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Gerald William Silke for the M.S. in Mechanical Engineering
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Title FORCED VIBRATION OF A BEAM-PENDULUM SYSTEM

Abstract approved [redacted]
(Major professor)

These studies are concerned with the forced motion of a two-degree-of-freedom system consisting of a "flexible beam" supported by pendulum type hangers.

Using an asymptotic method, an approximate solution to the nonlinear equations of motion was found. The solution takes the form of a series in powers of the amplitude of vibration ratio μ.

A mechanical model was built and data were taken for purposes of verifying the analytical steady state solution.

The results are presented in the form of maximum amplitude of pendulum swing and maximum "beam" deflection curves plotted as functions of the forcing frequency ratio.

Experimental results were found to compare quite favorably with the analytically predicted solution except for several resonant peaks thought to be a part of the higher order approximations not analyzed here. As μ was made smaller, agreement was better.

As a result of this work, some important phenomena have been

been discovered which do not appear in the solution to the linearized equations of motion.

FORCED VIBRATION OF A BEAM-PENDULUM SYSTEM

by

GERALD WILLIAM SILKE

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Associate Professor of Mechanical Engineering

In Charge of Major

[redacted]

Head of Department of Mechanical Engineering

[redacted]

Dean of Graduate School

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LIST OF SYMBOLS

A	A(a, b), coefficient in the series exapnsion of a'
B	B(a, b), coefficient in the series expansion of b'
C	C(a, b), coefficient in the series expansion of ψ'
D	D(a, b), coefficient in the series expansion of ϕ'
$F(\tau, x, x', y, y')$	Function defined in equation (8b)
M	Mass of rigid beam
T	Kinetic energy
U	$\lambda/(\alpha^2 - 1)$
V	$\delta/(1-\beta^2)$
V*	Potential energy
a, b	Variational parameters (See equations (9))
$f(\tau, x, x', y, y')$	Function defined in equation (8a)
f_0	$f_0(a, b, \psi, \phi, \tau)$, Function defined in equation (14a)
$f_{nps}(a, b)$	Coefficients in the Fourier series expansion of u (See equation (15a))
$f^{(0)}_{nps}(a, b)$	Coefficients in the Fourier series expansion of f_0 (See equations (18))
g	Acceleration due to gravity
$g(\tau, x, x', y, y')$	Function defined in equation (8c)
g_0	$g_0(a, b, \psi, \phi, \tau)$, Function defined in equation (14b)
$g_{nps}(a, b)$	Coefficients in the Fourier series expansion of w (See equation (15b))
$g^{(0)}_{nps}(a, b)$	Coefficients in the Fourier series expansion of g_0 (See equations (18))

h	$(1-\eta) \sin \zeta$
ℓ	Length of pendulum
m	Sprung mass
n, p, s	Integers
t	Time
u	$u(a, b, \psi, \phi, \tau)$, Function defined in equation (15a)
w	$w(a, b, \psi, \phi, \tau)$, Function defined in equation (15b)
x, y	See equations (6)
z	Deflection of sprung mass from equilibrium
z_m	Maximum value of z for a given frequency of vibration
$\bar{z}$	$\eta z / \ell$
$\bar{z}_{ss}$	Steady state solution for $\bar{z}$
Δ	β / α
Ω	Forcing frequency ratio $\omega / \sqrt{g/\ell}$
α	Pendulum frequency ratio $\sqrt{g/\ell \omega^2}$
β	Sprung mass frequency ratio $\sqrt{k/m\omega^2}$
δ	$\mu \sin \zeta$
ϵ	Amplitude of support vibration
ζ	Angle of support vibration
η	Mass ratio $m/(M+m)$
θ	Angle of pendulum swing
θ_m	Maximum value of θ for given frequency of vibration

$\bar{\theta}$	$\sqrt{\eta}\,\theta$
$\bar{\theta}_{ss}$	Steady state solution for $\bar{\theta}$
θ_{nps}	$n\tau + p\psi + s\phi$
λ	$\sqrt{\eta}\cos\zeta$
μ	Amplitude ratio ϵ/ℓ
τ	ωt
ϕ, ψ	Variational parameters (see equations (9))
ω	Forcing frequency

Dots denote differentiation with respect to t.

Primes denote differentiation with respect to τ.

Partial differentiation is indicated by subscripts.

FORCED VIBRATION OF A BEAM-PENDULUM SYSTEM

I. INTRODUCTION

The work presented in this paper is concerned with the study of forced motion of a beam-pendulum system. The system is an idealized model of a certain vibration absorber incorporating a flexible beam supported at each end by pendulum-type hangers. According to Sevin (2, p. 330), absorbers of this type "...appear to be gaining increased application.... The platform (or beam) may be viewed either as accommodating a single piece of equipment, or as an entire floor system supporting numerous pieces of equipment."

Analysis of free vibration of this system has been the object of studies undertaken by Sevin (2) and Struble and Heinbockel (3, 4). The primary interest in these investigations has been a peculiar type of "resonance" which occurs when the natural frequency of the beam is twice that of the pendulum action of the hangers. This "resonance" occurs only when the nonlinear effects are taken into account.

Since this system finds application as a vibration absorber, it is felt that its response to forced vibration would be of interest. No exact solution to this nonlinear, two-degree-of-freedom system exists, so an approximate method of solution is used in the analysis.

Results taken from the response of an actual mechanical model of the beam-pendulum absorber is used as a check on the approximate solution obtained. It is felt that the results obtained, analytically and experimentally, will provide both indications as to the type of response to expect and ideas for further study of this interesting system.

II. FREE VIBRATION

The beam-pendulum system introduced by Sevin (2) consists of a flexible beam supported by hangers at each end as illustrated in Figure 1. In his analysis Sevin considers massless hangers and a beam deflection which can be described as a finite sum of sine terms satisfying the end conditions of the beam and having coefficients which are functions of time.

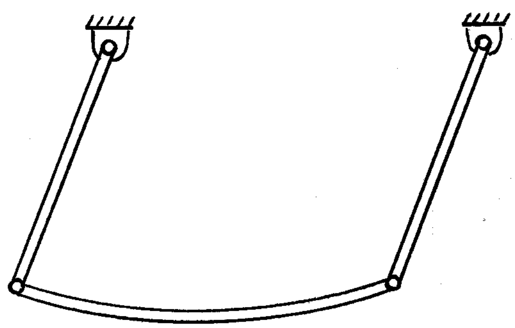

Figure 1. Idealized model of a beam-pendulum system.

Assuming small angular displacements, no damping and only the first mode[1] of beam vibration, he arrives at the nonlinear equations of motion for the system. Considering several possible initial conditions it was found that a type of "instability" occurs when the

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1 First mode refers to beam deflection described only by a half sine wave.

natural frequency of the beam is twice that of the pendulum action of the hangers ($\Delta = 2$). By numerical integration of the equations of motion, this "instability" was found to be a periodic energy transfer between beam vibrations and pendulum vibrations while the total energy of the system remained constant. This energy transfer, known as autoparametric excitation, does not appear as a solution when studying the linearized equations of motion.

Struble and Heinbockel (4, 5) used an asymptotic method (3, p. 220-262) in their analysis of the same system investigated by Sevin. The results give a more descriptive picture of the type of "instability" which occurs when $\Delta = 2$. According to their work, the motion may vary from no energy transfer, to periodic energy transfer, to total energy transfer, depending upon initial conditions. An approximate solution to the equations of motion for the "nonresonant" case and a procedure for carrying out higher approximations for both "resonant" and "nonresonant" cases are also presented by Struble and Heinbockel (5).

