AN EXPERIMENTAL NONDESTRUCTIVE METHOD FOR DETERMINING THE BUCKLING LOAD OF A COLUMN WITH ARBITRARY END CONDITIONS

by

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A THESIS

submitted to

OREGON STATE COLLEGE

in partial fulfillment of
the requirements for the
degree of

MASTER OF SCIENCE

June 1959

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Date thesis is presented 19 December 1958

Typed by Lilah N. Potter

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INTRODUCTION

Object

The computation of the critical load of an axially loaded column has always been a difficult problem. The classical method of approaching a stability problem of this kind is to write the differential equation of the problem, find the general solution, and solve for the eigenvalues corresponding to the buckling loads by substitution of the boundary values. For several reasons, which will be discussed later, this method does not give satisfactory results. It becomes necessary, then, to look for another method of determining the critical load of a column.

There appears to be a relationship between the frequency of vibration of a column, the axial load on the column, and the critical load of the column. This relationship is

$$\omega^2 = \omega_n^2 \left(1 - \frac{P}{P_{cr}}\right) ,$$

where ω is the frequency of vibration, ω_n the natural frequency of vibration, P the axial load, and P_{cr} the critical load of the column. The application of this relationship to the problem of finding the critical load of a column will be the subject of this discussion. In general, the columns under discussion will be limited to those that carry an axial compressive load only and have a slenderness ratio sufficiently large to classify as a "long column." There are no restrictions on the end conditions of the column, however, and it need not be of constant cross section along its length.

General

In 1757 Euler considered the case of a pin-ended column of uniform cross section, and found the critical load to be represented by the equation

$$P_{cr} = \frac{\pi^2 EI}{L^2} .$$

The derivation of this equation is given in the appendix, as it represents the classical method of approaching a column problem. Euler extended this equation to other than pin-ended columns by multiplying the right-hand side by a constant which was a function of the end conditions. One great disadvantage of this equation, and of any other theoretical equation, is that the end conditions of the column must be known. The difficulty arises from the fact that in actual practice, these end conditions are extremely difficult to determine. As every theoretical equation for determining the critical load of a column has this disadvantage, it would seem that a simple, non-destructive experimental method for finding the critical load would be quite valuable.

The relationship between frequency of vibration and axial load can be found by first considering the differential equation of a vibrating column. This equation is derived in the appendix, assuming very small deflections so that linear theory may be used, and is found to be

$$\frac{\partial^4 Y}{\partial x^4} + \frac{P}{EI}\frac{\partial^2 Y}{\partial x^2} = \frac{\rho A}{EI}\frac{\partial^2 Y}{\partial t^2}$$

for a column of constant cross section. By the separation of variables technique, the equation of X as a function of x is

$$\frac{d^4 X}{dx^4} + \frac{P}{EI}\frac{d^2 X}{dx^4} + \frac{\rho A}{EI}\omega^2 X = 0 \quad .$$

It can be proven that $\sin\frac{\pi}{L}x$ is a solution to this equation which meets the boundary values for a pin-ended column. Substituting this solution into the differential equation, the following relationship between load and frequency of vibration is obtained.

$$\left(\frac{\pi}{L}\right)^4 - \frac{P}{EI}\left(\frac{\pi}{L}\right)^2 = \frac{\rho A}{EI}\omega^2 \quad .$$

Rearranging, and substituting the value of the natural frequency, ω_n , for a pin-ended column, the final relationship obtained is

$$\omega^2 = \omega_n^2\left(1 - \frac{P}{P_{cr}}\right) \quad .$$

It must be remembered that this relationship was obtained by considering the simple case of a pin-ended column with a constant cross section.

The mathematics involved prohibit obtaining this relationship for more complex columns. A close investigation of the relationship between frequency of vibration and axial load will indicate that a plot of frequency of vibration squared versus axial load will be a straight line with negative slope, as shown in figure 1. It is important to notice that the intercept of the straight line on the abscissa is the critical load, and the intercept on the ordinate is the square of the natural frequency.

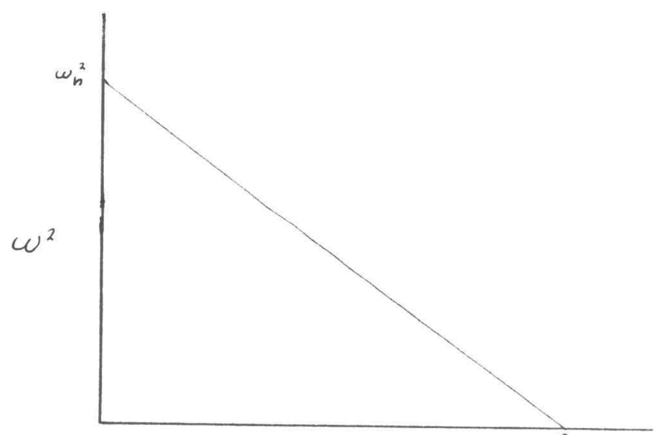

Figure 1

Theoretical Plot of Frequency Squared Versus Axial Load
For a Pin-Ended Column

In an effort to determine whether or not the above relationship between frequency of vibration and axial load is a relationship that is true in general, the problem was attacked by means of the energy methods. The results seemed to indicate that the relationship was general, but as the energy methods will only yield approximate solutions, the results were not conclusive. They were encouraging, however, and three different steel bars were built with five different end conditions available for each. Out of this assortment, eleven columns were constructed and tested. It is important to note at this point that of the eleven columns tested, the experimental data from every column plotted in a straight line similar to that shown in figure 1. This is quite significant, as many of the columns had a cross section that varied along the column, and the end conditions covered the complete range from the fixed-ended to the pin-ended conditions.

DESCRIPTION OF EQUIPMENT AND TESTS

Description of Test Columns

The columns that were used in the tests were made up of steel bars with various end conditions affixed to them, as was mentioned previously. Figure 2 shows a picture of these three bars, and they are drawn to scale in figure 3.

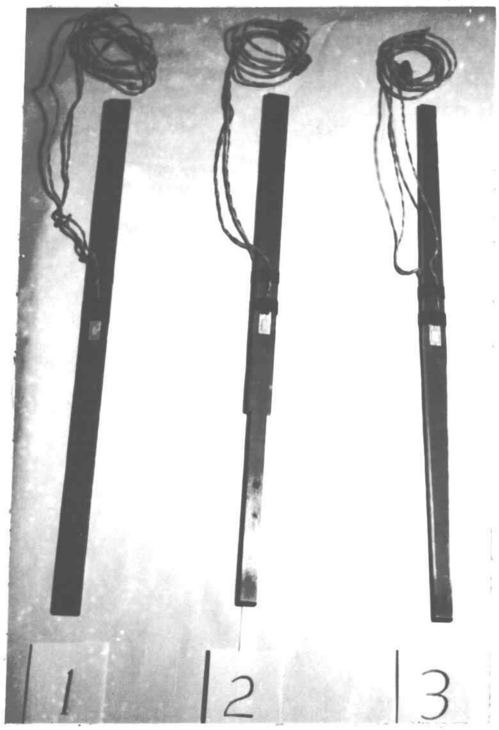

Figure 2

Steel Bars Used to Construct the Test Columns

All three bars are of cold-rolled steel, with the ends milled semi-circular and case-hardened to prevent local crushing. Bar number one is a simple bar of constant cross section. Bar number two has an abrupt change in its cross section one-third of the way along its length, while bar number three tapers toward both ends from a maximum width in the center. All three bars have approximately the same overall dimensions because of certain limitations of the

