}{4} .$$

Given arbitrary $\epsilon > 0$, let

$$U_1 = 2\sqrt{\epsilon}\, U, \quad V_1 = 2\sqrt{\epsilon}\, V .$$

If $\theta_1 \in U_1$, $\theta_2 \in V_1$, then

$$\theta_1 = 2\sqrt{\epsilon}\,\phi, \quad \phi \in U, \quad \theta_2 = 2\sqrt{\epsilon}\,\psi, \quad \psi \in V,$$

and

$$\begin{aligned} |<B(\theta_1, \theta_2)\xi, \xi>| &= |<B(2\sqrt{\epsilon}\,\phi, 2\sqrt{\epsilon}\,\psi)\xi, \xi>| \\ &= 4\epsilon\, |<B(\phi, \psi)\xi, \xi>| \\ &\leq 4\epsilon\, \frac{1}{4}\|\xi\|^2 \\ &= \epsilon\, \|\xi\|^2 . \end{aligned}$$

Therefore

$$\|L_\xi(\theta_1, \theta_2)\| \leq \epsilon ,$$

and

$$\|L_{\xi,\eta}(\theta_1, \theta_2)\| \leq \epsilon \quad \text{for all } \theta_1 \in U_1, \ \theta_2 \in V_1 .$$

But $\epsilon > 0$ is arbitrary. Hence B is continuous as asserted.

We extend by continuity the continuous map $B: \mathcal{D} \times \mathcal{D} \to B(H)$ to a <u>unique</u> map $B': \mathcal{S} \times \mathcal{S} \to B(H)$ which, in view of Lemma 2, is continuous.

<u>Lemma 3</u>. $\langle B'(\phi, \psi)\xi, \xi\rangle = \int \hat{\phi}\bar{\hat{\psi}}\, d\mu_\xi$ for all $(\phi, \psi) \in \mathcal{S} \times \mathcal{S}$ and every $\xi \in H$.

<u>Proof</u>. Let $\phi, \psi \in \mathcal{S}$ and take sequences ϕ_n and ψ_n such that $\phi_n \to \phi$ and $\psi_n \to \psi$ in the topology of $\mathcal{S}$. Note that $\psi_n^* \to \psi^*$ in the $\mathcal{S}$-topology. Then $\hat{\phi}_n \to \hat{\phi}$ and $\widehat{\psi_n^*} \to \widehat{\psi^*}$ in the $\mathcal{S}$-topology and therefore also $\hat{\phi}_n\widehat{\psi_n^*} \to \hat{\phi}\widehat{\psi^*}$ in $\mathcal{S}$. Hence,

$$\sup_{x \in R^n} \left|(1+|x|^2)^{p_1+p_2}(\hat{\phi}_n\widehat{\psi_n^*} - \hat{\phi}\widehat{\psi^*})(x)\right| \to 0,$$

where p_1 and p_2 are as in Lemma 2. Thus

$$\int \hat{\phi}_n\widehat{\psi_n^*}\, d\mu_\xi \to \int \hat{\phi}\widehat{\psi^*}\, d\mu_\xi = \int \hat{\phi}\bar{\hat{\psi}}\, d\mu_\xi .$$

Consequently,

$$\langle B'(\phi, \psi)\xi, \xi\rangle = \lim_{n\to\infty} \langle B'(\phi_n, \psi_n)\xi, \xi\rangle$$

$$= \lim_{n\to\infty} \int \hat{\phi}_n \overline{\hat{\psi}}_n = \int \hat{\phi}\overline{\hat{\psi}} d\mu_\xi$$

for all $(\phi, \psi) \in \mathcal{S} \times \mathcal{S}$ and every $\xi \in H$. Thus

$$\langle B'(\phi, \psi)\xi, \xi\rangle = \int \hat{\phi}\overline{\hat{\psi}} d\mu_\xi$$

on $\mathcal{S} \times \mathcal{S}$ for all $\xi \in H$.

Recall that R is the ring generated by the compact subsets of R^n. If $M \in R$, then M is bounded in R^n.

<u>Lemma 4</u>. The family $\{\mu_\xi : \xi \in H\}$ of positive tempered measures in R^n satisfies the following conditions:

(a) For each $M \in R$ there is a positive constant Q_M such that $\mu_\xi(M) \leq Q_M \|\xi\|^2$ for all $\xi \in H$.

(b) $[\mu_{\xi+\eta}(M)]^{1/2} \leq [\mu_\xi(M)]^{1/2} + [\mu_\eta(M)]^{1/2}$ for all $M \in R$ and all $\xi, \eta \in H$.

(c) $\mu_{\lambda\xi}(M) = |\lambda|^2 \mu_\xi(M)$ for all $M \in R$, $\lambda \in C$ and $\xi \in H$.

(d) $\mu_{\xi+\eta}(M) + \mu_{\xi-\eta}(M) = 2\mu_\xi(M) + 2\mu_\eta(M)$ for all $M \in R$ and all $\xi, \eta \in H$.

Proof. (a) Let $M \in R$ and take $\phi, \psi \in \mathcal{S}$ such that $\hat{\phi}, \hat{\psi} \in \mathcal{D}$ and $\hat{\phi}\overline{\hat{\psi}} \geq \chi_M$, the characteristic function of the set M. Then

$$\mu_\xi(M) = \int \chi_M d\mu_\xi \leq \int \hat{\phi}\overline{\hat{\psi}} d\mu_\xi$$

$$= \int (1+|x|^2)^{p_1}\hat{\phi}(x)(1+|x|^2)^{p_2}\overline{\hat{\psi}}(x)(1+|x|^2)^{-(p_1+p_2)} d\mu_\xi$$

$$\leq \sup_{x \in M} [(1+|x|^2)^{p_1+p_2}(\hat{\phi}\overline{\hat{\psi}})(x)] \int (1+|x|^2)^{-p} d\mu_\xi$$

$$\leq Q_M \|\xi\|^2 \quad \text{for all} \quad \xi \in H.$$

Here

$$Q_M = \sup_{x \in R^n} |(1+|x|^2)^{p_1+p_2}\hat{\phi}\overline{\hat{\psi}}(x)| AB ,$$

where A, B and $p = p_1 + p_2$ are as in Lemma 2.

(b) Let C be an arbitrary nonempty compact subset of R^n and take sequences $\phi_n, \psi_n \in \mathcal{S}(R^n)$ such that $\hat{\phi}_n, \overline{\hat{\psi}}_n \in \mathcal{D}(R^n)$, $\hat{\phi}_n \downarrow \chi_C$ and $\overline{\hat{\psi}}_n \downarrow \chi_C$. For all $\xi \in H$, we have

$$\langle B'(\phi_n, \psi_n)\xi, \xi\rangle = \int \hat{\phi}_n \overline{\hat{\psi}}_n d\mu_\xi \geq \int \chi_C d\mu_\xi \geq 0 .$$

Thus $B'(\phi_n, \psi_n)$ is a positive operator for each n. Since $\hat{\phi}_n\overline{\hat{\psi}}_n \downarrow \chi_C$, the Monotone Convergence Theorem implies that

(i) $\int \hat{\phi}_n \bar{\hat{\psi}}_n d\mu_{\xi+\eta} \downarrow \int \chi_C d\mu_{\xi+\eta} = \mu_{\xi+\eta}(C)$,

(ii) $\int \hat{\phi}_n \bar{\hat{\psi}}_n d\mu_{\xi} \downarrow \int \chi_C d\mu_{\xi} = \mu_{\xi}(C)$, and

(iii) $\int \hat{\phi}_n \bar{\hat{\psi}}_n d\mu_{\eta} \downarrow \int \chi_C d\mu_{\eta} = \mu_{\eta}(C)$.

By the positivity of the operator $B(\phi_n, \psi_n)$ for each n, $<B'(\phi_n, \psi_n)\xi, \xi> \geq 0$ for each vector $\xi \in H$. The generalized Cauchy-Schwartz inequality yields

$$<B'(\phi_n, \psi_n)(\xi+\eta), \xi+\eta>^{1/2} \leq <B'(\phi_n, \psi_n)\xi, \xi>^{1/2} + <B'(\phi_n, \psi_n)\eta, \eta>^{1/2} .$$

Letting $n \to \infty$ and taking (i), (ii) and (iii) above into account, we obtain

$$[\mu_{\xi+\eta}(C)]^{1/2} \leq [\mu_{\xi}(C)]^{1/2} + [\mu_{\eta}(C)]^{1/2}$$

for all vectors $\xi, \eta \in H$ and all compact subsets C of R^n. We invoke the regularity of the measures $\{\mu_\xi : \xi \in H\}$ to conclude that the preceding inequality holds for all sets M in R.