Although forced vibration is the main interest of this paper, free oscillations of an actual system might be of interest at this point. Using a mechanical model which is a slightly modified form of the idealized model introduced by Sevin and is described later in the text, some free vibration oscillations were observed. Only the qualitative results will be reproduced here since the primary interest of this

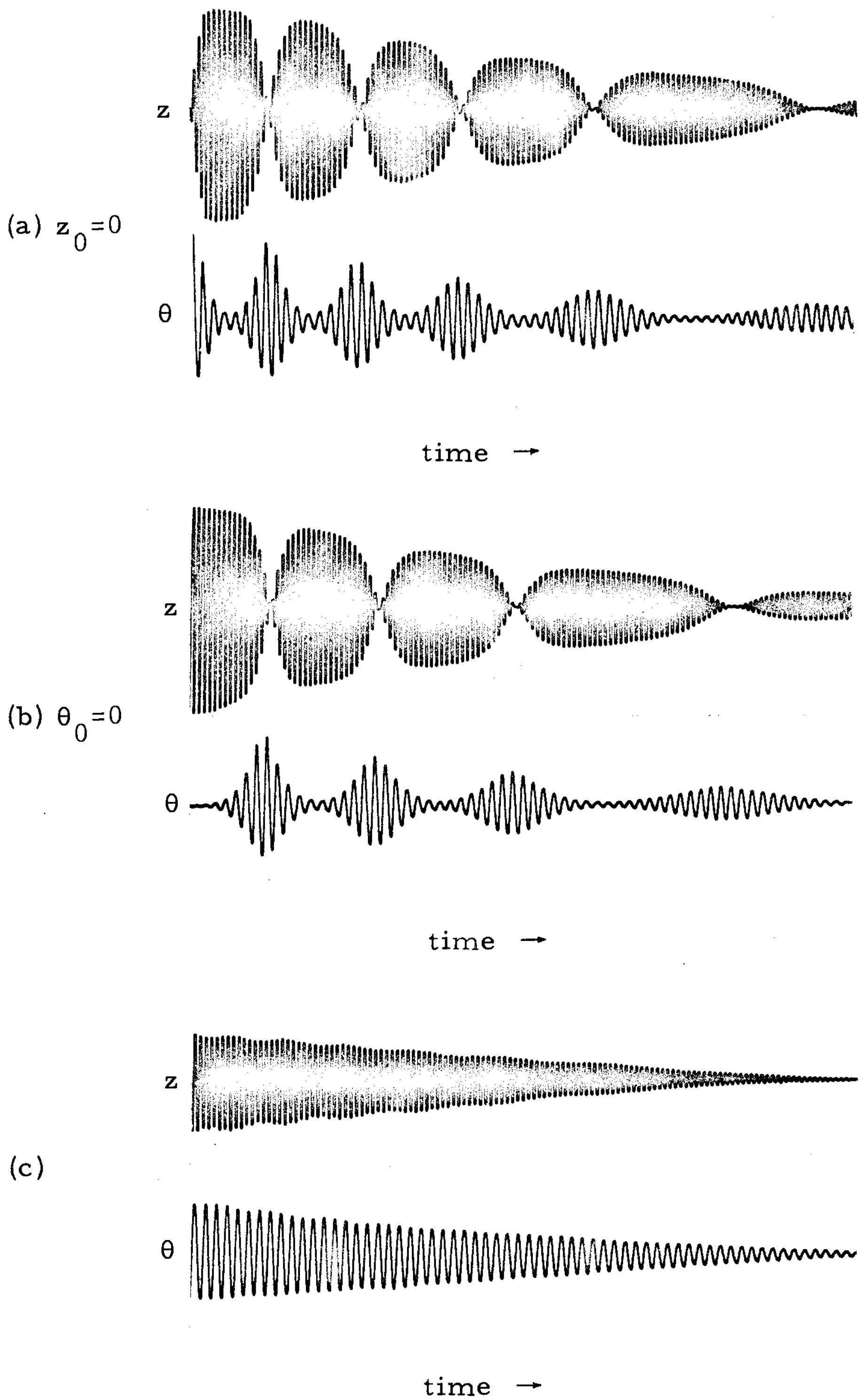


Figure 2. Beam deflection (z) and angle of pendulum swing (θ) as functions of time for various initial conditions when $\Delta = 2$.

paper is to study the forced vibrations. By adjusting the parameters of the mechanical system such that $\Delta = 2$, the condition of no energy transfer and periodic energy transfer were recorded and are shown in Figure 2. Case (a) respresents beam deflection (z) and angle of pendulum swing (θ) as functions of time when the initial beam deflection (z_0) is zero and the pendulum is given an initial displacement. Initial conditions in (b) are such that the beam has an initial displacement when the pendulum deflection is initially zero ($\theta_0 = 0$). The third case represents motion when the beam and pendulum displacements are "synchronized" such that no energy transfer occurs. Unavoidable damping in the system caused oscillations to gradually die out.

III. ANALYSIS OF FORCED VIBRATION

Idealized Model

A modified version of the system studied by Sevin (2) will be used in this analysis. The idealized model consists of a rigid beam supported by pendulum-type hangers at each end. A mass which is constrained to move vertically is suspended from the beam by means of a linear spring as shown in Figure 3. The same equations of motion govern both the flexible beam system and the spring mass system, provided only the first mode of beam vibration is considered. The latter was chosen because it lends itself to the building of a mechanical model with parameters that are easily varied.

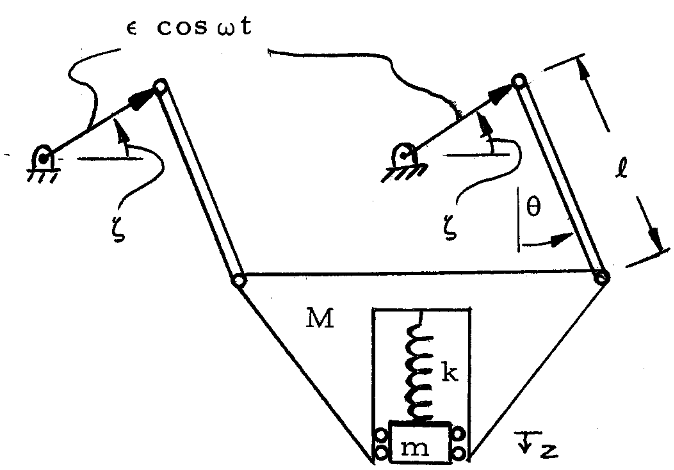


Figure 3. Forced beam-pendulum model.

Forced motion is introduced by means of a harmonically varying displacement applied at each support and moving at a fixed angle with the horizontal. Damping is neglected.

Equations of Motion

The kinetic energy (T) and the potential energy (V*) of the system in Figure 3 are given by the equations:

$$T = (M+m)/2[\ell^2 \dot\theta^2 - 2\epsilon\,\omega\ell\dot\theta \sin\omega t \cos(\theta-\zeta) + \epsilon^2\omega^2 \sin^2\omega t]$$
$$+ m/2[\dot z^2 - 2\ell\dot\theta\dot z \sin\theta + 2\epsilon\,\omega\dot z \sin\omega t \sin\zeta]$$
$$V^* = (M+m)g[\epsilon \cos\omega t \sin\zeta + \ell(1-\cos\theta)] + kz^2/2 \,. \tag{1}$$

Applying Lagrange's equations for time varying constraints, the equations of motion are found to be:

$$\ddot\theta + g/\ell \sin\theta = \eta/\ell\, \ddot z \sin\theta + \epsilon/\ell\, \omega^2 \cos\omega t \cos(\theta-\zeta) \tag{2a}$$

$$\ddot z + k/m\ z = \ell\ddot\theta \sin\theta + \ell\,\dot\theta^2 \cos\theta - \epsilon\omega^2 \cos\omega t \sin\zeta \tag{2b}$$

where $\eta = m/(M+m)$

Using (2b) to eliminate $\ddot z$ from (2a), and (2a) to eliminate $\ddot\theta$ from (2b), the following are obtained:

$$\ddot{\theta}(1-\eta\sin^2\theta)+g/\ell\,\sin\theta = \eta\dot{\theta}^2\sin\theta\,\cos\theta-k\eta/m\ell\;z\,\sin\theta$$
$$+\epsilon\,\omega^2/\ell\,\cos\omega t[\cos\theta\,\cos\zeta+(1-\eta)\sin\theta\sin\zeta] \tag{3}$$
$$\ddot{z}\,(1-\eta\sin^2\theta)+k/m\,z = \ell\dot{\theta}^2\cos\theta - g\sin^2\theta$$
$$+\epsilon\,\omega^2\cos\omega t\cos\theta[\sin\theta\cos\zeta-\cos\theta\,\sin\zeta]$$

If small angular displacements are assumed such that $\theta^2 << 1$, $\sin\theta \cong \theta$ and $\cos\theta \cong 1$ (note $\eta \leq 1$) equations (3) reduce to:

$$\ddot{\theta}+g/\ell\,\theta = \eta\dot{\theta}^2\theta-k\eta/m\ell\;z\theta+\epsilon\,\omega^2/\ell\,\cos\omega t[\cos\zeta+(1-\eta)\theta\,\sin\zeta]$$
$$\ddot{z}+k/m\,z = \ell\dot{\theta}^2-g\theta^2+\epsilon\,\omega^2\cos\omega t\,[\theta\cos\zeta-\sin\zeta] \tag{4}$$