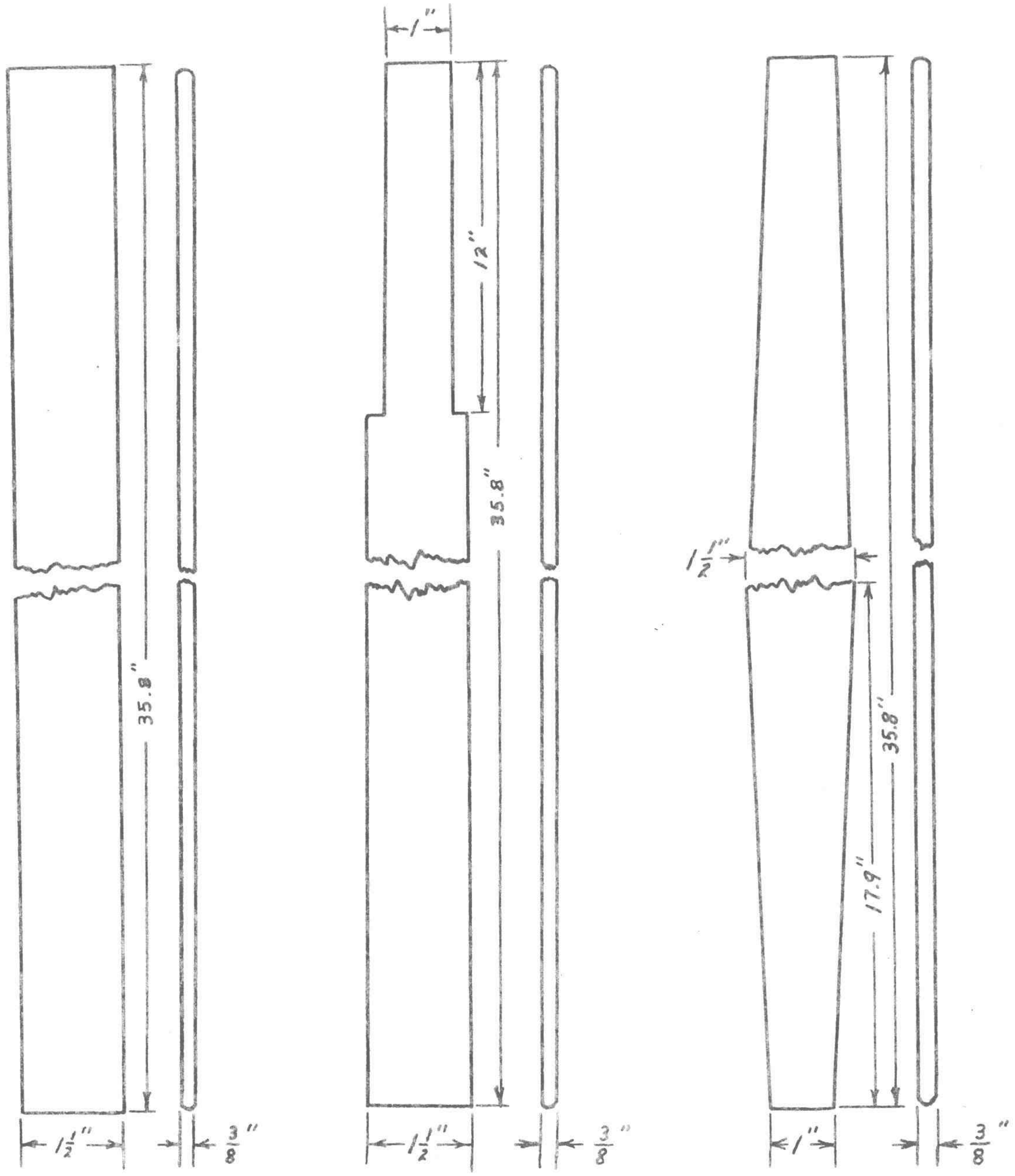

Figure 3

Details of the Basic Columns

test equipment available.

The three basic end conditions that were used in the tests are shown in figure 4. End condition number one

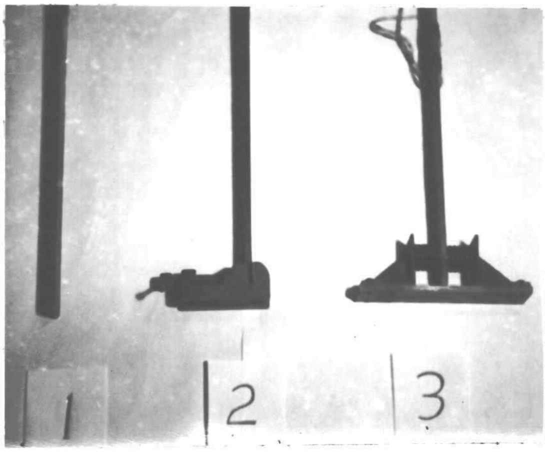

Figure 4

Basic Column End Conditions

represents a simple pin-ended configuration, and was the reason for milling and case-hardening the ends of the bars. End condition number two represents a fixed-ended condition. It is obtained by clamping small vises to both ends of the bars, thus restricting the ends of the bars against rotation while they are loaded in compression in a testing machine. This condition can be combined with end condition number one, giving a column with one end