(c) Let C, ϕ_n and ψ_n be as in (b) above so that $\hat{\phi}_n \bar{\hat{\psi}}_n \downarrow \chi_C$. Then

$$|\lambda|^2 \langle B'(\phi_n, \psi_n)\xi, \xi\rangle = \langle B'(\phi_n, \psi_n)\lambda\xi, \lambda\xi\rangle$$

$$= \int \hat{\phi}_n \overline{\hat{\psi}}_n d\mu_{\lambda\xi} \; .$$

Thus

$$|\lambda|^2 \langle B'(\phi_n, \psi_n)\xi, \xi\rangle = \int \hat{\phi}_n \overline{\hat{\psi}}_n d\mu_{\lambda\xi} \; .$$

Since

$$\langle B'(\phi_n, \psi_n)\xi, \xi\rangle = \int \hat{\phi}_n \overline{\hat{\psi}}_n d\mu_\xi \downarrow \int \chi_C d\mu_\xi = \mu_\xi(C),$$

we have

$$|\lambda|^2 \mu_\xi(C) = \mu_{\lambda\xi}(C)$$

for all compact subsets C of R^n and all complex numbers λ. By regularity the preceding equality holds for all M in R.

(d) Let C, ϕ_n, ψ_n be as above. Then

$$\int \hat{\phi}_n \overline{\hat{\psi}}_n d\mu_{\xi+\eta} \downarrow \mu_{\xi+\eta}(C) \; ,$$

and

$$\int \hat{\phi}_n \overline{\hat{\psi}}_n d\mu_{\xi-\eta} \downarrow \mu_{\xi-\eta}(C) \; .$$

We have

$$\langle B'(\phi_n, \psi_n)(\xi+\eta), \xi+\eta\rangle + \langle B'(\phi_n, \psi_n)(\xi-\eta), \xi-\eta\rangle$$

$$= \langle B'(\phi_n, \psi_n)\xi, \xi\rangle + \langle B'(\phi_n, \psi_n)\xi, \eta\rangle + \langle B'(\phi_n, \psi_n)\eta, \xi\rangle \quad +$$

$$+ \langle B'(\phi_n, \psi_n)\eta, \eta\rangle + \langle B'(\phi_n, \psi_n)\xi, \xi\rangle + \langle B'(\phi_n, \psi_n)\xi, -\eta\rangle$$

$$+ \langle B'(\phi_n, \psi_n)(-\eta), \xi\rangle + \langle B'(\phi_n, \psi_n)(-\eta), -\eta\rangle$$

$$= 2\langle B'(\phi_n, \psi_n)\xi, \xi\rangle + 2\langle B'(\phi_n, \psi_n)\eta, \eta\rangle .$$

Thus

$$\langle B'(\phi_n, \psi_n)(\xi+\eta), \xi+\eta\rangle + \langle B'(\phi_n, \psi_n)(\xi-\eta), \xi-\eta\rangle$$

$$= \quad 2\langle B'(\phi_n, \psi_n)\xi, \xi\rangle + 2\langle B'(\phi_n, \psi_n)\eta, \eta\rangle ,$$

so that

$$\int \hat{\phi}_n \overline{\hat{\psi}}_n d\mu_{\xi+\eta} + \int \hat{\phi}_n \overline{\hat{\psi}}_n d\mu_{\xi-\eta} = 2\int \hat{\phi}_n \overline{\hat{\psi}}_n d\mu_{\xi} + 2\int \hat{\phi}_n \overline{\hat{\psi}}_n d\mu_{\eta}$$

Applying the Monotone Convergence Theorem we obtain

$$\mu_{\xi+\eta}(C) + \mu_{\xi-\eta}(C) = 2\mu_{\xi}(C) + 2\mu_{\eta}(C)$$

for all vectors $\xi, \eta \in H$ and all compact subsets C of R^n. By regularity the result holds for all $M \in R$. This concludes the proof of Lemma 4.

In view of Lemma 4, there is a <u>unique</u> PO measure $E(\cdot)$ on R such that $\mu_{\xi}(M) = \langle E(M)\xi, \xi\rangle$ for all $M \in R$ and all $\xi \in H$. Lemma 1 implies that $E(\cdot)$ is <u>tempered</u>. Since

$$L_\xi(\phi, \psi) = \langle B(\phi, \psi)\xi, \xi \rangle$$

and

$$\langle B(\phi, \psi)\xi, \xi \rangle = \int \hat{\phi}\overline{\hat{\psi}}\, d\mu_\xi \quad \text{for all} \quad \xi \in H,$$

we have

$$\langle B(\phi, \psi)\xi, \xi \rangle = \langle \left(\int \hat{\phi}\overline{\hat{\psi}}\, dE\right)\xi, \xi \rangle \quad \text{for all} \quad \xi \in H.$$

Hence

$$B(\phi, \psi) = \int \hat{\phi}\overline{\hat{\psi}}\, dE \quad \text{for all} \quad (\phi, \psi) \in \mathcal{D}(R^n) \times \mathcal{D}(R^n).$$

It remains to show that the representation

$$B(\phi, \psi) = \int \hat{\phi}\overline{\hat{\psi}}\, dE$$

of B in terms of the tempered PO measure $E(\cdot)$ is unique.

To this end suppose that $F(\cdot)$ is another PO-measure on R such that

$$B(\phi, \psi) = \int \hat{\phi}\overline{\hat{\psi}}\, dE = \int \hat{\phi}\overline{\hat{\psi}}\, dF$$

for all $(\phi, \psi) \in \mathcal{D}(R^n) \times \mathcal{D}(R^n)$. For each $\xi \in H$ set $\nu_\xi(M) = \langle F(M)\xi, \xi \rangle$ for all $M \in R$. Then ν_ξ is a positive measure [2, pp. 8, 9] and

$$\int \hat{\phi}\bar{\hat{\psi}}\, d\mu_\xi = \int \hat{\phi}\bar{\hat{\psi}}\, d\nu_\xi$$

for each $\xi \in H$ and all functions $\hat{\phi}\bar{\hat{\psi}}$ with $\phi, \psi \in \mathcal{D}(R^n)$. Since $\mathcal{D}(R^n)$ is dense in $\mathcal{S}(R^n)$ and the Fourier transform maps $\mathcal{S}(R^n)$ onto $\mathcal{S}(R^n)$,

$$\int \hat{\phi}\bar{\hat{\psi}}\, d\mu_\xi = \int \hat{\phi}\bar{\hat{\psi}}\, d\nu_\xi$$

on $\mathcal{S}(R^n)$. But the Fourier transform is a continuous linear surjection of $\mathcal{S}(R^n)$ to itself. Therefore

$$\int \phi\bar{\psi}\, d\mu_\xi = \int \phi\bar{\psi}\, d\nu_\xi$$

holds on $\mathcal{S}(R^n)$ and thus also on $\mathcal{D}(R^n)$. In particular, if ψ is chosen in $\mathcal{D}(R^n)$ such that $\psi = 1$ on the support of ϕ, there follows

$$\int \phi\, d\mu_\xi = \int \phi\, d\nu_\xi$$

on $\mathcal{D}(R^n)$. But $\mathcal{D}(R^n)$ is dense in $C^0(R^n)$. Hence $\mu_\xi = \nu_\xi$, and the uniques in Proposition 2, Chapter 1, implies that $E(\cdot) = F(\cdot)$. Therefore the representation

$$B(\phi, \psi) = \int \hat{\phi}\bar{\hat{\psi}}\, dE, \quad \text{for all } (\phi, \psi) \in \mathcal{D}(R^n) \times \mathcal{D}(R^n),$$

is unique. This completes the proof of the theorem.

Corollary. Given any Hermitian bilinear distribution of operators B in $R^n \times R^n$ that is positive-definite and translation-invariant, there is a distribution T of operators such that

$$B(\phi, \psi) = T_{\phi * \psi^*} \quad \text{for all} \quad (\phi, \psi) \in \mathcal{D}(R^n) \times \mathcal{D}(R^n) .$$

Proof. The unique tempered PO-measure $E(\cdot)$ obtained in the proof of the above theorem determines T through

$$T_\theta = \int \theta dE \quad \text{for every} \quad \theta \in \mathcal{D}(R^n) .$$

4. CONDITIONALLY POSITIVE-DEFINITE DISTRIBUTIONS OF OPERATORS

The main theorem in this chapter yields a representation of a class of distributions of operators on $\mathcal{D}(R^n)$ satisfying a condition called conditionally positive-definite. The Fourier transform is applied to obtain the equivalent and more readily workable notion, that of multiplicative positivity, on the space Z of entire analytic functions of exponential type on C^n. We begin with a discussion of the space Z, its relation to $\mathcal{D}(R^n)$ and the basic definitions needed in the study of multiplicatively positive operator-valued distributions. Kritt's Proposition 2 is restated in terms of Z and applied in the proof of the main theorem in this chapter.