Now if the following nondimensional terms are introduced:

$\bar{\theta} = \sqrt{\eta}\,\theta$, $\bar{z} = \eta z/\ell$, $\tau = \omega t$, $\mu = \epsilon/\ell$, $a^2 = g/\omega^2\ell$, $\beta^2 = k/\omega^2 m$

$\lambda = \sqrt{\eta}\cos\zeta$, $h = (1-\eta)\sin\zeta$, $\delta = \eta\sin\zeta$, $\frac{d(\;)}{d\tau} = (\;)'$, $\frac{d^2(\;)}{d\tau^2} = (\;)''$,

the equations of motion become:

$$\bar{\theta}'' + a^2\bar{\theta} = \bar{\theta}'^2\bar{\theta} - \beta^2\bar{z}\bar{\theta} + \mu\,\bar{\theta}\,h\cos\tau + \mu\lambda\cos\tau$$
$$\bar{z}'' + \beta^2\bar{z} = \bar{\theta}'^2 - a^2\bar{\theta}^2 + \mu\,\bar{\theta}\lambda\cos\tau - \mu\,\delta\cos\tau \tag{5}$$

For later convenience the following substitution is made into equations (5).

$$\overline{\theta} = \mu x + \mu U \cos \tau$$
$$\overline{z} = \mu y + \mu V \cos \tau \qquad (6)$$

This results in:

$$x'' + \alpha^2 x = \mu f(\tau, x, x', y, y') + \mu^2 F(\tau, x, x', y, y')$$
$$y'' + \beta^2 y = \mu g(\tau, x, x', y, y'), \qquad (7)$$

where

$$f(\tau, x, x', y, y') = [(Uh-\beta^2 UV)/2 - \beta^2 xy + (xh-\beta^2 Uy-\beta^2 Vx)\cos \tau$$
$$+ (Uh-\beta^2 UV)/2 \cos 2\tau] , \qquad (8a)$$

$$F(\tau, x, x', y, y') = [xx'^2 + xU^2/2 - 2xx'U \sin \tau - x'U^2 \sin 2\tau + (x'^2 U + U^3/4)\cos \tau$$
$$-xU^2/2 \cos 2\tau - U^3/4 \cos 3\tau] \qquad (8b)$$

$$g(\tau, x, x', y, y') = [(x'^2 - \alpha^2 x^2) - 2x'U \sin \tau + (x\lambda - 2\alpha^2 xU)\cos \tau$$
$$+ (U\lambda - \alpha^2 U^2 - U^2)/2 \cos 2\tau] , \qquad (8c)$$

and where

$$U = \lambda / (\alpha^2 - 1), \qquad V = \delta / (1-\beta^2)$$

Method of Solution

The results obtained by Struble and Heinbockel suggest using an asymptotic method of analysis of the forced vibration equations. Minorsky (1, p. 356-367) presents a general asymptotic method for nonautonomous sytems. The method used here is extended to two degrees of freedom but the procedure will in general be the same.

Equations (7) are of the form discussed by Minorsky. A solution in the form of a power series in the small parameter μ, is sought:

$$\begin{aligned} x &= a\cos\psi + \mu\, u(a,b,\psi,\phi,T) + \mu^2[\quad] + \cdots \\ y &= b\cos\phi + \mu\, w(a,b,\psi,\phi,T) + \mu^2[\quad] + \cdots \end{aligned} \tag{9}$$

where a, b, ψ, ϕ, are variational parameters which are expected to vary slowly with time.

$$\begin{aligned} a' &= \mu A(a,b) + \mu^2[\quad] + \cdots \\ b' &= \mu B(a,b) + \mu^2[\quad] + \cdots \end{aligned} \tag{10}$$

$$\begin{aligned} \psi' &= \alpha + \mu C(a,b) + \mu^2[\quad] + \cdots \\ \phi' &= \beta + \mu D(a,b) + \mu^2[\quad] + \cdots \end{aligned} \tag{11}$$

The left side of equations (7) can then be expanded in powers of μ.[2]

---

[2] Subscripts denote partial differentiation (e.g. $u_{xx} = \partial^2 u/\partial x^2$).

$$x'' + \alpha^2 x = \mu[\alpha^2 u + \alpha^2 u_{\psi\psi} + \beta^2 u_{\phi\phi} + u_{\tau\tau} + 2\alpha\beta u_{\psi\phi} + 2\alpha u_{\psi\tau} + 2\beta u_{\phi\tau}$$

$$-2\alpha A \sin\psi - 2\alpha a C \cos\psi] + \mu^2[\quad] + \cdots$$

(12)

$$y'' + \beta^2 y = \mu[\beta^2 w + \beta^2 w_{\phi\phi} + \alpha^2 w_{\psi\psi} + w_{\tau\tau} + 2\alpha\beta w_{\psi\phi} + 2\beta w_{\phi\tau} + 2\alpha w_{\psi\tau}$$

$$-2\beta B \sin\phi - 2\beta b D \cos\phi] + \mu^2[\quad] + \cdots$$

The right side of the same equations take the form:

$$\mu f(\tau, x, x', y, y') + \mu^2 F(\tau, x, x', y, y') = \mu f_0(a, b, \psi, \phi, \tau) + \mu^2[\quad] + \cdots$$

(13)

$$\mu g(\tau, x, x', y, y') = \mu g_0(a, b, \psi, \phi, \tau) + \mu^2[\quad] + \cdots$$

where

$$f_0(a, b, \psi, \phi, \tau) = (Uh - \beta^2 UV)/2 - \beta^2 ab\cos\psi\cos\phi + (ha\cos\psi - \beta^2 Ub\cos\phi$$

(14a)

$$-\beta^2 Va\cos\psi)\cos\tau + (Uh - \beta^2 UV)/2\cos 2\tau$$

$$g_0(a, b, \psi, \phi, \tau) = (-a^2\alpha^2\cos 2\psi) + (2Ua\alpha\sin\psi)\sin\tau + (a\lambda\cos\psi - 2\alpha^2 aU\cos\psi)\cos\tau$$

(14b)

$$+ (U\lambda - U^2 - U^2\alpha^2)/2\cos 2\tau .$$

If $u(a, b, \psi, \phi, \tau)$ and $w(a, b, \psi, \phi, \tau)$ are expressed in triple Fourier series:[3]

---

[3] All summations are taken from $-\infty$ to $+\infty$.

$$u = \sum_n \sum_p \sum_s f_{nps}(a,b) \exp(i\theta_{nps}) \tag{15a}$$

$$w = \sum_n \sum_p \sum_s g_{nps}(a,b) \exp(i\theta_{nps}) \tag{15b}$$

$$\theta_{nps} = n\tau + p\psi + s\phi$$

and are substituted into (12) which together with equations (13) are inserted into (7), the result is:

$$\sum_n \sum_p \sum_s f_{nps}(a,b)[\alpha^2 - (p\alpha + s\beta + n)^2] \exp(i\theta_{nps})$$

$$= f_0 + 2\alpha\, aC\cos\psi + 2\alpha A \sin\psi$$

$$\sum_n \sum_p \sum_s g_{nps}(a,b)[\beta^2 - (p\alpha + s\beta + n)^2] \exp(i\theta_{nps})$$

$$= g_0 + 2\beta\, bD \cos\phi + 2\beta B \sin\phi \;.$$

Only terms of the first order of μ have been retained after substitution.