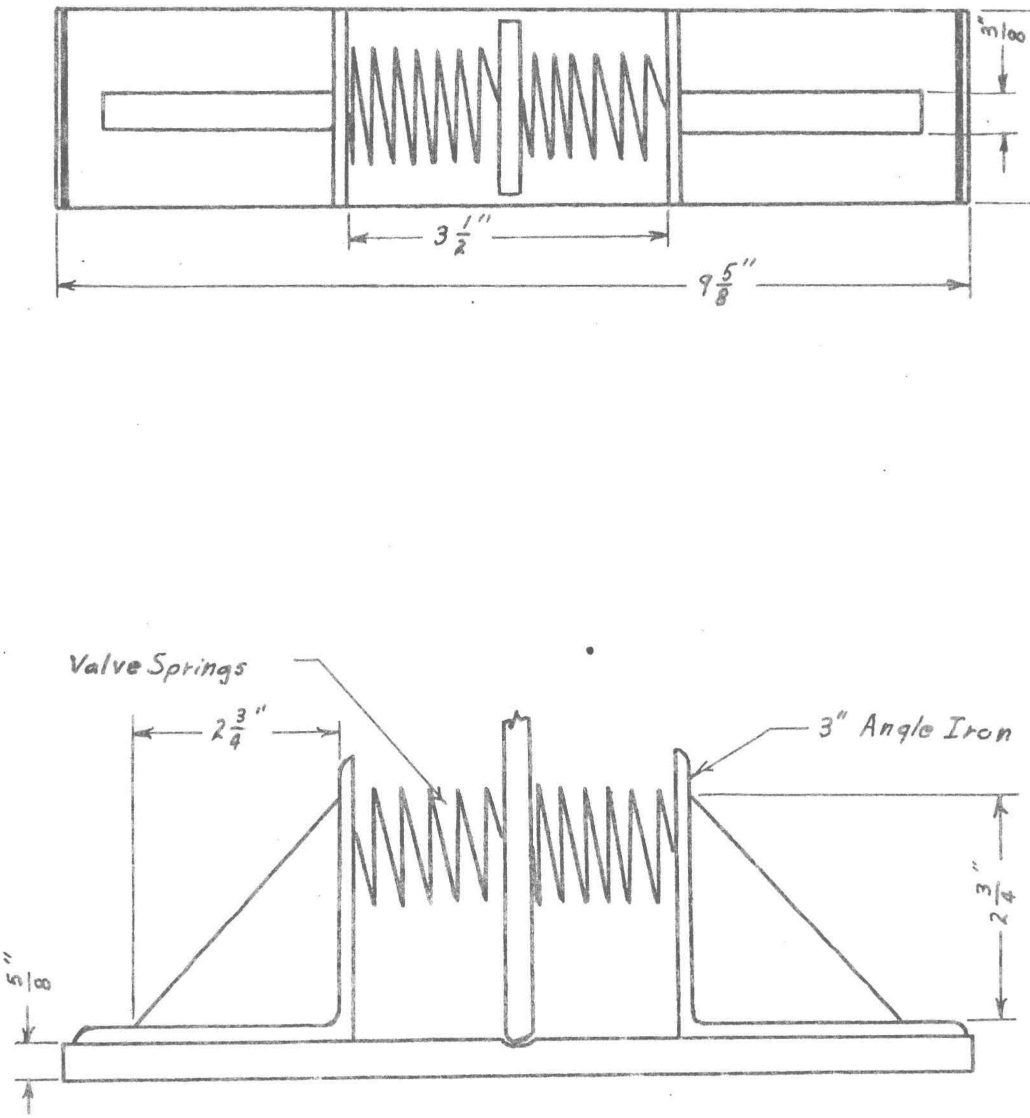

Figure 5

Details of the Elastic Restraint Jig

fixed and one end pinned. End condition number three is used in the simulation of an elastically restrained column, in which the moment on the end of the column is proportional to the slope at that point for small deflections of the column. Two sets of springs were used in the jig, which gave two constants of proportionality between the slope and moment. The jig used for this purpose is shown in detail in figure 5. The combinations of steel bars and end conditions that were tested are listed in figure 6.

TYPE OF BAR	END CONDITION
Uniform Cross Section	Pinned Pinned-Fixed Fixed Elastic (two spring constants)
Stepped Column	Pinned Fixed Elastic (one spring constant)
Tapered Column	Pinned Fixed Elastic (one spring constant)

Figure 6

Types of Columns Tested

Description of Test Instrumentation

In order to investigate the relationship between axial load and frequency of vibration, it is necessary to measure in some manner both the axial compressive load on the column and the frequency of vibration of the column. These measurements should be as accurate as possible, and should not interfere with the free vibration of the column. The test apparatus that was used to accomplish this is shown in figure 7.

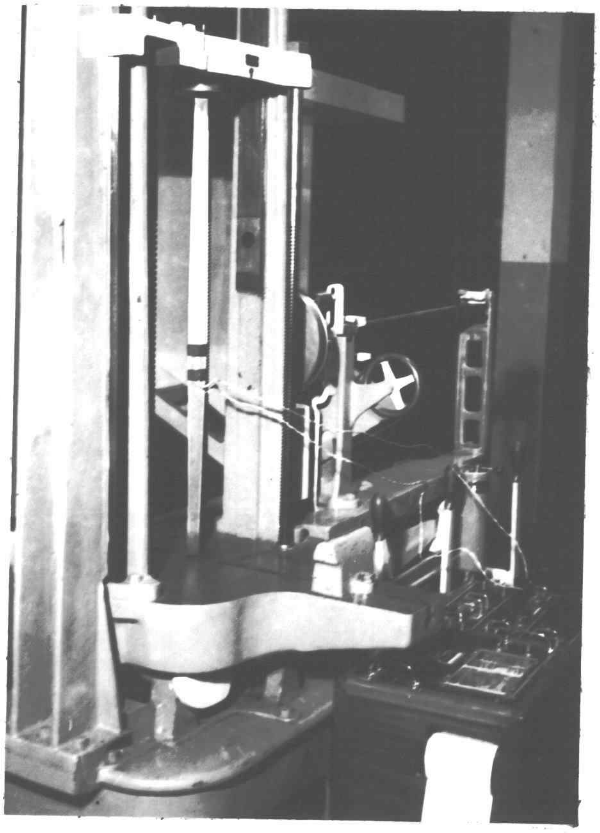

Figure 7
Column Testing Apparatus

The columns were loaded in compression on a Tinius Olsen 30,000 pound testing machine, the compressive loads being read directly from the machine. The Tinius Olsen machine has three load ranges available, of which the 0-3000 pound and the 0-15000 pound ranges were used, depending on the estimated critical load of the column being tested. Calibration data on the testing machine indicates that it is reliable to within about 2% of the dial reading on the 3000 pound load range, and to within about 0.6% of the dial reading on the 15000 pound range.

The frequency of vibration of the column being tested was picked up by means of two Baldwin SR-4 electric resistance strain gages mounted on both sides of the column at the midpoint of its length. The oscillating current output of these gages was amplified by a Sanborn Strain Gage Amplifier, and then used to drive a Sanborn Recorder. As the output speed of the recorder paper was known, the frequency could be counted over a relatively long period of time. This means of measuring frequency can be considered accurate to within 0.75% of the reading.

Description of Test Procedure

The actual test procedure used was quite simple. The column under investigation was placed in the Tinius Olsen testing machine as shown in figure 7, and the leads from the strain gages attached to the strain gage amplifier on the strain gage recorder. The recorder was then turned on and the column struck with a hammer. This gave a recording of the vibration of the column for several seconds. The axial compressive load on the column was then increased, and the procedure repeated. This series of steps was repeated for each column tested, the only variation being in the setting up of the various end conditions.

The pin-ended columns were simply set in the testing machine and given a slight compressive load to hold them in place. The semi-circular case-hardened ends minimized any frictional resistance to rotation caused by the compressive load. All three steel bars were tested on the 3000 pound range of the testing machine with this end condition.

The fixed-ended condition was simulated by clamping small bench vises to the ends of the columns, as was mentioned previously. For this series of tests, the 15000 pound range of the testing machine was utilized for all three steel bars. In order to investigate the effects of unsymmetrical end conditions, the steel bar with constant cross section (#1) was tested with one end pinned and one end fixed.

The simulation of an elastically restrained end involved the construction of a simple jig, shown in detail in figure 5. This jig was first used in conjunction with the constant cross section column with a spring constant of 70 pounds per inch. It was found that this end condition did not change the critical load of the column appreciably from the pin-ended configuration, and the springs were replaced with a new set, which had a spring constant of 160 pounds per inch.