A. The Space Z

C^n denotes, as usual, the set $C^1 \times \ldots \times C^1$, with n factors, and carries the standard topology. If $z \in C^n$, then $z = (z_1, \ldots, z_n)$, $z_j = x_j + iy_j$, $x_j, y_j \in R^1$, $1 \leq j \leq n$, and $i^2 = -1$. If $r = (r_1, \ldots, r_n)$ is an n-dimensional integer, then $z^r = z_1^{r_1} \ldots z_n^{r_n}$, and $rz = r_1 z_1 + \ldots + r_n z_n$, with $r_j z_j = r_j x_j + i r_j y_j$, $1 \leq j \leq n$.

Definition 1. A function ϕ defined on C^n is of exponential type if there are constants a and C such that

$$(1) \qquad |\phi(z)| \leq C \exp(a|z|) .$$

<u>Definition 2.</u> The set Z consists of all entire analytic functions ϕ of exponential type on C^n satisfying

$$(2) \qquad |z^r \phi(z)| \leq C_{\phi, r} \exp(a|y|), \quad z = x + iy ,$$

for all r. The constant a depends on the function ϕ and the constant $C_{\phi, r}$ depends on ϕ and r. If $\phi \in Z$ then

$$(3) \qquad |z^r (D^q \phi)(z)| \leq C \exp(a|y|)$$

for every r and q, where the constant C depends on r and q [6, p. 22].

Z is a linear space and the subset $Z(a)$ of Z consisting of all functions in Z satisfying (2) with a fixed value of a is a linear subspace of Z. We have $Z = \bigcup_a Z(a)$.

Let $\{\phi_m\}_{m=1}^{\infty}$ be a sequence of functions in Z. The sequence $\{\phi_m\}_{m=1}^{\infty}$ <u>converges to zero in Z</u> if every function ϕ_m in the sequence satisfies

$$(4) \qquad |z^r \phi_m(z)| \leq C \exp(a|y|)$$

for some constants C and a with a independent of m (that is, $\phi_m \in Z(a)$ for all m), and for every r and q,

$$(5) \qquad \lim_{m \to \infty} \sup_x \left| (1+|x|^2)^r (D^q \phi_m)(x) \right| = 0$$

is satisfied.

Letting $y = 0$, then $z = x$; that is $z = (z_1, z_2, \ldots, z_n) = (x_1, x_2, \ldots, x_n) \in R^n$, and (3) above reduces to

$$(6) \qquad |x^r (D^q \phi)(x)| \leq C \quad \text{for every } r \text{ and } q.$$

Thus the mapping $\phi \rightsquigarrow \phi|_{R^n} : Z \to \mathcal{S}(R^n)$ defines a continuous imbedding of Z into $\mathcal{S}(R^n)$.

The Fourier transform $\mathfrak{F}$ is a continuous bijection of $\mathcal{S}(R^n)$ onto itself and carries $\mathcal{D}(R^n)$ onto Z. In other words, there is a unique continuous mapping $\mathfrak{F}$ (the Fourier transform) such that

$$\begin{tikzcd}
Z \arrow[r, "\phi \rightsquigarrow \phi|_{R^n}"] & \mathcal{S} \\
\mathcal{D} \arrow[u, "\mathfrak{F}"] \arrow[r, "\text{dense}", "i(\phi) = \phi"'] & \mathcal{S} \arrow[u, "\mathfrak{F}"']
\end{tikzcd}$$

Thus Z is dense in $\mathcal{S}$. The relations between the spaces $\mathcal{D}$, Z and $\mathcal{S}$ relative to the Fourier transformation $\mathfrak{F}$ are summarized as follows:

$$\mathfrak{F}(\mathcal{D}) = Z, \quad \mathfrak{F}(Z) = \mathcal{D}, \quad \mathfrak{F}(\mathcal{S}) = \mathcal{S}.$$

Both $\mathcal{D}$ and Z are dense in $\mathcal{S}$ [6, pp. 22-25].

B. Multiplicatively Positive and Conditionally Positive-Definite Distributions of Operators

Definition 1. The distribution of operators $\phi \rightsquigarrow T_\phi : \mathcal{D}(R^n) \to B(H)$ is said to be conditionally positive-definite of order s if, for each vector ξ from H,

$$(D\overline{D}T^{\xi}, \phi * \phi^*) \geq 0 \tag{1}$$

holds for all $\phi \in \mathcal{D}(R^n)$ and all linear homogeneous constant coefficient differential operators D of order s (see [6, p. 176] for the scalar-valued case).

Let

$$D = \sum_{|k|=s} a_k \frac{d^k}{dx^k}, \quad \overline{D} = (-1)^{|k|} \sum_{|k|=s} \overline{a_k} \frac{d^k}{dx^k},$$

where

$$\frac{d^k}{dx^k} = \frac{\partial^{k_1+\ldots+k_n}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}, \quad |k| = k_1 + \ldots + k_n$$

and set $(T^{\xi}, \phi) = \langle T_\phi \xi, \xi \rangle$.

Associate with each D as above the polynomial

$$P(\lambda) = (2\pi)^{-n/2} \sum_{|k|=s} a_k \, (-i\lambda)^k .$$

Then

$$\overline{P}(\lambda) = (2\pi)^{-n/2} \sum_{|k|=s} \overline{a_k} \, (-i\lambda)^k$$

corresponds to $\overline{D}$. In view of the fact that

$$\widehat{\frac{d^k}{dx^k} T^\xi} = (2\pi)^{-n/2} (-i\lambda)^k \widehat{T^\xi} ,$$

where $(i\lambda)^k : \lambda \rightsquigarrow (i\lambda_1)^{k_1} (i\lambda_2)^{k_2} \ldots (i\lambda_n)^{k_n}$, there follows $\widehat{DT^\xi} = P\widehat{T^\xi}$. Since $\widehat{\phi * \phi^*} = (2\pi)^{n/2} \hat{\phi}\hat{\bar{\psi}}$, Definition 1 above is equivalent to the following:

Definition 2. The distribution of operators $\phi \rightsquigarrow T_\phi : \mathcal{D}(R^n) \to B(H)$ is conditionally positive-definite of order s if, for each vector ξ from H and any homogeneous polynomial P of degree s, $(P\overline{P}\widehat{T^\xi}, \psi\overline{\psi}) \geq 0$ for all $\psi \in Z$. (Recall that $\widehat{\mathcal{D}(R^n)} = Z$.)

Definition 3. A distribution of operators on $\mathcal{D}$ (or Z or $\mathcal{S}$) is multiplicatively positive if, for each vector ξ from H, $(T^\xi, \phi\overline{\phi}) \geq 0$ for every ϕ in $\mathcal{D}$ (or Z or $\mathcal{S}$).

It will be convenient in what follows to have the following equivalent statement of Kritt's Proposition 2 (see Chapter 2).

Proposition 2' (Kritt). Every multiplicatively positive distribution of operators S on Z is given by a tempered PO-measure $E(\cdot)$; that is, $S_\psi = \int \psi dE$ for all ψ in Z. This situation is described in the following diagram:

$$\psi \longmapsto S_\psi$$

$$\begin{tikzcd}
\hat{\phi} = \psi & Z \arrow[rr, "S"] & & B(H) \\
\phi \arrow[u, mapsto] & \mathcal{D} \arrow[u, "\mathcal{F}\ \text{1-1 cont.}", "\text{onto}"'] \arrow[urr, "T"'] & \phi \arrow[r, mapsto] & T_\phi = \int \phi dE
\end{tikzcd}$$

$$S_\psi = S_{\hat{\phi}} = T_\phi = \int \hat{\phi} dE = \int \psi dE$$

Thus,

$$S_\psi = \int \psi dE \quad \text{for every } \psi \text{ in } Z.$$

Remark. Let R_0 be the ring generated by the compact subsets of $R^n - \{0\}$ and let μ^ξ be a measure defined on R_0. We will say that μ^ξ is tempered if there is a nonnegative integer p and a positive constant K such that

$$\int_{|x| \geq 1} (1+|x|^2)^{-p} d\mu^\xi \leq K \|\xi\|^2 \quad \text{for all } \xi \text{ in } H.$$

If there is a PO-measure $F(\cdot)$ on R_0 such that for each $\xi \in H$

$$\mu^{\xi}(M) = \langle F(M)\xi, \xi \rangle$$

for all $M \in R_0$, and each of the measures μ^{ξ} is tempered in the above sense, then the PO-measure $F(\cdot)$ will also be called tempered (cf. Definition 2, Chapter 2, B).