The periodic functions f_0 and g_0 can also be expressed in triple Fourier series.

$$f_0 = \sum_n \sum_p \sum_s f^{(0)}_{nps}\ (a, b)\ \exp\ (i\ \theta_{nps})$$

$$g_0 = \sum_n \sum_p \sum_s g^{(0)}_{nps}\ (a, b)\ \exp\ (i\ \theta_{nps}) \tag{17}$$

where

$$f^{(0)}_{nps}(a, b) = 1/8\ \pi^3 \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} f_0 \exp\ (-i\ \theta_{nps}) d\tau d\psi d\phi$$

$$g^{(0)}_{nps}(a, b) = 1/8\pi^3 \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} g_0 \exp\ (-i\ \theta_{nps}) d\tau d\psi d\phi \tag{18}$$

Hence equations (16) become:

$$\sum_n \sum_p \sum_s f_{nps}(a, b)[\ \alpha^2 - (p\alpha + s\beta + n)^2]\ \exp\ (i\ \theta_{nps})$$

$$= 2\alpha\ aC\ \cos\psi + 2\alpha\ A\ \sin\psi + \sum_n \sum_p \sum_s f^{(0)}_{nps}(a, b)\ \exp\ (i\ \theta_{nps})$$

$$\sum_n \sum_p \sum_s g_{nps}(a, b)[\ \beta^2 - (p\alpha + s\beta + n)^2]\ \exp\ (i\ \theta_{nps}) \tag{19}$$

$$= 2\beta\ bD\ \cos\phi + 2\beta\ B\sin\phi + \sum_n \sum_p \sum_s g^{(0)}_{nps}\ (a, b)\ \exp\ (i\ \theta_{nps})\ .$$

If in equations (19) $f_{nps}(a,b)$ and $g_{nps}(a,b)$ are chosen such that

$$f_{nps}(a,b) = f^{(0)}_{nps}(a,b)/[\alpha^2 - (p\alpha + s\beta + n)^2]$$

$$\alpha^2 - (p\alpha + s\beta + n)^2 \neq 0$$

$$g_{nps}(a,b) = g^{(0)}_{nps}(a,b)/[\beta^2 - (p\alpha + s\beta + n)^2] \tag{20}$$

$$\beta^2 - (p\alpha + s\beta + n)^2 \neq 0 ,$$

then to eliminate resonant terms in (19), A, B, C, and D are chosen such that

$$2\alpha a C \cos\psi + 2\alpha A \sin\psi = -\sum_n \sum_p \sum_s f^{(0)}_{nps}(a,b) \exp(i\theta_{nps})$$

$$\alpha^2 - (p\alpha + s\beta + n)^2 = 0 \tag{21a}$$

$$2\beta b D \cos\phi + 2\beta B \sin\phi = -\sum_n \sum_p \sum_s g^{(0)}_{nps}(a,b) \exp(i\theta_{nps})$$

$$\beta^2 - (p\alpha + s\beta + n)^2 = 0 . \tag{21b}$$

Equation (21a), with $p = 1$, $s = n = 0$, is satisfied when $A = C = 0$. Equation (21b), letting $s = 1$, $p = n = 0$, is satisfied when $B = D = 0$. Using (15a,b), (20), (18), and (14a,b) to solve for u and w, after considerable manipulation, yields:

$$
\begin{aligned}
u = {} & [(Uh - \beta^2 UV)/2\alpha^2] + [(Uh - \beta^2 UV)/2(\alpha^2 - 4)] \cos 2\tau \\
& - [(ha - \beta^2 Va)/2(2\alpha+1)] \cos(\tau+\psi) + [(ha - \beta^2 Va)/2(2\alpha-1)] \cos(\tau-\psi) \\
& - [\beta^2 Ub/2(\alpha^2 - (1+\beta)^2)] \cos(\tau+\phi) - [\beta^2 Ub/2(\alpha^2 - (1-\beta)^2)] \cos(\tau-\phi) \\
& + [\beta ab/2(2\alpha+\beta)] \cos(\psi+\phi) - [\beta ab/2(2\alpha-\beta)] \cos(\psi-\phi)
\end{aligned}
\tag{22a}
$$

$$
\begin{aligned}
w = {} & [(U\lambda - U^2(1+\alpha^2))/2(\beta^2-4)] \cos 2\tau + [(a\lambda - 2\alpha aU(\alpha+1))/2(\beta^2 - (1+\alpha)^2)] \\
& \cos(\tau+\psi) + [(a\lambda - 2\alpha aU(\alpha-1))/2(\beta^2 - (1-\alpha)^2)] \cos(\tau-\psi) \\
& - [\alpha^2 a^2/(\beta^2 - 4\alpha^2)] \cos 2\psi .
\end{aligned}
\tag{22b}
$$

Equations (22) should be valid provided that parameters are such that a near resonance condition does not occur. If $\beta = 2\alpha$, the resonant terms in (22) can be removed by going back to equations (21) and solving once again for A, B, C, and D, by including terms with $p = 1$, $s = -1$, $n = 0$ in 21a and terms with $p = 2$, $n = s = 0$ in 21b. Doing this results in a set of equations almost identical to those solved by Struble and Heinbockel (5) and would yield similar results for the variational parameters.

Combining (22) with (9), (10), and (11) gives a solution for

x and y to the first order of μ. If this result and equations (6) are combined, the nondimensional parameters $\bar{\theta}$ and $\bar{z}$, which are of primary interest, follow as:

$$
\begin{aligned}
\bar{\theta} = {} & \mu(\eta \cos\zeta/(\alpha^2-1)] \cos \tau + \mu\, a \cos\psi + \mu^2\{[\sqrt{\eta} \sin 2\zeta((1-\eta)-\beta^2)/4\alpha^2(\alpha^2-1)(1-\beta^2)] \\
& +[\sqrt{\eta} \sin 2\zeta((1-\eta)-\beta^2)/4(\alpha^2-1)(\alpha^2-4)(1-\beta^2)] \cos 2\tau \\
& +[a \sin\zeta((1-\eta)-\beta^2)/2(2\alpha+1)(\beta^2-1)] \cos(\tau+\psi) \\
& +[a \sin\zeta((1-\eta)-\beta^2)/2(2\alpha-1)(\beta^2-1)] \cos(\tau-\psi) \\
& -[b\beta^2\sqrt{\eta} \cos\zeta/2(\alpha^2-1)(\alpha^2-(1+\beta)^2)] \cos(\tau+\phi) \\
& -[b\beta^2\sqrt{\eta} \cos\zeta/2(\alpha^2-1)(\alpha^2-(1-\beta^2)] \cos(\tau-\phi) \\
& +[\beta\, ab/2(2\alpha+\beta)] \cos(\psi+\phi)-[\beta\, ab/2(2\alpha-\beta)] \cos(\psi-\phi)\} \\
& +\mu^3\{\ \}+\cdots
\end{aligned}
\tag{23a}
$$

$$
\begin{aligned}
\bar{z} = {} & \mu[\eta \sin\zeta/(1-\beta^2)] \cos\tau + \mu b \cos\phi + \mu^2\{[\eta \cos^2\zeta/(\alpha^2-1)^2(4-\beta^2)] \cos 2\tau \\
& +[a\sqrt{\eta} \cos\zeta(\alpha+1)^2/2(1-\alpha^2)(\beta^2-(1+\alpha)^2)] \cos(\tau+\psi) \\
& +[a\sqrt{\eta} \cos\zeta(\alpha-1)/2(1-\alpha^2)(\beta^2-(1-\alpha)^2)] \cos(\tau-\psi) \\
& +[\alpha^2 a^2/(4\alpha^2-\beta^2)] \cos 2\psi\} + \mu^3\{\ \}+\cdots
\end{aligned}
\tag{23b}
$$

where

$$
\begin{aligned}
& a' = b' = 0 \\
& \psi' = \alpha \\
& \phi' = \beta.
\end{aligned}
\tag{24}
$$

Steady State Solution

When $\mu = 0$ in equations (7), the "zero" order solution (9) is given by:

$$x = a \cos \psi$$

$$y = b \cos \phi \ .$$

These are complementary solutions, where a and b are constants depending upon initial conditions. With a small amount of damping present this motion would gradually die out. According to the results obtained for the first order solution in μ, the variational parameters (a and b) are still constants provided a near resonance condition does not exist. It might be expected then, and will be assumed here, that damping in the system will cause terms containing a factor a or b to gradually die out. On this basis a steady state solution is constructed and will be compared later with experimental work. The simplified equations (23a, b) as $a \rightarrow 0$ and $b \rightarrow 0$ become:

$$\overline{\theta}_{ss} = \mu[\sqrt{\eta}\cos\zeta\,\Omega^2/(1-\Omega^2)]\;\cos\;\tau$$

$$\mu^2[\sqrt{\eta}\sin 2\zeta(\Delta^2+(\eta-1)\Omega^2)\Omega^4/2(1-\Omega^2)(\Delta^2-\Omega^2)(1-4\Omega^2)]\;(\cos^2\tau-2\Omega^2) \tag{25}$$

$$\overline{z}_{ss} = \mu[\eta\sin\zeta\,\Omega^2/(\Omega^2-\Delta^2)]\;\cos\;\tau+\mu^2[\eta\cos^2\zeta\,\Omega^6(1-\Omega^2)^2(4\Omega^2-\Delta^2)]\cos 2\tau,$$

where the substitution $\Omega = 1/a$, and $\Delta = \beta/a$, have been used.