RESULTS AND DISCUSSION

General Discussion

In an investigation of the relationship between frequency of vibration and axial load the ideal approach would be to consider the differential equation of a vibrating column, and determine theoretically the relationship between ω^2 and P without imposing end conditions. The nature of differential equations prohibits this, however, at least by ordinary mathematical procedures. By utilizing the Rayleigh Energy Method, the relationship in question may be approximated for several cases as shown on pages 31-35 in the appendix. In an effort to substantiate and enlarge upon the results of the energy method, the relationship between frequency of vibration and axial load was experimentally determined for eleven columns, and the results plotted in figures 8, 9, and 10.

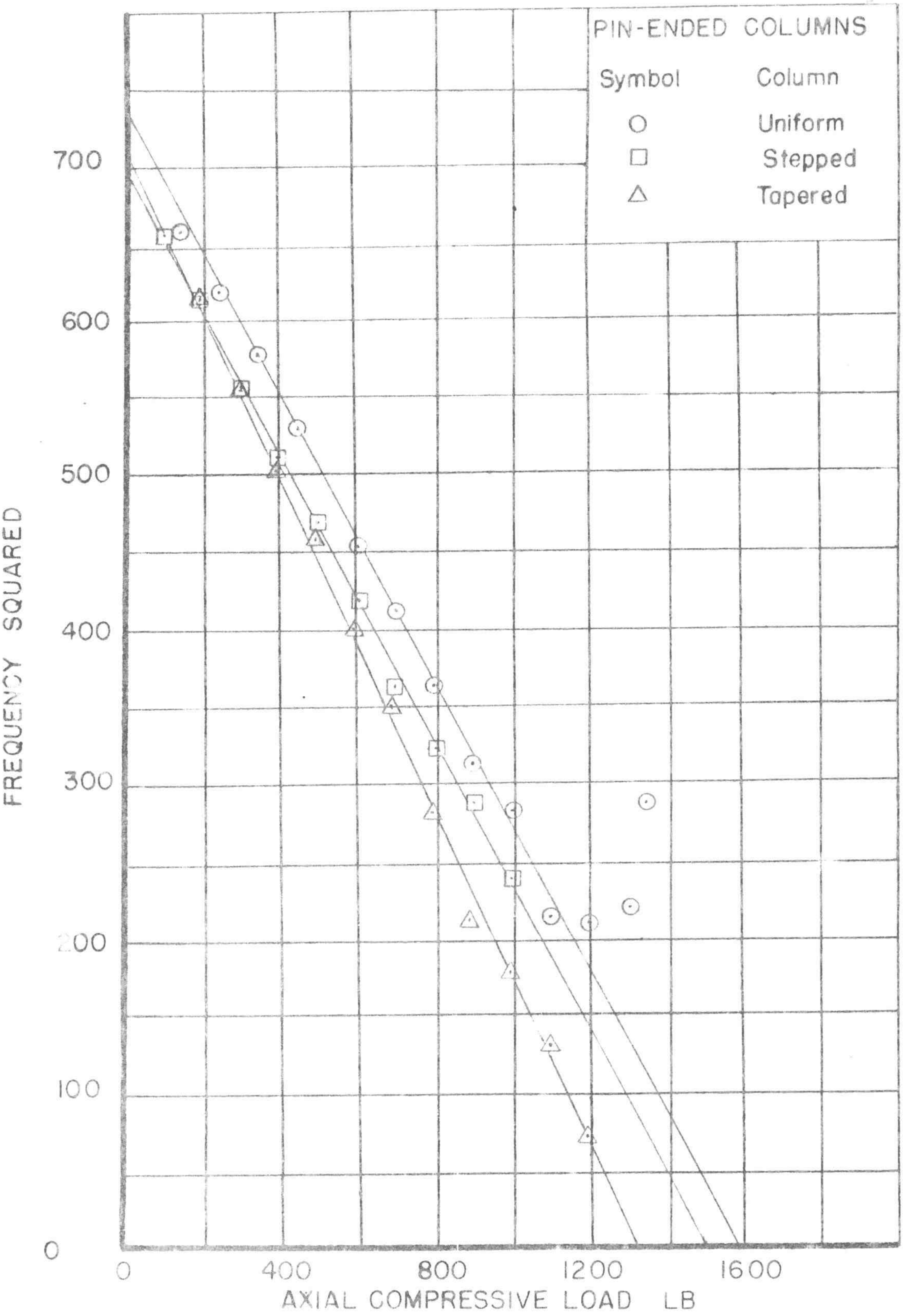
PIN-ENDED COLUMNS
Symbol Column
○ Uniform
□ Stepped
△ Tapered
FREQUENCY SQUARED
0
100
200
300
400
500
600
700
0
400
800
1200
1600
AXIAL COMPRESSIVE LOAD LB

FIGURE 8

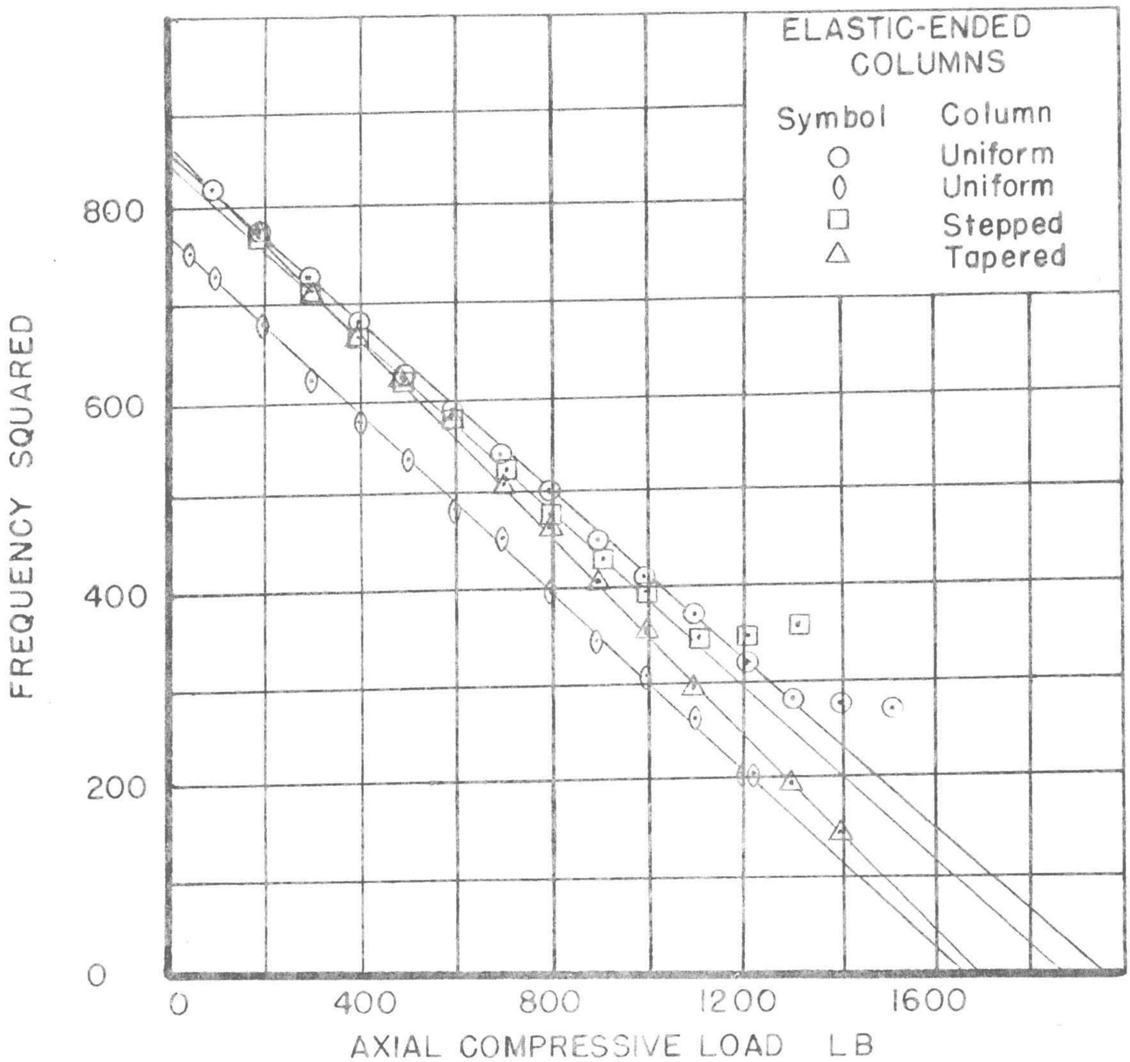
ELASTIC-ENDED COLUMNS
Symbol Column
○ Uniform
0 Uniform
□ Stepped
△ Tapered
FREQUENCY SQUARED
0
200
400
600
800
0
400
800
1200
1600
AXIAL COMPRESSIVE LOAD LB