<u>Theorem 3.</u> Let $T: Z \to B(H)$ be a distribution of operators which is conditionally positive of order s. Then there exists a unique tempered positive operator-valued measure $F(\cdot)$ on the ring R_0 generated by the compact subsets of $R^n - \{0\}$ such that

$$T_{\phi} = \int \left[\phi - \alpha \sum_{|k| \leq 2s-1} \frac{\phi^{(k)}(0)}{k!} x^k \right] dF + \sum_{|k| \leq 2s} \frac{\phi^{(k)}(0)}{k!} E_k ,$$

where α is a function in Z such that $\alpha - 1$ has a zero of order $2s+1$ at the origin, $E_k = T_{\alpha z^k}$, $|k| < 2s$, are fixed operators, and E_k, $|k| = 2s$, are certain fixed operators such that, for all complex numbers η_i, $|i| = s$,

$$\sum_{|i|=|j|=s} \eta_i \bar{\eta}_j E_{i+j} \geq 0 .$$

The tempered PO-measure $F(\cdot)$ is <u>controlled at zero</u>, in the sense that the corresponding measures μ^{ξ} given, for each $\xi \in H$, by

$$\mu^{\xi}(M) = \langle F(M)\xi, \xi\rangle, \quad M \in R_0 ,$$

satisfy the condition that there is a positive constant A such that

$$\int_{0<|x|<1} |x|^{2s} d\mu^{\xi} \leq A\|\xi\|^2 \quad \text{for all} \quad \xi \in H.$$

Proof. Let P be any homogeneous polynomial of degree s. By Proposition 2′, there exists a unique tempered PO-measure E_p such that

$$(1) \qquad T_{P\bar{P}\psi} = \int \psi(x) dE_p \quad \text{for all} \quad \psi \in Z.$$

For each $\xi \in H$, set $L_{\xi}(\phi) = \langle T_{\phi}\xi, \xi\rangle$, $\phi \in Z$. Then

$$(2) \qquad L_{\xi}(P\bar{P}\psi) = \langle T_{P\bar{P}\psi}\xi, \xi\rangle .$$

From (1) and (2) we have

$$L_{\xi}(P\bar{P}\psi) = \langle T_{P\bar{P}\psi}\xi, \xi\rangle = \langle (\int \psi(x) dE_p)\xi, \xi\rangle$$
$$= \int \psi(x) d\langle E_p\xi, \xi\rangle = \int \psi(x) d\nu_p^{\xi} .$$

Thus

$$(3) \qquad L_{\xi}(P\bar{P}\psi) = \int \psi(x) d\nu_p^{\xi} , \quad \psi \in Z ,$$

where ν_p^ξ is the tempered measure given by $\nu_p^\xi(\cdot) = \langle E_p(\cdot)\xi, \xi\rangle$.

Define ϕ on C^n by

$$\phi(z) = z^k \psi(z), \quad \psi \in Z, \quad |k| = 2s .$$

Since $z^k = z^i z^j$, $|i| = |j| = s$, and $z^k = (\frac{1}{2}z^i + \frac{1}{2}z^j)^2 - (\frac{1}{2}z^i - \frac{1}{2}z^j)^2$, letting $P_1(z) = \frac{1}{2}z^i + \frac{1}{2}z^j$, $P_2(z) = \frac{1}{2}z^i - \frac{1}{2}z^j$, it follows that $P_1 = \overline{P}_1$, $P_2 = \overline{P}_2$, and (3) above applied to $z^k\psi(z)$, $\psi \in Z$, yields

$$\begin{aligned} L_\xi(z^k\psi) &= L_\xi(P_1\overline{P}_1\psi - P_2\overline{P}_2\psi) \\ &= L_\xi(P_1\overline{P}_1\psi) - L_\xi(P_2\overline{P}_2\psi) \\ &= \int \psi(x)d\nu_{p_1}^\xi - \int \psi(x)d\nu_{p_2}^\xi \\ &= \int \psi(x)d\nu_k^\xi , \end{aligned}$$

where by definition the Radon measure $\psi \to \int \psi d\nu_k^\xi$ is given by

$$\int \psi d\nu_k^\xi = \int \psi d\nu_{p_1}^\xi - \int \psi d\nu_{p_2}^\xi .$$

Note that ν_k^ξ is not defined as a set function. Thus

$$(4) \qquad L_\xi(z^k\psi) = \int \psi(x)d\nu_k^\xi ,$$

where the measure

$$\psi \rightsquigarrow \int \psi d\nu_k^\xi$$

is not necessarily positive. Let $\Gamma_k = \{x: x^k = 0\}$ and set $\Omega_k = R^n - \Gamma_k$. Since $x^k \neq 0$ in Ω_k, the mapping

$$C_0^0(\Omega_k) \to \psi \rightsquigarrow \int \psi(x) d\mu_k^\xi \underset{\text{def}}{=} \int \psi(x) \frac{d\nu_k^\xi}{x^k}$$

determines a measure in Ω_k. But

$$L_\xi(\psi z^j z^k) = \int \psi z^j d\nu_k^\xi = \int \psi z^k d\nu_j^\xi ,$$

so that

$$\text{(4a)} \qquad \int \psi d\mu_j^\xi = \int \psi d\mu_k^\xi \quad \text{on} \quad \Omega_j \cap \Omega_k ;$$

that is, the measures $\psi \to \int \psi d\mu_k^\xi$ are compatible. Let $\Omega_0 = \bigcup_k \Omega_k$. Then

$$\Omega_0 = \bigcup_k \Omega_k = \bigcup_k (R^n - \Gamma_k)$$

$$= R^n - \bigcap_k \Gamma_k = R^n - \{0\} .$$

In view of the compatibility of the measures $\int \_\_ d\mu_k^\xi$, there exists a <u>unique</u> measure $\int \_\_ d\mu^\xi$ on Ω_0 such that

$$\int \_\_ d\mu^{\xi} = \int \_\_ d\mu_k^{\xi} \quad \text{on} \quad \Omega_k ,$$

as in [3, p. 59]. Returning to (4), it follows that

$$(5) \qquad L_{\xi}(z^k\psi) = \int_{R^n} \psi(x) d\nu_k^{\xi} = \int_{\Omega_k \cup \Gamma_k} \psi(x) d\nu_k^{\xi}$$

$$= \int_{\Omega_k} \psi(x) d\nu_k^{\xi} + \int_{\Gamma_k} \psi(x) d\nu_k^{\xi}$$

$$= \int_{\Omega_k} x^k \psi(x) d\mu_k^{\xi} + \int_{\Gamma_k} \psi(x) d\nu_k^{\xi}$$

$$= \int_{\Omega_k} x^k \psi(x) d\mu^{\xi} + \int_{\Gamma_k} \psi(x) d\nu_k^{\xi} .$$

Since $x^k\psi$ vanishes in the complement of Ω_k and $\Omega_k \subset \Omega_0$, we have

$$(6) \qquad \int_{\Omega_k} x^k \psi(x) d\mu^{\xi} = \int_{\Omega_0} x^k \psi(x) d\mu^{\xi} ,$$

For $|k| = 2s$, let $\Gamma'_{k,j}$ denote the set of points $x \in R^n$ such that $x^k = 0$ and $x_j = \delta_j$, where $\delta_j = (0, 0, \ldots, 1, 0, \ldots, 0)$, with 1 in the jth position. (Recall that $x^k = x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}$, where the k_j, $1 \leq j \leq n$, are nonnegative integers). Taking (4a) into account together with the fact that

$$0 = \int_{\Gamma_{k,j}} x^k d\nu^{\xi}_{\delta_j} = \int_{\Gamma'_{k,j}} x_j d\nu^{\xi}_k ,$$

we have

$$0 = \int_{\Gamma_{k,j}} \underline{\quad}\, x_j d\nu^{\xi}_k , \tag{7}$$

and since some $x_j \neq 0$ on $\Gamma_{k,j}$, there follows

$$\int_{\Gamma_{k,j}} \underline{\quad}\, d\nu^{\xi}_k = 0.$$

But the set Γ_k can be partitioned into the set $\{0\} = \{(0, 0, \ldots, 0)\}$ and a finite number of sets in each of which at least one of the factors $x_j \neq 0$. In view of the fact that

$$\int_{\Gamma'_{k,j}} \underline{\quad}\, d\nu^{\xi}_k = 0$$

for every j, it follows that the measure

$$\int_{\Gamma_k} \underline{\quad}\, d\nu^{\xi}_k$$

is concentrated on the set $\{0\}$. Returning to (5) we have

$$L_\xi(z^k\psi) = \int_{\Omega_k} x^k\psi(x)d\mu^\xi + \int_{\Gamma_k} \psi(x)d\nu_k^\xi$$

$$= \int_{\Omega_k} x^k\psi(x)d\mu^\xi + \int_{\{0\}} \psi(x)d\nu_k^\xi$$

$$= \int_{\Omega_k} x^k\psi(x)d\mu^\xi + \psi(0)\int_{\{0\}} d\nu_k^\xi .$$

Thus,

(8) $$L_\xi(z^k\psi) = \int_{\Omega_k} x^k\psi(x)d\mu^\xi + a_k\psi(0) ,$$

where

(8a) $$a_k = \int_{\{0\}} d\nu_k^\xi , \quad |k| = 2s .$$

Since $\phi(z) = z^k\psi(z)$, $\psi \in Z$, $|k| = 2s$, then $\psi(0) = \frac{\phi^{(k)}(0)}{k!}$, and therefore (8) may be written as