IV. EXPERIMENTATION

Apparatus

A logical means of checking analytical results is to construct a mechanical model and observe the motion. As a part of this study, a model was built in such a manner that harmonic amplitude variation could be applied to the supports. The mechanical model was patterned after the idealized model pictured in Figure 3. A variable speed motor supplied motion to the supports.

Such parameters as amplitude of vibration (ϵ), angle of vibration (ζ), pendulum natural frequency ($\sqrt{g/\ell}$), and spring mass natural frequency ($\sqrt{k/m}$), were made adjustable so that various combinations of these could be tested.

The completed apparatus is pictured in Figure 4. A straight line linkage constrained the sprung mass to move vertically. Small, lightly oiled ball bearings were incorporated at all moving joints to minimize friction and damping.

Instrumentation

It was decided upon to measure amplitude of pendulum swing (θ), and spring mass deflection (z) to verify analytical results. Photocells provided a simple and economical means of

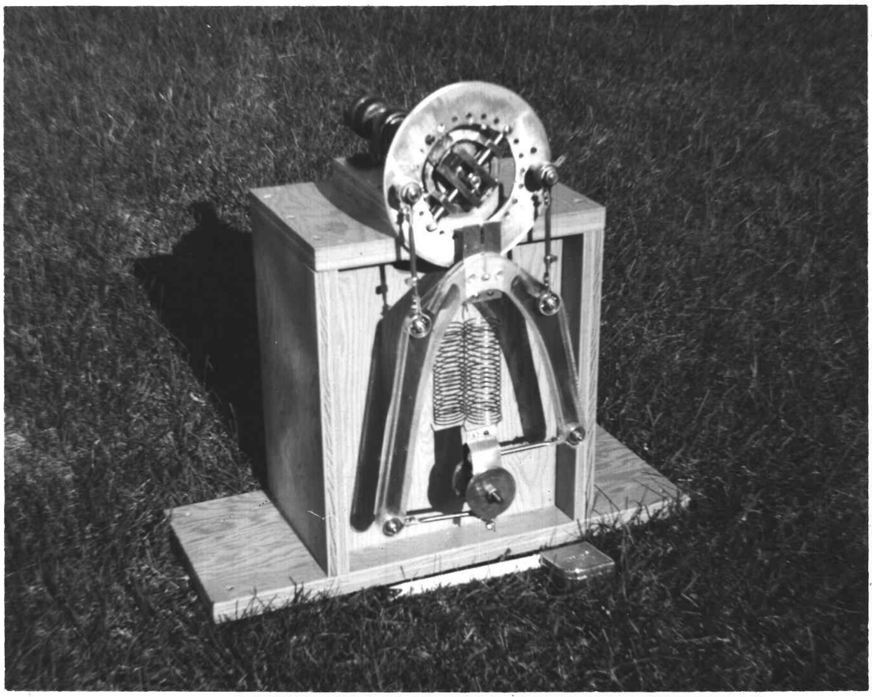


Figure 4. System used for experimentation.

instrumentation for this project. A shield fastened to the moving part, of which the motion is to be measured, was used to intercept light passing from a spotlight to a selenium photocell. As the shield varied in position the variation of illumination on the photocell caused the output current to change. By masking the area of the photocell, the output current could be made nearly proportional to deflection. A typical schematic of the circuit involved is shown in Figure 5.

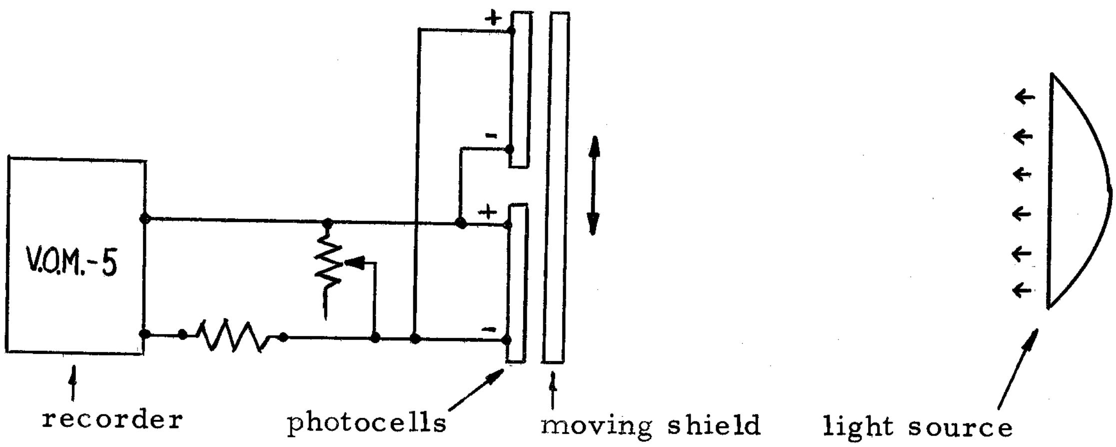


Figure 5. Typical instrumentation circuit.

The variable resistor provides a means of adjusting the current for scale purposes. Bausch and Lomb V. O. M. -5 strip chart recorders were used to make a permanent record of the sinusoidally varying output current. Both the pendulum and the sprung mass motions were later analyzed and measured from these records.

Procedure

Maximum amplitude as a function of frequency will be of main interest here. Frequency of vibration was variable from 0 to 6.7 cycles per second. For a given set of parameters (fixing μ, ℓ, Δ, η) the frequency was varied and the maximum amplitudes (θ_m, z_m) were measured. The resulting amplitude frequency curves are shown in Chapter V.

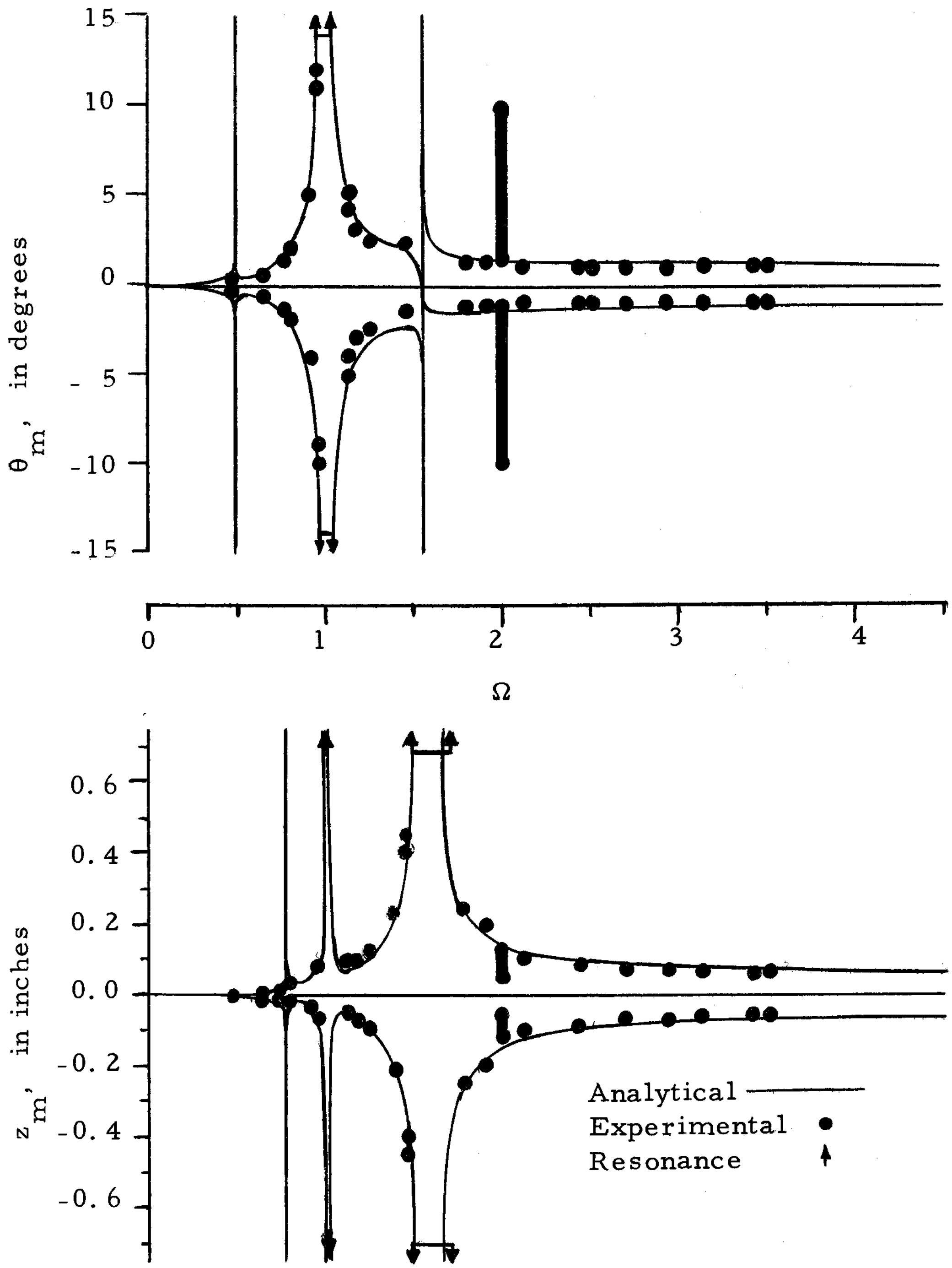
Pendulum swing was limited to ±20 degrees with an accuracy of about ±1 degree. The sprung mass deflection range was ±1 inch with an accuracy of about ±0.03 inch.