FIGURE 9

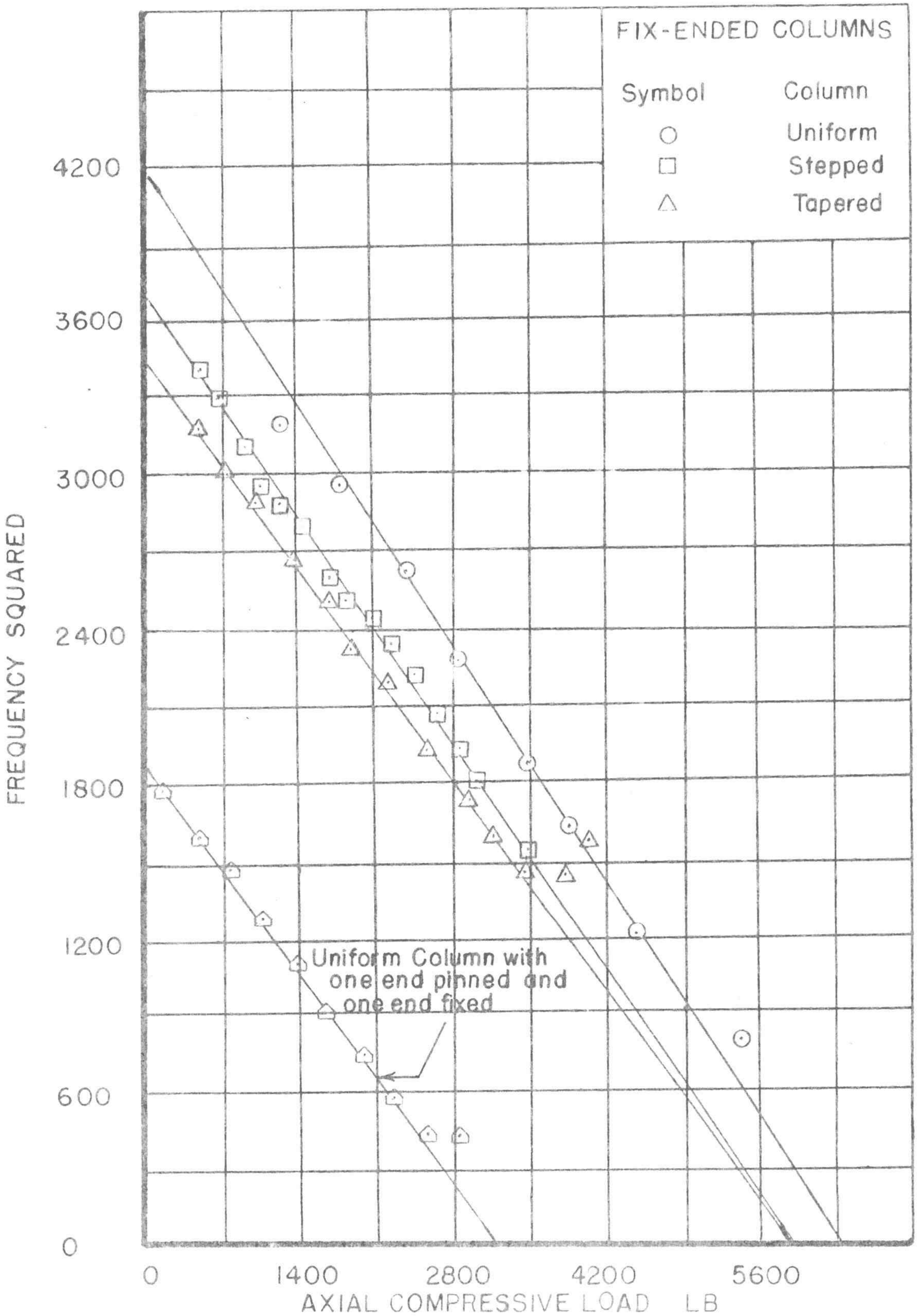
FIX-ENDED COLUMNS
Symbol Column
○ Uniform
□ Stepped
△ Tapered
4200
3600
3000
2400
1800
1200
600
0
FREQUENCY SQUARED
Uniform Column with one end pinned and one end fixed
0
1400
2800
4200
5600
AXIAL COMPRESSIVE LOAD LB

FIGURE 10

A study of these figures will immediately show that the columns that were tested did obey a linear relationship between ω^2 and P, at least over part of the range of P. A further study will also show that the frequency tends to increase at the higher values of P. This characteristic is especially noticeable in the case of the column of uniform cross section. The experimental data for this column also seems to fall away from the linear relationship at the higher values of ω^2, a phenomenon that does not appear in the data from the other columns. In regard to the high values of ω^2 at the higher loads, two basic assumptions should be considered. The column was assumed to be initially straight, and the vibration was assumed to be in the fundamental mode. In the normal range of loading the first assumption has very little effect, and the second assumption is valid. As the axial compressive load approaches the critical load, however, the column may start to deform, due to a small eccentricity, either in the column itself or in the manner of loading. When this happens the vibrating column no longer has enough inertia force along its length to vibrate in the fundamental mode, and thus begins to vibrate in a higher mode to one side of the unloaded position.

It should also be mentioned at this point that although all of the columns seem to obey the same linear relationship, the slope of the straight line is not the same for all cases.

Discussion of Pin-Ended Columns

The experimental critical load for the pin-ended columns taken from figure 8 compares favorably with theory. The experimental and theoretical critical loads for the pin-ended case are shown in figure 11.