(9) $$L_\xi(z^k\psi) = \int_{\Omega_0} \phi(x)d\mu^\xi + a_k \frac{\phi^{(k)}(0)}{k!} ,$$

whenever $\phi = z^k\psi$, $\psi \in Z$, $|k| = 2s$. The fact that each derivative of order j, $|j| = 2s$, of the function $\phi = z^k\psi$ vanishes at the origin if $j \neq k$ implies that (9) can be written as

$$L_\xi(\phi) = \int_{\Omega_0} \psi(x)\,d\mu^\xi + \sum_{|k|=2s} a_k \frac{\phi^{(k)}(0)}{k!} \tag{10}$$

for all functions ϕ of the form $\phi(z) = z^k\psi$, $|k| = 2s$, for some choice of $\psi \in Z$, and where

$$a_k = \int_{\{0\}} d\nu_k^\xi, \quad |k| = 2s .$$

Since the right member of (10) is independent of k (the dependence is on s), (10) holds by linearity for all functions ϕ of the form

$$\phi(z) = \sum_{|k|=2s} z^k\phi_k(z), \quad \phi_k \in Z. \tag{11}$$

<u>Lemma 1</u>. μ^ξ is a positive tempered measure for each $\xi \in H$.

<u>Proof</u>. Since

$$\int_{\overline{\Omega_k}} x^k d\mu_k = \int_{\overline{\Omega_k}} d\nu_k^\xi$$

and $\mu^\xi = \mu_k^\xi$ on Ω_k, it follows that

$$\int_{\_\_} x^k d\mu^\xi = \int_{\_\_} d\nu_k^\xi \qquad \text{on} \quad R^n - \{0\}$$

because both sides vanish on $\Gamma_k - \{0\}$. Choose $k = 2j$, $|j| = s$, so that $x^k = x_1^{2j_1} x_2^{2j_2} \cdots x_n^{2j_n}$. Since T is conditionally positive of order s, $z^{2j}T$ is multiplicitively positive. Proposition 2' implies the existence of a tempered PO-measure $E_{2j}(\cdot)$ which represents $z^{2j}T$ in the sense stated in that proposition. The associated family $\{\nu_{2j}^\xi : \xi \in H\}$ consists of positive tempered measures. Thus there is an integer $p_{2j} \geq 0$ and a constant $K_{2j} > 0$ such that

$$\int (1+|x|^2)^{-p_{2j}} d\nu_{2j}^\xi \leq K_{2j} \|\xi\|^2 \quad \text{for all} \quad \xi \in H.$$

The fact that $x^{2j} d\mu^\xi = d\nu_{2j}^\xi$ and $d\nu_{2j}^\xi$ is positive for every ξ and every j implies that the measure $d\mu^\xi$ is positive for every ξ.

To establish that $d\mu^\xi$ is tempered, choose $k = 2(0, 0, \ldots, 0, s, 0, \ldots, 0)$ with s in the ith position. Since

$$\{x: |x| \geq 1\} \subset \bigcup_{i=1}^{n} \{x: |x_i| \geq \frac{1}{\sqrt{n}}\},$$

$d\mu^\xi$ is positive and $x^{2j} = x_i^{2s} \geq 1/n^s$, we have

$$\int_{|x_i| \geq 1/\sqrt{n}} (1+|x|^2)^{-p_{2j}} d\mu^\xi \leq n^s \int_{|x_i| \geq 1/\sqrt{n}} (1+|x|^2)^{-p_{2j}} x_i^{2j} d\mu^\xi$$

$$= n^s \int_{|x_i| \geq 1/\sqrt{n}} (1+|x|^2)^{-p_{2j}} d\nu_{2j}^\xi$$

$$\leq n^s \int_{\Omega_0} (1+|x|^2)^{-p_{2j}} d\nu_{2j}^\xi$$

$$\leq n^s K_{2j} \|\xi\|^2$$

for all $\xi \in H$. Thus

$$\int_{\{x:\ |x| \geq 1\}} (1+|x|^2)^{-p_{2j}} d\mu^\xi \leq A\|\xi\|^2$$

for all $\xi \in H$, where A is a positive constant. Therefore the measure $d\mu^\xi$ is tempered.

Lemma 2. Let $B_0 = \{x: 0 < |x| < 1\}$. Then there exists a constant $K < \infty$ such that

$$\int_{B_0} x^{2j} d\mu^\xi \leq K\|\xi\|^2, \quad |j| = s, \quad \text{for all } \xi \in H.$$

Proof.

$$\int_{B_0} x^{2j} d\nu_{2j}^{\xi} = \int_{B_0} (1+|x^2|)^{p_{2j}}(1+|x|^2)^{-p_{2j}} x^{2j} d\nu_{2j}^{\xi}$$

$$\leq 2^{p_{2j}} \int_{B_0} (1+|x|^2)^{-p_{2j}} d\nu_{2j}^{\xi}$$

$$\leq 2^{p_{2j}} \int_{R^n} (1+|x|^2)^{-p_{2j}} d\nu_{2j}^{\xi}$$

$$\leq 2^{p_{2j}} K_{2j} \|\xi\|^2$$

for all ξ in H. Thus $K = 2^{p_{2j}} K_{2j}$.

Corollary.

$$\int_{B_0} |x|^{2s} d\mu^{\xi} \leq A\|\xi\|^2$$

for all $\xi \in H$, where A is a positive constant. This follows from the fact that the integral above is a finite linear combination of integrals of the form occurring in the statement of Lemma 2.

Recall that $a_k = \int_{\{0\}} d\nu_k^{\xi}$, $|k| = 2s$ (see (8a) above).

Lemma 3. The numbers a_k, $|k| = 2s$, are such that the Hermitian form $\sum_{|i|=|j|=s} a_{i+j}\eta_i\overline{\eta}_j$ is positive-definite (see [6, p.184]).

Proof. There corresponds to the homogeneous polynomial $P(z) = \sum_{|j|=s} \eta_j z^j$ of degree s a positive measure $d\nu_p^{\xi}$ (see (3) above) such that

(i) $L_\xi(P\overline{P}\phi) = \int \phi(x)d\nu_p^{\xi}$ for all $\phi \in Z$. We also have

$$\text{(ii)}\quad L_\xi(P\overline{P}\phi) = \sum_{|i|=|j|=s} L_\xi(z^{i+j}\phi)\eta_i\overline{\eta}_j$$

$$= \sum_{|i|=|j|=s} \eta_i\overline{\eta}_j \int \phi(x)d\nu_{i+j}^{\xi} \quad \text{for every } \xi \in H.$$

Since (i) and (ii) hold for every ϕ in Z,

(iii) $0 \leq \int_S 1d\nu_p^{\xi} = \sum_{|i|=|j|=s} \int_S 1d\nu_{i+j}^{\xi}\eta_i\overline{\eta}_j$ for any compact set S. In particular, (iii) holds for $S = \{0\}$. In view of the fact that $\int_{\{0\}} 1d\nu_{i+j}^{\xi} = a_{i+j}$, we have

$$0 \leq \sum_{|i|=|j|=s} a_{i+j}\eta_i\overline{\eta}_j$$

for all complex numbers η_i, η_j, $|i|=|j|=s$.

The preceding results may be summarized as follows:

For every vector $\xi \in H$,

$$L_\xi(\phi) = \int_{\Omega_0} \phi(x)d\mu^\xi + \sum_{|k|=2s} a_k \frac{\phi^{(k)}(0)}{k!}$$

for all functions ϕ of the form

$$\phi(z) = \sum_{|k|=2s} z^k \phi_k(z),$$

$\phi_k \in Z$. There is a nonnegative integer p and a positive constant K such that

$$\int_{|x| \geq 1} (1+|x|^2)^{-p} d\mu^\xi \leq K\|\xi\|^2$$

for all $\xi \in H$. (that is, μ^ξ is tempered) and

$$\int_{0<|x|<1} |x|^{2s} d\mu^\xi \leq A\|\xi\|^2$$

for all $\xi \in H$ (that is, μ^ξ is controlled at zero). In addition there is a nonnegative integer p and a constant $K > 0$ such that

$$\int_{\Omega_0} |x|^{2s}(1+|x|^2)^{-p} d\mu^\xi \leq K\|\xi\|^2$$

for all $\xi \in H$. Moreover, the numbers

$$a_k = \int_{\{0\}} d\nu_k^{\xi}, \quad |k| = 2s,$$

are such that the Hermitian form $\sum_{|i|=|j|=s} \eta_i \bar{\eta}_j a_{i+j}$ is positive-definite.