V. RESULTS

Experimental work was done to verify the steady-state solution presented in equations (25). Since maximum amplitude of vibration is usually of primary importance, a comparison is made of maximum amplitude results obtained by analytical and experimental procedure. Equations (25), along with the procedure described in the appendix, were used to calculate maximum positive and maximum negative values of θ and z as functions of the nondimensional frequency ratio Ω.

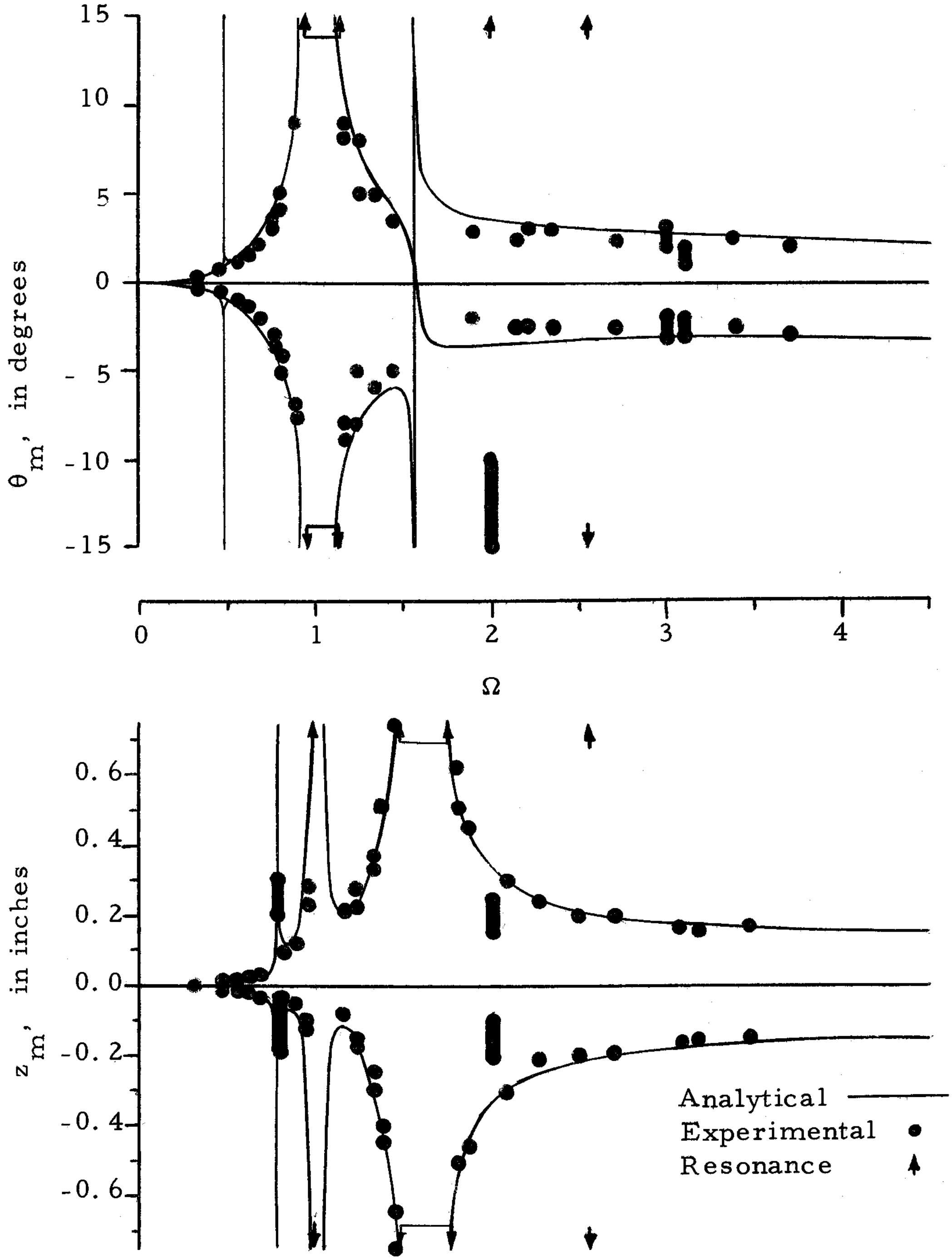
The most interesting cases of forced vibration occur when $\zeta \neq n\pi/2$, $n = 0, 1, 2, \cdots$, since all terms of equations (25) are retained. Choosing $\zeta = \pi/4$ and fixing parameters $\ell = 2.85$ inches, $\Delta = 1.56$, and $\eta = 0.53$, curves were plotted for the three different values of μ shown in Figures 6, 7, and 8. These parameters are within the limits of the mechanical system and will give some idea of the effect of the size of μ on the results. The analytical curves along with points of experimental data are shown in these figures.

Next choosing $\zeta = 0$, only the second order μ term in the expression for $\overline{z}_{ss}$ remains. With $\ell = 2.85$ inches, $\Delta = 1.56$, and $\eta = 0.53$, curves were plotted and data were taken as shown in Figure 9. The analytical curves are symmetric about the abscissa so the absolute values of θ_m and z_m are presented.