Column	Experimental P_{cr} (lb)	Theoretical P_{cr} (lb)
Uniform	1580	1465
Stepped	1390	1320
Tapered	1310	1360

Figure 11

Critical Loads for the Pin-Ended Columns

Figure 11 indicates that there was some frictional resistance to rotation at the ends of the columns allowing the columns to sustain a slightly higher load than they would in the ideally pin-ended case. This was also indicated in the actual testing of the pin-ended columns. Although the ends of the columns were case-hardened to prevent crushing, the loading surfaces on the testing machine were not, and the columns actually pressed a very slight groove in the loading surfaces. The presence of the grooves would certainly indicate that a small amount of end-fixity is present.

Discussion of Elastically Restrained Columns

The tests of the elastically restrained columns show an increase of between 25% and 27% in the critical load over the pin-ended case. The critical loads could not be determined readily by theoretical means, but the experimental results are certainly reasonable and consistent with the pinned-ended and fixed-ended cases. The experimental results for the elastically restrained columns are shown in figure 12. It should be noted that there are

Column	Experimental Critical load (lb)
Uniform	1650
Uniform	1950
Stepped	1870
Tapered	1690

Figure 12

Critical Loads for the Elastically Restrained Columns

two curves for a uniform cross section column at this end condition. The lower curve is from the first test using the elastic restraint jig, in which the springs in the jig had a spring constant of 70 pounds per inch. It was found that this did not raise the critical load from that obtained for the pin-ended case by a significant amount, so a stiffer set of springs with a spring constant of 160 pounds per inch was substituted. The other three curves were obtained using the stiff springs.

Discussion of Fixed-Ended Columns

The test results for the fix-ended columns are shown in figure 13. The uniform cross section column experimentally shows a critical load of 6300 pounds, corresponding to a theoretical critical load of 6640 pounds. This

Column	Experimental P_{cr} (lb)	Theoretical P_{cr} (lb)
Uniform	6300	6640
Stepped	5850	--
Tapered	5850	--

Figure 13

Critical Loads for the Fix-Ended Column

is reasonable, as the small bench vises that were used certainly were not infinitely rigid, although they were reasonably close. It is interesting to note that the stepped and tapered columns have approximately the same critical load for this end condition. Apparently fix-ended columns are not sensitive to changes in cross section near the ends, as they are held rigid at that point, but are quite sensitive to the amount of metal near the center of the column. It is certainly obvious that in the case of the fix-ended columns, the portion of the column near the ends will be deflected only slightly, therefore both the inertia force term and the Py term in the basic differential equation will be small and that portion of the column will contribute little to the stability of the column.

There is an additional curve on figure 10 that is not, strictly speaking, a fix-ended column, in that it is fixed on one end and pinned on the other. This column was made up, using the uniform cross section steel bar, in order to investigate the effects of unsymmetrical end conditions. The column exhibits in general the same characteristics as the other test columns, and agrees quite well with theory, the experimental critical load being about 3150 pounds and the theoretical critical load 3215 pounds. Apparently the slight friction on the pinned end compensates for the lack of complete rigidity at the fixed end.

CONCLUSION

The immediate conclusions that can be drawn from this investigation are:

1. The critical load of a homogenous, long slender column may be found by measuring the frequency of vibration of the column at two or more values of axial load. This data is then plotted in the form ω^2 versus P and a straight line drawn through the experimental points. The intercept of this line on the P-axis is then the critical load of the column.

2. This method will work for a column which is elastically restrained at the ends, the degree of end-fixity being anywhere from the completely pinned to the completely fixed condition.

3. This method will work for columns of varying cross section as well as for columns of uniform cross section.

4. It is possible to determine the critical load of a column after it has been fastened to its surrounding structure, by the use of strain gages or similar transducers to measure both the frequency of vibration and the axial load.

It is felt that the columns that were tested were representative of the great majority of columns in actual engineering applications. The accuracy of the results was somewhat limited by the test equipment available, but if this is kept in mind, the results are quite valid.

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APPENDIX

Notation

The following symbols and subscripts are used throughout this thesis.

P	Axial compressive force
L	Length of a column
M	Bending moment
E	Modulus of elasticity
I	Least moment of inertia of cross section of column
x	Distance measured along centroid of cross section of unloaded column
y	Distance from x-axis to centroid of cross section of loaded column
u	$x + \frac{1}{a}$
a	$\frac{-I_1 + I_0}{I_0 L}$
τ	$\sqrt{\frac{4P}{a^2 E I_0}}$
β	$\sqrt{\left(\frac{4P}{a E I_0}\right)\left(-L + \frac{1}{a}\right)}$
k	$\sqrt{\frac{P}{E I_0}}$
ρ	Mass density of material in column
A	Cross sectional area of column
ω	Frequency of vibration

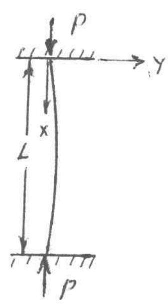
q	Inertia force per unit length
V	Strain energy
T	Kinetic energy of vibration
J_z	Bessel function of the first kind of order z
Y_z	Bessel function of the second kind of order z

Subscripts

n	Natural frequency
cr	Critical load
b	Energy due to bending
p	Energy due to axial load
m	Energy due to end moments

Derivation of the Euler Load Formula

Consider a pin-ended column as shown.



Assuming small deflections, a homogenous material, and an initially straight column, the moment differential equation is

$$\frac{d^2Y}{dx^2} = \frac{M}{EI} = -\frac{PY}{EI} \quad .$$

Substituting $K^2 = \frac{P}{EI}$, the equation may be written

$$\frac{d^2Y}{dx^2} + K^2Y = 0 \quad .$$

The solution to this equation is well known, and is

$$Y = A \sin Kx + B \cos Kx \quad .$$

The constants A and B must be determined from the end conditions, which are

$$Y = 0 \text{ at } x = 0 \,\&\, L \quad .$$

Substituting the first end condition,

$$0 = A \sin(0) + B \cos(0) \quad ,$$

therefore B = 0. Substituting the second end condition,

$$0 = A \sin KL$$

As setting A = 0 would lead to a trivial solution, sin KL must be zero. The only way for this to occur, however, is for KL to equal a multiple of π .

$$KL = \sqrt{\frac{P}{EI}} = n\pi$$

This may be written

$$P = \frac{n^2\pi^2EI}{L^2} \quad ,$$

which, for the critical load, reduces to

$$P_{cr} = \frac{\pi^2EI}{L^2} \quad .$$

It may be shown that for a column pinned on one end and fixed on the other,

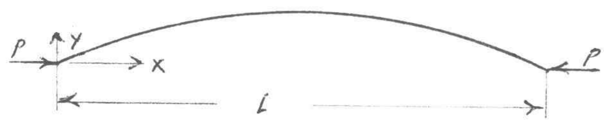
$$P_{cr} = \frac{2.05\pi^2 EI}{L^2} \quad ,$$

and for a column fixed at both ends,

$$P_{cr} = \frac{4\pi^2 EI}{L^2} \quad .$$

Derivation of Differential Equation of Vibrating Column

Assume a pin-ended, homogenous column of constant cross section, with small deflections.