The set of functions ϕ of the form

$$\text{(12)} \qquad \phi(z) = \sum_{|k|=2s} z^k \phi_k(z), \quad \phi_k \in Z,$$

is dense (in the Z-topology) in the set of functions in Z having a zero of order $2s$ at the origin [6, pp. 194, 195]. Let

$$\text{(13)} \qquad L'_{\xi}(\phi) = \int_{\Omega_0} \phi(x) d\mu^{\xi} + \sum_{|k|=2s} a_k \frac{\phi^{(k)}(0)}{k!}$$

for all functions ϕ of the form (12). Then L'_{ξ} is continuous in the topology of Z [6, p. 194]. Since $L'_{\xi} = L_{\xi}$ on the set of functions of the form (12) and this set of functions is Z-dense in the set of functions in Z having a zero of order $2s$ at the origin, it follows that (10) holds for all functions in Z having a zero of order $2s$ at the origin.

Let $\phi \in Z$ be arbitrary and let α be any fixed function in Z such that the function $\alpha - 1$ has a zero of order $2s + 1$ at the origin [6, p. 177, footnote 4]. Then

$$(14) \qquad \theta(z) = \phi(z) - \alpha(z) \sum_{|k| \leq 2s-1} \frac{\phi^{(k)}(0)}{k!} z^k$$

has a zero of order $2s$ at the origin. Hence,

$$(15) \qquad L_\xi(\theta) = \int_{\Omega_0} \theta(x) d\mu^\xi + \sum_{|k|=2s} a_k \frac{\theta^{(k)}(0)}{k!}$$

Since $\theta^{(k)}(0) = \phi^{(k)}(0)$, $|k| = 2s$, we have

$$(16) \qquad L_\xi(\theta) = \int_{\Omega_0} \theta(x) d\mu^\xi + \sum_{|k|=2s} a_k \frac{\phi^{(k)}(0)}{k!} .$$

Taking (14) into account there follows

$$(17) \qquad L_\xi(\phi) = L_\xi(\theta) + \sum_{|k| \leq 2s-1} \frac{\phi^{(k)}(0)}{k!} L_\xi(z^k \alpha) .$$

Set

$$(18) \qquad a_k = L_\xi(z^k \alpha), \quad 0 \leq |k| \leq 2s-1 .$$

Then we have

$$(19) \qquad L_\xi(\phi) = \langle T_\phi \xi, \xi \rangle$$

$$= \int_{\Omega_0} \left[\phi(x) - \alpha(x) \sum_{|k| \leq 2s-1} \frac{\phi^{(k)}(0)}{k!} x^k \right] d\mu^\xi(x) + \sum_{|k| \leq 2s} a_k \frac{\phi^{(k)}(0)}{k!} .$$

Lemma 4. Let $\phi \in Z$, $\eta > 0$ and $|\phi^{(k)}(0)| < \eta$, $0 \leq |k| \leq 2s$. Then there is a positive number M such that

$$(20) \qquad \left| \sum_{|k| \leq 2s} a_k \frac{\phi^{(k)}(0)}{k!} \right| \leq \eta M \|\xi\|^2 \quad \text{for all} \quad \xi \in H.$$

Proof. Since $a_k = L_\xi(z^k a) = \langle T_{z^k a}\xi, \xi\rangle$ for $0 \leq |k| \leq 2s-1$, it follows that

$$|a_k| = |\langle T_{z^k a}\xi, \xi\rangle| \leq \|T_{z^k a}\| \, \|\xi\|^2 = M_k \|\xi\|^2,$$

where $M_k = \|T_{z^k a}\| < \infty$ since $T_{z^k a} \in B(H)$. For $|k| = 2s$,

$$\begin{aligned} |a_k| &= \left|\int_{\{0\}} d\nu_k^\xi\right| = \left|\int_{\{0\}} d\nu_{p_1}^\xi - \int_{\{0\}} d\nu_{p_2}^\xi\right| \\ &= |\langle E_{p_1}(\{0\})\xi, \xi\rangle - \langle E_{p_2}(\{0\})\xi, \xi\rangle| \\ &\leq |\langle E_{p_1}(\{0\})\xi, \xi\rangle| + |\langle E_{p_2}(\{0\})\xi, \xi\rangle| \\ &\leq \|E_{p_1}(\{0\})\| \, \|\xi\|^2 + \|E_{p_2}(\{0\})\| \, \|\xi\|^2 \\ &= (\|E_{p_1}(\{0\})\| + \|E_{p_2}(\{0\})\|) \|\xi\|^2 = M_{2s}\|\xi\|^2, \end{aligned}$$

where

$$M_{2s} = \|E_{p_1}(\{0\})\| + \|E_{p_2}(\{0\})\|.$$

Set

$$M = \sum_{|k| \leq 2s} M_k .$$

Then

$$\Big| \sum_{|k| \leq 2s} \frac{\phi^{(k)}(0)}{k!} a_k \Big| \leq \sum_{|k| \leq 2s} \frac{|\phi^{(k)}(0)|}{k!} |a_k|$$

$$\leq \eta \sum_{|k| \leq 2s} |a_k| \leq \eta M \| \xi \|^2 .$$

<u>Lemma 5</u>. Let $\{\phi_m\}$ be a sequence converging to zero in Z and let

$$\theta_m(z) = \phi_m(z) - \alpha(z) \sum_{|k| \leq 2s-1} \frac{\phi_m^{(k)}(0)}{k!} z^k , \tag{21}$$

where α is a fixed function in Z such that $\alpha - 1$ has a zero of order $2s + 1$ at the origin. Then the sequence $\{\theta_m\}$ converges to zero in Z. This is clear and the proof is omitted.

The functions θ_m in the preceding lemma have a zero of order $2s$ at the origin and therefore

$$\theta_m(z) = \sum_{|k|=2s} z^k \phi_{m,k}(z), \quad \phi_{m,k} \in Z , \tag{22}$$

[6, p. 194]. Since θ_m converges to zero in the topology of Z, given any $\eta > 0$ there is an N such that for $m > N$,

$$(23) \qquad |\theta_m(x)| < \frac{\eta |x|^{2s}}{(1+|x|^2)^{p+s}} \qquad [6, p. 185].$$

Lemma 6. $\phi \rightsquigarrow T_\phi : Z \to B(H)$ is tempered. That is, T is continuous on Z in the relative topology as a subspace of $\mathcal{S}$.

Proof. Let ϕ_m be a sequence of functions of the form (22) converging to zero in the topology of $\mathcal{S}$, and let $\eta = \frac{1}{4(K'+M')}$, where $M' \geq M$ with M as in Lemma 4, and K' is a positive constant, $K' \geq K$, K as in the Corollary to Lemma 2, such that for some nonnegative integer p,

$$\int_{|x| \geq 1} (1+|x|^2)^{-p} d\mu^\xi \leq K' \|\xi\|^2 \quad \text{for all } \xi \in H.$$

Such a constant K' exists because μ^ξ is tempered (Lemma 1). Then

$$(24) \quad L_\xi(\phi_m) = \langle T_{\phi_m} \xi, \xi \rangle$$

$$= \int_{\Omega_0} \left[\phi_m(x) - a(x) \sum_{|k| \leq 2s-1} \frac{\phi_m^{(k)}(0)}{k!} x^k \right] d\mu^\xi + \sum_{|k| \leq 2s} a_k \frac{\phi_m^{(k)}(0)}{k!}.$$

Since $\phi_m \to 0$ in Z, given $\eta > 0$ there is an N_1 such that

$|\phi_m^{(k)}(0)| < \eta$, $|k| \leq 2s$, for all $m > N_1$. By Lemma 5, $\theta_m \to 0$ in Z. Given the same η as above, there is an N_2 such that (23) holds for all $m > N_2$. Let $N = \max\{N_1, N_2\}$. Then for all $m > N$,

$$|L_\xi(\phi_m)| = |\int_{\Omega_0} \theta_m(x)d\mu^\xi + \sum_{|k|=2s} \frac{\phi_m^{(k)}(0)}{k!} a_k|$$

$$\leq \int_{\Omega_0} |\theta_m(x)|d\mu^\xi + |\sum_{|k|\leq 2s} \frac{\phi_m^{(k)}(0)}{k!} a_k|$$

$$\leq \int_{\Omega_0} \eta|x|^{2s}(1+|x|^2)^{-p}d\mu^\xi + \eta M\|\xi\|^2$$

$$\leq \eta K\|\xi\|^2 + \eta M\|\xi\|^2 = \frac{1}{4(K'+M')}(K+M)\|\xi\|^2 \leq \frac{1}{4}\|\xi\|^2.$$