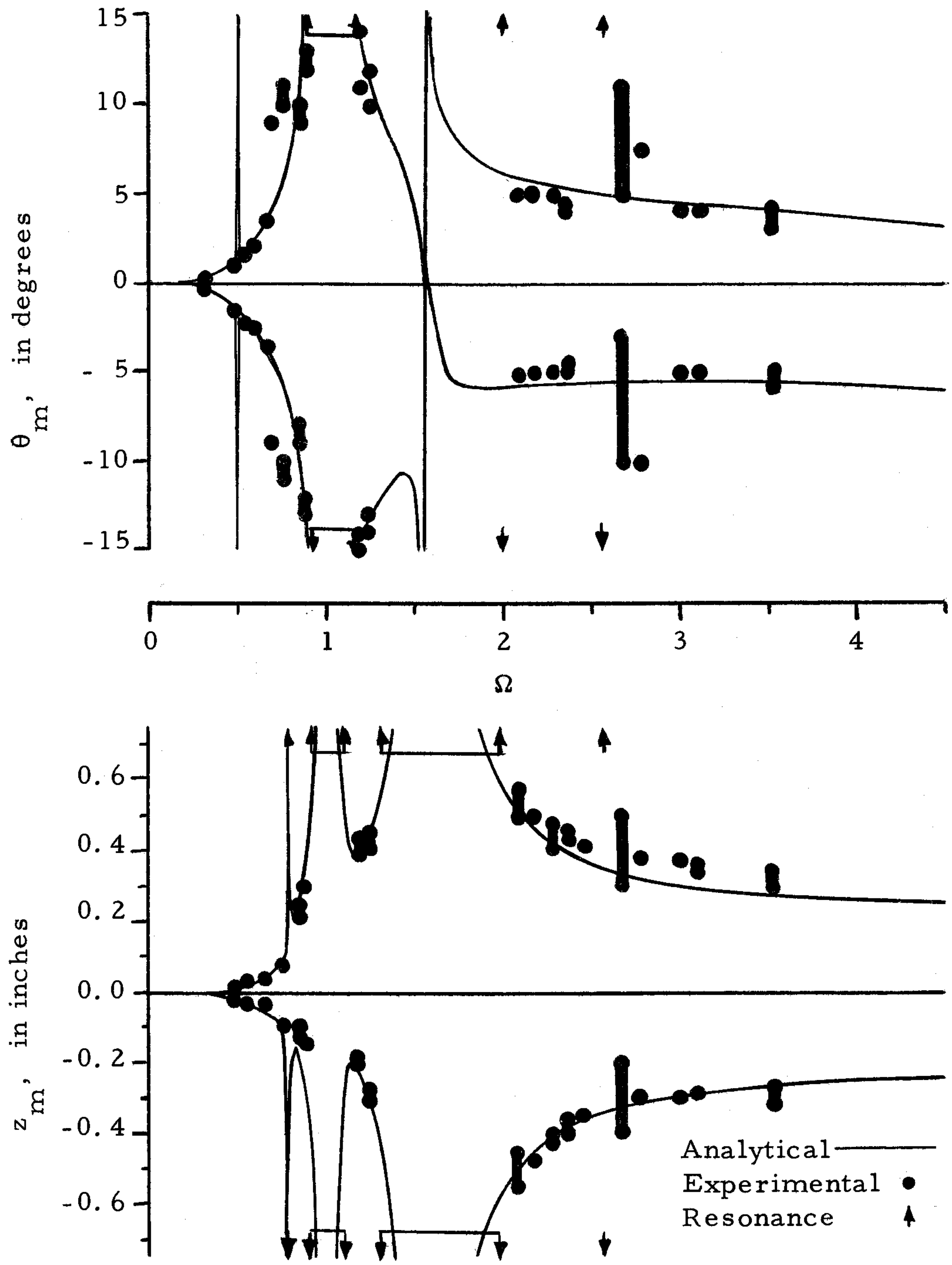
$\zeta = \pi/4$, $\Delta = 1.56$, $\mu = 0.0274$, $\ell = 2.85$ in.

Figure 6. Maximum amplitudes (θ_m, z_m) as functions of frequency ratio Ω.



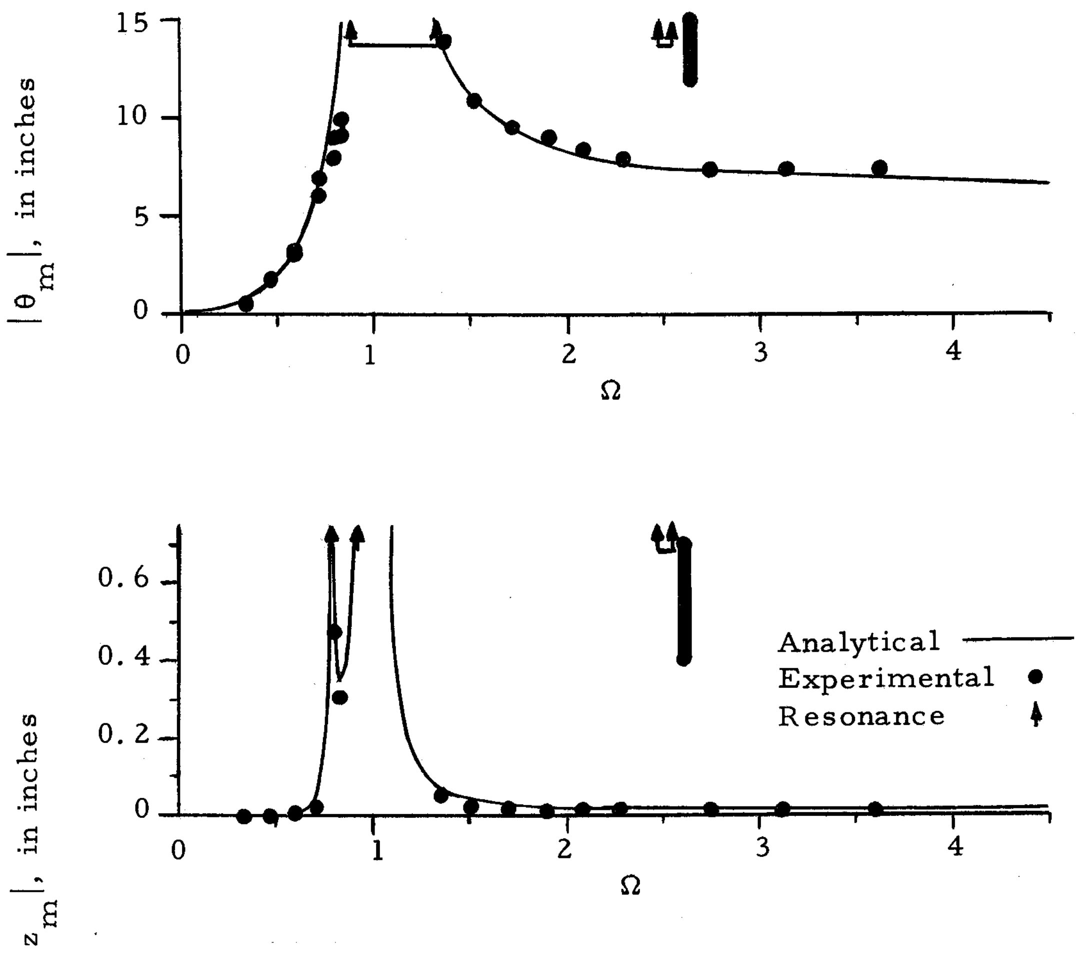
$\zeta = \pi/4$, $\Delta = 1.56$, $\mu = 0.0658$, $\ell = 2.85$ in.

Figure 7. Maximum amplitudes (θ_m, z_m) as functions of frequency ratio Ω.



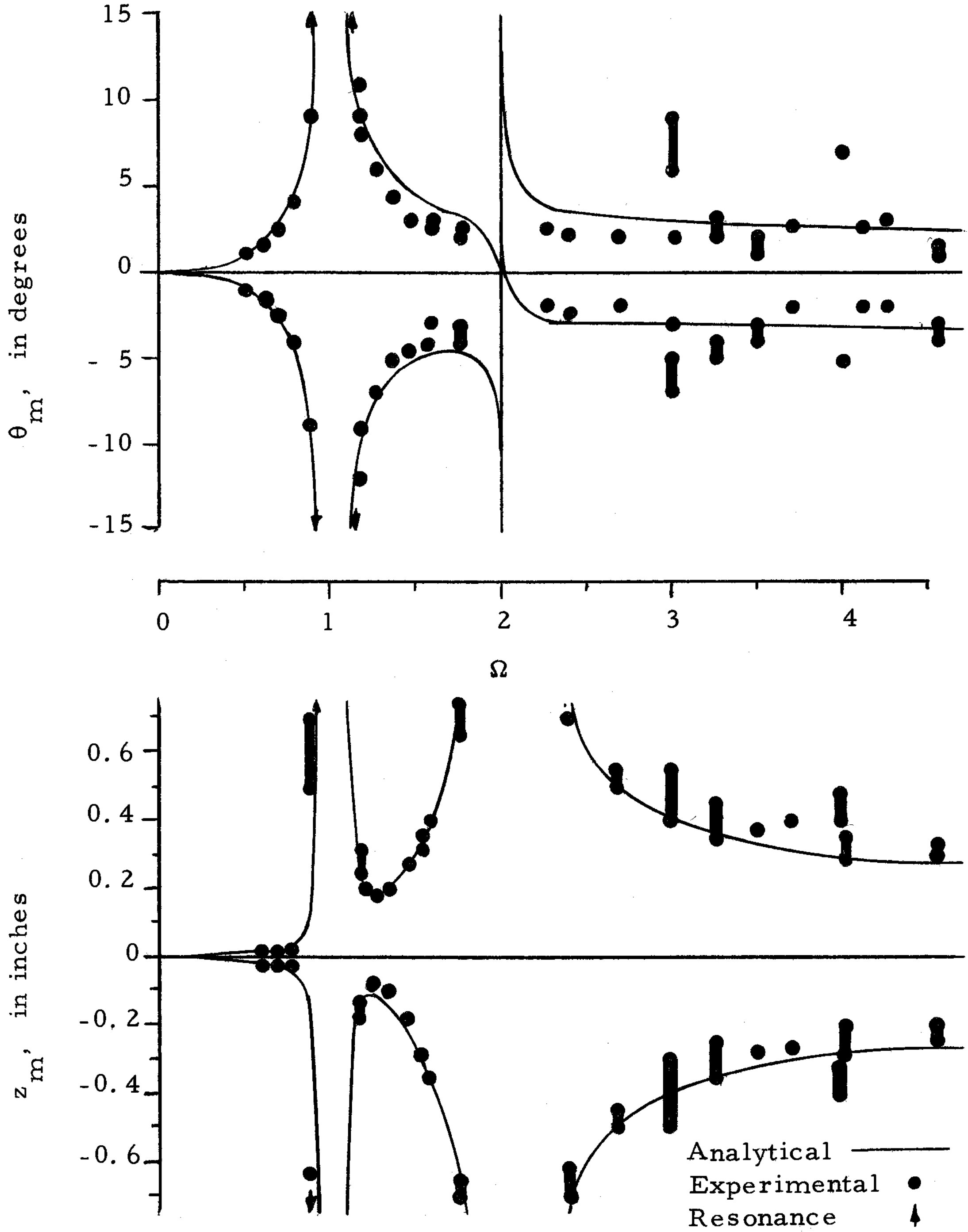
$\zeta = \pi/4$, $\Delta = 1.56$, $\mu = 0.1097$, $\ell = 2.85$ in.