From elementary strength of materials,

$$EI\frac{d^2y}{dx^2} = -Py \quad .$$

If the column is vibrating laterally, so that there is an inertia force acting along its length,

$$EI\frac{d^2y}{dx^2} = -Py + M$$

where M is the moment caused by the inertia of the column. Differentiating twice,

$$\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) = -P\frac{d^2y}{dx^2} + q$$

where q is the inertia force of the column per unit length,

$$\rho A\frac{d^2y}{dt^2} \quad .$$

For a prismatic homogenous column, the equation reduces to

$$\frac{\partial^4 Y}{\partial x^4} + \frac{P}{EI}\frac{\partial^2 Y}{\partial x^2} = \frac{\rho A}{EI}\frac{\partial^2 Y}{\partial t^2} .$$

Derivation of the Critical Load-Frequency Relationship

Assuming a solution to the above equation of the form

$$y = X(x)\,T(t) ,$$

and assuming periodic vibrations, such that

$$T(t) = \sin \omega t ,$$

by substitution,

$$X^{IV} \sin\omega t + \frac{P}{EI} X^{II} \sin\omega t = -\frac{\rho A}{EI}\omega^2 \sin\omega t .$$

Rewriting,

$$X^{IV} + \frac{P}{EI} X^{II} + \frac{\rho A}{EI}\omega^2 = 0 .$$

A solution to this equation is

$$X = \sin\frac{\pi}{L}x .$$

By substitution, eliminating sin terms,

$$\left(\frac{\pi}{L}\right)^4 + \frac{P}{EI}\left(\frac{\pi}{L}\right)^2 = \frac{-\rho A}{EI}\omega^2 .$$

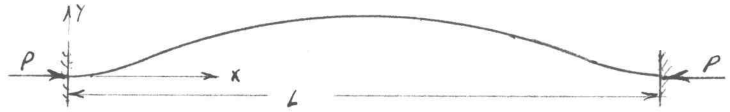
Substituting the value of the natural frequency, ω_n, the following is obtained.

$$\omega^2 = \omega_n^2 (1 - P/P_{cr})$$

Energy Methods Analysis

The above relationship may be found by Rayleigh's Approximation Method for a fixed-ended column and a cantilever column quite readily. The Rayleigh Method may also be applied to the case of a column with elastically restrained ends. The solution for a fixed-ended column is given below.

Consider a homogenous column built in at both ends, and initially perfectly straight.



Rayleigh's Method consists basically of assuming a deflection curve for the column, and then equating the potential and kinetic energy of the column.

For the column illustrated above, the strain energy of bending is

$$V_b = + \frac{EI}{2} \int_0^L \left(\frac{d^2 y}{dx^2} \right)^2 dx \quad .$$

The strain energy caused by the axial force is

$$V_P = - \frac{P}{2} \int_0^L \left(\frac{dy}{dx} \right)^2 dx \quad ,$$

and the kinetic energy of vibration is

$$T = + \frac{\rho \omega^2}{2} \int_0^L y^2 dx \quad .$$

Equating the potential and kinetic energy,

$$\frac{EI}{2}\int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx - \frac{P}{2}\int_0^L \left(\frac{dy}{dx}\right)^2 dx = \frac{\rho A^2}{2}\int_0^L y^2 dx \; .$$

Assuming

$$y = Y_0\left[1 - \cos\frac{2\pi}{L}x\right] \quad ,$$

by substitution,

$$EI\int_0^L Y_0^2\frac{16\pi^4}{L^4}\cos^2\frac{2\pi}{L}x \, dx \; - \; P\int_0^L Y_0^2\frac{4\pi^2}{L^2}\sin^2\frac{2\pi}{L}x \, dx$$

$$= \rho\omega^2\int_0^L Y_0^2\left[1 - \cos\frac{2\pi}{L}x\right] dx \; .$$

This reduces to

$$\omega^2 = \frac{4}{3\rho}\left(\frac{\pi}{L}\right)^2\left[4EI\left(\frac{\pi}{L}\right)^2 - P\right] \quad ,$$

or

$$\omega^2 = \frac{16\pi^4 EI}{3\rho L^4}\left[1 - P/P_{cr}\right] \; .$$

The above expression is about 1.3% in error if the relationship

$$\omega^2 = \omega_n^2\left(1 - P/P_{cr}\right)$$

is true in general. As the Rayleigh Method is an approximation method, 1.3% is well within the possible accuracy of the analysis.

Using a procedure similar to that outlined above, the relationship between frequency and critical load for a cantilever column can be found to be within a few percent of

$$\omega^2 = \omega_0^2 (1 - P/P_{cr}) \quad .$$

Again, the error is well within that which might be expected of the approximation method.

Consider now the case of a column with elastic end restraints, as shown below.

For an elastic restraint on the ends, the moment on the ends of the column will be assumed to be

$$M = \Gamma \frac{EI}{L} \theta \Big|_{x=0} \quad ,$$

where Γ is an arbitrary constant, depending on the stiffness of the end restraints. For this case, in addition to the strain energy of bending and strain energy of compression, there will be an additional potential energy term due to the moment on the ends of the column. This term is

$$V_m = 2M\left(\frac{dy}{dx}\right)_{x=0} = 2\Gamma \frac{EI}{L}\left(\frac{dy}{dx}\right)^2_{x=0} \quad .$$

The energy balance equation for this case is then

$$\frac{EI}{2}\int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx - \frac{P}{2}\int_0^L \left(\frac{dy}{dx}\right)^2 dx + 2\Gamma \frac{EI}{2}\left(\frac{dy}{dx}\right)^2_{x=0}$$

$$= \frac{\rho\omega^2}{2}\int_0^L y^2 dx \quad .$$

If the assumption is made that

$$y = \sin\frac{\pi}{L}x \quad ,$$

substitution in the energy balance equation yields

$$\frac{EI}{2}\frac{\pi^4}{L^3} - \frac{P}{2}\frac{\pi^2}{L} + 4\Gamma EI\frac{\pi^2}{L^3} = \frac{\rho\omega^2}{2}L \quad .$$

Rearranging the equation,

$$\omega^2 = \frac{EI}{\rho}\left(\frac{\pi}{L}\right)^4 - \frac{P}{\rho}\left(\frac{\pi}{L}\right)^2 + 8\Gamma\frac{EI}{\rho L^2}\left(\frac{\pi}{L}\right)^2 \quad ,$$

or

$$\omega^2 = \frac{EI}{\rho L^4}(8\Gamma\pi^2 + \pi^4) - \frac{1}{\rho}\left(\frac{\pi}{L}\right)^2 P \quad .$$

Rewriting,

$$\omega^2 = \frac{1}{\rho}\left(\frac{\pi}{L}\right)^2\left[\frac{EI}{L^2}(8\Gamma + \pi^2) - P\right] \quad .$$

This equation can be put in the form

$$\omega^2 = \frac{1}{\rho}\left(\frac{\pi}{L}\right)^2\left(\frac{EI}{L^2}\right)(8\Gamma + \pi^2)\left[1 - \frac{P}{(8\Gamma + \pi^2)\frac{EI}{L^2}}\right] \quad .$$

A close investigation of this equation will show that it is very similar to the previously derived relationships between frequency of vibration and axial load. Unfortunately, the mathematics involved prohibit the calculation of P_{cr} for an elastically restrained column in general.

The form

$$(8\Gamma + \pi^2)\frac{EI}{L^2}$$

is of the form that P_{cr} could be expected to be in. It should be noted that if the elastic restraint, Γ , is allowed to approach zero, the term becomes the critical load for a pin-ended column. As the elastic restraints become stiffer, the term increases, as would be expected of the critical load.