Thus

(25) $$|<T_{\phi_m}\xi, \xi>| \leq \frac{1}{4}\|\xi\|^2 \quad \text{for all } \xi \in H,$$

and therefore

(26) $$|<T_{\phi_m}\xi, \lambda>| \leq \|\xi\|\|\lambda\| \quad \text{for all } \xi, \lambda \text{ in } H,$$

from which it follows that

$$\|T_{\phi_m}\| = \sup\{|<T_{\phi_m}\xi, \lambda>|, \|\xi\|\leq 1, \|\lambda\|\leq 1\} = 1 .$$

Thus

$$\|T_{\phi_m}\| \leq 1 \quad \text{for all} \quad m > N. \tag{27}$$

If $\|T_{\phi_m}\| \not\to 0$, then there exists an $\epsilon > 0$ such that $\|T_{\phi_{m_j}}\| \geq \epsilon$ for $j = 1, 2, 3, \ldots$. But then $\|T_{\frac{2}{\epsilon}\phi_{m_j}}\| \geq 2$, contradicting (27) for the sequence $\frac{2}{\epsilon}\phi_{m_j} \to 0$. Therefore $\|T_{\phi_m}\| \to 0$ and T is tempered.

The situation now is as depicted below

$$\begin{tikzcd}
\mathscr{S} \arrow[rr, dashed, "\overline{T}"] & & B(H) \\
Z \arrow[u, "\leftarrow\text{dense}"'] \arrow[urr, "T"'] & &
\end{tikzcd}$$

where $\overline{T}$ is the <u>unique</u> extension of T.

<u>Lemma 7</u>. $\overline{T}$ has the same form on $\mathscr{S}$ that T has on Z. That is, for each $\xi \in H$,

$$\overline{L}_\xi(\phi) = \langle \overline{T}_\phi \xi, \xi \rangle = \int_{\Omega_0} \theta(x) d\mu^\xi + \sum_{|k| \leq 2s} \frac{\phi^{(k)}(0)}{k!} a_k$$

for all $\phi \in \mathscr{S}$. All other symbols have precisely the same meanings as before.

Proof. Choose any $\phi \in \mathcal{S}$ and let ϕ_m be a sequence from $\mathcal{S}$ converging to ϕ in the space $\mathcal{S}$. Then

$$\sup_x |(1+|x|^2)^{-p}(\phi_m-\phi)(x)| \to 0$$

and therefore

$$\sup_x |(1+|x|^2)^{-p}(\theta_m-\theta)(x)| \to 0 .$$

Since

$$|\theta_m(x)| \leq \eta |x|^{2s}(1+|x|^2)^{-p-s} ,$$

the Bounded Convergence Theorem implies that

$$\int_{\Omega_0} \theta_m d\mu^{\xi} \to \int_{\Omega_0} \theta d\mu^{\xi} \quad \text{for all } \xi \text{ in } H.$$

Therefore

$$\begin{aligned}
<\overline{T}_{\phi}\xi, \xi> &= \lim_{m\to\infty} <T_{\phi_m}\xi, \xi> \\
&= \lim_{m\to\infty}\left[\int_{\Omega_0}\theta_m(x)d\mu^{\xi} + \sum_{|k|\leq 2s}\frac{\phi_m^{(k)}(0)}{k!}a_k\right] \\
&= \lim_{m\to\infty}\int_{\Omega_0}\theta_m(x)d\mu^{\xi} + \sum_{|k|\leq 2s}\lim_{m\to\infty}\frac{\phi_m^{(k)}(0)}{k!}a_k =
\end{aligned}$$

$$= \int_{\Omega_0} \theta(x) d\mu^{\xi} + \sum_{|k| \leq 2s} \frac{\phi^{(k)}(0)}{k!} a_k$$

$$= \int_{\Omega_0} \left[\phi(x) - \alpha(x) \sum_{|k| \leq 2s-1} \frac{\phi^{(k)}(0)}{k!} x^k \right] d\mu^{\xi} + \sum_{|k| \leq 2s} \frac{\phi^{(k)}(0)}{k!} a_k .$$

Q. E. D.

<u>Lemma 8</u>. For every pair of vectors ξ, η in H and all M in the ring R_0 generated by the compact subsets of $\Omega_0 = R^n - \{0\}$,

(a) $[\mu^{\xi+\eta}(M)]^{1/2} \leq [\mu^{\xi}(M)]^{1/2} + [\mu^{\eta}(M)]^{1/2}$

(b) $\mu^{\xi+\eta}(M) + \mu^{\xi-\eta}(M) = 2\mu^{\xi}(M) + 2\mu^{\eta}(M)$

(c) $\mu^{c\xi}(M) = |c|^2 \mu^{\xi}(M)$ for all $c \in C$, and

(d) $\mu^{\xi}(M) \leq k_M \|\xi\|^2$ for all $\xi \in H$, where the positive constant k_M depends on the set M.

<u>Proof</u>. (a) Let C be an arbitrary compact subset of Ω_0 and let $r = d(\{0\}, C) > 0$ denote the distance between the sets $\{0\}$ and C. Let χ_C denote the characteristic function of the set C, and choose a sequence ϕ_m from $\mathcal{S}$ such that

(i) $\phi_m \in \mathcal{D}$ and $\operatorname{supp} \phi_m \subset Q$ for all m, where Q is a compact subset of Ω_0,

(ii) $\phi_m^{(k)}(0) = 0$, $|k| \leq 2s$, for all m, and

(iii) $\phi_m \downarrow \chi_C$.

Since

$$<\overline{T}_{\phi}\xi, \xi> = \int_{\Omega_0} \theta(x)d\mu^{\xi} + \sum_{|k| \leq 2s} \frac{\phi^{(k)}(0)}{k!} a_k$$

for all ϕ in $\mathcal{S}$, and therefore for all $\phi \in \mathcal{D}$, it follows that

$$<\overline{T}_{\phi_m}\xi, \xi> = \int_{\Omega_0} \phi_m(x)d\mu^{\xi} = \int_{Q} \phi_m(x)d\mu^{\xi}$$

$$\geq \int \chi_C d\mu^{\xi} = \mu^{\xi}(C) \geq 0 \quad \text{for all} \quad \xi \quad \text{in} \quad H.$$

Thus $<\overline{T}_{\phi_m}\xi, \xi> \geq 0$ for all m and each $\xi \in H$. Similarly, $<\overline{T}_{\phi_m}\eta, \eta> \geq 0$ and $<\overline{T}_{\phi_m}(\xi+\eta), \xi+\eta> \geq 0$ for all m and all vectors ξ, η in H. By the Monotone Convergence Theorem,

$$\int \phi_m d\mu^{\xi+\eta} \downarrow \mu^{\xi+\eta}(C), \quad \int \phi_m d\mu^{\xi} \downarrow \mu^{\xi}(C)$$

and

$$\int \phi_m d\mu^{\eta} \downarrow \mu^{\eta}(C) .$$

Applying the Generalized Cauchy-Schwartz inequality, we obtain

$$<\overline{T}_{\phi_m}(\xi+\eta), \xi+\eta>^{1/2} \leq <\overline{T}_{\phi_m}\xi, \xi>^{1/2} + <\overline{T}_{\phi_m}\eta, \eta>^{1/2}$$

for all m and all ξ, η in H. Letting $m \to \infty$ yields

$$[\mu^{\xi+\eta}(C)]^{1/2} \leq [\mu^{\xi}(C)]^{1/2} + [\mu^{\eta}(C)]^{1/2}$$

for any compact sets C in Ω_0. By the regularity of the measures involved, (a) holds for all $M \in R_0$.

Parts (b) and (c) of the lemma are established in a similar manner. To establish (d), fix $M \in R_0$. Since the ring R_0 is generated by the compact subsets of Ω_0, there is a compact set Q in Ω_0 such that $Q \supset M$. Choose a function ϕ satisfying

(i) $\phi \in \mathcal{D}$, $\operatorname{supp} \phi \subset Q$,

(ii) $\phi \geq \chi_Q$.