Figure 8. Maximum amplitudes (θ_m, z_m) as functions of frequency ratio Ω.



$\zeta = 0$, $\Delta = 1.56$ $\mu = 0.1097$, $\ell = 2.85$ in.

Figure 9. Maximum amplitudes $(|\theta_m|, |z_m|)$ as functions of frequency ratio Ω.



$\zeta = \pi/4, \quad \Delta = 2.00, \quad \mu = 0.0668, \quad \ell = 4.68$ in.

Figure 10. Maximum amplitudes (θ_m, z_m) as functions of frequency ratio Ω.

Because the case when $\Delta = 2$ is of so much interest in free vibration, it was chosen for investigation of forced vibration. With $\ell = 4.68$ inches, $\Delta = 2$, $\eta = 0.53$, and $\mu = 0.0668$, curves and data are shown in Figure 10.

Experimental values, represented as large dots, and experimental resonance, indicated by arrows, are shown superimposed on the analytical curves for the various parameters. In some instances the steady state amplitudes of vibration appeared in the form of "beats." This periodic variation in amplitude is represented on the curves as two dots connected by a solid line.

VI. DISCUSSION AND CONCLUSION

The experimental results presented in Chapter V compare quite favorably with those predicted by the steady state solution (25). The photocell instrumentation used was sensitive but difficult to keep calibrated in some cases. Results, however, are fairly well within the limits of instrument error mentioned in Chapter IV.

As can be seen from the experimental results presented, resonance did not occur everywhere predicted; and instability occurred sometimes where it was not predicted. According to equations (25), resonance should occur when $\Omega = 1/2$ and $\Omega = \Delta/2$. In Figures 8 and 9 an unstable condition was observed when $\Omega = \Delta/2$. As μ is decreased this instability becomes a large amplitude as in Figure 7 and nearly unnoticeable in Figure 6.

No resonance was observed when $\Omega = 1/2$ in any of the cases studied. A slight amount of distortion of harmonic motion was present, but there was little effect on the amplitude. The absence of these resonance effects might be explained by the fact that the resonant peaks are extremely narrow and the small amount of damping would cancel their effect.

Experimental results indicate resonance occurring where it was not expected according to equations (25). In particular, Figure 10 shows resonance at $\Omega = 3, 4$; Figure 9 at $\Omega = 1 + \Delta$; Figure 8 at

$\Omega = 2,\ 1 + \Delta$; and Figure 7 at $\Omega = 2,\ 1 + \Delta$. These peaks decrease in effect as μ is made smaller, which is to be expected. They are probably predictable by higher order μ terms if the approximate solution were carried out retaining these terms. If μ is kept small enough, these effects appear to be negligible when a slight amount of damping is present, as was the case in the experimental model used here.

It is interesting to note that the second term on the right side of top equation (25) indicates that $\overline{\theta}_{ss}$ becomes a very large negative number as Ω becomes large, provided $\eta \neq 1$. If $\eta = 1$ this term converges. This phenomenon seems reasonable since $\eta \neq 1$ indicates the presence of mass (M) in the rigid beam of the idealized system. For large frequencies of oscillation the acceleration of this mass becomes very large thus giving rise to large displacements. When $\eta = 1$ the only mass in the system appears as the mass isolated by the spring and the action of the hangers so that large accelerations are not present.

Large enough frequencies of oscillation were not available for doing extensive studies of this behavior; however, in Figure 7 and 8 a noticeable shift in this direction is observed. The effect could be made more apparent by either increasing the frequency ratio or decreasing η.

It would be difficult here to give a specific range of μ where the analytical and experimental results are in good agreement since only a few examples were tried. The results obtained, however, do show good agreement except near the unpredicted resonant peaks. For the smaller values of μ chosen, the agreement is even better, as would be expected.

VII. SUMMARY AND RECOMMENDATIONS

Using an asypmtotic method described in Minorsky (1, p. 356-367) an approximate solution to the equations of motion has been obtained for the beam-pendulum system. Damping was neglected and small angular deflections assumed. The approximation, involving only first and second order terms in the series development in powers of the amplitude ratio μ, was reduced to a steady state solution.

With the aid of a mechanical model, maximum amplitude of pendulum swing and mass deflection data were plotted as functions of frequency ratio Ω. Comparison of the experimental results with the analytical results showed good agreement except for what are thought to be higher order resonant effects. These resonant peaks occurred at forcing frequency ratios of $\Omega = 2,\ 1+\Delta,\ 2\Delta$. As μ was made smaller these effects diminished in intensity.

Further analytical work might be carried out to obtain higher order approximations. A stability analysis would also be interesting. In the experimental area pendulum deflection at higher frequency might prove interesting when $\eta \neq 1$. If more experimental work is carried out, it is recommended that a more accurate method of instrumentation be used.

A beam-pendulum natural frequency ratio (Δ) greater than two might have more practical interest than those ratios studied here.

It seems that the natural frequency of a beam would be much greater than that of the pendulum type hangers.

BIBLIOGRAPHY

1. Minorsky, Nicholas. Nonlinear oscillations. Princeton, Van Nostrand, 1962. 714 p.

2. Sevin, Eugene. On the parametric excitation of pendulum-type vibration absorber. Journal of Applied Mechanics 83: 330-334. 1961.

3. Struble, Raimond A. Nonlinear differential equations. New York, McGraw-Hill, 1962. 267 p.

4. Struble, R. A. and J. H. Heinbockel. Energy transfer in a beam-pendulum system. Journal of Applied Mechanics 84: 590-592. 1962.

5. ________________. Resonant oscillations of a beam pendulum system. Journal of Applied Mechanics 85: 181-188. 1963.

APPENDIX

APPENDIX

The maximum positive and negative values of the following functions of τ are listed below.

$$A \cos \tau + B \cos 2\tau$$

		Maximum positive	Maximum negative
$\lvert A \rvert \geq 4 \lvert B \rvert$		$\lvert A \rvert + B$	$B - \lvert A \rvert$
$\lvert A \rvert < 4 \lvert B \rvert$	$B > 0$	$\lvert A \rvert + B$	$-(A^2+8B^2)/8B$
	$B < 0$	$-(A^2+8B^2)/8B$	$B - \lvert A \rvert$

$$C \cos \tau + D(\cos^2 \tau - 2\Omega^2)$$

		Maximum positive	Maximum negative
$\lvert C \rvert \geq 2 \lvert D \rvert$		$\lvert C \rvert + D(1-2\Omega^2)$	$D(1-2\Omega^2) - \lvert C \rvert$
$\lvert C \rvert < 2 \lvert D \rvert$	$D > 0$	$\lvert C \rvert + D(1-2\Omega^2)$	$-(C^2+8D^2\Omega^2)/4D$
	$D < 0$	$-(C^2+8D^2\Omega^2)/4D$	$D(1-2\Omega^2) - \lvert C \rvert$