Computation of Critical Loads

The calculation of critical loads in the case of a column of constant cross section is quite simple if the end conditions are known.

Column of uniform cross section:

Pinned-ended column:

$$L = 35.8''$$

$$EI = 19.1 \times 10^4 \text{ lb}''$$

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{(9.86)(19.1 \times 10^4)}{(35.8)^2} = 1465 \text{ lb.}$$

Pinned-fixed ended column:

$$L = 34.6''$$

$$EI = 19.1 \times 10^4 \text{ lb.}''$$

$$P_{cr} = \frac{2.05\pi^2 EI}{L^2} = \frac{(2.05)(9.86)(19.1 \times 10^4)}{(34.6)^2} = 3215 \text{ lb.}$$

Fixed-ended column:

$$L = 33.7''$$

$$EI = 19.1 \times 10^4 \text{ lb}''$$

$$P_{cr} = \frac{4\pi^2 EI}{L^2} = \frac{(4)(9.86)(EI)}{(33.7)^2} = 6640 \text{ lb.}$$

No attempt will be made here to "guess" the values of end-fixity for the elastically restrained columns. It is possible to compute the critical load directly from the deflection equation, knowing the end conditions. It becomes quite tedious, however, and as the lengths of the columns are not readily determined because of the elastic restraint jig that was used, the calculated critical load is not of great value.

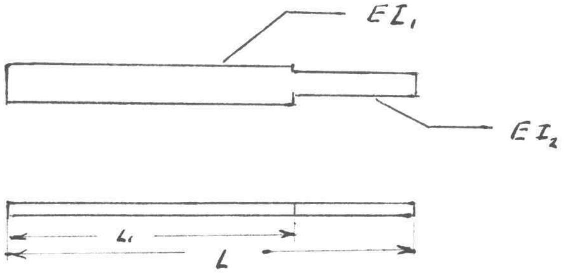
Column with stepped cross section:

The calculation of the critical load for the stepped column becomes extremely complex mathematically for other than the pin-ended case. The pin-ended case may be handled in a fairly straightforward way, however, and is a good example of the approach that is necessary in the more complex column problems.

From the physical dimensions of the column, as shown on page 7, the stiffness of the wide portion of the column is 3/2 the stiffness of the narrow portion, or $EI_1 = \frac{3}{2} EI_2$.

Likewise, the length of the wider portion is 2/3 the total length of the column.

$$L_1 = \frac{2}{3} L$$



The end conditions for the column are:

$$Y = 0 \ \& \ \frac{d^2Y}{dX^2} = 0 \quad at \ Y = 0, L \ .$$

The moment curve of the column may be represented by two differential equations.

$$0 \le X \le L: \quad EI\frac{d^2Y}{dx^2} + PY_1 = 0$$

$$L_1 \le X \le L: \quad EI\frac{d^2Y}{dx^2} + PY_2 = 0$$

The solutions to these equations are:

$$Y_1 = C_1 \sin kx + C_2 \cos kx \ ,$$

$$Y_2 = C_3 \sin kx + C_4 \cos kx \ .$$

The arbitrary constants can be evaluated by substituting the end conditions, and all but one eliminated. As $Y_1 = Y_2$ at $X = L_1$, the equations may be set equal to each other, and the final constant eliminated, leaving the following eigenfunction:

$$\frac{K_1}{K_2}\left[\frac{\tan K_2 L_1}{\tan K_2 L} - 1\right] - \left[\frac{\tan K_1 L}{\tan K_2 L} + \tan K_1 L_1 \tan K_2 L_1\right] = 0 .$$

This may be simplified to

$$\tan\left[K_2(L_1 - L)\right] = \frac{K_2}{K_1}\tan K_1 L .$$

As the physical relationships between $K_1 \& K_2$ and $L_1 \& L_2$ are known, this equation may be solved graphically. The lowest eigenvalue may be found to be $K_1 L = 2.978$.

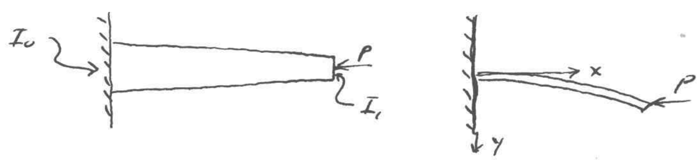
The critical load is then

$$P_{cr} = (2.978)^2 \frac{EI_1}{L^2} = 1320 \text{ lb.}$$

Column with tapered cross section:

The tapered column is extremely difficult to handle theoretically, but the critical load for the pinned-ended case can be found in the following manner.

Consider a column as shown:



The moment of inertia is a function of the distance along the column, and is represented by

$$I = I_o - \frac{I_o - I_1}{L}x = I_o - I_o a x$$

where $\quad a = \dfrac{I_o - I_1}{I_o L} .$

The differential equation of the column is then

$$EI_0(1-ax)\frac{d^2y}{dx^2} + Py = 0.$$

Making the variable change , the equation is

$$u\frac{d^2y}{du^2} + \frac{P}{aEI_0}y = 0.$$

From (2,13), the solution to this equation is

$$y = A\sqrt{\frac{1}{a}-x}\; J_1\sqrt{\frac{4P}{aEI_0}\left(\frac{1}{a}-x\right)} + B\sqrt{\frac{1}{a}-x}\; Y_1\sqrt{\frac{4P}{aEI_0}\left(\frac{1}{a}-x\right)},$$

Substituting the end conditions $y=0$ at $x=0$ and $\frac{d^2y}{dx^2}=0$ at $x=L$, the constants A and B may be evaluated. Making the substitutions

$$\tau = \sqrt{\frac{4P}{a^2EI_0}}, \quad \beta = \sqrt{\frac{4P}{aEI_0}\left(\frac{1}{a}-L\right)},$$

The eigenfunction that is obtained is

$$\frac{Y_1(\tau)}{J_1(\tau)} = -\frac{Y_{-1}(\beta) - Y_1(\beta)}{J_{-1}(\beta) - J_1(\beta)}.$$

Using the relationship

$$J_{-1}(x) = -J_1(x),$$

The eigenfunction takes the form

$$J_1(\tau)\, Y_1(\beta) = -Y_1(\tau)\, J_1(\beta).$$

By graphical methods, the solution to this equation may be found to be $\beta = 7.69$.

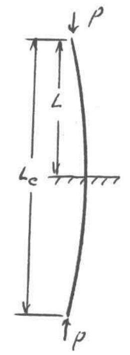
Writing this equation in the form

$$\sqrt{\frac{4P}{aEI_0}}\left(\frac{1}{a}-L\right) = 7.69 ,$$

The critical load is found to be

$$P_{cr} = 340 \text{ lb} .$$

This is the critical load for a tapered column pinned on one end and fixed on the other. To find the critical load for a column pinned on both ends, an effective column length may be used.



It may be seen that $L_e = 2L$

The length of the column appears squared in the denominator of the critical load equation.

$$a^2 = \frac{(I_0 - I_1)^2}{I_0^2 L^2}$$

Substituting the new column length, L_e , for L , the effect is to multiply the critical load by a factor of 4. The critical load for the pinned-ended column used in the tests is then $P_{cr} = 1360 \text{ lb}.$