Then

$$\begin{aligned}
\mu^{\xi}(M) &\leq \mu^{\xi}(Q) = \int \chi_Q d\mu^{\xi} \\
&\leq \int_Q \phi(x) d\mu^{\xi} = \int_Q \phi(x)(1+|x|^2)^{-p}|x|^{2s}(1+|x|^2)^p|x|^{-2s} d\mu^{\xi} \\
&\leq \sup_{x \in Q} |(1+|x|^2)^p x^{-2s}\phi(x)| \int_Q |x|^{2s}(1+|x|^2)^{-p} d\mu^{\xi} \\
&\leq k \int_{\Omega_0} |x|^{2s}(1+|x|^2)^{-p} d\mu^{\xi} \\
&\leq kK\|\xi\|^2
\end{aligned}$$

for all $\xi \in H$, (Lemmas 1 and 2), where

$$k = \sup_{x \in Q} |(1+|x|^2)^{-p} x^{-2s} \phi(x)| < \infty$$

since $0 \notin Q$. Thus

$$\mu^{\xi}(M) \leq k_M \|\xi\|^2 \quad \text{for all} \quad \xi \in H,$$

where $k_M = kK$. Q.E.D.

In view of the preceding lemma and Proposition 2 of Chapter 1, there is a unique PO-measure $F(\cdot)$ on the ring R_0 such that, for each $\xi \in H$,

$$\mu^{\xi}(M) = \langle F(M)\xi, \xi\rangle \quad \text{for all} \quad M \in R_0 .$$

Moreover, $F(\cdot)$ is tempered, by Lemma 1, and controlled at zero in the sense of Lemma 2.

Recall that the measure $d\nu_k^{\xi}$ was defined by $\psi \rightsquigarrow \int \psi d\nu_k^{\xi}$, with

$$\int d\nu_k^{\xi} = \int d\nu_{p_1}^{\xi} - \int d\nu_{p_2}^{\xi} ,$$

where p_1, p_2 refer to the polynomials $P_1(z) = \frac{1}{2}(z^i + z^j)$, $P_2(z) = \frac{1}{2}(z^i - z^j)$, $|i| = |j| = s$. Thus $\nu_k^{\xi}(M) = \nu_{p_1}^{\xi}(M) - \nu_{p_2}^{\xi}(M)$. The corresponding operator valued measure E_k, $|k| = 2s$, is defined by

$$<E_k(M)\xi, \xi> = <(E_{p_1}(M) - E_{p_2}(M))\xi, \xi> = \nu_k^{\xi}(M).$$

In particular,

$$<E_k(\{0\})\xi, \xi> = \nu_k^{\xi}(\{0\}) \ ,$$

and by Lemma 3,

$$\sum_{|i|=|j|=s} \eta_i \bar{\eta}_j E_{i+j} \geq 0$$

for all complex numbers η_i, η_j, $|i| = |j| = s$.

Recalling that $L_{\xi}(\phi) = <T_{\phi}\xi, \xi>$, where

$$L_{\xi}(\phi) = \int_{\Omega_0} \left[\phi - \alpha \sum_{|k| \leq 2s-1} \frac{\phi^{(k)}(0)}{k!} x^k \right] d\mu^{\xi} + \sum_{|k| \leq 2s} \frac{\phi^{(k)}(0)}{k!} a_k \ ,$$

and that $\mu^{\xi}(M) = <F(M)\xi, \xi>$, $M \in R_0$, for all $\xi \in H$, we have

$$<T_{\phi}\xi, \xi> = \left< \left(\int \left[\phi - \alpha \sum_{|k| \leq 2s-1} \frac{\phi^{(k)}(0)}{k!} x^k \right] dF + \sum_{|k| \leq 2s} \frac{\phi^{(k)}(0)}{k!} E_k \right) \xi, \xi \right>$$

for all $\xi \in H$. Thus

$$T_{\phi} = \int \left[\phi - \alpha \sum_{|k| \leq 2s-1} \frac{\phi^{(k)}(0)}{k!} x^k \right] dF + \sum_{|k| \leq 2s} \frac{\phi^{(k)}(0)}{k!} E_k \ ,$$

where $F(\cdot)$ is the PO-measure on R_0; α is a fixed function in Z such that $\alpha - 1$ has a zero of order $2s + 1$ at the origin; $E_k = T_{z^k \alpha}$, $|k| < 2s$, are fixed bounded operators on H, and for $|k| = 2s$, $E_k = E_k(\{0\})$ are bounded operators on H such that

$$\sum_{|i|=|j|=s} \eta_i \bar{\eta}_j E_{i+j} \geq 0$$

for all complex numbers η_i, η_j with $|i|=|j|=s$; that is,

$$\left\langle \left(\sum_{|i|=|j|=s} \eta_i \bar{\eta}_j E_{i+j} \right) \xi, \xi \right\rangle \geq 0$$

for all $\xi \in H$. This completes the proof of the theorem.

Theorem (Converse of Theorem 3). Let F be a positive tempered operator-valued measure on the ring R_0 generated by the compact subsets of $R^n - \{0\}$ such that

$$\int_{0 < |x| < 1} |x|^{2s} d\mu^{\xi} \leq K \| \xi \|^2$$

for all $\xi \in H$, where $\mu^{\xi}(M) = \langle F(M)\xi, \xi \rangle$, $\xi \in H$, for all $M \in R_0$; E_k ($|k| = 2s$) operators on H such that for all complex numbers η_i, $|i| = s$,

$$\sum_{|k|=|j|=s} \eta_i \bar{\eta}_j E_{i+j} \geq 0 ;$$

E_k, ($|k| < 2s$) certain fixed operators on H, and α a fixed function in Z such that the function $\alpha - 1$ has a zero of order $2s + 1$ at the origin. Then T defined by

$$T_\phi = \int \left[\phi(x) - \alpha(x) \sum_{|k| \leq 2s-1} \frac{\phi^{(k)}(0)}{k!} x^k\right] dF + \sum_{|k| \leq 2s} \frac{\phi^{(k)}(0)}{k!} E_k$$

is a distribution of operators on Z which is conditionally positive of order s.

<u>Proof</u>. It is clear that T is a distribution of operators. Let P be any homogeneous polynomial of degree s. We need to show that $T_{P\bar{P}\phi\bar{\phi}}$ is a positive operator for any $\phi \in Z$. Since $P\bar{P}\phi\bar{\phi}$ has a zero of order $2s$ at the origin, $(P\bar{P}\phi\bar{\phi})^{(k)}(0) = 0$, $|k| \leq 2s-1$. On the other hand for $k = i + j$ with $|k| = 2s$,

$$(P\bar{P}\phi\bar{\phi})^{(k)}(0) = \sum_{\substack{i+j=k \\ |i|=|j|=s}} (i+j)! \frac{(P\phi)^{(i)}(0)}{i!} \frac{\overline{(P\phi)^{(j)}(0)}}{j!}$$

and therefore

$$T_{P\overline{P}\phi\overline{\phi}} = \int |P\phi|^2 dF + \sum_{\substack{i+j=k \\ |i|=|j|=s}} \frac{(P\phi)^{(i)}(0)}{i!} \, \overline{\frac{(P\phi)^{(j)}(0)}{j!}} \, E_{i+j} .$$

Since $|P\phi|^2 \geq 0$ and F is a positive operator-valued measure, $\int |P\phi|^2 dF$ is a positive operator. Moreover the operator

$$\sum_{|i|=|j|=s} \frac{(P\phi)^{(i)}(0)}{i!} \, \overline{\frac{(P\phi)^{(j)}(0)}{j!}} \, E_{i+j}$$

is positive. Hence

$$<T_{P\overline{P}\phi\overline{\phi}}\xi, \xi> \geq 0$$

for every ξ in H which means, by definition, that T is conditionally positive of order s.

BIBLIOGRAPHY

1. Amrein, W.O., Georgescu, V., and Jauch, M., Stationary State Scattering Theory, Institute of Theoretical Physics, University of Geneva, Geneva, Switzerland (9.X11.70).

2. Berberian, S.K., Notes on Spectral Theory, Van Nostrand, Princeton (1966).

3. Friedman, A., Generalized Functions and Partial Differential Equations, Prentice-Hall, Inc. Englewood Cliffs, N.J. (1963).

4. Gel'fand, I.M., and Shilov, G.E., Generalized Functions, vol. 2, Academic Press, New York (1968).

5. ________________, Generalized Functions, vol. 3, Academic Press, New York (1967).

6. Gel'fand, I.M., and Vilenkin, N. Ya., Generalized Functions, vol. 4, Academic Press (1964).

7. Kōmura, T., Semigroups of Operators in Locally Convex Spaces, Journal of Functional Analysis 2, 258-296 (1968).

8. Kritt, B., Spectral Decomposition of Positive and Positive-Definite Distributions of Operators, Bulletin De L'Academie Polonaise Des Sciences, Serie des sciences math., astr. et phys., 16 (1968), 865-870. MR 39#3307.

9. ________________, Generalized Pseudo-Hermitian Operators, Proceedings of the American Mathematical Society, Volume 30, Number 2, (October, 1971).

10. Treves, F., Topological Vector Spaces, Distributions and Kernels, Academic Press, New York · London (1967).

11. Yosida, K., Functional Analysis, second edition, Springer-Verlag New York Inc. (1968).